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Ito formula and Girsanov theorem on a new Ito integral

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ITO FORMULA AND GIRSANOV THEOREM ON A NEW ITO INTEGRAL

A Dissertation

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Louisiana State University and
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in partial fulfillment of the
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in

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Abstract

The celebrated Itô theory of stochastic integration deals with stochastic integrals of adapted stochastic processes. The Itô formula and Girsanov theorem in this theory are fundamental results which are used in many applied fields, in particular, the finance and the stock markets, e.g. the Black-Scholes model. In chapter 1 we will briefly review the Itô theory

In recent years, there have been several extension of the Itô integral to stochastic integrals of non-adapted stochastic processes. In this dissertation we will study an extension initiated by Ayed and Kuo in 2008. In Chapter 2 we review this new stochastic integral and some results.

In chapter 3, we prove the Itô formula for the Ayed-Kuo integral.

In chapter 4, we prove the Girsanov theorem for this new stochastic integral.

In chapter 5, we present an application of our results.

Chapter 1

Introduction

1.1 History

The study of stochastic processes is mainly inspired from the subject of physics. The history can be traced to early 19th century .

In 1827, physicist Robert Brown [5] published his observation about micro-objects that pollen particles suspended on the surface of water will traverse continuously in an unpredictable way. This kind of motion is named the *Brownian Motion* to indicate its randomness and continuousness.

After 80 years, with the use of Brownian Motion, Albert Einstein developed a physics model to support his statement that atoms exist [7]. He also provide a mathematical description for the Brownian Motion and proved that the position of the particle should follow some normal distribution. However, his mathematical description is not very strict from the view of mathematicians.

Not Until 1923, did Wiener [20] finally provide a strict mathematical definition of the stochastic process observed by Brown and described by Einstein, which is the Brownian Motion that we define today.

For clarification, here we give the current definition of stochastic processes and Brownian Motion.

Definition 1.1. (Stochastic Process) Let Ω be a probability space, a function $f(t, \omega) : [0, +\infty] \times \Omega \rightarrow \mathbb{R}$ is called a *stochastic process* if

1. for any fixed t in \mathbb{R}^+ , $f(t, \omega) : \Omega \rightarrow \mathbb{R}$ is a random variable.
2. for any ω in Ω , $f(t, \omega) : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a function in t .

Definition 1.2. (Brownian Motion) A stochastic process $B(t, \omega)$ ($t \in [0, \infty]$ and $\omega \in \Omega$) is called a *Brownian Motion* if

- i. for any fixed $0 \leq s < t$, $B(t, \omega) - B(s, \omega)$ is a random variable with distribution $N(0, t - s)$;
- ii. $Prob\{\omega : B(0, \omega) = 0\} = 1$;
- iii. for each partition $0 \leq t_1 < t_2 \dots < t_n$, the random variables

$$B(t_n, \omega) - B(t_{n-1}, \omega), B(t_{n-1}, \omega) - B(t_{n-2}, \omega), \dots,$$

$$B(t_2, \omega) - B(t_1, \omega), B(t_1, \omega) - B(0, \omega)$$

are independent;

- iv. $Prob\{\omega : B(t, \omega) \text{ is continuous on } t \in [0, \infty)\} = 1$.

Since then, millions of studies on stochastic process had been published. In 1944, K. Itô [9] defined a new integral with the form

$$\int_a^b f(t, \omega) dB(t, \omega)$$

where $f(t, \omega)$ is a square integrable stochastic process and $B(t, \omega)$ is a Brownian Motion. This stochastic integral is defined in a way that the integral itself keeps the Martingale and Markov property. It leads to a whole new field of research, which is called stochastic calculus nowadays. The theory of stochastic calculus has a wide application on numerous industries like Information Theory and the modern Quantitative Finance .

To illustrate, we first introduce some concepts. In this whole context, we will use (Ω, \mathcal{F}, P) to represent a probability space where Ω is the sample space, \mathcal{F} is the σ -field and P is the probability measure.

1.2 Conditional Expectation

The property of conditional expectation will be used frequently.

Definition 1.3. Let (Ω, \mathcal{F}, P) be a probability space. Let X be a random variable in $\mathcal{L}^1(\Omega)$, and assume \mathcal{G} is a sub- σ -field of \mathcal{F} , then we say the *conditional expectation* of X on \mathcal{G} , denoted by $E[X|\mathcal{G}]$, is a random variable Y such that

1. Y is measurable w.r.t. \mathcal{G} .
2. For any Borel set A in \mathcal{G} , we have

$$\int_A Y dP = \int_A X dP.$$

Remark 1.4. The conditional expectation exists and is unique for any given X and \mathcal{G} , see [17, Section 1.4].

We mention some important properties that will be used later.

Theorem 1.5. Let (Ω, \mathcal{F}, P) be a probability space, and let X, Y be a random variables on Ω , assume \mathcal{G} is a sub- σ -field of \mathcal{F} , then we have following facts

1. $E[E[X|\mathcal{G}]] = E[X]$
2. $E[X + Y|\mathcal{G}] = E[X|\mathcal{G}] + E[Y|\mathcal{G}]$
3. If X is measurable on \mathcal{G} , then $E[X|\mathcal{G}] = X$
4. If X is independent w.r.t. \mathcal{G} , then $E[X|\mathcal{G}] = E[X]$
5. If Y is \mathcal{G} -measurable, and XY is integrable, then $E[XY|\mathcal{G}] = YE[X|\mathcal{G}]$
6. If \mathcal{G}_1 is a sub- σ -field of \mathcal{G} , then $E[E[X|\mathcal{G}]|\mathcal{G}_1] = E[X|\mathcal{G}_1]$

1.3 Martingale

Definition 1.6. (Filtration) Let (Ω, \mathcal{F}, P) be a probability space where Ω is the sample space, \mathcal{F} is the σ -field and P is the probability measure. A class of σ -fields $\{\mathcal{F}_t, 0 \leq t < \infty\}$ on Ω is called a *filtration* if for any $0 \leq s < t$, $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$.

Definition 1.7. (Adapt) Let $B(t, \omega)$ be a Brownian Motion on the probability space (Ω, \mathcal{F}, P) , $\{\mathcal{F}_t, 0 \leq t < \infty\}$ is some filtration, we say $B(t, \omega)$ is *adapted* to $\{\mathcal{F}_t, 0 \leq t < \infty\}$ if for any $t \geq 0$, $B(t, \omega)$ is measurable with respect to \mathcal{F}_t .

Remark 1.8. From here on, we will denote by $B(t)$ or B_t the Brownian Motion, $\{\mathcal{F}_t\}$ the filtration. And we say $\{\mathcal{F}_t\}$ is the underlying filtration for the Brownian Motion $B(t)$ if for any $t \geq 0$, $\mathcal{F}_t = \sigma\{B(s), 0 \leq s \leq t\}$.

Example 1.9. A Brownian Motion $B(t)$ is adapted to its *underlying filtration* $\{\mathcal{F}_t\}$ where $\mathcal{F}_t = \sigma\{B(s), 0 \leq s \leq t\}$.

The martingale property is one of the most useful properties with many applications.

Definition 1.10. Let (Ω, \mathcal{F}, P) be a probability space, and $\{\mathcal{F}_t\}$ is a filtration. Assume that $X(t)$ is a stochastic process that adapts to $\{\mathcal{F}_t\}$, then, we say $X(t)$ is a *martingale* if for any $0 \leq s < t$,

$$E[X(t)|\mathcal{F}_s] = X(s) \text{ almost surely}$$

Example 1.11. The Brownian Motion $B(t)$ is a martingale w.r.t. its underlying filtration $\{\mathcal{F}_t\}$

Proof.

$$\begin{aligned} E[B(t)|\mathcal{F}_s] &= E[B(t) - B(s) + B(s)|\mathcal{F}_s] \\ &= E[B(t) - B(s)|\mathcal{F}_s] + E[B(s)|\mathcal{F}_s] \end{aligned}$$

Since $B(t) - B(s)$ is independent with \mathcal{F}_s and $B(s)$ is adapted to \mathcal{F}_s (1.1)

$$\begin{aligned} &= E[B(t) - B(s)] + B(s) \\ &= B(s) \end{aligned}$$

□

1.4 Itô Integral

For the simplicity of demonstration, with out loss of generality, from here on we will use a fixed interval $[0, T]$ instead of $[a, b]$, all the results will be unchanged under this special interval.

We denote by $\mathcal{L}_{ad}^2(\Omega \times [0, \infty))$ the space of square integrable stochastic process on Ω that adapt to $\{\mathcal{F}_t\}$.

Definition 1.12. (Itô Integral) Let $B(t)$ be a Brownian Motion, and $f(t)$ be a stochastic process in $\mathcal{L}_{ad}^2(\Omega \times [0, \infty))$. If $[a, b]$ is a interval on the positive part of real line, then we call the stochastic integral

$$\int_a^b f(t) dB(t)$$

an *Itô Integral*.

This definition is composed of 3 steps.

Step 1:

When $f(t)$ is an adapted step stochastic process. i.e.

$$f(t) = \sum_{i=1}^n X_i \chi_{[t_{i-1}, t_i)}$$

where $0 = t_0 < t_1 < t_2 \cdots < t_n = T$, we difine

$$\int_0^T f(t)dB(t) = \sum_{i=1}^n X_i(B(t_i) - B(t_{i-1}))$$

Step 2:

We prove that for any function $f(t)$ in $\mathcal{L}_{ad}^2([0, T] \times \Omega)$, there is a sequence of adapted step stochastic functions $\{f_n(t)\}_{n=1}^\infty$ converging to $f(t)$ in $\mathcal{L}^2([0, T] \times \Omega)$.

Step 3:

We define

$$\int_0^T f(t)dB(t) = \lim_{n \rightarrow \infty} \int_0^T f_n(t)dB(t),$$

and prove the wellness of the definition.

Itô defines its integral in a strict way, but it's also very intuitive to view the integral as a limit of Riemann Sum.

Theorem 1.13. *Assuming that the $f(t)$ in above definition is left continuous, then the limit of the Riemman sum exist. I.e. let $\Delta = \{0 = t_0 < t_1 < \dots < t_n = T\}$ be any partition of the interval $[0, T]$, then*

$$\lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f(t_{i-1})(B(t_i) - B(t_{i-1})) \text{ exists.}$$

In this case, we also have

$$\int_0^T f(t) dB(t) = \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f(t_{i-1})(B(t_i) - B(t_{i-1}))$$

Remark 1.14. Notice the integrand of the Riemann sum is $f(t_{i-1})$, in fact it must be evaluated using left end point of each subinterval $[t_{i-1}, t_i]$, and this is the key idea in Itô's definition of this integral.

The Itô integral has many good properties. For detail proof, see [11].

Theorem 1.15. *(Itô Isometry) Let $X_T = \int_0^T f(t) dB(t)$ be an Itô integral defined before, then X_T is a random variable and we have*

1. $E\left(\int_0^T f(t) dB(t)\right) = 0$
2. $E\left(\int_0^T f(t) dB(t)\right)^2 = \int_0^T (E[f(t)])^2 dt$

Further more, if $f(t)$ is deterministic(non random), then X_T has a normal distribution.

Theorem 1.15 gives a way to compute the first and second moment of this random variable. If $f(t)$ is deterministic, then the distribution of X_T can be decided by the first two moments.

Theorem 1.16. (continuous martingale) Let X_t be a stochastic process defined by

$$X_t = \int_0^t f(s) dB(s)$$

i.e. it is an Itô integral from 0 to t . Then X_t has following properties,

1. X_t is continuous.
2. X_t is a martingale.

1.5 Quadratic Variation

Recall in Real Analysis, we define the variation of a function as below.

Definition 1.17. Let $f(t)$ be a measurable function on real line \mathbb{R} , let $\Delta = \{0 = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n = T\}$ be any partition of interval $[0, T]$, then the variation of $f(x)$ on $[0, T]$ is defined by

$$\bigvee_0^T f(t) = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n |f(t_i) - f(t_{i-1})|$$

provided the right hand side exists.

In the case of Brownian motion $B(t)$, one can prove that the variation of $B(t)$ on any interval is ∞ , see [11].

Similarly, we can define the Quadratic variation.

Definition 1.18. (Quadratic Variation) Let $f(t)$ be a stochastic process, let $\Delta = \{0 = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n = T\}$ be any partition of interval $[0, T]$, then

the *quadratic variation* of $f(x)$ on $[0, T]$ is defined by

$$[f]_T = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n (f(t_i) - f(t_{i-1}))^2$$

provided the right hand side exists.

Example 1.19. If $f(t)$ is any deterministic function with bounded variation. Then $[f]_t = 0$ for any interval $[0, t]$

Example 1.20. The quadratic variation $[B]_t$ of Brownian motion $B(t)$ is t almost surely for any $t > 0$.

1.6 Itô Formula

To make the Itô integral more applicable, we introduce several formulas that is useful in computing the integral. We first introduce the concept of Itô process.

Definition 1.21. (Itô process) Let $B(t)$ be a Brownian Motion, $f(t), g(t)$ be stochastic processes adapted to the underlying filtration of $B(t)$, then we call $X(t)$, defined by

$$X(t) = \int_0^t f(t) dB(t) + \int_0^t g(t) dt$$

an *Itô process*.

Remark 1.22. In the definition, the first term is defined by Itô integral, and the second term is defined almost surely for each ω in Ω . By Theorem 1.16, X_t is continuous. And we denote by $dX(t) = f(t) dB(t) + g(t) dt$ the stochastic differential of this process.

Theorem 1.23. (Itô Formula, [9]) Suppose that $\{X_t^{(i)} : i = 1, 2, \dots, n\}$ are Itô processes adapted to $\{\mathcal{F}_t : 0 \leq t \leq T\}$, the underlying filtration of Brownian Motion $B(t)$, and $f(x_1, x_2, \dots, x_n)$ is a twice continuously differentiable real function on

\mathbb{R}^n . Then

$$\begin{aligned} f(X_t^{(1)}, X_t^{(2)}, \dots, X_t^{(n)}) &= f(X_0^{(1)}, X_0^{(2)}, \dots, X_0^{(n)}) \\ &+ \int_0^t \sum_{i=1}^n \frac{\partial}{\partial x_i} f(X_s^{(1)}, X_s^{(2)}, \dots, X_s^{(n)}) dX_s^{(i)} \\ &+ \frac{1}{2} \int_0^t \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} f(X_s^{(1)}, X_s^{(2)}, \dots, X_s^{(n)}) (dX_s^{(i)})(dX_s^{(j)}). \end{aligned}$$

where $dX_s^{(i)}$ are the stochastic differential for the Itô process $X_s^{(i)}$.

Remark 1.24. We can denote it in the differential form,

$$\begin{aligned} df(X_t^{(1)}, X_t^{(2)}, \dots, X_t^{(n)}) &= \sum_{i=1}^n \frac{\partial}{\partial x_i} f(X_t^{(1)}, X_t^{(2)}, \dots, X_t^{(n)}) dX_t^{(i)} \\ &+ \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} f(X_t^{(1)}, X_t^{(2)}, \dots, X_t^{(n)}) (dX_t^{(i)})(dX_t^{(j)}) \end{aligned} \tag{1.2}$$

This form of equation has no realistic meaning as we did not define what the actual differential is. However, we can always translate this to the integral form in Theorem 1.23.

Theorem 1.25. *We have following differential identities:*

1. $dB(t) \times dB(t) = dt$
2. $dB(t) \times dt = 0$
3. $dt \times dt = 0$

Remark 1.26. The above facts come from the properties of quadratic variation.

Example 1.27. Let $X(t) = B(t)$, $f(x) = x^2$, then we have $f'(x) = 2x$ and $f''(x) = 2$, using Theorem 1.23, we have

$$B(t)^2 = B(0)^2 + \int_0^t 2B(t) dB(t) + \frac{1}{2} \int_0^t 2 dt \tag{1.3}$$

Since $B(0) = 0, a.s.$, we get

$$B(t)^2 = 2 \int_0^t B(s) dB(s) + t \quad (1.4)$$

or

$$\int_0^t B(s) dB(s) = \frac{1}{2}(B(t)^2 - t) \quad (1.5)$$

This provides a representation of the Itô integral $\int_0^t B(s) dB(s)$.

Example 1.28. (Exponential Process) Assume that $B(t)$ is a Brownian Motion on the probability space (Ω, \mathcal{F}, P) with underlying filtration $\{\mathcal{F}_t\}$, Let f_t be a stochastic process in $\mathcal{L}_{ad}^2(\Omega \times [0, \infty])$, define

$$\mathcal{E}_t(f) = \exp \left\{ \int_0^t f_s dB_s - \frac{1}{2} \int_0^t f_s^2 ds \right\}.$$

then, we have

$$d\mathcal{E}_t(f) = f(t)\mathcal{E}_t(f) dB(t)$$

Notice this is the analogue of exponential function in ODE, thus we call $\mathcal{E}_t(f)$ the *exponential process* for f_t .

Proof. Use Theorem 1.23. □

1.7 Girsanov Theorem

In the application of Itô calculus, Girsanov theorem get used frequently since it transforms a class of process to Brownian Motion with an equivalent probability measure transformation.

Theorem 1.29 (Girsanov, 1960, [8]). *Assume that $B(t)$ is a Brownian Motion on the probability space (Ω, \mathcal{F}, P) with underlying filtration $\{\mathcal{F}_t\}$, Let f_t be a square integrable stochastic process adapts to $\{\mathcal{F}_t\}$ such that $\mathbb{E}_P[\mathcal{E}_t(f)] < \infty$ for all $t \in [0, T]$. Then*

$$\tilde{B}(t) = B(t) - \int_0^t f_s ds$$

is a Brownian motion with respect to an equivalent probability measure Q given by

$$dQ = \mathcal{E}_T(f) dP = \exp \left\{ \int_0^T f_s dB_s - \frac{1}{2} \int_0^T f_s^2 ds \right\} dP. \quad (1.6)$$

Remark 1.30. Using differential form, we can also say, if $d\tilde{B}(t) = dB(t) - f_t dt$. Then $\tilde{B}(t)$ is a Brownian Motion w.r.t. the probability measure Q .

1.8 Risk Neutral Probability Measure in Quantitative Finance

In the modern Finance theory, the market is composed of two kind of asset: The risky assets (stocks, securities), and the risk free assets (bonds). The return rate for the risk free asset is called risk free interest rate, denoted by $r(t)$. this rate is assumed to be known at any time t .

According to the theorem of random walk, see [19, Shreve] for detail, price of each stock, denoted by $S(t)$ can be modeled by the stochastic differential equation

$$dS(t) = S(t)\alpha(t) dt + S(t)\sigma(t) dB(t) \quad (1.7)$$

where $\alpha(t)$ is the relative instantaneous return at time t , and $\sigma(t)$ is the volatility at time t . Here, we assume the probability measure of real world is (Ω, \mathcal{F}, P) .

The volatility usually can be obtained by history data since it is commonly unchanged for each individual stock. However, the expected instantaneous return $\alpha(t)$ is hard to achieve.

To overcome this problem, we notice that there is one known instantaneous return rate, namely the risk free interest rate $r(t)$. So, we can rewrite Equation (1.7) as

$$\begin{aligned} dS(t) &= S(t)r(t) dt + S(t)(\alpha(t) - r(t)) dt + S(t)\sigma(t) dB(t) \\ &= S(t)r(t) dt + S(t)\sigma(t) \left[dB(t) + \frac{\alpha(t) - r(t)}{\sigma(t)} dt \right] \end{aligned} \quad (1.8)$$

Here, since $B(t)$ is a Brownian Motion on (Ω, \mathcal{F}, P) , let

$$\tilde{B}(t) = B(t) + \int_0^t \frac{\alpha(t) - r(t)}{\sigma(t)} dt,$$

by Theorem 1.29 we have $\tilde{B}(t)$ is a Brownian Motion w.r.t. the probability measure

$$dQ = \mathcal{E}_t\left(\frac{\alpha(t) - r(t)}{\sigma(t)}\right)dP$$

where $\mathcal{E}_t\left(\frac{\alpha(t) - r(t)}{\sigma(t)}\right)$ is the exponential process defined in Example 1.28.

Thus, under the Q -measure, the stochastic differential equation becomes

$$dS(t) = S(t)r(t) dt + S(t)\sigma(t) d\tilde{B}(t) \tag{1.9}$$

Since $r(t)$ is known, with some regularity condition (see [11, chapter 10]), there is a unique solution for the stochastic differential equation (1.9). And because Q and P are equivalent, we conclude that any event that happens almost surely in Q will also happen almost surely in P .

Chapter 2

The New Stochastic Integral

2.1 Non-adapted Processes

Itô's integral deals with adapted processes as integrand. However, recently, more and more models involve components of non-adapted processes. We say a stochastic process is *anticipative* if it is not adapted to the filtration we are using.

For example, what is $\int_0^1 b(1) dB(t)$?

In 1978, Itô proposed one approach to define the anticipative stochastic integral $\int_0^1 b(1) dB(t)$, his idea is to enlarge the filtration, define

$$\mathcal{F}_t = \sigma\{B(1), B(s); s \in [0, t]\}$$

Then, $f(t)$ is adapted to $\{\mathcal{F}_t\}$ and $B(t)$ is a \mathcal{F}_t -quasimartingale with decomposition

$$B(t) = M(t) + A(t)$$

where

$$A(t) = \int_0^t \frac{B(T) - B(s)}{T - s} dB(s)$$

And one can define $\int_0^T f(t) dB(t)$ as integral w.r.t. a quasimartingale $B(t)$, see [11].

Example 2.1. Let $B(t)$ be a Brownian Motion, then using Itô's definition,

$$\int_0^t B(1) dB(t) = B(1)B(t).$$

The drawback of this definition is that $\int_0^t B(1) dB(t)$ will no longer be a martingale, nor will it satisfies the properties in Theorem 1.15.

2.2 The New Approach

In [1, 2], Ayed and Kuo proposed a new definition of stochastic integral for a certain class of anticipating stochastic processes. In their construction, the authors

exploit the independence of increments of Brownian motion. In order to do so, they decompose the integrands into sums of products of adapted and instantly independent processes (defined below). In this section, we will introduce this new definition.

Definition 2.2. We say that a stochastic process $\{\varphi_t\}$ is *instantly independent* with respect to the filtration $\{\mathcal{F}_t\}$ if for each $t \in [0, T]$, the random variable φ_t and the σ -field \mathcal{F}_t are independent.

Example 2.3. Let $\{\mathcal{F}_t\}$ be the underlying filtration of Brownian Motion $B(t)$, then $\varphi(B_1 - B_t)$ is instantly independent of $\{\mathcal{F}_t: t \in [0, 1]\}$ for any real measurable function $\varphi(x)$. However, $\varphi(B_1 - B_t)$ is adapted to $\{\mathcal{F}_t: t \geq 1\}$.

Theorem 2.4. *If a stochastic process $f(t)$ is both instantly independent and adapted to some filtration $\{\mathcal{F}_t\}$, then $f(t)$ must be non-random, i.e. it is a deterministic function.*

Proof. From Theorem 1.5

$$f(t) \text{ is adapted} \Rightarrow E[f(t)|\mathcal{F}_t] = f(t)$$

$$f(t) \text{ is instantly independent} \Rightarrow E[f(t)|\mathcal{F}_t] = E[f(t)]$$

Thus we have

$$f(t) = E[f(t)], \text{ for any } \omega \text{ in } \Omega$$

i.e. it is deterministic. □

Itô integral measures the integrand using left end point for each subinterval, see Theorem 1.13. For instantly independent part, if we also use the left end point to approximate, it will not keep the important properties as Example 2.1. However, after observations, if we measure the instantly independent part using right end

point, the outcome actually keeps all these properties in some sense. This lead to Ayed and Kuo's definition of the new integral.

Definition 2.5. Let B_t be a Brownian Motion with underlying filtration $\{\mathcal{F}_t\}$. If $f(t)$ is an adapted stochastic process with respect to the filtration $\{\mathcal{F}_t\}$ and $\varphi(t)$ is instantly independent with respect to the same filtration, we define the stochastic integral of $f(t)\varphi(t)$ as

$$\int_0^T f(t)\varphi(t) dB_t = \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f(t_{i-1})\varphi(t_i)\Delta B_i, \quad (2.1)$$

where $\Delta_n = \{0 = t_0 < t_1 < \dots < t_n = T\}$ is a partition of the interval $[0, T]$ and $\Delta B_i = B_{t_i} - B_{t_{i-1}}$ and $\|\Delta_n\| = \max\{t_i - t_{i-1} : i = 1, 2, \dots, n\}$, provided the limit exists in probability.

Example 2.6. For any $0 < t < T$,

$$\int_0^T B(T) - B(s) dB(s) = \frac{1}{2}(B(T)^2 - T)$$

In general, we have

$$\int_t^T B(T) - B(s) dB(s) = \frac{1}{2}((B(T) - B(t))^2 - (T - t))$$

Take the difference between above two equations, we have

$$\int_0^t B(T) - B(s) dB(s) = \frac{1}{2}(2B(T)B(t) - B(t)^2 - t)$$

and

$$\begin{aligned} \int_0^t B(T) dB(s) &= \int_0^t B(s) dB(s) + \int_0^t B(T) - B(s) dB(s) \\ &= B(T)B(t) - t \end{aligned} \quad (2.2)$$

Proof. We only proof the general equation

$$\int_t^T B(T) - B(s) dB(s) = \frac{1}{2}((B(T) - B(t))^2 - (T - t)).$$

All the rest will be easily induced by linearity of integration.

In fact, let $\Delta = \{t = t_0 < t_1 < \dots < t_{n-1} < t_n = T\}$ be any partition of interval $[t, T]$, and $\Delta B_i = B(t_i) - B(t_{i-1})$, by definition

$$\begin{aligned}
\int_t^T B(T) - B(s) dB(s) &= \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n (B(T) - B(t_i)) \Delta B_i \\
&= \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n B(T) \Delta B_i - \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n (B(t_{i-1}) + \Delta B_i) \Delta B_i \\
&= B(T) \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n \Delta B_i - \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n B(t_{i-1}) \Delta B_i \\
&\quad - \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n \Delta B_i^2 \\
&= B(T)(B(T) - B(t)) - \int_t^T B(s) dB(s) - [B]_t^T
\end{aligned} \tag{2.3}$$

Notice the second term uses Theorem 1.13 and the last term is the quadratic variation of $B(t)$ on interval $[t, T]$. Thus, we have

$$\begin{aligned}
\int_t^T B(T) - B(s) dB(s) &= B(T)(B(T) - B(t)) - \int_t^T B(s) dB(s) - [B]_t^T \\
&= B(T)(B(T) - B(t)) - \frac{1}{2} \left(B(T)^2 - B(t)^2 - T + t \right) - T + t \\
&= \frac{1}{2} \left((B(T) - B(t))^2 - (T - t) \right)
\end{aligned} \tag{2.4}$$

□

The New integral is a generalization of Itô integral, if we make the instantly independent part $\varphi(t) = 1$, it reduce to the Itô integral. Thus, lots of theorems for the Itô integral can be generalized to the new integral (in some sense).

2.3 Martingale and Near-martingale

We first recall some basic facts about martingales and their instantly independent counterpart, processes called near-martingales that were introduced and studied

by Kuo, Sae-Tang and Szozda in [14]. It is worth mentioning that the same kind of processes are studied in [3], however they serve a different purpose and are termed *increment martingales*.

Recall from chapter 1, Definition 1.10 of martingale is equivalent to the following three statements:

1. $\mathbb{E} |X_t| < \infty$ for all $t \in [0, T]$;
2. X_t is adapted;
3. $\mathbb{E}[X_t - X_s | \mathcal{F}_s] = 0$ for all $0 \leq s < t \leq T$.

In the case of instantly independent processes, clearly the condition (ii) will not be satisfied anymore, this means there is not martingale concept in the anticipative stochastic process space.

However, in [14], the authors propose to take the first and the last of the above conditions as the definition of a *near-martingale*. We recall this definition below.

Definition 2.7. We say that a process X_t is a *near-martingale* with respect to a filtration $\{\mathcal{F}_t\}$ if $E |X_t| < \infty$ for all $0 \leq t \leq T$ and $\mathbb{E}[X_t - X_s | \mathcal{F}_s] = 0$ for all $0 \leq s < t \leq T$.

It is a well-known fact that the Itô integral is a martingale, that is $X_t = \int_0^t f(s) dB_s$ is a martingale with respect to $\{\mathcal{F}_t\}$, for any adapted stochastic process $f(t)$ that is integrable with respect to B_t on the interval $[0, T]$. Similar result holds for the new stochastic integral

Theorem 2.8. ([14, Theorem 3.5]) *if $f(t)$ and $\varphi(t)$ are as in Definition 2.5 and the integral exists, then $X_t = \int_0^t f(s)\varphi(s) dB_s$ is a near-martingale with respect to $\{\mathcal{F}_t\}$*

Moreover, it is also a near-martingale with respect to a *natural backward filtration* $\{\mathcal{G}^{(t)}\}$ of B_t defined below.

Definition 2.9. (see [14, Theorem 3.7]) Let $B(t)$ be a Brownian motion, then we call $\{\mathcal{G}^{(t)}\}$, where

$$\mathcal{G}^{(t)} = \sigma\{B_T - B_s : t \leq s \leq T\},$$

the *Backward Filtration* of the Brownian motion $B(t)$

Remark 2.10. In general, a *backward filtration* is any decreasing family of σ -fields, *i.e.* $\{\mathcal{G}^{(t)}\}$ satisfies $\mathcal{G}^{(t)} \subseteq \mathcal{G}^{(s)}$ for any $0 \leq s \leq t \leq T$. A concept similar to that of the backward filtration is also used in [18].

Theorem 2.11. ([14, Theorem 3.7]) *if $f(t)$ and $\varphi(t)$ are as in Definition 2.5 and the integral exists, then $X_t = \int_0^t f(s)\varphi(s) dB_s$ is a near-martingale with respect to $\{\mathcal{G}^{(t)}\}$*

Finally, we recall another result from [14] that will be useful in establishing our results.

Theorem 2.12. *Let $\{X_t\}$ be instantly independent with respect to the filtration $\{\mathcal{F}_t\}$. Then $\{X_t\}$ is a near-martingale with respect to $\{\mathcal{F}_t\}$ if and only if $\mathbb{E}[X_t]$ is constant as a function of t .*

Proof. For the proof see [14, Theorem 3.1]. □

2.4 Zero Expectation and Itô Isometry

In Chapter 1 we mentioned that in Itô integral there is an important theory to compute the mean and variance of the integral, see Theorem 1.15. In [16], Kuo, SeaTang and Szozda proved that these property still hold for the stochastic integral on certain class of anticipative stochastic processes.

Theorem 2.13. (*Zero Expectation,[16]*) Let $f(t)$ and $\varphi(t)$ be defined as before, assume that $\int_0^T f(s)\varphi(s) dB(t)$ exists. In addition, suppose for any $t \in [0, T]$, $f(t)$ and $\varphi(t)$ are integrable in (Ω, \mathcal{F}, P) . Then we have,

$$E \left[\int_0^T f(t)\varphi(t) dB(t) \right] = 0$$

If the integrand is purely instantly independent with the form $\varphi(B(T) - B(t))$, then we have the isometry.

Theorem 2.14. (*Itô Isometry,[16]*) Assume that $\varphi(x)$ is a function with Maclaurin series on the whole real line. Let $B(t)$ be a Brownian motion, then we have

$$E \left(\int_0^T \varphi(B(T) - B(t)) dB(t) \right)^2 = \int_0^T E(\varphi(B(T) - B(t)))^2 dt$$

For the more general case, we have following theorem.

Theorem 2.15. (*[16]*) Assume that $f(t)$, $\varphi(x)$ are functions with Maclaurin series on the whole real line. Let $B(t)$ be a Brownian motion, then we have

$$\begin{aligned} E \left[\left(\int_0^T f(B(t))\varphi(B(T) - B(t)) dB(t) \right)^2 \right] &= \int_0^T E[(f(B(t))\varphi(B(T) - B(t)))^2] dt \\ &+ 2 \int_0^T \int_0^t E[f(B(s))\varphi'(B(T) - B(s))f'(B(t))\varphi(B(T) - B(t))] ds dt \end{aligned} \tag{2.5}$$

2.5 Itô Formula

In [15], Kuo, Sae-Tang and Szozda provide an series of Itô formulas for a certain class of anticipative processes. The simplest case in [15] is for functions of $B(t)$ and $B(T) - B(t)$ that are related to the results of the present paper, namely [15, Theorem 5.1].

Theorem 2.16. *Suppose that $f, \varphi \in C^2(\mathbb{R})$ and $\theta(x, y) = f(x)\varphi(y - x)$. Then for $0 \leq t \leq T$,*

$$\begin{aligned} \theta(B_t, B_T) &= \theta(B_0, B_T) + \int_0^t \frac{\partial \theta}{\partial x}(B_s, B_T) dB_s \\ &\quad + \int_0^t \left[\frac{1}{2} \frac{\partial^2 \theta}{\partial x^2}(B_s, B_T) + \frac{\partial^2 \theta}{\partial x \partial y}(B_s, B_T) \right] ds. \end{aligned}$$

This formula only deals with functions on Brownian motion $B(t)$ and its counterpart $B(T) - B(t)$. To find the analogue of Theorem 1.23, we need to introduce some new concept.

Definition 2.17. Recall in Definition 1.21, we defined the Itô process in the adapted situation. Here, we call $Y^{(t)} = \int_t^T h(s) dB_s + \int_t^T g(s) ds$ the counterpart of Itô process if both $h(t)$ and $g(t)$ are instantly independent. In this case, we have the differential notation $dY^{(t)} = -h(t) dB(t) - g(t) dt$

When $h(t)$ and $g(t)$ are deterministic, Kuo, SaeTang and Szozda proved the following Itô Formula.

Theorem 2.18. *Suppose that $h \in L^2[0, T]$, $g \in L^1[0, T]$ are integrable deterministic functions and*

$$Y^{(t)} = \int_t^T h(s) dB_s + \int_t^T g(s) ds.$$

Suppose also that $f \in C^2(\mathbb{R} \times [0, T])$. Then

$$\begin{aligned} f(Y^{(t)}, t) &= f(Y^{(0)}, 0) + \int_0^t \frac{\partial f}{\partial t}(Y^{(s)}, t) ds \\ &\quad + \int_0^t \frac{\partial f}{\partial x}(Y^{(s)}, t) dY^{(s)} - \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(Y^{(s)}, t) (dY^{(s)})^2. \end{aligned}$$

where $dY^{(t)} = -h(t) dB(t) - g(t) dt$

Remark 2.19. We can also write the above formula in differential form:

$$\begin{aligned} df(Y^{(t)}, t) &= \frac{\partial f}{\partial t}(Y^{(t)}, t) dt + \frac{\partial f}{\partial x}(Y^{(t)}, t) dY^{(t)} \\ &\quad - \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(Y^{(s)}, t) (dY^{(s)})^2. \end{aligned}$$

Remark 2.20. The following identity can still be applied in computation.

1. $dB(t) \times dB(t) = dt$
2. $dB(t) \times dt = 0$
3. $dt \times dt = 0$

Remark 2.21. Since deterministic function is a special kind of instantly independent function, we consider the above theorem a special case of analogue of the Itô Formula in Chapter 1. Comparing with Theorem 1.23, we can see the only difference is that the coefficient of the last term becomes negative in the anticipative case.

Probably the most interesting case should be the formula combining both adapted process and anticipative process. When the anticipative process is the special case as in Theorem 2.18, Kuo, SaeTang and Szozda provide the following theorem in [15].

Theorem 2.22. *Suppose that $\theta(x, y)$ is a function of the form $\theta(x, y) = f(x)\varphi(y)$, where f and φ are twice continuously differentiable real-valued functions. Suppose also that X_t is an Itô process and $Y^{(t)}$ are defined as in Theorem 2.18. Then*

$$\begin{aligned} \theta(X_T, Y^{(T)}) &= \theta(X_0, Y^{(0)}) + \int_0^t \frac{\partial \theta}{\partial x}(X_t, Y^{(t)}) dX_t + \frac{1}{2} \int_0^t \frac{\partial^2 \theta}{\partial x^2}(X_t, Y^{(t)}) (dX_t)^2 \\ &\quad + \int_0^t \frac{\partial \theta}{\partial y}(X_t, Y^{(t)}) dY^{(t)} - \frac{1}{2} \int_0^t \frac{\partial^2 \theta}{\partial y^2}(X_t, Y^{(t)}) (dY^{(t)})^2 \end{aligned}$$

where dX_t and $dY^{(t)}$ are corresponding differentials.

Remark 2.23. The differential form of this formula is

$$\begin{aligned} d\theta(X_t, Y^{(t)}) &= \frac{\partial \theta}{\partial x}(X_t, Y^{(t)}) dX_t + \frac{1}{2} \frac{\partial^2 \theta}{\partial x^2}(X_t, Y^{(t)}) (dX_t)^2 \\ &\quad + \frac{\partial \theta}{\partial y}(X_t, Y^{(t)}) dY^{(t)} - \frac{1}{2} \frac{\partial^2 \theta}{\partial y^2}(X_t, Y^{(t)}) (dY^{(t)})^2 \end{aligned}$$

In Chapter 3, We will generalize Theorem 2.18 and Theorem 2.22 such that the integrand $h(t)$ and $g(t)$ are no longer deterministic functions.

2.6 Stochastic Differential Equations

Differential Equations is another useful topic of the Itô integral, here we only introduce the linear case. In the adaptive situation, we can find the solution of Equation (1.9). In the anticipative situation, we can give a general solution for a certain class of new Itô integrals.

In the adaptive situation, we have

Theorem 2.24. ([11]) *Let (Ω, \mathcal{F}, P) be a probability space, and $B(t)$ is a Brownian motion with underlying filtration $\{\mathcal{F}_t\}$ on (Ω, \mathcal{F}, P) . Assume that $\alpha(t)$ and $\beta(t)$ are square integrable stochastic processes adapted to $\{\mathcal{F}_t\}$. If x_0 is a constant, then the S.D.E.*

$$\begin{cases} dX(t) = X(t)\alpha(t) dB(t) + X(t)\beta(t) dt \\ X(0) = x_0 \end{cases} \quad (2.6)$$

has a unique solution

$$X(t) = x_0 \exp \left\{ \int_0^t \alpha(t) dB(t) + \int_0^t \beta(t) - \frac{1}{2} \alpha(t)^2 dt \right\} \quad (2.7)$$

Example 2.25. The S.D.E. (1.9) of Quantitative Finance in Section 1.8 has a unique solution

$$S(t) = S_0 \exp \left(\int_0^t \sigma(t) d\tilde{B}(t) + \int_0^t r(t) - \frac{1}{2} \sigma(t)^2 dt \right) \quad (2.8)$$

where S_0 is the stock price at time 0, $\tilde{B}(t)$ is the Brownian motion in Q -measure.

In [10], Khalifa, Kuo, Ouerdiane and Szozda proved that if the initial condition $X(0)$ is anticipative with the form $p(B(T))$ where $p(x)$ is a rapidly decreasing function with Maclaurin Series on the whole real line, then we can also have the

solution for S.D.E.

$$\begin{cases} dX(t) = X(t)\alpha(t) dB(t) + X(t)\beta(t) dt \\ X(0) = p(B(T)) \end{cases} \quad (2.9)$$

Theorem 2.26. ([10]) *Let $B(t)$ be a Brownian motion on (Ω, \mathcal{F}, P) , $p(t)$ is a rapidly decreasing function in $\mathcal{S}(\mathbb{R})$ with Maclaurin Series on the whole real line.*

Then the S.D.E (2.9) has a solution given by

$$X(t) = (p(B(T)) - \xi(t, B(T)))Z(t) \quad (2.10)$$

where

$$\xi(t, y) = \int_0^t \alpha(s) p' \left(y - \int_s^t \alpha(u) du \right) ds \quad (2.11)$$

and

$$Z(t) = \exp \left\{ \int_0^t \alpha(s) dB(s) + \int_0^t \beta(s) - \frac{1}{2} \alpha(s)^2 ds \right\} \quad (2.12)$$

Later, with the new theorems stated below, we can deal with more general SDEs.

Chapter 3

Generalized Itô Formula

In this section, we present the anticipative version of the Itô formula.

3.1 Some Basic Lemmas

We begin with several results to be used later to extend the Itô formula itself and the Girsanov type theorems. First, we present two simple but crucial observations that allow us to use some of the results from classical Itô theory in our setting.

Theorem 3.1. *Suppose that B_t is a Brownian motion on (Ω, \mathcal{F}, P) , $\{\mathcal{F}_t\}$ and $\{\mathcal{G}^{(t)}\}$ are its natural filtrations, forward and backward respectively. Then the probability spaces $(\Omega, \mathcal{G}^{(0)}, P)$ and $(\Omega, \mathcal{F}_T, P)$ coincide, that is $\mathcal{G}^{(0)} = \mathcal{F}_T$.*

Proof. Recall that the probability space $(\Omega, \mathcal{F}_T, P)$ of B_t is a classical Wiener space with the σ -field \mathcal{F}_T generated by the cylinder sets. Notice that the σ -field $\mathcal{G}^{(0)}$ is generated by the same cylinder sets. Thus the result follows. \square

Using the above theorem, we can define a *backward Brownian motion*, that is a Brownian motion with respect to the backward filtration. Let

$$B^{(t)} = B_T - B_{T-t}.$$

By the argument above, we have the following fact.

Proposition 3.2. *Process $\{B^{(t)}\}$ is a Brownian motion with respect to the filtration $\{\bar{\mathcal{G}}^{(t)}\}$, where $\bar{\mathcal{G}}^{(t)} = \mathcal{G}^{(T-t)}$.*

Proof. According to Definition 1.2, we need to prove the four conditions of Brownian motion.

1. For any $0 < s < t$,

$$B^{(t)} - B^{(s)} = B(T - s) - B(T - t)$$

According to condition (i) in Definition 1.2, it is a random variable with distribution $N(0, (T - s) - (T - t)) = N(0, t - s)$

2. $B^{(0)} = B(T) - B(T) = 0$ for sure.

3. For any partition $0 \leq t_1 < t_2 \dots < t_n$, the random variables

$$B^{(t_n)} - B^{(t_{n-1})}, B^{(t_{n-1})} - B^{(t_{n-2})}, \dots,$$

$$B^{(t_2)} - B^{(t_1)}, B^{(t_1)} - B^{(0)}$$

can be written as

$$B(T - t_{n-1}) - B(T - t_n), B(T - t_{n-2}) - B(T - t_{n-1}), \dots,$$

$$B(T - t_1) - B(T - t_2), B(T) - B(T - t_1)$$

Notice that $0 \leq T - t_n < T - t_{n-1} \dots < T - t_1 \leq T$ is another partition of the interval $[0, T]$, thus these random variables are independent according to (iii) in Definition 1.2

4. Since for any t , $B(t)$ is continuous almost surely, we conclude that $B^{(t)} = B(T) - B(T - t)$ is also continuous almost surely.

□

As we have previously mentioned, all of the classic results on Brownian motion apply to $B^{(t)}$ and this will provide us with information on the integral for adapted and instantly independent processes.

Before we proceed with the proof of the Itô formula, we present a technical lemma. This lemma is used in the proof of the Itô formula as well as in the proof of the Girsanov theorem for the new stochastic integral.

Lemma 3.3. *Suppose that $\{B_t\}$ is a Brownian motion and $\{B^{(t)}\}$ is its backward Brownian motion, that is $B^{(t)} = B_T - B_{T-t}$ for all $0 \leq t \leq T$. Suppose also that $g(x)$ is a continuous function. Then the following two identities hold*

$$\int_t^T g(B_T - B_s) ds = \int_0^{T-t} g(B^{(s)}) ds \quad (3.1)$$

$$\int_t^T g(B_T - B_s) dB_s = \int_0^{T-t} g(B^{(s)}) dB^{(s)}. \quad (3.2)$$

Proof. We begin with the proof of Equation (3.1). Observe that upon a change of variables $\bar{s} = T - s$ in the right side of Equation (3.1), we have

$$\begin{aligned} \int_0^{T-t} g(B^{(s)}) ds &= \int_0^{T-t} g(B_T - B_{T-s}) ds \\ &= - \int_T^t g(B_T - B_{\bar{s}}) d\bar{s} \\ &= \int_t^T g(B_T - B_{\bar{s}}) d\bar{s}. \end{aligned}$$

Thus Equation (3.1) holds.

Now, we will show Equation (3.2). Writing out the right side of Equation (3.2) using the definition of the stochastic integral, we have

$$\begin{aligned} \int_0^{T-t} g(B^{(s)}) dB^{(s)} &= \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n g(B^{(t_{i-1})}) (B^{(t_i)} - B^{(t_{i-1})}) \\ &= \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n g(B_T - B_{T-t_{i-1}}) (B_{T-t_{i-1}} - B_{T-t_i}), \end{aligned} \quad (3.3)$$

where Δ_n is a partition of the interval $[0, T-t]$ and the convergence is understood to be in probability on the space $(\Omega, \bar{\mathcal{G}}^{(T)}, P)$ from Theorem 3.1. Applying the change of variables,

$$\bar{s} = T - s, \quad \bar{t}_i = T - t_i, \quad i = 1, 2, \dots, n,$$

we transform Equation (3.3) into

$$\begin{aligned} \int_0^{T-t} g(B^{(s)}) dB^{(s)} &= \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n g(B_T - B_{\bar{t}_{i-1}})(B_{\bar{t}_{i-1}} - B_{\bar{t}_i}) \\ &= \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n g(B_T - B_{\bar{t}_{i-1}})(B_{\bar{t}_{i-1}} - B_{\bar{t}_i}) \end{aligned} \quad (3.4)$$

Notice that $T = \bar{t}_0 > \bar{t}_1 > \bar{t}_2 > \dots > \bar{t}_n = t$, and that the probability space $(\Omega, \bar{\mathcal{G}}^{(T)}, P)$ is the same as $(\Omega, \mathcal{F}_T, P)$, we conclude that the last term in Equation (5.1) converges in probability to the new stochastic integral

$$\int_t^T g(B_T - B_s) dB_s.$$

Hence the Equation (3.2) holds. \square

Remark 3.4. As we can see, this lemma reveals the relation between an classic Itô integral and a new stochastic integral that has anticipative integrand. This provide us a key inspiration to translate the new integral into classic integrals which can be dealt with using present theorems.

3.2 First Step: Itô Formula For Anticipative Processes

Now we are ready to prove the First Itô formula for the new stochastic integral.

Theorem 3.5 (Itô formula). *Suppose that*

$$Y_i^{(t)} = \int_t^T h_i(B_T - B_s) dB_s + \int_t^T g_i(B_T - B_s) ds, \quad i = 1, 2, \dots, n,$$

where h_i, g_i , $i = 1, 2, \dots, n$ are continuous, square integrable functions. Then for any i , Y_i is instantly independent with respect to $\{\mathcal{F}_t\}$. Furthermore, let $f(x_1, x_2, \dots, x_n)$ be a function in $C^2(\mathbb{R}^n)$. Then

$$\begin{aligned} df(Y_1^{(t)}, Y_2^{(t)}, \dots, Y_n^{(t)}) &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(Y_1^{(t)}, Y_2^{(t)}, \dots, Y_n^{(t)}) dY_i^{(t)} \\ &\quad - \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(Y_1^{(t)}, Y_2^{(t)}, \dots, Y_n^{(t)}) (dY_i^{(t)})(dY_j^{(t)}). \end{aligned}$$

Remark 3.6. This theorem provides a basic insight that the new Itô integral will have almost identical form as the classic Itô integral. We notice the only difference between them is the sign of the second order term is changed from positive to negative.

Proof. We prove the one-dimensional case only as the multi-dimensional case follows the same line of reasoning. Let

$$Y^{(t)} = \int_t^T h(B_T - B_s) dB_s + \int_t^T g(B_T - B_s) ds$$

Define

$$X_t = \int_0^t h(B^{(s)}) dB^{(s)} + \int_0^t g(B^{(s)}) ds.$$

where $B^{(t)}$ is the associate backward Brownian motion defined above. Since $\{B^{(s)}\}$ is a Brownian motion, we can apply the standard Itô formula Theorem 1.23 to write

$$f(X_{T-t}) - f(X_0) = \int_0^{T-t} f'(X_s) dX(s) + \frac{1}{2} \int_0^{T-t} f''(X_s) (dX_s)^2 \quad (3.5)$$

Recall that $dX_s = h(B^{(s)}) dB^{(s)} + g(B^{(s)}) ds$, and by Theorem 1.25 we get

$$\begin{aligned} (dX_s)^2 &= \left(h(B^{(s)}) dB^{(s)} + g(B^{(s)}) ds \right)^2 \\ &= h^2(B^{(s)}) (dB^{(s)})^2 + g^2(B^{(s)}) ds^2 \\ &\quad + 2h(B^{(s)})g(B^{(s)}) dB^{(s)} ds \\ &= h^2(B^{(s)}) dt \end{aligned} \quad (3.6)$$

Hence Equation (3.5) can be written as

$$\begin{aligned} f(X_{T-t}) - f(X_0) &= \int_0^{T-t} f'(X_s) h(B^{(s)}) dB^{(s)} + \int_0^{T-t} f'(X_s) g(B^{(s)}) ds \\ &\quad + \frac{1}{2} \int_0^{T-t} f''(X_s) h^2(B^{(s)}) ds \end{aligned}$$

By Theorem 1.13, we can write out the integrals in the above equation as limits of the Riemann-like sums, we have

$$\begin{aligned}
f(X_{T-t}) - f(X_0) &= \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f'(X_{t_{i-1}})h(B^{(t_{i-1})})\Delta B^{(t_i)} \\
&\quad + \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f'(X_{t_i})g(B^{(t_i)})\Delta t_i \\
&\quad + \frac{1}{2} \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f''(X_{t_i})h^2(B^{(t_i)})\Delta t_i,
\end{aligned} \tag{3.7}$$

where $\Delta_n = \{0 = t_0 \leq t_1 \leq \dots \leq t_n = T - t\}$ is a partition of the interval $[0, T - t]$.

By Lemma 3.3, we can replace X_{T-t} with $Y^{(t)}$ in Equation (3.7) to obtain

$$\begin{aligned}
&f(Y^{(t)}) - f(Y^{(T)}) \\
&= \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f'(Y^{(T-t_{i-1})})h(B_T - B_{T-t_{i-1}})(B_{T-t_{i-1}} - B_{T-t_i}) \\
&\quad + \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f'(Y^{(T-t_i)})g(B_T - B_{T-t_i})\Delta t_i \\
&\quad + \frac{1}{2} \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f''(Y^{(T-t_i)})h^2(B_T - B_{T-t_i})\Delta t_i
\end{aligned}$$

Now we apply the change of variables, $\bar{t}_i = T - t_i$ for $i = 1, 2, \dots, n$ and notice that $\{t = \bar{t}_n < \bar{t}_{n-1} < \dots < \bar{t}_1 < \bar{t}_0 = T\}$ is a partition of the interval $[t, T]$. Notice that functions f, g and h are continuous, from definition of the integral we have

$$\begin{aligned}
f(Y^{(t)}) - f(Y^{(T)}) &= \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f'(Y^{(\bar{t}_{i-1})})h(B_T - B_{\bar{t}_{i-1}})(B_{\bar{t}_{i-1}} - B_{\bar{t}_i}) \\
&\quad + \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f'(Y^{(\bar{t}_i)})g(B_T - B_{\bar{t}_i})\Delta t_i \\
&\quad + \frac{1}{2} \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f''(Y^{(\bar{t}_i)})h^2(B_T - B_{\bar{t}_i})\Delta t_i \\
&= \int_t^T f'(Y^{(s)})h(B_T - B_s)dB_s \\
&\quad + \int_t^T f'(Y^{(s)})g(B_T - B_s)ds \\
&\quad + \frac{1}{2} \int_t^T f''(Y^{(s)})h^2(B_T - B_s)ds.
\end{aligned} \tag{3.8}$$

Writing Equation (3.8) in differential form and using the fact that $dY^{(t)} = -h(B_T - B_t) dB_t - g(B_T - B_t) dt$ and $(dY^{(t)})^2 = h^2(B_T - B_t) dt$, we obtain

$$\begin{aligned} df(Y^{(t)}) &= -f'(Y^{(t)})h(B_T - B_t) dB_t - f'(Y^{(t)})g(B_T - B_t) dt \\ &\quad - \frac{1}{2}f''(Y^{(t)})h^2(B_T - B_t) dt \\ &= f'(Y^{(t)}) dY^{(t)} - \frac{1}{2}f''(Y^{(t)}) (dY^{(t)})^2. \end{aligned}$$

□

Remark 3.7. In the above proof, we implicitly used Theorem 3.1 while taking limits. Notice that in Equation (3.7), the limit is taken in the probability space $(\Omega, \bar{\mathcal{G}}^{(T)}, P)$, which by definition is equal to $(\Omega, \mathcal{G}^{(0)}, P)$. On the other hand, in Equation and (3.24), the limit is taken in the probability space $(\Omega, \mathcal{F}_T, P)$, which by Theorem 3.1 is equivalent to $(\Omega, \mathcal{G}^{(0)}, P)$

Theorem 3.5 serves as the Itô formula for functions that do not explicitly depend on t , however, we can easily make this generalization using standard methods.

Corollary 3.8. *Let $\{Y_i^{(t)} : i = 1, 2, \dots, n\}$ be defined as in Theorem 3.5 and let $f(x_1, x_2, \dots, x_n, t)$ be a function in $C^2(\mathbb{R}^n \times [0, T])$. Then*

$$\begin{aligned} df(Y_1^{(t)}, Y_2^{(t)}, \dots, Y_n^{(t)}, t) &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(Y_1^{(t)}, Y_2^{(t)}, \dots, Y_n^{(t)}, t) dY_i^{(t)} \\ &\quad - \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(Y_1^{(t)}, Y_2^{(t)}, \dots, Y_n^{(t)}, t) (dY_i^{(t)}) (dY_j^{(t)}) \\ &\quad + \frac{\partial f}{\partial t}(Y_1^{(t)}, Y_2^{(t)}, \dots, Y_n^{(t)}, t) dt \end{aligned}$$

Proof. We also only prove the one-dimensional case, the multi-dimensional case follows with same argument. Let $f = f(x, t)$ in $C^2(\mathbb{R}^n \times [0, T])$. $Y^{(t)}$ be defined as in Theorem 3.5. Then, define

$$X_t = \int_0^t h(B^{(s)}) dB^{(s)} + \int_0^t g(B^{(s)}) ds.$$

From Theorem 1.23, we have

$$\begin{aligned}
f(X_{T-t}, T-t) - f(X_0, 0) &= \int_0^{T-t} f_x(X_s, T-s) dX(s) \\
&\quad + \frac{1}{2} \int_0^{T-t} f_{xx}(X_s, T-s) (dX_s)^2 \\
&\quad + \int_0^{T-t} f_t(X_s, T-s) ds \\
&= \int_0^{T-t} f_x(X_s, T-s) h(B^{(s)}) dB^{(s)} \\
&\quad + \int_0^{T-t} f_x(X_s, T-s) g(B^{(s)}) ds \\
&\quad + \frac{1}{2} \int_0^{T-t} f_{xx}(X_s, T-s) h^2(B^{(s)}) ds \\
&\quad - \int_0^{T-t} f_t(X_s, T-s) ds
\end{aligned} \tag{3.9}$$

Compare with the proof of Theorem 3.5, we just have one more extra term. For the last term, we have

$$\begin{aligned}
\int_0^{T-t} f_t(X_s, T-s) ds &= \int_0^{T-t} f_t(Y^{(T-s)}, T-s) ds \\
&= \int_t^T f_t(Y^{(s)}, s) ds
\end{aligned} \tag{3.10}$$

Thus by replacing X_{T-t} with $Y^{(t)}$, Equation 3.9 can be written as

$$\begin{aligned}
f(Y^{(t)}, t) - f(Y^{(T)}, T) &= \int_t^T f_x(Y^{(s)}, s) h(B_T - B_s) dB_s \\
&\quad + \int_t^T f_x(Y^{(s)}, s) g(B_T - B_s) ds \\
&\quad + \frac{1}{2} \int_t^T f_{xx}(Y^{(s)}, s) h^2(B_T - B_s) ds \\
&\quad - \int_t^T f_t(Y^{(s)}, s) ds
\end{aligned} \tag{3.11}$$

Writing Equation (3.11) in differential form, we have

$$\begin{aligned}
df(Y^{(t)}, t) &= -f_x(Y^{(t)}, t)h(B_T - B_t) dB_t \\
&\quad - f_x(Y^{(t)}, t)g(B_T - B_t) dt \\
&\quad - \frac{1}{2}f_{xx}(Y^{(t)}, t)h^2(B_T - B_t) dt \\
&\quad + f_t(Y^{(t)}, t) dt
\end{aligned} \tag{3.12}$$

Notice that $dY^{(t)} = -h(B_T - B_t) dB_t - g(B_T - B_t) dt$ and $(dY^{(t)})^2 = h^2(B_T - B_t) dt$, we obtain

$$\begin{aligned}
df(Y^{(t)}, t) &= f_x(Y^{(t)}, t) dY^{(t)} - \frac{1}{2}f_{xx}(Y^{(t)}, t) (dY^{(t)})^2 \\
&\quad + f_t(Y^{(t)}, t) dt
\end{aligned}$$

□

3.3 Examples for First Step Itô Formula

We illustrate the use of the Itô formula from Theorem 3.5 with a few examples.

Example 3.9. Let $Y^{(t)} = \int_t^T 1 dB_s = B_T - B_t$. Let also $f(x) = e^x$, $g(x) = x^n$ and $h(x, t) = \exp\{x + \frac{1}{2}t\}$. Application of Theorem 3.5 and Corollary 3.8 yields

$$\begin{aligned}
df(Y^{(t)}) &= -e^{B_T - B_t} dB_t - \frac{1}{2}e^{B_T - B_t} dt, \\
dg(Y^{(t)}) &= -n(B_T - B_t)^{n-1} dB_t - \frac{1}{2}n(n-1)(B_T - B_t)^{n-2} dt, \\
dh(Y^{(t)}, t) &= -e^{B_T - B_t + \frac{1}{2}t} dB_t.
\end{aligned}$$

On the other hand, the first two of the above equalities can be derived using Theorem 2.16. The last of the above equalities can be obtained using Theorem 2.18.

The following example shows a connection between the more general Theorem 3.5 and the original Theorem 2.18.

Example 3.10. Let $Z^{(t)} = \int_t^T (B_T - B_s) dB_s$ and let $f(x)$ and $g(x)$ be as in Example 3.9. Straightforward calculations based on Definition 2.5 yield

$$Z^{(t)} = \frac{1}{2}(B_T - B_t)^2 - \frac{1}{2}(T - t) \quad \text{and} \quad dZ^{(t)} = -(B_T - B_t) dB_t.$$

Applying Theorem 3.5 and Corollary 3.8, for any $0 \leq t \leq T$, we obtain

$$\begin{aligned} df(Z^{(t)}) &= -\exp\left[\frac{1}{2}(B_T - B_t)^2 - \frac{1}{2}(T - t)\right](B_T - B_t) dB_t \\ &\quad - \frac{1}{2}\exp\left[\frac{1}{2}(B_T - B_t)^2 - \frac{1}{2}(T - t)\right](B_T - B_t)^2 dt \end{aligned}$$

and

$$\begin{aligned} dg(Z^{(t)}) &= -n\left(\frac{1}{2}(B_T - B_t)^2 - \frac{1}{2}(T - t)\right)^{n-1}(B_T - B_t) dB_t \\ &\quad - \frac{1}{2}n(n-1)\left(\frac{1}{2}(B_T - B_t)^2 - \frac{1}{2}(T - t)\right)^{n-2}(B_T - B_t)^2 dt. \end{aligned}$$

On the other hand, notice that $Z^{(t)} = \frac{1}{2}(Y^{(t)})^2 - \frac{1}{2}(T - t)$, with $Y^{(t)}$ as in Example 3.9. Defining

$$f^*(x, t) = f\left(\frac{1}{2}x^2 - \frac{1}{2}(T - t)\right),$$

allows for an application of Theorem 2.18 to obtain the same result

$$\begin{aligned} df(Z^{(t)}) &= df\left(\frac{1}{2}(Y^{(t)})^2 - \frac{1}{2}(T - t)\right) \\ &= df^*(Y^{(t)}) \\ &= -\exp\left[\frac{1}{2}(B_T - B_t)^2 - \frac{1}{2}(T - t)\right](B_T - B_t) dB_t \\ &\quad - \frac{1}{2}\exp\left[\frac{1}{2}(B_T - B_t)^2 - \frac{1}{2}(T - t)\right](B_T - B_t)^2 dt. \end{aligned}$$

The above examples demonstrate that our results are generalizations of the results presented in [15] whose special cases were cited in Theorems 2.16 and 2.18. The following example illustrates how one can define an instantly independent counterpart to the exponential process.

Example 3.11. Let

$$\mathcal{E}^{(t)}(\theta) = \exp\left\{-\int_t^T \theta(B_T - B_s) dB_s - \frac{1}{2} \int_t^T \theta^2(B_T - B_s) ds\right\}.$$

Then

$$d\mathcal{E}^{(t)}(\theta) = \theta(B_T - B_s)\mathcal{E}^{(t)}(\theta) dB_t.$$

We call $\mathcal{E}^{(t)}(\theta)$ the exponential process of the instantly independent process $\theta(B_T - B_s)$.

Proof. Let $f(x) = e^x$ and define

$$Y_t = -\int_t^T \theta(B_T - B_s) dB_s - \frac{1}{2} \int_t^T \theta^2(B_T - B_s) ds.$$

Since $f(x) = f'(x) = f''(x)$ and $f(Y_t) = \mathcal{E}^{(t)}(\theta)$, application of Theorem 3.5 to $f(Y_t)$, yields

$$\begin{aligned} d\mathcal{E}^{(t)}(\theta) &= df(Y_t) \\ &= f'(Y_t) dY_t - \frac{1}{2} f''(Y_t) (dY_t)^2 \\ &= e^{Y_t} (\theta(B_T - B_t) dB_t + \frac{1}{2} \theta^2(B_T - B_t) dt) - \frac{1}{2} e^{Y_t} \theta^2(B_T - B_t) dt \\ &= \theta(B_T - B_s)\mathcal{E}^{(t)}(\theta) dB_t. \end{aligned}$$

Here we have used the fact that $dY(t) = \theta(B_T - B_t) dB_t + \frac{1}{2}\theta^2(B_T - B_t) dt$. \square

In the next example, we give a solution to a simple linear stochastic differential equation with terminal condition and anticipating coefficients.

Example 3.12. Suppose that $f, g: \mathbb{R} \rightarrow \mathbb{R}$ are continuous square integrable functions. Then the linear SDE

$$\begin{cases} dX_t &= f(B_T - B_t)X_t dB_t + g(B_T - B_t)X_t dt \\ X_T &= \xi_T, \end{cases}$$

where ξ_T is a real deterministic constant, has a solution given by

$$X_t = \xi_T \exp \left\{ - \int_t^T f(B_T - B_s) dB_s - \int_t^T \frac{1}{2} f^2(B_T - B_s) + g(B_T - B_s) ds \right\}.$$

Notice the above SDE contains both anticipative diffusion term $f(B(T)-B(t))X_t$ and drift term $g(B_T - B_t)X_t$, thus this is a whole new SDE that can not be dealt with using any results before this chapter.

3.4 Generalized Itô Formula For Backward-Adapted Processes

After a detail proof reading of the above section, the author realize that using same method we can generalize the above formula with larger class of integrands.

In this section we will first present a generalization of several results from Section 3.2. Before we proceed with proofs of the new results, let us recall in Theorem 3.5 and Corollary 3.8 in Section 3.2, where we proved Itô formula for the stochastic process $f(Y^{(t)}, t)$, where

$$Y^{(t)} = \int_t^T h(B_T - B_s) dB_s + \int_t^T g(B_T - B_s) ds$$

Here, we weaken the assumptions of Theorems 3.5. In general, the main improvement lies in the fact that we drop the explicit dependence on the tail of Brownian motion in favor of adaptedness to the backward filtration. That is, instead of representing the integrand function of $Y^{(t)}$ as $h(B(T) - B(t))$ and $g(B(T) - B(t))$, we assume that $f(t), g(t)$ are just *backward-adapted* stochastic processes.

Definition 3.13. (Backward Adaptedness) Let $B(t)$ be a Brownian Motion on (Ω, \mathcal{F}, P) with underlying forward filtration $\{\mathcal{F}_t\}$ and associate backward filtration $\{\mathcal{G}_t\}$ in Definition 2.9, we say a stochastic process $f(t)$ is *Backward adapted* w.r.t. $B(t)$ if for any given $0 \leq t \leq T$, $f(t)$ is measurable w.r.t. \mathcal{G}_t .

Theorem 3.14. *Backward adapted processes $f(t)$ are instantly independent, but the reverse are generally not true*

Proof. For any fixed t , Since $\mathcal{F}_t = \sigma B(s), 0 < s < t$ and $\mathcal{G}_t = \sigma B(T) - B(s), t < s < T$, we can see that each base of \mathcal{F}_t is independent with \mathcal{G}_t , thus \mathcal{F}_t is independent with \mathcal{G}_t . Then, any random variable $f(t)$ on \mathcal{G}_t will be independent of \mathcal{F}_t . Since this is true for all t , we conclude that $f(t)$ is instantly independent with $\{\mathcal{F}_t\}$.

For the reverse, define another Brownian Motion $B_1(t)$ that is totally independent with $B(t)$, i.e. for any $0 < s, t < T$, $B_1(s)$ is independent with $B(t)$ as two random variables. Then clear $B_1(t)$ is instantly independent with $\{\mathcal{F}_T\}$ but not backward-adapted. \square

Notice that this is in fact a generalization for the situation in Section 3.2 because that $h(B(T) - B(t))$ and $g(B(T) - B(t))$ is obviously adapted to the natural backward Brownian filtration $\{\mathcal{G}^{(t)}\}$. Moreover, it is a nontrivial generalization. A simple example that is not in the scope of Section 3.2 is the following. For a square-integrable real-valued function g define

$$\theta_t = \int_t^T g(B_T - B_s) dB_s.$$

Then $\{\theta_t\}$ is backward-adapted and (in general) cannot be expressed as $\theta_t = f(B_T - B_t)$.

To prove the Generalized Itô Formula for anticipative processes, we begin with the following technical lemma. It is a direct generalization of Lemma 3.3.

Lemma 3.15. *Suppose that $\{B_t\}$ is a Brownian motion and $\{B^{(t)}\}$ is its backward Brownian motion, that is $B^{(t)} = B_T - B_{T-t}$ for all $0 \leq t \leq T$. Suppose also that g_t is a square-integrable process adapted to $\{\mathcal{G}^{(t)}\}$,. Then the following two identities hold*

$$\int_t^T g_s ds = \int_0^{T-t} g_{T-s} ds \tag{3.13}$$

$$\int_t^T g_s dB_s = \int_0^{T-t} g_{T-s} dB^{(s)}. \tag{3.14}$$

Proof. Let us first show that Equation (3.13) holds. Note that application of a change of variables $\bar{s} = T - s$ in the right side of Equation (3.13) yields

$$\begin{aligned} \int_0^{T-t} g_{T-s} ds &= \int_0^{T-t} g_{T-s} ds \\ &= - \int_T^t g_{\bar{s}} d\bar{s} \\ &= \int_t^T g_{\bar{s}} d\bar{s}. \end{aligned}$$

Thus the validity of Equation (3.13) is proven.

Next, we show that Equation (3.14) holds. By the definition of the stochastic integral the right side of Equation (3.14) becomes

$$\begin{aligned} \int_0^{T-t} g_{T-s} dB^{(s)} &= \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n g_{T-t_{i-1}} (B^{(t_i)} - B^{(t_{i-1})}) \\ &= \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n g_{T-t_{i-1}} (B_{T-t_{i-1}} - B_{T-t_i}), \end{aligned} \tag{3.15}$$

where Δ_n is a partition of the interval $[0, T-t]$ and the convergence is understood to be in probability on the space $(\Omega, \bar{\mathcal{G}}^{(T)}, P)$. A change of variables, $\bar{t}_i = T - t_i$, $i = 1, 2, \dots, n$ transforms Equation (3.15) into

$$\begin{aligned} \int_0^{T-t} g_{T-s} dB^{(s)} &= \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n g_{\bar{t}_{i-1}} (B_{\bar{t}_{i-1}} - B_{\bar{t}_i}) \\ &= \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n g_{\bar{t}_{i-1}} (B_{\bar{t}_{i-1}} - B_{\bar{t}_i}). \end{aligned} \tag{3.16}$$

Since $T = \bar{t}_0 > \bar{t}_1 > \bar{t}_2 > \dots > \bar{t}_n = t$ can be chosen arbitrarily, and the probability space $(\Omega, \bar{\mathcal{G}}^{(T)}, P)$ coincides with $(\Omega, \mathcal{F}_t, P)$, by the definition of the new stochastic integral, the last term in Equation (3.16) converges in probability to the new stochastic integral

$$\int_t^T g_s dB_s.$$

Hence the Equation (3.14) holds. □

Now we are ready to prove the generalization of the Itô formula.

Theorem 3.16. *Suppose that*

$$Y_i^{(t)} = \int_t^T h_i^{(s)} dB(s) + \int_t^T g_i^{(s)} ds \quad i = 1, 2, \dots, n,$$

where $h_i^{(s)}, g_i^{(s)}$ for $i = 1, 2, \dots, n$ are continuous square-integrable stochastic processes that are adapted to $\{\mathcal{G}^{(t)}\}$. Then for any $i = 1, 2, \dots, n$, Y_i is instantly independent with respect to \mathcal{F}_t . Let furthermore $f(x_1, x_2, \dots, x_n)$ be a function in $C^2(\mathbb{R}^n)$, we have following Itô Formula,

$$\begin{aligned} df(Y_1^{(t)}, Y_2^{(t)}, \dots, Y_n^{(t)}) &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(Y_1^{(t)}, Y_2^{(t)}, \dots, Y_n^{(t)}) dY_i^{(t)} \\ &\quad - \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(Y_1^{(t)}, Y_2^{(t)}, \dots, Y_n^{(t)}) (dY_i^{(t)}) (dY_j^{(t)}). \end{aligned} \tag{3.17}$$

Proof. Since the only difference between the arguments establishing the one- and multi-dimensional cases is the amount of bookkeeping, we will only show that Equation (3.17) holds with $n = 1$. For the sake of clarity of notation, we let

$$Y^{(t)} = \int_t^T h_s dB_s + \int_t^T g_s ds.$$

Let us define

$$X_t = \int_0^t h_{T-s} dB^{(s)} + \int_0^t g_{T-s} ds.$$

Since h_s and g_s are adapted to $\{\mathcal{G}^{(t)}\}$, we can view X_t as an Itô integral on the probability space $(\Omega, \mathcal{G}^{(0)}, P)$. Application of the classic Itô Formula and the Itô table yield

$$\begin{aligned} f(X_{T-t}) - f(X_0) &= \int_0^{T-t} f'(X_s) dX_s + \frac{1}{2} \int_0^{T-t} f''(X_s) (dX_s)^2 \\ &= \int_0^{T-t} f'(X_s) h_{T-s} dB^{(s)} + \int_0^{T-t} f'(X_s) g_{T-s} ds \\ &\quad + \frac{1}{2} \int_0^{T-t} f''(X_s) h_{T-s}^2 ds. \end{aligned} \tag{3.18}$$

By Lemma 3.15 we have the following identities

$$\begin{aligned}
X_{T-t} &= Y^{(t)}, \\
\int_0^{T-t} f'(X_s) h_{T-s} dB^{(s)} &= \int_t^T f'(X_{T-s}) h_s dB_s, \\
\int_0^{T-t} f'(X_s) g_{T-s} ds &= \int_t^T f'(X_{T-s}) g_s ds, \\
\int_0^{T-t} f''(X_s) h_{T-s}^2 ds &= \int_t^T f''(X_{T-s}) h_s^2 ds.
\end{aligned} \tag{3.19}$$

Putting Equations (3.18) and (3.19) together gives

$$f(Y^{(t)}) - f(Y^{(T)}) = \int_t^T f'(Y^{(s)}) h_s dB_s + \int_t^T f'(Y^{(s)}) g_s ds \tag{3.20}$$

$$+ \frac{1}{2} \int_t^T f''(Y^{(s)}) h_s^2 ds. \tag{3.21}$$

Notice that $dY^{(t)} = -h_t dB_t - g_t dt$ and $(dY^{(t)})^2 = h_t^2 dt$. Using the above in Equation (3.21) and changing to the differential notation yields

$$df(Y^{(t)}) = f'(Y^{(t)}) dY^{(t)} - \frac{1}{2} f''(Y^{(t)}) (dY^{(t)})^2,$$

which ends the proof. □

Since it is not difficult to derive a corollary to Theorem 3.16 that covers the case when the function f depends explicitly on time, we state it without a proof.

Corollary 3.17. *Suppose that*

$$Y_i^{(t)} = \int_t^T h_i^{(s)} dB(s) + \int_t^T g_i^{(s)} ds \quad i = 1, 2, \dots, n,$$

where $h_i^{(s)}, g_i^{(s)}$ for $i = 1, 2, \dots, n$ are continuous square-integrable stochastic processes that are adapted to $\{\mathcal{G}^{(t)}\}$. Suppose also that $f(x_1, x_2, \dots, x_n, t)$ is a function twice continuously differentiable in the first n variables and once continuously

differentiable in the last variable. Then,

$$\begin{aligned}
& df(Y_1^{(t)}, Y_2^{(t)}, \dots, Y_n^{(t)}, t) \\
&= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(Y_1^{(t)}, Y_2^{(t)}, \dots, Y_n^{(t)}, t) dY_i^{(t)} \\
&\quad - \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(Y_1^{(t)}, Y_2^{(t)}, \dots, Y_n^{(t)}, t) (dY_i^{(t)}) (dY_j^{(t)}) \\
&\quad + \frac{\partial f}{\partial t}(Y_1^{(t)}, Y_2^{(t)}, \dots, Y_n^{(t)}, t).
\end{aligned}$$

Using Theorem 3.16, we can easily find the counterpart to the exponential process for any process θ_t adapted to the backward filtration $\{\mathcal{G}^{(t)}\}$.

Example 3.18. Suppose that θ_t is a square-integrable stochastic process adapted to $\{\mathcal{G}^{(t)}\}$ and let

$$\mathcal{E}^{(t)}(\theta) = \exp \left\{ - \int_t^T \theta_s dB_s - \frac{1}{2} \int_t^T \theta_s^2 ds \right\}.$$

Then

$$d\mathcal{E}^{(t)}(\theta) = \theta_t \mathcal{E}^{(t)}(\theta) dB_t.$$

The process $\mathcal{E}^{(t)}(\theta)$ is called an *exponential process* of the backward-adapted process θ_t .

Proof. Let $f(x) = e^x$ and define

$$Y_t = - \int_t^T \theta_s dB_s - \frac{1}{2} \int_t^T \theta_s^2 ds.$$

Since $f(x) = f'(x) = f''(x)$ and $f(Y_t) = \mathcal{E}^{(t)}(\theta)$, application of Theorem 3.16 to $f(Y_t)$, yields

$$\begin{aligned}
d\mathcal{E}^{(t)}(\theta) &= df(Y_t) \\
&= f'(Y_t) dY_t - \frac{1}{2} f''(Y_t) (dY_t)^2 \\
&= e^{Y_t} (\theta_t dB_t + \frac{1}{2} \theta_t^2 dt) - \frac{1}{2} e^{Y_t} \theta_t^2 dt \\
&= \theta_t \mathcal{E}^{(t)}(\theta) dB_t.
\end{aligned}$$

Above we have used the fact that $dY_t = \theta_t dB_t + \frac{1}{2}\theta_t^2 dt$. □

We also will have the solution for the following SDE.

Example 3.19. Suppose that $f(t), g(t)$ are continuous square integrable processes that adapted to the backward filtration $\{\mathcal{G}_t\}$. Then the linear SDE

$$\begin{cases} dX_t &= f(t)X_t dB_t + g(t)X_t dt \\ X_T &= \xi_T, \end{cases}$$

where ξ_T is a real deterministic constant, has a solution given by

$$X_t = \xi_T \exp\left\{-\int_t^T f(s) dB_s - \int_t^T \frac{1}{2}f^2(s) + g(s) ds\right\}.$$

Proof. Let $f(x) = e^x$, then $f'(x) = f''(x) = e^x = f(x)$. Let

$$Y^{(t)} = -\int_t^T f(s) dB_s - \int_t^T \frac{1}{2}f^2(s) + g(s) ds,$$

then,

$$dY^{(t)} = f(t) dB(t) + \frac{1}{2}f^2(t) dt + g(t) dt$$

and

$$\left(dY^{(t)}\right)^2 = f^2(t) dt.$$

Apply Theorem 3.16 to $f(Y^{(t)})$ to get

$$\begin{aligned} df(Y^{(t)}) &= f'(Y^{(t)}) dY^{(t)} - \frac{1}{2}f''(Y^{(t)}) (dY^{(t)})^2 \\ &= f(Y^{(t)})(f(t) dB(t) + \frac{1}{2}f^2(t) dt + g(t) dt) - \frac{1}{2}f(Y^{(t)}) (f^2(t) dt)^2 \quad (3.22) \\ &= f(Y^{(t)})f(t) dB(t) + f(Y^{(t)})g(t) dt \end{aligned}$$

Thus $f(Y^{(t)})$ is the solution of SDE

$$dX_t = f(t)X_t dB_t + g(t)X_t dt$$

In addition, notice $f(Y^{(T)}) = 1$, thus we have $\xi_T f(Y^{(t)})$ is the solution for the SDE 3.19.

This proves our claim. □

3.5 Generalized Itô Formula for Mixed Terms

In Section 3.4, we proved the Itô formula for the backward-adapted Itô processes. The obvious limitation of the Itô formulas above is the fact that it can only treat functions that depend on the backward-adapted processes. In the present section, we prove a more general result that is applicable to function depending on adapted and backward-adapted Itô processes.

In this section, we will have following notations.

The *adapted Itô process* is a stochastic process of the form

$$X_t = \int_0^t h_s dB(s) + \int_0^t g_s ds, \quad (3.23)$$

where h_t, g_t are adapted square-integrable processes.

The *backward-adapted Itô process* is a stochastic process of the form

$$Y^{(t)} = \int_t^T \eta_s dB(s) + \int_t^T \zeta_s ds, \quad (3.24)$$

where η_t, ζ_t are backward-adapted processes.

The classic Itô formula is applicable to functions of X_t , while Theorem 3.16 is applicable to functions of $Y^{(t)}$. The next theorem constitutes an Itô formula for functions that depend on both types of processes. It is a first step towards a general Itô formula and it only applies to functions of the form $\theta(X_t, Y^{(t)})$, where $\theta(x, y) = f(x)\varphi(y)$. The first Itô formula of this type was introduced in [15, Theorem 5.1], where authors treated only the case when η, ζ are deterministic functions. Thus, while our arguments are similar to those of [15], our result extends the result of [15] substantially.

Theorem 3.20. *Suppose that $\theta(x, y)$ is a function of the form $\theta(x, y) = f(x)\varphi(y)$, where f and φ are twice continuously differentiable real-valued functions. Assume also that X_t and $Y^{(t)}$ are defined as in Equations (3.23) and (3.24) respectively such that $h_t, g_t, \eta_t, \zeta_t$ are all square integrable. Then the general Itô formula for $\theta(X_T, Y^{(T)})$ is*

$$\begin{aligned} \theta(X_T, Y^{(T)}) &= \theta(X_0, Y^{(0)}) + \int_0^t \frac{\partial \theta}{\partial x}(x_t, y^{(t)}) dx_t + \frac{1}{2} \int_0^t \frac{\partial^2 \theta}{\partial x^2}(x_t, y^{(t)}) (dx_t)^2 \\ &\quad + \int_0^t \frac{\partial \theta}{\partial y}(x_t, y^{(t)}) dy^{(t)} - \frac{1}{2} \int_0^t \frac{\partial^2 \theta}{\partial y^2}(x_t, y^{(t)}) (dy^{(t)})^2. \end{aligned}$$

provided the right hand side exist in probability.

Proof. we begin by writing out $\theta(x_t, y^{(t)}) - \theta(x_0, y^{(0)})$ as a telescoping sum. for any partition $\delta_n = \{0 = t_0 < t_1 < \dots < t_{n-1} < t_n = t\}$, we have

$$\begin{aligned} \theta(x_t, y^{(t)}) - \theta(x_0, y^{(0)}) &= \sum_{i=1}^n [\theta(x_{t_i}, y^{(t_i)}) - \theta(x_{t_{i-1}}, y^{(t_{i-1})})] \\ &= \sum_{i=1}^n [f(x_{t_i})\varphi(y^{(t_i)}) - f(x_{t_{i-1}})\varphi(y^{(t_{i-1})})]. \end{aligned} \quad (3.25)$$

now, we apply the taylor expansion to f and φ , to obtain

$$\begin{aligned} f(x_i) &= \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_{t_{i-1}}) (\Delta x_i)^k \\ \varphi(y^{(t_{i-1})}) &= \sum_{k=0}^{\infty} \frac{1}{k!} \varphi^{(k)}(y^{(t_i)}) (-\Delta y_i)^k, \end{aligned}$$

where $\Delta X_i = X_{t_i} - X_{t_{i-1}}$ and $\Delta Y_i = Y^{(t_i)} - Y^{(t_{i-1})}$. Using the standard approximation results for the Brownian motion and adapted Itô processes, we obtain the following approximations

$$\begin{aligned} \Delta X_i &\approx h_{t_{i-1}} \Delta B_i + g_{t_{i-1}} \Delta t_i. \\ (\Delta X_i)^2 &\approx h_{t_{i-1}}^2 \Delta t_i \\ (\Delta X_i)^k &= o(\Delta t_i) \quad \text{for } k \geq 3 \end{aligned} \quad (3.26)$$

To obtain a result for ΔY_i analogous to the first of Equations (3.26), we employ Lemma 3.15

$$\begin{aligned}
\Delta Y_i &= \int_{t_i}^T \eta_s dB_s + \int_{t_i}^T \zeta_s ds - \int_{t_{i-1}}^T \eta_s dB_s - \int_{t_{i-1}}^T \zeta_s ds \\
&= \int_0^{T-t_i} \eta_{T-s} dB^{(s)} - \int_0^{T-t_{i-1}} \eta_{T-s} dB^{(s)} - \int_{t_{i-1}}^{t_i} \zeta_s ds \quad (3.27) \\
&= - \int_{T-t_i}^{T-t_{i-1}} \eta_{T-s} dB^{(s)} - \int_{t_{i-1}}^{t_i} \zeta_s ds.
\end{aligned}$$

Now, the first term of the integrals in Equation (3.27) can be viewed as a standard Itô integral of an adapted process with respect to a Brownian motion $B^{(t)}$ in its natural filtration $\{\overline{\mathcal{G}}^{(t)}\}$. Thus we can approximate this integral with the left end point value. For the second term, since it is defined pathwisely, and for each continuous path, its a Lebesgue integral. Thus we can use either end to approximate. Notice that $T - t_{i-1} > T - t_i$, so Equation (3.27) can be approximated as

$$\Delta Y_i \approx -\eta_{T-(T-t_i)} \Delta B_i - \zeta_{t_i} \Delta t_i = -\eta_{t_i} \Delta B_i - \zeta_{t_i} \Delta t_i.$$

Thus,

$$(\Delta Y_i)^2 \approx \eta_{t_i}^2 \Delta t_i \quad \text{and} \quad (\Delta Y_i)^k \approx o(\Delta t_i) \text{ for } k \geq 3. \quad (3.28)$$

Putting Equations (3.25) and (3.27)–(3.28) together yields

$$\begin{aligned}
&\theta(X_T, Y^{(T)}) - \theta(X_0, Y^{(0)}) \\
&= \sum_{i=1}^n \left\{ f'(X_{t_{i-1}}) \varphi(Y^{(t_i)}) [h_{t_{i-1}} \Delta B_i + g_{t_{i-1}} \Delta t_i] + \frac{1}{2} f''(X_{t_{i-1}}) \varphi(Y^{(t_i)}) h_{t_{i-1}}^2 \Delta t_i \right. \\
&\quad \left. + f(X_{t_{i-1}}) \varphi'(Y^{(t_i)}) [-\eta_{t_i} \Delta B_i - \zeta_{t_i} \Delta t_i] - \frac{1}{2} f(X_{t_{i-1}}) \varphi''(Y^{(t_i)}) \eta_{t_i}^2 \Delta t_i \right\}.
\end{aligned}$$

Using Definition 2.5 of the new stochastic integral, the definition of the Itô integral and letting n go to ∞ , we obtain

$$\begin{aligned}
& \theta(X_T, Y^{(T)}) - \theta(X_0, Y^{(0)}) \\
&= \int_0^T f'(X_t)\varphi(Y^{(t)})h_t dB_t + \int_0^T f'(X_t)\varphi(Y^{(t)})g_t dt \\
&\quad + \frac{1}{2} \int_0^T f''(X_t)\varphi(Y^{(t)})h_t^2 dt - \int_0^T f(X_t)\varphi'(Y^{(t)})\eta_t dB_t \\
&\quad - \int_0^T f(X_t)\varphi'(Y^{(t)})\zeta_t dt - \frac{1}{2} \int_0^T f(X_t)\varphi''(Y^{(t)})\eta_t^2 dt \\
&= \int_0^T \frac{\partial\theta}{\partial x}(X_t, Y^{(t)}) dX_t + \frac{1}{2} \int_0^T \frac{\partial^2\theta}{\partial x^2}(X_t, Y^{(t)}) (dX_t)^2 \\
&\quad + \int_0^T \frac{\partial\theta}{\partial y}(X_t, Y^{(t)}) dY^{(t)} - \frac{1}{2} \int_0^T \frac{\partial^2\theta}{\partial y^2}(X_t, Y^{(t)}) (dY^{(t)})^2.
\end{aligned}$$

This proves our claim. □

Chapter 4

Generalized Girsanov Theorem

As we have seen from Section 1.7 to Section 1.8, the Girsanov Theory plays a crucial role in transforming the underlying probability measure as well as the application in the Quantitative Finance. As an analogue, the Girsanov Theorem in the new stochastic integral will also play crucial role in applying the new integral. In this chapter, we provide several theorems that extend the classic Girsanov Theorem.

4.1 Some Basic Lemmas

In this section we will first prove several lemmas that can be used to prove a special case of instantly independent counterpart of the Girsanov theorem. In the classical case of adapted stochastic processes, Girsanov theorem states that a translated Brownian motion is again a Brownian motion in some equivalent probability measure (see Theorem 1.29.) The prove of Theorem 1.29 concerns the using of so-called Lévy characterization theorem that we recall below.

Theorem 4.1 (Lévy characterization). *A stochastic process $\{X_t\}$ is a Brownian motion if and only if there exists a probability measure Q and a filtration $\{\mathcal{F}_t\}$ such that*

1. $\{X_t\}$ is a continuous Q -martingale
2. $Q(X_0 = 0) = 1$
3. the Q -quadratic variation of $\{X_t\}$ on the interval $[0, t]$ is equal to t .

As discussed in Section 2.3, there is no martingale in the anticipating setting. Thus the first item above cannot be satisfied. However, as we have indicated earlier, we can relax the requirement of adaptedness and consider near-martingales instead

of martingales. Thus we will show below that certain translations of Brownian motion produce continuous Q -near-martingales (condition (1) from Theorem 4.1) whose Q -quadratic variation is equal to t (condition (3) from Theorem 4.1). Here, the measure Q is a probability measure equivalent to measure P and will be introduced later. This equivalence immediately takes care of item (2) from Theorem 4.1.

Let us first present a technical result that is used in the proof of the theorems presented later in this section.

Theorem 4.2. *Suppose that $\{B_t\}$ is a Brownian motion and $\{\mathcal{F}_t\}$, $\{\mathcal{G}^{(t)}\}$ are its forward and backward natural filtrations, respectively. Suppose also that stochastic processes $\{X_t^{(i)}, i = 1, 2, \dots, n\}$ are adapted to $\{\mathcal{F}_t\}$ and $\{Y_t^{(i)}, i = 1, 2, \dots, n\}$ are processes that are instantly independent with respect to $\{\mathcal{F}_t\}$ and adapted to $\{\mathcal{G}^{(t)}\}$.*

Let

$$S_t = \int_0^t \sum_{i=1}^n X_s^{(i)} Y_s^{(i)} dB_s$$

and assume that $E|S_t| < \infty$ for all $0 \leq t \leq T$. Then S_t is a near-martingale with respect to both $\{\mathcal{F}_t\}$ and $\{\mathcal{G}^{(t)}\}$.

Proof. It is enough to show that the above holds for $n = 1$. The general case follows by the linearity of the conditional expectation. Thus we will show that

$$\mathbb{E}[S_t - S_s | \mathcal{F}_s] = 0, \tag{4.1}$$

$$\mathbb{E}[S_t - S_s | \mathcal{G}^{(t)}] = 0, \tag{4.2}$$

where

$$S_t = \int_0^t X_s Y_s dB_s.$$

Proof of Equation (4.1): For all $0 \leq s \leq t \leq T$ we have

$$\begin{aligned} \mathbb{E}[S_t - S_s | \mathcal{F}_s] &= \mathbb{E} \left[\int_s^t X_s Y_s dB_s \middle| \mathcal{F}_s \right] \\ &= \mathbb{E} \left[\lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n X_{t_{i-1}} Y_{t_i} \Delta B_i \middle| \mathcal{F}_s \right] \\ &= \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n \mathbb{E} [X_{t_{i-1}} Y_{t_i} \Delta B_i | \mathcal{F}_s]. \end{aligned}$$

Thus we see that it is enough to show that $\mathbb{E} [X_{t_{i-1}} Y_{t_i} \Delta B_i | \mathcal{F}_s] = 0$ for all $i = 1, 2, \dots, n$. In fact, using the tower property of the conditional expectation to condition on \mathcal{F}_{t_i} and the fact that $X_{t_{i-1}} \Delta B_i$ is \mathcal{F}_{t_i} measurable and Y_{t_i} is independent of \mathcal{F}_{t_i} , we obtain

$$\begin{aligned} \mathbb{E} [X_{t_{i-1}} Y_{t_i} \Delta B_i | \mathcal{F}_s] &= \mathbb{E} [\mathbb{E} [X_{t_{i-1}} Y_{t_i} \Delta B_i | \mathcal{F}_{t_i}] | \mathcal{F}_s] \\ &= \mathbb{E} [X_{t_{i-1}} \Delta B_i \mathbb{E} [Y_{t_i}] | \mathcal{F}_s] \\ & \quad \cdot \end{aligned}$$

Conditioning on $\mathcal{F}_{t_{i-1}}$ and using the fact that $X_{t_{i-1}}$ is measurable with respect to $\mathcal{F}_{t_{i-1}}$ and ΔB_i is independent of $\mathcal{F}_{t_{i-1}}$, we have

$$\begin{aligned} \mathbb{E} [X_{t_{i-1}} Y_{t_i} \Delta B_i | \mathcal{F}_s] &= \mathbb{E} [Y_{t_i}] \mathbb{E} [\mathbb{E} [X_{t_{i-1}} \Delta B_i | \mathcal{F}_{t_{i-1}}] | \mathcal{F}_s] \\ &= \mathbb{E} [Y_{t_i}] \mathbb{E} [X_{t_{i-1}} \mathbb{E} [\Delta B_i] | \mathcal{F}_s] \\ &= 0. \end{aligned}$$

The last equality above follows from the fact that $\mathbb{E} [\Delta B_i] = 0$.

Proof of Equation (4.2): Now we turn our attention to the second claim of the theorem. For the same reasons as above, it is enough to show that $E [X_1(t_{i-1}) Y_1(t_i) \Delta B_i | \mathcal{G}^{(t)}] = 0$ for all $i = 1, 2, \dots, n$. We start by using the tower property, independence of $X_{t_{i-1}}$

and $\mathcal{G}^{(t_{i-1})}$, and measurability of $Y_{t_i}\Delta B_i$ with respect to $\mathcal{G}^{(t_{i-1})}$.

$$\begin{aligned} E [X_{t_{i-1}}Y_{t_i}\Delta B_i|\mathcal{G}^{(t)}] &= E [E [X_{t_{i-1}}Y_{t_i}\Delta B_i|\mathcal{G}^{(t_{i-1})}]|\mathcal{G}^{(t)}] \\ &= E [E [X_{t_{i-1}}] Y_{t_i}\Delta B_i|\mathcal{G}^{(t)}] \end{aligned}$$

Finally, since ΔB_i is independent of $\mathcal{G}^{(t_i)}$ and Y_{t_i} is measurable with respect to $\mathcal{G}^{(t_i)}$, application of the tower property and the fact that $\mathbb{E} [\Delta B_i] = 0$, yields

$$\begin{aligned} E [X_{t_{i-1}}Y_{t_i}\Delta B_i|\mathcal{G}^{(t)}] &= E [X_{t_{i-1}}] E [E [Y_{t_i}\Delta B_i|\mathcal{G}^{(t_i)}]|\mathcal{G}^{(t)}] \\ &= E [X_{t_{i-1}}] E [Y_{t_i}E [\Delta B_i]|\mathcal{G}^{(t)}] \\ &= 0. \end{aligned}$$

Thus the proof is complete. □

Now, we turn our attention to the study of the translated Brownian motion $\tilde{B}_t = B_t + \int_0^t \theta(B_T - B_s) ds$. As in the classical theory of the Itô calculus, this is the process that is described by the Girsanov theorem, with the exception that $B_T - B_s$ is substituted by B_s in the classical case. The crucial role in the construction of the measure Q , in which $\{\tilde{B}_t\}$ is a near-martingale, is played by the exponential process $\mathcal{E}^{(t)}(\theta)$. The next result gives a very useful representation of the exponential process that is applied in the proofs of our main results.

Lemma 4.3. *Suppose that $\theta(x)$ is a real-valued square integrable function. Then the exponential process of $\theta(B_T - B_t)$ given by*

$$\mathcal{E}^{(t)}(\theta) = \exp\left\{-\int_t^T \theta(B_T - B_s) dB_s - \frac{1}{2} \int_t^T \theta^2(B_T - B_s) ds\right\}$$

has the following representation

$$\mathbb{E} [\mathcal{E}^{(0)}(\theta)|\mathcal{G}^{(t)}] = \mathcal{E}^{(t)}(\theta),$$

where $\{B_t\}$ is a Brownian motion, $\{\mathcal{G}^{(t)}\}$ is its natural backward filtration.

Proof. By Example 3.11, we have

$$\mathcal{E}^{(t)}(\theta) - \mathcal{E}^{(0)}(\theta) = \int_0^t \theta(B_T - B_t) \mathcal{E}^{(t)}(\theta) dB_t.$$

Note that $\theta(B_T - B_t) \mathcal{E}^{(t)}(\theta)$ is instantly independent of $\{\mathcal{F}_t\}$ and adapted to $\mathcal{G}^{(t)}$, thus by Theorem 4.2, $\mathcal{E}^{(t)}(\theta)$ is a near-martingale relative to $\{\mathcal{G}^{(t)}\}$, that is

$$\mathbb{E} [\mathcal{E}^{(0)}(\theta) - \mathcal{E}^{(t)}(\theta) | \mathcal{G}^{(t)}] = 0.$$

Equivalently, we have

$$\mathbb{E} [\mathcal{E}^{(0)}(\theta) | \mathcal{G}^{(t)}] = \mathbb{E} [\mathcal{E}^{(t)}(\theta) | \mathcal{G}^{(t)}].$$

Note that $\mathcal{E}^{(t)}(\theta)$ is measurable with respect to $\mathcal{G}^{(t)}$. Hence

$$E [\mathcal{E}^{(0)}(\theta) | \mathcal{G}^{(t)}] = \mathcal{E}^{(t)}(\theta),$$

and the proof is complete. □

4.2 Anticipative Girsanov Theorem

With above lemmas, we can now show the anticipative version of Girsanov Theorem. Namely, we will prove the analogue of condition (1) and condition (3) of Theorem 4.1. Condition 2 is trivial, thus omitted.

The next theorem proves the analogue of condition (1) of Theorem 4.1.

Theorem 4.4. *Suppose that $\{B_t\}$ is a Brownian motion on $(\Omega, \mathcal{F}_T, P)$ and $\varphi(x)$ is a square integrable function on \mathbb{R} s.t. $E[\mathcal{E}^{(t)}(\varphi(B(T) - B(t)))] < \infty$. Let*

$$\tilde{B}_t = B_t + \int_0^t \varphi(B_T - B_s) ds.$$

Then \tilde{B}_t is a continuous near-martingale with respect to the probability measure Q given by

$$\begin{aligned} dQ &= \mathcal{E}^{(0)}(\varphi) dP \\ &= \exp \left\{ - \int_0^T \varphi(B_T - B_s) dB_s - \frac{1}{2} \int_0^T \varphi^2(B_T - B_s) ds \right\} dP. \end{aligned}$$

As a remark, let us note that the measure Q that was used in Theorem 4.4 is the same as the one derived in [6] where the author uses methods of Malliavin calculus to study anticipative Girsanov transformations. For more details on this approach see [6] and references therein.

Proof. The continuity of the process $\{\tilde{B}_t\}$ is trivial. In order to clearly present the remainder of the proof, we proceed in several steps.

Step 1: First, we simplify the problem at hand. We will show that it is enough to verify that the expectation of a certain process is constant as a function of t . Define

$$\hat{B}_t = B_T - B_t + \int_t^T \varphi(B_T - B_s) ds.$$

Then for any $0 \leq s \leq t \leq T$ we have

$$\tilde{B}_t - \tilde{B}_s = \hat{B}_s - \hat{B}_t. \quad (4.3)$$

This implies that $\mathbb{E}[\tilde{B}_t - \tilde{B}_s | \mathcal{F}_s] = 0$ if and only if $\mathbb{E}[\hat{B}_t - \hat{B}_s | \mathcal{F}_s] = 0$. Thus it is enough to show that \hat{B}_t is a Q -near-martingale. Note that \hat{B}_t is an instantly independent process, hence by Theorem 2.12, it suffices to show that $\mathbb{E}_Q[\hat{B}_t]$ is constant.

Step 2: In this step, we show that $\mathbb{E}_Q[\hat{B}_t]$ is constant. First, by the property of the conditional expectation, we have

$$\begin{aligned} \mathbb{E}_Q[\hat{B}_t] &= \mathbb{E} \left[\hat{B}_t \mathcal{E}^{(0)}(\varphi) \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\hat{B}_t \mathcal{E}^{(0)}(\varphi) \middle| \mathcal{G}^{(t)} \right] \right] \end{aligned}$$

Since \hat{B}_t is measurable with respect to $\mathcal{G}^{(t)}$, Lemma 4.3 yields

$$\begin{aligned} \mathbb{E}_Q[\hat{B}_t] &= \mathbb{E} \left[\hat{B}_t \mathbb{E} \left[\mathcal{E}^{(0)}(\varphi) \middle| \mathcal{G}^{(t)} \right] \right] \\ &= \mathbb{E} \left[\hat{B}_t \mathcal{E}^{(t)}(\varphi) \right]. \end{aligned} \quad (4.4)$$

Next, we apply the Itô Formula (Theorem 3.5) to $\widehat{B}_t \mathcal{E}^{(t)}(\varphi)$. In order to do so we let $f(x, y) = xy$, thus

$$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x, \quad \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y} = 1.$$

We have

$$df \left(\widehat{B}_t, \mathcal{E}^{(t)}(\varphi) \right) = \widehat{B}_t d\mathcal{E}^{(t)}(\varphi) + \mathcal{E}^{(t)}(\varphi) d\widehat{B}_t - (d\mathcal{E}^{(t)}(\varphi)) \left(d\widehat{B}_t \right). \quad (4.5)$$

Using the facts that $d\widehat{B}_t = -dB_t - \varphi(B_T - B_t) dt$ and $d\mathcal{E}^{(t)}(\varphi) = \varphi(B_T - B_t) \mathcal{E}^{(t)}(\varphi) dB_t$, Equation (4.5) becomes

$$d \left(\widehat{B}_t \mathcal{E}^{(t)}(\varphi) \right) = \left(\mathcal{E}^{(t)}(\varphi) \varphi(B_T - B_t) \widehat{B}_t - \mathcal{E}^{(t)}(\varphi) \right) dB_t,$$

Thus we have

$$\left(\widehat{B}_T \mathcal{E}^{(T)}(\varphi) \right) - \left(\widehat{B}_t \mathcal{E}^{(t)}(\varphi) \right) = \int_t^T \left(\mathcal{E}^{(t)}(\varphi) \varphi(B_T - B_t) \widehat{B}_t - \mathcal{E}^{(t)}(\varphi) \right) dB_t, \quad (4.6)$$

Notice $\widehat{B}_T = 0$, we have

$$\left(\widehat{B}_t \mathcal{E}^{(t)}(\varphi) \right) = - \int_t^T \left(\mathcal{E}^{(t)}(\varphi) \varphi(B_T - B_t) \widehat{B}_t - \mathcal{E}^{(t)}(\varphi) \right) dB_t, \quad (4.7)$$

Observe that $\mathcal{E}^{(t)}(\varphi) \varphi(B_T - B_t) \widehat{B}_t - \mathcal{E}^{(t)}(\varphi)$ is instantly independent with respect to $\{\mathcal{F}_t\}$, hence by Theorem 4.2, $\widehat{B}(t) \mathcal{E}^{(t)}(\varphi)$ is a near-martingale with respect to \mathcal{F}_t . Thus by Theorem 2.12, $\mathbb{E}[\widehat{B}(t) \mathcal{E}^{(t)}(\varphi)]$ is constant therefore, by Equation 4.4, $\mathbb{E}_Q[\widehat{B}(t)]$ is constant as desired. \square

Next, we present a result on the process \widehat{B}_t that we introduced in the proof of Theorem 4.4. As it turns out, the properties of this process are crucial in the proof of the condition (3) as well. Moreover, in the Corollary 4.8 we will present some more properties of this process as it is interesting on its own.

Theorem 4.5. *Suppose that $\{B_t\}$ is a Brownian motion on $(\Omega, \mathcal{F}_T, P)$. Suppose also that $\varphi(x)$ is a square integrable function on \mathbb{R} s.t. $E[\mathcal{E}^{(t)}(\varphi(B(T) - B(t)))] < \infty$, and Q is the probability measure introduced in Theorem 4.4. Let*

$$\widehat{B}_t = B_T - B_t + \int_t^T \varphi(B_T - B_s) ds. \quad (4.8)$$

Then $\widehat{B}_t^2 - (T - t)$ is a continuous Q -near-martingale.

Proof. As previously, the continuity of the process $\widehat{B}_t^2 - (T - t)$ is obvious. Since $\widehat{B}^2(t) - (T - t)$ is instantly independent with respect to $\{\mathcal{F}_t\}$, by Theorem 2.12, we only need to show that $\mathbb{E}_Q[\widehat{B}_t^2 - (T - t)]$ is constant. In fact, using the same methods as in the proof of Theorem 4.4, we have

$$\begin{aligned} \mathbb{E}_Q [\widehat{B}_t^2 - (T - t)] &= \mathbb{E} [(\widehat{B}_t^2 - (T - t))\mathcal{E}^{(0)}(\varphi)] \\ &= \mathbb{E} \left[\mathbb{E} [(\widehat{B}^2(t) - (T - t))\mathcal{E}^{(0)}(\varphi) | \mathcal{G}^{(t)}] \right] \\ &= \mathbb{E} [(\widehat{B}_t^2 - (T - t))\mathbb{E} [\mathcal{E}^{(0)}(\varphi) | \mathcal{G}^{(t)}]] \\ &= \mathbb{E} [(\widehat{B}_t^2 - (T - t))\mathcal{E}^{(t)}(\varphi)]. \end{aligned}$$

In the last equality above we have used Lemma 4.3. Note that now it is enough to show that $\mathbb{E}[(\widehat{B}_t^2 - (T - t))\mathcal{E}^{(t)}(\varphi)]$ is constant by Theorem 2.12.

Next, we apply the Itô formula (see Corollary 3.8) to $f(x, y, t) = (x^2 - (T - t))y$ with $x = \widehat{B}_t$ and $y = \mathcal{E}^{(t)}(\varphi)$ to obtain

$$\begin{aligned} df(\widehat{B}_t, \mathcal{E}^{(t)}(\varphi), t) &= \frac{\partial f}{\partial x}(\widehat{B}_t, \mathcal{E}^{(t)}(\varphi), t) d\widehat{B}_t + \frac{\partial f}{\partial y}(\widehat{B}_t, \mathcal{E}^{(t)}(\varphi), t) d\mathcal{E}^{(t)}(\varphi) \\ &\quad + \frac{\partial f}{\partial t}(\widehat{B}_t, \mathcal{E}^{(t)}(\varphi), t) dt - \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(\widehat{B}_t, \mathcal{E}^{(t)}(\varphi), t) (d\widehat{B}_t)^2 \\ &\quad - \frac{\partial^2 f}{\partial x \partial y}(\widehat{B}_t, \mathcal{E}^{(t)}(\varphi), t) (d\widehat{B}_t)(d\mathcal{E}^{(t)}(\varphi)) \end{aligned} \quad (4.9)$$

Since partial derivatives of $f(x, y, t)$ are given by

$$\frac{\partial f}{\partial x} = 2xy, \quad \frac{\partial f}{\partial y} = x^2 - (T - t), \quad \frac{\partial f}{\partial t} = y, \quad \frac{\partial^2 f}{\partial x^2} = 2y, \quad \frac{\partial^2 f}{\partial x \partial y} = 2x,$$

and the stochastic differentials in Equation (5.5) are given by

$$d\widehat{B}(t) = -dB_t - \varphi(B_T - B_t) dt, \quad d\mathcal{E}^{(t)}(\varphi) = \varphi(B_T - B_t)\mathcal{E}^{(t)}(\varphi) dB_t,$$

we obtain

$$\begin{aligned} & d\left(\left(\widehat{B}_t^2 - (T - t)\right)\mathcal{E}^{(t)}(\varphi)\right) \\ &= \left(\mathcal{E}^{(t)}(\varphi)\left(\widehat{B}_t^2\varphi(B_T - B_t) - 2\widehat{B}_t\right) - (T - t)\varphi(B_T - B_t)\right) dB_t, \end{aligned}$$

Thus we have

$$\left(\widehat{B}_T^2 - (T - T)\right)\mathcal{E}^{(T)}(\varphi) - \left(\widehat{B}_t^2 - (T - t)\right)\mathcal{E}^{(t)}(\varphi) \tag{4.10}$$

$$= \int_t^T \left(\mathcal{E}^{(t)}(\varphi)\left(\widehat{B}_t^2\varphi(B_T - B_t) - 2\widehat{B}_t\right) - (T - t)\varphi(B_T - B_t)\right) dB_t, \tag{4.11}$$

Notice that $\widehat{B}_T^2 - (T - T) = 0$, thus we have

$$\left(\widehat{B}_t^2 - (T - t)\right)\mathcal{E}^{(t)}(\varphi) \tag{4.12}$$

$$= - \int_t^T \left(\mathcal{E}^{(t)}(\varphi)\left(\widehat{B}_t^2\varphi(B_T - B_t) - 2\widehat{B}_t\right) - (T - t)\varphi(B_T - B_t)\right) dB_t, \tag{4.13}$$

It is straightforward, although tedious, to show that the integrand under the integral above, is instantly independent with respect to $\{\mathcal{F}_s\}$. Therefore, by Theorem 4.2, $\left(\widehat{B}_t^2 - (T - t)\right)\mathcal{E}^{(t)}(\varphi)$, as an integral of an instantly independent process, is a near-martingale with respect to \mathcal{F}_t . And thus by Theorem 2.12, $E\left[\left(\widehat{B}^2(t) - (T - t)\right)\mathcal{E}^{(t)}(\varphi)\right]$ is constant, so the theorem holds. \square

Now we are ready to present the proof of condition (3) for the process \widetilde{B}_t .

Theorem 4.6. *Suppose that $\{B_t\}$ is a Brownian motion in the probability space $(\Omega, \mathcal{F}_T, P)$. Let Q be a measure introduced in Theorem 4.4. Then the Q -quadratic variation of*

$$\widetilde{B}_t = B_t + \int_0^t \varphi(B_T - B_s) ds$$

on the interval $[0, t]$ is equal to t .

Remark 4.7. In the proof of this theorem, we will first make a detour to define a stochastic process w.r.t. the backward Brownian motion. Then use similar argument as in Theorem 4.4 to prove our statement.

Proof. We know that under measure P , the process $\{B^{(t)}\}$ defined by $B^{(t)} = B_T - B_{T-t}$ is a Brownian motion relative to filtration $\{\bar{\mathcal{G}}^{(t)}\}$ (see Proposition 3.2.) By the classical Girsanov theorem (see Theorem 1.29), define

$$\tilde{B}^{(t)} = B^{(t)} + \int_0^t \varphi(B^{(s)}) ds \quad (4.14)$$

Then $\tilde{B}^{(t)}$ is a Brownian motion under measure \tilde{Q} given by

$$d\tilde{Q} = \exp\left\{-\int_0^T \varphi(B^{(s)}) dB^{(s)} - \frac{1}{2} \int_0^T \varphi^2(B^{(s)}) ds\right\} dP \quad (4.15)$$

Using Lemma 3.3 (where $t = 0$), we have

$$\begin{aligned} d\tilde{Q} &= \exp\left\{-\int_0^T \varphi(B_T - B_s) dB(s) - \frac{1}{2} \int_0^T \varphi^2(B_T - B_s) ds\right\} dP \\ &= dQ. \end{aligned}$$

This means that $\tilde{Q} = Q$, i.e. $\{\tilde{B}^{(t)}\}$ is a Brownian motion under Q . Therefore the Q -quadratic variation of $\{\tilde{B}^{(t)}\}$ on the interval $[T-t, T]$ is equal to t . In other words, for any partition $\Delta_n = \{T-t = t_0 \leq t_1 \leq \dots \leq t_n = T\}$ of the interval $[T-t, T]$, we have that

$$\lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n (\tilde{B}^{(t_i)} - \tilde{B}^{(t_{i-1})})^2 = t,$$

where the limit is taken in probability under measure Q . Changing the variables $\bar{t}_i = T - t_{n-i}$, $i = 0, 1, \dots, n$, yields a partition $\bar{\Delta}_n$ of the interval $[0, t]$ and (still under Q),

$$\lim_{\|\bar{\Delta}_n\| \rightarrow 0} \sum_{i=1}^n (\tilde{B}^{(T-\bar{t}_i)} - \tilde{B}^{(T-\bar{t}_{i-1})})^2 = t. \quad (4.16)$$

Now notice that by Equation (4.14), the definition of $B^{(s)}$, we have

$$\begin{aligned}\tilde{B}^{(T-\bar{t}_i)} &= B^{(T-\bar{t}_i)} + \int_0^{T-\bar{t}_i} \varphi(B^{(s)}) ds \\ &= B_T - B_{\bar{t}_i} + \int_0^{T-\bar{t}_i} \varphi(B^{(s)}) ds\end{aligned}$$

by definition of $B^{(t)}$,

$$= B_T - B_{\bar{t}_i} + \int_{\bar{t}_i}^T \varphi(B_T - B_s) ds$$

by Lemma 3.3,

$$= \widehat{B}_{\bar{t}_i}.$$

Hence Equation (4.16) becomes

$$\lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n (\widehat{B}_{\bar{t}_i} - \widehat{B}_{\bar{t}_{i-1}})^2 = t, \quad (4.17)$$

where the limit is understood as a limit in probability under measure Q and $0 = \bar{t}_0 \leq \bar{t}_1 \leq \dots \leq \bar{t}_n = t$. Finally, Equations (4.3) and (4.17) yield

$$\lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n (\tilde{B}_{\bar{t}_i} - \tilde{B}_{\bar{t}_{i-1}})^2 = t,$$

establishing the desired result. Tracing back the argument above, we can choose the points \bar{t}_i arbitrarily, hence Equation (4.17) implies that the Q -quadratic variation of \widehat{B}_t on the interval $[0, t]$ is equal to t . Hence the proof is complete. \square

Corollary 4.8. *Under the assumptions of Theorem 4.6, the Q -quadratic variation of the stochastic process $\{\widehat{B}_t\}$ in Equation (4.8) on the interval $[0, t]$ is equal to t .*

Proof. This follows immediately from Equation (4.17). \square

4.3 Examples for Anticipative Girsanov Theorem

Below we give several examples as the application of Theorems 4.4 to 4.6. The first one is the case with a simple instantly independent process driving the drift term of the translated Brownian motion.

Example 4.9. Let $X_t = B_t + \int_0^t B_T - B_s ds$. The exponential process of $\varphi(B_T - B_t) = B_T - B_t$ is given by

$$\begin{aligned}\mathcal{E}^{(t)}(\varphi) &= \exp\left\{-\int_t^T B_T - B_s dB_s - \frac{1}{2}\int_t^T (B_T - B_s)^2 ds\right\} \\ &= \exp\left\{-\frac{1}{2}(B_T - B_t)^2 + \frac{1}{2}(T - t) - \frac{1}{2}\int_t^T (B_T - B_s)^2 ds\right\},\end{aligned}$$

where the last equality follows from the Itô formula (see Theorem 3.5.)

Thus, by Theorems 4.4 and 4.6, we conclude that X_t is a continuous Q -near-martingale with Q -quadratic variation on the interval $[0, t]$ equal to t , where the measure Q is defined by

$$dQ = \mathcal{E}^{(0)}(\varphi) dP = \exp\left\{-\frac{1}{2}B_T^2 + \frac{1}{2}T - \frac{1}{2}\int_0^T (B_T - B_s)^2 ds\right\} dP.$$

In addition, the quadratic variation $[X]_t$ of X_t is t almost surely for any interval $[0, t]$.

The second example serves as a comparison of our results and the classical Girsanov theorem. The only case that we are able to compare right now is the one where the function $\varphi(x) = a$ for some real number $a \neq 0$.

Example 4.10. Suppose that $\varphi(x) = a$. By Theorems 4.4 and 4.6, the stochastic process

$$X_t = B_t + \int_0^t a ds = B_t + at$$

is a continuous \bar{Q} -near-martingale with \bar{Q} -quadratic variation on the interval $[0, t]$ being equal to t , where measure \bar{Q} is given by

$$d\bar{Q} = \mathcal{E}^{(0)}(\varphi) dP = \exp\{-aB_t - \frac{1}{2}a^2T\} dP.$$

Note that since the translation is deterministic, the process $\{X_t\}$ is in fact adapted to the underlying filtration $\{\mathcal{F}_t\}$. Hence, as an adapted near-martingale, $\{X_t\}$ is a

martingale with respect to $\{\mathcal{F}_t\}$. Therefore, by the Lévy characterization theorem (see Theorem 4.1), process $\{X_t\}$ is a Brownian motion under \bar{Q} .

On the other hand, the drift term of the process $\{X_t\}$ is deterministic, therefore adapted, so we can use the classical Girsanov theorem (see Theorem 1.29 with $f(s) = -a$) to conclude that X_t is a Q -Brownian motion, where Q is given by

$$dQ = \mathcal{E}_T(\varphi) dP = \exp\left\{-aB_t - \frac{1}{2}a^2T\right\} dP.$$

Notice that Q and \bar{Q} are actually the same measure. Thus, as expected, our results lead to the same conclusion in the case when both theorems are applicable, that is in the case when the translation is in fact deterministic. Later, we will provide a more general Girsanov Theorem that unifies the theorems in last section and the classical Girsanov Theorem.

4.4 Generalized Anticipative Girsanov Theorem

In this section, we will borrow the same idea from Section 3.4, that is, to generalize the results in Section 4.2 to the stochastic integrals with backward adapted integrand instead. Recall in Section 4.2, the drift term is only allowed to take forms like $\varphi(B(T) - B(t))$. While $\varphi(B(T) - B(t))$ is special form is backward adapted, we can not deal with following problem

Question 4.11. Let

$$X_t = \int_t^T B(T) - B(s) dB(s),$$

and that

$$\tilde{B}_t = B_t + \int_0^t X_s ds.$$

Then, what is the Girsanov Transformation of \tilde{B}_t ?

To deal with this problem, we need to utilize the Itô Formula in Section 3.4, and then use similar argument as the proof of Theorem 4.4 to 4.6.

We first generalize Lemma 4.3.

Lemma 4.12. *Suppose that $\theta(t)$ is a backward adapted square integrable process.*

Then the exponential process of $\theta(t)$ given by

$$\mathcal{E}^{(t)}(\theta) = \exp \left\{ - \int_t^T \theta(s) dB_s - \frac{1}{2} \int_t^T \theta^2(s) ds \right\}$$

has the following representation

$$\mathbb{E} [\mathcal{E}^{(0)}(\theta) | \mathcal{G}^{(t)}] = \mathcal{E}^{(t)}(\theta),$$

where $\{B_t\}$ is a Brownian motion, $\{\mathcal{G}^{(t)}\}$ is its natural backward filtration.

Proof. By Example 3.18, we have

$$\mathcal{E}^{(t)}(\theta) - \mathcal{E}^{(0)}(\theta) = \int_0^t \theta(s) \mathcal{E}^{(s)}(\theta) dB_t.$$

Note that $\theta(t)\mathcal{E}^{(t)}(\theta)$ is instantly independent of $\{\mathcal{F}_t\}$ and adapted to $\mathcal{G}^{(t)}$, thus by Theorem 4.2, $\mathcal{E}^{(t)}(\theta)$ is a near-martingale relative to $\{\mathcal{G}^{(t)}\}$, that is

$$\mathbb{E} [\mathcal{E}^{(0)}(\theta) - \mathcal{E}^{(t)}(\theta) | \mathcal{G}^{(t)}] = 0.$$

Equivalently, we have

$$\mathbb{E} [\mathcal{E}^{(0)}(\theta) | \mathcal{G}^{(t)}] = \mathbb{E} [\mathcal{E}^{(t)}(\theta) | \mathcal{G}^{(t)}].$$

Note that $\mathcal{E}^{(t)}(\theta)$ is measurable with respect to $\mathcal{G}^{(t)}$. Hence

$$E [\mathcal{E}^{(0)}(\theta) | \mathcal{G}^{(t)}] = \mathcal{E}^{(t)}(\theta),$$

and the proof is complete. □

Next, we state the generalization of Theorem 4.4 to Theorem 4.6.

Theorem 4.13. *Suppose that $\{B_t\}$ is a Brownian motion on $(\Omega, \mathcal{F}_T, P)$ and φ_t is a square-integrable real-valued stochastic process adapted to $\{\mathcal{G}^{(t)}\}$ s.t. $E[\mathcal{E}^{(t)}(\varphi)] < \infty$ for all $t > 0$. Let*

$$\tilde{B}_t = B_t + \int_0^t \varphi(s) ds. \tag{4.18}$$

Then \tilde{B}_t is a continuous near-martingale with respect to the probability measure Q given by

$$\begin{aligned} dQ &= \mathcal{E}^{(0)}(\varphi) dP \\ &= \exp\left\{-\int_0^T \varphi_s dB_s - \frac{1}{2} \int_0^T \varphi_s^2 ds\right\} dP. \end{aligned} \tag{4.19}$$

Proof. The continuity of the process $\{\tilde{B}_t\}$ is trivial. The proof is very similar to Theorem 4.4.

Step 1: First, define

$$\hat{B}_t = B_T - B_t + \int_t^T \varphi(s) ds.$$

Then for any $0 \leq s \leq t \leq T$ we have

$$\tilde{B}_t - \tilde{B}_s = \hat{B}_s - \hat{B}_t. \tag{4.20}$$

This implies that $\mathbb{E}[\tilde{B}_t - \tilde{B}_s | \mathcal{F}_s] = 0$ if and only if $\mathbb{E}[\hat{B}_t - \hat{B}_s | \mathcal{F}_s] = 0$. Thus it is enough to show that \hat{B}_t is a Q -near-martingale. Note that \hat{B}_t is an instantly independent process, hence by Theorem 2.12, it suffices to show that $\mathbb{E}_Q[\hat{B}_t]$ is constant.

Step 2: In this step, we show that $\mathbb{E}_Q[\hat{B}_t]$ is constant. First, by the property of the conditional expectation, we have

$$\begin{aligned} \mathbb{E}_Q[\hat{B}_t] &= \mathbb{E}\left[\hat{B}_t \mathcal{E}^{(0)}(\varphi)\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\hat{B}_t \mathcal{E}^{(0)}(\varphi) \middle| \mathcal{G}^{(t)}\right]\right] \end{aligned}$$

Since \hat{B}_t is measurable with respect to $\mathcal{G}^{(t)}$, Lemma 3.15 yields

$$\begin{aligned} \mathbb{E}_Q[\hat{B}_t] &= \mathbb{E}\left[\hat{B}_t \mathbb{E}\left[\mathcal{E}^{(0)}(\varphi) \middle| \mathcal{G}^{(t)}\right]\right] \\ &= \mathbb{E}\left[\hat{B}_t \mathcal{E}^{(t)}(\varphi)\right]. \end{aligned} \tag{4.21}$$

Next, we apply the Generalized Itô Formula for anticipative processes (Theorem 3.16) to $\widehat{B}_t \mathcal{E}^{(t)}(\varphi)$. In order to do so we let $f(x, y) = xy$, thus

$$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x, \quad \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y} = 1.$$

We have

$$df \left(\widehat{B}_t, \mathcal{E}^{(t)}(\varphi) \right) = \widehat{B}_t d\mathcal{E}^{(t)}(\varphi) + \mathcal{E}^{(t)}(\varphi) d\widehat{B}_t - (d\mathcal{E}^{(t)}(\varphi)) \left(d\widehat{B}_t \right). \quad (4.22)$$

Using the facts that $d\widehat{B}_t = -dB_t - \varphi(t) dt$ and $d\mathcal{E}^{(t)}(\varphi) = \varphi(t)\mathcal{E}^{(t)}(\varphi) dB_t$, Equation (4.22) becomes

$$d \left(\widehat{B}_t \mathcal{E}^{(t)}(\varphi) \right) = \left(\mathcal{E}^{(t)}(\varphi)\varphi(t)\widehat{B}_t - \mathcal{E}^{(t)}(\varphi) \right) dB_t,$$

Thus we have

$$\left(\widehat{B}_T \mathcal{E}^{(T)}(\varphi) \right) - \left(\widehat{B}_t \mathcal{E}^{(t)}(\varphi) \right) = \int_t^T \left(\mathcal{E}^{(s)}(\varphi)\varphi(s)\widehat{B}_s - \mathcal{E}^{(s)}(\varphi) \right) dB_s, \quad (4.23)$$

Notice $\widehat{B}_T = 0$, we have

$$\left(\widehat{B}_t \mathcal{E}^{(t)}(\varphi) \right) = - \int_t^T \left(\mathcal{E}^{(s)}(\varphi)\varphi(s)\widehat{B}_s - \mathcal{E}^{(s)}(\varphi) \right) dB_s, \quad (4.24)$$

Observe that $\mathcal{E}^{(t)}(\varphi)\varphi(t)\widehat{B}_t - \mathcal{E}^{(t)}(\varphi)$ is instantly independent with respect to $\{\mathcal{F}_t\}$, hence by Theorem 4.2, $\widehat{B}(t)\mathcal{E}^{(t)}(\varphi)$ is a near-martingale with respect to \mathcal{F}_t . Thus by Theorem 2.12, $\mathbb{E}[\widehat{B}(t)\mathcal{E}^{(t)}(\varphi)]$ is constant therefore, by Equation 4.21, $\mathbb{E}_Q[\widehat{B}(t)]$ is constant as desired. \square

Theorem 4.14. *Suppose that $\{B_t\}$ is a Brownian motion on $(\Omega, \mathcal{F}_T, P)$. Suppose also that φ_t is a square-integrable real-valued stochastic process adapted to $\{\mathcal{G}^{(t)}\}$ s.t. $E[\mathcal{E}^{(t)}(\varphi)] < \infty$ for all $t > 0$ and Q is the probability measure given by Equation (4.19). Let*

$$\widehat{B}_t = B_T - B_t + \int_t^T \varphi(s) ds.$$

Then $\widehat{B}_t^2 - (T - t)$ is a continuous Q -near-martingale.

Proof. As previously, the continuity of the process $\widehat{B}_t^2 - (T - t)$ is obvious. Since $\widehat{B}^2(t) - (T - t)$ is instantly independent with respect to $\{\mathcal{F}_t\}$, by Theorem 2.12, we only need to show that $\mathbb{E}_Q[\widehat{B}_t^2 - (T - t)]$ is constant. In fact,

$$\begin{aligned}\mathbb{E}_Q \left[\widehat{B}_t^2 - (T - t) \right] &= \mathbb{E} \left[(\widehat{B}_t^2 - (T - t)) \mathcal{E}^{(0)}(\varphi) \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[(\widehat{B}^2(t) - (T - t)) \mathcal{E}^{(0)}(\varphi) \middle| \mathcal{G}^{(t)} \right] \right] \\ &= \mathbb{E} \left[(\widehat{B}_t^2 - (T - t)) \mathbb{E} \left[\mathcal{E}^{(0)}(\varphi) \middle| \mathcal{G}^{(t)} \right] \right] \\ &= \mathbb{E} \left[(\widehat{B}_t^2 - (T - t)) \mathcal{E}^{(t)}(\varphi) \right].\end{aligned}$$

In the last equality above we have used Lemma 3.15. Note that now it is enough to show that $\mathbb{E}[(\widehat{B}_t^2 - (T - t))\mathcal{E}^{(t)}(\varphi)]$ is constant by Theorem 2.12.

Next, we apply the Itô formula (see Corollary 3.17) to $f(x, y, t) = (x^2 - (T - t))y$ with $x = \widehat{B}_t$ and $y = \mathcal{E}^{(t)}(\varphi)$ to obtain

$$\begin{aligned}df(\widehat{B}_t, \mathcal{E}^{(t)}(\varphi), t) &= \frac{\partial f}{\partial x}(\widehat{B}_t, \mathcal{E}^{(t)}(\varphi), t) d\widehat{B}_t + \frac{\partial f}{\partial y}(\widehat{B}_t, \mathcal{E}^{(t)}(\varphi), t) d\mathcal{E}^{(t)}(\varphi) \\ &\quad + \frac{\partial f}{\partial t}(\widehat{B}_t, \mathcal{E}^{(t)}(\varphi), t) dt - \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(\widehat{B}_t, \mathcal{E}^{(t)}(\varphi), t) (d\widehat{B}_t)^2 \quad (4.25) \\ &\quad - \frac{\partial^2 f}{\partial x \partial y}(\widehat{B}_t, \mathcal{E}^{(t)}(\varphi), t) (d\widehat{B}_t) (d\mathcal{E}^{(t)}(\varphi))\end{aligned}$$

Since partial derivatives of $f(x, y, t)$ are given by

$$\frac{\partial f}{\partial x} = 2xy, \quad \frac{\partial f}{\partial y} = x^2 - (T - t), \quad \frac{\partial f}{\partial t} = y, \quad \frac{\partial^2 f}{\partial x^2} = 2y, \quad \frac{\partial^2 f}{\partial x \partial y} = 2x,$$

and the stochastic differentials in Equation (4.25) are given by

$$d\widehat{B}(t) = -dB_t - \varphi(t) dt, \quad d\mathcal{E}^{(t)}(\varphi) = \varphi(t)\mathcal{E}^{(t)}(\varphi) dB_t,$$

we obtain

$$\begin{aligned}&d \left((\widehat{B}_t^2 - (T - t)) \mathcal{E}^{(t)}(\varphi) \right) \\ &= \left(\mathcal{E}^{(t)}(\varphi) (\widehat{B}_t^2 \varphi(t) - 2\widehat{B}_t) - (T - t) \varphi(t) \right) dB_t,\end{aligned}$$

Thus we have

$$(\widehat{B}_T^2 - (T - T))\mathcal{E}^{(T)}(\varphi) - (\widehat{B}_t^2 - (T - t))\mathcal{E}^{(t)}(\varphi) \quad (4.26)$$

$$= \int_t^T \left(\mathcal{E}^{(t)}(\varphi)(\widehat{B}_t^2 \varphi(t) - 2\widehat{B}_t) - (T - t)\varphi(t) \right) dB_t, \quad (4.27)$$

Notice that $\widehat{B}_T^2 - (T - T) = 0$, thus we have

$$(\widehat{B}_t^2 - (T - t))\mathcal{E}^{(t)}(\varphi) \quad (4.28)$$

$$= - \int_t^T \left(\mathcal{E}^{(t)}(\varphi)(\widehat{B}_t^2 \varphi(t) - 2\widehat{B}_t) - (T - t)\varphi(t) \right) dB_t, \quad (4.29)$$

Notice that the integrand above,

$$\overline{\mathcal{E}}_\varphi^{(s)}(\widehat{B}_s^2 \varphi(s) - 2\widehat{B}_s) - (T - s)\varphi(s)$$

is instantly independent with respect to $\{\mathcal{F}_s\}$. Therefore, by Theorem 4.2, $(\widehat{B}_t^2 - (T - t))\mathcal{E}^{(t)}(\varphi)$, as an integral of an instantly independent process, is a near-martingale with respect to \mathcal{F}_t . And thus by Theorem 2.12, $E[(\widehat{B}^2(t) - (T - t))\mathcal{E}^{(t)}(\varphi)]$ is constant. \square

Theorem 4.15. *Suppose that $\{B_t\}$ is a Brownian motion in the probability space $(\Omega, \mathcal{F}_T, P)$, Q is a measure given by Equation (4.19) and \widetilde{B} be given by Equation (4.18). Then the quadratic variation of \widetilde{B} on the interval $[0, t]$ is equal to t .*

Proof. We know that under measure P , the process $\{B^{(t)}\}$ defined by $B^{(t)} = B_T - B_{T-t}$ is a Brownian motion relative to filtration $\{\overline{\mathcal{G}}^{(t)}\}$ (see Proposition 3.2.) By the classical Girsanov theorem (see Theorem 1.29), define

$$\widetilde{B}^{(t)} = B^{(t)} + \int_0^t \varphi(T - s) ds \quad (4.30)$$

Then $\widetilde{B}^{(t)}$ is a Brownian motion under measure \widetilde{Q} given by

$$d\widetilde{Q} = \exp \left\{ - \int_0^T \varphi(T - s) dB^{(s)} - \frac{1}{2} \int_0^T \varphi^2(T - s) ds \right\} dP \quad (4.31)$$

Using Lemma 3.3 (where $t = 0$), we have

$$\begin{aligned} d\tilde{Q} &= \exp\left\{-\int_0^T \varphi(s) dB(s) - \frac{1}{2} \int_0^T \varphi^2(s) ds\right\} dP \\ &= dQ. \end{aligned}$$

This means that $\tilde{Q} = Q$, i.e. $\{\tilde{B}^{(t)}\}$ is a Brownian motion under Q . Therefore the Q -quadratic variation of $\{\tilde{B}^{(t)}\}$ on the interval $[T-t, T]$ is equal to t . In other words, for any partition $\Delta_n = \{T-t = t_0 \leq t_1 \leq \dots \leq t_n = T\}$ of the interval $[T-t, T]$, we have that

$$\lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n (\tilde{B}^{(t_i)} - \tilde{B}^{(t_{i-1})})^2 = t,$$

where the limit is taken in probability under measure Q . Changing the variables $\bar{t}_i = T - t_{n-i}$, $i = 0, 1, \dots, n$, yields a partition $\bar{\Delta}_n$ of the interval $[0, t]$ and (still under Q),

$$\lim_{\|\bar{\Delta}_n\| \rightarrow 0} \sum_{i=1}^n (\tilde{B}^{(T-\bar{t}_i)} - \tilde{B}^{(T-\bar{t}_{i-1})})^2 = t. \quad (4.32)$$

Now notice that by Equation (4.30), the definition of $B^{(s)}$, we have

$$\begin{aligned} \tilde{B}^{(T-\bar{t}_i)} &= B^{(T-\bar{t}_i)} + \int_0^{T-\bar{t}_i} \varphi(T-s) ds \\ &= B_T - B_{\bar{t}_i} + \int_0^{T-\bar{t}_i} \varphi(T-s) ds \end{aligned}$$

by definition of $B^{(t)}$,

$$\begin{aligned} &= B_T - B_{\bar{t}_i} + \int_{\bar{t}_i}^T \varphi(s) ds \\ &= \hat{B}_{\bar{t}_i}. \end{aligned}$$

Hence Equation (4.32) becomes

$$\lim_{\|\bar{\Delta}_n\| \rightarrow 0} \sum_{i=1}^n (\hat{B}_{\bar{t}_i} - \hat{B}_{\bar{t}_{i-1}})^2 = t, \quad (4.33)$$

where the limit is understood as a limit in probability under measure Q and $0 = \bar{t}_0 \leq \bar{t}_1 \leq \dots \leq \bar{t}_n = t$. Finally, Equations (4.3) and (4.33) yield

$$\lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n (\tilde{B}_{\bar{t}_i} - \tilde{B}_{\bar{t}_{i-1}})^2 = t,$$

establishing the desired result. Tracing back the argument above, we can choose the points \bar{t}_i arbitrarily, hence Equation (4.17) implies that the Q -quadratic variation of \widehat{B}_t on the interval $[0, t]$ is equal to t . Hence the proof is complete. \square

Now we can answer Question 4.11.

Example 4.16. Let B_t be a Brownian motion on (Ω, \mathcal{F}, P) . Assume

$$\tilde{B}_t = B_t + \int_0^t X_s ds.$$

where X_t is defined in Question 4.11. Then, \tilde{B}_t is a continuous near-martingale on probability space (Ω, \mathcal{F}, Q) , where Q is defined as

$$\begin{aligned} dQ &= \mathcal{E}^{(0)}(X) dP \\ &= \exp\left\{-\int_0^T X_s dB_s - \frac{1}{2} \int_0^T X_s^2 ds\right\} dP. \end{aligned} \tag{4.34}$$

4.5 Girsanov Theorem for Mixture of Adapted and Anticipative Drifts

The main result of this section follows from application of the classic Girsanov Theorem 1.29 as well as Theorems 4.13–4.15 that constitute an anticipative version of the Girsanov theorem. The improvement of Theorems 4.17–4.20 over Theorems 4.13–4.15 lays in the fact that we allow for the translations of Brownian motion that can be decomposed into a sum of processes that are either adapted to $\{\mathcal{F}_t\}$ or adapted to $\{\mathcal{G}^{(t)}\}$. In this setting, we find that the Girsanov type results have the exact same form as Theorem 1.29.

Theorem 4.17. *Suppose that $\{B_t\}$ is a Brownian Motion and $\{\mathcal{F}_t\}$ is its natural filtration on probability space (Ω, \mathcal{F}, P) . Let f_t and g_t be continuous square-integrable*

stochastic processes such that f_t is adapted to $\{\mathcal{F}_t\}$ and g_t is adapted to $\{\mathcal{G}^{(t)}\}$ s.t. $E[\mathcal{E}^{(t)}(g)] < \infty$ and $E[\mathcal{E}_t(f)] < \infty$ for all $t > 0$., i.e. the backward Brownian filtration. Let

$$\tilde{B}_t = B_t + \int_0^t (f_s + g_s) ds.$$

Then \tilde{B}_t is a near-martingale with respect to (Ω, \mathcal{F}, Q) , where

$$dQ = \exp \left\{ - \int_0^T (f_t + g_t) dB_t - \frac{1}{2} \int_0^T (f_t + g_t)^2 dt \right\} dP. \quad (4.35)$$

Remark 4.18. This theorem can be proved with methods similar to the ones used in 4.4, that is by defining the exponential process for a sum of processes f and g adapted to $\{\mathcal{F}_t\}$ and $\{\mathcal{G}^{(t)}\}$ respectively, and using the results of the preceding sections to repeat the calculations done in [12]. However, since the results that are applicable to translations of Brownian motion by $\int_0^t f(s) ds$ and $\int_0^t g(s) ds$ separately already exist, we can apply them to obtain a shorter proof.

Proof. First, let us rewrite \tilde{B}_t as

$$\tilde{B}_t = B_t + \int_0^t f_s ds + \int_0^t g_s ds$$

and define $W_t = B_t + \int_0^t f_s ds$. Thus $\tilde{B}_t = W_t + \int_0^t g_s ds$. Since f_t is adapted, application of the original Girsanov theorem yields that W_t is a Brownian motion with respect to $(\Omega, \mathcal{F}_T, Q_1)$ where

$$dQ_1 = \exp \left\{ - \int_0^T f_t dB_t - \frac{1}{2} \int_0^T f_t^2 dt \right\} dP \quad (4.36)$$

Now, since W_t is a Brownian motion on $(\Omega, \mathcal{F}_T, Q_1)$ and g_t is adapted to the backward filtration $\{\mathcal{G}^{(t)}\}$, we can apply Theorem 4.15. Therefore, $\tilde{B}(t)$ is a near-martingale with respect to $(\Omega, \mathcal{F}_T, Q)$, where

$$dQ = \exp \left\{ - \int_0^T g_t dW_t - \frac{1}{2} \int_0^T g_t^2 dt \right\} dQ_1 \quad (4.37)$$

with dQ_1 given by Equation (4.36).

It remains to show that the measure Q in Equation (4.37) coincides with the measure Q in Equation (4.35). To this end, we put together the identity $dW_t = dB_t + f_t dt$, Equation (4.36) and Equation (4.37) obtain

$$\begin{aligned}
dQ &= \exp\left\{-\int_0^T g_t dW_t - \frac{1}{2} \int_0^T g_t^2 dt\right\} dQ_1 \\
&= \exp\left\{-\int_0^T g_t dW_t - \frac{1}{2} \int_0^T g_t^2 dt - \int_0^T f_t dB_t - \frac{1}{2} \int_0^T f_t^2 dt\right\} dP \\
&= \exp\left\{-\int_0^T g_t dB_t - \int_0^T g_t f_t dt - \frac{1}{2} \int_0^T g_t^2 dt - \int_0^T f_t dB_t - \frac{1}{2} \int_0^T f_t^2 dt\right\} dP \\
&= \exp\left\{-\int_0^T (g_t + f_t) dB_t - \frac{1}{2} \int_0^T (g_t + f_t)^2 dt\right\} dP.
\end{aligned}$$

Thus the theorem holds. □

Next we state generalization of Theorem 4.14.

Theorem 4.19. *Suppose that assumptions of Theorem 4.17 hold. Let*

$$\widehat{B}_t = B_T - B_t + \int_t^T (f_s + g_s) ds.$$

Then $\widehat{B}_t^2 - (T - t)$ is a continuous Q -near-martingale.

Finally, we give the generalization of Theorem 4.15.

Theorem 4.20. *Suppose that the assumptions of Theorem 4.17 hold. Then the Q -quadratic variation of \widetilde{B} on the interval $[0, t]$ is equal to t .*

Note that the proofs of Theorems 4.19 and 4.20 follow the same reasoning as the proof of Theorem 4.17, that is one first applies the adapted version of the Girsanov theorem (see Theorem 1.29) and then applies one of Theorems 4.14 or 4.15. We omit these proofs for the sake of brevity.

Remark 4.21. Using the relationship between probability measures Q and Q_1 given by Equation (4.37) from the proof of Theorem 4.17 we can deduce an interesting

stochastic differential equation. To this end we will follow the lines of Example 3.11.

From Equation (4.37) we have

$$dQ = \exp\left\{-\int_0^T g_t dW_t - \frac{1}{2} \int_0^T g_t^2 dt\right\} dQ_1.$$

Let us define

$$\theta^{(t)}(g) = \exp\left\{-\int_t^T g_s dW_s - \frac{1}{2} \int_t^T g_s^2 ds\right\},$$

Clearly, according to Example 3.11, $\theta^{(t)}(g)$ is a backward exponential process for the backward-adapted stochastic process g_t in the space $(\Omega, \mathcal{F}_T, Q_1)$. Thus we have the following SDE

$$\begin{aligned} d\theta^{(t)}(g) &= g_t \theta^{(t)}(g) dW_t \\ &= g_t \theta^{(t)}(g) (dB_t + f_t dt) \\ &= g_t \theta^{(t)}(g) dB_t + f_t g_t \theta^{(t)}(g) dt. \end{aligned}$$

The above equation may give some insight into Itô formulas for processes that are adapted to neither $\{\mathcal{F}_t\}$ nor $\{\mathcal{G}^{(t)}\}$ as the last term in the above equation is a stochastic process of the form

$$X_t = \int_0^t f_s \varphi_s ds,$$

with f and φ being adapted to $\{\mathcal{F}_t\}$ and $\{\mathcal{G}^{(t)}\}$ respectively.

We conclude this section with two examples. The first one is a special interesting case.

Example 4.22. Let

$$X_t = B_t + \int_0^t B_1 dB_s,$$

where B_s is a Brownian motion on the probability space $(\Omega, \mathcal{F}_T, P)$. Define the equivalent probability measure Q by

$$dQ = \exp\left\{-\int_0^T B_1 dB_t - \frac{1}{2} \int_0^T B_1^2 dt\right\}.$$

Using Theorems 4.17–4.20, we conclude that X_t is a near-martingale in the probability space $(\Omega, \mathcal{F}_T, Q)$, its quadratic variation on the interval $[0, t]$ is equal to t and if

$$\tilde{X}_t = X_T - X_t = B_T - B_t + \int_t^T B_1 dB_s,$$

then $\tilde{X}_t^2 - (T - t)$ is a near martingale on $(\Omega, \mathcal{F}_T, Q)$.

Note that the conclusions of this example cannot be obtained with the classic Girsanov Theorem 1.29 as the B_1 is not adapted to $\{\mathcal{F}_t\}$. It is also not possible to approach this example with results of [12] or of Section 3.4 of the present paper because B_1 is not adapted to $\{\mathcal{G}^{(t)}\}$. However, we can rewrite B_1 as

$$B_1 = (B_1 - B_t) + B_t,$$

where $(B_1 - B_t)$ is adapted to $\{\mathcal{G}^{(t)}\}$ and B_t is adapted to $\{\mathcal{F}_t\}$. In the view of the above equation, Theorems 4.17–4.20 are applicable.

The second one is the a general case.

Example 4.23. Let $B(t)$ be a Brownian motion on (Ω, \mathcal{F}, P) with forward and backward filtration $\{\mathcal{F}_t\}$ and $\{\mathcal{G}_t\}$ respectively. Assume

$$X = \int_0^T h(B(t)) dB(t) + \int_0^T \xi(B(t)) dt,$$

where $h(x), \xi(x)$ are square integrable functions on \mathbb{R} , then the drifted stochastic process $\tilde{B}(t)$,

$$\tilde{B}(t) = B(t) + \int_0^t X ds, \tag{4.38}$$

is a continuous near-martingale with quadratic variation $[\tilde{B}]_t = t$, w.r.t. (Ω, \mathcal{F}, Q)

where

$$dQ = \exp\left\{-\int_0^T X dB(t) - \frac{1}{2}\int_0^T X^2 dt\right\} dP \tag{4.39}$$

Proof. To prove the statement, we decompose the random variable X as sum of two processes.

$$X = \int_0^t h(B(t)) dB(t) + \int_0^t \xi(B(t)) dt + \int_t^T h(B(t)) dB(t) + \int_t^T \xi(B(t)) dt. \quad (4.40)$$

Define

$$f(t) = \int_0^t h(B(t)) dB(t) + \int_0^t \xi(B(t)) dt$$
$$g(t) = \int_t^T h(B(t)) dB(t) + \int_t^T \xi(B(t)) dt$$

Then we have $f(t)$ is adapted to $\{\mathcal{F}_t\}$, $g(t)$ is adapted to $\{\mathcal{G}_t\}$ and that $X = f(t) + g(t)$. Thus by Theorem 4.17 to Theorem 4.20, We get the conclusion. \square

Chapter 5

An Application

5.1 Black–Scholes Equation in the Backward Case

In this section we discuss a simple scenario of Black–Scholes model in the backward-adapted setting. The outline of this approach comes from [4, Chapter 7]. In our setting, the market is composed of two assets. The first asset is a risk-free bond whose price D_t is driven by a deterministic differential equation

$$dD_t = rD_t dt,$$

where r is the risk-free interest rate. The second asset is a stock (or some security) S_t , whose price is dependent on the information right after time t and driven by a stochastic differential equation

$$dS_t = S_t\alpha_t dt + S_t\sigma_t dB_t,$$

where α_t and σ_t are both adapted to $\{\mathcal{G}^{(t)}\}$. This can be viewed as a special case of “insider information”, where the “insider” uses only the knowledge unavailable to the rest of the market as the processes α_t and σ_t are completely out of the scope of the natural forward Brownian filtration, but instead are adapted to the natural backward Brownian filtration. The backward Brownian filtration describes exactly the future information generated by the driving Brownian process that is independent of the current or past state of the market.

We assume there is a contingent claim $\Phi(S_T)$, which is tradable on the market and whose price process is given by

$$\Pi_t = F(t, S_t)$$

for some smooth function $F(x, y)$. Our goal is to find a function F such that the market is arbitrage-free. Using Corollary 3.17, we have

$$\begin{aligned}
d\Pi_t &= dF(t, S_t) \\
&= \frac{\partial F}{\partial y}(t, S_t) dS_t - \frac{1}{2} \frac{\partial^2 F}{\partial y^2}(t, S_t) (dS_t)^2 + \frac{\partial F}{\partial x}(t, S_t) dt \\
&= \frac{\partial F}{\partial y}(t, S_t) (S_t \alpha_t dt + S_t \sigma_t dB_t) - \frac{1}{2} \frac{\partial^2 F}{\partial y^2}(t, S_t) (S_t^2 \sigma_t^2 dt) + \frac{\partial F}{\partial x}(t, S_t) dt \\
&= \left(\frac{\partial F}{\partial y}(t, S_t) S_t \alpha_t - \frac{1}{2} \frac{\partial^2 F}{\partial y^2}(t, S_t) S_t^2 \sigma_t^2 + \frac{\partial F}{\partial x}(t, S_t) \right) dt \\
&\quad + \left(\frac{\partial F}{\partial y}(t, S_t) S_t \sigma_t \right) dB_t \\
&= \alpha_t^\Pi \Pi_t dt + \sigma_t^\Pi \Pi_t dB_t,
\end{aligned}$$

where

$$\begin{aligned}
\alpha_t^\Pi &= \frac{\frac{\partial F}{\partial y}(t, S_t) S_t \alpha_t - \frac{1}{2} \frac{\partial^2 F}{\partial y^2}(t, S_t) S_t^2 \sigma_t^2 + \frac{\partial F}{\partial x}(t, S_t)}{F(t, S_t)}, \\
\sigma_t^\Pi &= \frac{\frac{\partial F}{\partial y}(t, S_t) S_t \sigma_t}{F(t, S_t)}.
\end{aligned} \tag{5.1}$$

We now form a relative portfolio consisting of the stock and the contingent claim. We denote by u_t^S the percentage of stock in the portfolio at time t and by u_t^Π the percentage of the contingent claim in our portfolio. Thus the portfolio is given by (u_t^S, u_t^Π) with the restriction that $u_t^S + u_t^\Pi = 1$ for all t . Assuming that our portfolio is self-financing and without consumption or transaction costs, we obtain the following SDE for the dynamics of the value of the portfolio V

$$\begin{aligned}
dV_t &= V_t u_t^S \frac{dS_t}{S_t} + V_t u_t^\Pi \frac{d\Pi_t}{\Pi_t} \\
&= V_t \left(u_t^S [\alpha_t dt + \sigma_t dB_t] + u_t^\Pi [\alpha_t^\Pi dt + \sigma_t^\Pi dB_t] \right) \\
&= V_t \left([u_t^S \alpha_t + u_t^\Pi \alpha_t^\Pi] dt + [u_t^S \sigma_t + u_t^\Pi \sigma_t^\Pi] dB_t \right).
\end{aligned}$$

In order to obtain a risk-free portfolio, we need to ensure that there is no stochastic part in the equation above. Moreover, in order to ensure that the new financial

instrument does not introduce the arbitrage to the market, the interest rate of the value process of the risk-free portfolio needs to coincide with the interest rate of the risk-free bond, namely r . That is, together with the structural constraints on the portfolio, we have

$$u_t^S + u_t^\Pi = 1 \quad (5.2)$$

$$u_t^S \alpha_t + u_t^\Pi \alpha_t^\Pi = r \quad (5.3)$$

$$u_t^S \sigma_t + u_t^\Pi \sigma_t^\Pi = 0. \quad (5.4)$$

Equations (5.2) and (5.4) yield

$$u_t^S = -\frac{\sigma_t^\Pi}{\sigma_t - \sigma_t^\Pi}, \quad u_t^\Pi = \frac{\sigma_t}{\sigma_t - \sigma_t^\Pi}. \quad (5.5)$$

Putting together Equations (5.5) and (5.1), we obtain

$$u_t^S = \frac{\frac{\partial F}{\partial y}(t, S_t) S_t}{\frac{\partial F}{\partial y}(t, S_t) S_t - F(t, S_t)}, \quad u_t^\Pi = \frac{F(t, S_t)}{F(t, S_t) - \frac{\partial F}{\partial y}(t, S_t) S_t} \quad (5.6)$$

Now, together with Equation (5.3) and the terminal condition that comes from the form of the contingent claim Π , Equation (5.6) yields

$$\begin{cases} \frac{\partial F}{\partial x}(t, S_t) + \frac{\partial F}{\partial y}(t, S_t) r S_t - \frac{1}{2} \frac{\partial^2 F}{\partial y^2}(t, S_t) S_t^2 \sigma_t^2 - F(t, S_t) r = 0 \\ F(T, s) = \Phi(s). \end{cases}$$

Observe that unlike with the classic Black–Scholes formula, in the above PDE we have a minus in front of the term with $\frac{\partial^2 F}{\partial y^2}$. This change of sign enters through the Itô formula for the backward-adapted processes. Intuitively this can be explained by the fact that the difference between the classic Black–Scholes model and our example is that of a different point of view. That is the former model looks forward with the information on the past and the latter looks backward with the information from the future. Thus the influence of the volatility (σ_t) will have opposite effects in the two models.

Of course the above example is rather simple and not realistic on its own, however one might use it together with the classic Black–Scholes model to study the influence of the insider information on the market.

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Vita

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