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Some Results About Value Sets of Quadratic Forms Over Fields.

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SOME RESULTS ABOUT VALUE SETS OF QUADRATIC FORMS OVER FIELDS

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ABSTRACT

In 1979, Solow defined the square class invariant of a quadratic form \( q \) over a field \( F \) to be a function from the square classes of \( F \) into the integers. For each square class of \( F \), this function indicates the maximum number of coefficients in all diagonalized quadratic forms equivalent to \( q \) that lie in that square class. The intent of Chapter I is to determine the fields over which the square class invariant classifies quadratic forms. It will be proved that if the level of the field is at most two and if the square class invariant classifies the quadratic forms, then the field must be a C-field. Also, it will be shown that if the level of the field is at least four, then the square class invariant does not classify the quadratic forms.

In 1969, Kaplansky showed that a field over which the binary quadratic form value sets have maximum index two in the multiplicative group of the field has exactly two quaternion algebras. In Chapter II a characterization will be found for all fields over which the binary form value sets have maximum index four in the multiplicative group of the field. With one exceptional case, the answer will be that the field has exactly four quaternion algebras.
INTRODUCTION

A quadratic form over a field $F$ is defined as a homogeneous polynomial $f$ of degree 2, $f(x_1, x_2, \ldots, x_n) = \sum a_{ij} x_i x_j$. In this paper we will assume that all fields have characteristic not equal to 2, and thus we may assume $a_{ij} = a_{ji}$ and write $f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} a_{ii} x_i^2 + 2 \sum_{i<j} a_{ij} x_i x_j$. Two quadratic forms $f(x_1, \ldots, x_n)$ and $g(y_1, \ldots, y_m)$ over $F$ are equivalent (denoted $f \equiv g$) if $m = n$ and if there exist $c_{ij} \in F$ such that the matrix $[c_{ij}]$, $1 \leq i, j \leq n$ is invertible and if $f(x_1, \ldots, x_n) = g(y_1, \ldots, y_n)$ where $y_i = \sum_{j=1}^{n} c_{ij} x_j$, $1 \leq i \leq n$. This forms an equivalence relation on the quadratic forms over $F$.

A quadratic map on a vector space $V$ over a field $F$ is a map $Q : V \to F$ such that $Q(\alpha v) = \alpha^2 Q(v)$ for every $v \in V$ and $\alpha \in F$, and $B(u,v) = \frac{1}{2}[Q(u+v) - Q(u) - Q(v)]$ is a symmetric bilinear form. The pair $(V, Q)$ is called a quadratic space. Two quadratic spaces $(V, Q)$ and $(V', Q')$ are isometric (denoted $V \cong V'$) if there exists a bijective linear transformation $\sigma : V \to V'$ such that $Q(v) = Q'(\sigma(v))$ for every $v \in V$. 

v
Given a quadratic form $f(x_1, \ldots, x_n)$, we associate with it a quadratic space. Let $V$ be any $n$-dimensional vector space over $F$. Let $\{v_1, \ldots, v_n\}$ be any basis for $V$ over $F$. Define $Q: V \to F$ by $Q(\alpha_1 v_1 + \ldots + \alpha_n v_n) = f(\alpha_1, \ldots, \alpha_n)$. Then $(V, Q)$ is a quadratic space. Any two quadratic spaces associated with $f$ are isometric.

Given a quadratic space $(V, Q)$, we associate with it a quadratic form. Let $\{v_1, \ldots, v_n\}$ be a basis for $V$ over $F$. Define $f(x_1, \ldots, x_n) = \sum_{i,j} B(v_i, v_j) x_i x_j$. Any two quadratic forms associated with $(V, Q)$ are equivalent.

**Theorem 0.1:** Let $(V, Q)$ and $(V', Q')$ be quadratic spaces over a field $F$ and $f$ and $f'$ be respective associated quadratic forms. Then $V = V'$ if and only if $f = f'$.

All of the definitions to follow will be stated in terms of quadratic forms. Similar definitions exist in terms of quadratic space.

Let $f(x_1, \ldots, x_n) = \sum_{i,j} a_{ij} x_i x_j$ and $g(y_1, \ldots, y_m) = \sum_{k,l} b_{k,l} y_k y_l$ be quadratic forms over a field $F$. The sum $f+g$ is defined as the form $(f+g)(x_1, \ldots, x_n, y_1, \ldots, y_m) = f(x_1, \ldots, x_n) + g(y_1, \ldots, y_m)$, and the product $fg$ is defined as
as the form \((f \cdot g)(z_1, z_2, \ldots, z_m, \ldots, z_n, z_1, z_2, \ldots, z_n) = \sum a_{ij}b_{k\ell}z_i z_j\). If \(n\) is a positive integer, we will denote by \(n \cdot g\) the form \(q + q + \ldots + q\) (\(n\) times). If \(q = q_1 + q_2\), \(q_1\) and \(q_2\) are called subforms of \(q\).

\(f = a_1 x_1^2\) is a 1-dimensional form. Every form \(f\) is equivalent to a sum of 1-dimensional forms denoted by \(f = \langle a_1 \rangle + \langle a_2 \rangle + \ldots + \langle a_n \rangle = \langle a_1, a_2, \ldots, a_n \rangle\). This diagonal representation is not unique. We will only consider forms \(f = \langle a_1, a_2, \ldots, a_n \rangle\) with \(a_i \neq 0\) for all \(i\). These forms are called regular or non-singular. Equivalently, a quadratic space \((V, Q)\) with symmetric bilinear form \(B\) is regular if for \(x \in V\), \(B(x, y) = 0\) for all \(y \in V\) implies that \(x = 0\).

If \(f = \langle a_1, a_2, \ldots, a_n \rangle\) is a regular form over \(F\), then the dimension of \(f\) is defined as \(\dim(f) = n\). The determinant of \(f\) is defined as \(\det f = \prod_{i=1}^{n} a_i F^2 \in \hat{F}/\hat{F}^2\) and is well-defined modulus \(\hat{F}^2\).

Let \(f\) be an \(n\)-dimensional form over \(F\). If there exist \(b_1, b_2, \ldots, b_n \in F\) such that \(f(b_1, b_2, \ldots, b_n) = b\), then it is said that \(b\) is a value of \(f\) or that \(f\) represents \(b\) over \(F\). The value set of \(f\), denoted \(D(f)\), is the set of non-zero values of \(f\). Equivalent forms have the same value set, and \(D(f)\) is a union of cosets of \(\hat{F}^2\) called square classes. If \(b \in D(f)\), then there exist \(b_2, b_3, \ldots, b_n \in F\) such that \(f = \langle b, b_2, b_3, \ldots, b_n \rangle\). If \(D(f) = \hat{F}\) (the multiplicative group of non-zero elements of \(F\)), \(f\) is called universal.
An n-dimensional form $f$ is isotropic over $F$ if there exist $a_1, a_2, \ldots, a_n \in F$, not all zero, such that 
$$f(a_1, a_2, \ldots, a_n) = 0.$$ Otherwise, $f$ is anisotropic over $F$. 
For example $x^2 - y^2 = \langle 1, -1 \rangle$ is isotropic over any field $F$. 
This quadratic form is called the hyperbolic plane. A form $f$ is isotropic if and only if $\langle 1, -1 \rangle$ is a subform of $f$. The following is the Witt Decomposition Theorem.

**Theorem 0.2:** Any regular form $f$ over $F$ can be written 
$$f = n\langle 1, -1 \rangle + f'$$ where $f'$ is an anisotropic form and $n$ is a non-negative integer. Moreover, $n$ is unique and $f'$ is unique (up to equivalence).

Also, $b \in D(f)$ if and only if $f + \langle -b \rangle$ is isotropic, and if $f$ is isotropic then $f$ is universal.

The following is another important theorem of Witt.

**Theorem 0.3 (Witt Cancellation Theorem):** Let $f_1, f_2$ be quadratic forms over a field $F$ and let $g_1, g_2$ be regular quadratic forms over $F$ with $g_1 \cong g_2$. If $f_1 + g_1 \cong f_2 + g_2$, then $f_1 \cong f_2$.

The set of similarity factors of a form $f$ over $F$, denoted $G(f)$, is defined by $G(f) = \{ a \in F \mid a \cdot f \cong f \}$. $G(f)$ is actually a group. An anisotropic form $f$ is called multiplicative if $G(f) = D(f)$. An isotropic form is called
multiplicative if it is a sum of hyperbolic planes. So the value set of a multiplicative form is a group. The form \(<1,a_1>\cdot<1,a_2>\cdots<1,a_n>\) is called an n-fold Pfister form. A 2-fold Pfister form is also called a quaternion form. A Pfister form is a multiplicative form.

A field \(F\) is called non-real if \(-1 \in D(n<1>)\) for some positive integer \(n\). Otherwise the field \(F\) is real. A field \(F\) is real if and only if \(F\) is ordered. If \(F\) is a non-real field, the level of \(F\) is defined as \(s(F) = \min\{n \in \mathbb{N} \mid -1 \in D(n<1>)\}\), and the u-invariant of \(F\) is defined as \(u(F) = \min\{n \in \mathbb{N} \mid \text{every form of dimension at least } n+1 \text{ is isotropic}\}\), which may be \(\infty\). Clearly \(s(F) \leq u(F)\).

A field \(F\) is called a C-field if there is a 1-1 correspondence between the anisotropic forms over \(F\) and their value sets. Thus if \(F\) is a C-field, there exists a unique anisotropic universal form over \(F\). Cordes showed in [4] that if \(F\) is a field with \(u(F) < \infty\) which has exactly one anisotropic universal form and if \(D(\phi) \subseteq D(\psi)\) for anisotropic \(\phi\) and \(\psi\), then \(\psi\) is a subform of \(\psi\). In particular, \(F\) is a C-field.

Another useful result is

Lemma 0.4: If \(F\) is a field with \(a,b \in F\), then \(D<1,a> \cap D<1,b> \subseteq D<1,-ab>\).
An algebra $A$ over a field $F$ is a vector space over $F$ which also has a ring structure with multiplicative identity $1_A$ satisfying $\alpha(xy) = (\alpha x)y = x(\alpha y)$ for every $x,y \in A$ and for every $\alpha \in F$. An algebra $A$ is a division algebra if $A$ is a division ring. The center of $A$ is $\{x \in A | xy = yx, \text{for every } y \in A\}$. $A$ is said to be central if the center of $A$ is equal to $F \cdot 1_A$. $A$ is said to be simple if $A$ as a ring contains no two-sided ideals other than $\{0\}$ and $A$. $M_n(F) = \{nxn \text{ matrices over } F\}$ is a central, simple algebra of $F$.

**Theorem 0.5 (Wedderburn):** Let $A$ be a finite dimensional central, simple algebra over $F$. Then there exists a central, simple division algebra $D$ and an integer $n$ such that $A \cong M_n(F) \otimes D$ (algebra isomorphism). Moreover, $n$ is unique and $D$ is unique up to isomorphism.

$D$ is called the division algebra component of $A$. Two central, simple algebras over $F$ are defined to be equivalent if their division algebra components are isomorphic. This is an equivalence relation, and these equivalence classes form a group under tensor product $\otimes$ called the Brauer group.

Let $a,b \in F$ and $A = [a,b]$ be the quaternion $F$-algebra on two generators $i,j$ with the defining relations: $i^2 = a$, $j^2 = b$, and $ij = -ji$. Then $A$ is four-dimensional over $F$ with basis $\{1, i, j, k = ij\}$. For $x = a + bi + yj + \delta k \in A$, define the conjugate of $x$ to be $\overline{x} = a - (bi + yj + \delta k)$. Define the trace
T: A → F and the norm N: A → F as follows: T(x) = x + \overline{x} and N(x) = x \cdot \overline{x}, for every x ∈ A. B: A × A → F defined by B(x, y) = \frac{1}{2} T(xy) is a symmetric bilinear form on A. So (A, Q) is a quadratic space where Q: A → F is defined by Q(x) = B(x, x) for every x ∈ A. The quadratic form associated with (A, Q) is Q(x) = B(x, x) = \frac{1}{2} T(xx) = \frac{1}{2} 2xx = xx = N(x). So the norm is a quadratic form on A. The quadratic space (A, Q) is regular and the associated form is equivalent to <1, -a, -b, ab>.

Quaternion algebras are central, simple algebras and thus are elements of the Brauer group. The following are a few results concerning quaternion algebras in the Brauer group:

(1) [a, b] [a, c] = [a, bc].
(2) [a, b] = 1 if and only if 1 ∈ D(a, b).
(3) If [a, a'] = [b, b'], then there exists x ∈ F such that [a, a'] = [a, x] and [b, b'] = [b, x].

Let m(F) denote the number of quaternion algebras in the Brauer group.

Kaplansky in [8] defined the radical of a field F, denoted R(F), as the set of all a ∈ F such that 1 ∈ D(a, b) for all b ∈ F. Equivalent definitions include:

(1) R(F) = \{a ∈ F | D<1, -a> = F\}.
(2) R(F) = \{a ∈ F | [a, b] = 1 for every b ∈ F\}.
(3) R(F) = \bigcap_{a ∈ F} D<1, -a>.
$R(F)$ is a subgroup of $\hat{F}$ containing $\hat{F}^2$. Also, the value set of a form of dimension greater than one is the union of cosets of the radical. Kaplansky also introduced a generalized Hilbert field as a field $F$ in which there exist at most two 2-dimensional forms of determinant $d$ for every $d \in \hat{F}$ and at least two 2-dimensional forms of determinant $d$ for some $d \in \hat{F}$. Kaplansky showed that if $F$ is a generalized Hilbert field with $|\hat{F}/R(F)| = 2$, then $F$ is a real field with positive elements $P(F) = R(F) = D<1,1>$. Let $q$ be a form with $\dim q \geq 2$ over such a field $F$. If $D(q) \cap P(F) = \emptyset$, then $1 \in D(q)$ and if $D(q) - P(F) \neq \emptyset$, then $-1 \in D(q)$. Kaplansky also showed that if $F$ is a generalized Hilbert field with $|\hat{F}/R(F)| > 2$, then $F$ is a non-real field with $|\hat{F}/D<1,a>| \leq 2$ for every $a \in \hat{F}$. This implies that $m(F) = 2$. Other results for such a field $F$ include:

1. There exists a unique 3-dimensional anisotropic form $q$ of determinant $d$ for every $d \in \hat{F}$, and $D(q) = \hat{F}-(-d R)$.

2. There exists an anisotropic 4-dimensional form of determinant $d$ if and only if $d \in R(F)$.

Let $q(F)$ denote the number of square classes of $F$, i.e., $q(F) = |\hat{F}/\hat{F}^2|$ which may be $\infty$. $F$ is quadratically closed if every element is a square, i.e., $q(F) = 1$.

Let $f = <a_1,a_2,\ldots,a_n>$ be a regular form. The Hasse invariant of $f$ is defined to be $S(f) = \prod [a_i, a_j]$ where the $i < j$.
multiplication is taken in the Brauer group. If $f=g$, then $S(f) = S(g)$.

An element $x \in F$ is defined to be totally positive if $x$ is positive in all possible (if any) orderings of $F$. $x$ is totally positive if and only if $x$ is a sum of squares in $F$.

A field is said to be pythagorean if every sum of squares is a square. A field $F$ is pythagorean and non-real if and only if $F$ is quadratically closed. A real field $F$ satisfies the Strong Approximation Property (SAP) if for any two disjoint closed sets $A, B$ of orderings of $F$, there exists $a \in F$ such that $a>0$ at all orderings of $A$ and $a<0$ at all orderings of $B$. A real field $F$ is said to be super-pythagorean if $q(F) = 2^n < \infty$ and $F$ has $2^{n-1}$ orderings (the maximum number of orderings).

The classification question is an important problem in the study of quadratic forms. That is, given a field $F$, what are the invariants that classify the quadratic forms over $F$? Although the question has not been answered in general, the classification problem has been solved for several specific types of fields, as evidenced by the following theorems.

**Theorem 0.6:** Quadratic forms over a field $F$ are classified by the dimension if and only if $F$ is quadratically closed.

The following is a generalization of Theorem 0.6.
Theorem 0.7: Quadratic forms over a field $F$ are classified by the dimension and the total signature (with respect to all orderings of $F$) if and only if $F$ is pythagorean.

Theorem 0.8: Quadratic forms over a field $F$ are classified by dimension and determinant if and only if the $u$-invariant is at most 2 if and only if there is exactly one quaternion algebra.

Again, a generalization follows.

Theorem 0.9: Quadratic forms over $F$ are classified by dimension, determinant, and the total signature if and only if all binary forms which represent 1, represent all totally positive elements of $F$.

Theorem 0.10: Quadratic forms over $F$ are classified by dimension, determinant, and the Hasse invariant if and only if all quaternion forms are universal.

Finally, a generalization of Theorem 0.10 follows.

Theorem 0.11: Quadratic forms over $F$ are classified by dimension, determinant, Hasse invariant, and the total signature if and only if all quaternion forms represent all totally positive elements of $F$. 
In [13], Solow defined the square class invariant for quadratic forms over a field. In [13], [14], and [15] she showed that the square class invariant classifies quadratic forms over each of the following classes of fields:

1. Quadratically closed fields.
2. Real closed fields.
3. Finite fields.
4. Pythagorean fields which satisfy the strong approximation property and which have a finite number of square classes.
5. Superpythagorean fields.
6. Iterated power series fields over a field $K$, where $K$ is of type (4) or (5) above.
8. C-fields.

Solow also showed that the square class invariant and determinant classify quadratic forms over a field $F$ with $I^2F = 0 \Rightarrow u(F) \leq 2$ and over dyadic p-adic fields. Shapiro and Lam showed in [12] that if $F$ is a pythagorean field with a finite number of square classes then the square class invariant classifies quadratic forms over $F$ if and only if $F$ is a field of type (4), (5), or (6) above. In Chapter 1, we will try to determine exactly when the square class invariant classifies quadratic forms over a field, depending on the level of the field. We will also examine whether the
square class invariant and determinant classify quadratic forms over a generalized Hilbert field, a field with four quaternion algebras, and a field with finite u-invariant equal to half of the number of square classes.

It is known that the number of quaternion algebras of a field $F$ serves as an upper bound for $|\bar{\mathbb{F}}/D<1,-\alpha>|$ for every $\alpha \in \bar{\mathbb{F}}$. It is then easy to see that a field $F$ has exactly one quaternion algebra if and only if $|\bar{\mathbb{F}}/D<1,-\alpha>| = 1$ for every $\alpha \in \bar{\mathbb{F}}$ if and only if the u-invariant of $F$ is at most 2.

Kaplansky showed in [8] that a field $F$ has exactly two quaternion algebras if and only if $|\bar{\mathbb{F}}/D<1,-\alpha>| \leq 2$ for every $\alpha \in \bar{\mathbb{F}}$ and equality holding for some $\alpha \in \bar{\mathbb{F}}$. Thus the next reasonable question to ask is the following: If $F$ is a field with $|\bar{\mathbb{F}}/D<1,-\alpha>| \leq 4$ for every $\alpha \in \bar{\mathbb{F}}$ and equality holding for some $\alpha \in \bar{\mathbb{F}}$, does $F$ have exactly four quaternion algebras? We will examine this question in Chapter 2.
CHAPTER I
The Square Class Invariant

Solow introduced the square class invariant in [13]. She showed that the square class invariant or the square class invariant and the determinant classify quadratic forms over some particular fields. In this chapter, we will determine the fields over which the square class invariant classifies quadratic forms.

Definition 1.1. Let $q$ be a quadratic form over a field $F$. The square class invariant for $q$ is a function $m_q : F/F^2 \to \mathbb{Z}$ given by $m_q(aF^2) = n$ where $q \cong n\langle a \rangle + p$ and $a \notin D(p)$.

Theorem 1.2. Let $F$ be a field with $s(F) = 1$. Then the square class invariant classifies forms over $F$ if and only if $F$ is a C-field.

Proof: Assume the square class invariant classifies forms over $F$. Suppose $f$ and $g$ are anisotropic forms with $D(f) = D(g)$. If $m_f(a) \geq 2$ for some $a \in F$, then $\langle a, a \rangle = \langle a, -a \rangle = \langle 1, -1 \rangle$ is a subform of $f$ and thus $f$ is isotropic. So $m_f \leq 1$ and similarly $m_g \leq 1$ (that is, $m_f(a) \leq 1$ and $m_g(a) \leq 1$, for all $a \in F$). Thus $D(f) = \{a \in F \mid m_f(a) = 1\}$ and $D(g) = \{a \in F \mid m_g(a) = 1\}$. But $D(f) = D(g)$. Hence, $m_f = m_g$ which implies $f \cong g$. Thus $F$ is a C-field.
Now assume \( F \) is a \( C \)-field. Suppose \( f \) and \( g \) are forms over \( F \) with \( m_f \neq m_g \). Write \( f \equiv 2n<1> + f' \) where \( n \) is a nonnegative integer, and \( \dim f' = 0 \) or \( \dim f' \geq 1 \) and \( f' \) is anisotropic. If \( \dim f' = 0 \), then assume \( m_{f'} = 0 \). If \( \dim f' \geq 1 \), then \( m_{f'} \leq 1 \) since \( s=1 \) and \( f' \) is anisotropic. Similarly write \( g \equiv 2k<1> + g' \) where \( k \) is a non-negative integer, and \( \dim g' = 0 \) or \( \dim g' \geq 1 \) and \( g' \) is anisotropic. Again \( m_{g'} \leq 1 \). Without loss of generality, assume \( k \leq n \).

Then for every \( \alpha \in \mathbb{F} \), \( m_f(\alpha) = 2n + m_{f'}(\alpha) \leq 2n + 1 \) and \( m_g(\alpha) = 2k + m_{g'}(\alpha) \leq 2k + 1 \). So \( 0 = m_f(\alpha) - m_g(\alpha) = 2(n-k) + m_{f'}(\alpha) - m_{g'}(\alpha) \). Thus \( m_{g'}(\alpha) = 2(n-k) + m_{f'}(\alpha) \leq 1 \). Hence \( n=k \) and \( m_{g'}(\alpha) = m_{f'}(\alpha) \) for every \( \alpha \in \mathbb{F} \). Thus \( D(f') = D(g') \).

But \( F \) is a \( C \)-field, so \( f' \equiv g' \). Thus since \( n=k \) and \( f' \equiv g' \), \( f \equiv g \). Hence the square class invariant classifies forms over \( F \). □

**Theorem 1.3.** Let \( F \) be a \( C \)-field with \( s(F) = 2 \) and \( u(F) < \infty \). Then the square class invariant classifies quadratic forms over \( F \).

**Proof:** Suppose \( \psi \) and \( \phi \) are forms over \( F \) with \( m_\psi \equiv m_\phi \).

First, suppose \( \psi \) and \( \varphi \) are both anisotropic. Since \( D(\psi) = \{ \alpha \in \mathbb{F} | m_\psi(\alpha) > 1 \} \) and \( D(\phi) = \{ \alpha \in \mathbb{F} | m_\phi(\alpha) > 1 \} \), \( D(\psi) = D(\phi) \). Thus \( \psi \equiv \phi \) since \( F \) is a \( C \)-field.

Now suppose \( \psi \) is isotropic and \( \phi \) is anisotropic. Since \( \psi \) is isotropic, \( \psi \) is universal and \( m_\psi(\alpha) \geq 1 \) for every \( \alpha \in \mathbb{F} \). If \( m_\phi(\alpha) \geq 3 \) for some \( \alpha \in \mathbb{F} \), then \( <\alpha, \alpha, \alpha> \)
is a subform of \( \phi \). But \( \langle \alpha, \alpha, \alpha \rangle = \langle \alpha, -\alpha, -\alpha \rangle = \langle 1, -1, -\alpha \rangle \) since \( s = 2 \). But this contradicts \( \phi \) being anisotropic. Thus \( m_\phi(\alpha) \leq 2 \) for every \( \alpha \in F \). Then \( 1 \leq m_\psi(\alpha) = m_\phi(\alpha) \leq 2 \) for every \( \alpha \in F \).

Now suppose there exists a form \( \psi' \) such that \( \psi = \langle 1, -1 \rangle + \psi' \) and let \( \alpha \in D(\psi') \). Then \( \langle 1, -1, \alpha \rangle \equiv \langle \alpha, -\alpha, \alpha \rangle \equiv \langle -\alpha, -\alpha, -\alpha \rangle \) is a subform of \( \psi \). But \( m_\psi(\alpha) \leq 2 \). So \( \psi \equiv \langle 1, -1 \rangle \) and \( m_\psi \equiv m_\phi \equiv 1 \). Since \( F \) is a C-field, \( \phi \) must be the unique anisotropic universal form. So \( \langle 1, 1 \rangle \) is a subform of \( \phi \), since \( u(F) < \infty \). This follows from Proposition 5.3 in [4].

But this contradicts \( m_\phi \equiv 1 \). Thus it cannot occur that \( \psi \) is isotropic and \( \phi \) is anisotropic.

Finally, suppose both \( \psi \) and \( \phi \) are isotropic. Write \( \psi \equiv 2n\langle 1, -1 \rangle + \psi' \) and \( \phi \equiv 2k\langle 1, -1 \rangle + \phi' \) where \( n \) and \( k \) are non-negative integers, \( \dim \psi' = 0 \) or \( \dim \psi' \geq 1 \) and \( m_\psi'(1) \leq 3 \), and \( \dim \phi' = 0 \) or \( \dim \phi' \geq 1 \) and \( m_\phi'(1) \leq 3 \). If \( \dim \psi' = 0 \), let \( m_\psi' \equiv 0 \) and if \( \dim \phi' = 0 \), let \( m_\phi' \equiv 0 \). In any case, \( m_\psi'(\alpha) \leq 3 \) and \( m_\phi'(\alpha) \leq 3 \) for every \( \alpha \in F \). Thus \( m_\psi(\alpha) = 4n + m_\psi'(\alpha) \leq 4n + 3 \) and \( m_\phi(\alpha) = 4k + m_\phi'(\alpha) \leq 4n + 3 \) for all \( \alpha \in F \).

Without loss of generality, assume \( k \leq n \). Then for every \( \alpha \in F \), \( 0 = m_\psi(\alpha) - m_\phi(\alpha) = 4(n-k) + m_\psi'(\alpha) - m_\phi'(\alpha) \). Thus \( m_\phi'(\alpha) = 4(n-k) + m_\psi'(\alpha) \leq 3 \). Hence \( n = k \) and \( m_\psi' = m_\phi' \).

If \( \psi' \) and \( \phi' \) are anisotropic, then \( \psi' \equiv \phi' \) by the first part of this proof. Hence \( \psi \equiv \phi \). Also by the first part of this proof, it cannot occur that one of \( \psi' \) and \( \phi' \) is isotropic and the other is anisotropic. Now suppose that \( \psi' \) and \( \phi' \) are both isotropic. Since \( m_\psi' \leq 3 \), \( m_\phi' \leq 3 \), and \( s(F) = 2 \),
\[ \psi' = \langle 1, -1 \rangle + f \text{ and } \phi' = \langle 1, -1 \rangle + g, \] where \( f \) and \( g \) are anisotropic (or \( \dim f = 0 \) or \( \dim g = 0 \)). Now, \( a \in D(f) \) iff 
\[ \langle 1, -1 \rangle + \langle a \rangle \text{ is a subform of } \psi' \text{ iff } \langle a, -a \rangle + \langle a \rangle \equiv \langle -a, -a, -a \rangle \text{ is a subform of } \psi'. \] Thus \( D(f) = \{ a \in F | m_{\psi'}(-a) = 3 \} \).

Similarly, \( D(g) = \{ a \in F | m_{\phi'}(-a) = 3 \} \). But since \( m_{\psi'} = m_{\phi'} \), 
\( D(f) = D(g) \). Thus \( f \equiv g \) since \( F \) is a C-field. Thus \( \psi' \equiv \phi' \) and \( \psi \equiv \phi \).

**Theorem 1.4.** Let \( F \) be a field with \( s(F) = 2 \). If the square class invariant classifies the quadratic forms over \( F \), then \( F \) is a C-field.

**Proof:** Suppose \( F \) is not a C-field. Then there exists anisotropic forms \( f \neq g \) with \( D(f) = D(g) \). Let \( f' = \langle 1, -1 \rangle + f \) and 
\[ g' = \langle 1, -1 \rangle + g. \] Consider \( m_{f'} \). If there exists \( a \in F \) such that \( m_{f'}(a) \geq 4 \), then \( 4\langle a \rangle \equiv 2\langle 1, -1 \rangle \) is a subform of \( f' \).

Hence by Witt's Cancellation Theorem, \( \langle 1, -1 \rangle \) is a subform of \( f \). But \( f \) is anisotropic. Thus \( m_{f'} \leq 3 \).

\[ m_{f'}(a) = 3 \text{ iff } f' \equiv 3\langle a \rangle + \phi, \text{ for some form } \phi \text{ iff } f' \equiv \langle a, -a, -a \rangle + \phi \equiv \langle 1, -1 \rangle + \langle -a \rangle + \phi \text{ iff } f \equiv \langle -a \rangle + \phi \text{ by Witt's Cancellation Theorem iff } -a \in D(f). \]

\[ m_{f'}(a) = 2 \text{ iff } f' \equiv 2\langle a \rangle + \phi \text{ and } a \in D(\phi), \text{ for some form } \phi \text{ iff } \langle -a \rangle + f \equiv \langle a \rangle + \phi \text{ and } a \in D(\phi) \text{ iff } a \in D[\langle -a \rangle + f] \text{ and } -a \in D(\phi) \text{ iff } a \in D[\langle -a \rangle + f] - D(-f). \]

\[ m_{f'}(a) = 1 \text{ iff } a \not\in D[\langle -a \rangle + f]. \]
Thus
\[
m_{f'}(a) = \begin{cases} 
3 & \text{if } -a \in D(f) \\
2 & \text{if } a \in D([-a]+f) - D(-f) \\
1 & \text{if } a \not\in D([-a]+f)
\end{cases}
\]

Similarly,
\[
m_{g'}(a) = \begin{cases} 
3 & \text{if } -a \in D(g) \\
2 & \text{if } a \in D([-a]+g) - D(-g) \\
1 & \text{if } a \not\in D([-a]+g)
\end{cases}
\]

Since $D(f) = D(g)$, $D(-f) = D(-g)$ and $D([-a]+f) = D([-a]+g)$,
for every $a \in F$. So $m_{f'} \neq m_{g'}$, but $f' \neq g'$ since $f \neq g$. But
this contradicts the fact that the square class invariant
classifies forms over $F$. Hence $F$ must be a $C$-field. □

Solow [13] showed that the square class invariant classifies forms over the following pythagorean fields:

1. Pythagorean fields $F$ which satisfy the strong approximation property and for which $q(F) < \infty$.
2. Superpythagorean fields.
3. Iterated power series fields over a field $K$, where $K$ is of type 1 or 2 above.

Shapiro and Lam [12] showed that Solow's results include all the pythagorean fields with finite square class group for which the square class invariant classifies forms. Furthermore, Shapiro and Lam extended the results to pythagorean fields with finitely many real-valued places. They proved that if $K$ is a pythagorean field with only finitely many
real-valued places, then the square class invariant classifies forms over \( K \) if and only if \( K \) is equivalent to an iterated power series field over \( k \), where \( k \) is some SAP pythagorean field.

The following considers non-pythagorean field with \( s(F) \geq 4 \).

**Theorem 1.5.** Let \( F \) be a non-pythagorean field with \( s(F) \geq 4 \). Then the square class invariant does not classify forms over \( F \).

**Proof:** Since \( F \) is not pythagorean, \( D<1,1> \not\supseteq F^2 \). Let \( a \in D<1,1> - F^2 \). Consider \( f = <1,-1,-1> \) and \( g = <1,-1,-a> \).

Both \( f \) and \( g \) are isotropic and thus universal, so \( m_f \geq 1 \) and \( m_g \geq 1 \). Suppose \( m_f(a) = 3 \) for some \( a \in F \).

\[ <1,-1,-1> \equiv <a,a,a> \]

\[ \Rightarrow \det(<1,-1,-1>) = \det(<a,a,a>) \]

\[ \Rightarrow a \in F^2. \]

So \( <1,-1,-1> \equiv <1,1,1> \). Thus by Witt's Cancellation Theorem, \( <-1,-1> \equiv <1,1> \) and \(-1 \in D<1,1>\). This contradicts \( s(F) \geq 4 \). Thus \( 1 \leq m_f(a) \leq 2 \), for every \( a \in F \).

Let us determine when \( m_f \) is equal to 2.

\[ m_f(a) = 2 \Leftrightarrow <1,-1,-1> \equiv <a,a,1> \]

\[ \Leftrightarrow <-1,-1> \equiv <a,a> \]

\[ \Leftrightarrow -a \in D<1,1> \]

Thus \( m_f(a) = \begin{cases} 2 & \text{if } -a \in D<1,1> \\ 1 & \text{otherwise} \end{cases} \)
Now suppose $m_g(a) = 3$, for some $a \in F$. Then $\langle 1, -1, -a \rangle \neq \langle a, a, a \rangle$. Thus $\det(\langle 1, -1, -a \rangle) = \det(\langle a, a, a \rangle)$. So $a \in F^2$ and $\langle 1, -1, -a \rangle \neq \langle a, a, a \rangle$. This implies that $\langle a, -a, -a \rangle \neq \langle a, a, a \rangle$ and that $-1 \in D(1, 1)$. This contradicts $s(F) \geq 4$. Thus $1 \leq m_g(a) \leq 2$, for every $a \in F$.

Let us determine when $m_g$ is equal to 2.

$m_g(a) = 2 \iff \langle 1, -1, -a \rangle \neq \langle a, a, a \rangle$

$\iff \langle a, -a, -a \rangle \neq \langle a, a, a \rangle$

$\iff \langle -a, a \rangle \neq \langle a, a \rangle$

$\iff -a \in D(1, 1)$

$\iff -a \in D(1, 1)$ since $a \in D(1, 1)$

Thus

$$m_g(a) = \begin{cases} 2 & \text{if } -a \in D(1, 1) \\ 1 & \text{otherwise} \end{cases}$$

Hence $m_f = m_g$. But $f \neq g$ since $\det f \neq \det g$. So the square class invariant does not classify forms over $F$. □

We have just shown that the square class invariant alone fails to classify forms over non-pythagorean fields with $s \geq 4$. The following theorem shows that the square class invariant coupled with the determinant does classify forms over a specific type of field.

**Theorem 1.6.** Let $F$ be a generalized Hilbert field. Assume also that if $s(F) = 1$, then $H(F) = F^2$ and that if $s(F) = 2,
then \(-1 \not\in R(F)\). Then the square class invariant and determinant classify quadratic forms over \(F\).

**Proof:** Case 1: Suppose \(|F/R| = 2\). Then \(F\) is a real field with \(P(F) = R = D<1,1>\). Let \(f\) be a quadratic form over \(F\) with \(m_f(1) = k\), \(m_f(-1) = j\), and \(\det f = d\).

(i) Assume \(k = j = 0\). Thus \(\pm 1 \not\in D(f)\). Since \(D(\phi) \cap \{1,-1\} \neq \emptyset\) for any binary form \(\phi\), \(\dim f = 1\) and \(f \equiv <d>\) with \(d \neq \pm 1\).

(ii) Assume \(j = 0\) and \(k > 0\). Then \(f \equiv k<1> + f'\) where \(\pm 1 \not\in D(f')\). Again, since \(D(\phi) \cap \{1,-1\} \neq \emptyset\) for any binary form \(\phi\), \(\dim f' \leq 1\). If \(\dim f' = 1\), then \(f' \equiv <d>\). But \(\pm 1 \not\in D(f')\). So \(\dim f' = 1\) if and only if \(d \neq 1\). Also if \(\dim f' = 1\) then \(d > 0\), otherwise, \(D<1,d> \cap -P \neq \emptyset \Rightarrow -1 \in D<1,d> \subseteq D(f)\).

Summarizing,

\[
f \equiv \begin{cases} 
k<1>, & \text{if } d = 1 \\
<1> + <d>, & \text{if } d \neq 1
\end{cases}
\]

and in either case, \(d > 0\).

(iii) Assume \(k = 0\) and \(j > 0\). Then \(f \equiv j<-1> + f'\) where \(\pm 1 \not\in D(f')\). Since \(D(\phi) \cap \{1,-1\} \neq \emptyset\) for any binary form \(\phi\), \(\dim f' \leq 1\). If \(\dim f' = 1\), then \(f' \equiv j<(-1)d>\). But since \(\pm 1 \not\in D(f')\), \(d\) must not be \(\pm 1\). So \(\dim f' = 1 \Leftrightarrow d \neq \pm 1\).

Therefore,

\[
f \equiv \begin{cases} 
<j<-1>>, & \text{if } d = \pm 1 \\
<j<-1> + <(-1)d>, & \text{if } d \neq \pm 1
\end{cases}
\]
Consider \( \dim f' = 1 \) and \( f' = \langle(-1)^j d\rangle \). If \( (-1)^j d > 0 \), then \( \emptyset \neq D(f') \cap P \subseteq D(f) \cap P \). But \( \dim f \geq 2 \). So \( 1 \in D(f) \) which is a contradiction. Thus \( (-1)^j d < 0 \).

(iv) Assume \( k \geq 1 \) and \( j \geq 1 \). Write \( f = k\langle 1 \rangle + \langle b_1, \ldots, b_n \rangle \) with \( 1 \notin D\langle b_1, \ldots, b_n \rangle \). Then \( n \geq 1 \), otherwise \( D(f) \subseteq P \). If all the \( b_i \)'s are positive, then again \( D(f) \subseteq P \). So some \( b_i \) is negative, say \( b_1 < 0 \). Now suppose some other \( b_j \) is positive, say \( b_2 > 0 \). Then \( D\langle b_1, b_2 \rangle = b_1 D\langle 1, b_1 b_2 \rangle = \hat{F} \) since \( -b_1 b_2 \in P = R(F) \). Then \( 1 \notin D\langle b_1, b_2 \rangle \subseteq D\langle b_1, \ldots, b_n \rangle \) which is a contradiction. So \( b_1 < 0 \), for all \( 1 \leq i \leq n \). So \( D\langle b_1, \ldots, b_n \rangle \subseteq -P \). By the result mentioned in the Introduction, \(-1 \notin D\langle b_1, \ldots, b_n \rangle \), if \( n \geq 2 \). Thus \( \langle b_1, \ldots, b_n \rangle \equiv \langle -1, b'_2, \ldots, b'_n \rangle \) and \( D\langle b'_2, \ldots, b'_n \rangle \subseteq -P \). So again \(-1 \notin D\langle b'_2, \ldots, b'_n \rangle \). Continuing, we can write \( f \equiv k\langle 1 \rangle + (n-1)\langle -1 \rangle + \langle b \rangle \) for some \( b < 0 \). Since \( -b > 0 \), \( -b \in R \) and \( D\langle 1, b \rangle = \hat{F} \). Thus \( \langle 1, b \rangle \equiv \langle -1, -b \rangle \). So \( f \equiv (k-1)\langle 1 \rangle + n\langle -1 \rangle + \langle -b \rangle \) and \(-1 \notin D[(k-1)\langle 1 \rangle + \langle -b \rangle] \subseteq P \). Thus \( n = j \) and \( f \equiv k\langle 1 \rangle + (j-1)\langle -1 \rangle + \langle b \rangle \). Since \( \det f = d, b = (-1)^{j-1} d \). Therefore, \( f \equiv k\langle 1 \rangle + (j-1)\langle -1 \rangle + \langle (-1)^{j-1} d \rangle \). Note that if \( j=1 \) and \( k \geq 1 \), then \( d < 0 \). For if \( d > 0 \), then \( D(f) = D[k\langle 1 \rangle + \langle d \rangle] \subseteq P \) which contradicts \( j=1 \).

There are two possible overlaps in the above results. Let us consider them.

First, suppose \( f \) and \( g \) are forms with \( m_f(1) = m_g(1) = k > 0, \ m_f(-1) = 0, \ m_g(-1) = 1 \), and \( \det f \neq 1 \). Then by cases (ii) and (iv), \( f \equiv k\langle 1 \rangle + \langle \det f \rangle \) and \( g \equiv k\langle 1 \rangle + \langle \det g \rangle \).
But it was also shown that in these situations, \( \det f > 0 \) and \( \det g < 0 \). So \( f \not\equiv g \) and there is no overlap here.

Secondly, suppose \( f \) and \( g \) are forms with \( m_f(1) = 0 \), \( m_f(-1) = m_g(-1) = j > 0 \), \( m_g(1) = 1 \), \( \det f \neq \pm 1 \), and \( \det g = (-1)^j \). Then by cases (iii) and (iv), \( f \cong j^{<-1>} + <(-1)^j \det f> \) and \( g \cong <1> + (j-1)^<-1> + <(-1)^j-1 \det g> \cong <1> + (j-1)^<-1> + <1> \equiv j^{<-1>} + <1> \). But it was also shown that \( (-1)^j \det f < 0 \). So \( f \not\equiv g \) and there is no overlap here.

Case 2: Suppose \( |\hat{F}/\hat{R}| > 2 \). Then \( F \) is a nonreal field with \( u(F) = 4 \) and \( m(F) = 2 \). Also assume that if \( s(F) = 1 \), then \( R = \hat{F}^2 \), and if \( s(F) = 2 \), then \(-1 \notin R \). In the Introduction, several results pertaining to these generalized Hilbert fields were recorded. These results will be used below.

(i) Assume \( s = 1 \) and \( R = \hat{F}^2 \).

First, suppose \( \det f = 1 \). Write \( f \cong m_f(1)<1> + f' \) and \( m_f(1) = 2n + k \) where \( n \) is a non-negative integer and \( k = 0 \) or \( 1 \). Since \( \det f = 1 \), either \( \dim f' = 0 \) or \( \dim f' > 1 \) and \( \det f' = 1 \). If \( \dim f' = 0 \), then \( f \cong m_f(1)<1> \) and \( m_f(\alpha) = 2n \) for every \( \alpha \notin \hat{F}^2 \). Now consider \( \dim f' > 1 \). Since \( u = 4 \) and \( 1 \notin D(f') \), \( \dim f' \leq 3 \). If \( \dim f' = 1 \), then \( f' \cong (\det f') \equiv <1> \). But \( 1 \nin D(f') \). So \( \dim f' \neq 1 \). If \( \dim f' = 2 \), then \( f' \equiv <a, a> \) since \( \det f' = 1 \). But then \( 1 \nin D(f') \) since \( s = 1 \). So \( \dim f' \neq 2 \). Can \( \dim f' = 3 \)? Yes, there does exist a unique 3-dimensional anisotropic form \( f' \) with \( \det f' = 1 \) and \( D(f') = \hat{F} - \hat{F}^2 \). Let \( \alpha \in \hat{F} - \hat{F}^2 \). Then \( f' \equiv <a, \beta, \alpha \beta> \) with \( \beta \nin D<1, \alpha> \) and \( f \equiv (2n+k)<1> + <a, \beta, \alpha \beta> \). Then \( m_f(\alpha) = 2n + 1 \) since
α∉D<β,αβ>(if k=0) and α∉D<1,β,αβ>(if k=1). In summary, if det f = 1, then either f = mf<1><1> = (2n + k)<1> or
f = mf(1)<1> + f' = (2n+k)<1> + f' where f' is the unique anisotropic 3-dimensional form of determinant 1. But for α∉F^2, mf(α) = 2n in the first case and mf(α) = 2n + 1 in the second case. So the square class invariant distinguishes these two possibilities.

Now suppose det f = d ≠ 1. Write f = mf(d)<d> + f' and
mf(d) = 2n + k where n is a non-negative integer and k=0 or 1. If k=0, then dim f' > 1 and det f' = det f. If dim f' = 1, then f' = <det f'> = <d> which contradicts d∉D(f'). So dim f' ≠ 1. If dim f' = 2, then f' = <a,ad> with a∉D<1,d>. There exists a unique 3-dimensional anisotropic form f' of determinant d with d∉D(f'). In fact, D(f') = F - dF^2. so
f' = <1,a,ad> with a∉D<1,d>. So if det f = d ≠ 1 and mf(d) is even, then f' = <a,ad> with a∉D<1,d> or f' = <1,a,ad> with a∉D<1,d>. We will now use mf(1) to distinguish between these two possibilities for f'. If mf(1) = 2n, then
f' = <a,ad> and if mf(1) = 2n + 1, then f' = <1,a,ad>. In summary, if det f≠1 and mf(d) = 2n, then either f = mf(d)<d> + <a,ad> or f = mf(d)<d> + <1,a,ad>, for a∉D<1,d>. But mf(1) = 2n in the first case and mf(1) = 2n + 1 in the second case. So the square class invariant distinguishes these two possibilities.

Now if k=1, then dim f' = 0 or dim f' ≥ 1 and det f' = 1. If dim f' = 1, then f' = <1>. If dim f' = 2, then
f' = <a,a> = <1,1> which is universal since s=1. But this
contradicts \( d \not\equiv D(f') \). So \( \dim f' \neq 2 \). If \( \dim f' = 3 \), then \( D(f') = F - F^2 \). But again this contradicts \( d \not\equiv D(f') \). So \( \dim f' \neq 3 \).

So if \( \det f = d \not\equiv F^2 \) and \( m_f(d) \) is odd, then \( f \equiv m_f(d) <d> \equiv (2n+1) <d> \) or \( f \equiv m_f(d) <d> + <1> \equiv (2n+1) <d> + <1> \). Again we will use \( m_f(1) \) to distinguish between these two possibilities for \( f \). If \( m_f(1) = 2n \), then \( f \equiv m_f(d) <d> \equiv (2n+1) <d> \) and if \( m_f(1) = 2n + 1 \), then \( f \equiv m_f(d) <d> + <1> \equiv (2n+1) <d> + <1> \).

Let us list again the possibilities for \( f \) if \( s = 1 \) and \( R = F^2 \). If \( \det f = 1 \), then \( f \equiv m_f(1) <1> \) if \( m_f(a) \) is even for every \( a \not\equiv F^2 \), and \( f \equiv m_f(1) <1> + f' \), where \( f' \) is the unique 3-dimensional anisotropic form of determinant 1, if \( m_f(a) \) is odd for every \( a \not\equiv F^2 \). If \( d = \det f \neq 1 \), there are four possibilities for \( f \). \( f \equiv m_f(d) <d> + <\alpha, \alpha d> \), where \( \alpha \not\equiv D <1, d> \), if \( m_f(d) \) is even and \( m_f(1) \) is even; \( f \equiv m_f(d) <d> + <1, \alpha, \alpha d> \), where \( \alpha \not\equiv D <1, d> \), if \( m_f(d) \) is even and \( m_f(1) \) is odd; \( f \equiv m_f(d) <d> \) if \( m_f(d) \) is odd and \( m_f(1) \) is even; \( f \equiv m_f(d) <d> + <1> \) if \( m_f(d) \) is odd and \( m_f(1) \) is odd. Thus the square class invariant and determinant classify forms in this case.

(ii) Assume \( s = 2 \) and \(-1 \not\equiv R\).

First, assume \( \det f = 1 \). Write \( f \equiv m_f(1) <1> + f' \).

Since \( \det f = 1 \), \( \dim f' = 0 \) or \( \dim f' \geq 1 \) and \( \det f' = 1 \). If \( \dim f' = 1 \), then \( f' \equiv <\det f'> \equiv <1> \). But \( 1 \not\equiv D(f') \). So \( \dim f' \neq 1 \). If \( \dim f' = 3 \), then \( f' \) must be the unique anisotropic 3-dimensional form of determinant 1 and \( D(f') = F - (-R) \). Then \( 1 \not\equiv D(f') \) which is a contradiction. So \( \dim f' \neq 3 \).

Thus \( \dim f' = 2 \) since \( u = 4 \). So \( f' \equiv <\alpha, \alpha> \) where \( \alpha \not\equiv D <1,1> \).
Hence \( f = m_f(1)\langle 1 \rangle \) or \( f = m_f(1)\langle 1 \rangle + \langle \alpha, \alpha \rangle \). We will distinguish these two possibilities by examining \( m_f(\alpha) \) for \( \alpha \notin D<1,1> \). Write \( m_f(1) = 4n + k \) where \( n \) is a non-negative integer and \( k = 0, 1, 2, 3 \). For \( k = 0, 1, 2 \), \( m_f(\alpha) = 4n \) if \( f = m_f(1)\langle 1 \rangle \) and \( m_f(\alpha) = 4n + 2 \) if \( f = m_f(1)\langle 1 \rangle + \langle \alpha, \alpha \rangle \). This follows since \( \alpha \notin D(1,1) \) and \( s = 2 \). Now consider \( k = 3 \). Then

\[
\langle 1,1,1 \rangle = \langle 1,-1,-1 \rangle = \langle \alpha,-\alpha,-1 \rangle \text{ and } \alpha \notin D<-\alpha,-1> \text{ since } \\

\alpha \notin D<1,1>. \text{ Thus } m_f(\alpha) = 4n + 1 \text{ if } f = m_f(1)\langle 1 \rangle \text{ and } m_f(\alpha) = 4n + 3 \text{ if } f = m_f(1)\langle 1,1 \rangle + \langle \alpha, \alpha \rangle. \text{ Summarizing, if } \det f = 1, \text{ then } f = m_f(1)\langle 1 \rangle \text{ if } m_f(\alpha) \equiv 0,1 \pmod{4} \text{ for every } \alpha \notin D<1,1> \text{ and } f = m_f(1)\langle 1 \rangle + \langle \alpha, \alpha \rangle \text{ if } m_f(\alpha) \equiv 2,3 \pmod{4} \text{ for every } \alpha \notin D<1,1>.

Now assume \( d = \det f \neq 1 \). Write \( f = m_f(d)\langle d \rangle + f' \).

Suppose \( m_f(d) \) is even. Then \( \dim f' \geq 1 \) and \( \det f' = d \). But \( \dim f' = 1 \), otherwise \( \det D(f') = D<d> \). If \( \dim f' = 3 \), then \( f' \) is the unique anisotropic 3-dimensional form of determinant \( d \) and \( D(f') = F - (-dR) \). Then \( \det D(f') \), which is a contradiction. So \( \dim f' = 3 \). If \( \dim f' = 2 \), then \( f' = \langle \alpha, ad \rangle \) where \( \alpha \notin D<1,d> \), (which can occur if and only if \( -d \notin R \)). Hence, if \( m_f(d) \) is even, then \( f = m_f(d)\langle d \rangle + \langle \alpha, ad \rangle \) where \( \alpha \notin D<1,d> \).

Suppose \( m_f(d) = 4n + 1 \) where \( n \) is a non-negative integer and \( d \notin D<1,1> \). Then \( \dim f' = 0 \) or \( \dim f' \geq 1 \) and \( \det f' = 1 \). If \( \dim f' = 1 \), then obviously \( f' = \langle 1 \rangle \). If \( \dim f' = 2 \), then \( f' = \langle \alpha, \alpha \rangle \) for \( \alpha \notin D<1,1> \). If \( \dim f' = 3 \), then \( f' \) is the unique anisotropic 3-dimensional form of determinant 1 and \( D(f') = F - (-R) \). Since \( d \notin D(f') \), this can
occur if and only if \(-d \in \mathbb{R}\). Then \(f' = <1, a, a>\) where \(a \notin D<1,1>\).

Hence there are four possibilities for \(f\):

- \(f = (4n+1)<d>\)
- \(f = (4n+1)<d> + <1>\)
- \(f = (4n+1)<d> + <a, a>, where a \notin D<1,1>\)
- \(f = (4n+1)<d> + <1, a, a>, where a \notin D<1,1> and -d \in \mathbb{R}\).

We can distinguish these four possibilities by considering \(m_f(-d)\) and \(m_f(a)\) for \(a \notin D<1,1>\). Obviously, \(-d, a \notin D<d>\). So \(m_f(-d) = 4n = m_f(a)\) if \(f = (4n+1)<d>, d \in D<1,1> \Rightarrow -d \in D<1,1> \Rightarrow -1 \in D<1,d> \Rightarrow -d \in D<1,d>\). So \(m_f(-d) = 4n + 1\) if \(f = (4n+1)<d> + <1>\). Also \(m_f(a) = 4n + m_{<1,d>}(a) \leq 4n + 1\) if \(f = (4n+1)<d> + <1>\). \(-d \in D<d,a,a> \Rightarrow <d,a,a> \Rightarrow <d,-d,-d> = <-d,-1,-1> \Rightarrow <-d,d,-d> \Rightarrow <d,a,a> \Rightarrow <-d,-d> = <1,1> = \alpha \in D<1,1>\).

So \(-d \in D<d,a,a>\) and \(m_f(-d) = 4n\) if \(f = (4n+1)<d> + <a,a>\). Obviously, \(m_f(a) = 4n + 2\) if \(f = (4n+1)<d> + <a,a>\). Since \(s = 2\), \(m_{<d,1,a,a>}(-d) \geq 2\) if and only if \(m_{<d,1,a,a>}(-d) \geq 2\).

But then \(d \in D<1,a,a>\), which will not occur if \(-d \in \mathbb{R}\). So \(m_f(-d) = 4n + 1\) if \(f = (4n+1)<d> + <1,a,a>\) and \(-d \in \mathbb{R}\). Also \(m_f(a) = 4n + 2 + m_{<1,d>}(a) = 4n + 3\) if \(f = (4n+1)<d> + <1,a,a>\) and \(-d \in \mathbb{R}\). Therefore the following chart indicates the possibilities for \(f\) and the corresponding values of \(m_f(-d)\) and \(m_f(a)\), where \(a \notin D<1,1>\), if \(d = \det f \neq 1, d \in D<1,1>\), and \(m_f(d) = 4n + 1\).
Now suppose $m_f(d) = 4n + 3$ where $n$ is a non-negative integer and $d \in D<1,1>$. As above, there are four possibilities for $f$:

- $f \equiv (4n+3)d$
- $f \equiv (4n+3)d + 1$
- $f \equiv (4n+3)d + \alpha$, where $\alpha \notin D<1,1>$
- $f \equiv (4n+3)d + 1, \alpha$, where $\alpha \notin D<1,1>$ and $-d \in R$.

Again we will distinguish these four possibilities by considering $m_f(-d)$ and $m_f(\alpha)$ for $\alpha \notin D<1,1>$. $<d,d,d> \equiv <d,-d,-d>$

- $<1,1,1> \equiv <\alpha,-\alpha,-d>$. $\alpha \in D<1,1>$ if $f \equiv (4n+3)d$. $d \in D<1,1> \implies -d \in D<1,d> \implies -d \in D<1,1>$.

But $\alpha \notin D<1,1>$. So $m_f(-d) = 4n + 2$ and $m_f(\alpha) = 4n + 1$ if $f \equiv (4n+3)d$. $d \in D<1,1> \implies -d \in D<1,d> \implies -d \in D<1,1>$.

So $<d,d,d,1> \equiv <\alpha,\alpha,\alpha>$. Obviously $m_f(\alpha) = 4n + m<1,d,d,d>(\alpha)$ if $f \equiv (4n+3)d + 1$. Note: $-d \in D<1,d,d,d>(\alpha) \iff <d,d,d> \equiv <\alpha,\alpha> \equiv <d,d> \equiv <1,1>$. But $\alpha \notin D<1,1>$.

Thus $m_f(-d) = 4n + 2$ if $f \equiv (4n+3)d + \alpha$. If $f \equiv (4n+3)d + \alpha$, then $m_f(\alpha) =$
\[ m_f(a) = 4n + 2 + m_3\langle d \rangle(a). \] But \( m_3\langle d \rangle(a) \) was shown above to equal 1. So \( m_f(a) = 4n + 3. \) Consider \( \langle d, d, 1, \alpha, \alpha \rangle \) with \(-d \in \mathbb{R}.\) Then \( \langle d, d, d, 1, \alpha, \alpha \rangle \equiv \langle -d, -d, -d, -1, \alpha, \alpha \rangle. \) \(-d \in \mathbb{R} \langle -1, \alpha, \alpha \rangle \equiv \langle -1, \alpha, \alpha \rangle \equiv \langle -1, -d, -d, -d, -1, \alpha, \alpha \rangle \equiv \langle -1, -d, \beta, \beta d \rangle \) for some \( \beta. \) Then \( \langle -1, \alpha, \alpha \rangle \equiv \langle -1, 1, 1 \rangle \equiv \langle \alpha, \alpha \rangle \equiv \langle 1, 1 \rangle. \) But \( \alpha \notin \mathbb{D} <1, 1> \). So \(-d \notin \mathbb{D} <1, 1, \alpha, \alpha \rangle \) and \( m_f(-d) = 4n + 3 \) if \( f \equiv (4n + 3)\langle d \rangle + <1, \alpha, \alpha \rangle \) and \(-d \notin \mathbb{R}. \) Obviously, \( m_f(a) = 4n + 2 + m\langle 1, d, d, d \rangle(a) \geq 4n + 3 \) if \( f \equiv (4n + 3)\langle d \rangle + <1, \alpha, \alpha \rangle. \) Still assuming \(-d \notin \mathbb{R}, \) \( m\langle 1, d, d, d \rangle(a) = 3 \equiv \langle 1, 1, d \rangle \equiv \langle 1, 1 \rangle \equiv \langle \alpha, \alpha, \alpha, \alpha \rangle \equiv \langle \alpha, \alpha, 1, d \rangle \equiv \langle \alpha, \alpha, \alpha \rangle \equiv \langle 1, 1 \rangle \equiv \langle \alpha, \alpha \rangle. \) But \( \alpha \notin \mathbb{D} <1, 1> \). So \( m\langle 1, d, d, d \rangle(a) \leq 2 \) if \(-d \notin \mathbb{R}.\)

Therefore, the following chart indicates the possibilities for \( f \) and the corresponding values of \( m_f(-d) \) and \( m_f(a), \) where \( \alpha \notin \mathbb{D} <1, 1> \), if \( d = \det f \neq 1, \) de \( \mathbb{D} <1, 1> \), and \( m_f(d) = 4n + 3. \)

<table>
<thead>
<tr>
<th>( f )</th>
<th>( m_f(-d) )</th>
<th>( m_f(a) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (4n + 3)\langle d \rangle )</td>
<td>4n+2</td>
<td>4n+1</td>
</tr>
<tr>
<td>( (4n + 3)\langle d \rangle + &lt;1&gt; )</td>
<td>4n+3</td>
<td>4n+2+m\langle 1, d, d, d \rangle(a) \leq 4n+3</td>
</tr>
<tr>
<td>( (4n + 3)\langle d \rangle + \langle \alpha, \alpha \rangle )</td>
<td>4n+2</td>
<td>4n+3</td>
</tr>
<tr>
<td>( (4n + 3)\langle d \rangle + &lt;1, \alpha, \alpha \rangle )</td>
<td>4n+3</td>
<td>4n+2+m\langle 1, d, d, d \rangle(a) \leq 4n+3</td>
</tr>
</tbody>
</table>

So \( m_f(-d) \) and \( m_f(a) \) distinguish the four possibilities for \( f.\)

Now suppose \( m_f(d) = 4n + 1 \) where \( n \) is a non-negative integer and \( \alpha \notin \mathbb{D} <1, 1> \). Then \( \dim f' = 0 \) or \( \dim f' \geq 1 \) and \( \det f' = 1. \) If \( \dim f' = 1, \) then \( f' = <1>. \) If \( \dim f' = 2, \) then \( f' = <1, 1> \)
since $d \notin D\langle 1,1 \rangle$. If $\dim f' = 3$, then $f'$ is the unique anisotropic 3-dimensional form of determinant 1 and $D(f') = F - (-R)$. But $d \notin D\langle 1,1 \rangle \Rightarrow -d \notin D\langle 1,1 \rangle \Rightarrow -d \notin R \Rightarrow d \notin R$. So $d \in D(f')$. Thus $\dim f' \neq 3$. Thus there are three possibilities for $f$:

1. $f \cong \langle 4n+1 \rangle_d$ for which $m_f(1) = 4n$
2. $f \cong \langle 4n+1 \rangle_d + \langle 1 \rangle$ for which $m_f(1) = 4n + 1$
3. $f \cong \langle 4n+1 \rangle_d + \langle 1,1 \rangle$ for which $m_f(1) = 4n + 2$.

Thus $m_f(1)$ distinguishes these three possibilities.

Suppose $m_f(d) = 4n + 3$ where $n$ is a non-negative integer and $d \notin D\langle 1,1 \rangle$. As above, there are three possibilities for $f$:

1. $f \cong \langle 4n+3 \rangle_d$
2. $f \cong \langle 4n+3 \rangle_d + \langle 1 \rangle$
3. $f \cong \langle 4n+3 \rangle_d + \langle 1,1 \rangle$.

Thus $m_f(1) = 4n + 1$ if $f \cong \langle 4n+3 \rangle_d$

$m_f(1) = 4n + 2$ if $f \cong \langle 4n+3 \rangle_d + \langle 1 \rangle$

$m_f(1) = 4n + 3$ if $f \cong \langle 4n+3 \rangle_d + \langle 1,1 \rangle$

Therefore $m_f(1)$ distinguishes the three possibilities for $f$.

(iii) assume $s = 4$.

Let $\alpha \in F$. Since $u = 4$, $1 \in D(4\langle \alpha \rangle)$. So $4\langle \alpha \rangle$ is a Pfister form. Thus $G(4\langle \alpha \rangle) = D(4\langle \alpha \rangle) = F$ and $4\langle \alpha \rangle \cong 4\langle \beta \rangle$ for every $\beta \in F$. Thus $m_4\langle \alpha \rangle(\beta) = 4$, for every $\alpha, \beta \in F$. 


Suppose \( \det f = 1 \). Write \( f = m_f(1)\langle 1 \rangle + f' \) with \( m_f(1) = 4n + k \) where \( n \) is a non-negative integer and \( k=0,1,2, \) or \( 3 \). Then \( \dim f' = 0 \) or \( \dim f' > 1 \) and \( \det f' = 1 \). If \( \dim f' = 1 \), then \( f' = \langle \det f' \rangle = \langle 1 \rangle \). But \( 1 \notin D(f') \). So \( \dim f' 
eq 1 \).

If \( \dim f' = 3 \), then \( f' \) is the unique anisotropic 3-dimensional form of determinant 1 and \( D(f') = \hat{F} - (-R) \).

But then \( 1 \in D(f') \) which is a contradiction. So \( \dim f' 
eq 3 \).

If \( \dim f' = 2 \), then \( f' = \langle -1,-1 \rangle \) since \( 1 \notin D<-1,-1> \). So there are two possibilities for \( f \):

\[
f = m_f(1)\langle 1 \rangle
\]

\[
f = m_f(1)\langle 1 \rangle + \langle -1,-1 \rangle
\]

These two possibilities can be distinguished by considering \( m_f(-1) \). Since \( -1 \notin D<1,1,1> \), \( m_f(-1) = 4n \) if \( f = m_f(1)\langle 1 \rangle \) and \( m_f(-1) = 4n + 2 \) if \( f = m_f(1)\langle 1 \rangle + \langle -1,-1 \rangle \).

Now suppose \( \det f = d \neq 1 \). Write \( f = m_f(d)\langle d \rangle + f' \) and \( m_f(d) = 4n + k \) where \( n \) is a non-negative integer and \( k=0,1,2, \) or \( 3 \). If \( m_f(d) = 4n \) or \( 4n + 2 \), then \( \dim f' > 1 \) and \( \det f' = d \). If \( \dim f' = 1 \), then \( f' = \langle \det f' \rangle = \langle d \rangle \). But this contradicts \( d \notin D(f') \). So \( \dim f' 
eq 1 \). If \( \dim f' = 3 \), then \( f' \) is the unique anisotropic form of determinant \( d \) and \( D(f') = \hat{F} - (-dR) \). In particular, \( d \notin D(f') \) which is a contradiction. So \( \dim f' 
eq 3 \). If \( \dim f' = 2 \), then \( f' = \langle a,ad \rangle \) since \( \det f' = d \). Since \( d \notin D(f') \), \( -d \notin R \) and \( a \notin D<1,d> \).

Now suppose \( m_f(d) = 4n + 1 \) or \( 4n + 3 \) and \( d \notin D<1,1> \).

Then \( \dim f' = 0 \) or \( \dim f' > 1 \) and \( \det f' = 1 \). If \( \dim f' = 1 \), then \( f' = \langle \det f' \rangle = \langle 1 \rangle \). If \( \dim f' = 2 \), then \( f' = \langle -1,-1 \rangle \) since \( d \notin D<-1,-1> \). If \( \dim f' = 3 \), then \( f' \) is the
unique anisotropic 3-dimensional form of determinant 1 and
\[ D(f') = \hat{f} - (-R). \]
But \( d \in D < 1, 1 > \implies -d \in D < 1, 1 > \implies -d \not\in R \implies d \not\in R \implies d \in D(f') \) which is a contradiction. So \( \dim f' \neq 3. \) So there are three possibilities for \( f: \)

\[ f = m_f(d) < d > \]
\[ f = m_f(d) < d > + <1> \]
\[ f = m_f(d) < d > + <-1, -1> \]

If \( m_f(d) = 4n + 1, \) these three possibilities for \( f \) can be distinguished by considering \( m_f(1) \) and \( m_f(-d). \) If

\[ f = (4n+1)<d>, \text{ then } m_f(1) = 4n = m_f(-d). \]
\[-1 \in D < d, d > \implies d \in D < 1, 1 > \]. So \( -d \not\in D < 1, 1 >. \) Thus \( m_f(-d) = 4n \) and \( m_f(1) = 4n + 1 \) if \( f = (4n+1)<d> + <1>. \) \( d \in D < 1, 1 > \implies -d \in D < 1, 1 > \implies 1 \in D < d, -1 >. \) So \( < d, -1, -1 > \equiv < 1, -d, -1 >. \)

\[ 1 \in D < -d, -1 > \implies -1 \in D < 1, d > \implies -d \in D < 1, 1 > \implies -d \in D < 1, 1 >. \]

As above, \( < d, -1, -1 > \equiv < 1, -d, -1 > \equiv < -d, -d, d >. \) So \( m_f(-d) = 4n + 2 \) if \( f = (4n+1)<d> + <-1, -1>. \)

Therefore, the following chart indicates the possibilities for \( f \) and the corresponding values of \( m_f(1) \) and \( m_f(-d), \) if \( d = \det f \neq 1, \) \( d \in D < 1, 1 >, \) and \( m_f(d) = 4n + 1. \)

<table>
<thead>
<tr>
<th>( f )</th>
<th>( m_f(1) )</th>
<th>( m_f(-d) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((4n+1)&lt;d&gt;)</td>
<td>4n</td>
<td>4n</td>
</tr>
<tr>
<td>((4n+1)&lt;d&gt; + &lt;1&gt;)</td>
<td>4n+1</td>
<td>4n</td>
</tr>
<tr>
<td>((4n+1)&lt;d&gt; + &lt;-1,-1&gt;)</td>
<td>4n+1</td>
<td>4n+2</td>
</tr>
</tbody>
</table>

Thus \( m_f(1) \) and \( m_f(-d) \) distinguish the possibilities for \( f. \)
If $m_f(d) = 4n + 3$, then these three possibilities for $f$ can be distinguished by considering $m_f(-d)$ alone.

$-d \epsilon D <d,d,d> \Rightarrow <d,d,d> = <-d,\beta,-\gamma> = <-d,d,-d> = <d,d> =$

$<-d,-d> \Rightarrow -1 \epsilon D <d,d> = D<1,1>$ which contradicts $s=4$. So

$m_f(-d) = 4n$ if $f \equiv (4n+3)<d>$. Since $u=4$, $-d \epsilon D <1,d,d,d>$. So

$<1,d,d,d> \equiv <-d,\beta,\gamma,-\beta \gamma>$ for some $\beta,\gamma \epsilon F$. If $<\beta,\gamma,-\beta \gamma>$ is isotropic, then certainly $-d \epsilon D <\beta,\gamma,-\beta \gamma>$. If $<\beta,\gamma,-\beta \gamma>$ is anisotropic, then $D <\beta,\gamma,-\beta \gamma> = F - R$. But $d \epsilon D <1,1> \Rightarrow$

$-d \epsilon D <1,1> \Rightarrow -d \epsilon R$. So $-d \epsilon D <\beta,\gamma,-\beta \gamma>$. In either case,

$<1,d,d,d> \equiv <-d,-d,\delta,\delta d>$ for some $\delta \epsilon F$. If $\delta \epsilon D <1,d>$, then

$<1,d,d,d> \equiv <-d,-d,1,d> \Rightarrow <d,d> \equiv <-d,-d>$ which contradicts

$-1 \epsilon D <1,1>$. So $\delta \not\epsilon D <1,d>$. Note: $-d \epsilon D <1,d> \equiv -1 \epsilon D <1,d> =$

$-d \epsilon D <1,1> \Rightarrow -1 \epsilon D <1,1>$. So $-d \not\epsilon D <1,d>$. Thus $<1,d,d,d> \equiv$

$<-d,-d,-d,-1>$ and $m_f(-d) = 4n + 3$ if $f \equiv (4n+3)<d> + <1>$. $<d,d,d,-1,-1> \equiv <1,1,d,-d,-d>$ since $d \epsilon D <1,1>$ and $-d \epsilon D <-1,-1>$. $-d \epsilon D <1,1> \Rightarrow <1,d> \equiv <-d,1,-1> \Rightarrow <1,d> \equiv <-1,-d> =$

$-1 \epsilon D <1,d> \Rightarrow -d \epsilon D <1,1> \Rightarrow -1 \epsilon D <1,1>$ which contradicts $s=4$. So

$-d \not\epsilon D <1,1,d>$ and $m_f(-d) = 4n + 2$ if $f \equiv (4n+3)<d> + <-1,-1>$. Therefore, the following chart indicates the possibilities for $f$ and the corresponding values of $m_f(-d)$, if $d = det f \neq 1$, $d \epsilon D <1,1>$, and $m_f(d) = 4n + 3$.

<table>
<thead>
<tr>
<th>$f$</th>
<th>$m_f(-d)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(4n+3)&lt;d&gt;$</td>
<td>$4n$</td>
</tr>
<tr>
<td>$(4n+3)&lt;d&gt; + &lt;1&gt;$</td>
<td>$4n + 3$</td>
</tr>
<tr>
<td>$(4n+3)&lt;d&gt; + &lt;-1,-1&gt;$</td>
<td>$4n + 2$</td>
</tr>
</tbody>
</table>
Thus \( m_f(-d) \) distinguishes the three possibilities of \( f \).

Finally, suppose \( m_f(d) = 4n + 1 \) or \( 4n + 3 \) and \( d \notin D <1,1> \). Then \( d \in D <-1,-1> \). Thus \( \dim f' = 0 \) or \( \dim f' \geq 1 \) and \( \det f' = 1 \). If \( \dim f' = 1 \), then obviously \( f' = \langle \det f' \rangle = \langle 1 \rangle \). If \( \dim f' = 2 \), then \( f' \equiv \langle 1,1 \rangle \) since \( d \notin D <1,1> \). If \( \dim f' = 3 \), then \( f' \) is the unique anisotropic \( 3 \)-dimensional form of determinant \( 1 \) and \( D(f') = \mathbb{P}^{*}(-R) \). Since \( d \notin D(f') \), we must assume \( d \in -R \Rightarrow -d \in R \). Then \( 1 \in D(f') \) and \( f' = \langle 1,\alpha,\alpha \rangle \equiv \langle 1,1,1 \rangle \) since \( f' \) is anisotropic. Hence there are four possibilities for \( f \):

\[
\begin{align*}
  f & \equiv m_f(d) <d> \\
  f & \equiv m_f(d) <d> + \langle 1 \rangle \\
  f & \equiv m_f(d) <d> + \langle 1,1 \rangle \\
  f & \equiv m_f(d) <d> + \langle 1,1,1 \rangle \text{ with } -d \in R.
\end{align*}
\]

If \( m_f(d) = 4n + 1 \), these four possibilities can be distinguished by considering \( m_f(1) \). Obviously,

\[
\begin{align*}
  m_f(1) & = 4n \quad \text{if } f \equiv (4n+1)<d> \\
  m_f(1) & = 4n + 1 \quad \text{if } f \equiv (4n+1)<d> + \langle 1 \rangle \\
  m_f(1) & = 4n + 2 \quad \text{if } f \equiv (4n+1)<d> + \langle 1,1 \rangle \\
  m_f(1) & = 4n + 3 \quad \text{if } f \equiv (4n+1)<d> + \langle 1,1,1 \rangle \\
\end{align*}
\]

If \( m_f(d) = 4n + 3 \), these four possibilities can also be distinguished by considering \( m_f(1) \). Since \( -d \notin D(d,d) \), \( <d,d,d> \) is anisotropic and \( D(d,d,d) = \mathbb{P}^{*}(-dR) \). In particular, if \( -d \notin R \) then \( 1 \in D(d,d,d) \). So \( <d,d,d> \equiv \langle 1,\alpha,\alpha d \rangle \). Also, \( d \notin D(1,d) \) otherwise \( <d,d> \equiv \langle 1,1 \rangle \) which contradicts \( d \notin D <1,1> \). Thus \( m_{<d,d,d>}(1) = 1 \), if \( -d \notin R \). If \( -d \in R \), then \( 1 \notin D(d,d,d) \).
Therefore the following chart indicates the possibilities for $f$ and the corresponding values of $m_f(1)$, if $d = \det f \neq 1$, $d \notin \mathcal{D}\langle 1,1 \rangle$, and $m_f(d) = 4n + 3$.

<table>
<thead>
<tr>
<th>$f$</th>
<th>$m_f(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(4n+3)d$</td>
<td>\begin{cases} 4n &amp; \text{if } -d \in \mathbb{R} \ 4n+1 &amp; \text{if } -d \notin \mathbb{R} \end{cases}</td>
</tr>
<tr>
<td>$(4n+3)d + 1$</td>
<td>\begin{cases} 4n+1 &amp; \text{if } -d \in \mathbb{R} \ 4n+2 &amp; \text{if } -d \notin \mathbb{R} \end{cases}</td>
</tr>
<tr>
<td>$(4n+3)d + 1,1$</td>
<td>\begin{cases} 4n+2 &amp; \text{if } -d \in \mathbb{R} \ 4n+3 &amp; \text{if } -d \notin \mathbb{R} \end{cases}</td>
</tr>
<tr>
<td>$(4n+3)d + 1,1,1$</td>
<td>$4n+3$</td>
</tr>
</tbody>
</table>

Thus $m_f(1)$, depending on whether $-d \in \mathbb{R}$ or $-d \notin \mathbb{R}$, distinguishes the four possibilities for $f$. □

Note: The assumptions in Theorem 1.6 that if $s(F) = 1$, then $R(F) = \hat{F}^2$, and that if $s(F) = 2$, then $-1 \in \mathcal{R}(F)$ are necessary as evidenced by the following examples.

Example 1: Suppose $F$ is a generalized Hilbert field with $s(F) = 1$ and $R(F) \neq \hat{F}^2$. Let $d \in \mathcal{R}(F) - \hat{F}^2$ and let $f = \langle 1, d \rangle$. Then $m_f \neq 1$. Let $g$ be the unique anisotropic 4-dimensional form of determinant $d \in \mathcal{R}(F)$. Since $g$ is anisotropic and $s = 1$, $m_g \leq 1$. But $u(F) = 4$. So $g$ is universal. Thus $m_g \neq 1$. But $f \neq g$.

Example 2: Suppose $F$ is a generalized Hilbert field with $s(F) = 2$ and $-1 \in \mathcal{R}(F)$. Let $f = \langle 1, -1 \rangle$. Then $m_f \neq 1$. Let $g$ be the unique anisotropic 4-dimensional form with $\det g = -1 \in \mathcal{R}$. Suppose there exists $a \in \hat{F}$ such that $m_g(a) \geq 2$. 
Then $g = \langle \alpha, \alpha, \beta, -\beta \rangle$ for some $\beta \in \mathbb{F}$ and $g$ is thus isotropic. So $m_g \leq 1$. But again since $u=4$, $g$ is universal. Thus $m_g = 1$. But $f \neq g$.

We have just shown that the square class invariant and determinant classify forms over a nonreal generalized Hilbert field $\mathbb{F}$ (assuming if $s=1$, then $\mathbb{R} = \mathbb{F}^2$, and if $s=2$, then $-1 \not\in \mathbb{F}$). Also, a nonreal field $\mathbb{F}$ is a generalized Hilbert field if and only if $m(\mathbb{F}) = 2$. So now it is natural to ask whether the square class invariant and determinant classify forms over a nonreal field $\mathbb{F}$ with $m(\mathbb{F}) = 4$.

**Theorem 1.7.** If $\mathbb{F}$ is a nonreal field with $m(\mathbb{F}) = 4$, then the square class invariant and determinant do not classify quadratic forms over $\mathbb{F}$.

**Proof:** If $m(\mathbb{F}) = 4$, then $u(\mathbb{F}) = 4$ as shown in [3]. Also, it is well known that if $u(\mathbb{F}) = 4$, then the quaternion algebras form a group. If $\phi$ is a quaternion form, then $m_\phi(\alpha) = m_\phi(1)$ for every $\alpha \in \mathbb{F}$ since $\phi$ is universal and $G(\phi) = D(\phi)$. Thus $m_\phi$ is a constant function, and, in particular, $m_\phi = m_\phi(1)$ which is a power of 2.

Let $\phi_1, \phi_2, \phi_3$ be the anisotropic quaternion forms. Then $\det \phi_i = 1$ for $i=1,2,3$. If $s(\mathbb{F}) \leq 2$, then $m_{\phi_i}(1) = m_{\phi_j}(1)$ for some $i \neq j$. Thus $m_{\phi_i} = m_{\phi_j}$. So the square class invariant and determinant do not classify quadratic forms.

Now suppose $s(\mathbb{F}) = 4$. Then $m_{\phi_i} \equiv 1, 2, 4$ for $1 \leq i \leq 3$. Say $m_{\phi_1} = 1$, $m_{\phi_2} = 2$, and $m_{\phi_3} = 4$. Let $A_i$ be the quaternion algebra associated with $\phi_i$. Since $m_{\phi_2}(1) = 2$, $A_2 = [-1, -a]$
for some \( \alpha \in \mathbb{F} - D\langle 1,1 \rangle \). Since \( m_{\phi_3}(1) = 4 \), \( A_3 = [-1,-1] \). But \( A_1 = A_2 \text{ and } A_3 = [-1,-\alpha] \), \( [-1,-1] = [-1,\alpha] \). Thus \( m_{\phi_1}(1) \geq 2 \) which contradicts \( m_{\phi_1} = 1 \). Thus \( m_{\phi_i} = m_{\phi_j} \) for some \( i \neq j \). So the square class invariant and determinant do not classify quadratic forms over \( \mathbb{F} \). \( \Box \)

Solow showed in [15] that if \( \mathbb{F} \) is a field with \( u(\mathbb{F}) = q(\mathbb{F}) < \infty \), then the square class invariant classifies forms over \( \mathbb{F} \). Cordes showed in [4] that a field \( \mathbb{F} \) with \( u(\mathbb{F}) = q(\mathbb{F}) < \infty \) is indeed a \( C \)-field with \( s(\mathbb{F}) \leq 2 \) and \( u(\mathbb{F}) < \infty \), and we have previously shown in a more general result that the square class invariant classifies quadratic forms over these fields. Now let us consider a field \( \mathbb{F} \) with \( u(\mathbb{F}) = q(\mathbb{F})/2 < \infty \).

**Proposition 1.8:** Suppose \( \mathbb{F} \) is a field with \( s(\mathbb{F}) = 4 \) and \( u(\mathbb{F}) = q(\mathbb{F})/2 < \infty \). Then the square class invariant and determinant classify forms over \( \mathbb{F} \) if and only if \( u(\mathbb{F}) = 4 \).

**Proof:** Cordes and Ramsey showed in [5] that if \( s(\mathbb{F}) = 4 \) and \( u(\mathbb{F}) = q(\mathbb{F})/2 < \infty \), then \( \mathbb{F} \) is equivalent to the 2-adic numbers \( \mathbb{Q}_2 \) or an iterated power series extension of \( \mathbb{Q}_2 \).

Springer showed in [16] that a quadratic form \( \phi \) over a field \( K=\mathbb{F}(x) \) can be written \( \phi = k\langle 1,-1 \rangle + g + x h \), where \( k \) is a non-negative integer and \( g \) and \( h \) are anisotropic forms over \( \mathbb{F} \). Then \( m_{\phi}(\alpha) = m_k\langle 1,-1 \rangle + g(\alpha) \) and \( m_{\phi}(x\alpha) = m_k\langle 1,-1 \rangle + h(\alpha) \)
for every $a \in F$. Since the square class representatives of $K$ consist of $\{\text{square class representatives of } F\} \cup x \{\text{square class representatives of } F\}$, $m_\phi$ is now known for every square class in $K$. Using this result of Springer's and induction, a similar result is obtained for $K = F((x_1))(x_2)) \ldots ((x_n))$. Write a form $\phi$ over $K$ as

$$\phi = k <1,-1> + f + \sum_{i_1=1}^{n} x_{i_1} f_{i_1} + \sum_{1 \leq i_1 < i_2 \leq n} x_{i_1} x_{i_2} f_{i_1 i_2} + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} x_{i_1} x_{i_2} x_{i_3} f_{i_1 i_2 i_3} + \ldots + x_1 x_2 \ldots x_n f_{12} \ldots n$$

where $k$ is a non-negative integer and $f$ and all of the $f_{i_1 i_2} \ldots i_m$'s are anisotropic forms over $F$. Then

$$m_\phi(ax_{i_1} \ldots x_{i_m}) = m_{k<1,-1>} f_{i_1} \ldots i_m(a)$$

for every $a \in F$.

In [14], Solow showed that the square class invariant and determinant classify forms over $Q_2$. Now consider $K = Q_2((x_1))(x_2)) \ldots ((x_n))$ for some positive integer $n$. Let $f \equiv <1,-1,-1> + <1,-1,-2 >$ over $Q_2$. Solow showed that $m_f \neq m_g$ over $Q_2$ but obviously $f \neq g$. Now consider $\phi \equiv <1,-1> + <1,-1> + x_1 <1,-1> + <2> + x_1 <2> \ldots x_1 <2> \ldots x_n <2>$ over $K$. Then for every $a \in Q_2$, $m_\phi(a) = m_{<1,-1> + <1,-1>} (a) = m_f(a)$, and $m_\phi(a x_1 \ldots x_{i_m}) = m_{<1,-1> + <1,-1>} (a) = m_{<1,-1>} (a) = 1$ for $m \geq 1$, $1 \leq i_1 \leq i_2 \leq \ldots i_m \leq n$, and $\{i_1, i_2, \ldots, i_m\} \neq \{1\}$. Similarly, $m_\psi(a) = m_g(a)$, and $m_\psi(ax_{i_1} \ldots x_{i_m}) = m_{<1,-1>} (a) = 1$ for $m \geq 1$, $1 \leq i_1 \leq i_2 \ldots i_m \leq n$ and $\{i_1, i_2, \ldots, i_m\} \neq \{1\}$. Since $m_f \neq m_g$ over $Q_2$, $m_\phi \neq m_\psi$ over $K$. Also $\det \phi = -x_1 = \det \psi$. But $\phi \neq \psi$ over $K$ since $f \neq g$ over
Q_2. Hence the square class invariant and determinant do not classify forms over Q_2((x_1))((x_2))...((x_n)) for n≥1.
Therefore, if F is a field with s(F) = 4 and u(F) = q(F)/2<ω, the square class invariant and determinant classify forms over F if and only if F=Q_2 if and only if u(F) = 4. □

**Proposition 1.9:** Let F be a field with R(F) = F^2, s(F)=4, and u(F)=4. If the square class invariant and determinant classify quadratic forms over F, then F is a C-field.

**Proof:** Let φ_1,φ_2,...,φ_n be the quaternion forms over F.
Thus m_{φ_i} = m_{φ_i}(1) = 1,2, or 4 since φ_i is universal (u=4) and G(φ_i) = D(φ_i). Also det φ_i = 1 for each i. Since the square class invariant and determinant classify forms over F, n=m(F) ≤ 3. But since u(F) = 4, the quaternion algebras form a group. So m(F) = 2. In [2] Cordes showed that the number of anisotropic 4-dimensional forms of determinant d which represent 1 is equal to m(F) - |F/D<1,-d>| = 2 - |F/D<1,-d>| = 0, if d∉R = F^2 , 1, if d∈R = F^2 . Since u=4, every 4-dimensional form represents 1. Hence there exists a unique anisotropic 4-dimensional form. Cordes showed in [4] that this implies that F is a C-field. □
CHAPTER 2

Fields with Binary Form Value Sets
Having Index at Most Four

In this chapter we will consider a field \( F \) with

\[ |\hat{F}/D<1,-\alpha>| \leq 4 \]
for every \( \alpha \in \hat{F} \) and equality holding for some \( \alpha \in \hat{F} \). Question: Is \( m(F) = 4 \)? In [4], Cordes showed that there exists a field \( F \) with \( q(F) = 8 = u(F) \). In this field, \( |D<1,-\alpha>/\hat{F}^2| = 2 \) for every \( \alpha \notin \hat{F}^2 \); hence, \( |\hat{F}/D<1,-\alpha>| = 4 \) for every \( \alpha \notin \hat{F}^2 \). Also \( m(F) = 8 \). So the answer to the above question is no. What we will show is that \( m(F) = 4 \) or \( |\hat{F}/R| \leq 8 \).

Can \( |\hat{F}/R| = 4 \) for such a field \( F \)? First suppose \( F \) is non-real. Since \( D<1,-\beta> \) has index 4 in \( \hat{F} \) for some \( \beta \in \hat{F} \), \( u(F) \geq 8 \). Thus \( |\hat{F}/R| = 4 = u(F) \). Then each non-universal binary form represents exactly two cosets of \( R \) (see[1]). Then

\[ |\hat{F}/D<1,-\alpha>| \leq 2 \]
for every \( \alpha \in \hat{F} \) which is a contradiction. So no such non-real field \( F \) exists. Now suppose \( F \) is a real field with some fixed ordering such that \( \hat{F} = <\alpha,-1>R \) and \( P(F) = <\alpha>R \). Since \( \pm \alpha \notin R \), \( D<1,-\alpha> \neq \hat{F} \) and \( D<1,\alpha> \neq \hat{F} \). So

\[ D<1,-\alpha> = <-\alpha>R \] and \( D<1,\alpha> = <\alpha>R = P \). Now \( D<1,1> \subseteq P = D<1,\alpha> \). So \( D<1,1> = D<1,1> \cap D<1,\alpha> \subseteq D<1,-\alpha> = <-\alpha>R \).

Thus \( R \subseteq D<1,1> \subseteq <-\alpha>R \), and either \( D<1,1> = R \) or \( D<1,1> = <-\alpha>R \). If \( D<1,1> = <-\alpha>R \), then \( -\alpha \in D<1,1> \subseteq P = <\alpha>R \). This implies that \( -1 \in P \) which is a contradiction. So \( D<1,1> = R \).

Thus 1, \([-1\alpha], [-1,-\alpha] \), and \([-1,-1] \) are distinct quaternion
algebras. The only other possible non-split quaternion algebras are \([\alpha,\alpha]\) and \([-\alpha,-\alpha]\), but \([\alpha,\alpha] = [-1,\alpha]\) and \([-\alpha,-\alpha] = [-1,-\alpha]\). So \(m(F) = 4\).

Thus we will show that if \(F\) is a field with 
\(|\hat{F}/\hat{D}<1,-\alpha>| \leq 4\) for every \(\alpha \in \hat{F}\) and equality holding for some \(\alpha \in \hat{F}\), then \(m(F) = 4\) or \(|\hat{F}/\hat{R}| = 8\) (and \(u(F) = 8\)).

In many cases, results which hold in terms of \(\hat{F}^2\) can be strengthened by replacing \(\hat{F}^2\) with \(R(F)\). To simplify matters for a while, we will assume in the remainder of this chapter that \(R(F) = \hat{F}^2\).

Let \(F\) be a field with 
\(|\hat{F}/\hat{D}<1,-\alpha>| \leq 4\) for every \(\alpha \in \hat{F}\) and equality holding for some \(\alpha \in \hat{F}\). In [2] Cordes showed that if \(K\) is a field with \(m(K) < \infty\) and \(d \in K\), then there are exactly 
\(m - |\hat{K}/\hat{D}<1,-d>|\) anisotropic forms of determinant \(d\) and dimension 4 which represent 1. So \(m(F) = 4\) if and only if there exists \(d \in \hat{F}\), with \(|\hat{F}/\hat{D}<1,-d>| = 4\), for which there exists no anisotropic 4-dimensional form with determinant \(d\) which represents 1. \(<1,-x,-y,xyd>\) is isotropic for every 
\(x,y \in \hat{F}\) if and only if 
\(D<1,-x> \cap yD<1,-xd> \neq \emptyset\) for every \(x,y \in \hat{F}\).

\(D<1,-x> = \bigcup \{ D<1,-x> \cap yD<1,-xd> \mid y \in \{\text{coset representatives of } D<1,-xd>\}\}\)

Lemma 2.1. Suppose \(A\) and \(B\) are subgroups of a group \(G\). If \(a,b \in G\) and if \(aA \cap bB \neq \emptyset\), then there exists \(c \in G\) such that 
\(aA \cap bB = c(\overline{A} \cap \overline{B})\).
Proof: Suppose \( x \in aA \cap bB \). Consider \( x(A \cap B) \). If \( y \in x(A \cap B) \), then \( y = xz \) for some \( z \in aA \cap bB \). Now \( x \in aA \Rightarrow a^{-1}x \in aA \Rightarrow a^{-1}y = a^{-1}xz \in aA \Rightarrow y \in aA \). Similarly \( y \in bB \) and so \( y \in aA \cap bB \). Thus \( x(A \cap B) \subseteq aA \cap bB \). Now suppose that \( y \in aA \cap bB \). Then \( y = \alpha \beta = bB \) for some \( \alpha \in aA \) and \( \beta \in bB \). Also there exists \( \alpha_1 \in aA \) and \( \beta_1 \in bB \) such that \( x = \alpha \alpha_1 = bB \). So \( b^{-1}a = \beta \alpha_1 = b \beta_1 \alpha_1^{-1} \Rightarrow \alpha_1^{-1}a = \beta_1^{-1}b \in aA \cap bB \). Then \( y = \alpha \alpha_1 \alpha_1^{-1}a = x(\alpha_1^{-1}a) \in x(A \cap B) \). Thus \( aA \cap bB \subseteq x(A \cap B) \). Let \( c = x \). \( \Box \)

So if \( D<1,-x> \cap yD<1,-xd> \neq \emptyset \), then \( D<1,-x> \cap yD<1,-xd> \) is a coset of \( D<1,-x> \cap D<1,-xd> \). But

\[
D<1,-x> = \bigcup_{y \in \{ \text{coset representatives of } D<1,-xd> \}} [D<1,-x> \cap yD<1,-xd>]
\]

So \( |D<1,-x>/ (D<1,-x> \cap D<1,-xd>)| = \text{number of cosets of } D<1,-x> \cap D<1,-xd> \) in \( D<1,-x> \) = number of \( y \in \{ \text{coset representatives of } D<1,-xd> \} \) for which \( D<1,-x> \cap yD<1,-xd> \neq \emptyset \). Thus \( D<1,-x> \cap yD<1,-xd> \neq \emptyset \) for each \( y \in \{ \text{coset representatives of } D<1,-xd> \} \) if and only if

\[
|D<1,-x>/ (D<1,-x> \cap D<1,-xd>)| = |\hat{F}/D<1,-xd>|. \quad \text{Then each } D<1,-x> \cap yD<1,-xd> \neq \emptyset \text{ if and only if }
\]

\[
|\hat{F}/(D<1,-x> \cap D<1,-xd>)| = |\hat{F}/D<1,-x> \cdot |D<1,-x>/ (D<1,-x> \cap D<1,-xd>)|
\]

Thus \( m(F) = 4 \) if and only if there exists \( d \in F \) with \( |\hat{F}/D<1,-d>| = 4 \) such that

\[
|\hat{F}/D<1,-x> \cdot |\hat{F}/D<1,-xd> | = |\hat{F}/(D<1,-x> \cap D<1,-xd>)|
\]

for every \( x \in \hat{F} \).
Given $dF$ with $|\hat{F}/D<1,-d>| = 4$, let us consider under what circumstances

$|\hat{F}/D<1,-x| \cdot |\hat{F}/D<1,-xd| = |\hat{F}/(D<1,-x \cap D<1,-xd>|$, for some $x \in \hat{F}$.

(1) Suppose $D<1,-x> \text{ and } D<1,-xd>$ both have index 2 in $\hat{F}$. Then $|\hat{F}/(D<1,-x \cap D<1,-xd>)|$ must not equal 4. So $D<1,-x> = D<1,-xd>$ and $D<1,-x> = D<1,-x> \cap D<1,-xd> \subseteq D<1,-d>$ which contradicts $|\hat{F}/D<1,-d>| = 4$.

(2) Suppose $|\hat{F}/D<1,-x>| = 2 \text{ and } |\hat{F}/D<1,-xd>| = 4$. Then $|\hat{F}/(D<1,-x \cap D<1,-xd>)|$ must not equal 8. Thus $D<1,-xd> \subseteq D<1,-x>$ and $D<1,-xd> = D<1,-xd> \cap D<1,-x> \subseteq D<1,-d>$. So $D<1,-xd> = D<1,-d> \supseteq D<1,-x> \supseteq \hat{F}$.

(3) Suppose $D<1,-x>$ and $D<1,-xd>$ both have index 4 in $\hat{F}$. Then $|\hat{F}/(D<1,-x \cap D<1,-xd>)|$ must not equal 16. Thus $D<1,-x> = D<1,-xd>$ or $D<1,-x> \cap D<1,-xd>$ has index 8 in $\hat{F}$. If $D<1,-x> = D<1,-xd>$, then $D<1,-x> = D<1,-xd> = D<1,-d>$.

Thus $m(F) \neq 4$ if and only if for every $dF$ with $|\hat{F}/D<1,-d>| = 4$, there exists $x \in \hat{F}$ such that one of the following is true:

(1) $D<1,-xd> = D<1,-d> \supseteq D<1,-x> \supseteq \hat{F}$.

(2) $D<1,-x> = D<1,-xd> = D<1,-d>$.

(3) $D<1,-x>$ and $D<1,-xd>$ also have index 4 in $\hat{F}$ and $D<1,-x> \cap D<1,-xd> = D<1,-x> \cap D<1,-d> = D<1,-xd> \cap D<1,-d>$ has index 8 in $\hat{F}$.

Now we will consider each of these three cases individually and show that either no such $\hat{F}$ exists or $|\hat{F}/\hat{F}^2| = 8$. 
Case I: Let $F$ be a field with $|\hat{F}/D<1,-\alpha| \leq 4$ for every $\alpha \in \hat{F}$, and equality holding for some $\alpha \in \hat{F}$. Also assume that $R(F) = \hat{F}^2$. Suppose there exists $a, b \in \hat{F}$ such that $D<1,-a> = D<1,-b> \not\subseteq D<1,-ab> \not\subseteq \hat{F}$.

First, assume $-1 \in D<1,-a>$. Let $y \in D<1,-ab> - D<1,-a>$. Lemma 2.2: If $F$ is a field as above, then there exists $x \in \hat{F}$ such that $x \not\in D<1,-ab> \cup D<1,-y> \cup aD<1,-aby> \cup bD<1,-aby>$.

Proof: (1) Assume $D<1,-y> \not\subseteq D<1,-ab>$. Then $D<1,-y> \not\subseteq D<1,-ab>$ since $a \in D<1,-ab> - D<1,-y>$. So $D<1,-ab> = \langle a \rangle D<1,-y>$ (the group generated by $a$ and $D<1,-y>$) and $D<1,-y> = D<1,-y> \cap D<1,-ab> \not\subseteq D<1,-aby>$.

(i) Suppose $D<1,-y> \not\subseteq D<1,-aby>$. Choose $x \not\in D<1,-ab>$. Then obviously $x \not\in D<1,-y> = D<1,-aby>$ and $ax, bx \not\in D<1,-aby>$. So $x \not\in D<1,-y> \cup D<1,-aby> \cup aD<1,-aby> \cup bD<1,-aby>$.

(ii) Suppose $D<1,-y> \not\subseteq D<1,-aby>$. Then $D<1,-aby>$ has index 2 in $\hat{F}$ and $D<1,-aby> \not= D<1,-ab>$. Choose $x \in D<1,-aby> - D<1,-ab>$. Then $x \not\in D<1,-y>$. If $ax \in D<1,-aby>$, then $a \in D<1,-aby>$ and $aby \in D<1,-a>$. Thus $y \in D<1,-a>$ which is a contradiction.

So $ax \not\in D<1,-aby>$ and $x \not\in aD<1,-aby>$. If $bx \in D<1,-aby>$, then $b \in D<1,-aby>$ and $aby \in D<1,-b>$. Thus $y \in D<1,-b>$ which is a contradiction. So $bx \not\in D<1,-aby>$. Thus $x \not\in D<1,-y> \cup D<1,-aby> \cup aD<1,-aby> \cup bD<1,-aby>$.

(2) Now assume $D<1,-y> \not\subseteq D<1,-ab>$. Choose $x' \in D<1,-y> - D<1,-ab>$. Let $x = ax'$. Then $x \not\in D<1,-ab>$ since $a \in D<1,-ab>$, and $x \not\in D<1,-y>$ since $a \not\in D<1,-y>$. If $x' = ax \in D<1,-aby>$, then $x' = ax \in D<1,-aby> \cap D<1,-y> \subseteq D<1,-ab>$.
which is a contradiction. So $ax \notin D<1,-aby>$. If $abx' = bx \in D<1,-aby>$, then $abx' \in D<1,-aby> \cap D<1,-y> \subseteq D<1,-ab>$. So $x' \in D<1,-ab>$ which is a contradiction. So $bx \notin D<1,-aby>$. Thus $x \notin D<1,-ab> \cup D<1,-y> \cup aD<1,-aby> \cup bD<1,-aby>$. □

So from Lemma 2.2, there exists $x \in F$ such that $x \notin D<1,-ab> \cup D<1,-y> \cup aD<1,-aby> \cup bD<1,-aby>$. Thus $y \notin D<1,-x>$ and $aby \notin D<1,-ax> \cup D<1,-bx>$. But $D<1,-x>$, $D<1,-ax>$, and $D<1,-bx>$ each have index 4 in $F$ with coset representatives $\{1,a,b,ab\}$. Thus $ay \in D<1,-x>$, $a'aby \in D<1,-ax>$, and $a''aby \in D<1,-bx>$ for some $\alpha, \alpha', \alpha'' \in \{a, b, ab\}$. If $\alpha' = \alpha ab$, then $ay \in D<1,-x> \cap D<1,-ax> \subseteq D<1,-a>$. Thus $y \in D<1,-a>$ which is a contradiction. If $\alpha' = \alpha b$, then $ay \in D<1,-x>$ and $aay \in D<1,-ax>$. Thus $-axy \in D<1,-x> \cap D<1,-ax> \subseteq D<1,-a> \implies x \in yD<1,-a>$ which is a contradiction. So $\alpha' = \alpha$ or $\alpha a$.

If $\alpha'' = \alpha ab$, then $ay \in D<1,-x> \cap D<1,-bx> \subseteq D<1,-b>$. Thus $y \in D<1,-b>$ which is a contradiction. If $\alpha'' = \alpha a$, then $ay \in D<1,-x>$ and $aby \in D<1,-bx>$. Thus $-axy \in D<1,-x> \cap D<1,-bx> \subseteq D<1,-ab> \implies x \in yD<1,-b>$ which is a contradiction. So $\alpha'' = \alpha$ or $\alpha b$.

Suppose $\alpha' = \alpha a$ and $\alpha'' = \alpha b$. Then $aby \in D<1,-ax>$ and $aay \in D<1,-bx>$. Thus $-aabxy \in D<1,-ax> \cap D<1,-bx> \subseteq D<1,-ab> \implies x \in D<1,-ab>$ which is a contradiction.

So either $\alpha' = \alpha$ or $\alpha'' = \alpha$. If $\alpha' = \alpha$, then there exists $\alpha \in D<1,-a>$ such that $ay \in D<1,-x>$ and $aaby \in D<1,-ax>$. So $x \in D<1,-ay> \cap aD<1,-aby>$. Thus $<1,-ay> - a<1,-aby> = <1,-a> - ay<1,-b>$ is isotropic. But $D<1,-a> = D<1,-b>$. 
Hence \( ay \in D<1,-a> \). Thus \( y \in D<1,-a> \) which is a contradiction. A similar argument shows that \( a''=a \) also leads to a contradiction.

Thus we have shown the following:

**Proposition 2.3:** Let \( F \) be a field with \( |\hat{F}/D<1,-a>| \leq 4 \) for every \( a \in F \) and equality holding for some \( a \in \hat{F} \). Then there does not exist \( a, b \in F \) such that \(-1 \in D<1,-a> \) and \( D<1,-a> = D<1,-b> \supset D<1,-ab> \subset F \).

Now assume \(-1 \notin D<1,-a> \) and either \( D<1,1> \not\subset D<1,-ab> \) or \( D<1,ab> \not\subset D<1,-ab> \).

**Lemma 2.4:** There exists \( x \in \hat{F} \) such that \( x \notin D<1,a> \cup D<1,b> \cup D<1,-ab> \).

**Proof (1)** Suppose \( D<1,1> \not\subset D<1,-ab> \). Then there exists \( x' \in D<1,1> - D<1,-ab> \). Let \( x=ax' \). Then \( x \notin D<1,-ab> \) since \( a \in D<1,-ab> \), and \( x \notin D<1,1> \) since \( a \notin D<1,1> \). If \( x = ax' \in D<1,a> \), then \( x' \in D<1,a> \). Thus \( x' \in D<1,a> \supset D<1,1> \subseteq D<1,-a> \subseteq D<1,-ab> \) which is a contradiction. So \( x \notin D<1,a> \).

If \( x = ax' \in D<1,b> \), then \( x' \in D<1,b> \) since \( a \in D<1,b> \).

Hence \( x' \in D<1,b> \cap D<1,1> \subseteq D<1,-b> \subseteq D<1,-ab> \) which is a contradiction. So \( x \notin D<1,b> \). If \( x' = ax \in D<1,ab> \), then \( x' \in D<1,ab> \cap D<1,1> \subseteq D<1,-ab> \) which is a contradiction.

So \( ax \notin D<1,ab> \). Thus, in all, \( x \notin D<1,a> \cup D<1,b> \cup D<1,-ab> \cup D<1,1> \cup aD<1,ab> \).
(2) Suppose \( D<1,ab> \supsetneq D<1,-ab> \). Then there exists \( x \in D<1,ab> - D<1,-ab> \). \( x \notin D<1,1> \), otherwise \( x \in D<1,1> \cap D<1,ab> \subseteq D<1,-b> \subseteq D<1,-ab> \) which is a contradiction. So \( x \notin D<1,a> \). Similarly \( x \notin D<1,b> \). If \( ax \in D<1,ab> \), then \( a \in D<1,ab> b \in D<1,-a> = D<1,-b> \rightarrow -1 \in D<1,-a> \) which is a contradiction. So \( ax \notin D<1,ab> \). Thus there exists \( x \in \mathbb{F} \) such that \( x \notin D<1,a> \cup D<1,b> \cup D<1,-ab> \cup D<1,1> \cup aD<1,ab> \). □

Let \( x \in \mathbb{F} \) be as in Lemma 2.4. Hence \( -1 \notin D<1,-x> \) and \( -ab \notin D<1,-ax> \). But \( D<1,-x> \) and \( D<1,-ax> \) each have index 4 in \( \mathbb{F} \) with coset representatives \( \{1,-a,-b,ab\} \). So \( -a \notin D<1,-x> \) and \( -a'ab \notin D<1,-ax> \) for some \( a,a' \in \{-a,-b,ab\} \). Since \( -a \in D<1,-x> \), \( x \in D<1,a> \subseteq D<1,-ab> \) for \( a=-a \) or \( -b \). But \( x \notin D<1,-ab> \). So \( a=ab \). Since \( -a'ab \in D<1,-ax> \), \( ax \in D<1,a'ab \subseteq D<1,-ab> \) for \( a'=a-a \) or \( -b \). But \( ax \notin D<1,-ab> \) since \( a \in D<1,-ab> \) and \( x \notin D<1,-ab> \). So \( a'=ab \). Thus \( -ab \in D<1,-x> \) and \( -1 \in D<1,-ax> \). \( x \in D<1,1> \cap aD<1,1> \). Hence \( \langle 1,ab \rangle - a<1,1> = \langle 1,-a \rangle - a<1,-b> \) is isotropic. But \( D<1,-a> = D<1,-b> \). So \( a \in D<1,-a> \) which implies that \( -1 \in D<1,-a> \) which is a contradiction.

Thus we have shown the following:

**Proposition 2.5:** Let \( \mathbb{F} \) be a field with \( |\mathbb{F}/D<1,-a>| \leq 4 \) for every \( a \in \mathbb{F} \) and equality holding for some \( a \in \mathbb{F} \). Then there
does not exist $a, b \in \mathbb{F}$ with $-1 \not \in D<1,-a>$ and either $D<1,1> \not \subseteq D<1,-ab>$ or $D<1,a> \not \subseteq D<1,-a>$, such that $D<1,-a> = D<1,-b> \not \subseteq D<1,-ab> \not \subseteq \mathbb{F}$.

Finally, assume $-1 \not \in D<1,-a>$ and $D<1,1> \cup D<1,ab> \subseteq D<1,-ab>$. Then $D<1,1> \cap D<1,-ab> \subseteq D<1,ab>$ and $D<1,ab> = D<1,ab> \cap D<1,-ab> \subseteq D<1,1>$. So $D<1,1> = D<1,ab>$. Since $-1 \not \in D<1,-a>$, $a \not \in D<1,1>$. Thus $D<1,1> \not = D<1,-ab>$. Hence $D<1,1> = D<1,ab> \not \subseteq D<1,-ab> \not \subseteq \mathbb{F}$.

**Proposition 2.6:** Let $\mathbb{F}$ be a field as described in the above paragraph. If $z \in \mathbb{F}$ such that $z \in D<1,1>$, then the index of $D<1,-z>$ in $\mathbb{F}$ is at most 2.

**Proof.** Pick $x \in D<1,-ab> - D<1,1>$ and $y \in \mathbb{F} - D<1,-ab>$. Then $G_1 = \{1,[y,-1],[y,-ab],[y,ab]\}$ is the group of quaternion algebras for $<1,-y>$, and $G_2 = \{1,[xy,-1],[xy,-ab],[xy,ab]\}$ is the group of quaternion algebras for $<1,-xy>$.

**Notes:**

1. $[y,ab] = [xy,ab]$ since $x \in D<1,-ab>$.
2. $[y,-1] \not = [xy,-1]$ since $x \not \in D<1,1>$.
3. $[y,-ab] \not = [xy,-ab]$ since $x \not \in D<1,ab>$.
4. $[y,-ab] \not = [xy,-1]$ for if so, there would exist $z \in \mathbb{F}$ such that $[y,-ab] = [z,-ab]$ and $[xy,-1] = [z,-1]$. This follows from the result stated in the introduction. Then $z \in D<1,-ab> \cap yD<1,ab> = \emptyset$.
(5) $[y, -1] \neq [xy, -ab]$. The argument is similar to that in (4).

So $G_1 \cap G_2 = \{1, [y, ab]\}$.

Let $z \in D<1,1>$. Now

$[zy, -1] = [z, -1][y, -1] = [y, -1],
[zy, -ab] = [z, -ab][y, -ab] = [y, -ab], \text{ and}
[zy, ab] = [z, ab][y, ab] = [y, ab].$

So $G_1$ is the group of quaternion algebras for $<1, -zy>$.

Similarly $G_2$ is the group of quaternion algebras for $<1, -zxy>$.

Consider any $a \in \hat{F}$. Then $[z, a] = [zy, a] [y, a] \in G_1 \cdot G_1 = G_1$. Also $[z, a] = [zxy, a] [xy, a] \in G_2 \cdot G_2 = G_2$. So $[z, a] \in G_1 \cap G_2 = \{1, [y, ab]\}$. Thus $D<1, -z>$ has index in $\hat{F}$ at most 2.

\textbf{Corollary 2.7:} $-1 \notin D<1,1>$. 

\textbf{Proof.} Suppose $-1 \in D<1,1>$. Then $-ab \in D<1,1>$. By Proposition 2.6, $D<1,ab>$ has index in $\hat{F}$ at most 2. But $|\hat{F}/D<1,ab>| = 4$. So $-1 \notin D<1,1>$. □

\textbf{Proposition 2.8:} Let $F$ be a field as described in the paragraph before Proposition 2.6. If $x, y \in \hat{F}$ such that $x, y \in D<1,1>$ and $D<1, -x> = D<1, -y>$, then $x = y$.

\textbf{Proof.} Suppose $x \neq y$. Then $D<1, -x> = D<1, -y> = D<1, -xy>$ has index 2 in $\hat{F}$ by Proposition 2.6. If $a \notin D<1, -x>$, then
[x,α], [y,α], and [xy,α] are all non-split, and by the proof of Proposition 2.6, [x,α] = [y,α] = [xy,α]. So [xy,α] = [x,α][y,α] = 1 which is a contradiction. So x=y. □

**Proposition 2.9:** Let F be a field as described in the paragraph before Proposition 2.6. Then |F/F²| ≤ 8.

**Proof.** Suppose |𝔽/𝔽²| > 8. Then |D<1,1>/𝔽²| > 2 and there exists x ∈ D<1,1> - <ab>𝔽². Notice that if D<1,1> ⊆ D<1,-x>, then D<1,-x> = <-x>D<1,1> = D<1,-ab> since -1 ∈ D<1,-ab>. But this contradicts Proposition 2.8. So D<1,1> ∉ D<1,-x>. By Proposition 2.6, D<1,-x> has index 2 in F. Thus K = D<1,1> ∩ D<1,-x> has index 8 in F. So there exists e ∈ F such that D<1,1> = K<e>. Then D<1,-x> = K<-1,f> where F = K<-1,e,f>. Also D<1,1>,{-1} ⊆ D<1,-ab>. So D<1,-ab> = K<e,-1>. D<1,-x>, D<1,-ab>, and D<1,-xab> must be distinct subgroups of index 2 in F, and K<-1> = D<1,-x> ∩ D<1,-ab> ⊆ D<1,-xab>. So D<1,-xab> = K<-1,ef>. We now have established that D<1,1> = K<e>, D<1,-ab> = K<e,-1>, D<1,-x> = K<f,-1>, and D<1,-xab> = K<ef,-1>. Also K = D<1,1> ∩ D<1,-x> ⊆ D<1,x> and K = D<1,1> ∩ D<1,-xab> ⊆ D<1,xab>.

Since |𝔽/D<1,x>| ≤ 4, D<1,x> ∉ K. Thus there exists 1 ≠ t ∈ D<1,x> ∩ <-1,e,f>. Then D<1,x> ∩ D<1,1> ⊆ D<1,-x> = K<-1,f>. Hence t ≠ e. Also D<1,x> ∩ D<1,-x> ⊆ D<1,1> = K<e>. Hence t≠-1, ±f. Similarly, D<1,x> ∩ D<1,-xab> ⊆ D<1,ab> = D<1,1> = K<e>. Hence t≠-1, ±ef. Therefore t=-e and D<1,x> = K<-e>. The same argument works to show that D<1,xab> = K<-e>. 
Thus $D^{1,x} = D^{1, xab} \subseteq D^{1, -ab}$. By the proof of Proposition 2.6, $|\tilde{F}/D^{1,-z}| \leq 2$ for every $z \in D^{1,x} = K^{<e}$. Moreover, all nonsplit quaternion algebras of the form $[z,w]$, where $z \in D^{1,x}$ and $w \in \tilde{F}$, are equal.

Since we have assumed that $|D^{1,1}/F^2| > 2$, we must have $|K/F^2| > 1$. Let $z_0 \in K - F^2$. Then there exists $w_0 \in \tilde{F}$ such that $[z_0,w_0] \neq 1$. Thus all non-split quaternion algebras of the form $[z,w]$, for $z \in D^{1,x}$ and $w \in \tilde{F}$, must be equal to $[z_0,w_0]$. Similarly, all non-split quaternion algebras of the form $[z',w']$, for $z' \in D^{1,1}$ and $w' \in \tilde{F}$, must be equal to $[z_0,w_0]$. Therefore, all nonsplit quaternion algebras $[z,w]$, where $z \in D^{1,x} \cup D^{1,1} = K \cup K^{<e}$ and $w \in \tilde{F}$, are equal.

Let $w \in \tilde{F} - (D^{1,e} \cup D^{1,-e})$. Then $[-e,w]$ and $[e,w]$ are nonsplit and hence must be equal. Thus $1 = [-e,w][e,w] = [-1,w]$. So $w \in D^{1,1}$. Now $e \in D^{1,1}$. By Proposition 2.6, $D^{1,-e}$ has index 2 in $\tilde{F}$. Also $-e \in D^{1,1}$. Again by Proposition 2.6, $D^{1,e}$ has index 2 in $\tilde{F}$. Also $D^{1,-e} \cap D^{1,e} \subseteq D^{1,1}$. So $D^{1,-e} \cap D^{1,e} = D^{1,1}$. Hence $w \in D^{1,1} \subseteq D^{1,-e} \cup D^{1,e}$ which is a contradiction. So $|\tilde{F}/F^2| \leq 8$. □

Propositions 2.3, 2.5, and 2.9 imply the following theorem.

Theorem 2.10: Let $F$ be a field such that $|\tilde{F}/D^{1,-\alpha}| \leq 4$ for every $\alpha \in \tilde{F}$ and equality holding for some $\alpha \in \tilde{F}$. Also
assume that \( R(F) = F^2 \). If there exist \( a, b \in F \) such that
\( D<1, -a> = D<1, -b> \nsubseteq D<1, -ab> \nsubseteq F \), then \( |F/F^2| = 8 \). □

**Case II:** Let \( F \) be a field with \( |F/D<1, -a>| \leq 4 \) for every \( a \in F \) and equality holding for some \( a \in F \). Also assume that \( R(F) = F^2 \). Suppose there exist \( a, b \in F \) such that
\( D<1, -a> = D<1, -b> = D<1, -ab> \) has index 4 in \( F \). Then \(-1 \in D<1, -a>\). Let \( y \in F - D<1, -a>\).

**Proposition 2.11:** There exists \( x \in F - <y>D<1, -a> \) such that
\( x \notin D<1, -y> \cup aD<1, -aby> \cup bD<1, -aby> \).

**Proof.** (1) Assume \( D<1, -y> \subseteq <y>D<1, -a> = <y>D<1, -ab> \).

Note: \( ab \in <y>D<1, -a> - D<1, -y> \). So \( <y>D<1, -a> = <ab>D<1, -y> \). \( D<1, -y> = D<1, -y> \cap <y>D<1, -a> \subseteq <-y>D<1, -aby> \) and \( <-y>D<1, -aby> \nsubseteq F \) since \( a, b, ab \notin D<1, -aby> \). \( ab \in <-y>D<1, -aby> - D<1, -y> \). So \( <-y>D<1, -aby> = <ab>D<1, -y> \).

Thus \( <y>D<1, -a> = <-y>D<1, -aby> \Rightarrow a \in <-y>D<1, -aby> \Rightarrow a \in D<1, -aby> \) or \(-ay \in D<1, -aby> \Rightarrow a \in D<1, -aby> \) or \( b \in D<1, -aby> \Rightarrow aby \in D<1, -a> \) or \( aby \in D<1, -b> = D<1, -a> \Rightarrow y \in D<1, -a> \) since \( ab \in D<1, -a> \). But this is a contradiction.

(2) Thus \( D<1, -y> \notin <y>D<1, -a> = <y>D<1, -ab> \).

Choose \( x' \in D<1, -y> - <y>D<1, -ab> \). Let \( x = ax' \). Then
\( x \notin <y>D<1, -ab> \) since \( a \in <y>D<1, -ab> \) and \( x \notin D<1, -y> \) since \( a \notin D<1, -y> \). If \( ax = x' \in D<1, -aby> \), then \( x' \in D<1, -aby> \cap D<1, -y> \subseteq D<1, -ab> \) which is a contradiction. So
ax \notin D<1,-aby>. If bx = abx' \in D<1,-aby>, then x' \in D<1,-y>
\cap ab D<1,-aby> \subseteq -y D<1,-ab> = y D<1,-ab> which is a contradiction. So bx \notin D<1,-aby>. So there exists x \in 
\hat{F} - <y>D<1,-a> such that x \notin D<1,-y> \cup aD<1,-aby> \cup bD<1,-aby>. □

For an x \in \hat{F} as in Proposition 2.11, y \notin D<1,-x> and aby 
\notin D<1,-ax> \cup D<1,-bx>. But D<1,-x>, D<1,-ax>, and D<1,-bx>
each have index 4 in \hat{F} with coset representatives \{1,a,b,ab\}. So ay \in D<1,-x>, a'aby \in D<1,-ax>, and a"aby \in D<1,-bx>, for
some a,a',a" \in \{a,b,ab\}. If a' = ab, then ay \in D<1,-x> \cap 
D<1,-ax> \subseteq D<1,-a> \implies y \in D<1,-a> which is a contradiction.
So a' \neq ab. If a' = ab, then ay \in D<1,-x> and aay \in 
D<1,-ax>. \implies -axy \in D<1,-x> \cap D<1,-ax> \subseteq D<1,-a> \implies x \in 
yD<1,-a> which is a contradiction. So a' \neq ab. Thus a' = a
or a". If a" = ab, then ay \in D<1,-x> \cap D<1,-bx> \subseteq D<1,-b>
\implies y \in D<1,-b> which is a contradiction. So a" \neq ab. If
a" = a, then ay \in D<1,-x> and aby \in D<1,-bx>. \implies -axy 
\in D<1,-x> \cap D<1,-bx> \subseteq D<1,-b>. \implies x \in yD<1,-b> which is a
contradiction. So a" \neq a. Thus a" = a or ab.

Now suppose a' = a and a" = ab. Then aby \in D<1,-ax> and 
\text{aay} \in D<1,-bx>. \implies -aabyxy \in D<1,-ax> \cap D<1,-bx> \subseteq D<1,-ab>
\implies x \in yD<1,-ab> which is a contradiction. So either a' = a
or a" = a. Suppose a' = a. Then ay \in D<1,-x> and aaby \in 
D<1,-ax>. \implies x \in yD<1,-a> \cup \text{aD}<1,-aaby>. So
\langle 1,-\text{ay} \rangle - a \langle 1,-\text{aby} \rangle = \langle 1,-a \rangle - \text{ay}<1,-b> is isotropic. But
D<1,-a> = D<1,-b>. So ay \in D<1,-b> and thus y \in D<1,-b>
which is a contradiction. So $a' \neq a$. A similar argument shows $a'' \neq a$.

Thus we have shown the following theorem:

**Theorem 2.12:** Let $F$ be a field such that $|\dot{F}/D<1,-a>| \leq 4$ for every $a \in F$ and equality holding for some $a \in F$. Also assume that $R(F) = \dot{F}^2$. Then there does not exist $a,b \in \dot{F}$ such that $D<1,-a> = D<1,-b> = D<1,-ab>$ has index 4 in $F$. □

**Case III:** Let $F$ be a field with $|\dot{F}/D<1,-a>| \leq 4$ for every $a \in F$ and equality holding for some $a \in F$. Also assume that $R(F) = \dot{F}^2$. Suppose there exist $a,b \in \dot{F}$ such that $D<1,-a>$, $D<1,-b>$, and $D<1,-ab>$ each has index 4 in $F$ and $H = D<1,-a> \cap D<1,-b> = D<1,-a> \cap D<1,-ab> = D<1,-b> \cap D<1,-ab>$ has index 8 in $F$.

(A) Let us suppose that there exist $e,f \in \dot{F}$ such that $D<1,-a> = H<e>$, $D<1,-b> = H<f>$, and $D<1,-ab> = H<ef>$.

**Proposition 2.13:** If $x \in \dot{F} - H<e,f>$, then $H \cap D<1,-x> = H \cap D<1,-ax> = H \cap D<1,-bx> = H \cap D<1,-abx> = D<1,-x> \cap D<1,-ax> \cap D<1,-bx> \cap D<1,-abx>$.

**Proof.** Let $G = D<1,-x> \cap D<1,-ax> \cap D<1,-bx> \cap D<1,-abx>$. Then $G \subseteq D<1,-x> \cap D<1,-ax> \subseteq D<1,-a>$ and $G \subseteq D<1,-x> \cap D<1,-bx> \subseteq D<1,-b>$. So $G \subseteq D<1,-a> \cap D<1,-b> = H$. Thus clearly $G \subseteq H \cap D<1,-ax>$ for $a \in <a,b>$. Now suppose $h \in H \cap D<1,-ax>$ for some $a \in <a,b>$. Then $h \in H<e> = D<1,-a>$, $h \in H<f> = D<1,-b>$, and $h \in H<ef> = D<1,-ab>$. So
Proposition 2.14: \(-1 \in D<1,-a> \cup D<1,-b> \cup D<1,-ab> = H<e,f>.

Proof. Note that \(-a \in H<e>, -b \in H<f>, \) and \(-ab \in H<ef>.

If \(-a \in H<e> \) and \(-b \in H<f> \), then \(-ab \in H<ef> \). But \(-ab \in H<ef> \).

So either \(-e \) or \(-f \) must be in \(H\). Hence, we could have \(-1 \in eH \cup fH \subseteq D<1,-a> \cup D<1,-b> \).

If \(-a \in H<e> \) and \(-b \in H<f> \), then \(-ab \in H<ef> \). But \(-ab \in H<ef> \).

So \(-ef \) or \(-1 \in H<e,f> \). Hence, we could have \(-1 \in efH \cup H = D<1,-ab> \).

If \(-a,-b \in H \), then \(-ab \in H<ef> \). But \(-ab \in H<ef> \).

So \(-1 \) or \(-ef \in H<ef> \) and \(-1 \in D<1,-ab> \).

Therefore \(-1 \in D<1,-a> \cup D<1,-b> \cup D<1,-ab> = H<e,f> \). □

(1) First, let us assume \(-1 \in H<e> \) and \(a,b,ab \notin H \). Then \(D<1,-a> = H<a> \), \(D<1,-b> = H<b> \), and \(D<1,-ab> = H<ab> \). Consider any \(x \in \hat{F} - H<1,b> \). Then \(D<1,-x> \) must have index 4 in \(\hat{F} \) with coset representatives \(\{1,a,b,ab\} \). Let \(G = H \cap D<1,-x> \).

Since \(|\hat{F}/D<1,-x>| = 4 \) and \(|\hat{F}/H| = 8 \), \(|H/G| = n = 1,2,4 \).

(i) Assume \(n=1 \). Then \(H \subseteq D<1,-x> \). \(\hat{F} = H<1,a,b,x> \). Thus \(D<1,-x> = H<x> \). Fix \(\alpha \in \{a,b,ab\} \). Then for \(\beta \in \{a,b,ab\} \), \(\beta \in D<1,-ax> = ax \in D<1,-\beta> \Rightarrow x \in D<1,-\beta> \). But
x ⋄ H<a,b> = D<1,-a> U D<1,-b> U D<1,-ab>. So a,b,ab ⋄ D<1,-ax> for α ∈ {a,b,ab} and hence |F/D<1,-ax>| = 4, for α ∈ {a,b,ab}. By Proposition 2.13, H = H ∩ D<1,-x> ⋐ D<1,-ax> for α ∈ {a,b,ab}. Thus D<1,-ax> = H<ax>, D<1,-bx> = H<bx>, and D<1,-abx> = H<abx>. So D<1,-a>, D<1,-b>, D<1,-ab>, D<1,-x>, D<1,-ax>, D<1,-bx>, and D<1,-abx> are all of the index 4 subgroups of F which contain H. Suppose there exists yeF such that y ⋄ H<a,b> U x·<a,b> and H ⋐ D<1,-y>. Since y ⋄ H<a,b> = D<1,-a> U D<1,-b> U D<1,-ab>, |F/D<1,-y>| = 4. So D<1,-y> is an index 4 subgroup of F which contains H and must be equal to one of the seven listed above. If D<1,-y> = D<1,-α> for α ∈ {a,b,ab}, then -y ∈ D<1,-α> ⇒ y ∈ D<1,α> ⋐ H<a,b>. But y ⋄ H<a,b>. So D<1,-y> = D<1,-ax> for α ∈ {a,b}. But this has been shown in Cases I and II to imply that |F/F^2| = 8. Then either F = H<a,b> U x·<a,b> which implies that |F/F^2| = 8, or there exists yeF such that y ⋄ H<a,b> U x·<a,b> and H ⋐ D<1,-y> which is considered in (ii) and (iii).

(ii) Assume n = |H/G| = 2. Let h ∈ H - H ∩ D<1,-x>. Then ah ∈ D<1,-x> for some α ∈ {a,b,ab}. Then D<1,-x> = (H ∩ D<1,-x>)·<ah,-x> = G<ah,-x>.

Note: ah ∈ D<1,-x> ∩ D<1,-α> ⋐ D<1,-ax>. By Proposition 2.13, H ∩ D<1,-x> = H ∩ D<1,-ax> = G. So we have D<1,-ax> = G<ah,-ax>. Now choose α ∈ {a,b,ab} and consider D<1,-bx>. By Proposition 2.13, H ∩ D<1,-x> = H ∩ D<1,-bx> = G. So h ⋄ D<1,-bx>. Since D<1,-bx> has index 4 in F with coset representatives {1,a,b,ab}, yh∈D<1,-bx> for some y ∈ {a,b,ab}. 


Thus as above, $D<1,-\beta x> = G<\gamma h,-\beta x>$ and $D<1,-\beta Yx> = G<\gamma h,-\beta Yx>$.

So $D<1,-x> = G<\alpha h,-x>$ \hspace{1cm} (1)

$D<1,-\alpha x> = G<\alpha h,-\alpha x>$ \hspace{1cm} (2)

$D<1,-\beta x> = G<\gamma h,-\beta x>$ \hspace{1cm} (3)

$D<1,-\beta Yx> = G<\gamma h,-\beta Yx>$ \hspace{1cm} (4)

Claim: $\gamma = \alpha$. If not, then $\gamma = \alpha \beta$ or $\beta$.

If $\gamma = \alpha \beta$, then from (2) and (4), $D<1,-\alpha x>$ contains $\alpha h$ and $\alpha \beta h$ and hence $\beta$, which is a contradiction. If $\gamma = \beta$, then from (1) and (4), $D<1,-x>$ contains $\alpha h$ and $\beta h$ and hence $\alpha \beta$, which is a contradiction. So $\gamma = \alpha$ and we have $\alpha h \in D<1,-x> \cap D<1,-\alpha x> \cap D<1,-\alpha \beta x> = G \subseteq H$. Thus $\alpha \in H$, which is a contradiction. Thus $n \neq 2$.

(iii) Assume $n = |H/G| = 4$. Let $h,k \in H$ such that $H = G<h,k>$. Since $h,k \notin D<1,-x>$, there exist $\alpha,\beta \in \{a,b,ab\}$ such that $\alpha h,\beta k \in D<1,-x>$. If $\alpha = \beta$, then $\alpha h,\alpha k \in D<1,-x>$ and $\alpha k \in D<1,-x>$, which is a contradiction. So $\alpha \neq \beta$.

$\alpha h \in D<1,-x> \cap D<1,-\alpha> \subseteq D<1,-\alpha x>$. By Proposition 2.13, $G = D<1,-x> \cap H = D<1,-\alpha x> \cap H$. So $k \notin D<1,-\alpha x>$. Thus $\alpha k, \beta k, \alpha \beta k \in D<1,-\alpha x>$. Again, if $\alpha k \in D<1,-\alpha k>$, then $hk \in D<1,-\alpha x>$ and thus $hk \in D<1,-\alpha x> \cap H = D<1,-x> \cap H = G$ which is a contradiction. If $\beta k \in D<1,-\alpha k>$, then $\beta k \in D<1,-x> \cap D<1,-\alpha x> \subseteq D<1,-\alpha>$

$\Rightarrow \beta \in D<1,-\alpha>$ since $k \in H \subseteq D<1,-\alpha>$.

$\Rightarrow \beta \in D<1,-\alpha> \cap D<1,-\beta> = H$ since $\alpha \neq \beta$.

But $\{a,ab,ab\} \cap H = \emptyset$. So $\beta k \notin D<1,-\alpha x>$. Hence $\alpha \beta k \in D<1,-\alpha x>$. Then $\alpha \beta k \in D<1,-\alpha x> \cap D<1,-\alpha \beta> \subseteq D<1,-\beta x>$. Again
by Proposition 2.13, \( G = D<1,-x> \cap H = D<1,-\beta x> \cap H \). Since 
\( h \in H - G, h \notin D<1,-\beta x> \). Thus \( ah, bh, \) or \( abh \in D<1,-\beta x> \). If 
\( ah \in D<1,-\beta x>, \) then \( ah \in D<1,-\beta x> \cap D<1,-x> \subseteq D<1,-\beta> \rightarrow 
\alpha \in D<1,-\beta> \) since \( k \in H \subseteq D<1,-\beta> \).

\( \alpha \in D<1,-\beta> \cap D<1,-\alpha> = H \) since \( \alpha \neq \beta \). But \( \{a,b,ab\} \cap H = \emptyset \).

So \( ah \notin D<1,-\beta x> \). If \( abh \in D<1,-\beta x> \), then \( abh, abk \in 
D<1,-\beta x> \) and hence \( hk \in D<1,-\beta x> \rightarrow hk \in D<1,-\beta x> \cap H = 
D<1,-x> \cap H = G, \) which is a contradiction. Thus \( bh \in 
D<1,-\beta x> \). Hence \( bh \in D<1,-\beta x> \cap D<1,-\beta> \subseteq D<1,-x> 
\rightarrow \beta \in D<1,-x> \) since \( h \in H \subseteq D<1,-x> \). \( \Rightarrow x \in D<1,-\beta> \subseteq 
H\langle a,b \rangle \) which is a contradiction. Thus \( n \neq 4 \).

(2) Secondly, let us assume that \(-1 \in H \) and either \( a, b, \) or 
\( ab \in H \). Say \( a \in H \). If \( b \notin H \), then \( b \in fH \Rightarrow ab \in fH \). But \( ab 
\in H \langle e,f \rangle \). So \( b \in H \langle e,f \rangle \). Thus \( a, b, \) and \( ab \) in any case are either 
all in \( H \) or none is in \( H \).

Let \( x \in F = H \langle e,f \rangle \). Then \( a,b,ab \notin D<1,-x> \). Then \(|H/G| = 4 \) where 
\( G = H \cap D<1,-x> \), and \( F = \langle e,f,x \rangle H = \langle e,f,x,a,b \rangle G \).

Also \( |D<1,-x>/G| = 8 \). Since \( a,b,ab \notin D<1,-x> \), \( D<1,-x> = 
\langle \alpha e, \beta f, -x \rangle G \) for some \( \alpha, \beta \in \langle a,b \rangle \). \( \alpha e \in D<1,-x> \cap D<1,-a> \subseteq 
D<1,-ax> \) and \( G \subseteq D<1,-ax> \) by Proposition 2.13. So \( D<1,-ax> 
= \langle \alpha e, \delta f, -ax \rangle G \) for some \( \delta \in \langle a,b \rangle \). If \( \beta = \delta \), then \( \beta f \in 
D<1,-ax> \cap D<1,-x> \subseteq D<1,-a> \Rightarrow f \in D<1,-a> \) which is a 
contradiction. So \( \delta \neq \beta \). \( \beta f \in D<1,-x> \cap D<1,-b> \subseteq D<1,-bx> 
\). So similarly \( D<1,-bx> = \langle \gamma e, \beta f, -bx \rangle G \) for some \( \gamma \in \langle a,b \rangle \). If 
\( \gamma = a \), then \( \alpha e \in D<1,-bx> \cap D<1,-x> \subseteq D<1,-b> \).
\[ e \in D<1,-b> \] which is a contradiction. So \( Y \neq a \). \( Ye \in D<1,-bx> \cap D<1,-a> \subseteq D<1,-abx> \) and \( \delta f \in D<1,-ax> \cap D<1,-b> \subseteq D<1,-abx> \). Thus \( D<1,-abx> = <Ye,\delta f,-abx>G \). Thus we have established \( D<1,-x> = <ae,\beta f,-x>G \), \( D<1,-ax> = <ae,\delta f,-ax>G \), \( D<1,-bx> = <Ye,\beta f,-bx>G \), and \( D<1,-abx> = <Ye,\delta f,-abx>G \) for some \( a,\beta,\gamma,\delta \in <a,b> \) and \( Y \neq a \) and \( \delta \neq b \).

If \( Y=ab \), then \( -ax \in D<1,-bx> \cap D<1,-x> \subseteq D<1,-b> \).
\[ \Rightarrow \ ex \in D<1,-b> \Rightarrow x \in eD<1,-b> \subseteq H<e,f> \] which is a contradiction. So \( Y \neq ab \). If \( Y=ab \), then \( -ax \in D<1,-abx> \cap D<1,-x> \subseteq D<1,-ab> \Rightarrow ex \in D<1,-ab> \Rightarrow x \in eD<1,-ab> \subseteq H<e,f> \) which is a contradiction. So \( Y \neq ab \). Thus \( Y=aa \).

If \( \delta = ba \), then \( -bx \in D<1,-ax> \cap D<1,-x> \subseteq D<1,-a> \).
\[ \Rightarrow fx \in D<1,-a> \Rightarrow x \in fD<1,-a> \subseteq H<e,f> \] which is a contradiction. So \( \delta \neq ba \). If \( \delta = bab \), then \( -bx \in D<1,-abx> \cap D<1,-x> \subseteq D<1,-ab> \) \( \Rightarrow fx \in D<1,-ab> \Rightarrow x \in fD<1,-ab> \subseteq H<e,f> \) which is a contradiction. So \( \delta \neq bab \). Thus \( \delta = b \).

So \( Y=aa \) and \( \delta = b \). Then \( -b \in D<1,-x> \cap D<1,-abx> \subseteq D<1,-ab> \). \[ \Rightarrow ef \in D<1,-ab> \Rightarrow x \in efD<1,-ab> \subseteq H<e,f> \] which is a contradiction.

Thus we have shown that it is not possible for \(-1 \in H \) and at least one of \( a, b, ab \) to also be in \( H \).

(3) Finally, let us assume \(-1 \notin H \). Thus by Proposition 2.14, \(-1 \) is in exactly one of \( D<1,-a> \), \( D<1,-b> \), and \( D<1,-ab> \). Without loss of generality, say \(-1 \in D<1,-a> \). Thus \( D<1,-a> = H<-1> \) and either \( a \) or \(-a \in H \).
First, suppose $-a \in H$. Then $D\langle 1, -a \rangle = H\langle -1 \rangle$, $D\langle 1, -b \rangle = H\langle f \rangle$, and $D\langle 1, -ab \rangle = H\langle f \rangle$. If $-b \in H$, then $-ab \in H$. But $-ab \in H\langle -f \rangle$. So $-1$ or $f \in H$ which is a contradiction. Thus $-b \notin H$ and then $-ab \notin H$. So $D\langle 1, -a \rangle = H\langle -1 \rangle = H\langle a \rangle$, $D\langle 1, -b \rangle = H\langle -b \rangle = H\langle ab \rangle$, and $D\langle 1, -ab \rangle = H\langle -ab \rangle = H\langle b \rangle$.

Let $x \in F - H\langle -1, f \rangle = F - H\langle -1, b \rangle = F - H\langle a, b \rangle$. Then $F = \langle -1, f, x \rangle H = \langle -1, b, x \rangle H = \langle a, b, x \rangle H$ and $D\langle 1, -x \rangle$ has index 4 in $F$ with coset representatives $\{1, a, b, ab\}$. Let $G = H \cap D\langle 1, -x \rangle$. Then $|H/G| = n = 1, 2,$ or 4.

(i) Suppose $n = 1$. Then $H \subseteq D\langle 1, -x \rangle$ and $D\langle 1, -x \rangle = H\langle -x \rangle = H\langle ax \rangle$. By Proposition 2.13, $H \subseteq D\langle 1, -ax \rangle$ for each $\alpha \in \{a, b, ab\}$. Also, for $\beta \in \{a, b, ab\}$, $\beta \in D\langle 1, -ax \rangle = \alpha x \in D\langle 1, -\beta \rangle \subseteq \alpha H\langle a, b \rangle = H\langle a, b \rangle$. But $x \in F - H\langle a, b \rangle$. So $\beta \notin D\langle 1, -ax \rangle$ and $|F/D\langle 1, -ax \rangle| = 4$ for $\alpha \in \{a, b, ab\}$. Thus $D\langle 1, -ax \rangle = H\langle -ax \rangle = H\langle x \rangle$, $D\langle 1, -bx \rangle = H\langle -bx \rangle = H\langle abx \rangle$, and $D\langle 1, -abx \rangle = H\langle -abx \rangle = H\langle bx \rangle$. Then $D\langle 1, -a \rangle$, $D\langle 1, -b \rangle$, $D\langle 1, -ab \rangle$, $D\langle 1, -x \rangle$, $D\langle 1, -ax \rangle$, $D\langle 1, -bx \rangle$, and $D\langle 1, -abx \rangle$ are all of the index 4 subgroups of $F$ which contain $H$. Suppose there exists $y \in F$ such that $y \notin H\langle a, b \rangle \cup x\langle a, b \rangle$ and $H \subseteq D\langle 1, -y \rangle$. Since $y \notin H\langle a, b \rangle = D\langle 1, -a \rangle \cup D\langle 1, -b \rangle \cup D\langle 1, -ab \rangle$, $|F/D\langle 1, -y \rangle| = 4$. So $D\langle 1, -y \rangle$ is an index 4 subgroup of $F$ which contains $H$ and thus must be equal to one of the seven listed above. If $D\langle 1, -y \rangle = D\langle 1, -\alpha \rangle$ for $\alpha \in \{a, b, ab\}$, then $-y \in D\langle 1, -\alpha \rangle = -1 \cdot D\langle 1, -\alpha \rangle \subseteq -1 \cdot H\langle a, b \rangle = H\langle a, b \rangle$. But $y \notin H\langle a, b \rangle$. So $D\langle 1, -y \rangle \neq D\langle 1, -\alpha \rangle$ for $\alpha \in \{a, b, ab\}$. Hence $D\langle 1, -y \rangle = D\langle 1, -ax \rangle$ for $\alpha \in \langle a, b \rangle$. But this has been shown in Cases I and II to imply that $|\hat{F}/\hat{F}^2| = 8$. 
Then either $F = H(a,b) \cup x \cdot (a,b)$ which implies that $|\hat{F}/\hat{F}^2| = 8$, or there exists $y \in F$ such that $y \notin H(a,b) \cup x \cdot (a,b)$ and $H \not\subset D^{<1,-y>}$ which is considered in (ii) and (iii).

(ii) Suppose $n=2$. Let $h \in H - G$. Then $F = \langle -1,b,x \rangle H = \langle -1,b,x,h \rangle G$. Since $h \notin D^{<1,-x>}$, $ah \in D^{<1,-x>}$ for some $a \in \{a,b,ab\}$. Suppose $a=a$. Then $ah \in D^{<1,-x>}$ and $D^{<1,-x>} = \langle ah,-x \rangle G$. Now choose $b \in \{b,ab\}$. Then $h \notin D^{<1,-b-x> \cup D^{<1,-ab-x>}}$ by Proposition 2.13. So $\gamma h \in D^{<1,-b-x>}$ by Proposition 2.13. So $\gamma h \in D^{<1,-b-x> \cup D^{<1,-ab-x>}}$ by Proposition 2.13. So $\gamma h \in D^{<1,-b-x>}$ which is a contradiction. So $\gamma \neq a$. Similarly, $\delta \neq a$. If $\gamma = \delta$, then $\gamma h \in D^{<1,-b-x> \cup D^{<1,-ab-x>}}$ which is a contradiction. So $\gamma \neq \delta$. Thus $\delta = ay$ and $-byhx \in D^{<1,-b-x> \cup D^{<1,-ab-x>}} \subseteq D^{<1,-a>}$. So $byhx \in D^{<1,-a>}$.

But $by = 1$ or $a$, so $x \in D^{<1,-a>}$ which is a contradiction. Therefore, $a \neq a$.

Thus $a \in \{b,ab\}$ and $D^{<1,-x>} = \langle ah,-x \rangle G$. If $ah \in D^{<1,-ax>}$, then $ah \in D^{<1,-ax> \cup D^{<1,-x>}} \subseteq D^{<1,-a>}$ which is a contradiction. So $ah \notin D^{<1,-a>}$. So $D^{<1,-ax>} = \langle bh,-ax \rangle G$ for some $\beta \neq a \in \{a,b,ab\}$. Thus $\beta \in \{a,a\}$.

Suppose $\beta = a$. Then $D^{<1,-ax>} = \langle ah,-ax \rangle G$ and $ah \in D^{<1,-ax> \cup D^{<1,-a>} \subseteq D^{<1,-ax>}}$. So $D^{<1,-ax>}, D^{<1,-ax>} = \langle ah,-ax \rangle G$. Now proceed as in the $a \neq a$ case above with $a \in x$ in place of $x$. We can do this because of Proposition 2.13. This shows that $\beta \neq a$.

Suppose $\beta = a\alpha$. Then $D^{<1,-ax>} = \langle a\beta h,-ax \rangle G$. Consider $D^{<1,-ax>} = \langle \delta h,-ax \rangle G$ for some $\delta \in \{a,b,ab\} = \{a,\alpha,\alpha a\}$. If
\( \delta = a \), then \( ah \in D \langle 1, -ax \rangle \cap D \langle 1, -a \rangle \subseteq D \langle 1, -x \rangle \) which is a contradiction. If \( \delta = a \), then \( ah \in D \langle 1, -ax \rangle \cap D \langle 1, -x \rangle \subseteq D \langle 1, -a \rangle \) which is a contradiction. If \( \delta = a\alpha \), then \( a\alpha h \in D \langle 1, -ax \rangle \cap D \langle 1, -ax \rangle \subseteq D \langle 1, -\alpha a \rangle \) implies \( -1 \in D \langle 1, -\alpha a \rangle \) which is a contradiction.

Thus \( n \neq 2 \).

(iii) Suppose \( n = 4 \). Let \( h, k \in H \) such that \( H = \langle h, k \rangle G \). Then \( \hat{G} = \langle -1, b, x, h, k \rangle G \). Since \( h, k \notin D \langle 1, -x \rangle \), \( ah, \beta k \in D \langle 1, -x \rangle \) for some \( \alpha, \beta \in \{a, b, ab\} \).

Note: \( \alpha \neq \beta \) since otherwise \( hk \in D \langle 1, -x \rangle \). Suppose \( \{\alpha, \beta\} = \{b, ab\} \), say \( \alpha = b \) and \( \beta = ab \). Then \( bh, abk \in D \langle 1, -x \rangle \Rightarrow ahk \in D \langle 1, -x \rangle \). \( H = \langle h, k \rangle G \) can be written \( H = \langle h, \beta k \rangle G \). Thus we may assume that either \( \alpha \) or \( \beta \) is equal to \( a \). Say \( \alpha = a \). Then \( ah \in D \langle 1, -x \rangle \cap D \langle 1, -a \rangle \subseteq D \langle 1, -ax \rangle \). There exist \( Y, \delta \in \{a, b, ab\} \) such that \( Yh \in D \langle 1, -bx \rangle \) and \( \delta h \in D \langle 1, -abx \rangle \). If \( Y = a \), then \( ah \in D \langle 1, -bx \rangle \cap D \langle 1, -a \rangle \subseteq D \langle 1, -abx \rangle \).

\( \Rightarrow ah \in D \langle 1, -abx \rangle \cap D \langle 1, -ax \rangle \subseteq D \langle 1, -b \rangle \) which is a contradiction. So \( Y \neq a \). Similarly \( \delta \neq a \). If \( Y = \delta \), then \( Yh \in D \langle 1, -bx \rangle \cap D \langle 1, -abx \rangle \subseteq D \langle 1, -a \rangle \). Thus \( Y \in D \langle 1, -a \rangle = H \langle a \rangle \Rightarrow Y = a \) since \( b, ab \notin H \). But we have just shown \( Y \neq a \). Thus \( \delta = aY \). Then \( -Yhb \in D \langle 1, -bx \rangle \cap D \langle 1, -abx \rangle \subseteq D \langle 1, -a \rangle \).

\( \Rightarrow Ybx \in D \langle 1, -a \rangle \). But \( Yb = 1 \) or \( a \) and \( a \in D \langle 1, -a \rangle \). So \( x \in D \langle 1, -a \rangle \) which is a contradiction. Thus \( a = a \) leads to a contradiction and \( \alpha \neq a \). Similarly \( \beta \neq a \). Hence \( n \neq 4 \).

Now assume \( a \in H \). Then \( D \langle 1, -a \rangle = H \langle -a \rangle = H \langle -1 \rangle \), \( D \langle 1, -b \rangle = H \langle f \rangle \), and \( D \langle 1, -ab \rangle = H \langle -f \rangle \). If \( -b \notin H \), then \( -b \in H \) and \( -ab \notin H \). But \( -ab \in H \langle -f \rangle \). So \( -b \in H \) and \( -ab \in H \). Let
x ∈ \mathbb{F} \setminus H\langle -1, f \rangle$. Then $D\langle 1, -x \rangle$ has index 4 in $\mathbb{F}$ with coset representatives \{1, a, b, ab\}. Let $G = H \cap D\langle 1, -x \rangle$. Since $a \in H - G$, $|H/G| = n = 2$ or 4.

(i) Suppose $n = 2$. Then $H = \langle a \rangle G$ and $\mathbb{F} = \langle -1, f, x \rangle H = \langle -1, f, x, a \rangle G$. Then $D\langle 1, -x \rangle = \langle \alpha, -x \rangle G$ where $\alpha \in \{±f, ±af, -1, -a\}$. Suppose $\alpha = 1$ or $-a$. Then $\alpha \in D\langle 1, -x \rangle \cap D\langle 1, -a \rangle \subseteq D\langle 1, -ax \rangle$. So $D\langle 1, -ax \rangle = \langle \alpha, -ax \rangle G$. Now $\alpha \notin D\langle 1, -bx \rangle$, otherwise $\alpha \in D\langle 1, -b \rangle$. So $D\langle 1, -bx \rangle = \langle \beta, -bx \rangle G$ for some $\beta \in \{±f, ±af\}$. If $\beta = f$ or $af$, then $\beta \in D\langle 1, -b \rangle \cap D\langle 1, -bx \rangle \subseteq D\langle 1, -x \rangle = \langle \alpha, -x \rangle G \subseteq \langle \alpha, -x \rangle H = \langle -1, x \rangle H$. Then $f \in \langle -1, x \rangle H$ which is a contradiction. If $\beta = -f$ or $-af$, then $\beta \in D\langle 1, -ab \rangle \cap D\langle 1, -bx \rangle \subseteq D\langle 1, -ax \rangle = \langle \alpha, -ax \rangle G \subseteq \langle \alpha, -ax \rangle H = \langle -1, x \rangle H$. Then again $f \in \langle -1, x \rangle H$ which is a contradiction. So $\alpha \neq 1$ or $-a$.

Suppose $\alpha = f$ or $af$. Then $\alpha \in D\langle 1, -x \rangle \cap D\langle 1, -b \rangle \subseteq D\langle 1, -bx \rangle$. So $D\langle 1, -bx \rangle = \langle \alpha, -bx \rangle G$. Now $\alpha \notin D\langle 1, -ax \rangle$, otherwise $\alpha \in D\langle 1, -a \rangle$. Hence $D\langle 1, -ax \rangle = \langle \beta, -ax \rangle G$ for some $\beta \in \{-1, -a, -f, -af\}$. If $\beta = -1$ or $-a$, then $\beta \in D\langle 1, -ax \rangle \cap D\langle 1, -a \rangle \subseteq D\langle 1, -x \rangle = \langle \alpha, -x \rangle G \subseteq \langle \alpha, -x \rangle H = \langle f, -x \rangle H \Rightarrow -1 \in \langle f, -x \rangle H$ which is a contradiction. If $\beta = -f$ or $-af$, then $\beta \in D\langle 1, -ax \rangle \cap D\langle 1, -ab \rangle \subseteq D\langle 1, -bx \rangle = \langle \alpha, -bx \rangle G \subseteq \langle \alpha, -bx \rangle H = \langle f, x \rangle H \Rightarrow -1 \in \langle f, x \rangle H$ which is a contradiction. So $\alpha \neq f$ or $af$.

A similar argument shows $\alpha \neq -f$ or $-af$.

Hence $n = 2$.

(ii) Suppose $n = 4$. Recall that $-b, -ab \in H$ and $D\langle 1, -a \rangle = H\langle -1 \rangle$, $D\langle 1, -b \rangle = H\langle f \rangle$, and $D\langle 1, -ab \rangle = H\langle -f \rangle$. Also recall that $x \in \mathbb{F} \setminus H\langle f, -1 \rangle = \mathbb{F} \setminus [D\langle 1, -a \rangle U D\langle 1, -b \rangle U D\langle 1, -ab \rangle]$. 


Note: \(bf \in bfH = -fH \subseteq D<1,-ab>\) and \(abf \in abfH = -fH \subseteq D<1,-ab>\).

Suppose \(f \in D<1,-x>\). Then \(f \in D<1,-x> \cap D<1,-b> \subseteq D<1,-bx>\). Also, \(f \notin D<1,-ax>\) since \(f \notin D<1,-a>\), and \(f \notin D<1,-abx>\) since \(f \notin D<1,-ab>\). Thus there exist \(\alpha, \beta \in \{a,b,ab\}\) such that \(\alpha f \in D<1,-ax>\) and \(\beta f \in D<1,-abx>\). If \(\alpha = a\), then \(\alpha f \in D<1,-ax>\). \(\Rightarrow -fx \in D<1,-ax> \cap D<1,-x> \subseteq D<1,-a> \Rightarrow x \in fD<1,-a>\) which is a contradiction. So \(\alpha \neq a\).

If \(\alpha = ab\), then \(abf \in D<1,-ax>\). \(\Rightarrow -bf \in D<1,-ax> \cap D<1,-bx> \subseteq D<1,-ab> \Rightarrow x \in fD<1,-ab>\) which is a contradiction. So \(\alpha \neq ab\). Hence \(\alpha = b\). If \(\beta = a\), then \(af \in D<1,-abx>\).

\(\Rightarrow -bf \in D<1,-abx> \cap D<1,-bx> \subseteq D<1,-a> \Rightarrow x \in fD<1,-a>\) which is a contradiction. So \(\beta \neq a\). If \(\beta = ab\), then \(bf \in D<1,-abx>\). \(\Rightarrow x \in D<1,-bx>\) which is a contradiction. So \(\beta \neq ab\). Hence \(\beta = b\). Now consider \(\alpha = b\) and \(\beta = b\). Then \(bf \in D<1,-ax> \cap D<1,-abx> \subseteq D<1,-b> \Rightarrow b \in D<1,-b>\) which is a contradiction. Therefore, \(f \notin D<1,-x>\).

Now suppose \(af \in D<1,-x>\). Then \(af \in D<1,-x> \cap D<1,-b> \subseteq D<1,-bx>\). Also, \(af \notin D<1,-ax>\) since \(af \notin D<1,-a>\), and \(af \notin D<1,-abx>\) since \(af \notin D<1,-ab>\). Thus there exist \(\alpha, \beta \in \{1,b,ab\}\) such that \(af \in D<1,-ax>\) and \(\beta f \in D<1,-abx>\). In an argument similar to that above, it can be shown that each of the possible values of \(\alpha\) and \(\beta\) leads to a contradiction. Therefore, \(af \notin D<1,-x>\).

Now suppose \(bf \in D<1,-x>\). Then from the above note, \(bf \in D<1,-x> \cap D<1,-abx> \subseteq D<1,-abx>\). Also, \(bf \notin D<1,-ax>\).
bf \not\in D<1,-a>$, and $bf \not\in D<1,-b>$ since $bf \not\in D<1,-b>$. Thus there exist $\alpha, \beta \in \{1, a, ab\}$ such that $af \in D<1,-ax>$ and $bf \in D<1,-bx>$. Again, in an argument similar to that above, it can be shown that each possible value of $\alpha$ and $\beta$ leads to a contradiction. Therefore, $bf \not\in D<1,-x>$.

Finally suppose $abf \in D<1,-x>$. Then $abf \in D<1,-x> \cap D<1,-ab> \subseteq D<1,-abx>$. Also, $abf \notin D<1,-ax>$ since $abf \notin D<1,-a>$, and $abf \notin D<1,-bx>$ since $abf \notin D<1,-b>$. Thus there exist $\alpha, \beta \in \{1, a, b\}$ such that $af \in D<1,-ax>$ and $bf \in D<1,-bx>$. Once again in an argument similar to that above it can be shown that each possible value of $\alpha$ and $\beta$ leads to a contradiction. Therefore, $abf \not\in D<1,-x>$.

Hence $n=4$.

(B) Now let us suppose that there exist $e, f, g \in \mathbb{F}$ such that $D<1,-a> = H\langle e \rangle$, $D<1,-b> = H\langle f \rangle$, and $D<1,-ab> = H\langle g \rangle$ and $\mathbb{F} = H\langle e, f, g \rangle$.

$G_1 = \{1, [ef, a], [ef, b], [ef, ab]\}$ is the group of quaternion algebras for $<1,-ef>$. $G_2 = \{1, [eg, a], [eg, b], [eg, ab]\}$ is the group of quaternion algebras for $<1,-eg>$. $G_3 = \{1, [fg, a], [fg, b], [fg, ab]\}$ is the group of quaternion algebras for $<1,-fg>$.

Note: (1) $[eg, ab] = [e, ab][g, ab]$

$= [e, ab]$ since $g \in D<1,-ab>$

$= [e, a][e, b]$

$= [e, b]$ since $a \in D<1,-e>$. 52
Also, 

\[ [ef, b] = [e, b][f, b] = [e, b] \text{ since } f \in D<1, -b> \]

So \([eg, ab] = [ef, b] \]

(2) \([eg, a] \neq [ef, a] \text{ since } fg \not\in D<1, -a> \]

(3) \([eg, a] \neq [ef, ab], \text{ otherwise, there would exist } z \in F \]
such that \([eg, a] = [z, a] \text{ and } [ef, ab] = [z, ab] \). Then
\[ z \in D<1, -b> = H<f> \text{ and } z \in eg D<1, -a> = eg H<e> = g H<e> \]
But \(H<f> \cap g H<e> = \emptyset \)

(4) \([eg, b] \neq [ef, a], \text{ otherwise, there would exist } z \in F \]
such that \([eg, b] = [z, b] \text{ and } [ef, a] = [z, a] \). Then
\[ z \in D<1, -ab> = H<g> \text{ and } z \in eg D<1, -b> = eg H<e>. \]
But \(H<g> \cap eg H<e> = \emptyset \)

(5) \([eg, b] \neq [ef, ab], \text{ otherwise, there would exist } z \in F \]
such that \([eg, b] = [z, b] \text{ and } [ef, ab] = [z, ab] \). Then
\[ z \in D<1, -a> = H<e> \text{ and } z \in eg D<1, -b> = eg H<e>. \]
But \(H<e> \cap eg H<e> = 0 \).

So (1)-(5) show that \(G_1 \cap G_2 = \{1, [eg, ab] = [ef, b] = [e, b]\} \).

Now consider \(G_1 \cap G_2 \cap G_3 \).

(1) \([fg, a] \neq [e, b], \text{ otherwise, there would exist } z \in F \]
such that \([fg, a] = [z, a] \text{ and } [e, b] = [z, b] \). Then
\[ z \in D<1, -ab> = H<g> \text{ and } z \in fg D<1, -a> = fg H<e>. \]
But \(H<g> \cap fg H<e> = \emptyset \).

(2) \([fg, b] \neq [e, b], \text{ otherwise, } [efg, b] = 1 \text{ and } efg \in D<1, -b> = H<f> \).

(3) \([fg, ab] \neq [e, b], \text{ otherwise, there would exist } z \in F \]
such that \([fg, ab] = [z, ab] \text{ and } [e, b] = [z, b] \). Then \(z \in \)}
D<1,-a> = H<e> and z ∈ fg D<1,-ab> = fg H<g> = f H<g>. But
H<g> ∩ f H<g> = ∅.
Thus G_1 ∩ G_2 ∩ G_3 = {1}.

Let z ∈ H. Then
[zef,a] = [z,a][ef,a] = [ef,a]
[zef,b] = [z,b][ef,b] = [ef,b]
[zef,ab] = [z,ab][ef,ab] = [ef,ab].

So G_1 is the group of quaternion algebras for <1,-zef>.
Similarly, G_2 and G_3 are the groups of quaternion algebras
for <1,-zeg> and <1,-zfg> respectively.

Now let α ∈ F, then [z,a] = [zef,a][ef,a] ∈ G_1 · G_1 = G_1;
[z,a] = [zeg,a][eg,a] ∈ G_2 · G_2 = G_2; [z,a] = [zfg,a][fg,a]
∈ G_3 · G_3 = G_3. Thus [z,a] ∈ G_1 G_2 G_3 = {1}. So z ∈ R = F^2. Hence H = F^2 and | F/F^2 | = 8.

Therefore we have shown the following theorem.

**Theorem 2.15:** Let F be a field such that | F/D<1,-a> | ≤ 4
for every α ∈ F and equality holding for some α ∈ F. Also
assume that R(F) = F^2. If there exist a,b ∈ F such that
D<1,-a>, D<1,-b>, and D<1,-ab> each have index 4 in F and
D<1,-a> ∩ D<1,-b> = D<1,-a> ∩ D<1,-ab> = D<1,-b> ∩ D<1,-ab>
has index 8 in F, then | F/F^2 | = 8.

All of the results in this chapter are in fact true if
F^2 is replaced by R(F). Therefore Theorems 2.10, 2.12, and
2.15 prove the following theorem.
Theorem 2.16: Let $F$ be a field such that $|\hat{F}/D<1,-\alpha>| \leq 4$ for every $\alpha \in \hat{F}$ and equality holding for some $\alpha \in \hat{F}$. Then $m(F) = 4$ or $|\hat{F}/R(F)| = 8$. 

BIBLIOGRAPHY


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