1984

The Velocity and Shape of Gas Slugs Rising in Vertical Tubes and Rectangular Slots.

Henry Valma Nickens
Louisiana State University and Agricultural & Mechanical College

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THE VELOCITY AND SHAPE OF GAS SLUGS RISING IN VERTICAL TUBES AND RECTANGULAR SLOTS

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THE VELOCITY AND SHAPE OF GAS SLUGS RISING IN VERTICAL TUBES AND RECTANGULAR SLOTS

A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy in

The Department of Mechanical Engineering

by

Henry Valma Nickens
B.S., Louisiana State University, 1969
M.S., University of Southern Mississippi, 1972
M.S., Carnegie Institute of Technology, 1976
December 1984
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Bubble Shape Z(x,0)
7.62 cm Square Channel
Air/Water

Bubble Shape Z(x,0)
7.62 cm x 5.08 cm Channel
Air/Water

Bubble Shape Z(0,y)
7.62 cm x 5.08 cm Channel
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Air Slug in 7.62 cm x 1.27 cm Channel
View from 7.62 cm Side

Air Slug in 7.62 cm x 1.27 cm Channel
View from 1.27 cm Side

Flow in 3D Channel - 2L = 7.62 cm
Measured Slip Velocity for Air/Water

Flow in 3D Channel - 2L = 7.62 cm
Measured Froude Number for Air/Water

Coordinate Systems Defined for
Derivation of Equation A-13

Laminar Liquid Film Flowing Down
Inner Surface of a Vertical Tube of
Inner Radius R
NOMENCLATURE

English Upper Case

A^1, A^2  arbitrary constants
A^i j  coefficients of series expansion
B^1, B^2  arbitrary constants
B^i j  Chapter 2 - coefficients of series expansion
Chapter 3 - function defined by Equation 3-2
C^1, C^2  arbitrary constants
C^i j  coefficients of series expansion
C^b^a  combinatorial function = \frac{b!}{a!(b-a)!}
D  tube inner diameter
de f f  effective diameter for viscous liquid defined by Equation 3-40
D^e q  equivalent equiperipheral diameter of a channel as defined in Section 2.4
Eo  Eotvos Number defined by equation 2-24
Eo eff  effective Eotvos Number for viscous liquid defined by Equation 3-42
F  arbitrary function
Fr  Froude Number defined by equation 2-23
Fr eff  effective Froude Number for viscous liquid defined by Equation 3-41
G  arbitrary function
H  level function for bubble surface, H = 0
J^1  first order Bessel Function
K  function of radii of curvature at arbitrary point on bubble surface, defined by Equation 2-15
L  half-length of rectangular slot
N parameter denoting the highest order term included in the truncated series for stream function (Chapter 3) or velocity potential (Chapter 4)

ND parameter denoting the highest order terms included in the Taylor Series approximation to the bubble surface

NF parameter defined in section 5.C.

$N_{LB}$ liquid viscosity number defined by Equation 3-38

$N_P$ liquid property number defined by Equation 3-45

P pressure

R inner radius of tube

$R_c$ radius of curvature at bubble nose

$R_i$ local radius of curvature on bubble surface in direction of principal axis $i$.

$R_{\text{eff}}$ effective radius for viscous liquid defined in Equation 3-35

$R_N$ normalized effective radius defined by Equation 3-46

$T_1, T_2$ functions which describe bubble surface defined in Equations 2-15 and 2-18

$\bar{U}$ average liquid velocity upstream of bubble

$U_c$ liquid velocity of centerline of channel upstream of bubble

$U_s$ velocity of bubble relative to liquid

$V$ magnitude of liquid velocity vector

W half-width of rectangular slot

X arbitrary function of $x$

Y arbitrary function of $y$

Z($r$) bubble surface in tube

Z($x,y$) bubble surface in rectangular slot

**English Lower Case**

$a_i$ arbitrary constants

d$_i$ coefficients of Bessel series expansion defined by Equation 3-1
function representing bubble surface

parameter defined in Section 5.C.

gravitational constant

arbitrary function

ith zero of \( J_1 \) Bessel function

position coordinate in radial direction for tube

time

liquid velocity in length direction for rectangular slot
or axial direction for tube

liquid velocity in width direction for rectangular slot or
radial direction for tube

liquid velocity in axial direction for rectangular slot

position coordinate in length direction for rectangular slot

position coordinate in width direction for rectangular slot

position coordinate in axial direction for rectangular slot
and tube

Greek Lower Case

constant determined by separation of variables - Chapter 2

constant determined by separation of variables - Chapter 2

constant determined by separation of variables - Chapter 2

laminar film thickness in tube flow

constants in Bessel Series Expansion

normalized film thickness for tube flow

liquid parameter defined as \( \rho g/\mu \)

position coordinate in angular direction for tube

constants of integration defined in Equations E-6, E-7

nondimensional parameter defined in Equation 3-14

viscosity
\( \nu \) parameter defining an effective radius for viscous liquids in Equation 3-35

\( \rho \) density

\( \sigma \) surface tension

\( \tau \) shear stress

\( \phi \) velocity potential function

\( \phi_0 \) arbitrary function

\( \Psi \) Stokes Stream Function

**Greek Upper Case**

\( \Delta \) denotes change in value of a parameter

\( \Lambda \) parameter defined in Equation 2-15

\( \Gamma_1, \Gamma_2 \) parameters defined by Equations A-4, A-5

**Subscripts**

\( B \) denotes arbitrary point on duct wall

\( N \) denotes normal component of a vector

\( f \) denotes laminar flow in tube flow

\( g \) denotes gaseous phase

\( i \) denotes x-direction in Cartesian Coordinates

\( j \) denotes y-direction in Cartesian Coordinates

\( k \) denotes z-direction in Cartesian Coordinates

\( l \) denotes liquid phase

\( m \) general indexing parameter

\( n \) general indexing parameter

\( o \) denotes origin at bubble nose

\( p \) denotes arbitrary point in space

\( r \) denotes radial component of vector in Cylindrical Coordinates

\( w \) denotes value at the tube wall

\( z \) denotes axial component of vector in Cylindrical Coordinates
Superscripts

* denotes non-dimensional quantity

' denotes coordinate system with origin at arbitrary point on bubble surface

n general iteration index

Vectors

\( \mathbf{V} \) local velocity in liquid phase

\( \mathbf{X} \) arbitrary vector

\( \mathbf{i} \) unit vector in x-direction in Cartesian Coordinates

\( \mathbf{j} \) unit vector in y-direction in Cartesian Coordinates

\( \mathbf{k} \) unit vector in z-direction in Cartesian Coordinates

\( \mathbf{r} \) arbitrary point in space
The rise of a large gas bubble, or slug, through a closed, vertical, liquid-filled channel of infinite length has been investigated by means of potential flow theory. Three channel geometries are considered: 1) the circular tube, 2) the two-dimensional rectangular channel and 3) the three-dimensional rectangular slot. The effect of interfacial surface tension is explicitly accounted for by application of the Kelvin-LaPlace equation, thus making the bubble shape an integral part of the solution.

For the circular tube of radius R and diameter D, the solution is expressed in terms of the Stokes Stream Function which consists of an infinite Bessel Function series. The resultant equations have been solved numerically for the first six terms in the series.

For negligible surface tension and ideal liquid, the bubble rise velocity is given by \( U_s = 0.352 \sqrt{gD} \) and the radius of curvature at the bubble nose \( R_c = 0.75R \). For air/water and \( D = 2.54 \, \text{cm} \), the inclusion of surface tension gives \( U_s = 0.346 \sqrt{gD} \) and \( R_c / R = 0.71 \) which is consistent with experimental observation.

It is also shown for the tube case that the potential flow solution may be applied with good results to liquids of moderate viscosity if an effective tube radius \( R_{\text{eff}} \equiv R - \nu \delta \) is used, where \( \nu \) is a function of the liquid properties and \( \delta \) is the laminar wall film thickness.

For rectangular channels, the solution is expressed in terms of the velocity potential function which consists of an infinite series of trigonometric functions.
In the case of a two-dimensional rectangular channel, of finite length 2L and infinite width, the solution does not converge as rapidly as for the tube case. Solutions were obtained for the first six terms in the velocity potential expansion without obtaining convergence, although the solution appears to be converging to $Fr = 0.289$ and $\frac{R_c}{L} = 0.65$ for negligible surface tension. This value of $Fr$ is consistent with other theoretical analyses but the value of $\frac{R_c}{L}$ is significantly different and is in closer agreement with experimental data. Measured data for slots of very large aspect ratio, which are assumed to approximate the physically impossible two-dimensional channel, give $Fr = 0.31$ and $\frac{R_c}{L} = 0.62$.

For a three-dimensional rectangular slot of length 2L and width 2W, the solution converges even more slowly than the two-dimensional channel case, especially at large aspect ratios and significant surface tension. For negligible surface tension, the theoretical solution compares well with measured data for aspect ratios up to 6. For a square channel and negligible surface tension, a value of $Fr = 0.333$ is calculated compared to the measured value of 0.330.

Including surface tension in the three-dimensional solution decreases the maximum aspect ratio at which the solution gives good agreement with data, for the same number of terms in the series expansion. For a 7.62 cm square channel and an air/water system, a value of $Fr = 0.330$ is calculated compared to the measured value of 0.330.
1.1. Introduction

The study of the rise of gas bubbles in liquids is of importance in many engineering fields; for example, nuclear engineering (loss of coolant accidents), chemical and mechanical engineering (heat exchangers) and petroleum engineering (artificial gas lift).

Considerations of bubble rise may be classified into two broad groups: 1) bubbles rising in infinite media and 2) bubbles rising in restricted media. An infinite medium is here defined as a liquid which is sufficiently large in volume relative to the bubble such that the effect of the liquid containing wall on the flow field around the bubble is negligible. Conversely, a restricted medium is defined as a liquid through which a bubble is rising such that the containing walls of the liquid have a significant effect on the flow field around the bubble.

If gas bubbles of varying volumes are allowed to rise in a vertical restricted medium of a given size, it is found that a limiting bubble volume is attained such that the rise velocity of bubbles of volumes greater than the limiting value is constant (O'Brien and Gosline, 1935). This limiting condition where the rise velocity of the gas bubble remains constant as the gas volume increases is called "slug flow" and can occur only in restricted media. In unrestricted media, the rise velocity increases without bound as the gas bubble volume increases.
This dissertation will consider only the rise of gas slugs in vertical restricted media. The walls of the liquid volume will be constrained to be either a vertical pipe, a rectangular slot of finite length and infinite width or a rectangular channel of both finite length and width.

Because of the extensive literature on the general subject of gas bubbles in liquid media, it is not possible in this limited space to adequately summarize all research of the past sixty years. The following discussion will therefore concentrate on work considered directly relevant to the subject of this report, slug flow in vertical channels.

1.2. Slug Flow in Vertical Pipes

The most extensive effort for slug flow, both theoretical and experimental, has been directed toward flow in vertical pipes.

Gibson (1913) performed experiments on large air bubbles in water and obtained the empirical correlation

\[ U_s (\text{cm/sec}) = 34.6 - \frac{114.0}{D^2 + 3.012} \]  

\[ (1-1) \]

O'Brien and Gosline (1935) performed experiments on air bubbles in water and two petroleum oils for three sizes of vertical tubing (3.00, 5.69, and 15.24 cm). They found that the limiting velocity was higher than Gibson's equation 1-1 for large diameter tubes and lower for small diameter tubes.

Dumitrescu (1943) performed the first theoretical calculation of a gas slug in a vertical tube. From dimensional analysis for an ideal liquid
and no surface tension, he showed that the Froude number for slug flow in a vertical pipe filled with a stagnant liquid is constant. That is,

\[ Fr \equiv \frac{U_s}{\sqrt{gD}} = \text{constant.} \quad (1-2) \]

He approached the theoretical solution to the problem by solving for the potential flow of an ideal liquid around a stationary surface of revolution of radius of curvature \( R_c \) in a pipe of diameter \( D \). The velocity potential in the liquid was shown to be

\[ \phi(r,z) = \sum_{n=1}^{\infty} j_n(\beta_n r/R) e^{-\beta_n z/R} \quad (1-3) \]

where \( R = D/2 \) and \( r,z \) are the radial and axial positions relative to the bubble nose.

In addition, under the assumption of potential flow, Bernoulli's Equation at the bubble surface is written, in terms of the velocity potential \( \phi \), as

\[ \left( \frac{\partial \phi}{\partial r} \right)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2 + 2gz = 0 \quad (1-4) \]

where the pressure terms are eliminated by the assumption of equal pressure everywhere in the gas bubble and thus on the surface also.

Dumitrescu expanded the Bessel and exponential functions of 1-3 in their respective series forms and substituted into 1-4. Representing the bubble surface by the expansion
\[ Z(r) = - \sum_{n=1}^{\infty} a_n r^{2n} \]

he wrote 1-4 as a power series in the radius \( r \). Equating the coefficients of equal powers of \( r \) then yields algebraic equations for the \( \beta_n \)'s in terms of the coefficient \( a_1 \). Since

\[ R_c = - \frac{1}{\left( \frac{\partial^2 Z}{\partial r^2} \right)} \]

then

\[ a_1 = - \frac{1}{2R_c} \]

and the solution depends only on the bubble shape at the nose.

In order to complete the solution Dumitrescu assumed a hemispherical shape for the bubble nose and calculated the radius required so that the liquid velocity downstream of the nose, as calculated from a one-dimensional mass balance on the liquid, merged asymptotically with the velocity calculated from 1-4.

Using this procedure, Dumitrescu determined the first three terms in 1-3 and calculated

\[ \frac{R_c}{R} = 0.75 \quad \text{Fr} = 0.351 \]  \hspace{1cm} (1-5)

Experiments performed for air in water-filled tubes of diameter 0.99, 2.00, 3.76 and 7.00 cm yielded the results of Table 1-1. It is clear that the simple assumption of potential flow gives very good results for air/water in vertical pipes of diameter \( D > 3.0 \) cm. For diameters less
than 3.0 cm, surface tension effects become more significant as the radius of curvature of the bubble also decreases.

Davies and Taylor (1950) approached the same problem by considering only the first term in the series expansion 1-3 and requiring that Bernoulli's Equation 1-4 be satisfied at the arbitrary radius \( r = R/2 \). This method, much simpler than that of Dumitrescu, yields \( Fr = 0.328 \).

A drawback to this method is that a different choice for the arbitrary radius would yield a different solution. Dumitrescu's procedure is superior in that it is not dependent upon an arbitrary radius, but relates the solution to the shape of the bubble as defined by the radius of curvature at the bubble nose.

Birkhoff (1955) reported velocity and shape measurements for an air slug in a tube of diameter 10.32 cm. He reported

\[
\frac{U}{\sqrt{gD}} = 0.35 \quad \frac{R_c}{R} = 0.70
\]

The radius of curvature was determined by performing a least squares fit to four measured points on the bubble surface, 1.524 cm or less below the vertex. The actual \( R_c \) at the nose would therefore be slightly larger than reported.

Laird and Chisholm (1956) measured the velocities of bubbles of varying length in an open tube of 5.08 cm diameter. The length/diameter (L/D) ratios ranged from 1 to 23.5. Their data show an increasing slip velocity as the length increases. The measured \( Fr \) ranged from 0.337 for L/D = 1 to 0.380 for L/D = 23.5.
Harmathy (1960) introduced the name Eotvos Number for the dimensionless quantity

\[ Eo = \frac{(\rho_2 - \rho_g)gD}{\sigma} \]

He plotted experimentally determined Fr as a function of Eo for data of previous investigations (Hattori (1935), O'Brien and Gosline (1935), Dumitrescu (1943), Davies and Taylor (1950), Laird and Chisholm (1956)) and for data taken by himself. The resulting graph, reproduced in Figure 1-1, shows a definite variation of Fr with Eo for liquids of low viscosity. For the two liquids of high viscosity, glycerine and liquid paraffin, a different curve results, but appears to approach the low viscosity curve for very high (>100) or very low (~4) values of Eo.

This suggests that for negligible viscosity, Fr depends solely upon the Eotvos Number.

Bretherton (1961) considered the case of gas slugs in tubes of very small diameter where the surface tension forces dominate the inertial forces \((\mu U/a \ll 1)\). Under these conditions, the bubble shape is determined by the balance of gravitational and surface tension forces. Bretherton found that for \(\rho_2g\sqrt{r^2/\sigma} > 0.842\) there was no vertical tangent plane at any point on the bubble profile whereas for \(\rho_2g\sqrt{r^2/\sigma} < 0.842\) there was such a vertical tangent plane. Thus for \(\rho_2g\sqrt{r^2/\sigma} < 0.842\), the bubble surface tended to curve back upon itself. Bretherton hypothesized that the limiting value of Eo, below which a gas slug could not rise in a vertical tube, was
\[(Eo)_{\text{min}} = 4(.842) = 3.37.\]

This hypothesis is supported by Harmathy's (1960) data (Figure 1-1) which suggests a limiting Eo value of \(3 < (Eo)_{\text{min}} < 6.\)

White and Beardmore (1962) performed experiments of air slugs rising in liquids of varying properties. For negligible viscosity (Figure 1-2), they found good agreement with Harmathy's (1960) data (Figure 1-1). For liquids of significant viscosity (Figure 3-18), the curves are separated but approach the zero viscosity curve for very large Eo. For the case of no slug rise, viscous forces are not involved and all curves pass through the point \(Eo = (Eo)_{\text{min}}.\)

There is some degree of scatter about the zero velocity points on both Figures 1-1 and 1-2. White and Beardmore hypothesized that the scatter is due to incomplete wetting of the tube surface due to impurities or surface roughness.

Based on their work, White and Beardmore proposed a limiting value of the slug rise velocity as \(Fr = 0.345\) for negligible viscosity \((\rho L^2gD^3/\mu^2 > 3 \times 10^5)\) and negligible surface tension \((Eo > 70).\)

Wallis (1962) summarized previous work on gas slugs in vertical tubes for stagnant flow. He presented the correlation

\[
Fr = 0.35 \left[ 1 - e^{\frac{3.37-Eo}{10}} \right]
\]

for negligible viscosity, defined by
Nicklin, et. al. (1962) studied the rise of long gas bubbles in vertical tubes. They found that the slug rise velocity is given by

\[ U_s = 1.2 \bar{U} + 0.35\sqrt{gD} \]

where \( \bar{U} \) is the average liquid velocity in the tube.

Their work showed that the increase of slug rise velocity with slug length is due to the expansion of the rising bubble. If the bubble is rising in an open tube, the bubble expansion produces a net liquid velocity above the bubble and the slug is effectively rising in a non-stagnant liquid column. If the tube is closed, bubble expansion displaces the liquid below the bubble and the liquid above the slug remains stagnant.

Brown (1965) performed experiments for air bubbles in viscous liquids and found that the effects of viscosity could be approximated by replacing the tube radius by an effective radius

\[ R_{\text{eff}} = R - \delta \]

where \( \delta \) is the liquid film thickness on the tube wall downstream of the bubble nose. Thus, Brown proposed
\[ U_s = 0.350 \sqrt{g(D-2\delta)} . \quad (1-6) \]

Assuming laminar flow in the film, Brown derived an expression for the film thickness as a function of the slip velocity \( U_s \). Eliminating \( \delta \) by using 1-6, he obtained

\[ U_s = 0.350 \sqrt{gD} \left[ 1 - \frac{2(\sqrt{1 + \frac{N_{LB}}{N_{LB}}} - 1)}{N_{LB}} \right]^{1/2} \quad (1-7) \]

where

\[ N_{LB} = \left( \frac{116 R^3 \rho^2 g \mu^{1/3}}{\mu^2} \right) . \]

Comparison of 1-7 to experimental data gave good results for

\[ N_{LB} > 60 \]
\[ Eo > \frac{20}{(1 - \delta/R)^2} . \]

Stewart and Davidson (1967) solved for the bubble shape near the nose by solving the streamline equation

\[ \nabla^2 \psi = 0 \]

via finite difference methods for negligible surface tension under the assumed known condition that
They calculated the radius of curvature $\frac{R_c}{R}$ as

$$\frac{R_c}{R} = 0.69$$

at the bubble nose. This value compared well with measured values of 0.71 (Dumitrescu (1943)), 0.75 (Brown (1965), and their own data of 0.70 for air/toluene.

Zukoski (1966) investigated the effects of viscosity and surface tension in closed tubes. By varying the liquids used, he succeeded in producing cases where viscous effects were similar (i.e., similar Reynolds Numbers), but the surface tension effects were significantly different (i.e., different Eotvos Numbers). He concluded from his data that the velocity decrease observed as the tube diameter is decreased is due primarily to the surface tension effect and is only weakly dependent upon viscous effects. Only when liquids of very high viscosity are used does the viscous effect significantly affect the bubble motion.

Collins (1967) represented the surface of a gas bubble as uniform flow past a doublet on the axis of a vertical tube. The strength of the doublet was adjusted as a function of the doublet position on the axis so that the origin remained at the bubble nose in all cases.

From this model, Collins calculated the bubble rise velocity as a function of the radius of curvature of the bubble nose. For slug flow, the model predicts the limiting values
Fr = 0.361 \quad \frac{R_c}{R} = 1.044 .

Comparison to experimental data for various sized bubbles gave good agreement for small bubbles, where the wall effect is negligible. For slugs, however, the maximum radius of curvature observed was $R_c/R = 0.71$, where $Fr = 0.346$, consistent with previous measurements.

For $R_c/R = 0.71$, Collins' model predicts $Fr = 0.344$. However, the model is unable to predict the limiting value adequately.

Collins, et al. (1978) considered the motion of gas slugs rising through a liquid flowing in a vertical tube under both laminar and turbulent flow conditions. For laminar flow, they showed that the Stokes Stream Function is given by

$$
\Psi(r,z) = \frac{1}{2} U_s r^2 - \frac{1}{2} U_c r^2 [1 - \frac{1}{2}(\frac{r}{R})^2]
$$

$$
- (U_s - U_c) \sum_{n=1}^{\infty} \frac{d_n}{k_n} \frac{r k_n}{R} J_1 \left( \frac{r k_n}{R} \right) e^{-\frac{z k_n}{R}}
$$

where $k_n =$ $n^{th}$ zero of $J_1$ Bessel Function

$U_c =$ centerline velocity upstream of bubble.

Two methods of solution were presented. In their first solution method, the Bernoulli Equation (1-4) is expanded in a Taylor series about the bubble nose. Since each coefficient in the expansion must be identically zero, then as a first approximation to the solution, Collins required that
\[
\frac{\partial^2}{\partial r^2} (u^2 + v^2 + 2gz) \equiv 0
\]

where

\[
v = \frac{1}{r} \frac{\partial \psi}{\partial z} \quad \quad u = -\frac{1}{r} \frac{\partial \psi}{\partial r}.
\]

Calculating \(\frac{\partial^2 z}{\partial r^2}\) by differentiating the defining equation for the bubble surface, \(\psi \equiv 0\), Collins showed that

\[
U_s = U_c + \sqrt{gR} \sum_{n=1}^{\infty} d_n k_n \Omega(s)^{1/2}
\]

where

\[
\Omega(s) = \begin{cases} 
2 \cos(\arccos(s)/3)/\sqrt{3}, & s \leq 1 \\
2 \cosh(\arccosh(s)/3)/\sqrt{3}, & s > 1 
\end{cases}
\]

\[
s \equiv \frac{2U_c}{\sqrt{gR}} \left\{ \sum_{n=1}^{\infty} d_n k_n \right\}^{3/2}
\]

\[\sum_{n=1}^{\infty} d_n = 1.\]

Collins considered only the first term in the series expansion of the stream function and obtained, after simplification,
For stagnant flow, $U_c = 0$, then

$$U_s = .361 \sqrt{gD}.$$

The second solution was based on a previous paper (Collins (1967)) discussed above. The bubble surface was modeled as flow around a dipole located on the tube axis. Utilizing Dumitrescu's method of selecting the radius of curvature required to match the solution to the flow in the film downstream of the nose for $U_c = 0$, Collins found that

$$U_s = 1.08 U_c + .347 \sqrt{gD}$$

and for stagnant flow

$$U_s = .347 \sqrt{gD}$$

Collins also found the radius of curvature to the bubble nose to be

$$\frac{R_c}{R} = 1.044 \quad \text{, first solution}$$

$$\frac{R_c}{R} = 0.75 \quad \text{, second solution}.$$

Collins et al (1978) also reported experimental measurement of the radius of curvature for air slugs in water and in a water/glycerol mixture in a tube of 5.14 cm diameter. For stagnant flow $R_c/R$ was found to be .71. As $U_c$ increased, still maintaining laminar flow, $R_c/R$ decreased roughly parallel to the behavior predicted by the second solution. Thus, the second solution correctly predicts the variation of the radius of curvature as $U_c$ increases.

1.3. Slug Flow in Vertical Rectangular Channels

Garabedian (1957) presented a theory for the rise of a gas slug in a rectangular slot of length $2L$ and infinite width. He asserted that the solution consists of a family of bubble shapes, each uniquely determined by the Froude Number $U_s/\sqrt{gL}$ associated with it. In other words, instead of a single unique solution, determined by the slot length, Garabedian asserted that a family of bubble shapes exists, each determined by the slot length and bubble velocity.

He further asserted that the bubble shape and velocity observed by experiment is the most stable of the family of solutions and corresponds to the maximum velocity $U_s$. He estimated the lower limit of the maximum velocity as

$$\frac{U_s}{\sqrt{gL}} \geq 0.3342$$

Birkhoff and Carter (1955, 1957) considered the same problem as Garabedian. They approached the solution by conformally mapping the region of
liquid flow onto a unit semicircle in the complex plane and applying complex potential theory.

Their solution is dependent upon a parameter C having the range \( 0 < C < .5 \). Solving the bubble free boundary condition

\[
|\vec{V}|^2 + 2gZ = 0,
\]

where \( \vec{V} \) is the velocity at the bubble surface, for a range of values for C allowed them to estimate

\[
\frac{U_s}{\sqrt{g} L} = 0.318 \pm 0.007 .
\]

The calculated radius of curvature at the bubble nose was not as consistent giving values ranging from \( R_c/L = 0.260 \) to \( R_c/L = 0.902 \).

They also reported measurements for an air slug in a 10.32 cm x 2.54 cm rectangular channel. Their data give

\[
\frac{U_s}{\sqrt{g} L} = 0.410 \quad \frac{R_c}{L} = 0.52
\]

where \( 2L = 10.32 \) cm.

The radius of curvature was measured by performing a least squares fit to four points on the bubble surface 1.524 cm or less from the nose. Therefore, the actual \( R_c \) at the nose would be slightly larger than the calculated value.
Collins (1965) modelled the two-dimensional slug flow in an infinite slot of length $2L$ as potential flow around an infinitely long circular cylinder positioned in the slot so that the circular surface of the cylinder represents the bubble surface. Thus, the bubble is assumed to have a circular cap or nose. He calculated

\[
\frac{U}{\sqrt{g} L} = 0.326 \quad \text{and} \quad \frac{R_c}{L} = 0.955.
\]

Experiments were performed in channels of width $0.635$ cm and varying length up to $83$ cm. Slugs were observed in channels of length $2L \sim 7.6$ cm with the results

\[
\frac{U}{\sqrt{g} L} = 0.35 \quad \text{and} \quad \frac{R_c}{R} = 0.62.
\]

The measured velocity agrees reasonably with predictions, but the radius of curvature is significantly different.

The predicted velocity is 9 per cent higher than measured. Collins attributed this discrepancy to a residual three-dimensional effect.

Griffith (1964) reported experimental measurements of air slugs in five rectangular channels of varying dimensions as shown in Table 1-2. The liquids used were water, pure glycerin, and a water/glycerin mixture (25% by weight glycerin). The Froude numbers given in Table 1-2 are for the air/water slugs and were estimated from Figure 1 of the original publication.
For one of the channels, the width is small enough so that surface tension was not negligible. For this case (Channel E), the velocity was reduced in addition to the velocity reduction due to the channel dimensions alone. For the remaining four channels, the Froude number is roughly a linear function of the ratio W/L.

Hills (1975) reconsidered the two-dimensional plane bubble analyzed by Collins (1965). Based on observations of bubbles in slots of 9 mm width, he concluded that the bubble cap was more elliptical than circular. Accordingly, he modeled the slug flow as flow between two parallel plates produced by a source and sink of equal strength separated vertically by a small but finite distance.

Applying an analysis similar to Collins' to the elliptical gap bubble thus generated, Hills calculated rise velocities within 2 per cent of measured values. He attributed the discrepancy to three-dimensional effects unaccounted for in the two-dimensional analysis.

Sadatomi, et al. (1982), reported measurements of slug flow in 20 vertical channels of various cross-sections. He considered circular, rectangular, annular, and triangular cross-sectional areas, as well as a nuclear reactor rod bundle configuration. Defining the equiperipheral diameter as

\[ D_{eq} = \frac{\text{wetted perimeter}}{\pi} \]

he found that for all but two cases,
\[ 0.31 \leq \frac{U_s}{\sqrt{gD_{eq}}} \leq 0.36. \]

The two exceptions were a rod bundle configuration \((Fr = 0.39)\) and a rectangular channel of width 7 mm where surface tension would be significant \((Fr = 0.38)\).

1.4. Statement of Current Problem

The above sections briefly summarize the major contributions to the problem of gas slugs rising in vertical channels. In all cases where the problem has been approached analytically, four common factors are apparent.

Firstly, potential flow has been assumed. This requires that the liquid be ideal \((\mu = 0)\) which is a reasonable approximation for low viscosity fluids such as water. For more viscous fluids this assumption is, of course, not valid. The effects of viscosity are to introduce boundary layers along the channel wall and, to a lesser extent, at the bubble interface.

Even in a viscous fluid, however, potential theory is still valid in the flow region outside the boundary layer. For an initially stagnant liquid, therefore, the effect of a rising slug is to generate a laminar boundary layer at the channel wall. The boundary layer would begin forming slightly ahead of the bubble and merge asymptotically into the wall film as the liquid flows around the bubble.
The boundary layer at the bubble interface can be assumed of negligible thickness, even for a liquid of high viscosity, due to the tendency of the liquid to "pull the gas along" as it flows around the interface. This is in contrast to the no-slip conditions at the walls.

Secondly, the effect of surface tension has been neglected entirely. Therefore, the results of the analyses are valid only in channels of sufficient size such that the curvature of the interface is large enough so that the pressure drop across the interface is negligible. The primary effect of surface tension is felt at the bubble nose where the radius of curvature is the smallest. For small channels, surface tension is clearly significant.

Thirdly, when an analytical solution is attempted, the solution is invariably in the form of an infinite series. In order to obtain a solution that can be reasonably handled, the series must be truncated after a very few terms. Dumitrescu (1943) truncated after three terms in his solution, whereas in all other cases discussed above, only the first term is utilized. A one-term solution has been shown to be a fairly good approximation for negligible surface tension and negligible liquid viscosity in the cases considered thus far.

Fourthly, although the velocity is reasonably approximated by a one term solution, the measured radius of curvature differs significantly from that predicted from a one term solution.

It will be the objective of this dissertation to apply potential flow theory to slug flow in a vertical tube and a vertical rectangular
channel, including the effects of surface tension on the curvature of the interface. With the inclusion of surface tension, the bubble shape becomes a parameter of the solution.
Table 1-1

Summary of Dumitrescu's (1943) Data for Air Slugs in Water-Filled Tubes

a) Average Velocity (11 to 14 measurements for each tube)

\[
Fr \equiv \frac{U_g}{\sqrt{gD}}
\]

<table>
<thead>
<tr>
<th>D (cm)</th>
<th>Fr</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.99</td>
<td>.203</td>
</tr>
<tr>
<td>2.00</td>
<td>.335</td>
</tr>
<tr>
<td>3.76</td>
<td>.349</td>
</tr>
<tr>
<td>7.00</td>
<td>.346</td>
</tr>
</tbody>
</table>

b) Average Values of Measured Bubble Profile for All Four Tubes

<table>
<thead>
<tr>
<th>r/R</th>
<th>z/R</th>
<th>R_c /R</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>.268</td>
<td>-.051</td>
<td>.730</td>
</tr>
<tr>
<td>.372</td>
<td>-.103</td>
<td>.723</td>
</tr>
<tr>
<td>.503</td>
<td>-.207</td>
<td>.715</td>
</tr>
<tr>
<td>.653</td>
<td>-.413</td>
<td>.723</td>
</tr>
<tr>
<td>.735</td>
<td>-.620</td>
<td>.746</td>
</tr>
</tbody>
</table>

*Measured relative to bubble nose.

**Calculated from \( \frac{R_c}{R} = \frac{(\frac{r}{R})^2 + (\frac{z}{R})^2}{2|\frac{z}{R}|} \)**
Table 1-2

Experimental Measurements of Air Slugs in Rectangular Channels (Griffith 1964) for Negligible Viscosity

<table>
<thead>
<tr>
<th>Dimensions</th>
<th>Us/\sqrt{gL}</th>
<th>W/L</th>
</tr>
</thead>
<tbody>
<tr>
<td>2L</td>
<td>2W</td>
<td>*</td>
</tr>
<tr>
<td>A</td>
<td>5.21</td>
<td>5.16</td>
</tr>
<tr>
<td>B</td>
<td>5.03</td>
<td>3.76</td>
</tr>
<tr>
<td>C</td>
<td>3.78</td>
<td>.89</td>
</tr>
<tr>
<td>D</td>
<td>13.34</td>
<td>1.10</td>
</tr>
<tr>
<td>E**</td>
<td>4.52</td>
<td>.33</td>
</tr>
</tbody>
</table>

* Estimated from graphical presentation of data, Figure 1 of Griffith (1964)

** May be affected by surface tension for small width dimension.
Figure 1-1 Slug flow in vertical tubes for various liquids
Harmathy (1960)
Figure 1-2 Slug flow in vertical tubes for negligible viscosity
White and Beardmore (1962)
CHAPTER 2

General Potential Theory of
Vertical Gas/Liquid Slug Flow

As discussed in Chapter 1, the vertical motion of large gas bubbles, or gas slugs, has been treated historically by potential flow theory. This approach should be valid provided that the liquid through which the gas bubble rises has a sufficiently negligible viscosity. A low viscosity fluid would ensure that the boundary layer at the duct wall is of negligible thickness near the bubble nose. Therefore, the potential flow theory would apply throughout the liquid flow region under the condition of low liquid viscosity.

The following discussion will develop the general theory of potential flow, including surface tension effects, for gas slug flow in a vertical duct of arbitrary cross section. The general theory will then be applied to two specific geometries, a circular pipe and a rectangular slot, in Chapters 3 and 4, respectively. Chapter 3 will also introduce a viscous approximation to allow liquids of moderate viscosity to be modeled.

2.1. Derivation of General Form of Velocity Potential

Consider a vertical duct of infinite length with constant cross section (Figure 2-1). Suppose the duct to be filled with an ideal liquid ($\mu = 0$) which is initially stagnant ($U_c = 0$) before the gas is introduced as a gas slug. As the gas slug rises, the liquid will fall around the bubble forming a liquid film on the duct wall. If the duct is
closed at \( z = +0 \), then the liquid will remain stagnant far from the bubble nose. If the duct is open at \( z = +0 \), then the liquid far from the bubble nose will have some net velocity imparted to it as the bubble rises. The bubble would therefore be rising in a flowing liquid column.

It will be assumed in this dissertation that the duct is closed at \( z = 0 \) so that the liquid far above the bubble nose will remain stagnant.

The origin of the coordinate system to be used will be attached to the nose of the rising gas bubble. In this frame of reference, the liquid velocity at \( z = 0 \) will be \( u(0) = -u_s \) where \( u_s \) is the steady-state bubble slip velocity relative to the stagnant liquid (Figure 2-1) in the laboratory frame.

Under the assumption of potential flow in the liquid, the governing equation will be the Laplacian

\[
\nabla^2 \Phi(x,y,z) = 0
\]  

(2-1)

where \( \Phi(x,y,z) \) is the velocity potential function (Appendix E.1.). The components of the velocity \( \mathbf{u}(x,y,z) \) are then

\[
\begin{align*}
u(x,y,z) &= \frac{\partial \Phi}{\partial x} \\
v(x,y,z) &= \frac{\partial \Phi}{\partial y} \\
w(x,y,z) &= \frac{\partial \Phi}{\partial z}
\end{align*}
\]  

(2-2)

in the \( x, y, \) and \( z \) directions, respectively.
Equation 2-1 will be solved subject to the boundary conditions

1) \( \lim_{z \to \infty} w(x,y,z) = -U_s \)

2) \( \vec{V}_N(x_B, y_B, z) = 0 \) \hspace{1cm} (2-3)

3) \( w(0,0,0) = 0 \)

where \((x_B, y_B, z)\) is an arbitrary point on the duct wall and \(\vec{V}_N\) denotes the normal component of the velocity vector.

Expressing the potential function as

\[ \Phi(x,y,z) = F(x,y) G(z) \]

and applying separation of variables to 2-1 it is readily seen that

\[ \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \alpha^2 F(x,y) = 0 \]

and

\[ \frac{\partial^2 G}{\partial z^2} - \alpha^2 G(z) = 0 \] \hspace{1cm} (2-4)

where \(\alpha > 0\) is a constant.

From 2-4

\[ G(z) = C_1 e^{-\alpha z} + C_2 e^{\alpha z} \]

so that 2-1 is satisfied by

\[ \Phi(x,y,z) = F(x,y) \left[ C_1 e^{-\alpha z} + C_2 e^{\alpha z} \right] . \]
However, any function $\phi_0(x,y,z)$ of the form

$$
\phi_0(x,y,z) = a_0 + a_1 x + a_2 y + a_3 xy + a_4 z + a_5 xz + a_6 yz
$$

where the $a_i$, $i = 0, 1, \ldots, 6$ are constants, also satisfies $\nabla^2 \phi_0(x,y,z) = 0$. The most general solution to the governing equation 2-1 is therefore

$$
\phi(x,y,z) = F(x,y) \left[ C_1 e^{-az} + C_2 e^{az} \right] + a_0 + a_1 x
$$

$$
+ a_2 y + a_3 xy + a_4 z + a_5 xz + a_6 yz .
$$

(2-5)

The axial velocity component $w(x,y,z)$ is then

$$
w = \frac{\partial \phi}{\partial z} = F(x,y) \left[ -a_1 C_1 e^{-az} + a_4 C_2 e^{az} \right] + a_4 + a_5 x + a_6 y .
$$

(2-6)

Application of boundary condition (1) to 2-6 requires that

$$
C_2 = 0 , \quad a_4 = -U_s , \quad a_5 = 0 , \quad a_6 = 0
$$

so that

$$
\phi(x,y,z) = C_1 F(x,y) e^{-az} - U_s z + a_0 + a_1 x + a_2 y + a_3 xy .
$$

(2-7)

The function $F(x,y)$ will depend upon the particular coordinate system chosen for the radial cross-section of the vertical duct. Without loss of generality, a Cartesian system may be used and $F(x,y)$ may be written as

$$
F(x,y) = X(x) Y(y) .
$$
Application of separation of variables to $F(x,y)$ will give

$$\frac{\partial^2 X}{\partial x^2} + \beta^2 X(x) = 0$$

$$\frac{\partial^2 Y}{\partial y^2} + \gamma^2 Y(y) = 0$$

(2-8)

where $\beta$, $\gamma$ are constants and

$$\gamma^2 = \alpha^2 - \beta^2$$

The general solution to 2-8 is

$$X(x) = A_1 \cos(\beta x) + A_2 \sin(\beta x)$$

$$Y(y) = B_1 \cos(\gamma y) + B_2 \sin(\gamma y)$$

(2-9)

Application of the boundary conditions 2-3 to 2-8 will determine the constants $\beta$, $\gamma$. In general, because of the periodicity of the trigonometric functions, one may write

$$X(x) = \sum_{n=-\infty}^{\infty} A_n \cos(\beta_n x) + A_{2n} \sin(\beta_n x)$$

$$Y(y) = \sum_{m=-\infty}^{\infty} B_m \cos(\gamma_m y) + B_{2m} \sin(\gamma_m y)$$

so that

$$\Phi(x,y,z) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} F_{nm}(x,y)e^{-\alpha \frac{z}{nm}} - U_s z$$

$$+ a_0 + a_1 x + a_2 y + a_3 xy$$

(2-10)
where

\[ F_{nm}(x,y) = C_1 [A_{1n} \cos(\beta_n x) + A_{2n} \sin(\beta_n x)] \]

\[ \cdot [B_{1m} \cos(\gamma_m y) + B_{2m} \sin(\gamma_m y)] \]

and

\[ a_{nm} = (\beta_n^2 + \gamma_m^2)^{1/2} \]

### 2.2. Derivation of Defining Equations for Bubble Shape

Under the conditions of potential flow, the surface of the bubble must be a flow streamline. One may therefore write Bernoulli's Equation in the liquid, at the bubble surface, as

\[ P_l + \frac{1}{2} \rho_l \nabla^2_l + \rho_l g \cdot \nabla = P_{l0} \quad (2-11) \]

where \( \nabla \) is the magnitude of the velocity in the liquid and the subscript 0 refers to the bubble nose at the origin.

A similar equation may be written for the gas inside the bubble and adjacent to the interface. Thus

\[ P_g + \frac{1}{2} \rho_g \nabla^2_g + \rho_g g \cdot \nabla = P_{g0} \quad (2-12) \]

Equations 2-11 and 2-12 may be combined by employing the Kelvin-LaPlace Equation

\[ \frac{P_g - P_l}{\sigma} = \frac{1}{R_1} + \frac{1}{R_2} \quad (2-13) \]
where $\sigma$ is the interfacial surface tension and $R_1$, $R_2$ are the principal radii of curvature at a given point on the interface. Combining 2-11, 2-12, and 2-13 gives

$$\frac{1}{2} \rho_l \nu^2 + \frac{1}{2} \rho_g v^2 + (\rho_l - \rho_g) gz + \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - \left(\frac{1}{R_1} + \frac{1}{R_2}\right) = 0. \quad (2-14)$$

Across the interface, the gas and liquid must have the same velocity at the interface boundary. Thus $V_L = V_g$ and 2-14 becomes

$$T_L(x,y,z) = u^2 + v^2 + w^2 + 2gz + 2\Lambda(K - K_0) = 0 \quad (2-15)$$

where

$$\Lambda \equiv \left(\frac{\sigma}{\rho_l - \rho_g}\right)$$

$$V_L^2 = u^2 + v^2 + w^2$$

$$K \equiv -\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

and the subscript 0 denotes evaluation at the stagnation point.

Equation 2-15 is the defining equation for the bubble surface under the condition of potential flow. Note that the bubble shape is, in effect, a variable in 2-15 due to the local radii of curvature $R_1$, $R_2$.

In order to determine a complete solution of the liquid flow and the bubble shape, another equation relating the liquid flow at the interface to the bubble shape is required. This equation can be written from the physical requirement that no liquid may cross the interfacial boundary. That is, the liquid flow at the bubble surface is always parallel to the
surface. This is simply the physical representation of the fact that
the bubble surface is a flow streamline.

Using this condition, one may write for the liquid

\[ \vec{v} \cdot \nabla H = 0 \]  \hspace{1cm} (2-16)

where \( H(x,y,z) = 0 \) is a level function for the bubble shape. The sim-
plest form for \( H(x,y,z) \) is

\[ H(x,y,z) = f(x,y) - z = 0 \]

and (2-16) becomes

\[ u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} - w = 0 \] \hspace{1cm} (2-17)

Since \( z = f(x,y) \), then 2-17 may be written as

\[ T_2(x,y,z) = u \frac{\partial z}{\partial x} + v \frac{\partial z}{\partial y} - w = 0 \] \hspace{1cm} (2-18)

Equations 2-15 and 2-18 relate the liquid velocity at the bubble surface
to the bubble shape. In order to complete the formulation, one must be
able to express the principal radii of curvature, \( R_1 \) and \( R_2 \), in terms of
the bubble shape \( z = f(x,y) \).

This latter requirement is accomplished in Appendix A-1. The final
results are given below:
The principal axes denoted by subscripts 1 and 2 are defined in Appendix A.

The sum of the radii of curvature is

\[
\left( \frac{1}{R_1} + \frac{1}{R_2} \right) = \frac{\frac{\partial^2 z}{\partial x^2}}{\left[ 1 + \left( \frac{\partial z}{\partial x} \right)^2 \right]^{1/2}} - \frac{\frac{\partial^2 z}{\partial y^2}}{\left[ 1 + \left( \frac{\partial z}{\partial y} \right)^2 \right]^{1/2}} - 2 \frac{\frac{\partial^2 z}{\partial x \partial y}}{\left[ 1 + \left( \frac{\partial z}{\partial x} \right)^2 \right]^{1/2}} \] (2-20)

and is invariant with regard to coordinate rotations. Equations 2-9, 2-15, 2-18, and 2-20 constitute the equations necessary to define the liquid flow in a closed vertical duct of infinite length through which a gas slug is rising with slip velocity \( U_s \). The actual duct geometry determines the velocity potential function \( \Phi \) which in turn determines the local velocity components. A solution cannot be determined until a duct geometry is specified and the function \( \Phi \) determined.
However, one may formulate the solution procedure in terms of the velocity components and their spatial derivatives. The components $u$, $v$, $w$ are found from 2-9 and 2-2.

Although the complete solution consists of an infinite series, one can consider only a finite number of terms in the series expansion. In order to determine the coefficients in 2-9, a series of equations must be developed, one for each of the unknown coefficients. Note that the slip velocity $U_s$ is treated here as one of the coefficients.

2.3. General Procedure for Development of Equations to Determine Series Coefficients

The necessary equations may be derived from the two independent equations 2-15 and 2-18 repeated here for convenience.

\[ T_1(x,y,z) = u^2 + v^2 + w^2 + 2gz + 2A(K - K_0) = 0 \]  

\[ T_2(x,y,z) = u \frac{\partial z}{\partial x} + v \frac{\partial z}{\partial y} - w = 0 \]  

Both equations are identically satisfied at the stagnation point.

With the coordinate system referenced to the bubble nose, symmetry of the bubble requires that all odd-ordered derivatives of $z(x,y)$ must be identically zero when evaluated at the stagnation point, or origin of the coordinate system. Thus all derivatives of the form

\[ \frac{\partial^{m+n}z}{\partial x^m \partial y^n} \]

are zero for either $m$ or $n$ odd.
The functions $T_1, T_2$ are identically zero everywhere on the bubble surface. One may thus write, for $i = 1, 2$,

$$
\frac{dT_i}{dt} = \left( \frac{\partial T_i}{\partial x} + \frac{\partial T_i}{\partial z} \frac{\partial z}{\partial x} \right) dx + \left( \frac{\partial T_i}{\partial y} + \frac{\partial T_i}{\partial z} \frac{\partial z}{\partial y} \right) dy = 0
$$

for the total derivative of $T_i$ along the interface. Independence of the $x$ and $y$ variables thus requires that the expressions in parentheses also be identically zero everywhere on the bubble surface. Introducing the operator

$$
\frac{D}{Dn} = \frac{\partial}{\partial n} + \frac{\partial z}{\partial n} \frac{\partial}{\partial z} , \quad n = x, y
$$

one may then write

$$
\frac{DT_i}{Dx} \equiv 0 \quad \frac{DT_i}{Dy} \equiv 0
$$

Successive differentiations of $T_1, T_2$ along the bubble surface will then require that

$$
\frac{D^n T_i}{Dx^n} = 0 \text{ and } \frac{D^n T_i}{Dy^n} = 0 \quad (2-21)
$$

for all $n \geq 0$, $i = 1, 2$.

Carrying out the differentiation as indicated in 2-21 and evaluating the resultant expressions at the origin will give equations for the even-ordered derivatives
Equating the same order derivative as calculated from $T_1$ and $T_2$ will result in an equation involving only the coefficients of 2-9.

An example of this procedure is provided in Appendix B along with the resultant equations derived in this manner.

Closure for the system of equations is provided from the condition that the axial velocity be zero at the stagnation point,

$$
\omega_0 \equiv \left( \frac{\partial \phi}{\partial z} \right)_0 = 0
$$

2.4. Non-Dimensional Form of Defining Equation

Equation 2-15 may be put into a non-dimensional form by making the definitions

$$
\begin{align*}
\hat{u}^* &= \frac{u}{U_s} \\
\hat{z}^* &= \frac{z}{D_{eq}} \\
\hat{w}^* &= \frac{w}{U_s} \\
\lambda &= D_{eq} \left( K - K_c \right)
\end{align*}
$$

With these definitions, 2-15 becomes
\[(u^*)^2 + (v^*)^2 + (w^*)^2 + \frac{2}{Fr^2} z^* + \frac{2\lambda}{E_o Fr^2} = 0 \]  

(2-22)

where

\[
Fr \equiv \frac{U_g}{\sqrt{gD_{eq}}} \equiv \text{Froude Number} \quad (2-23)
\]

and

\[
E_o = \frac{gD_{eq}^2}{\Lambda} = \frac{gD_{eq}^2 (\rho_d - \rho)}{\sigma} \equiv \text{Eotvos Number}. \quad (2-24)
\]

The variable \(D_{eq}\) is a reference length for the system and is defined, after Sadatomi, et al. (1982), as

\[
D_{eq} = \frac{\text{wetted perimeter}}{\pi}.
\]

\(D_{eq}\) is then the equiperipheral diameter for the flow channel under consideration.

For the channels to be considered in this dissertation, \(D_{eq}\) has the following form:

- **Circular pipe**
  \[
  D_{eq} = \frac{\pi D}{\pi} = D
  \]

- **Two-Dimensional Rectangular Slot**
  \[
  D_{eq} = \frac{(2L + 2L)}{\pi} = \frac{4L}{\pi}
  \]

- **Three-Dimensional Rectangular Slot**
  \[
  D_{eq} = \frac{2(2L + 2W)}{\pi} = \frac{4(L+W)}{\pi}
  \]  

(2-25)
where $L, W$ are the half-length and half-width of the rectangular channel, respectively.
Figure 2-1  Coordinate system chosen for gas slug rising in a vertical duct of infinite length filled with a stagnant liquid
This chapter will apply the potential flow theory, including surface tension effects, to the case of gas slug flow in a vertical circular pipe. In addition, the effects of viscosity will be approximated by an "effective radius" concept which will allow slug flow in liquids of moderate viscosity to be modeled with reasonable accuracy.

3.1. Basic Theory

Consider a vertical duct of circular cross section with radius R. The general theory developed in Chapter 2 is valid for an arbitrary cross section, but due to the radial symmetry for a circular duct, the solution may be obtained more readily in terms of the Stokes Stream Function.

Collins, et al. (1978) have considered the case of slug flow in a vertical pipe with no surface tension or viscous effects. In their work, the potential flow solution for the Stokes Stream Function \( \psi(r,z) \) in a vertical pipe was shown to be

\[
\psi(r,z) = \left[ \frac{1}{2} r^2 - \frac{1}{2} r^2 (1 - \frac{r^2}{R^2}) \frac{U_c}{U_s} \right] - (1 - \frac{U_c}{U_s}) r \sum_{n=1}^{\infty} J_n(\delta r) e^{-\delta z/n} \]

(3-1)
for a parabolic velocity profile at $z = 0$, where

$$U_c = \text{centerline velocity of liquid},$$

$$d_n, \delta_n = \text{constants},$$

$$J_1 = \text{first-order Bessel Function.}$$

Collins, et al. (1978) considered only the $n = 1$ term in 3-1. This chapter will extend 3-1 out to $n = 6$ and also include surface tension and viscous effects.

Making the convenient definition

$$B_{ij} = \sum_{n=1}^{\infty} d_n \delta_j^n J_i(\delta_n r)e^{-\delta_n z}$$  \hspace{1cm} (3-2)

then 3-1 becomes

$$\psi(r,z) = [\frac{1}{2} r^2 - \frac{1}{2} r^2(1 - (\frac{r}{R})^2)(\frac{U}{U_s}) - (1 - \frac{U}{U_s})r B_{10}^i]U_s .$$  \hspace{1cm} (3-3)

For stagnant flow, $U_c = 0$, 3-3 simplifies to

$$\psi(r,z) = [\frac{1}{2} r^2 - r B_{10}^i]U_s .$$  \hspace{1cm} (3-4)

This equation will be the starting point for the discussions of this chapter.
The radial and axial velocity components are given respectively by

\[ v(r,z) = \frac{1}{r} \frac{\partial \psi}{\partial z} \]

\[ u(r,z) = \frac{1}{r} \frac{\partial \psi}{\partial r} \]

or from 3-4

\[ v(r,z) = -\frac{\partial B_{10}}{\partial z} U_s \]

\[ u(r,z) = \left[ \frac{B_{10}}{r} + \frac{\partial B_{10}}{\partial r} - 1 \right] U_s \]

From the properties of Bessel Functions, it may be shown that

\[ \frac{\partial B_{i+1}}{\partial r} = B_{i-1,j+1} - \left( \frac{i}{r} \right) B_{i,j} \]

\[ \frac{\partial B_{i+1}}{\partial z} = -B_{i,j+1} \]

so that 3-5 becomes

\[ v(r,z) = B_{11} U_s \]

\[ u(r,z) = (B_{01} - 1) U_s \]

The boundary conditions appropriate for the radial and axial velocities are

\[ v(R,z) = 0 \]

\[ u(0,0) = 0 \]
so that 3-7 requires that

\[(B_{11})_0 = 0\]
\[(B_{01})_0 = 1 .\]  \hspace{1cm} (3-9)

By definition,

\[B_{11} = \sum_{n=1}^{\infty} d_n \delta_n J_1(\delta r) e^{-\delta z} \]
\[(3-10)\]
\[B_{01} = \sum_{n=1}^{\infty} d_n \delta_n J_0(\delta r) e^{-\delta z} \]

and 3-9 becomes

\[\sum_{n=1}^{\infty} d_n \delta_n J_1(\delta r) e^{-\delta z} = 0\]  \hspace{1cm} (3-11)

\[\sum_{n=1}^{\infty} d_n \delta_n = 1 .\]  \hspace{1cm} (3-12)

In order for 3-11 to be satisfied for any arbitrary \(z\), one must require that

\[J_1(\delta r) = 0\]

and thus

\[\delta_n = \frac{k_n}{R}\]

where \(k_n\) is the \(n\)th zero of \(J_1\). For reference, any mathematical handbook will provide
\[ k_1 = 3.83171 \quad k_4 = 13.32369 \]
\[ k_2 = 7.01559 \quad k_5 = 16.47063 \]
\[ k_3 = 10.17347 \quad k_6 = 19.61586 \]

From 2-22, it is apparent that the solution for the Froude number is some function of the Eotvos number,

\[ Fr = Fr(Eo) \]

and for \( Eo \gg 1 \), \( Fr \) should be a constant. Such behavior is indeed observed experimentally (White and Bearmore, 1961; Harmathy, 1960; Wallis, 1962) and the data can be correlated well for a large variety of gas/liquid systems by

\[
Fr = 0.345 \left[ 1 - e^{-10} \right] \quad (3-13)
\]

for negligible liquid viscosity (Wallis, 1962).

Since the stream function \( \psi \) consists of an infinite Bessel series, the solution must be approximated by truncating the series after a finite number of terms. An \( N^{th} \) order approximation will be defined as the solution resulting from considering only the first \( N \) terms in the series. Since the slip velocity \( u_s \) is also an unknown, an \( N^{th} \) order approximation will require \( (N+1) \) simultaneous equations in order to uniquely determine the \( N \) coefficients \( d_n \) and the slip velocity.

The required equations for a circular pipe are derived in Appendix C. The equations for the derivatives of the bubble shape are provided in Table C-2 where
\[ z_m = \frac{\partial^m z}{\partial r^m} \]

and
\[ u_{mn} = \frac{\partial^{(m+n)} u}{\partial r^m \partial z^n} \]

(3-14)

\[ v_{mn} = \frac{\partial^{(m+n)} v}{\partial r^m \partial z^n} \]

From 3-7, making use of 3-6, one may show that

\[ (u_{mn})_0 = (-1)^{\frac{m}{2} + n} U_s C_{m/2} (B_0, m+n+1)^0 / 2^n \]

(3-15)

\[ (v_{mn})_0 = (-1)^{\frac{m-1}{2} + n} U_s C_{(m+1)/2} (B_0, m+n+1)^0 / 2^n \]

where
\[ C_a^b = \frac{b!}{a!(b-a)!} \]

From Table C-2, note that \((Z_4)_0\) is a function of \((Z_2)_0\), \((Z_6)_0\) a function of \((Z_2)_0\) and \((Z_4)_0\), and so forth, so that one could in principle express all the higher order derivatives in terms of \((Z_2)_0\) and the velocity derivatives.

Thus, even for the first order approximation \(N = 1\), all derivatives of the shape function could be evaluated and the shape near the bubble nose expressed as a Taylor expansion.
Because the higher order derivatives are functions of \((Z_2^0)\), it is clear that the accuracy of the higher order derivatives is dependent upon the accuracy of the approximation to \((Z_2^0)\). It would also be expected that the higher the order, the more the error introduced.

Due to the complexity of the equations, more than one real mathematical solution may exist. One must then decide which solution to accept.

For the simplest case \(N = 1\), there is only one solution as will be shown below. If the series in equation 3-4 is to converge to the actual stream function, one would expect that the values of the coefficients from the solution for \(N\) should be a good approximation for \(N+1\). In addition, one would expect that the absolute value of the coefficients should be monotonically decreasing with \(N\). These two observations will be the criteria used to accept a solution.

3.2. Solution for Negligible Surface Tension (\(\Lambda = 0\)) and Ideal Fluid (\(\mu = 0\))

3.2.1. Slug Velocity \(U_s\)

For the simplest case, \(N = 1\), one has from 3-12

\[
d_1 \delta_1 = 1
\]

or

\[
d_1 = \frac{1}{\delta_1} = \frac{R}{k_1}.
\]
In addition, from Tables C-1, C-2, and C-3, one obtains for $\Lambda = 0$

$$[v_{10}^2 + g z_2] = 0 \quad (3-16)$$

and

$$z_2 = \left[ \frac{u_{20}}{2v_{10} - u_{01}} \right]_0 \quad (3-17)$$

The derivatives of the velocity components are required to evaluate 3-14

and may be obtained from 3-15. Thus

$$v(r,z) = B_{11} U_s$$

$$u(r,z) = (B_{01} - 1)U_s$$

so that

$$v_{10} = \frac{\partial B_{11}}{\partial r} U_s = (B_{02} - \frac{B_{11}}{r})U_s$$

$$u_{20} = \frac{\partial^2 B_{01}}{\partial r^2} U_s = (\frac{B_{12}}{r} - B_{03})U_s \quad (3-19)$$

$$u_{01} = \frac{\partial B_{01}}{\partial z} U_s = -B_{02} U_s$$

Evaluating 3-19 at the stagnation point, and realizing that

$$\lim_{r \to 0} \left( -\frac{l+1}{r} \right) = \frac{B_{02} i + 1}{2}$$
then one has

\[ (v_{10})_0 = \left( \frac{1}{2} B_{02} U_s \right)_0 \]

\[ (u_{20})_0 = -\left( \frac{1}{2} B_{03} U_s \right)_0 \]

\[ (u_{01})_0 = -\left( B_{02} U_s \right)_0 . \]

Thus, 3-17 becomes

\[ (Z_2)_0 = -\left( \frac{B_{03}}{4 B_{02}} \right)_0 \]

and 3-16 becomes

\[ \frac{U_s^2 B_{02}^2}{4} - \frac{g B_{03}}{4 B_{02}} = 0 . \]  \hspace{1cm} (3-20)

Solving for \( U_s \), one obtains

\[ U_s = \left( \frac{g B_{03}}{B_{02}} \right)^{1/2} . \]  \hspace{1cm} (3-21)

From 3-2, for \( N = 1 \),

\[ (B_{03})_0 = d_1 \delta_{11}^3 = \left( \frac{k_1}{R} \right)^2 \]  \hspace{1cm} (3-22)

\[ (B_{02})_0 = d_1 \delta_{11}^2 = \left( \frac{k_1}{R} \right) . \]
and 3-21 becomes,

\[ U_s = \left( \frac{gR}{k_1} \right)^{1/2} = 0.511 \sqrt{gR} = 0.361 \sqrt{gD} \]  \hspace{1cm} (3-23)

Equation 3-23 is the same result determined by Collins, et al. (1978) and agrees well with the experimentally determined velocity

\[ U_s = 0.345 \sqrt{gD} \]  \hspace{1cm} (3-24)

valid for negligible surface tension \((E_o \geq 100)\) and negligible liquid viscosity. However, 3-23 predicts a constant Froude Number

\[ Fr \equiv \frac{U_s}{\sqrt{gD}} = 0.361 \]

independent of tube radius or surface tension. The assumption of negligible surface tension must, therefore, be relaxed if one hopes to predict slip velocities for \(E_o < 100\).

For higher order approximations to the stream function \(\psi\), an analytic solution is untenable. It is necessary to resort to a numerical solution for \(N > 1\).

The velocity derivatives required, derived by successive differentiation of 3-7, are provided by 3-15.

The solution of the equations of Table C-3 is obtained by means of a multi-dimensional Newton-Raphson procedure as summarized in Appendix D.
The results for $\Lambda = 0$ are summarized in Figure 3-1. The nomenclature employed is as follows:

$N \equiv$ order of approximation to the Stokes Stream Function and is equal to the number of terms included in the finite truncation of the infinite Bessel Series, equation 3-2.

$ND \equiv$ order of approximation of the bubble shape and is equal to the order of the highest derivative included in the finite Taylor Series expansion. ND will always be even because the odd-order derivatives are identically zero at the bubble nose.

Since the slug velocity $U_s$ is also an unknown in the system of equations, the number of equations required is $N+1$.

Because of numerical difficulty in solving the equations for $\Lambda$ identically zero, the equations were solved for $\sigma = 5$ dynes/cm. Since $\sigma = 72$ dynes/cm for an air/water system, $\sigma = 5$ dynes/cm should be sufficiently small to adequately represent the case of $\Lambda = 0$.

From Figure 3-1, the Froude Number $Fr$ converges to a value $Fr = 0.352$, which is consistent with the experimental observation of $Fr = 0.345$ as $N$ increases and ND increases accordingly. This is consistent with the previous observation that the higher order derivatives of $Z(r)$ are functions of $(Z_2)_0$ only since the inclusion of additional terms in the finite series approximation to $\psi$ should result in a better value for $(Z_2)_0$ and thus the higher derivatives also.
The solution for $N = 2$ is conspicuously absent from Figure 3-1. No solution consistent with the criteria presented above for the acceptance of a solution could be found. It can be shown that for $N = 2$ and $\Lambda = 0$, the resultant equations can be reduced to a cubic in the unknown $d_1$ that has only one real solution. The only real solution is then found to be, for a one inch diameter tube,

$$
\begin{align*}
    d_1 &= 0.1717 \\
    d_2 &= -0.0225 \\
    U_s &= 14.448 \text{ cm/sec}
\end{align*}
$$

These results are not consistent with the values of the coefficients for $N = 1, 3, 4, 5, 6$ (Table 3-1). The equations were checked thoroughly and no errors were found.

3.2.2. Normalized Radius of Curvature at Bubble Nose, $\Lambda = 0$

The radius of curvature at the bubble nose is given by

$$
R_c = -\left(\frac{1}{Z_2}\right)
$$

so that one would expect that as more terms are included in the stream function approximation ($N$), $R_c$ should tend toward a limiting value as the estimate of $(Z_2)_0$ is improved. This is indeed the case as demonstrated by Figure 3-2.

Figure 3-2 presents the radii of curvature normalized to the tube radius for the same cases as Figure 3-1. It is apparent that for negligible surface tension, the normalized radius of curvature converges to
for stagnant flow in a vertical tube.

3.3. Solution for Non-Zero Surface Tension ($\Lambda > 0$) and Ideal Fluid ($\mu = 0$)

When the surface tension is included, $\Lambda > 0$, the Eotvos Number $Eo$ becomes a significant parameter in the solution. The maximum value of $ND$ is then $(2N+2)$.

3.3.1. Slug Velocity $U_s$

The analytic solution for $U_s$ for $N = 1$ and $\Lambda > 0$ will be derived below. For $N > 1$, a numerical solution will be required.

From Table C-3, one has

$$[v_{10}^2 + g Z_2 + \Lambda K_2]_0 = 0$$

(3-25)

where

$$(K_2)_0 = \left[\frac{4}{3} Z_4 - 4Z_2^3\right]_0.$$

Recalling that

$$(Z_2)_0 = \left(-\frac{B_0}{4B_{02}}\right)_0$$

$$\left(v_{10}\right)_0 = \frac{1}{2} B_{02} U_s$$
then 3-25 becomes

\[
\frac{U_s^2 B_{02}}{4} - \frac{g B_{03}}{4 B_{02}} + \Lambda \left( \frac{4Z_4}{3} + \frac{B_{03}^3}{16 B_{02}} \right) = 0
\]  \(3-26\)

The bubble shape \(Z(r)\) may be approximated as

\[
Z(r) = \frac{(Z_2)_0 r^2}{2}, \text{ ND } = 2
\]  \(3-27a\)

or

\[
Z(r) = \frac{(Z_2)_0 r^2}{2} + \frac{(Z_4)_0 r^4}{24}, \text{ ND } = 4
\]  \(3-27b\)

If 3-27a is chosen, then \((Z_4)_0 = 0\) and 3-26 becomes

\[
\frac{U_s^2 B_{02}}{4} - \frac{g B_{03}}{4 B_{02}} + \Lambda \frac{B_{03}^3}{16 B_{02}} = 0
\]  \(3-28\)

Substituting \((B_{02})_0\) and \((B_{03})_0\) from 3-22 and solving for \(U_s\) yields

\[
U_s = \left[ \frac{g R}{k_1} - \frac{\Lambda k_1}{4R} \right]^{1/2}
\]  \(3-29\)

Recalling that

\[
\Lambda = \frac{\sigma}{\rho_2 - \rho_g}, \quad E_0 = \frac{4 g R^2 (\rho_2 - \rho_g)}{\sigma}
\]
then $\Lambda = \frac{4gR^2}{E_0}$ and 3-29 becomes

$$U_s = \left[ \frac{gR}{k_1} \left( 1 - \frac{k_1^2}{E_0} \right) \right]^{1/2}$$

or

$$U_s = 0.361 \sqrt{gR} \left( 1 - \frac{14.68}{E_0} \right)^{1/2}$$

(3-30)

for $ND = 2$.

In the second case (3-27b), $(Z_4)_0 \neq 0$ and Table C-2 gives

$$(Z_4)_0 = \frac{u_{40} + (6u_{21} - 4v_{30})Z_2 + (3u_{02} - 12v_{11})Z_2^2}{4v_{10} - u_{01}}$$

(3-31)

From Table C-1,

$$(u_{40})_0 = (3 \frac{B_{05}}{8})_0$$

$$(u_{21})_0 = (\frac{B_{04}}{2})_0$$

$$(u_{02})_0 = (\frac{B_{03}}{2})_0$$

(3-32)

$$(v_{30})_0 = (-3 \frac{B_{04}}{8})_0$$

$$(v_{11})_0 = (-3 \frac{B_{03}}{2})_0$$
where

\[
\begin{align*}
(B_{04})_0 &= \left(\frac{k_1}{R}\right)^3 \\
(B_{05})_0 &= \left(\frac{k_1}{R}\right)^4
\end{align*}
\]  \tag{3-33}

Substituting 3-22, 3-31, 3-32, and 3-33 into 3-26 and solving for \( U_s \) will yield

\[
U_s = 0.361 \sqrt{gD} \left(1 + \frac{4.89}{E_o}\right)^{1/2}
\]  \tag{3-34}

for ND = 4.

A comparison of 3-30 and 3-34 to the experimental correlation (Wallis, 1962)

\[
U_s = 0.345 \sqrt{gD} \left[1 - e^{-\frac{3.37-E_o}{10}}\right]
\]

is provided in Figure 3-3 for air/water. It is apparent that neither case gives very good agreement with the experimental data for small \( E_o \), but 3-30 does reproduce the correct trend of decreasing slip velocity with decreasing \( E_o \) for ND = 2.

The numerical solutions for higher order approximations are presented in Figures 3-4 through 3-7 for varying shape approximations (ND). Figure 3-8 shows the effect of increasing the order of approximation of the stream function while maintaining the highest shape approximation permitted at each value of \( N \).
Figure 3-9 summarizes the results for the case of air/water in a one inch diameter tube. The Froude number converges very well to the experimentally observed value of $Fr = 0.345$.

3.3.2. Normalized Radius of Curvature at Bubble Nose

Figures 3-10 through 3-15 summarize the results for the normalized radius of curvature for air/water.

Figure 3-16 summarizes the results for air/water in a one inch diameter tube for varying degrees of approximation. The solution converges to a normalized radius of curvature of

$$\frac{R_{C}}{R} = 0.71$$

3.3.3. Solution for $Eo >> 1$

For very large values of $Eo$, corresponding to negligible surface tension and/or large diameter tubes, one obtains the results of Figure 3-17. For air/water an $Eo = 2500$ corresponds to a tube of approximately 5 inch diameter.

The solution converges rapidly to

$$Fr = 0.352$$

$$\frac{R_{C}}{R} = 0.76$$

which agrees with the previous results for negligible surface tension in a one inch diameter tube (Figures 3-1 and 3-2).
3.4. Effect of Viscosity on Slip Velocity ($\mu \neq 0$)

The above results are independent of viscosity due to the assumption of potential flow throughout the liquid flow field. In actuality, there will be a boundary layer on the tube wall, developing into a fully developed liquid film downstream of the bubble nose.

Potential theory requires that the bubble surface approach the tube wall asymptotically as $z$ increases. However, the actual bubble is confined within the radius $R - \delta$ where $\delta$ is the thickness of the fully developed liquid film. The actual region where potential flow theory applies is therefore that volume of liquid outside the boundary layer and liquid film where viscous effects are negligible.

3.4.1. The Effective Tube Radius

A detailed treatment of the viscous effects on the bubble slip velocity is outside the scope of this report. However, one may approximate the effects of viscosity over a limited range by defining an effective tube radius as

$$R_{\text{eff}} \equiv R - \nu \delta$$

(3-35)

where $\nu$ is a constant and

$$0 \leq \nu \leq 1 .$$

If $\nu > 1$, then the effective tube radius would be smaller than $R - \delta$, indicating that the region of non-zero vorticity has diffused into a significant portion of the liquid flow field and that potential flow theory is no longer valid.
If \( V < 1 \), then the effective radius would be greater than \( R - \delta \) and potential theory should be valid throughout the liquid flow field for \( r < R - \delta \).

It is shown in Appendix E.2 that for stagnant flow

\[
\frac{U_s}{g} = \frac{2\eta R^2}{3} \frac{\varepsilon^3}{(1-\varepsilon)}
\]  

(3-36)

where

\[ \varepsilon = \frac{\delta}{R} \]

and

\[ \eta = \frac{\rho g}{\mu} \]

so that once the liquid properties (i.e., \( \eta \)) have been specified, 3-36 provides an additional constraint on the system of equations and imposes an additional unknown, the film thickness \( \delta \).

Equation 3-36 was also obtained by Brown (1965) who correlated the bubble velocity as

\[
U_s = 0.350 \left[ g(D - 2\delta) \right]^{1/2}
\]  

(3-37)

Brown calculated the film thickness from experimental data using 3-36 and correlated against a liquid viscosity number \( N_{LB} \) as

\[
\delta = \frac{2R[(1 + N_{LB})^{1/2} - 1]}{N_{LB}}
\]  

(3-38)
where

\[ N_{LB} \equiv \left( \frac{116 \mu^2 R^3}{g} \right)^{1/3}. \]

Brown proposed limits on equations 3-37 of

\[ E_0 > \frac{20}{(1-\varepsilon)^2} \quad N_{LB} > 60. \]

Below either of these values, he considered the potential flow model to be invalid.

White and Beardmore (1961) obtained data for several air/liquid systems for liquids of varying viscosities. Their results are presented in Figure 3-18 and the liquid properties summarized in Table 3-1. These data will be used to test the validity of the effective radius concept discussed above.

3.4.2. The Effective Froude Number

It is unrealistic to expect that a constant value for \( \nu \) could be found that would be valid for all fluids. Water, a nearly ideal fluid, should have \( \nu \approx 0 \), whereas a highly viscous fluid such as glycerine should have \( \nu \gg 1 \). One would therefore expect that a maximum viscosity can be found such that \( \nu = 1 \).

From 3-35, one obtains

\[ D_{\text{eff}} = 2R_{\text{eff}} = 2R(1 - \nu \varepsilon) \quad (3-39) \]
so that one may define an effective Froude number and an effective Eotvos number as

\[
Fr_{\text{eff}} \equiv \frac{U_s}{\sqrt{gD_\text{eff}}} = \left(\frac{R}{R_{\text{eff}}}\right)^{1/2} Fr \\
(3-40)
\]

and

\[
Eo_{\text{eff}} \equiv \frac{gD_\text{eff}^2 (\rho_2 - \rho_1)}{\sigma} = \left(\frac{R_{\text{eff}}}{R}\right)^2 Eo \\
(3-41)
\]

The theoretical curve for an ideal fluid using the N = 5, ND = 6 approximation (Figure 3-6) may be approximated by the equation

\[
Fr = .364 \left(1 - \frac{3.18}{Eo} - \frac{14.77}{Eo^2}\right) \\
(3-42)
\]

where Fr = .364 is the limiting value for this approximation for large Eo. If one assumes that the shape of the curve is correct, an adjustment may be made for the N = 6, ND = 12 approximation for large Eo by replacing the coefficient of 3-42 with the limiting Fr for the higher order approximation, Fr = .352. Thus one obtains for the theoretical curve for ideal fluids,

\[
Fr = .352 \left(1 - \frac{3.18}{Eo} - \frac{14.77}{Eo^2}\right) \\
(3-43)
\]
The preceding arguments then allow one to write

\[
Fr_{\text{eff}} = 0.352 \left(1 - \frac{3.18}{Eo_{\text{eff}}} - \frac{14.77}{Eo_{\text{eff}}^2}\right) \quad (3-44)
\]

In order to apply 3-44 the film thickness \( \delta \) must be known. Combining the definitions of \( Fr_{\text{eff}} \) and \( Eo_{\text{eff}} \) with equations 3-36, 3-39, and 3-44 yields, after algebraic manipulation,

\[
F(\varepsilon)G(\varepsilon)(1 - \varepsilon) - 1.339N\varepsilon^3 = 0 \quad (3-45)
\]

where

\[
F(\varepsilon) = (1 - \nu \varepsilon)^{1/2}
\]

\[
G(\varepsilon) = 1 - \frac{3.18}{Eo_F^4(\varepsilon)} - \frac{14.77}{Eo_F^8(\varepsilon)}
\]

\[
N_p = \left(\frac{\rho \mu g R^3}{\mu^2}\right)^{1/2}
\]

Once the fluid properties, tube diameter and \( \nu \) have been specified, 3-45 will permit calculation of the dimensionless film thickness \( \varepsilon \) from which \( Eo_{\text{eff}} \) may be calculated. Using 3-44 to calculate \( Fr_{\text{eff}} \), equation 3-40 then yields the corresponding \( Fr \). Equation 3-43 may be cast into a more general form for viscous liquids as

\[
Fr = 0.352 (R_N)_{1/2} \left[1 - \frac{3.18}{EoR_N^2} - \frac{14.77}{EoR_N^4}\right] \quad (3-46)
\]
where $R_N$ is the normalized effective radius,

$$ R_N = \frac{R_{\text{eff}}}{R} = 1 - \nu \varepsilon. $$

It is now postulated that $\nu$ is of the form

$$ \nu = a_1 N_p^{a_2}. $$

The coefficients $a_1$, $a_2$ are determined empirically by applying 3-45 and 3-46 to the data of Table 3-3 for Tellus Oil at $E_0 = 30$ and $E_0 = 100$. This procedure yields

$$ \nu = 6.40 N_p^{-.60}. \quad (3-47) $$

The minimum $N_p$ for which the above theory applies is then $N_p = 22$ (i.e., $\nu = 1$).

3.4.3. Application of the Viscous Theory

Applying this analysis to several of the liquids of Table 3-3 produces the results of Figure 3-19, which compare favorably with the experimental data of White and Beardmore (1962), reproduced in Figure 3-18, for Eotvos numbers as low as 8 for the low viscosity liquids. As intuitively expected, the Eotvos number below which good agreement is obtained increases with viscosity. The solid bullets on Figure 3-19 denote the value of $E_0$ at which $N_p = 22$ for each liquid if this value exceeds the minimum $E_0$ at which $Fr = 0$. It can be seen that agreement is very good for $E_0$ above each bullet but begins to deteriorate at lower values of $E_0$. 
Values of $R_N$ have been calculated for a variety of liquid properties, and the results are given in Figure 3-21 as a function of $E_0$. If the liquid properties and tube radius are given, Figure 3-21 and 3-46 may be easily utilized to predict the bubble velocity.

As an example, consider a 1.27 cm diameter tube and a liquid of 58% sucrose solution. From the data of Table 3-3, one may calculate

$$E_0 = 26.5 \quad N_p = 50$$

so that, from Figure 3-20, $R_N = .875$ and from 3-47,

$$F_r = .352(.875)^{1/2}[1 - \frac{3.18}{26.5(.875)^2} - \frac{14.77}{(26.5)^2(.875)^4}] = .265.$$  

Then

$$U_s = .265 \sqrt{(981)(1.27)} = 9.35 \text{ cm/sec.}$$

A computer program to solve equations 3-45 through 3-47 for viscous liquids is provided in Appendix F.2.

3.5. Bubble Shape

The shape of the bubble interface is determined by the condition that $\psi = 0$ on the bubble surface. Therefore, from 3-4,

$$\frac{1}{2} r^2 - r B_{10}(r,z) = 0$$

or

$$2B_{10}(r,z) = r.$$
From the definition of $B_{10}$,

$$\sum_{n=1}^{N} d_n J_1(\delta_n r) e^{-\delta_n z(r)} = r \quad (3-48)$$

and once the coefficients $d_n$ are determined, the shape function $z(r)$ may be found from the solution of 3-48. It is most convenient to choose a radius $r_s$ and solve 3-48 for $z(r_s)$ by application of Newton's Method.

3.5.1. Bubble Shape for Negligible Surface Tension ($\Lambda = 0$)

For the case of $\Lambda = 0$, Figures 3-1 and 3-2 summarize the slug velocity and radius of curvature for varying degrees of approximation.

The corresponding bubble shape for the same cases as Figures 3-1 and 3-2 are shown in Figure 3-22. Figure 3-22 was generated by solving 3-48 for the appropriate set of coefficients $d_n$.

The higher order approximations yield a bubble shape nearly identical to the spherical cap for $r \leq 0.5R$. For larger $r$, the $N = 6$, $ND = 12$ solution is nearly identical to the spherical cap over the range for which a solution to 3-48 exists ($r \leq 0.7R$).

3.5.2. Bubble Shape for Non-Zero Surface Tension ($\Lambda > 0$)

Figure 3-23 shows the bubble shape for air/water in a 1 inch diameter tube for increasing order of approximation. The shapes are compared to the limiting value of the normalized radius of curvature, $R_c = 0.71$, for a 2.54 cm diameter tube (Figure 3-16).
The results are consistent with Figure 3-22. The higher order approximations agree very well with the spherical cap for \( r \leq 0.6R \). The \( N = 6, ND = 12 \) solution is nearly identical to the spherical cap for the entire range of solution.

3.5.3. Spherical Cap Approximation to Bubble Shape

Figures 3-22 and 3-23 strongly support the experimental observation that the nose of the bubble can be very closely approximated by a spherical cap. It can, in fact, be shown that if the surface tension has no effect on the slug velocity, then the bubble surface must of necessity be spherical in the vicinity of the nose. The following discussion demonstrates this fact.

From Table C-1, the terms involving surface tension are

\[
(k_2)_0 = -[4z_2^3 - \frac{4z_4^2}{3}]_0
\]

\[
(k_4)_0 = -[36z_2^2z_4 - 54z_2^5 - \frac{6}{5}z_6^2]_0
\]

\[
(k_6)_0 = -[72z_2^2z_6 + 240z_2z_4^2 + 1800z_2]_0
\]

\[
- 1800z_2^4z_4 - \frac{8}{7}z_8]_0
\]

and so forth.
If the bubble slip velocity is indeed independent of surface tension, then one must require that

\[(K_2)_0 = 0 , \quad (K_4)_0 = 0 , \quad (K_6)_0 = 0 , \quad \ldots\]

so that 3-49 gives

\[(Z_4)_0 = 3(Z_2)_0^3\]
\[(Z_6)_0 = 45(Z_2)_0^5\] \hspace{1cm} (3-50)
\[(Z_8)_0 = 1575(Z_2)_0^7 .\]

Similar equations may be obtained for the higher order derivatives.

The radius of curvature at the nose, \(R_c\), is given by

\[(Z_2)_0 = -\frac{1}{R_c}\] \hspace{1cm} (3-51)

so that the Taylor Series expansion at the nose

\[Z(r) = \frac{(Z_2)_0}{2} r^2 + \frac{(Z_4)_0}{24} r^4 + \frac{(Z_6)_0}{720} r^6 + \ldots\]

becomes, after substituting 3-50 and 3-51,

\[Z(r) = -R_c \left[ \frac{1}{2} \left( \frac{r}{R_c} \right)^2 + \frac{1}{8} \left( \frac{r}{R_c} \right)^4 + \frac{1}{16} \left( \frac{r}{R_c} \right)^6 + \frac{5}{128} \left( \frac{r}{R_c} \right)^8 + \ldots \right] \] \hspace{1cm} (3-52)

Utilizing the series expansion

\[ (1 - x)^{1/2} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \ldots \]
Equation 3-52 becomes

\[
Z(r) = -R_c \left[ (1 - \left( \frac{r}{R_c} \right)^2 \right]^{1/2} - 1
\]

which after some manipulation becomes

\[
r^2 + (R_c - Z(r))^2 = R_c^2.
\]  

Equation 3-53 is recognized as the equation of a circle with center on the z-axis a distance \( R_c \) below the bubble nose.

Therefore, if the bubble slip velocity in a stagnant column is independent of the surface tension, the bubble interface must be hemispherical so long as potential flow is valid. Conversely, if the slip velocity is observed to be a function of surface tension, then of necessity at least one of the equations 3-49 must be nonzero and the bubble interface cannot be hemispherical. As the deviation of 3-49 from zero increases, the interface deviates more and more from a perfect hemisphere.

One may then conclude that for \( Eo \gtrsim 100 \) the spherical cap approximation should be very good but is not valid for small \( Eo (Eo \lesssim 70) \) where surface tension begins to dominate the bubble motion.

3.6. Summary and Conclusions for Vertical Tube Flow

This chapter demonstrates that potential flow theory can adequately model the slip velocity and radius of curvature of a gas slug rising in a closed vertical tube filled with a stagnant liquid of moderate viscosity.
For large $Eo (Eo > 100)$ and negligible liquid viscosity, the solution converges to $Fr = 0.352$, $Rc/R = 0.762$ as the number of terms $N$ and the shape approximation $ND$ increases (Figures 3-1, 3-2 and 3-17). These results are in excellent agreement with observed values for gas/liquid systems where viscous effects are negligible. Table 3-5 summarizes data reported in the literature.

For $Eo > 100$, the $ND=2N$ solution diverges from the observed values as $Eo$ decreases (Figure 3-8); however, the degree of divergence decreases as $N$ increases, suggesting that the solution is converging toward the actual observed data and would converge exactly as more terms are included in the series approximation.

The solution for $N=7$, $ND=14$ was attempted without success. The equations were derived and programmed, but no solution could be found consistent with the previous solutions for lower $N$ (e.g., Table 3-2). It is postulated that truncation and round off errors become very severe for $N>7$ because of two effects.

Firstly, as $N$ increases the equations become algebraically more complex so that more arithmetic operations are required to evaluate the derivatives in the Newton-Raphson numerical procedure. This would be conducive to an increasing error in the Jacobian Matrix and thus make any solution obtained suspect.

Secondly, as $N$ increases, higher order velocity and shape derivatives are required ($N=7$ requires spatial derivatives of order 14) and the magnitude of the derivatives increase rapidly. For example, for $N=7$,
\(Z_{14}^0 = 0(10^{20})\). In double precision, the IBM 3033 computer has a maximum precision of only sixteen digits so that complex equations involving numbers with more than sixteen significant digits may not be evaluated with sufficient precision. The situation is made even more severe as \(E_o\) decreases below a value of \(E_o = 100\) because such values of \(E_o\) require decreasing tube diameters thereby further increasing the magnitude of the derivatives. Table 3-4 illustrates these effects for \(N = 6, \ ND = 12\).

The radius of curvature at the bubble nose closely approximates a spherical cap as \(N\) and \(ND\) increase (Figures 3-20 and 3-21). Little data on the radius of curvature is reported in the literature but what is available is consistent with \(R_c / R = 0.762\) for \(E_o > 100\) (Table 3-5). For \(E_o < 100\), the predicted \(R_c / R\) varies greatly depending upon \(N\) and \(ND\). The behavior predicted by the \(N = 5, \ ND = 6\) solution would be expected to best represent the actual variation of \(R_c / R\) with decreasing tube diameter based on the ability of this solution to adequately predict the slip velocity (Figure 3-6). For higher approximations, truncation and round off errors as discussed above are postulated to introduce errors into the solution.

The effects of moderate liquid viscosity may be approximated by introducing an effective tube radius, equation 3-35. Using the effective radius concept and the \(N=5, \ ND=6\) solution, good results are obtained for liquid property numbers \(N_p > 22\) (Figure 3-19). For this range of properties, a viscous fluid in a tube of radius \(R\) may be approximated as an ideal fluid in a tube of radius \(R_{eff} = R(1 - \nu \epsilon)\) for the purpose of calculating gas slip velocity.
### TABLE 3-1

Coefficients* of Stokes Stream Function for Slug Flow in a Vertical Tube with Negligible Surface Tension

<table>
<thead>
<tr>
<th>n</th>
<th>1/2</th>
<th>2/4</th>
<th>3/6</th>
<th>4/8</th>
<th>5/10</th>
<th>6/12</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1.3159</td>
<td>.6622</td>
<td>.8137</td>
<td>.8026</td>
<td>.8112</td>
</tr>
<tr>
<td>2</td>
<td>.3159</td>
<td>.2111</td>
<td>.1459</td>
<td>.1551</td>
<td>.1408</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>.1267</td>
<td>.0336</td>
<td>.0359</td>
<td>.0461</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>.0068</td>
<td>.0061</td>
<td>.0004</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td>.0002</td>
<td>.0017</td>
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</tr>
<tr>
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<td></td>
<td></td>
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</tr>
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<td>.289</td>
<td>.335</td>
<td>.353</td>
<td>.352</td>
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</table>

* Tabulated values are \( \frac{d_n k_n}{R} \)
TABLE 3-2

Coefficients* of Stokes Stream Function for Slug Flow in a 2.54 cm Diameter Vertical Tube for Air/Water

\[ \text{Eo} = 88 \]

<table>
<thead>
<tr>
<th>n</th>
<th>1/2</th>
<th>2/4</th>
<th>3/6</th>
<th>4/8</th>
<th>5/10</th>
<th>6/12</th>
</tr>
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<td>.0836</td>
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<td>.2144</td>
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</tbody>
</table>

* Tabulated values are \( \frac{d_n k_n}{R} \)

** No compatible solution could be found for \( N = 2, \ ND = 4 \).
TABLE 3-3
Details of Fluids Used by White and Beardmore (1962)

<table>
<thead>
<tr>
<th>Fluid</th>
<th>Temp. (°C)</th>
<th>ρ (g/ml)</th>
<th>μ (cP)</th>
<th>σ (dyne/cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distilled water</td>
<td>26</td>
<td>0.997</td>
<td>0.87</td>
<td></td>
</tr>
<tr>
<td>40% sucrose soln.</td>
<td>26</td>
<td>1.172</td>
<td>5.65</td>
<td></td>
</tr>
<tr>
<td>58% sucrose soln.</td>
<td>25</td>
<td>1.272</td>
<td>40.50</td>
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</tr>
<tr>
<td>ethylene glycol</td>
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<td>1.113</td>
<td>19.9</td>
<td></td>
</tr>
<tr>
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<td>0.803</td>
<td>1.385</td>
<td></td>
</tr>
<tr>
<td>Tellus oil</td>
<td>25</td>
<td>0.864</td>
<td>52.0</td>
<td></td>
</tr>
<tr>
<td>Voluta oil</td>
<td>25</td>
<td>0.902</td>
<td>294</td>
<td></td>
</tr>
<tr>
<td>glycerol</td>
<td>24</td>
<td>1.260</td>
<td>712</td>
<td></td>
</tr>
<tr>
<td>90% glycerol soln.</td>
<td>24</td>
<td>1.234</td>
<td>154</td>
<td></td>
</tr>
<tr>
<td>95% glycerol soln.</td>
<td>25</td>
<td>1.246</td>
<td>323</td>
<td></td>
</tr>
<tr>
<td>sugar syrup</td>
<td>27</td>
<td>1.42</td>
<td>20,900</td>
<td></td>
</tr>
<tr>
<td>diluted sugar syrup</td>
<td>27</td>
<td>1.40</td>
<td>2650</td>
<td></td>
</tr>
<tr>
<td>rediluted sugar syrup</td>
<td>26</td>
<td>1.39</td>
<td>1610</td>
<td></td>
</tr>
</tbody>
</table>
TABLE 3-4

Variation of Spatial Derivatives for N = 6/ND = 12 Solution for Air/Water Tube Diameter (cm)/Eo

<table>
<thead>
<tr>
<th>Tube Diameter (cm)/Eo</th>
<th>2.54/88</th>
<th>1.91/49</th>
<th>1.27/22</th>
</tr>
</thead>
<tbody>
<tr>
<td>($Z_2^0$)</td>
<td>-.28(01)*</td>
<td>-.40(01)</td>
<td>-.64(01)</td>
</tr>
<tr>
<td>($Z_4^0$)</td>
<td>-.66(02)</td>
<td>-.20(03)</td>
<td>-.82(03)</td>
</tr>
<tr>
<td>($Z_6^0$)</td>
<td>-.65(04)</td>
<td>-.45(05)</td>
<td>-.47(06)</td>
</tr>
<tr>
<td>($Z_8^0$)</td>
<td>-.12(07)</td>
<td>-.20(08)</td>
<td>-.56(09)</td>
</tr>
<tr>
<td>($Z_{10}^0$)</td>
<td>-.23(09)</td>
<td>-.13(11)</td>
<td>-.10(13)</td>
</tr>
<tr>
<td>($Z_{12}^0$)</td>
<td>.28(10)</td>
<td>-.90(13)</td>
<td>-.20(16)</td>
</tr>
</tbody>
</table>

* The number in parentheses denotes the exponent of ten.
Thus a(b) represents $a \times 10^b$. 
<table>
<thead>
<tr>
<th>Author</th>
<th>Diameter (cm)</th>
<th><em>Eo</em></th>
<th>Fr</th>
<th>( \frac{R_c}{R} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>O'Brien and Gosline (1935)</td>
<td>1.18</td>
<td>122</td>
<td>.337</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.24</td>
<td>441</td>
<td>.337</td>
<td></td>
</tr>
<tr>
<td>Dumitriscu (1943)</td>
<td>.99</td>
<td>13</td>
<td>.202</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.00</td>
<td>55</td>
<td>.335</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3.76</td>
<td>193</td>
<td>.350</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.00</td>
<td>670</td>
<td>.347</td>
<td>.75</td>
</tr>
<tr>
<td>Davies and Taylor (1949)</td>
<td>.49</td>
<td>21</td>
<td>.287</td>
<td></td>
</tr>
<tr>
<td></td>
<td>.85</td>
<td>64</td>
<td>.310</td>
<td></td>
</tr>
<tr>
<td>Laird and Chisholm (1956)</td>
<td>2.00</td>
<td>352</td>
<td>.341</td>
<td></td>
</tr>
<tr>
<td>Griffith and Wallis</td>
<td>.50</td>
<td>22</td>
<td>.304</td>
<td></td>
</tr>
<tr>
<td></td>
<td>.75</td>
<td>49</td>
<td>.340</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>88</td>
<td>.350</td>
<td></td>
</tr>
<tr>
<td>White and Beardmore (1961)</td>
<td>1.00</td>
<td>88</td>
<td>.335</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.52</td>
<td>203</td>
<td>.342</td>
<td></td>
</tr>
<tr>
<td>Nicklin, et al. (1962)</td>
<td>1.00</td>
<td>88</td>
<td>.347</td>
<td></td>
</tr>
<tr>
<td>Ormiston, et al.</td>
<td>1.00</td>
<td>88</td>
<td>.350</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.25</td>
<td>445</td>
<td>.354</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5.50</td>
<td>2660</td>
<td>.352</td>
<td></td>
</tr>
<tr>
<td>Brown (1965)</td>
<td>1.038</td>
<td>95</td>
<td>.346</td>
<td>.75</td>
</tr>
<tr>
<td>Collins, et al. (1978)</td>
<td>2.024</td>
<td>360</td>
<td>.347</td>
<td>.71</td>
</tr>
</tbody>
</table>

* *Eo calculated assuming \( \rho = 1 \text{ gm/ml}, \sigma = 72 \text{ dyne/cm} \).
FIGURE 3-1
SLUG VELOCITY IN VERTICAL TUBE
DIAM = 2.54 CM  EO = 1257

Legend
△ FR = .352
× N = 1
□ N = 3
■ N = 4
□ N = 5
× N = 6

TERMS IN SHAPE APPROXIMATION - ND
FIGURE 3-2
FLOW IN VERTICAL TUBE
RADIUS OF CURVATURE
DIAM = 2.54 CM  EO = 1257
FIGURE 3-3
FLOW IN VERTICAL TUBE
AIR/WATER
N = 1

LEGEND

\[ \text{EXP CORRELATION} \]

\[ \text{ND = 4} \]

\[ \text{ND = 2} \]
FIGURE 3-4
FLOW IN VERTICAL TUBE
AIR/WATER
N = 3

Legend
△ EXP CORRELATION
× ND = 6
□ ND = 4
Figure 3-5
Flow in Vertical Tube
Air/Water
N = 4

Legend
\(\Delta\) EXP CORRELATION
\(\times\) ND = 8

Froude Number

Eotvos Number

0.5
0.4
0.3
0.2
0.1
0.0
0 10 20 30 40 50 60 70 80 90 100
FIGURE 3-6
FLOW IN VERTICAL TUBE
AIR/WATER
N = 5

Legend

△ EXP CORRELATION
× ND = 10
□ ND = 6
※ ND = 4
FIGURE 3-7
FLOW IN VERTICAL TUBE
AIR/WATER
N = 6
FIGURE 3-8
FLOW IN VERTICAL TUBE
AIR/WATER

FROUDE NUMBER

EOTVOS NUMBER

Legend
△ EXP CORRELATION
× N=6 ND=12
□ N=5 ND=10
★ N=4 ND=8
☆ N=3 ND=6
× N=1 ND=2
FROUDE NUMBER

FIGURE 3-9
FLOW IN VERTICAL TUBE
SLUG VELOCITY
AIR/WATER
DIAM = 2.54 CM  EO = 88

Legend
△ FR = .346
× N = 1
□ N = 3
♦ N = 4
☒ N = 5
× N = 6

TERMS IN SHAPE APPROXIMATION - ND

FROUDE NUMBER

0.40

0.38

0.36

0.34

0.32

0.30

0 2 4 6 8 10 12 14
TERMS IN SHAPE APPROXIMATION - ND
FIGURE 3-10
FLOW IN VERTICAL TUBE
NORMALIZED RADIUS OF CURVATURE
AIR/WATER    N = 1

Legend
△ ND = 4
× ND = 2
FIGURE 3-11
FLOW IN VERTICAL TUBE
NORMALIZED RADIUS OF CURVATURE
AIR/WATER N = 3

Legend
△ ND = 6
× ND = 4
FIGURE 3-12
FLOW IN VERTICAL TUBE
NORMALIZED RADIUS OF CURVATURE
AIR/WATER  N = 4

Legend
△ ND = 8
FIGURE 3-13
FLOW IN VERTICAL TUBE
NORMALIZED RADIUS OF CURVATURE
AIR/WATER  N = 5

Legend
△ ND = 10
× ND = 6
□ ND = 4
FIGURE 3-14
FLOW IN VERTICAL TUBE
NORMALIZED RADIUS OF CURVATURE
AIR/WATER N = 6

Legend
△ ND = 12
× ND = 10
□ ND = 8
★ ND = 6
Figure 3-15
Flow in Vertical Tube
Normalized Radius of Curvature
Air/Water

Legend
△ N=6 ND=12
× N=5 ND=10
□ N=4 ND=8
■ N=3 ND=6
✦ N=1 ND=2

Normalized Radius of Curvature - RC/R
EOTVOS Number
Figure 3-16
Flow in Vertical Tube
Normalized Radius of Curvature
Air/Water
Diam = 2.54 cm  EO = 88

Legend
△ RC = .71
× N = 1
□ N = 3
★ N = 4
★★ N = 5
★ ★ N = 6

Normalized Radius of Curvature - RC/R

Terms in Shape Approximation - ND
FIGURE 3-17
FLOW IN VERTICAL TUBE
AIR/WATER
N = 6  ND = 12

Legend
△ FROUDE NUMBER
X NORMALIZED RC

RC = .762
FR = .352
Figure 3-18 Results for cylindrical air bubbles rising in vertical tubes (White and Beardmore, 1962)
FIGURE 3-19
SLUG FLOW IN VERTICAL TUBE
VISCOUS LIQUID APPROXIMATION

NOTE:
SEE TABLE 3-3
FOR FLUID PROPERTIES

Legend

Legend

FROUDE NUMBER

EOTVOS NUMBER

WATER
AQUEOUS ETHANOL
40 0/0 SUCROSE
ETHYLENE GLYCOL
58 0/0 SUCROSE
TELLUS OIL
90 0/0 GLYCEROL
95 0/0 GLYCEROL
VOLUTA OIL
NP = 22
FIGURE 3-20
NORMALIZED EFFECTIVE RADIUS

Legend
△ EO = 1000
× EO = 100
□ EO = 50
☒ EO = 30
☒ EO = 20
☒ EO = 15
☒ EO = 10
☒ EO = 8

Normalized Effective Radius - RN

[Normalized Effective Radius - RN]

NP

Legend
△ EO = 1000
× EO = 100
□ EO = 50
☒ EO = 30
☒ EO = 20
☒ EO = 15
☒ EO = 10
☒ EO = 8

Normalized Effective Radius - RN

NP
FIGURE 3-21
SLUG FLOW IN VISCOUS LIQUIDS

Legend
△ NP = 10000
× NP = 1000
□ NP = 100
◆ NP = 30
● NP = 20
Figure 3-22
Bubble shape $Z(r)$ for negligible surface tension
$R = 1.27 \text{ cm}$  $\varepsilon_0 = 1257$

Legend
- $\triangle$ Spherical cap
- $\times$ $N=1$  $\text{ND}=2$
- $\Box$ $N=3$  $\text{ND}=6$
- $\Join$ $N=4$  $\text{ND}=8$
- $\widthhat{XX}$ $N=5$  $\text{ND}=10$
- $\amalg$ $N=6$  $\text{ND}=12$

Note:
Spherical cap for $R_c = 0.76R$
FIGURE 3-23
BUBBLE SHAPE $Z(r)$
FOR AIR/WATER
$R = 1.27 \text{ cm} \quad EO = 88$

NOTE:
SPHERICAL CAP
FOR $RC = .71R$

Legend
- △ SPHERICAL CAP
- × $N=1 \ ND = 2$
- □ $N=3 \ ND = 6$
- ◯ $N=4 \ ND = 8$
- ✖ $N=5 \ ND = 10$
- ✗ $N=6 \ ND = 12$
CHAPTER 4

Slug Flow in Vertical Rectangular Channels

This chapter will apply the potential flow theory, including surface
tension effects, to the case of gas slug flow in a vertical rectangular
channel. Both two-dimensional and three-dimensional slots will be con­sidered. In all cases throughout this chapter, the Froude number will
be calculated from the equiperipheral diameter $D_{eq}$ (equation 2-25).

4.1. Basic Theory

Consider a three-dimensional vertical duct of rectangular cross section
with length $2L$ and width $2W$. From Chapter 2, recall that the velocity
potential is of the general form

$$\Phi(x, y, z) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} F_{nm}(x, y) e^{-\alpha_{nm} z} - U_s z + a_0 + a_1 x + a_2 y + a_3 xy$$  \hspace{1cm} (4-1)

where

$$F_{nm}(x, y) \equiv C_1 \left[ A_{1n} \cos(\beta_n x) + A_{2n} \sin(\beta_n x) \right] \cdot$$

$$\left[ B_{1m} \cos(\gamma_m y) + B_{2m} \sin(\gamma_m y) \right]$$

and

$$\alpha_{nm} = (\beta_n^2 + \gamma_m^2)^{1/2},$$
subject to the boundary conditions

\[ \frac{\partial \Phi}{\partial x}(\pm L, y, z) = 0 \]

\[ \frac{\partial \Phi}{\partial y}(x, \pm W, z) = 0 , \]

where \( \frac{\partial \Phi}{\partial x} \) and \( \frac{\partial \Phi}{\partial y} \) are the velocity components \( u \) and \( v \) in the \( x \) and \( y \) directions, respectively. The origin of the Cartesian coordinate system is placed at the bubble nose.

By symmetry of the rectangular channel, one must require that \( a_1 = 0, a_2 = 0, \) and \( a_3 = 0 \) so that (4-1) reduces to

\[ \Phi(x,y,z) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} F_{nm}(x,y)e^{-n^2 m^2} - U z + a_0 \].

Similarly, symmetry requires that \( A_{2n} = 0 = B_{2m} \) for all \( n,m \) so that one has

\[ F_{nm}(x,y) = C'_{nm} \cos(\beta_n x) \cos(\gamma_m y) \]

where \( C'_{nm} \equiv C_{1,n} A_{1,m} B_{1,m} \). Thus, one may write finally that

\[ \Phi(x,y,z) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} C'_{nm} \cos(\beta_n x) \cos(\gamma_m y)e^{-n^2 m^2} - U z + a_0 \]. (4-3)
Differentiating 4-3, one obtains

\[ u = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \beta_n c' \sin(\beta_n x) \cos(\gamma_m y) e^{-\alpha_{nm} z} \]

\[ v = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \gamma_m c' \cos(\beta_n x) \sin(\gamma_m y) e^{-\alpha_{nm} z} \]

Application of boundary condition 2) of 2-3 requires that

\[ \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \beta_n c' \sin(\beta_n x) \cos(\gamma_m y) = 0 \]

\[ \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \gamma_m c' \cos(\beta_n x) \sin(\gamma_m y) = 0 \]

In addition, since each \( F_{nm} \) is a solution of the governing equation, one must require that

\[ \beta_n c' \sin(\beta_n x) \cos(\gamma_m y) = 0 \]

\[ \gamma_m c' \cos(\beta_n x) \sin(\gamma_m y) = 0 \]

Therefore,

\[ \sin(\beta_n x) = 0 \]

\[ \sin(\gamma_m y) = 0 \]

and

\[ \beta_n x = n\pi, \quad \gamma_m y = m\pi \]
so that

$$\beta_n = \left(\frac{n\pi}{L}\right), \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\gamma_m = \left(\frac{m\pi}{W}\right), \quad m = 0, \pm 1, \pm 2, \ldots$$

Defining $\beta \equiv \pi/L$ and $\gamma \equiv \pi/W$, Eq. 4-3 then becomes

$$\Phi(x, y, z) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} C_{nm}^1 \cos(n\beta x) \cos(m\gamma y) e^{-\alpha nm z} - \frac{a_0}{s} z.$$ 

Noting that $\cos(\theta) = \cos(-\theta)$, one may define $C_{nm} \equiv 2C_{nm}^1 / U$ to finally obtain

$$\Phi(x, y, z) = U \left[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{nm} \cos(n\beta x) \cos(m\gamma y) e^{-\alpha nm z} - z \right] \quad (4-4)$$

where the constant $a_0$ has been omitted because only the derivatives of $\Phi$ will be required.

Differentiating 4-4, one obtains

$$u(x, y, z) = -U \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} n \beta C_{nm} \sin(n\beta x) \cos(m\gamma y) e^{-\alpha nm z}$$

$$v(x, y, z) = -U \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} m \gamma C_{nm} \cos(n\beta x) \sin(m\gamma y) e^{-\alpha nm z} \quad (4-5)$$

$$w(x, y, z) = -U \left[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \alpha C_{nm} \cos(n\beta x) \cos(m\beta y) e^{-\alpha nm z} + 1 \right].$$
In order to obtain a solution, the coefficients \( C_{nm} \) and the slip velocity \( U_s \) must be determined. From the equations derived in Chapter 2 and Appendix B, the derivatives of the velocity components, evaluated at the stagnation point, are required.

Because of the symmetry of the potential function in a rectangular duct, it can be shown that the non-zero derivatives are

\[
(u_{ijk})_0 = (-1)^{\frac{i+j+2k+1}{2}} U_s \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{nm} (n\beta)^i (m\gamma)^j \alpha^k_{nm} \\
(v_{ijk})_0 = (-1)^{\frac{i+j+2k+1}{2}} U_s \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{nm} (n\beta)^i (m\gamma)^j \alpha^k_{nm} \\
(w_{ijk})_0 = (-1)^{\frac{i+j+2k+2}{2}} U_s \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{nm} (n\beta)^i (m\gamma)^j \alpha^{k+1}_{nm} 
\]

Derivatives of the form

\[
(u_{ijk})_0 , \text{ i even} \\
(v_{ijk})_0 , \text{ j even} \\
(w_{ijk})_0 , \text{ i or j odd}
\]

are identically zero.

Although the solution for the velocity potential consists of an infinite series, the exact solution may be approximated by truncating the series
after a finite number of terms. For consistency, terms of the same order of magnitude must be included in any finite expansion.

It can readily be seen that the total number of terms in an \( N \)th order expansion is \( N(N+3)/2 \). Since the slip velocity \( U_s \) is also an unknown, an \( N \)th order expansion will therefore require

\[
\frac{N(N+3)}{2} + 1 = \frac{(N+1)(N+2)}{2}
\]
equations.

The required equations for a rectangular duct are derived in Appendix B. The equations for the derivatives of the bubble shape are provided in Table B-2 where

\[
Z_{mn} \equiv \frac{\partial^{(m+n)}Z}{\partial x^m \partial y^n}.
\]

The bubble shape in the vicinity of the bubble nose may be approximated by the Taylor Series expansion

\[
Z(x,y) = \left[ (Z_{20})_0 x^2 + (Z_{02})_0 y^2 \right] / 2
\]

\[
+ \left[ (Z_{40})_0 x^4 + 6(Z_{22})_0 x^2 y^2 + (Z_{04})_0 y^4 \right] / 24
\]

\[
+ \left[ (Z_{60})_0 x^6 + 15(Z_{42})_0 x^4 y^2 + 15(Z_{24})_0 x^2 y^4 + (Z_{06})_0 y^6 \right] / 720
\]

\[
+ \ldots
\]

(4-7)

From Table B-2, one sees that the fourth order derivatives are functions of the second order derivatives, the sixth order derivatives are func-
tions of the fourth order and second order, and so forth. One could therefore in principle express all the higher order derivatives in terms of the second order derivatives and the velocity derivatives. As more terms are included in the expansion approximation, 4-4, it is clear that the accuracy of the higher order derivatives is dependent upon the accuracy of the values for \((Z_{20}^0)\) and \((Z_{02}^0)\).

4.2. Criteria for Acceptable Solutions

Because of the algebraic complexity of the equations to be solved, it is possible to obtain multiple solutions. The criteria used to determine the acceptance of a given mathematical solution are as follows:

1. As more terms are included in the series approximation 4-4, the coefficients \(C_{nm}\) should exhibit a relatively smooth transition from the \(N\) order solution to the \(N + 1\) order solution. That is, including the \((N + 1)\) order coefficients should not significantly affect the \(N\) order coefficients.

2. As \(n\) and \(m\) are increased, the higher order coefficients should exhibit a generally decreasing absolute value. That is, the higher order coefficients should have a smaller contribution to the total series solution.

3. If surface tension is included, the solution for a given order of approximation should approach asymptotically the zero surface tension solution as the channel dimensions are increased. That is, the effect of surface tension should decrease asymptotically as the channel dimensions are increased.
4.3. Two-Dimensional Rectangular Slot

Before considering the full three-dimensional rectangular channel, first consider the two-dimensional case of a slot of length $2L$ and infinite width $W = \infty$.

For this simplification, $\gamma = 0$, $v = 0$, and

$$\varphi(x, z) = U_s \left[ \sum_{n=0}^{\infty} C_n \cos(n\beta x) e^{-n\beta z} - z \right]. \quad (4-8)$$

The velocity derivatives are given by 4-6 for $j \equiv 0$. The derivatives of the functions $T_1$ and $T_2$ may be found from either Appendix B, for $v \equiv 0$, or from Appendix C by substituting $w$ for $u$ and $u$ for $v$. The derivatives of the function $K$ may be found from either Appendix B, by setting all the cross derivatives of the shape function to zero, or from Appendix C by the relation

$$\left( K_m^0 \right) = \left( \frac{m + 1}{m + 2} \right) \left( K_m^0 \right). \quad (4-9)$$

4.3.1. Slip Velocity $U_s$ for Negligible Surface Tension ($\Lambda = 0$)

To illustrate the solution procedure, consider the simplest case: a first order expansion, $N = 1$, with negligible surface tension, $\Lambda = 0$.

For this case, two equations are required to solve for the unknowns $C_1$ and $U_s$. The first equation is simply the condition that $\omega_0 = 0$. Thus from 4-5,

$$\omega_0 = 0 = -U_s \left[ \sum_{n=0}^{1} C_n n\beta + 1 \right]$$
or,

\[ C_{10} \beta + 1 = 0 \]

and

\[ C_{10} = -1/\beta. \quad (4-10) \]

From Tables B-2 and B-3, one obtains for \( \Lambda = 0 \),

\[ [u_{100}^2 + gZ_{20}]_0 = 0 \quad (4-11) \]

where

\[ (Z_{20})_0 = \left( \frac{\omega_{200}}{2u_{100} - \omega_{001} 0} \right). \quad (4-12) \]

From 4-6, one obtains for \( \gamma = 0 \)

\[ (u_{100})_0 = -u_s C_{10} \beta^2 = u_s \beta \]

\[ (\omega_{001})_0 = u_s C_{10} \beta^2 = -u_s \beta \]

\[ (\omega_{200})_0 = u_s C_{10} \beta^3 = -u_s \beta^2 \]

so that 4-12 becomes

\[ (Z_{20})_0 = -\frac{\beta}{3} \]

and 4-11 becomes

\[ u_s^2 \beta^2 - \frac{g\beta}{3} = 0 \]
which yields

\[ U_s = \left( \frac{L}{3 \beta} \right)^{1/2} \]

Since \( \beta = \pi / L \) and \( D_{eq} = 4L/\pi \) for a two-dimensional slot, then

\[ U_s = \left( \frac{L}{3\pi} \right)^{1/2} \]

\[ = 0.289 \sqrt{D_{eq}} \]

or

\[ Fr = \frac{U_s}{\sqrt{gD_{eq}}} = 0.289 \quad (4-13) \]

Equation 4-13 is identical to theoretical results obtained by Collins (1965) and is consistent with Garabedian's (1957) estimate of

\[ Fr = 0.299, \] and Birkhoff and Carter's (1955) value of 0.282.

For higher order approximations, \( N > 1 \), a Newton-Raphson procedure was utilized to solve the appropriate equation set. The results for those solutions satisfying the criteria of Section 4.2, are presented in Figure 4-1. The nomenclature employed is:

\( N \) \( \equiv \) order of approximation to the velocity potential and is equal to the maximum value of \( n \) in equation 4-4.

\( ND \equiv \) order of approximation of the bubble shape and is equal to the highest order derivative in the finite Taylor Series expansion,
equation 4-7, in the neighborhood of the stagnation point. ND will always be even because the odd-ordered derivatives are identically zero at the bubble nose.

4.3.2. Radii of Curvature at Bubble Nose for Negligible Surface Tension $(\Lambda = 0)$

Figure 4-2 presents the radius of curvature at the bubble nose, $R_c = -1/(Z_2)'$, for the cases of Figure 4-1 normalized to the channel half-length $L$.

4.3.3. Slip Velocity $U_s$ for Non-zero Surface Tension $(\Lambda > 0)$

For $\Lambda > 0$, the appropriate equations for $N = 1$ are

$$C_{10} \beta + 1 = 0 \quad (4-14)$$

and

$$\left( u_{100}^2 + g Z_{20} + \Lambda K_2 \right) = 0 \quad (4-15)$$

where

$$K_2 = (Z_{40} - 3 Z_{20})_0 \quad (4-16)$$

$$Z_{20}' = \left( \frac{w_{200}}{2 u_{100} - \omega_{001}} \right) \quad (4-17)$$

$$Z_{40}' = \left( \frac{w_{400} + (6 \omega_{201} - 4 u_{300}) Z_{20} + (3 \omega_{002} - 12 u_{101}) Z_{20}^2}{4 u_{100} - \omega_{001}} \right)_0 . \quad (4-18)$$
Thus $C_{10} = -1/\beta$ and from 4-6, one finds for $\gamma = 0$

\[
\begin{align*}
(u_{100})_0 &= U_s \beta^3 \\
(u_{300})_0 &= -U_s \beta^3 \\
(\omega_{001})_0 &= -U_s \beta \\
(u_{111})_0 &= -U_s \beta^2 \\
(\omega_{200})_0 &= -U_s \beta^2 \\
(\omega_{400})_0 &= U_s \beta^4 \\
(\omega_{201})_0 &= U_s \beta^3 \\
(\omega_{002})_0 &= U_s \beta^2.
\end{align*}
\]

Then 4-17 and 4-18 become

\[
(Z_{20})_0 = -\frac{\beta}{3}
\]

\[
(Z_{40})_0 = \frac{-2\beta^3}{15}.
\]

The value of $(K_2)_0$ will depend on the shape parameter $ND$. If $ND = 2$, then $(Z_{40})_0$ is assumed to be zero and

\[
(K_2)_0 = -3Z_{20}^3 = \frac{\beta^3}{9} \quad \text{ND} = 2.
\]

If $ND = 4$, then

\[
(K_2)_0 = \frac{-2\beta^3}{15} + \frac{\beta^3}{9} \quad \text{ND} = 4
\]

\[
= \frac{-\beta^3}{45}.
\]

Consider now the $ND = 2$ case. Then 4-15 becomes
\[
(U_s \beta)^2 + g(-\frac{\beta}{3}) + \Lambda(\frac{\beta^3}{g}) = 0
\]

which yields

\[
U_s = (\frac{g \beta}{3 \beta})^{1/2} (1 - \frac{\beta^2}{3g} \Lambda)^{1/2}.
\]  \hspace{1cm} (4-19)

From 2-24, the two-dimensional Eotvos Number is

\[
Eo \equiv \frac{(\rho_\ell - \rho_g) g D^2}{\sigma Eq} = \frac{16 g L^2}{\pi^2 \Lambda}
\]

and 4-19 becomes

\[
Fr = .289 \left(1 - \frac{5.33}{Eo}\right)^{1/2}, \hspace{0.5cm} ND = 2.
\]

For ND = 4, similar calculations will yield

\[
Fr = .289 \left(1 + \frac{1.07}{Eo}\right)^{1/2}, \hspace{0.5cm} ND = 4.
\]

The solutions for higher order approximations satisfying the criteria of 4.2, are presented in Figures 4-3 through 4-6 for varying Eotvos numbers.
4.3.4. Radii of Curvature at Bubble Nose for Non-Zero Surface Tension

Figure 4-7 through 4-10 present the radius of curvature at the bubble nose, \( R_c = -1/(Z_2)_0 \), normalized to the channel half-length \( L \), for the cases of Figures 4-3 through 4-6.

4.3.5. Summary and Conclusions for Two-Dimensional Channel

Comparison of Figures 4-1 and 4-2 with Figures 3-1 and 3-2 suggests that the trigonometric series representation for the two-dimensional channel converges slower than the Bessel Function series for the tube solution. For negligible surface tension and viscosity, the tube solution has essentially converged for \( N > 3 \) as shown by the converged values of \( Fr \) and \( R_c/R \). For the rectangular channel, there is no definite convergence even for \( N = 6 \). It appears, however, that the two-dimensional channel is converging to \( Fr = 0.285 \) and \( R_c/L = 0.65 \).

Because the channel as herein described is impossible to construct physically, no measured data is available to compare with the above calculations. However, one may approximate the theoretical geometry by a rectangular slot of very narrow width \( (L/W ~>> 1) \) so that a slug generated in such a geometry may be considered as a slice taken from the theoretical channel of infinite width. Such a geometry is referred to in the literature as a "two-dimensional slot" and the three-dimensional effects are assumed negligible.

Collins (1965) measured the rise velocity of gas bubbles in such a geometry. He utilized a channel width of 0.635 cm and varying length up to \( L = 41.5 \) cm. For slugs, Collins reported measured values of \( Fr = 0.31 \).
and $R_c/L = 0.62$ in channels of $L = 3.8$ cm. (This value of the half-length $L$ is deduced from published photographs and is not reported directly by Collins.) These results are consistent with the theoretical results of $Fr = 0.285$ and $R_c/L = 0.65$, although the calculated velocity is 9 percent lower than measured.

Collins' theoretical analysis, summarized in 1.3., gives $Fr = 0.289$ and $R_c/L = 0.955$ which is equivalent to the $N=1$ solution above. Collins assumed a spherical cap for the bubble in his analysis and attributed the 9 percent discrepancy between his theoretical and measured velocities to a residual three-dimensional effect. Hills (1975) showed that the assumption of an elliptical bubble cap gives predicted velocities only 2 percent lower than measured.

By the same procedure of 3.5.3., it may be shown that in the absence of surface tension effects, the bubble cap must be spherical. Thus Collins' analysis is more applicable for very large channels where the surface tension, although present, is negligible.

Measured data for slots of half-length $L = 3.68$ cm and varying widths (Table 5-1, Figures 5-4 and 5-5) strongly suggest that the slip velocity is independent of surface tension for three-dimensional $Eo > 413$, with $Fr = 0.337$, but decreases rapidly as $Eo$ is decreased below 413. Thus Collins' assumption of negligible surface tension is not valid for his slug flow data ($L = 3.8$ cm, $W = .32$ cm, $Eo = 377$).

The increasing effect of surface tension as $Eo$ is reduced below 413 results in deformation of the actual three-dimensional surface. Thus
Hills' assumption of an elliptical cap in slots of very narrow width simply reflects the effects of surface tension.

When surface tension is included in the two-dimensional solution for a 7.62 cm channel (L = 3.68 cm) for air/water (Eo = 321) there is little effect on the calculated velocity (Table 4.2). As Eo is decreased below 321, the two-dimensional slot solutions (Figures 4-3 through 4-6) behave similar to the tube solutions for N=1 and N=3. There were no N=4 solutions in this range, however, unlike the tube case. For N=5 and N=6, the ND=8 solution gives the best solution, reflecting the expected velocity decrease as Eo decreases.

Like the tube solution, the N,ND = 2N solutions were nearly constant, decreasing slightly at low Eo, suggesting that more terms are required to obtain a truly converged solution for the rectangular case.

The radii of curvature (Figures 4-7 through 4-10) show more erratic behavior, even at large Eo, than for the tube solutions. However, the N=5,ND=10 and N=6,ND=12 solutions are very similar, as they were for the tube.

The erratic behavior of the radii of curvature and the apparently slow convergence of Fr for the two-dimensional channel suggest that the rectangular two-dimensional solution converges more slowly than the tube solution and that more than six terms in the series expansion will be required to obtain the same degree of convergence as in the tube case.

This conclusion does not make the prospect for obtaining a three-dimensional rectangular slot solution look very promising.
4.4. Three-Dimensional Rectangular Slot

This section will summarize the solutions for the three-dimensional slot. Due to the asymmetry of the slot geometry cross-section, cross terms are required in the series solution, 4-4. Thus for an $N^{th}$ order approximation, $N(N+3)/2$ terms are required so that an $N = 6$ solution would require 27 terms in the series expansion.

Because of the increasing complexity of the three-dimensional equations and the observation that the two-dimensional rectangular solution converges slowly, it was not considered practical to attempt a three-dimensional solution approximation to the same degree as the two-dimensional solution.

Accordingly, the highest order approximation considered for the three-dimensional case was $N = 4$, requiring 14 terms in the series approximation to the velocity potential.

4.4.1. Slip Velocity $U_s$ for Negligible Surface Tension ($\Lambda = 0$)

To illustrate the solution procedure, consider first the simplest case: a first order expansion, $N = 1$, with negligible surface tension ($\Lambda = 0$).

For this simplest case,

$$\frac{(1+1)(1+2)}{2} = 3$$

equations are required in order to solve for the three unknowns $C_{01}$, $C_{10}$, and $U_s$. 
The first equation is simply the requirement that the axial velocity be zero at the stagnation point, \( w_0 = 0 \). From 4-5 one then obtains

\[
\omega_0 = -U \sum_{n=0}^{1} \sum_{m=0}^{1} C_{nm} \alpha_n \alpha_m + 1 = 0
\]

or,

\[
C_{01} \alpha_{01} + C_{10} \alpha_{10} + 1 = 0
\] (4-20)

where

\[
\alpha_{01} = (\beta_0^2 + \gamma_1^2)^{1/2}
\]

\[
\alpha_{10} = (\beta_1^2 + \gamma_0^2)^{1/2}
\]

\[
\beta_0 = 0 \quad \beta_1 = \frac{\pi}{L} = \beta
\]

\[
\gamma_0 = 0 \quad \gamma_1 = \frac{\pi}{w} = \gamma
\]

From Tables B-2 and B-3, one finds for \( \Lambda = 0 \)

\[
[u_{100}^2 + g z_{20}^2]_0 = 0
\] (4-21)

\[
[v_{010}^2 + g z_{02}^2]_0 = 0
\]

where

\[
(z_{20})_0 = \left( \frac{w_{200}}{2u_{100} - w_{001}} \right)
\] (4-22)
\[
(z_{02})_0 = \left(\frac{\omega_{020}}{2\nu_{010} - \omega_{001}}\right).
\]  
\hspace{2cm} (4-23)

In addition, from 4-6, one obtains

\[
(u_{100})_0 = -u_s [c_{10} \beta^2]
\]

\[
(v_{010})_0 = -u_s [c_{01} \gamma^2]
\]

\[
(\omega_{001})_0 = u_s [c_{01} \alpha_{01}^2 + c_{10} \alpha_{10}^2]
\]

\[
= u_s [c_{01} \gamma^2 + c_{10} \beta^2]
\]  
\hspace{2cm} (4-24)

\[
(\omega_{200})_0 = u_s [c_{10} \beta^2 \alpha_{10}^2] = u_s [c_{10} \beta^3]
\]

\[
(v_{020})_0 = u_s [c_{01} \gamma^2 \alpha_{01}] = u_s [c_{01} \gamma^3]
\]

Substitution of 4-4 into 4-22 and 4-23 yields

\[
(z_{20})_0 = \left(\frac{-c_{10} \beta^3}{3c_{10} \beta^2 + c_{01} \gamma^2}\right)
\]  
\hspace{2cm} (4-25)

\[
(z_{02})_0 = \left(\frac{-c_{01} \gamma^3}{3c_{01} \gamma^2 + c_{10} \beta^2}\right)
\]
and 4-21 becomes

\[ u^2 c^2 \frac{\beta^4}{s^3 10^6} - \frac{g_{10} \beta^3}{3c_{10} \beta^2 + c_{01}^2} = 0 \]  

(4-26)

\[ u^2 c^2 \frac{\gamma^4}{s^3 01^6} - \frac{g_{01} \gamma^3}{3c_{01} \gamma^2 + c_{10}^2} = 0 \]

where, from 4-20,

\[ c_{01} \gamma + c_{10} \beta + 1 = 0 \]  

(4-27)

Solving 4-26 and 4-27, one obtains

\[ C_{10} = \begin{cases} 
-(\beta - 7\gamma) \pm \sqrt{\beta^2 + 34\beta\gamma + \gamma^2} \\
8\beta(\beta - \gamma) 
\end{cases} , \quad \beta \neq \gamma 
\]

\[ C_{10} = \begin{cases} 
-\frac{1}{2\beta} \\
, \quad \beta = \gamma 
\end{cases} 
\]

(4-28)

\[ C_{01} = \frac{-1 - \beta C_{10}}{\gamma} . \]

Taking the limit of \( C_{10} \) as \( \gamma \) becomes zero yields

\[ C_{10} = \frac{-1 \pm 1}{8\beta} \]

so that the negative sign must be taken in order for \( C_{10} \) to be non-zero.

Therefore
\[ U_g = \begin{cases} \sqrt{\frac{g}{(3\beta - \gamma)\beta^2c^2_{10} - \beta c_{10}}} & , \quad \beta \neq \gamma \\ (\frac{g}{\beta})^{1/2} = 0.354 \sqrt{\frac{gD_{eq}}{g}} & , \quad \beta = \gamma \end{cases} \]  

(4-29)

where

\[ c_{10} = \begin{cases} \frac{-(\beta - 7\gamma) - \sqrt{\beta^2 + 34\beta\gamma + \gamma^2}}{8\beta(\beta - \gamma)} & , \quad \beta \neq \gamma \\ -\frac{1}{2\beta} & , \quad \beta = \gamma \end{cases} \]

\[ c_{01} = \frac{-1 - \beta c_{10}}{\gamma} . \]

For a 7.62 cm square duct

\[ \beta = \gamma = \frac{\pi}{3.81} = 0.825 \]

and 4-29 gives \( U_g = 34.48 \text{ cm/sec} \) (Fr = 0.354), 7 percent greater than the measured value of 32.16 cm/sec (Fr = 0.330).

For a 7.62 cm x 2.46 cm rectangular duct,

\[ \beta = \frac{\pi}{3.81} = 0.825 \quad \gamma = \frac{\pi}{1.23} = 2.554 \]
and 4-29 gives $U_s = 25.74 \text{ cm/sec (Fr} = 0.324)$, 3 percent less than the measured value of $26.55 \text{ cm/sec (Fr} = 0.335)$.

For higher order approximations, $N > 1$, a Newton-Raphson procedure was utilized to solve the appropriate equation set. The results for $N = 1, 2, 3, 4$ are presented in Figure 4-11 for $A = 0$ for a channel of length $2L = 7.62 \text{ cm}$.

Most of the data points presented in Figure 4-11 are from experiments performed by the author and are described in Chapter 5. Other data points are from Birkhoff and Carter (1957), Griffith (1964), and Sadatomi, et al. (1982) for slots of large width so that surface tension is negligible. The velocities for these last three data points were adjusted to a channel length of $7.62 \text{ cm}$, under the assumption of negligible surface tension, by the relation

$$U_s(L=3.81) = \left( \frac{L}{3.81} \right)^{1/2} U_s(L).$$

The slot widths were similarly adjusted to maintain the same aspect ratio. All data are for air/water systems.

4.4.2. Radii of Curvature at Bubble Nose for Negligible Surface Tension

For a rectangular slot, one must specify two radii of curvature, with orthogonal axes, to completely describe the surface curvature at any given point on the bubble surface. The axes chosen were the $x$-$y$ Cartesian axes, parallel to the slot length and width, respectively.
The radii of curvature at the bubble nose in the respective directions are then given by

\[ R_{cx} = - \left( \frac{1}{Z_{20}} \right)_0 \quad \text{and} \quad R_{cy} = - \left( \frac{1}{Z_{02}} \right)_0. \]

Figures 4-12 and 4-13 present the normalized radii of curvature, \( R_{cx}/L \) and \( R_{cy}/W \), respectively, for the same cases considered in Figure 4-11.

4.4.3. Slip Velocity \( U_s \) for Non-Zero Surface Tension (\( \Lambda > 0 \))

For \( N = 1 \) and \( \Lambda > 0 \), Tables B-1, B-2, and B-3 give

\[ [u^2_{100} + gZ_{20} + \Lambda K_{20}]_0 = 0 \]  \hspace{1cm} (4-30)

\[ [v^2_{010} + gZ_{02} + \Lambda K_{02}]_0 = 0 \]

where for \( ND = 4 \)

\[ (K_{20})_0 = [Z_{40} + Z_{22} - 3Z_{20}^3 - Z_{20}^2 Z_{02}]_0 \]  \hspace{1cm} (4-31)

\[ (K_{02})_0 = [Z_{04} + Z_{22} - 3Z_{02}^3 - Z_{02}^2 Z_{20}]_0 \]

and for \( ND = 2 \)

\[ (K_{20})_0 = [-3Z_{20}^3 - Z_{20}^2 Z_{02}]_0 \]  \hspace{1cm} (4-32)

\[ (K_{02})_0 = [-3Z_{02}^3 - Z_{02}^2 Z_{20}]_0. \]
In addition, from Table B-2,

\[
(Z_{40})_0 = \left[ \frac{\omega_{400} + (6\omega_{201} - 4\omega_{300})Z_{20} + (3\omega_{002} - 12\omega_{101})Z_{20}^2}{4\omega_{010} - \omega_{001}} \right]_0
\]

\[
(Z_{04})_0 = \left[ \frac{\omega_{040} + (6\omega_{021} - 4\omega_{030})Z_{02} + (3\omega_{002} - 12\omega_{011})Z_{02}^2}{4\omega_{010} - \omega_{001}} \right]_0
\]

where from 4-6,

\[
(\omega_{400})_0 = U_s [C_{10} \beta^5]
\]

\[
(\omega_{040})_0 = U_s [C_{01} \gamma^5]
\]

\[
(\omega_{021})_0 = -U_s [C_{01} \gamma^4 + C_{11} \gamma^2 \alpha_{11}^2]
\]

\[
(\omega_{201})_0 = -U_s [C_{10} \beta^4 + C_{11} \beta^2 \alpha_{11}^2]
\]

\[
(\omega_{300})_0 = U_s [C_{10} \beta^4]
\]

\[
(\omega_{030})_0 = U_s [C_{01} \gamma^4]
\]

An analytic solution of 4-18 through 4-22 for \( \Lambda > 0 \) is not feasible. A numerical solution is therefore employed utilizing the Newton-Raphson procedure of Appendix B.

The results for \( \Lambda > 0 \) and \( N = 1, 2, 3, 4 \) are provided in Figure 4-14, compared to the same experimental data as in Figure 4-8 (\( \Lambda = 0 \)), for air/water in a 7.62 cm channel.
4.4.4. Radii of Curvature at Bubble Nose for Non-Zero Surface Tension

\( (\Lambda > 0) \)

Figures 4-15 and 4-16 present the normalized radii of curvature in the \( x \) and \( y \) directions for the same conditions as Figure 4-14.

4.4.5. Summary and Conclusions for Three-Dimensional Rectangular Channel

Figure 4-11 shows the same general trends as Figures 3-1 and 4-1. The \( N = 1 \) solution correctly predicts the general trend of the slip velocity as the width decreases. As \( N \) increases, the agreement of the theoretical prediction to the observed velocity improves. As for the tube and two-dimensional channel calculations, the \( N = 2 \) solution is an anomaly and gives the worst agreement with data.

The \( N = 4, ND = 8 \) solution for \( \Lambda = 0 \) gives excellent agreement of the calculated slip velocity with the observed data for a width as low as 1.27 cm (aspect ratio = 6).

For \( \Lambda > 0 \) (Figure 4-14), the same general trends as in Figure 4-11 are observed, but the range of the channel width for which a solution could be found is decreased in all cases. The \( N = 1 \) solution is not significantly changed when surface tension is included, whereas the higher order solutions are more significantly affected.

As \( N \) increases, the solution converges very well to the observed \( Fr = 0.330 \) for the square 7.62 cm x 7.62 cm channel. For \( N = 4 \), the agreement with observation is excellent.
Neglecting the $N = 2$ solution, which appears to be an anomaly for both circular pipe and rectangular channel, including surface tension results in the prediction of the correct trend of decreasing velocity with increasing aspect ratio. In addition, the predicted curve appears to be converging to the observed data as $N$ increases.

The same general trend is observed for the three-dimensional solution as in the tube and two-dimensional channel. A one term solution ($N = 1$) gives reasonably good results even for large aspect ratios, but overpredicts the measured velocity for large channels, where surface tension is negligible, and underpredicts as surface tension becomes more significant.

As more terms are included in the series approximations, the solution deviates significantly from measured data for $N = 2$, but tends to improve with increasing $N$ as $N$ is increased beyond $N = 2$.

The inclusion of surface tension in the three-dimensional calculations appears to distort the solution significantly, even for those cases where the effects of surface tension should be negligible. For example, for a 7.62 cm square channel, the effects of surface tension should be negligible and inclusion of surface tension in the equations should not significantly alter the coefficients in the series expansion for a given $N,ND$ solution. This was indeed the case for the tube and two-dimensional channel solutions.

However, the $N=3,ND=6$ and $N=4,ND=8$ coefficients are significantly different as shown by Tables 4-3 and 4-5. Furthermore, it is not possible
to obtain one set of coefficients from the other by a gradual increase or decrease of the surface tension value. That is, if the zero surface tension solution is used as the initial guess and the surface tension gradually increased in increments from zero towards the air/water value, a value of surface tension is eventually reached at which the solution fails before the air/water value is attained. Similarly, a decreasing surface tension solution fails before zero is reached.

Similar observations may be made of the 7.62 cm x 3.81 cm solutions (Tables 4-4 and 4-6).

Although the solution coefficients are drastically changed by the inclusion of surface tension, the calculated slip velocities are not significantly altered. It is also observed that as the number of terms in the velocity potential expansion increases, the calculated velocities appear to approach the experimental curve and the maximum aspect ratio at which the solution fails also increases (Figure 4-14).

This leads one to speculate that this behavior is a manifestation of the slow convergence of the three-dimensional solution and that as more terms are included in the finite series expansion, the bubble shape would be better calculated and the anomalous behavior of the coefficients would disappear.

As observed in the tube and two-dimensional channel solutions, the slug slip velocity converges faster than the bubble shape, as Figures 4-12, 4-13, 4-15 and 4-16 demonstrate. Although convergence of the radii of curvature is not obtained for N=4, the values do become more consistent
as $N$ increases, again suggesting that the inclusion of additional terms in the finite series expansion for the velocity potential would improve the solution.

From Figures 4-15 and 4-16, which include the effects of surface tension, the normalized radius of curvature in the length direction ($R_{cx}/L$) decreases as the aspect ratio increases while the width normalized radius of curvature ($R_{cy}/L$) increases for $N=3$ and $N=4$. In addition, the length radius of curvature appears to be decreasing to a value of approximately 0.6, which is consistent with that calculated for the two-dimensional channel ($R_c/L = 0.65$).

4.5. Bubble Shape

Since the rectangular slot is not axisymmetric, a Stokes Stream Function does not exist. One must therefore employ a different procedure than for the circular pipe case in order to calculate the bubble shape.

Consider the curve in the $x$-$z$ plane produced by the intersection of the $y = 0$ plane with the bubble surface. At any point on this curve, the slope is given by

$$ \frac{dz}{dx}_{y=0} = \frac{\partial y}{u}_{y=0} $$

(4-35)
where \( u(x,0,z) \) and \( w(x,0,z) \) are, from 4-5,

\[
\begin{align*}
  u(x,0,z) &= -U \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \beta C_n \sin(n\beta x) e^{-\alpha nm} z^m \\
  w(x,0,z) &= -U [ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \alpha C_n \cos(n\beta x) e^{-\alpha nm}+1 ] z^m.
\end{align*}
\]

One may then write the differential equation for the bubble shape, for \( y \equiv 0 \), as

\[
\frac{dz}{dx} = f(x,z) \quad (4-36)
\]

where

\[
f(x,z) \equiv \frac{w(x,0,z)}{u(x,0,z)}.
\]

Similarly, for \( x \equiv 0 \), one has

\[
\frac{dz}{dy} = h(y,z) = \frac{w(0,y,z)}{v(0,y,z)} \quad (4-37)
\]

where

\[
v(0,y,z) = -U \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \gamma C_n \sin(m\gamma y) e^{-\alpha nm} z^m
\]

\[
\omega(0,y,z) = -U [ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \alpha C_n \cos(m\gamma y) e^{-\alpha nm}+1 ] z^m
\]
Once the coefficients $C_{nm}$ are found, any of several numerical methods may be used to solve 4-36 and 4-37. The method used here is the Runge-Kutta Order Four (Burden, Faires and Reynolds, 1978, p 244) with initial conditions $f(0,0) = 0 = h(0,0)$.

Figures 4-17 through 4-19 present the calculated bubble shapes for a 7.62 cm square channel and a 7.62 cm x 5.08 cm slot including surface tension.
Table 4-1

Coefficients* of Velocity Potential Function
for Slug Flow in a 2-D Rectangular Slot of
Length 7.62 cm for Negligible Surface Tension ($\Lambda = 0$)

<table>
<thead>
<tr>
<th>n</th>
<th>1/2</th>
<th>2/4</th>
<th>3/6</th>
<th>4/8</th>
<th>5/10</th>
<th>6/12</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1.0000</td>
<td>**</td>
<td>-.9992</td>
<td>**</td>
<td>-.9000</td>
<td>-.8414</td>
</tr>
<tr>
<td>2</td>
<td>.0429</td>
<td></td>
<td>-.0632</td>
<td></td>
<td>-.0924</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-.0437</td>
<td></td>
<td>-.0354</td>
<td></td>
<td>-.0794</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td>-.0005</td>
<td></td>
<td>.0187</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td>-.0009</td>
<td></td>
<td>-.0059</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>.0004</td>
<td></td>
</tr>
</tbody>
</table>

Fr  
.289  
.299  
.291  
.285

* Tabulated values are $C_{no}$ $\alpha_{no}$

** No solution for this case

*** Values given are for $Eo = 1500$.
The solution fails for $Eo > 1500$. 
Table 4-2

Coefficients* of Velocity Potential Function for Slug Flow in a 2-D Rectangular Slot of Length 7.62 cm for Air/Water $\text{Eo} = 321$

<table>
<thead>
<tr>
<th>$n$</th>
<th>1/2</th>
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* Tabulated values are $C_{no}$, $\alpha_{no}$

** No solutions for these cases
Table 4-3

Coefficients* of Velocity Potential Function for Slug Flow in a 7.62 cm Square Vertical Channel for Negligible Surface Tension ($\Lambda = 0$)

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</table>

Fr  | .354  | .280  | .308  | .333 |

* Tabulated values are $C_{nm \alpha}$ for $\Lambda = 0$.
Table 4-4

Coefficients* of Velocity Potential Function
for Slug Flow in a 7.62 cm x 3.81 cm Rectangular
Channel for Negligible Surface Tension (Λ = 0)

<table>
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Fr | 0.342 | 0.276 |       | 0.329 |

* Tabulated values are \( C_{nm} \alpha_{nm} \)

** No solution for this case
Table 4-5

Coefficients* of Velocity Potential Function for Slug Flow in a 7.62 cm Square Vertical Channel for Air/Water

<table>
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<th>N/ND</th>
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Fr   | .351   | .298   | .317   | .330

* Tabulated values are $C_{nm} \alpha_{nm}$
Table 4-6

Coefficients* of Velocity Potential Function
for Slug Flow in a 7.62 cm x 3.81 cm
Rectangular Channel for Air/Water

<table>
<thead>
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</table>

* Tabulated values are $C_{nm} \alpha_{nm}$

** No solution for this case
FIGURE 4-1
FLOw IN 2-D CHANNEL
SLIP VELOCITY
NO SURFACE TENSION

Legend
△ N = 1
× N = 3
□ N = 4
◆ N = 5
★ N = 6
FIGURE 4-2
FLOW IN 2-D CHANNEL
NORMALIZED RADIUS OF CURVATURE
NO SURFACE TENSION

Legend

- △ N = 1
- × N = 3
- □ N = 4
- ● N = 5
- × N = 6

RC/L vs. ND
FIGURE 4-3
FLOW IN 2-D CHANNEL
AIR/WATER
N = 1

FROUDE NUMBER

EOTVOS NUMBER

Legend
\( \Delta \) ND = 2
\( \times \) ND = 4
FIGURE 4-4
FLOW IN 2-D CHANNEL
AIR/WATER
N = 3

Legend
△ ND = 4
× ND = 6
□ ND = 8
FIGURE 4-5
FLOW IN 2-D CHANNEL
AIR/WATER
N = 5

FROUDE NUMBER

EOTVOS NUMBER

Legend

\(\triangle ND = 4\)
\(\times ND = 6\)
\(\square ND = 8\)
\(\blacklozenge ND = 10\)
FIGURE 4-6
FLOW IN 2-D CHANNEL
AIR/WATER
N = 6

Legend
△ ND = 4
× ND = 6
□ ND = 8
★ ND = 10
★ ND = 12
× ND = 14
FIGURE 4-7
AIR/WATER FLOW IN 2-D CHANNEL
RADIUS OF CURVATURE FOR
N = 1

Legend
△ ND = 2
× ND = 4
FIGURE 4-8
AIR/WATER FLOW IN 2-D CHANNEL
RADIUS OF CURVATURE FOR
N = 3

Legend
Δ ND = 4
× ND = 6
□ ND = 8
FIGURE 4-9
AIR/WATER FLOW IN 2-D CHANNEL
RADIUS OF CURVATURE FOR
N = 5

Legend
△ ND = 4
× ND = 6
□ ND = 8
★ ND = 10
FIGURE 4-10
AIR/WATER FLOW IN 2-D CHANNEL
RADIUS OF CURVATURE FOR
N = 6
FIGURE 4-11
FLOW IN 3D CHANNEL - 2L = 7.62 CM
SLIP VELOCITY
NO SURFACE TENSION

FROUDE NUMBER

WIDTH (CM)

Legend
○ DATA, THIS REPORT
■ DATA B & C
+ DATA GRIFFITH
× DATA SADATOMI
○ N = 1 ND = 2
△ N = 2 ND = 4
□ N = 3 ND = 6
◇ N = 4 ND = 8
FIGURE 4-12
FLOW IN 3D CHANNEL - 2L = 7.62 CM
NORMALIZED RADII OF CURVATURE - LENGTH
NO SURFACE TENSION

Legend
△ N = 1 ND = 2
× N = 2 ND = 4
□ N = 3 ND = 6
★ N = 4 ND = 8

RCX/L

WIDTH (CM)

0 1.27 2.54 3.81 5.08 6.35 7.62
0 1 2 3 4 5
FIGURE 4-13
FLOW IN 3D CHANNEL - 2L = 7.62 CM
NORMALIZED RADII OF CURVATURE - WIDTH
NO SURFACE TENSION

Legend
△ N = 1 ND = 2
× N = 2 ND = 4
□ N = 3 ND = 6
☒ N = 4 ND = 8
FROUDE NUMBER

FIGURE 4-14
FLOW IN 3D CHANNEL - 2L = 7.62 CM
SLIP VELOCITY
AIR/WATER WITH ST = 71.5 DYNES/CM

Legend
- DATA, THIS REPORT
- DATA B & C
+ DATA GRIFFITH
× DATA SADATOMI
○ N = 1 ND = 2
△ N = 2 ND = 4
□ N = 3 ND = 6
◆ N = 4 ND = 8

FROUDE NUMBER

WIDTH (CM)
FIGURE 4-15
FLOW IN 3D CHANNEL - 2L = 7.62 CM
NORMALIZED RADIUS OF CURVATURE - LENGTH
AIR/WATER WITH ST = 71.5 DYNES/CM

Legend
△ N = 1 ND = 2
× N = 2 ND = 4
□ N = 3 ND = 6
☆ N = 4 ND = 8
FIGURE 4-16
FLOW IN 3D CHANNEL - 2L = 7.62 CM
NORMALIZED RADII OF CURVATURE - WIDTH
AIR/WATER WITH ST = 71.5 DYNES/CM

Legend
△ N = 1 ND = 2
× N = 2 ND = 4
□ N = 3 ND = 6
★ N = 4 ND = 8
NORMALIZED Z = Z(x,0)/L

NORMALIZED X = x/L

FIGURE 4-17
BUBBLE SHAPE Z(x,0) FOR
7.62 CM SQUARE 3D CHANNEL
AIR/WATER

Legend
△ N = 1 ND = 2
× N = 2 ND = 4
□ N = 3 ND = 6
☆ N = 4 ND = 8
FIGURE 4-18
BUBBLE SHAPE \( Z(x,0) \) FOR 7.62 CM X 5.08 CM CHANNEL
AIR/WATER

Legend
\( \triangle N = 1 \) ND = 2
\( \times N = 2 \) ND = 4
\( \square N = 4 \) ND = 8

\[ \text{NORMALIZED } Z = Z(x,0)/L \]

\[ \text{NORMALIZED } X = x/L \]
Figure 4-19
Bubble shape $Z(0,y)$ for 7.62 cm x 5.08 cm channel air/water

Normalized $Z = \frac{Z(0,y)}{L}$

Normalized $Y = \frac{y}{L}$

Legend
- $\triangle N = 1$ ND = 2
- $\times N = 2$ ND = 4
- $\square N = 4$ ND = 8
CHAPTER 5

Measurement of Slip Velocity of Gas
Slugs in a Vertical Rectangular Slot

This chapter will describe the experimental procedure and results obtained for the measurement of the slip velocity of gas slugs (air) in a vertical rectangular slot of stagnant liquid (water).

5.1. Construction of Experimental Apparatus

It was desired to measure the velocity of gas slugs in channels of variable aspect ratio. A maximum dimension of 7.62 cm was decided upon as a reasonable compromise between a size large enough to allow ease of measurement yet small enough to allow the construction of an apparatus that could be easily assembled and manipulated. In addition, based on the observations of slug flow in vertical tubes summarized in Chapter 1, a 7.62 cm square duct would have negligible surface tension effects so that the effects of surface tension could be observed as the channel width is decreased.

The apparatus of Figure 5-1 was constructed of two 2.44 meter long by 7.62 cm wide sections of 6.35 mm plexiglas. Each section was braced by attaching two 2.44 meter lengths of 3.18 mm aluminum angle iron lengthwise and flush to the outer edges. The attaching screws were recessed into the plexiglas and the surface leveled with clear silicone sealant. These sections comprise the inner surfaces of the constant length, \(2L = 7.62\) cm channel dimension.
The narrow width dimension for the channel was constructed by separating the 7.62 cm sections by spacers of the desired width, then cementing a 2.44 m x .61 m section of plexiglas to the exposed angle iron on either end. A thin layer of clear silicone sealant was used as both cement and sealant.

To provide additional strength and to prevent bowing of the plexiglas, one meter sections of 3.18 mm steel angle iron were bolted approximately every half-meter across the .61 m widths of plexiglas.

The top of the channel was sealed with a 10 cm square piece of plexiglas with a small hole in the center to allow air to escape as the channel was filled with liquid. The inlet for the liquid consisted of a 5.0 cm length of 6.35 mm diameter glass tubing cemented into a hole drilled approximately 2.5 cm from the bottom of one of the 7.62 cm sections.

The apparatus was completed by placing it upright, the bottom lying across a common household metal garbage can. A piece of 6.35 cm aluminum sheet metal was clamped to the bottom, sealed by flexible insulating "rope caulk".

When completed, the final apparatus formed a vertical watertight 2.44 meter high rectangular duct of length 7.62 cm and variable width.

After taking data for a given channel width as described in the following section, the apparatus was disassembled. The silicone sealant was removed by first softening it with paint thinner and then scraping with a razor blade paint scraper.
The apparatus was then reassembled with a different channel width and allowed to set overnight to allow the silicone to cure.

5.2. Procedure for Measuring Slip Velocity

After clamping the lower aluminum plate into place, water was pumped into the channel through the inlet at the lower end until the channel was full. The hole at the top of the channel was then firmly sealed with a rubber stopper thereby producing an air-tight, water-tight, closed rectangular channel.

The air bubble was introduced by simultaneously removing the clamps holding the lower aluminum plate in place. Some practice was required to obtain a smooth rapid release, but this method proved to be very simple and effective.

As the water drained from the channel, an air bubble was generated. Its motion was both photographed by still camera and recorded on video tape. The video camera was focused on the 7.62 cm length side of the channel at a height of approximately 1.5 meters. This height was high enough for the bubble to attain steady state yet low enough to be easily observed.

For each case, several preliminary bubbles were generated before recording on video tape in order to insure that the system was indeed air tight and water tight and also to remove any traces of paint thinner or loose silicone sealant that may have adhered to the inner surface. This procedure was adopted because it was noticed that in the initial data sets several runs were required before the measured velocity stabilized.
Figures 5-2 and 5-3 are representative of the bubbles generated by this procedure.

5.3. Calculation of Slip Velocity

The video tape system operated at 60 frames per second. A scale marked in tenths of a centimeter was attached to the vertical channel within the field of view of the camera so that the number of frames required for the bubble to travel a given distance could be determined.

The time elapsed during the bubble traverse between the reference points on the centimeter scale is then

\[ \Delta t = \frac{N_F - f_r}{60} \text{ seconds} \]

where

\( N_F = \) number of complete frames between initial and final bubble position.

\( f_r = \) fraction of a frame represented by the distance the bubble nose was displaced on the video screen.

The parameter \( f_r \) is a correction required because the video image is produced by an electron beam scanning from top to bottom of the video screen. One complete scan takes \( \frac{1}{60} \) second. The time required for the beam to reposition after a complete scan is assumed negligible.

The average bubble velocity is then simply

\[ U_s = \frac{\Delta x}{\Delta t} \]
where $\Delta x$ is the distance the bubble moved during the elapsed time $\Delta t$.

The data for the various channel widths investigated is summarized in Table 5-1. The slip velocity in cm/sec is then given by

$$U_s = \frac{60 \Delta x}{NF-fr}.$$

Figures 5-4 and 5-5 present the measured velocities in both dimensional and non-dimensional form. Neglecting the 7.62 cm x .61 cm slot, where surface tension would be expected to have a significant effect, the average three-dimensional Froude number is found to be

$$Fr = 0.337 \pm 0.006 \text{ cm/sec}.$$  

This value is consistent with Sadatomi et al (1982), who found that the three-dimensional Froude number for various channel geometries was essentially constant in the absence of surface tension and viscosity effects. He reported a value of $Fr = 0.35$. 
Table 5-1

Experimental Data for Calculation of Slug Slip Velocity in Rectangular Channels

<table>
<thead>
<tr>
<th>Dimensions (2L x 2W) (cm)</th>
<th>Δx (cm)</th>
<th>NF</th>
<th>fr</th>
<th>U_s (cm/sec)</th>
<th>U̅_s (cm/sec)</th>
<th>Fr</th>
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* Froude Number defined by \[ Fr = \frac{\bar{U}_s}{24.990\sqrt{2L + 2W}} \]
Figure 5-1 Photographs of Experimental Apparatus
Figure 5-2 Air Slug in 7.62 cm x 1.27 cm Channel
View from 7.62 cm Side

Figure 5-3 Air Slug in 7.62 cm x 1.27 cm Channel
View from 1.27 cm Side
FIGURE 5-4
FLOW IN 3D CHANNEL - 2L = 7.62 CM
MEASURED SLIP VELOCITY FOR AIR/WATER
FIGURE 5-5
FLOW IN 3D CHANNEL - 2L = 7.62 CM
MEASURED FROUDE NUMBER
FOR AIR/WATER
BIBLIOGRAPHY


APPENDIX A

Derivation of Radii of Curvature
for an Arbitrary Convex Bubble Surface

Consider the convex bubble surface \( z = f(x, y) \) in Figure A-1. The origin of the coordinate system is located at the stagnation point. Consider also an arbitrary point \( \vec{r}_p \) on the surface.

Defining a primed coordinate system with \( \vec{r}_p \) as the origin, then

\[
\vec{r} = \vec{r}_p + \vec{r}'
\]

where

\[
\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}
\]

\[
\vec{r}' = x'\hat{i}' + y'\hat{j}' + z'\hat{k}'
\]

\[
\vec{r}_p = x_p\hat{i} + y_p\hat{j} + z_p\hat{k}
\]

Then one may write,

\[
x = x_p + x' (\hat{i}' \cdot \hat{i}) + y' (\hat{j}' \cdot \hat{i}) + z' (\hat{k}' \cdot \hat{i})
\]

\[
y = x_p + x' (\hat{i}' \cdot \hat{j}) + y' (\hat{j}' \cdot \hat{j}) + z' (\hat{k}' \cdot \hat{j}) \tag{A-1}
\]

\[
z = z_p + x' (\hat{i}' \cdot \hat{k}) + y' (\hat{j}' \cdot \hat{k}) + z' (\hat{k}' \cdot \hat{k})
\]

Since the unit vectors \( \hat{i}', \hat{j}', \hat{k}' \) are arbitrary subject only to the requirement of mutual orthogonality, one may choose the primed coordinates as follows, where

\[
z_x = \frac{\partial z(x, y)}{\partial x},
\]
\[ \hat{\mathbf{i}}' = \left[ -\hat{\mathbf{i}} + \frac{Z_x \hat{\mathbf{k}}}{\sqrt{1 + Z_x^2}} \right] \]

\[ \hat{\mathbf{k}}' = \frac{\nabla T}{\left| \nabla T \right|} \] \hspace{1cm} (A-2)

\[ \hat{\mathbf{j}}' = (\hat{\mathbf{k}}' \times \hat{\mathbf{i}}') \]

The \( \hat{\mathbf{i}}' \) is the unit vector tangential to the bubble surface and parallel to the x-z plane. The \( \hat{\mathbf{k}}' \) is the normal unit vector to the surface where \( T \) is a level function for the surface, \( T = z - f(x,y) = 0 \).

With the definition in A-2, one may write

\[ \hat{\mathbf{k}}' = \frac{\partial T}{\partial x} \hat{\mathbf{i}} + \frac{\partial T}{\partial y} \hat{\mathbf{j}} + \frac{\partial T}{\partial z} \hat{\mathbf{k}} \]

\[ \left[ \left( \frac{\partial T}{\partial x} \right)^2 + \left( \frac{\partial T}{\partial y} \right)^2 + \left( \frac{\partial T}{\partial z} \right)^2 \right]^{1/2} \] \hspace{1cm} (A-3)

Substituting \( T = z - f(x,y) \) into (A-3) yields

\[ \hat{\mathbf{k}}' = -\left( \frac{Z_x}{\Gamma_1} \right) \hat{\mathbf{i}} - \left( \frac{Z_y}{\Gamma_1} \right) \hat{\mathbf{j}} + \left( \frac{1}{\Gamma_1} \right) \hat{\mathbf{k}} \]

where \( \Gamma_1 \equiv [Z_x^2 + Z_y^2 + 1]^{1/2} \).

Therefore, \( \hat{\mathbf{j}}' = \hat{\mathbf{k}}' \times \hat{\mathbf{i}}' \) becomes

\[ \hat{\mathbf{j}}' = -\left( \frac{Z_x Z_y}{\Gamma_1^{1/2}} \right) \hat{\mathbf{i}} + \left( \frac{1 + Z_x^2}{\Gamma_1^{1/2}} \right) \hat{\mathbf{j}} + \left( \frac{Z_y}{\Gamma_1^{1/2}} \right) \hat{\mathbf{k}} \]

(A-5)
where $\Gamma_2 \equiv \left(1 + \frac{Z^2}{x_p}\right)^{1/2}$.

Substitution of A-3, A-4, A-5 into A-1 then gives

$$x = x_p + \left(\frac{1}{\Gamma_2}\right) x' - \left(\frac{Z}{\Gamma_{1/2}}\right) y' - \left(\frac{Z}{\Gamma_1}\right) z'$$

$$y = y_p + \left(\frac{1}{\Gamma_2}\right) y' - \left(\frac{Z}{\Gamma_1}\right) z'$$ (A-6)

$$z = z_p + \left(\frac{Z}{\Gamma_1}\right) x' + \left(\frac{Z}{\Gamma_{1/2}}\right) y' + \left(\frac{1}{\Gamma_2}\right) z'$$

In the primed system, the axes are chosen to be the $x'$ and $y'$ axes.

Then one may write,

$$\left(\frac{1}{R_1}\right)_p = \frac{\left(\frac{\partial^2 z'}{\partial x'^2}\right)_p}{\left[1 + \left(\frac{\partial z'}{\partial x'}\right)^2\right]_p^{3/2}} \quad \left(\frac{1}{R_2}\right)_p = \frac{\left(\frac{\partial^2 z'}{\partial y'^2}\right)_p}{\left[1 + \left(\frac{\partial z'}{\partial y'}\right)^2\right]_p^{3/2}}$$ (A-7)

where

$$(R_1)_p = \text{radius of curvature in the } x' \text{ direction},$$

$$(R_2)_p = \text{radius of curvature in the } y' \text{ direction},$$

$$(z') = f'(x', y') \text{ is the equation of the bubble surface in the primed system.}$$

Since the bubble surface is everywhere convex, then
\[
\left( \frac{\partial z'}{\partial x'} \right) = 0 \quad \left( \frac{\partial z'}{\partial y} \right) = 0 \\
p \quad p
\]
\[
\left( \frac{\partial^2 z'}{\partial x'^2} \right) < 0 \quad \left( \frac{\partial^2 z'}{\partial y'^2} \right) < 0 \\
p \quad p
\]
so that A-7 becomes
\[
\left( \frac{1}{R_1} \right) p = - \left( \frac{\partial^2 z'}{\partial x'^2} \right) \quad \left( \frac{1}{R_2} \right) p = - \left( \frac{\partial^2 z'}{\partial y'^2} \right) . \quad \quad \text{(A-8)}
\]

Realizing that the bubble surface may be written as a function of either the primed coordinates \( z = z(x', y', z') \) or the unprimed coordinates \( z = z(x, y) \), then one may write
\[
\frac{\partial z}{\partial x'} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial z'} \frac{\partial z'}{\partial x'} = (\frac{\partial z}{\partial x})(\frac{\partial x'}{\partial x} + \frac{\partial x'}{\partial z'} \frac{\partial z'}{\partial x'}) + (\frac{\partial z}{\partial y})(\frac{\partial y'}{\partial x'} + \frac{\partial y'}{\partial z'} \frac{\partial z'}{\partial x'}) \quad \text{(A-9)}
\]

Differentiating (A-9) again, one obtains
\[
\frac{\partial^2 z}{\partial x'^2} + 2 \frac{\partial^2 z}{\partial x' \partial z'} \frac{\partial z'}{\partial x'} + \frac{\partial z}{\partial z'} \frac{\partial^2 z'}{\partial x'^2} + \frac{\partial^2 z}{\partial z'^2} \left( \frac{\partial z'}{\partial x'} \right)^2
\]
\[
= \frac{\partial z}{\partial x} \left( \frac{\partial^2 x}{\partial x'^2} + \frac{\partial x}{\partial z'} \frac{\partial^2 z}{\partial x'^2} + 2 \frac{\partial^2 x}{\partial x' \partial z'} \frac{\partial z'}{\partial x'} + \frac{\partial^2 x}{\partial z'^2} \left( \frac{\partial z'}{\partial x'} \right)^2 \right)
\]
\[
\left( \frac{\partial^2 z}{\partial x^2} \right) \left( \frac{\partial x'}{\partial x'} + \frac{\partial x'}{\partial z'} \frac{\partial z'}{\partial x'} \right)^2
\]
Substitution of A-6 into A-10 and evaluating at \( \mathbf{r}_p \) yields, after algebraic manipulation,

\[
\left( \frac{\partial^2 z}{\partial x^2} \right)_p = \left( \frac{Z_{xx}}{l_1^2} \right)_p . \tag{A-11}
\]

Similarly, differentiating with respect to \( y' \) yields

\[
\left( \frac{\partial^2 z}{\partial y'^2} \right)_p = \left[ \frac{Z_{xx}Z_{yy} - 2Z_{xy}Z_{xy}}{1/2} \right] \quad \text{p} . \tag{A-12}
\]

From A-8, A-11, A-12, and the definition

\[
K \equiv -\left( \frac{1}{R_1} + \frac{1}{R_2} \right)
\]

one thus obtains

\[
K(x,y) = \frac{\left[ Z_{xx}(1 + Z_x^2) - 2Z_{xy}Z_{xy} + Z_{yy}(1 + Z_y^2) \right]}{(1 + Z_x^2 + Z_y^2)^{3/2}} . \tag{A-13}
\]
If the coordinate system being employed is the Cylindrical system, then A-13 may be written in that system by using the relationships

\[ x = r \cos \theta \quad y = r \sin \theta \quad r = (x^2 + y^2)^{1/2} \]

so that one has

\[ Z_{xx} = \cos^2 \theta Z_{rr} + \frac{\sin^2 \theta}{r} Z_r \]
\[ Z_{xy} = \cos \theta \sin \theta (Z_{rr} - \frac{Z_r}{r}) \]
\[ Z_{yy} = \sin^2 \theta Z_{rr} + \frac{\cos^2 \theta}{r} Z_r \]
\[ Z_x = \cos \theta Z_r \]
\[ Z_y = \sin \theta Z_r \]

For a circular pipe, there is \( \theta \)-symmetry in the cylindrical system. The expression

\[ \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \]

must then be constant for any value of \( \theta \) for arbitrary \( r \). Thus one may choose any convenient \( \theta \) to evaluate A-14. Choosing \( \theta = 0 \), then

\[ Z_{xx} = Z_{rr} \quad Z_{yy} = \frac{Z_r}{r} \quad Z_{xy} = 0 \]
\[ Z_x = Z_r \quad Z_y = 0 \]
and A-13 becomes

\[ K(r) = \frac{rZ_{rr} + (1 + Z_r^2)Z_r}{r(1 + Z_r^2)^{3/2}} \]  \hspace{1cm} (A-15)

in the cylindrical coordinate system for a circular pipe.
\[ \mathbf{r} = \mathbf{r}_p + \mathbf{r}' \]

Figure A-1 Coordinate systems defined for derivation of Equation A-13
APPENDIX B

Derivation of Equations Required to Solve for Series Coefficients in Cartesian Coordinates

From Chapter 2, the equations required for the solution of the velocity potential are derived by differentiating the two equations

\[ T_1(x, y, z) = u^2 + v^2 + w^2 + 2gz + 2\Lambda(K - K_0) = 0 \]
\[ T_2(x, y, z) = u\frac{\partial Z}{\partial x} + v\frac{\partial Z}{\partial y} - w = 0 \]  

where

\[ K \equiv -\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \]
\[ \Lambda = \frac{\sigma}{(\rho_2 - \rho) g} \]

and the radii of curvature \( R_1, R_2 \) are derived in Appendix A. The resultant even-ordered derivatives are then evaluated at the stagnation point.

In order to facilitate the derivations, let \( T(x, y, z) = 0 \) denote any arbitrary surface with the condition that \( z = f(x, y) \) and

\[ \left[ \frac{\partial^{(m+n)}Z}{\partial x^m \partial y^n} \right] \equiv 0 \]

for either \( m \) odd or \( n \) odd.
In addition, the following nomenclature will be used:

\[
T_{mpn} = \frac{\partial (m+n+p)T}{\partial x^m \partial y^n \partial z^p}
\]

\[
z_{mn} = \frac{\partial (m+n)Z}{\partial x^m \partial y^n}
\]

\[
h_{mpn} = \frac{\partial (m+n+p)h}{\partial x^m \partial y^n \partial z^p}
\]

for \( h = u, v, w \).

Successive differentiations of \( T \) with respect to \( x \), evaluated at the origin, will yield:

\[
\left( \frac{D^2 T}{dx^2} \right)_0 = [T_{200} + T_{001} Z_{20}]_0 = 0 \tag{B-2}
\]

\[
\left( \frac{D^4 T}{dx^4} \right)_0 = [T_{400} + 6T_{201} Z_{20} + 3T_{002} Z_{20}^2 + T_{001} Z_{40}]_0 = 0 \tag{B-3}
\]

\[
\left( \frac{D^6 T}{dx^6} \right)_0 = [T_{600} + 15T_{401} Z_{20} + 15T_{201} Z_{40}^2 + 45T_{202} Z_{20}^2 + 15T_{002} Z_{20} Z_{40} + 15T_{003} Z_{20}^3 + T_{001} Z_{60}]_0 = 0 \tag{B-4}
\]
The y-derivatives of $T$ can be found by simply interchanging the $x$ and $y$ subscripts in B-2 through B-6.
The cross-derivatives may also be evaluated and are found to be

\[
\left( \frac{D^4 T}{Dx^2 Dy^2} \right)_0 = [T_{220} + T_{201}Z_{02} + T_{001}Z_{22} + T_{021}Z_{20} + T_{002}Z_{20}Z_{02}]_0 = 0
\]  
\text{(8-7)}

\[
\left( \frac{D^6 T}{Dx^4 Dy^2} \right)_0 = [T_{420} + T_{401}Z_{02} + 6T_{201}Z_{22} + 6T_{221}Z_{20} + 6T_{202}Z_{20}Z_{02} + 6T_{002}Z_{20}Z_{22} + 3T_{022}Z_{02}^2 + 3T_{003}Z_{20}^2Z_{20} + T_{001}Z_{42} + T_{021}Z_{40} + T_{002}Z_{40}Z_{02}]_0 = 0
\]  
\text{(8-8)}

\[
\left( \frac{D^6 T}{Dx^2 Dy^4} \right)_0 = [T_{240} + T_{041}Z_{20} + 6T_{021}Z_{22} + 6T_{221}Z_{02} + 6T_{022}Z_{02}Z_{20} + 6T_{002}Z_{02}Z_{22} + 3T_{202}Z_{02}^2 + 3T_{003}Z_{02}^2Z_{20} + T_{001}Z_{24} + T_{201}Z_{04} + T_{002}Z_{04}Z_{20}]_0 = 0
\]  
\text{(8-9)}
\[
\left( -\frac{D^8 T}{Dx^6 Dy^2} \right)_0 = [T_{620} + T_{601}Z_{02} + 15T_{401}Z_{22}
\]
\[+ 15T_{421}Z_{20} + 15T_{402}Z_{20}Z_{02}\]
\[+ 15T_{201}Z_{42} + 15T_{221}Z_{40} + 15T_{202}Z_{40}Z_{02}\]
\[+ 90T_{202}Z_{20}Z_{22} + 45T_{222}Z_{20}^2 + 45T_{203}Z_{20}Z_{02}^2\]
\[+ 45T_{003}Z_{20}Z_{22} + 15T_{023}Z_{20}^3 + 15T_{004}Z_{20}Z_{02}^3 ] = 0 \quad (B-10)\]

\[
\left( -\frac{D^8 T}{Dx^2 Dy^6} \right)_0 = [T_{260} + T_{061}Z_{20} + 15T_{041}Z_{22} + 15T_{241}Z_{02}\]
\[+ 15T_{042}Z_{02}Z_{20} + 15T_{021}Z_{24} + 15T_{221}Z_{04}\]
\[+ 15T_{022}Z_{04}Z_{20} + 90T_{022}Z_{02}Z_{22} + 45T_{222}Z_{02}^2\]
\[+ 45T_{023}Z_{02}Z_{22} + 45T_{003}Z_{02}Z_{22}\]
\[+ 15T_{203}Z_{02}^3 + 15T_{004}Z_{02}Z_{20}^3 ] = 0 \quad (B-11)\]
\[
\left( \frac{\partial^8 T}{\partial x^4 \partial y^4} \right)_0 = \left[ T_{440} + 6T_{421}Z_{02} + T_{401}Z_{04} + 3T_{402}Z_{02}^2 \right. \\
+ 6T_{201}Z_{24} + 36T_{221}Z_{22} + 36T_{202}Z_{22}Z_{02} \\
+ 6T_{241}Z_{20} + 36T_{222}Z_{20}Z_{02} + 6T_{202}Z_{20}^2Z_{04} \\
+ 18T_{203}Z_{20}^2Z_{02} + 6T_{002}Z_{20}Z_{24} + 18T_{002}Z_{22}^2 \\
+ 36T_{022}Z_{20}Z_{22} + 36T_{003}Z_{02}Z_{20}Z_{22} \\
+ 3T_{042}Z_{20}^2 + 18T_{023}Z_{20}^2Z_{02} + 3T_{002}Z_{20}Z_{04} \\
+ 9T_{004}Z_{20}Z_{02} + T_{001}Z_{044} + 3T_{021}Z_{42} \\
+ 6T_{002}Z_{02}Z_{42} + 3T_{021}Z_{42} + T_{041}Z_{40} \\
+ 6T_{022}Z_{40}Z_{02} + T_{002}Z_{40}Z_{04} + 3T_{002}Z_{40}Z_{02}^2 \right]_0 = 0 \quad (B-12)
\]

Note that for \( T = T_1(x,y,z) \), the derivatives of the function \( K \) are required. From A-13 one has

\[
K = \frac{Z_{20}(1 + Z_{02}^2) - Z_{10}Z_{01}Z_{11} + Z_{02}(1 + Z_{10}^2)}{(1 + Z_{10}^2 + Z_{01}^2)^{3/2}} \quad (B-13)
\]

The required derivatives of \( K \) can then be evaluated from B-13 and are provided in Table B-1.
Consider now the function $T = T_1(x,y,z)$. Then

$$T_{200} = 2[uu_{200} + u_{100}^2 + vv_{200} + v_{100}^2 + \omega_{200} + \omega_{100}^2] + 2\Lambda K_{20}$$

and

$$T_{001} = 2[uu_{001} + vv_{001} + \omega_{001}] + 2g .$$

At the stagnation point, $\bar{v}_0 = 0$, $\bar{v}_{100} = 0$, and $\bar{v}_{100}' = 0$ so that $\text{B-2}$ becomes, for $T = T_1$,

$$[u_{100}^2 + gZ_{20} + \Lambda K_{20}]_{0} = 0 . \tag{B-14}$$

Now letting $T = T_2(x,y,z)$, then one obtains

$$(T_{200})_0 = [2u_{100}Z_{20} - \omega_{200}]_0$$

$$(T_{001})_0 = - (\omega_{001})_0$$

so that $\text{B-2}$ becomes, for $T = T_2$,

$$[2u_{100} - \omega_{20} - \omega_{001}Z_{20}]_0 = 0 . \tag{B-15}$$

Combining $\text{B-14}$, $\text{B-15}$ and $(\Lambda_{20})_0$ from Table $\text{B-1}$, one has

$$(Z_{20})_0 = \left[ \frac{\omega_{200}}{2u_{100} - \omega_{100}} \right]_0 \tag{B-16}$$

and

$$[u_{100}^2 + gZ_{20} + \Lambda(Z_{40} + Z_{22} - 3Z_{20}^3 - Z_{20}^2Z_{202})]_0 = 0$$
The similar equation obtained from \( (D^2 T/Dy^2) = 0 \) for \( T = T_1 \) and \( T = T_2 \) is, by symmetry,

\[
(z_{02})_0 = \frac{\omega_{020}}{2\nu_{010} - \omega_{001}} \quad (B-17)
\]

and

\[
[\nu_{010}^2 + g z_{02} + \Lambda[z_{04} + z_{22} - 3z_{02}^3 - z_{02}^2 z_{20}]]_0 = 0
\]

Additional equations may be obtained by evaluation of the higher order derivatives. The results are summarized in Tables B-2 and B-3.

The equations presented in this appendix were derived in part by use of the algebraic programming system REDUCE-2 (Hearn, 1973).
TABLE B-1

Derivatives of Function $K(x,y)$ Evaluated at Stagnation Point (Bubble Nose)

$$K \equiv \left[ \frac{1}{R_1} + \frac{1}{R_2} \right]$$

1. $(k_{20})_0 = [z_{22}^3 - 3z_{20}^2z_{20}^02 - z_{20}^2z_{20}^02]_0$

2. $(k_{02})_0 = [z_{42}^3 - 3z_{20}^2z_{20}^02 - z_{20}^2z_{20}^02]_0$

3. $(k_{40})_0 = [z_{42}^3 - 4z_{20}^2z_{20}^02 - 6z_{20}^2z_{20}^2 - 30z_{20}^2z_{40}^0 + 9z_{20}^4z_{20}^02 + 45z_{20}^5]_0$

4. $(k_{04})_0 = [z_{42}^3 - 4z_{20}^2z_{20}^02 - 6z_{20}^2z_{20}^2 - 30z_{20}^2z_{40}^0 + 9z_{20}^4z_{20}^02 + 45z_{20}^5]_0$

5. $(k_{22})_0 = [z_{42}^3 - z_{20}^2z_{20}^04 + z_{20}^2z_{20}^04 + z_{20}^2z_{20}^04 + 9z_{20}^2z_{20}^02 + 9z_{20}^3z_{20}^02 - 9z_{20}^2z_{20}^2 - z_{20}^2z_{20}^02 - 12z_{20}^2z_{20}^02]_0$
6. \((K_{60})_0 = \left[ z_{80} - 6z_{02}z_{20}z_{06} - 60z_{20}z_{40}z_{22} \right. \\
- 10z_{02}^2z_{40} - 15z_{20}^2z_{42} + z_{62} - 63z_{20}z_{60} \\
+ 180z_{02}^3z_{40} + 135z_{20}^4z_{22} - 225z_{02}z_{20}^6 \\
+ 1575z_{20}^4z_{40} - 210z_{20}^2z_{40}^2 - 1575z_{20}^7z_{02}^0 \right]

7. \((K_{06})_0 = \left[ z_{08} + z_{26} - 6z_{20}z_{02}z_{06} - 60z_{02}z_{04}z_{22} \right. \\
- 10z_{20}^2z_{04} - 15z_{02}^2z_{24} - 63z_{02}z_{06} \\
+ 180z_{20}^3z_{02}z_{04} + 135z_{02}^4z_{22} - 225z_{20}^6z_{02} \\
+ 1575z_{02}^4z_{04} - 210z_{02}^2z_{04}^2 - 1575z_{02}^7z_{02}^0 \right]

8. \((K_{42})_0 = \left[ z_{62} + z_{44} - 22z_{20}z_{02}z_{42} - 32z_{40}z_{02}z_{22} \right. \\
- z_{02}^2z_{60} - 66z_{20}^2z_{22} - 6z_{20}^2z_{24} - 9z_{02}^2z_{42} \\
- 54z_{02}^2z_{22} - 30z_{20}^2z_{42} - 60z_{20}^2z_{40}z_{22} \\
+ 90z_{20}^2z_{02}z_{40} + 36z_{20}^3z_{02}z_{40} - 225z_{20}^4z_{22} \\
+ 288z_{20}^3z_{02}z_{22} + 162z_{20}^2z_{20}^2z_{22} - 4z_{04}z_{20}z_{40} \\
+ 9z_{20}^4z_{04} - 225z_{20}^5z_{02} - 135z_{20}^4z_{02}^3 \right]
TABLE B-1 (Cont)

9. \( (K_{24})_0 = \left[ Z_{26} + Z_{44} - 22Z_{02}Z_{20}Z_{24} - 32Z_{04}Z_{20}Z_{22} \right.

\quad - Z_{20}Z_{06} - 66Z_{02}Z_{22} - 6Z_{02}Z_{42} - 9Z_{20}Z_{24} \]

\quad - 54Z_{20}Z_{22} - 30Z_{02}Z_{24} - 60Z_{02}Z_{04}Z_{22} \]

\quad + 90Z_{02}Z_{22}Z_{04} + 36Z_{22}Z_{20}Z_{04} + 225Z_{02}Z_{22} \]

\quad + 288Z_{02}Z_{20}Z_{22} + 162Z_{02}Z_{20}Z_{22} - 4Z_{40}Z_{02}Z_{04} \]

\quad + 9Z_{02}Z_{40} - 225Z_{20}Z_{02} - 135Z_{02}Z_{20} \right]_0 \]

10. \( (K_{80})_0 = \left[ Z_{10,0} + Z_{82} - 840Z_{40}^3 - 280Z_{40}Z_{22} \right.

\quad + 2520Z_{40}Z_{20}Z_{20}^2 + 37800Z_{40}Z_{20}^3 + 5040Z_{40}^2Z_{20}Z_{20} \]

\quad - 12600Z_{40}^2Z_{20}Z_{20}^5 - 56Z_{40}Z_{02}Z_{60} - 132300Z_{40}Z_{20}Z_{20} \]

\quad - 1512Z_{40}Z_{20}Z_{20}Z_{60} - 280Z_{40}Z_{20}Z_{42} - 6300Z_{22}Z_{20}Z_{20} \]

\quad - 168Z_{22}Z_{20}Z_{60} + 11025Z_{02}Z_{20}Z_{80} + 504Z_{02}Z_{20}Z_{60} \]

\quad - 8Z_{02}Z_{20}Z_{80} + 99225Z_{20}Z_{60} + 5670Z_{20}Z_{60} \]

\quad + 630Z_{20}Z_{42} - 108Z_{20}Z_{80} - 28Z_{20}Z_{62} \right]_0
TABLE B-1 (Cont)

11. $(k_{08})_0 = \left[ z_{0,10} + z_{28} - 840z_{04}^3 - 280z_{04}^2 z_{22} \right.
+ 2520z_{04}^2 z_{20} z_{02} + 37800z_{04}^2 z_{02}^3 + 5040z_{04} z_{22} z_{02}^3
- 12600z_{04} z_{20} z_{02}^5 - 56z_{04} z_{20}^5 - 132300z_{04} z_{02}^6
- 1512z_{04} z_{02} z_{06} - 280z_{04} z_{02} z_{24} - 6300z_{22} z_{02}^6
- 168z_{22} z_{02} z_{06} - 11025z_{20} z_{02}^8 + 504z_{20} z_{02}^3 z_{06}
- 8z_{20} z_{02} z_{08} + 99225z_{02}^9 + 5670z_{02} z_{02}^7
+ 630z_{02} z_{24} - 108z_{02} z_{08} - 28z_{02} z_{26} \right]_0
TABLE B-1 (Cont)

12. \( (K_{62})_0 = \left[ z_{64}^2 + z_{82}^2 - \frac{210z_{40}^2z_{22}}{z_{20}} + \frac{90z_{40}^2z_{02}}{z_{20}} \right. \\
\left. + \frac{630z_{40}^2z_{02}^2}{z_{20}^2} - \frac{10z_{40}^2}{z_{04}} - \frac{390z_{40}^2z_{22}}{z_{20}^2} \right. \\
\left. + \frac{1620z_{40}^2z_{22}^2}{z_{02}^2z_{20}} + \frac{5400z_{40}^2z_{22}z_{02}^2}{z_{20}^2} \right. \\
\left. + \frac{6300z_{40}^2z_{22}z_{02}^3}{z_{20}^3} - \frac{2700z_{40}^2z_{02}^3}{z_{20}^3} \right. \\
\left. - \frac{7875z_{40}^2z_{02}^4}{z_{20}^4} - \frac{130z_{40}^2z_{02}z_{42}}{z_{20}} \right. \\
\left. + \frac{180z_{40}^3z_{20}}{z_{04}} - \frac{60z_{40}^2z_{20}z_{24}}{z_{04}} - \frac{420z_{40}^2z_{20}z_{42}}{z_{04}} \right. \\
\left. - \frac{270z_{22}^3}{z_{02}} + \frac{2430z_{22}^2z_{02}^2}{z_{20}^2} + \frac{3510z_{22}^2z_{20}^3}{z_{20}^3} \right. \\
\left. - \frac{6075z_{22}^2z_{02}^4}{z_{20}^4} - \frac{13500z_{22}^2z_{02}^5}{z_{20}^5} \right. \\
\left. - \frac{60z_{22}z_{20}z_{60}}{z_{22}z_{02}z_{42}} - \frac{270z_{22}z_{02}z_{42}}{z_{20}^2} - \frac{11025z_{22}z_{20}^6}{z_{22}z_{02}z_{42}} \right. \\
\left. - \frac{126z_{22}z_{20}^2}{z_{60}^2} - \frac{480z_{22}z_{20}z_{42}}{z_{02}^2z_{20}} + \frac{4725z_{3}z_{6}^2}{z_{02}^2z_{20}} \right. \\
\left. + \frac{54z_{02}z_{20}z_{60}}{z_{22}z_{02}z_{42}} + \frac{11025z_{2}z_{4}^2}{z_{02}^2z_{20}} + \frac{189z_{02}z_{20}z_{60}}{z_{22}z_{02}z_{42}} \right. \\
\left. + \frac{405z_{02}z_{20}z_{42}}{z_{22}z_{02}z_{42}} - \frac{z_{2}z_{02}z_{60}}{z_{22}z_{02}z_{42}} - \frac{9z_{02}z_{62}}{z_{22}z_{02}z_{42}} \right. \\
\left. + \frac{1170z_{02}z_{60}}{z_{22}z_{02}z_{42}} + \frac{32z_{02}z_{20}z_{60}}{z_{22}z_{02}z_{42}} - \frac{225z_{6}^2}{z_{22}z_{02}z_{42}} \right. \\
\left. + \frac{135z_{20}^4}{z_{22}z_{02}z_{42}} + \frac{1575z_{20}^4}{z_{02}^2z_{20}} - \frac{15z_{20}^4z_{44}}{z_{22}z_{02}z_{42}} \right. \\
\left. - \frac{63z_{20}^4z_{62}}{z_{22}z_{02}z_{42}} - \frac{6z_{20}^4z_{60}z_{04}}{z_{22}z_{02}z_{42}} \right]_0
TABLE B-1 (Cont)

13. \( (K_{26})_0 = [Z_{46} + Z_{28} - 210Z_{04}^2Z_{22} + 90Z_{04}^2Z_{20}^2 + 630Z_{04}^2Z_{20}Z_{02} - 10Z_{04}^2Z_{40} - 390Z_{04}^2Z_{22}^2 + 1620Z_{04}^2Z_{22}^2Z_{20}Z_{02} + 5400Z_{04}^2Z_{22}^2Z_{20}Z_{02}^2 + 6300Z_{04}^2Z_{22}^3Z_{02} - 2700Z_{04}^3Z_{20}Z_{02}^3 - 7875Z_{04}^2Z_{20}Z_{02}^4 - 130Z_{04}^2Z_{20}Z_{24}^5 + 180Z_{04}^3Z_{02}^2Z_{40} - 60Z_{04}^2Z_{02}Z_{42} - 420Z_{04}Z_{02}Z_{24}^3 + 270Z_{22}^3 + 2430Z_{22}^2Z_{20}Z_{02} - 2430Z_{22}^2Z_{20}Z_{02}^2 + 3510Z_{22}^3Z_{02}^2 - 6075Z_{22}^2Z_{20}Z_{02}^4 - 13500Z_{22}^2Z_{20}Z_{24}^5 + 60Z_{22}Z_{20}Z_{06} - 270Z_{22}Z_{20}Z_{24} - 11025Z_{22}Z_{20}Z_{24} - 126Z_{22}Z_{02}Z_{06} - 480Z_{22}Z_{02}Z_{24} + 475Z_{22}Z_{02}Z_{24}^2 + 54Z_{20}^2Z_{40} + 11025Z_{20}^2Z_{02}^2 + 189Z_{20}^2Z_{02}Z_{06} + 405Z_{20}^2Z_{24} - 405Z_{20}^2Z_{24} + 1170Z_{20}^3Z_{24} - 322Z_{20}Z_{02}Z_{24} - 225Z_{02}Z_{24}^2 + 135Z_{02}Z_{42} + 1575Z_{02}Z_{24} - 15Z_{02}Z_{44}^3 - 6Z_{02}Z_{24}^4 - 6Z_{02}Z_{06}Z_{40}^4 - 10Z_{02}Z_{06}Z_{40}^4] \)
TABLE B-1 (Cont)

14. \( (K_{44})_0 = \left[ z_{64}^2 + z_{46}^2 - 180z_{40}^2z_{22}^2 + 720z_{40}^2z_{22}^2z_{02}^3 \right. \)

\[ + 1080z_{40}^2z_{22}^2z_{02}z_{20} - 80z_{40}^2z_{22}z_{04} \]

\[ - 900z_{40}^2z_{02}z_{20}^5 - 1350z_{40}^2z_{02}z_{20}^{z_4}z_{02}^2 \]

\[ + 360z_{40}^2z_{02}z_{20}z_{04} + 360z_{40}^2z_{02}z_{20}z_{04}^2 - 60z_{40}^2z_{02}z_{20} \]

\[ - 42z_{40}^2z_{20}z_{06}^2 - 60z_{40}^2z_{20}z_{24} - 360z_{22}^3 \]

\[ + 2700z_{22}^2z_{02}^3 + 4860z_{22}^2z_{02}z_{20} \]

\[ + 4860z_{22}^2z_{02}z_{20}^2 + 2700z_{22}^2z_{02}z_{20}^3 - 180z_{22}^2z_{04}^2 \]

\[ - 6750z_{22}^2z_{02}z_{20}^4 - 10800z_{22}^2z_{02}z_{20}^2 \]

\[ - 6750z_{22}^2z_{02}z_{20}^4 + 1080z_{22}^2z_{02}z_{20}^2z_{04} \]

\[ - 360z_{22}^2z_{02}z_{20}^4 - 300z_{22}^2z_{02}z_{42} + 720z_{22}^2z_{20}z_{04} \]

\[ - 300z_{22}^2z_{20}z_{24}^2 - 360z_{22}^2z_{20}z_{24} + 4725z_{02}^5z_{20}^4 \]

\[ + 4725z_{20}^4z_{02} + 9z_{02}^4z_{20}^6 + 225z_{02}^4z_{20}^4 \]

\[ + 540z_{02}^3z_{20}z_{04} + 1350z_{02}^3z_{02}z_{20}z_{04} \]

\[ + 540z_{02}^3z_{20}z_{24} + 900z_{02}^3z_{20}z_{02} + 540z_{02}^3z_{20}z_{24} \]

\[ - 6z_{02}^2z_{62} + 540z_{02}^3z_{20}z_{42} - 30z_{02}^2z_{44} \]

\[ - 40z_{02}^2z_{20}z_{44} + 4z_{02}^2z_{60}z_{04} - 60z_{02}^2z_{42}z_{04} \]

\[ + 9z_{20}^4z_{06} + 225z_{20}^4z_{24} - 30z_{20}^4z_{44} \]
TABLE B-1 (Cont)

\[-6z_{20}^2 z_{26} - 60z_{20} z_{42} z_{04} \]
TABLE B-2

Equations for Derivatives of Bubble Shape $Z(x,y)$

Evaluated at Stagnation Point

1. \( (Z_{20})_0 = \left[ \frac{\omega_{200}}{2u_{100} - \omega_{001}} \right]_0 \)

2. \( (Z_{02})_0 = \left[ \frac{\omega_{020}}{2u_{010} - \omega_{001}} \right]_0 \)

3. \( (Z_{40})_0 = \left[ \frac{\omega_{040} + (6\omega_{021} - 4u_{030})Z_{20} + (3\omega_{002} - 12u_{101})Z_{20}^2}{4u_{100} - \omega_{001}} \right]_0 \)

4. \( (Z_{04})_0 = \left[ \frac{\omega_{040} + 6(\omega_{021} - 4u_{030})Z_{02} + (3\omega_{002} - 12u_{011})Z_{02}^2}{4u_{010} - \omega_{001}} \right]_0 \)

5. \( (Z_{22})_0 = -\left[ \omega_{220} + 3\omega_{002}Z_{20}Z_{02} + (\omega_{021} - 2u_{120})Z_{20}^2 + (\omega_{201} - 2u_{210})Z_{02} \right] \)

\[ \frac{1}{3\omega_{001}} \]

6. \( (Z_{60})_0 = \left[ \omega_{600} + (15\omega_{401} - 6u_{500})Z_{20} + (15\omega_{201} - 20u_{300})Z_{40} + (15\omega_{002} - 90u_{101})Z_{20}^2 \right] \)

\[ \frac{1}{(6u_{100} - \omega_{001})}_0 \]
7. \( (z_{06})_0 = \left[ w_{060} + (15w_{041} - 6v_{050})z_{02} + (15w_{021} - 20v_{030})z_{04} \right. \\
+ (15w_{002} - 90v_{011})z_{02}z_{04} \\
\left. + (45w_{022} - 60v_{031})z_{02}^2 + (15w_{003} - 90v_{012})z_{02}^3 \right]_0 \\
\frac{(6v_{010} - w_{001})_0}{0} \\
\]

8. \( (z_{42})_0 = \left[ w_{420} + (w_{401} - 2v_{410})z_{02} + (6w_{221} - 4u_{320})z_{20} \right. \\
+ (6w_{202} - 4u_{301} - 12v_{211})z_{20}z_{02} \\
\left. + (6w_{201} - 4u_{300} - 12v_{210})z_{22} + (w_{021} - 4u_{120})z_{40} \right. \\
\left. + (w_{002} - 4u_{101} - 2v_{011})z_{40}z_{02} \right. \\
\left. + (6w_{002} - 24u_{101} - 12v_{011})z_{22}z_{20} \right. \\
\left. + (3w_{003} - 12u_{102} - 6v_{012})z_{20}^2 \right. \\
\left. + (3w_{022} - 12u_{121}z_{20}^2 \right]_0 \frac{(4u_{100} + 2v_{010} - w_{001})_0}{0}
TABLE B-2 (Cont)

9. \( \langle Z_{24} \rangle_0 = [\omega_{240} + (\omega_{041} - 2u_{140})Z_{20} + (6\omega_{221} - 4v_{230})Z_{02} \]
\[+ (6\omega_{022} - 4v_{031} - 12u_{121})Z_{02}Z_{20} \]
\[+ (6\omega_{021} - 4v_{030} - 12u_{120})Z_{22} + (\omega_{201} - 4v_{210})Z_{04} \]
\[+ (\omega_{002} - 4v_{011} - 2u_{101})Z_{04}Z_{20} \]
\[+ (6\omega_{002} - 24v_{011} - 12u_{101})Z_{22}Z_{02} \]
\[+ (3\omega_{003} - 12v_{012} - 6u_{102})Z_{02}^2Z_{20} \]
\[+ (3\omega_{202} - 12v_{211})Z_{20}^2 \]
\[\frac{1}{(4v_{010} + 2u_{100} - \omega_{001})_0} \]

10. \( \langle Z_{80} \rangle_0 = [\omega_{800} + (28\omega_{201} - 56u_{300})Z_{60} + (28\omega_{002} - 224u_{101})Z_{20} - 60 \]
\[+ (70\omega_{401} - 56u_{500})Z_{40} + (420\omega_{202} - 840u_{301})Z_{20}Z_{40} \]
\[+ (35\omega_{002} - 280u_{101})Z_{40}^2 + (210\omega_{003} - 1680u_{102})Z_{20}^2Z_{40} \]
\[+ (210\omega_{402} - 168u_{501})Z_{20}^2 + (420\omega_{203} - 840u_{302})Z_{20}^3 \]
\[+ (105\omega_{004} - 840u_{103})Z_{20}^4 + (28\omega_{601} - 8u_{700})Z_{20} \]
\[\frac{1}{(8u_{100} - \omega_{001})_0} \]
\( (Z_{08})_0 \) = \[ w_{080} + (28w_{021} - 56v_{030})Z_{06} + (28w_{002} - 224v_{011})Z_{02}Z_{06} \]

\[ + (70w_{041} - 56v_{050})Z_{40} + (420w_{022} - 840v_{031})Z_{02}Z_{04} \]

\[ + (35w_{002} - 280v_{011})Z_{04}^2 + (210w_{003} - 1680v_{012})Z_{02}Z_{04} \]

\[ + (210w_{042} - 168v_{051})Z_{02}^2 + (420w_{023} - 840v_{032})Z_{02}^3 \]

\[ + (105w_{004} - 840v_{013})Z_{02}^4 + (28w_{061} - 8v_{070})Z_{02} \]

\[ (8v_{010} - w_{001})_0 \]
TABLE B-2 (Cont)

12. \( (Z_{62})_0 = [w_{620} + (15w_{201} - 20u_{300} - 30v_{210})Z_{42} + (15w_{002} - 90u_{101} - 30v_{011})Z_{20}Z_{42} + (15w_{003} - 6u_{500} - 30v_{410})Z_{22} + (90w_{202} - 120u_{301} - 180v_{211})Z_{20}Z_{22} + (45w_{003} - 270u_{102} - 90v_{012})Z_{20}^2Z_{22} + (w_{001} - 2v_{610})Z_{02} + (15w_{402} - 6u_{301} - 30v_{411})Z_{20}^2Z_{02} + (15w_{202} - 20u_{301} - 30v_{211})Z_{40}Z_{20}Z_{02} + (45w_{203} - 60u_{302} - 90v_{212})Z_{20}^2Z_{02} + (15w_{002} - 90u_{101} - 30v_{011})Z_{40}Z_{22} + (w_{002} - 6u_{101} - 2v_{011})Z_{60}Z_{02} + (15w_{003} - 90u_{102} - 30v_{012})Z_{40}Z_{20}Z_{02} + (15w_{004} - 90u_{103} - 30v_{013})Z_{20}^3Z_{02} + (w_{021} - 6u_{120})Z_{60} + (15w_{221} - 20u_{320})Z_{40} + (15w_{022} - 90u_{121})Z_{20}Z_{40} + (45w_{222} - 60u_{311})Z_{20}^2 + (15w_{421} - 6u_{520})Z_{20}Z_{20} + (15w_{023} - 90u_{122})Z_{20}^3 \]

\[
\frac{(6u_{100} + 2v_{010} - w_{002})}{0}
\]
TABLE B-2 (Cont)

13. $(Z_{26})_0 = \{\omega_{260} + (15\omega_{021} - 20\nu_{030} - 30u_{120})Z_{24}

+ (15\omega_{002} - 90\nu_{011} - 30u_{101})Z_{02}Z_{24}

+ (15\omega_{041} - 6\nu_{050} - 30u_{140})Z_{22}

+ (90\omega_{022} - 120\nu_{031} - 180u_{121})Z_{02}Z_{22}

+ (45\omega_{003} - 270\nu_{012} - 90u_{102})Z_{02}Z_{22}

+ (\omega_{061} - 2u_{160})Z_{20} + (15\omega_{042} - 6\nu_{051} - 30u_{141})Z_{02}Z_{20}

+ (15\omega_{022} - 20\nu_{031} - 30u_{121})Z_{04}Z_{20}

+ (45\omega_{023} - 60\nu_{032} - 90u_{122})Z_{02}Z_{20}

+ (15\omega_{002} - 90\nu_{011} - 30u_{101})Z_{04}Z_{22}

+ (\omega_{002} - 6\nu_{011} - 2u_{101})Z_{06}Z_{20}

+ (15\omega_{003} - 90\nu_{012} - 30u_{102})Z_{04}Z_{02}Z_{20}

+ (15\omega_{004} - 90\nu_{013} - 30u_{103})Z_{03}Z_{02}Z_{20}

+ (\omega_{201} - 6\nu_{210})Z_{06} + (15\omega_{221} - 20\nu_{230})Z_{04}

+ (15\omega_{202} - 90\nu_{211})Z_{02}Z_{04} + (45\omega_{222} - 60\nu_{131})Z_{02}^2

+ (15\omega_{241} - 6\nu_{250})Z_{02} + (15\omega_{203} - 90\nu_{212})Z_{03}^2

\frac{(6\nu_{010} - 2u_{100} - \nu_{001})Z_{02}}{(6\nu_{010} + 2u_{100} - \nu_{001})Z_{02}}$}
TABLE B-2 (Cont)

14. \((Z_{44})_0 = [w_{440} + (6w_{021} - 24u_{120} - 4v_{030})Z_{42} + (6w_{002} - 24u_{101} - 24v_{101})Z_{02}Z_{42} + (w_{041} - 4u_{140})Z_{40} + (w_{002} - 4u_{101} - 4v_{101})Z_{40}Z_{04} + (6w_{022} - 24u_{121} - 4v_{031})Z_{40}Z_{02} + (3w_{003} - 12u_{102} - 12v_{012})Z_{40}Z_{02}Z_{22} + (18w_{002} - 72u_{101} - 72v_{011})Z_{22} + (6w_{002} - 24u_{101} - 24v_{011})Z_{24}Z_{20} + (36w_{022} - 144u_{121} - 24v_{031})Z_{20}Z_{22} + (36w_{002} - 144u_{102} - 144v_{012})Z_{20}Z_{02}Z_{22} + (3w_{042} - 12u_{141})Z_{20} + (18w_{023} - 72u_{122} - 12v_{032})Z_{20}Z_{02} + (3w_{003} - 12u_{102} - 12v_{012})Z_{20}Z_{04} + (9w_{004} - 36u_{103} - 36v_{013})Z_{20}Z_{02} + (6w_{021} - 4u_{300} - 24v_{210})Z_{24} + (36w_{221} - 24u_{320} - 24v_{230})Z_{22} + (36w_{202} - 24u_{301} - 144v_{211})Z_{02}Z_{22}\)
TABLE B-2 (Cont)

\[ + (6\omega_{241} - 4u_{340})Z_{20} \]
\[ + (36\omega_{222} - 24u_{321} - 24v_{231})Z_{20}Z_{02} \]
\[ + (6\omega_{202} - 4u_{301} - 24v_{211})Z_{20}Z_{04} \]
\[ + (18\omega_{203} - 12u_{302} - 72v_{212})Z_{20}Z_{02}^2 \]
\[ + (6\omega_{421} - 4v_{430})Z_{02} + (3\omega_{402} - 12v_{411})Z_{02}^2 \]
\[ + \frac{(\omega_{401} - 4v_{410})Z_{04}}{(4u_{100} + 4v_{010} - \omega_{001})} \]
TABLE B-3

Derivatives of the Function

\[ T_2(x,y,z) = u^2 + v^2 + w^2 + 2gz + 2\Lambda(K - K_0) = 0 \]

Evaluated at the Stagnation Point

1. \[ [u_{100}^2 + g^2_{20} + \Lambda K_{20}]_0 = 0 \]

2. \[ [v_{010}^2 + g^2_{02} + \Lambda K_{02}]_0 = 0 \]

3. \[ [4u_{100}u_{300} + 3w_{200}^2 + (12u_{100}u_{101} + 6w_{200}w_{001} + 3w_{001}^2 z_{20})z_{20} \]
   \[ + g^2_{40} + \Lambda K_{40}]_0 = 0 \]

4. \[ [4v_{010}v_{030} + 3w_{020}^2 + (12v_{010}v_{011} + 6w_{020}w_{001} + 3w_{001}^2 z_{02})z_{02} \]
   \[ + g^2_{04} + \Lambda K_{04}]_0 = 0 \]

5. \[ [2(u_{100}u_{120} + v_{010}v_{210}) + w_{20}w_{02} + w_{001}^2 z_{20}z_{02} \]
   \[ + (2u_{100}u_{101} + w_{200}w_{001})z_{20} + (2v_{010}v_{011} + w_{020}w_{001})z_{20} \]
   \[ + g^2_{22} + \Lambda K_{22}]_0 = 0 \]
TABLE B-3 (Cont)

6. \[ (6u_{100}^{100}u_{300}^{100} + 10u_{300}^{100} + 15w_{200}^{100}w_{400}^{100}) \]
\[ + 15(4u_{100}^{100}u_{300}^{100} + 4u_{300}^{100}u_{101}^{100} + 6w_{200}^{100}w_{201}^{100} + w_{400}^{100}w_{001}^{100})z_{20} \]
\[ + 45(2u_{100}^{100}u_{102}^{100} + 2u_{101}^{100} + w_{200}^{100}w_{002}^{100} + 2w_{001}^{100}w_{201}^{100})z_{20}^2 \]
\[ + 45w_{001}^{100}w_{002}^{100}z_{20}^3 + 15(2u_{100}^{100}u_{101} + w_{200}^{100}w_{001}^{100})z_{40} + 15w_{001}^{100}z_{20}z_{40}^{20} \]
\[ + gZ_{60} + \Lambda K_{60} \cdot 0 = 0 \]

7. \[ (6v_{010}^{010}v_{050}^{010} + 10v_{030}^{010} + 15w_{020}^{010}w_{040}^{010}) \]
\[ + 15(4v_{010}^{010}v_{030}^{010} + 4v_{030}^{010}v_{011}^{010} + 6w_{020}^{010}w_{021}^{010} + w_{040}^{010}w_{001}^{100})z_{02} \]
\[ + 45(2v_{010}^{010}v_{012}^{010} + 2v_{011}^{010} + w_{020}^{010}w_{002}^{100} + 2w_{001}^{100}w_{021}^{100})z_{02}^2 \]
\[ + 45w_{001}^{010}w_{002}^{010}z_{02}^3 + 15(2v_{010}^{010}v_{011} + w_{020}^{010}w_{001}^{100})z_{04} \]
\[ + 15w_{001}^{010}z_{02}z_{04} + gZ_{06} + \Lambda K_{06} \cdot 0 = 0 \]
TABLE B-3 (Cont)

8. \[ [(\nu_{100} u_{320} + \nu_{300} u_{120} + 2\nu_{010} v_{410} + 6\nu_{210} + \nu_{400} v_{020} + 6\nu_{200} v_{220}) + 6(2\nu_{100} u_{121} + \nu_{120} u_{101} + 2\nu_{010} v_{211} + 2\nu_{210} v_{011} + \nu_{020} w_{201} + \nu_{200} v_{021} + \nu_{220} w_{001})]Z_{20}

+ (4\nu_{100} u_{301} + 4\nu_{300} u_{101} + 6\nu_{200} w_{201} + \nu_{400} w_{001})Z_{02}

+ 9\nu_{001} w_{002} Z_{20}^2 Z_{02} + 6(2\nu_{100} u_{101} + \nu_{200} w_{001})Z_{22}

+ 6\nu_{001} w_{002} Z_{20}^2 + 3(2\nu_{010} v_{012} + 2\nu_{011} + 2\nu_{001} w_{021} + \nu_{020} w_{002})Z_{20}^2

+ (2\nu_{010} v_{011} + \nu_{020} w_{001})Z_{40} + \nu_{001} Z_{02} Z_{40}

+ 6(2\nu_{100} u_{102} + \nu_{101} + \nu_{200} w_{002} + 2\nu_{001} w_{201})Z_{20} Z_{02}

+ 8Z_{42} + \Lambda \chi_{42} = 0 \]
TABLE B-3 (Cont)

9. \[(4v_{010}v_{230} + 4v_{030}v_{210} + 2u_{100}u_{140} + 6u_{120}^2 + w_{040}w_{200} + 6w_{020}w_{220}) + 6(2v_{010}v_{211} + 2v_{210}v_{011} + 2u_{100}u_{121}) + 2u_{120}u_{101} + w_{200}w_{021} + w_{020}w_{201} + w_{220}w_{001})Z_{02}
\] + \[(4v_{010}v_{031} + 4v_{030}v_{011} + 6w_{020}w_{021} + w_{040}w_{001})Z_{20} + 9w_{001}w_{002}Z_{02}^2Z_{20} + 6(2v_{010}v_{011} + w_{020}w_{002})Z_{22}
\] + \[6w_{001}w_{02}Z_{02}^2 + 3(2u_{100}u_{102} + 2u_{101}^2 + 2w_{001}w_{201} + w_{200}w_{002})Z_{02}^2
\] + \[(2u_{100}u_{101} + w_{200}w_{001})Z_{04} + w_{001}Z_{20}Z_{04}
\] + \[6(2v_{010}v_{012} + 2v_{011}^2 + w_{020}w_{002} + 2w_{001}w_{021})Z_{02}Z_{20}
\] + \[gZ_{24} + \Lambda k_{24}'_0 = 0
\]
10. \[ (8u_{100}u_{700} + 56u_{300}u_{500} + 28\omega_{200}u_{600} + 35\omega_{400}^2) + (56u_{100}u_{101} + 28\omega_{001}u_{200})u_{20}^2 + 28\omega_{001}^2u_{20}u_{20}^2 + 70(4u_{100}u_{301} + 4u_{300}u_{101} + 6\omega_{200}u_{201} + \omega_{400}u_{001})u_{40}^2 + 28(6u_{100}u_{501} + 20u_{300}u_{301} + 6u_{101}u_{500} + 15\omega_{200}u_{401} + 15\omega_{400}u_{201} + 6u_{600}u_{001})u_{20}^2 + 420(2u_{100}u_{102} + 2u_{101}^2 + 2\omega_{001}u_{201} + \omega_{002}u_{200})u_{20}^3u_{40}^2 + 630\omega_{001}u_{002}u_{20}^2u_{40}^2 + 35\omega_{001}^2u_{40}^3 + 420(2u_{100}u_{103} + 6u_{101}u_{102} + 3\omega_{001}u_{202} + 3\omega_{002}u_{201} + \omega_{003}u_{200})u_{20}^3u_{20}^3 + 210(4u_{100}u_{302} + 8u_{101}u_{301} + 4u_{300}u_{102} + 6\omega_{200}u_{202} + 6\omega_{201}^2 + \omega_{400}u_{002} + 2\omega_{401}u_{001})u_{20}^2 + 105(4\omega_{001}u_{003} + 3\omega_{002}u_{20}^2 + \omega_{80}^2 + \omega_{80}^3) = 0 \]
TABLE B-3 (Cont)

11. \[
\left( 8\nu_{010}^0\nu_{070} + 56\nu_{030}^0\nu_{050} + 28\nu_{020}^0\nu_{060} + 35\nu_{040}^0 \right) \\
+ 56\nu_{010}^0\nu_{011} + 28\nu_{001}^0\nu_{020}^2 \nu_{06} + 28\nu_{001}^0\nu_{02}^2 \nu_{06} \\
+ 70(4\nu_{010}^0\nu_{031} + 4\nu_{030}^0\nu_{011} + 6\nu_{020}^0\nu_{021} + \nu_{040}^0\nu_{001}^0 \nu_{04} \\
+ 28(6\nu_{010}^0\nu_{051} + 20\nu_{030}^0\nu_{031} + 6\nu_{011}^0\nu_{050} + 15\nu_{020}^0\nu_{041} \\
+ 15\nu_{040}^0\nu_{021} + \nu_{060}^0\nu_{001}^0 \nu_{04} \\
+ 420(2\nu_{010}^0\nu_{012} + 2\nu_{011}^0 + 2\nu_{001}^0\nu_{021} + \nu_{002}^0\nu_{020}^0 \nu_{02}^2 \nu_{04} \\
+ 630 \nu_{001}^0\nu_{002}^0 \nu_{02}^2 \nu_{04} + 35\nu_{001}^0\nu_{02}^2 \nu_{04} \\
+ 420(2\nu_{010}^0\nu_{013} + 6\nu_{011}^0\nu_{012} + 3\nu_{001}^0\nu_{022} + 3\nu_{002}^0\nu_{021} \\
+ \nu_{003}^0\nu_{020}^0 \nu_{02}^3 + 210(4\nu_{010}^0\nu_{032} + 8\nu_{011}^0\nu_{031} \\
+ 4\nu_{030}^0\nu_{012} + 6\nu_{020}^0\nu_{022} + 6\nu_{021}^0 + \nu_{040}^0\nu_{002} \\
+ 2\nu_{041}^0\nu_{001}^0 \nu_{02}^2 + 105(4\nu_{001}^0\nu_{003} + 3\nu_{002}^0 \nu_{02}^2 \nu_{04} \\
+ 8\nu_{08}^0 + \nu_{08}^0 \nu_{08}^0 \nu_{08}^0 \nu_{08}^0 \nu_{08}^0 = 0
\]
TABLE B-3 (Cont)

12. \[
\begin{align*}
&\left((4u_{100}u_{340} + 24u_{120}u_{320} + 4u_{300}u_{140} + 4v_{010}v_{430} + 24v_{210}v_{230} + 4v_{030}v_{410} + 6\omega_{020}\omega_{420} + 6\omega_{200}\omega_{240} + 18\omega_{220}^2 + \omega_{400}\omega_{040}) + 36(2u_{100}u_{121} + 2u_{120}u_{101}
+ 2v_{010}v_{211} + 2v_{210}v_{011} + \omega_{020}\omega_{201} + \omega_{200}\omega_{021} + \omega_{220}\omega_{001}\right)Z_{22}^2
+ 6(2u_{100}u_{141} + 12u_{120}u_{121} + 2u_{140}u_{101} + 4v_{010}v_{231} + 4v_{030}v_{211}
+ 4v_{210}v_{031} + 4v_{230}v_{011} + 6\omega_{020}\omega_{221} + \omega_{040}\omega_{201}
+ \omega_{200}\omega_{041} + 6\omega_{220}\omega_{021} + \omega_{240}\omega_{001}\right)Z_{20}^2
+ 6(2v_{010}v_{411} + 12v_{210}v_{211} + 2v_{410}v_{011} + 4u_{100}u_{321} + 4u_{300}u_{121}
+ 4u_{120}u_{301} + 4u_{320}u_{101} + 6\omega_{200}\omega_{221} + \omega_{400}\omega_{021}
+ \omega_{020}\omega_{401} + 6\omega_{220}\omega_{201} + \omega_{420}\omega_{001}\right)Z_{02}^2
+ 6(2u_{100}u_{101} + \omega_{200}\omega_{001})Z_{24}^2 + 6(2v_{010}v_{011} + \omega_{020}\omega_{001})Z_{42}^2
+ \left((4u_{100}u_{301} + 4u_{300}u_{101} + 6\omega_{200}\omega_{201} + \omega_{400}\omega_{001}\right)Z_{04}^2
+ \left((4v_{010}v_{031} + 4v_{030}v_{011} + 6\omega_{020}\omega_{021} + \omega_{040}\omega_{001}\right)Z_{40}^2
+ 6\omega_{001}\omega_{20}\omega_{24}^2 + 6\omega_{001}\omega_{02}\omega_{42}^2 + 18\omega_{001}\omega_{22}^2
+ 36(2v_{010}v_{012} + 2v_{011} + \omega_{020}\omega_{002} + \omega_{001}\omega_{021})Z_{20}Z_{22}^2
+ 36(2u_{100}u_{102} + 2u_{101} + \omega_{200}\omega_{020} + 2\omega_{001}\omega_{201})Z_{02}Z_{22}^2
\end{align*}
\]
TABLE B-3 (Cont)

\[ + w_{001}^2z_{40}z_{04} + 9w_{001}w_{002}^2z_{20}z_{04} + 9w_{001}w_{002}z_{02}z_{40} \]
\[ + 6(v_{010}v_{012} + v_{011}^2 + w_{020}w_{002} + 2w_{001}w_{021})z_{02}z_{04} \]
\[ + 6(u_{100}u_{102} + u_{101}^2 + w_{200}w_{002} + 2w_{001}w_{201})z_{20}z_{04} \]
\[ + 36(u_{100}u_{122} + 4u_{101}u_{121} + 2u_{120}u_{102} + 2v_{010}v_{212} \]
\[ + 4v_{011}v_{211} + 2v_{012}v_{210} + 2w_{001}w_{221} + w_{020}w_{202} \]
\[ + 2w_{021}w_{201} + w_{002}w_{220} + w_{022}w_{200})z_{20}z_{02} \]
\[ + 108w_{001}w_{002}z_{20}z_{02}z_{22} \]
\[ + 18(2v_{010}v_{013} + 6v_{011}v_{012} + w_{020}w_{003} + 3w_{001}w_{022} \]
\[ + 3w_{002}w_{021})z_{20}z_{02} \]
\[ + 18(u_{100}u_{103} + 6u_{101}u_{102} + w_{200}w_{003} + 3w_{001}w_{202} \]
\[ + 3w_{002}w_{021})z_{20}z_{20} \]
\[ + 9(4w_{001}w_{003} + 3w_{002})z_{20}z_{02}^2 \]
\[ + 3(4u_{100}u_{302} + 8u_{101}u_{301} + 4u_{300}u_{102} + 6w_{200}w_{202} \]
\[ + 6w_{201} + w_{400}w_{002} + 2w_{401}w_{001})z_{02}^2 \]
\[ + 3(4v_{010}v_{032} + 8v_{011}v_{031} + 4v_{030}v_{012} + 6w_{020}w_{022} \]
\[ + 6w_{021} + w_{040}w_{002} + 2w_{041}w_{001})z_{20}^2 \]
\[ + gZ_{44} + \Lambda_K_{44} \] = 0
13. \[
15(3^{2}\omega_{002}^{2} + 4\omega_{001}^{003})z_{20}^{3}z_{02}^{3}
+ 15(3\omega_{002}^{001} + 3\omega_{001}^{002} + 2\omega_{013}^{010} + \omega_{003}^{020})
+ 6\omega_{012}^{011})z_{20}^{3}
+ 45(2u_{100}^{u,103} + 3\omega_{002}^{201} + 6u_{102}^{u,101} + 3\omega_{001}^{022})
+ \omega_{200}^{u,003})z_{20}^{2}z_{02}^{2}
+ 135\omega_{001}^{002}z_{20}^{2}z_{22} + 15\omega_{001}^{2}z_{20}^{2}z_{42}
+ 45(2u_{100}^{u,122} + \omega_{002}^{220} + 2u_{102}^{u,120} + 2\omega_{201}^{021})
+ 2\omega_{001}^{221} + \omega_{200}^{u,022} + 4u_{101}^{u,121} + 2\omega_{212}^{v,010}
+ \omega_{202}^{u,020} + 2\omega_{102}^{v,210} + 4\omega_{101}^{v,211})z_{20}^{2}
+ 45\omega_{001}^{002}z_{20}^{2}z_{20}^{40}
+ 15(4u_{100}^{u,302} + \omega_{002}^{400} + 4u_{102}^{u,300} + 6\omega_{201}^{2})
+ 8u_{301}^{u,101} + 2\omega_{001}^{401} + 6\omega_{200}^{202})z_{20}^{2}z_{02}^{2}
+ 15(\omega_{002}^{002} + 2\omega_{001}^{002} + 2\omega_{012}^{010} + 2\omega_{011}^{2}z_{20}^{2}z_{40}
+ 90(2u_{100}^{u,102} + \omega_{002}^{200} + 2\omega_{201}^{001} + 2u_{101}^{2})z_{20}^{2}z_{22}
+ 15(4u_{100}^{u,321} + \omega_{400}^{u,201} + 6\omega_{201}^{u,220} + 4u_{300}^{u,121})
+ 4u_{301}^{u,120} + \omega_{001}^{u,420} + 6\omega_{200}^{u,221} + 4u_{101}^{u,320}.
TABLE B-3 (Cont)

\[ + 2v_{011}v_{410} + 2v_{411}v_{010} + v_{401}w_{020} + 12v_{210}v_{211} )Z_{20} \]

\[ + 15(u_{100}u_{102} + w_{002}w_{200} + 2w_{001}w_{201} + 2u_{101}^2 )Z_{02}Z_{40} \]

\[ + w_{001}^2 Z_{02}^2 Z_{60} + 15w_{001}Z_{40}Z_{22} \]

\[ + (6u_{100}u_{501} + 15w_{201}w_{400} + 20u_{300}u_{301} + w_{001}w_{600} \]

\[ + 15w_{200}w_{401} + 6u_{101}u_{500} )Z_{02} \]

\[ + 15(2u_{100}u_{121} + w_{201}w_{020} + w_{001}w_{220} + w_{200}w_{021} \]

\[ + 2u_{101}u_{120} + 2v_{011}v_{210} + 2v_{010}v_{211} )Z_{40} \]

\[ + 15(4u_{100}u_{301} + w_{400}w_{001} + 6w_{201}w_{200} + 4u_{101}u_{300} )Z_{22} \]

\[ + (w_{001}w_{020} + 2v_{010}v_{011} )Z_{60} \]

\[ + 15(2u_{100}u_{101} + w_{001}w_{200} )Z_{42} \]

\[ + (6u_{100}u_{520} + 15w_{400}w_{220} + 20u_{300}u_{320} \]

\[ + 15w_{200}w_{420} + 2v_{010}v_{610} + 30v_{410}v_{210} + w_{600}w_{020} \]

\[ + 6u_{500}u_{120} ) + gZ_{62} + \Lambda K_{62}^0 \]
TABLE B-3 (Cont)

14. \[ 15(3\omega_{002}^2 + 4\omega_{001}^003)^3Z_{02}Z_{20} \]

\[ + 15(3\omega_{002}^0201 + 3\omega_{001}^0202 + 2u_{103}^0u_{100} + \omega_{003}^3\omega_{200} \]

\[ + 6u_{102}^0u_{101}^2Z_{02} \]

\[ + 45(2\nu_{010}^0013 + 3\omega_{002}^0021 + 6\nu_{012}^0011 + 3\omega_{001}^0202 \]

\[ + \omega_{020}^0003Z_{02}^2Z_{20} \]

\[ + 135\omega_{001}^0002Z_{02}^2Z_{22} + 15\omega_{001}^2001Z_{02}Z_{24} \]

\[ + 45(2\nu_{010}^0212 + \omega_{002}^0220 + 2\nu_{012}^0210 + 2\omega_{021}^0201 \]

\[ + 2\omega_{001}^0221 + \omega_{020}^0202 + 4\nu_{011}^0211 + 2u_{122}^0u_{100} \]

\[ + \omega_{022}^0200 + 2u_{012}^0u_{120} + 4u_{011}^0u_{121}Z_{02}^2 \]

\[ + 45\omega_{001}^0002Z_{02}Z_{02}Z_{04} \]

\[ + 15(4\nu_{010}^0032 + \omega_{002}^0040 + 4\nu_{012}^0030 + 6\omega_{021}^2 \]

\[ + 8\nu_{031}^0011 + 2\omega_{001}^0041 + 6\omega_{020}^0022)Z_{02}Z_{20} \]

\[ + 15(\omega_{002}^0200 + 2\omega_{001}^0002 + 2u_{102}^0u_{100} + 2u_{101}^2Z_{02}Z_{04} \]

\[ + 90(2\nu_{010}^0012 + \omega_{002}^0020 + 2\omega_{021}^0001 + 2\nu_{101}^2)Z_{02}Z_{22} \]

\[ + 15(4\nu_{010}^0231 + \omega_{040}^0021 + 6\omega_{021}^0220 + 4\nu_{030}^0211 \]

\[ + 4\nu_{031}^0210 + \omega_{001}^0240 + 6\omega_{020}^0221 + 4\nu_{011}^0230 \]
TABLE B-3 (Cont)

\[ + 2u_{101}u_{140} + 2u_{141}u_{100} + w_{041}w_{200} + 12u_{120}u_{121}Z_{02} \]
\[ + 15(2v_{010}v_{012} + w_{002}w_{020} + 2w_{001}w_{021} + 2v_{011}^2)Z_{20}Z_{04} \]
\[ + \frac{2}{w_{001}}w_{20}Z_{06} + 15w_{001}Z_{04}Z_{22} \]
\[ + (6v_{010}v_{051} + 15w_{021}w_{040} + 20v_{030}v_{031} + w_{001}w_{060} \]
\[ + 15w_{020}w_{041} + 6v_{011}v_{050})Z_{20} \]
\[ + 15(2v_{010}v_{211} + w_{021}w_{200} + w_{001}w_{220} + w_{020}w_{201} \]
\[ + 2v_{011}v_{210} + 2u_{101}u_{120} + 2u_{100}u_{121}Z_{04} \]
\[ + 15(4v_{010}v_{031} + w_{040}w_{001} + 6w_{021}w_{020} + 4v_{011}v_{030})Z_{22} \]
\[ + (w_{001}w_{200} + 2u_{100}u_{011})Z_{06} \]
\[ + 15(2v_{010}v_{011} + w_{001}w_{020})Z_{24} \]
\[ + (6v_{010}v_{250} + 15w_{040}w_{220} + 20v_{030}v_{230} \]
\[ + 15w_{020}w_{240} + 2u_{100}u_{160} + 30v_{140}v_{120} + w_{060}w_{200} \]
\[ + 6v_{050}v_{210} + g^2Z_{26} + \Lambda K_{26}^2 \]
APPENDIX C

Derivation of Equations Required to Solve for Series Coefficients in Cylindrical Coordinates

From Chapter 2 and Appendix B, the two basic equations defining the bubble surface are

\[ T_1(x,y,z) = V_x^2 + 2gZ + 2\Lambda(K - K_0) = 0 \]
\[ T_2(x,y,z) = V_x Z_x + V_y Z_y - V_z = 0 \]

where \( \vec{V} = (V_x, V_y, V_z) \).

In cylindrical coordinates, the equations may be written as

\[ T_1(r,z) = u^2 + v^2 + 2gZ + 2\Lambda(K - K_0) = 0 \]
\[ T_2(r,z) = vZ_r - u = 0 \quad \text{(C-1)} \]

where \( u(r,z), v(r,z) \) are respectively the axial and radial components of the velocity vector. The function \( K \) is given by A-15 for the cylindrical coordinate system.

Letting \( D \) denote the tube diameter, the Eotvos Number \( E_o \) is given by

\[ E_o = \frac{gD^2(\rho_\ell - \rho_a)}{\sigma} \]

or

\[ E_o = \frac{gD^2}{\Lambda} \]
Then C-1 becomes for a circular duct

\[ T_1(r,z) = u^2 + v^2 + 2g z + \frac{2\pi D^2}{E_o} (K - K_0) = 0 \]

\[ T_2(r,z) = v Z_r - u = 0 \]  \hspace{1cm} (C-2)

As in Appendix B, let \( T(r,z) = 0 \) represent any arbitrary surface with the restriction that \( z = f(r) \) and

\[ \left( \frac{\partial Z}{\partial r} \right)^m \equiv 0 \quad , \quad m \text{ odd} \]

Defining

\[ T_{mn} \equiv \frac{\partial (m+n) T}{\partial r^m \partial z^n} \]

\[ Z_m \equiv \frac{\partial^m Z}{\partial r^m} \]

then one has

\[ \left( \frac{D^2 T}{Dr^2} \right)_0 = [T_{20} + T_{01} Z_2]_0 = 0 \]  \hspace{1cm} (C-3)

\[ \left( \frac{D^4 T}{Dr^4} \right)_0 = [T_{40} + 6T_{21} Z_2 + 3T_{02} Z_2^2 + T_{01} Z_4]_0 = 0 \]  \hspace{1cm} (C-4)
\[
\begin{align*}
\left( \frac{\partial^6 T}{\partial r^6} \right)_0 &= \left[ T_{60} + 15 T_{41} z_2 \z_4 + 15 T_{21} z_4^2 + 45 T_{22} z_2^2 \right. \\
&\quad + 15 T_{02} z_2 z_4 + 15 T_{03} z_2^3 + T_{01} z_6^6 \biggr]_0 = 0 \quad \text{(C-5)}
\end{align*}
\]

\[
\begin{align*}
\left( \frac{\partial^8 T}{\partial r^8} \right)_0 &= \left[ T_{80} + 70 T_{41} z_4 + 28 T_{61} z_2 + 210 T_{42} z_2^2 \right. \\
&\quad + 28 T_{21} z_6 + 420 T_{22} z_2 z_4 + 420 T_{23} z_2^3 \\
&\quad + 28 T_{02} z_2 z_6 + 210 T_{03} z_2 z_4 + 35 T_{02} z_2^2 \\
&\quad + 105 T_{04} z_2^4 + T_{01} z_8 \biggr]_0 = 0 \\
&\quad \text{(C-6)}
\end{align*}
\]

\[
\begin{align*}
\left( \frac{\partial^{10} T}{\partial r^{10}} \right)_0 &= \left[ T_{10,0} + 45 T_{81} z_2 + 210 T_{41} z_6 + 210 T_{61} z_4 \right. \\
&\quad + 3150 T_{42} z_2^2 z_4 + 630 T_{62} z_2^2 + 3150 T_{43} z_2^3 \\
&\quad + 45 T_{21} z_8 + 1260 T_{22} z_2 z_6 + 1575 T_{22} z_2^2 \\
&\quad + 9450 T_{23} z_2^2 z_4 + 4725 T_{24} z_2^4 + 45 T_{02} z_2 z_8 \\
&\quad + 210 T_{02} z_4 z_6 + 630 T_{03} z_2 z_6 + 1575 T_{03} z_2 z_4^2 \\
&\quad + 3150 T_{04} z_2^3 z_4 + 9450 T_{05} z_2^5 + T_{01} z_10^6 \biggr]_0 = 0 \quad \text{(C-7)}
\end{align*}
\]
\[
\left( \frac{D_2^{12}}{D_{12}} \right)_0 = \left[ T_{12,0} + 462Z^2_{6T02} + 13860Z^3_{6T03} + 13860Z^4_{6T22} + 13860Z^5_{6T04} + 41580Z^2_{2T23} + 13860Z^2_{6T42} + 924Z^3_{6T61} + 5775Z^3_{03} + 1485Z^2_{8T03} + 2970Z^2_{8T22} + 495Z^2_{8T41} + 51975Z^2_{4T04} + 103950Z^2_{4T22} + 17325Z^2_{4T42} + 495Z^2_{8T02} + 51975Z^2_{4T05} + 207900Z^3_{4T24} + 103950Z^2_{4T43} + 13860Z^2_{4T62} + 495Z^2_{8T81} + 66Z^2_{10T02} + 66Z^2_{10T21} + 10395Z^2_{2T06} + 62370Z^2_{2T25} + 51975Z^2_{2T44} + 13860Z^3_{2T63} + 1485Z^2_{8T82} + 66Z_{10T10,1} + Z^2_{12T01,0} \right] \]

(C-8)
\[
\left( \frac{D_{\text{14T}}}{\text{Dr}_{\text{14}}} \right)_{0} = \left[ T_{14,0} + 1001 T_{41} Z_{10} + 42042 T_{22} Z_{6}^2 + 45045 T_{22} Z_{4} Z_{8} \right. \\
\left. + 6006 T_{22} Z_{2} Z_{10} + 42042 T_{10} Z_{2}^2 Z_{2} + 105105 T_{03} Z_{2}^2 Z_{8} \right. \\
\left. + 45045 T_{03} Z_{4} Z_{2} Z_{8} + 3003 T_{03} Z_{2}^2 Z_{10} + 3003 T_{02} Z_{6} Z_{8} \right. \\
\left. + 630630 T_{04} Z_{6} Z_{4} Z_{2} Z_{2} + 1261260 Z_{6} Z_{2} Z_{4} Z_{23} + 210210 Z_{6} Z_{4} T_{42} \right. \\
\left. + 315315 Z_{6} Z_{4}^2 T_{20} + 1261260 Z_{6} Z_{2}^3 T_{24} + 630630 Z_{6} Z_{2}^2 T_{43} \right. \\
\left. + 84084 Z_{6} Z_{2}^2 T_{62} + 3003 Z_{6}^8 + 91 Z_{12} T_{21} \right. \\
\left. + 1001 Z_{4}^3 Z_{10} T_{02} + 91 Z_{2} Z_{12} T_{02} + 525525 Z_{4} Z_{2}^3 T_{04} \right. \\
\left. + 525525 Z_{4}^3 T_{23} + 1576575 Z_{4} Z_{2}^3 T_{05} + 4729725 Z_{2} Z_{2}^2 T_{24} \right. \\
\left. + 1576575 Z_{4}^2 T_{23} + 105105 Z_{4} T_{62} + 945945 Z_{4}^2 T_{06} \right. \\
\left. + 4729725 Z_{4}^2 Z_{2} T_{25} + 3153150 Z_{4} Z_{2}^3 T_{44} + 630630 Z_{4}^2 Z_{2}^6 T_{63} \right. \\
\left. + 45045 Z_{4} Z_{2}^2 T_{82} + 1001 Z_{4}^7 T_{10} + 135135 Z_{4}^7 T_{07} \right. \\
\left. + 945945 Z_{4}^6 T_{26} + 945945 Z_{4}^5 T_{45} + 315315 Z_{2}^4 T_{64} \right. \\
\left. + 45045 Z_{2}^3 T_{83} + 45045 Z_{2}^3 Z_{8} T_{04} + 3003 Z_{2} T_{10} + 2 \right. \\
\left. + 135135 Z_{2} Z_{8} T_{23} + 91 Z_{2} T_{12} + 45045 Z_{2} Z_{8} T_{42} \right. \\
\left. + 3003 Z_{8} T_{61} + Z_{14} T_{01} + \right]_{0}
\]
Applying C-3 through C-9 to $T_1$ and $T_2$ will yield the results in Tables C-2 and C-3, where the required derivatives of

$$K \equiv \left[ \frac{rZ_2 + (1 + Z_1^2)Z_1}{r(1 + Z_1^2)^{3/2}} \right]$$

are provided in Table C-1.

The equations presented in this appendix were derived in part by use of the algebraic programming system REDUCE-2 (Hearn, 1973).
TABLE C-1

Derivatives of Function $K(r)$ Evaluated at Stagnation Point (Bubble Nose)

$$K = -\left[ \frac{1}{R_1} + \frac{1}{R_2} \right]$$

1. $(K_2)_0 = -\left[ 4Z_2^3 - \frac{4}{3}Z_4 \right]_0$

2. $(K_4)_0 = -\left[ 36Z_2^2Z_4 - 54Z_2^5 - \frac{6}{5}Z_6 \right]_0$

3. $(K_6)_0 = -\left[ 72Z_2^2Z_6 + 240Z_2Z_4^2 + 1800Z_2^7 - 1800Z_2^4Z_4 - \frac{8}{7}Z_8 \right]_0$

4. $(K_8)_0 = -\left[ 120Z_2Z_8 + 1680Z_2^2Z_4Z_6 + \frac{2800}{3}Z_4^3 - 6300Z_2Z_6^4 \right.
   - 42000Z_2^3Z_4^2 - 110250Z_2^9 + 147000Z_2^6Z_4 - \frac{10}{9}Z_8 \left. \right]_0$

5. $(K_{10})_0 = -\left[ 180Z_2^2Z_{10} - 16200Z_2^4Z_8 - 453600Z_2^3Z_4Z_6 \right.
   - 756000Z_2^2Z_4^3 + 793800Z_2^6Z_6 + 793800Z_2^5Z_4^2
   + 10716300Z_2^{11} - 17860500Z_2^8Z_4 + 4320Z_2Z_4Z_8
   + 4536Z_2Z_6^2 + 15120Z_4^2Z_6 - \frac{12}{11}Z_8 \left. \right]_0$
TABLE C-1 (Cont)

6. \( (K_{12})_0 = -[252Z^2_{Z_{12}} - 1746360Z_{6Z_2}^2 + 116424Z_{6Z_4}^2 \]
   \(- 137525850Z_{6Z_2^8} + 122245200Z_{6Z_4Z_2^5} \]
   \(- 17463600Z_{6Z_4Z_2^2} + 33264Z_{6Z_2Z_8} \]
   \(- 1512784350Z_{2Z_4}^{13} + 3025568700Z_{2Z_4}^{10Z_4} \]
   \(- 1833678000Z_{2Z_4}^7 + 2910600Z_{2Z_8}^6 \]
   \(+ 33957000Z_{2Z_4}^3 - 34650Z_{2Z_10}^4 \]
   \(- 1663200Z_{2Z_4Z_8}^3 - 9702000Z_{2Z_4}^4 \]
   \(+ 9240Z_{2Z_4Z_10} + 55440Z_{4Z_8}^2 - \frac{14}{13} Z_{14Z_4} \]
TABLE C-1 (Cont)

7. \( (K_{14})_0 = -[336Z_{14}^2 Z_{14} + 403603Z_6^3 + 635675040Z_6^2 Z_{14}^2 + 181621440Z_{14}^2 Z_{14}^2 + 31465914480Z_6 Z_{12}^{10} \]

\[-38140502400Z_6 Z_{12}^{10} + 10594584000Z_6 Z_{12}^2 Z_{12}^2 - 17297280Z_6 Z_{12}^3 - 403603200Z_6 Z_{12}^3 Z_{12}^2 \]

\[+ 96096Z_6 Z_{12}^2 + 1153152Z_6 Z_{12}^4 \]

\[+ 292183491600Z_6 Z_{12}^{15} - 681761480400Z_6 Z_{12}^{12} \]

\[+ 524431908000Z_6^2 Z_{12}^6 - 681080400Z_6 Z_{12}^8 \]

\[-148324176000Z_6^2 Z_{12}^6 + 8408400Z_6 Z_{12}^6 \]

\[+ 605404800Z_6^2 Z_{12}^5 - 65520Z_{12} Z_{12}^4 \]

\[+ 117717600000Z_{12}^3 Z_{12}^3 - 4804800Z_{12} Z_{12}^3 Z_{12}^2 \]

\[86486400Z_6 Z_{12}^2 Z_{12}^2 + 17472Z_{12} Z_{12}^3 Z_{12}^2 \]

\[+ 82368Z_6 Z_{12}^2 Z_{12} - 67267200Z_{12}^5 \]

\[+ 160160Z_{12} Z_{12}^2 Z_{12} - \frac{16}{15} Z_{16}^{10} \]

\[+ \frac{16}{15} Z_{16}^{10} \]
TABLE C-2

Equations for Derivatives of Bubble Shape $Z(r)$

Evaluated at Stagnation Point

1. $(Z_2)_0 = \left[ \frac{u_20}{2v_{10} - u_{01}} \right]$

2. $(Z_4)_0 = \left[ \frac{u_40 + (6u_{21} - 4v_{30})Z_2 + (3u_{02} - 12v_{11})Z_2^2}{4v_{10} - u_{01}} \right]$

3. $(Z_6)_0 = \left[ u_{60} + (15u_{41} - 6v_{50})Z_2 + (15u_{21} - 20v_{30})Z_4 + (15u_{02} - 90v_{11})Z_2Z_4 + (45u_{22} - 60v_{31})Z_2^2 + (15u_{03} - 90v_{12})Z_2^3 \right]_0$

4. $(Z_8)_0 = \left[ u_{80} + (28u_{21} - 56v_{30})Z_2 + (28u_{02} - 224v_{11})Z_2Z_6 + (70u_{41} - 56v_{50})Z_4 + (420u_{22} - 840v_{31})Z_2Z_4 + (35u_{02} - 280v_{11})Z_4^2 + (210u_{03} - 1680v_{12})Z_2Z_4 + (210u_{42} - 168v_{31})Z_2^2 + (420u_{23} - 840v_{32})Z_2^2 + (105u_{04} - 840v_{13})Z_2^2 + (28u_{61} - 8v_{70})Z_2 \right]_0$

\[ \frac{(8v_{10} - u_{01})_0}{(8v_{10} - u_{01})_0} \]
5. \((Z_{10})_0 = \left[ u_{10,0} + (45u_{21} - 120v_{30})Z_8 + (210u_{41} - 252v_{50})Z_6 \right.
\]
\[+ (210u_{61} - 120v_{70})Z_4 + (45u_{81} - 10v_{90})Z_2 \]
\[+ (45u_{02} - 450v_{11})Z_2^2Z_8 + (1260u_{22} - 3360v_{31})Z_2^2Z_6 \]
\[+ (3150u_{42} - 3780v_{51})Z_2Z_4^2 + (630u_{62} - 360v_{71})Z_2^2Z_4^2 \]
\[+ (210u_{02} - 2100v_{11})Z_4Z_6 + (1575u_{22} - 4200v_{31})Z_4^2 \]
\[+ (1575u_{03} - 15750v_{12})Z_2Z_6^2 + (9450u_{23} - 25200v_{32})Z_2^2Z_6^2 \]
\[+ (630u_{03} - 6300v_{12})Z_2^2Z_6 + (4725u_{24} - 12600v_{33})Z_2^4 \]
\[+ (3150u_{43} - 3780v_{52})Z_2^3Z_4 + (3150u_{04} - 31500v_{13})Z_2^3Z_4^2 \]
\[+ (945u_{05} - 9450v_{14})Z_2^5 \]
\[\frac{1}{(10v_{10} - u_{01})Z_2^0} \]
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(T6Ai7 _ Z8n6)^zg9I + (U a ZI - COn)8ZZZS8l7T +

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(ETazi _ *°n)*zZZS£6TS + (Zi*9 - E9nZ) ^Z086I +
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(EEa8 _ *Vng) ZZ56£0T + ( ^ 3 1 - fi0n)*ZZZSZ6TS +

(VCaot - £Zr>£) ZZ06Z03 + (STa3T -9°n)gZS6£0T] = °(2TZ)

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7. \((Z_{14})_0 = [105105Z_{4}^{2}Z_{6}^{2}(u_{03} - 14v_{12}) + 15015Z_{4}^{2}(7u_{62} - 8v_{71})
+ 210210Z_{4}^{2}(u_{42} - 2v_{51}) + 45045Z_{4}^{2}(u_{22} - 4v_{31})
+ 1001Z_{4}^{2}Z_{10}(u_{02} - 14v_{11}) + 91Z_{4}^{2}(11u_{10,1} - 4v_{11,0})
+ 42042Z_{6}^{2}(u_{22} - 4v_{31}) + 3003Z_{6}^{2}(u_{02} - 14v_{11})
+ 1001Z_{6}^{2}(3u_{81} - 2v_{90}) + 429Z_{8}(7u_{61} - 8v_{70})
+ 1001Z_{10}(u_{41} - 2v_{50}) + 91Z_{12}(u_{21} - 4v_{30})
+ 135135Z_{2}^{7}(u_{07} - 14v_{16}) + 945945Z_{2}^{6}(u_{26} - 4v_{35})
+ 945945Z_{2}^{5}Z_{4}(u_{06} - 14v_{15}) + 945945Z_{2}^{5}(u_{45} - 2v_{54})
+ 4729725Z_{2}^{4}(u_{25} - 4v_{34}) + 315315Z_{2}^{4}Z_{6}(u_{05} - 14v_{14})
+ 45045Z_{2}^{4}(7u_{64} - 8v_{73}) + 1576575Z_{2}^{3}Z_{4}(u_{05} - 14v_{14})
+ 3153150Z_{2}^{3}Z_{4}(u_{25} - 2v_{35}) + 1261260Z_{2}^{3}Z_{6}(u_{24} - 4v_{53})
+ 45045Z_{2}^{3}Z_{8}(u_{04} - 14v_{13}) + 15015Z_{2}^{3}(3u_{83} - 2v_{92})
+ 4729725Z_{2}^{2}Z_{4}(u_{24} - 4v_{34}) + 630630Z_{2}^{2}Z_{6}(u_{04} - 14v_{13})
+ 90090Z_{2}^{2}Z_{8}(7u_{63} - 8v_{72}) + 630630Z_{2}^{2}Z_{6}(u_{43} - 2v_{52})
+ 135135Z_{2}^{2}(u_{23} - 4v_{32}) + 3003Z_{2}^{2}Z_{10}(u_{03} - 14v_{12})
+ 273Z_{2}^{2}(11u_{10,2} - 4v_{11,1}) + 525525Z_{2}^{2}Z_{4}^{3}(u_{04} - 14v_{13})
\]
TABLE C-2 (Cont)

\[ + 1576575z_2^2z_4^2 (u_{43} - 2v_{52}) + 1261260z_2^4z_6^5 (u_{23} - 4v_{32}) \]

\[ + 45045z_2^2z_4^8 (u_{03} - 14v_{12}) + 15015z_2^4z_6^4 (3u_{82} - 2v_{91}) \]

\[ + 42042z_2^2z_6^2 (u_{03} - 14v_{12}) + 12012z_2^2z_6^2 (7u_{62} - 8v_{71}) \]

\[ + 45045z_2^2z_8^2 (u_{42} - 2v_{51}) + 6006z_2^2z_{10}^2 (u_{22} - 4v_{31}) \]

\[ + 91z_2^2z_{12}^2 (u_{02} - 14v_{11}) + 7z_2^2 (13u_{12,1} - 2v_{13,0}) \]

\[ + 525525z_4^3 (u_{23} - 4v_{32}) + u_{14,0} \]

\[ (14v_{10} - u_{01})_0 \]
TABLE C-3

Derivatives of the Function

\[ T_2(r, z) = u^2 + v^2 + 2gz + 2\Lambda(K - K_0) = 0 \]

Evaluated at the Stagnation Point

1. \[ [v_{10}^2 + gZ_2 + \Lambda K_2]_0 = 0 \]

2. \[ [4v_{10}v_{30} + 3u_{20}^2 + (12v_{10}v_{11} + 6u_{20}u_{01} + 3u_{01}^2)Z_2 + gZ_4 + \Lambda K_4]_0 = 0 \]

3. \[ [(6v_{10}v_{50} + 10v_{30}^2 + 15u_{20}u_{40}) + 15(4v_{10}v_{31} + 4v_{30}v_{11} + 6u_{20}u_{21} + u_{40}u_{01})Z_2 + 45(2v_{10}v_{12} + 2v_{11}^2 + u_{20}u_{02} + 2u_{01}u_{21})Z_2^2 + 45u_{01}u_{02}Z_2^3 + 15(2v_{10}v_{11} + u_{20}u_{01})Z_4 + 15u_{01}^2Z_4^2 + gZ_6 + \Lambda K_6]_0 = 0 \]
### TABLE C-3 (Cont)

4. \( [(8v_{10}v_{70} + 56v_{30}v_{50} + 28u_{20}u_{60} + 35u_{40}^2)] \\
+ (56v_{10}v_{11} + 28u_{01}u_{20})Z_6 + 28u_{01}^2Z_2Z_6 \\
+ 70(4v_{10}v_{31} + 4v_{30}v_{11} + 6u_{20}u_{21} + u_{40}u_{01})Z_4 \\
+ 28(6v_{10}v_{51} + 20v_{30}v_{31} + 6v_{11}v_{50} + 15u_{20}u_{41} \\
+ 15u_{40}u_{21} + u_{60}u_{01})Z_2 \\
+ 420(2v_{10}v_{12} + 2v_{11}^2 + 2u_{01}u_{21} + u_{02}u_{20})Z_2Z_4 \\
+ 630u_{01}u_{02}Z_2Z_4 + 35u_{01}^2Z_4^2 \\
+ 420(2v_{10}v_{13} + 6v_{11}v_{12} + 3u_{01}u_{22} + 3u_{02}u_{21} + u_{03}u_{20})Z_3Z_2 \\
+ 210(4v_{10}v_{32} + 8v_{11}v_{31} + 4v_{30}v_{12} + 6u_{20}u_{22} \\
+ 6u_{21} + u_{40}u_{02} + 2u_{41}u_{01})Z_2^2 \\
+ 105(4u_{01}u_{03} + 3u_{02}^2)Z_2^4 + gZ_8 + \lambda\kappa Z_0 = 0 \\
+ 105(4u_{01}u_{03} + 3u_{02}^2)Z_2^4 + gZ_8 + \lambda\kappa Z_0 = 0 \\
+ 120v_{10}v_{70} + 1980Z_2^2Z_6 \\
+ 120u_{20}u_{02} + 2u_{01}u_{21} + 2v_{12}v_{10} + 2v_{11}^2)Z_2Z_6 \\
+ 210(6u_{20}u_{21} + u_{01}u_{40} + 4v_{31}v_{10} + 4v_{30}v_{11})Z_6 \\
+ 210u_{20}^2Z_2Z + 4725u_{10}u_{20}Z_2^2Z \)

5. \( [(45u_{20}u_{80} + 126v_{50}^2 + 210u_{60}u_{40} + 10v_{90}v_{10} \\
+ 120v_{30}v_{70} + 1980Z_2^2Z_6 \\
+ 120(u_{20}u_{02} + 2u_{01}u_{21} + 2v_{12}v_{10} + 2v_{11}^2)Z_2Z_6 \\
+ 210(6u_{20}u_{21} + u_{01}u_{40} + 4v_{31}v_{10} + 4v_{30}v_{11})Z_6 \\
+ 210u_{20}^2Z_2Z + 4725u_{10}u_{20}Z_2^2Z \)
TABLE C-3 (Cont)

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\[ + 1575(u_{20}u_{02} + 2u_{01}u_{21} + 2v_{12}v_{10} + 2v_{11}^2)z_4^2 \]

\[ + 3150(4u_{03}u_{01} + 3u_{02}^2)z_2^3z_4 \]

\[ + 9450(u_{20}u_{03} + 3u_{02}u_{21} + 3u_{01}u_{22} + 2v_{13}v_{10} \]

\[ + 6v_{12}v_{11})z_2^2z_4 \]

\[ + 3150(6u_{20}u_{22} + u_{02}u_{40} + 2u_{01}u_{41} + v_{32}v_{10} \]

\[ + 8v_{31}v_{11} + 4v_{30}v_{12} + 6u_{21}^2)z_2^2z_4 \]

\[ + 210(15u_{20}u_{41} + u_{01}u_{60} + 6v_{51}v_{10} + 6v_{50}v_{11} \]

\[ + 20v_{31}v_{30} + 15u_{40}u_{21})z_4 \]

\[ + 45u_{01}^2z_2z_8 + 45(u_{20}u_{01} + 2v_{11}v_{10})z_8 \]

\[ + 4725(u_{04}u_{01} + 2u_{03}u_{02})z_2^5 \]

\[ + 4725(u_{20}u_{04} + 4u_{03}u_{21} + 6v_{12}^2)z_2^4 \]

\[ + 2v_{14}v_{10} + 8v_{13}v_{11} + 6v_{12}v_{12}z_2^2 \]

\[ + 3150(6u_{20}u_{23} + u_{03}u_{40} + 3u_{02}u_{41} + 3u_{01}u_{42} \]

\[ + 4v_{33}v_{10} + 12v_{32}v_{11} + 12v_{31}v_{12} + 4v_{30}v_{13} \]
\begin{align*}
+ 18u_{22}u_{21}Z_2^3 \\
+ 630(15u_{20}u_{42} + u_{02}u_{60} + 2u_{01}u_{61} + 6v_{52}v_{10} \\
+ 12v_{51}v_{11} + 6v_{50}v_{12} + 20v_{32}v_{30} + 20v_{31}^2 \\
+ 30u_{41}u_{21} + 15u_{40}u_{22})Z_2^2 \\
+ 45(28u_{20}u_{61} + u_{01}u_{80} + 56v_{51}v_{30} + 56v_{50}v_{31} \\
+ 28u_{60}u_{21} + 70u_{41}u_{40} + 8v_{71}v_{10} + 8v_{71}v_{70})Z_2 \\
+ gZ_{10} + \Lambda K_{10} = 0
\end{align*}
APPENDIX D

Multi-Dimension Newton-Raphson Solution Technique

In one-dimension the well-known Newton-Raphson procedure for the solution of the nonlinear algebraic equation

\[ h(x) = 0 \]

takes the form

\[ x^{n+1} = x^n - \frac{h(x^n)}{h'(x^n)} \]  \hspace{1cm} (D-1)

where \( n \) is an iterative index and the prime indicates differentiation.

For example, if \( h(x) = x^2 - 2 \), then \( h'(x) = 2x \) and

\[ x^{(n+1)} = x^n - \frac{(x^n)^2 - 2}{2x^n} \]

or

\[ x^{(n+1)} = \frac{(x^n)^2 + 2}{2x^n} \]  \hspace{1cm} (D-2)

It is readily seen that (D-2) converges rapidly to

\[ x = \pm 1.414... \approx \pm \sqrt{2} \]

depending on the initial guess for \( x^{(0)} \).

If one now considers a system of \( N \) nonlinear equations
\begin{align*}
h_1(\bar{x}) &= 0 \\
h_2(\bar{x}) &= 0 \\
&\quad \vdots \\
h_N(\bar{x}) &= 0
\end{align*}

in the \( N \) unknowns \( x_1, x_2, \ldots, x_N \), where

\[ \bar{x} \equiv (x_1, x_2, \ldots, x_N), \]

then one may write the analogous \( N \)-dimensional form of D-1 (Burden, Faires and Reynolds, 1978, pp 443-448) as

\[ \bar{x}^{n+1} = \bar{x}^n - \text{JAC}^{-1}(\bar{x}) \cdot \bar{F}(\bar{x}^n) \quad (D-3) \]

where

\[ \bar{F}(\bar{x}) \equiv (h_1(\bar{x}), h_2(\bar{x}), \ldots, h_N(\bar{x})) \]

and \( \text{JAC}(\bar{x}) \equiv \) the Jacobian Matrix,

\[ \text{JAC}_{ij}(\bar{x}) \equiv \frac{\partial h_i(\bar{x})}{\partial x_j}. \]

If the functions \( h_i(\bar{x}) \) are simple enough, the Jacobian may be calculated analytically. Otherwise, a numeric differentiation will be required.
The equations to be solved in this paper are not readily differentiable and the numeric differentiation

\[ \frac{\partial h_i}{\partial x_j} = \frac{h_i(x_1, \ldots, x_j + \Delta_j, \ldots, x_n) - h_i(x_1, \ldots, x_n)}{\Delta_j} \]

was employed where \( \Delta_j = .0001 x_j \).
APPENDIX E

Derivation of Equations for Potential Flow and Laminar Film Thickness in a Vertical Tube

This appendix presents the derivation of the defining equation of the flow field in the liquid for potential flow and the derivation of the relationship between the slug slip velocity and the fully developed laminar boundary layer thickness.

E.1. Derivation of Governing Equation for Potential Flow

Consider an ideal liquid of constant density $\rho_\ell$. The Navier-Stokes Equation for such a liquid is then

$$\frac{D\vec{V}}{Dt} = -\frac{\nabla P}{\rho_\ell} + \vec{F}_b$$ \hspace{1cm} (E-1)

where $\vec{V}$ is the fluid velocity and $\vec{F}_b$ is the force per unit mass due to external forces.

For the problem considered in this dissertation, the only external force to be considered is that due to the gravitational force field. Since the gravitational force field is conservative, then $\vec{F}_b$ may also be written as the gradient of a scalar,

$$\vec{F}_b = \nabla \phi_g .$$

Then E-1 becomes
\[ \frac{D\vec{V}}{Dt} = -\frac{\nabla p}{\rho_\ell} + \nabla \phi \]  \hspace{1cm} (E-2)

Taking the curl of E-2 yields

\[ \frac{D(\nabla \times \vec{V})}{Dt} \equiv 0 \]  \hspace{1cm} (E-3)

where use has been made of the vector identity that the curl of the gradient of a scalar function is identically zero.

Equation E-3 thus requires that the quantity \( \nabla \times \vec{V} \) remain constant at all times. Since in the problem considered here the flow was initially at rest before the gas bubble was introduced, then

\[ \nabla \times \vec{V} \equiv 0 \]  \hspace{1cm} (E-4)

The velocity field is therefore conservative and may be expressed as the gradient of a scalar function. Thus,

\[ \vec{V} = \nabla \phi \]  \hspace{1cm} (E-5)

where \( \phi \) is called the velocity potential function.

The equation of mass continuity for a constant density liquid is simply

\[ \nabla \cdot \vec{V} \equiv 0 \]  \hspace{1cm} (E-6)

Substitution of E-5 into E-6 then yields the governing equation of potential flow

\[ \nabla^2 \phi = 0 \]
E.2. Fully Developed Laminar Film Thickness in a Vertical Tube

Consider a liquid of viscosity \( \mu \) and density \( \rho_L \) flowing down the inner surface of a vertical tube of radius \( R \) (Figure E-1).

The steady state axial component of the Navier-Stokes Equation, in Cylindrical coordinates is

\[
u \frac{\partial u}{\partial z} + v \frac{\partial u}{\partial r} + g = \frac{\mu}{\rho_L} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right)
\]

(E-7)

where the pressure gradient is that in the gas core and is negligible, i.e., the gas density is negligible).

The continuity equation is

\[
\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} + \frac{v}{r} = 0
\]

(E-8)

In fully developed steady-state flow, the velocity is in the axial direction only so that \( v \equiv 0 \) and E-8 requires that \( u = u(r) \). Then E-7 becomes

\[
\frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) = \left( \frac{\rho_L}{\mu} \right) r
\]

(E-9)
Integrating E-9 twice yields

\[ u(r) = k_1 \ln r + \frac{1}{4} \eta r^2 + k_2 \]  \hspace{1cm} (E-10)

where

\[ \eta \equiv \frac{\rho g}{\mu} \]

and \( k_1, k_2 \) are constants to be determined.

Referring to Figure E-1, a force balance written on the element of liquid in the film of height \( dz \) and thickness \( \delta \) yields

\[ 2\pi r \tau_w dz = \pi (R^2 - (R-\delta)^2) \rho g \ dz \]

or

\[ \tau_w = \frac{\pi}{2} (R^2 - (R-\delta)^2) \frac{\rho g}{R} \]  \hspace{1cm} (E-11)

where the interfacial shear stress at the gas/liquid interface has been assumed negligible compared to the wall stress \( \tau_w \).

Substituting E-11 and the definition

\[ \tau_w = (\mu \frac{\partial u}{\partial r})_{r=R} \]

into E-10 yields, after algebraic manipulation,
\[ \kappa_1 = - \frac{1}{2} \eta R^2 (1 - \frac{\delta}{R})^2 . \] (E-12)

At the wall \( u(R) = 0 \), and E-10 yields

\[ \kappa_2 = - [\kappa_1 2nR + \frac{\eta R^2}{4}] \] (E-13)

The average film velocity \( \bar{V}_f \) is then given by

\[ \bar{V}_f = \int_R^{R_\tau} u(r) rdR/ \int_R^{R_\tau} r dr \]

which, upon substitution of E-10, gives

\[ \bar{V}_f = \left[ 4\kappa_1 (R^2 (2lnR - 1) - (R-\delta)^2 (2ln(R-\delta) - 1)) \right. \]

\[ + \left. 8\kappa_2 (R^2 - (R-\delta)^2) + \eta (R^4 - (R-\delta)^4) \right] \frac{8(R^2 - (R-\delta)^2)}{8(R^2 - (R-\delta)^2)} \] (E-14)

The average film velocity is related to the bubble slip velocity through the continuity equation. In the bubble reference frame,

\[ \pi R^2 (\bar{U} - U_s) = (\bar{V}_f - U_s) \pi (R^2 - (R-\delta)^2) \] (E-15)

where \( \bar{U} \) is the average fluid velocity at \( z = +\infty \) in the laboratory reference (\( \bar{U} = 0 \) for stagnant flow). From E-15, one obtains,
The average film velocity \( \bar{V}_f \) may be simplified by making the approximation

\[
\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \quad \ldots \quad (-1 \leq x \leq 1)
\]

so that

\[
\ln(R - \delta) = \ln R + \ln(1 - \varepsilon)
\]

\[
= \ln R - \varepsilon - \frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{3} - \frac{\varepsilon^4}{4} \quad \ldots \quad (E-17)
\]

where \( \varepsilon = \delta/R \). Substituting (E-17) into (E-14) and neglecting terms higher than third order in \( \varepsilon \) yields

\[
\bar{V}_f = - \frac{2nR^2}{3} \frac{\varepsilon^2(1-\varepsilon)}{(2-\varepsilon)} \quad (E-18)
\]

which is identical to Brown's (1965) result.

Then for stagnant flow, from (E-16) one has

\[
U_s = - \frac{\bar{V}_f \varepsilon(2-\varepsilon)}{(1-\varepsilon)^2}
\]

so that
Once the fluid properties have been specified, E-19 provides the relation between the slip velocity $U_s$ and the film thickness $\delta = R \varepsilon$. 

\[ U_s = \frac{2\eta R^2}{3} \frac{\varepsilon^3}{(1-\varepsilon)}. \]  

(E-19)
Figure E-1  Laminar liquid film flowing down inner surface of a vertical tube of inner radius $R$
APPENDIX F

Listings of Computer Programs Used in This Report

This appendix presents the source code listings for the computer programs utilized in the calculations summarized in this report. The codes are written in Fortran IV for the IBM 3030 computer.
APPENDIX F.1

Slug Flow in Vertical Tube

IMPLICIT REAL *(A-H,O-Z)
DOUBLE PRECISION LAM
REAL EO,FR,RC,EOEFF,FREFF,RCEFF
DIMENSION D(8),B(17),ZERO(7),C(8),CNEW(8),
& ARRAY(8,9),DUMMY(8),COLD(8),FUN(16)
COMMON FUN,C,COLD,D,B,DUMMY,ST,UCL,RAD,LAM,
& U01,U02,U03,U04,U05,U06,U20,U21,U22,U23,U24,U25,
& U40,U41,U42,U43,U44,U60,U61,U62,U63,U80,U81,U82,U100,
& U101,U120,U07,U26,U45,U64,U83,U102,U121,U140,
& V10,V11,V12,V13,V14,V15,V30,V31,V32,V33,V34,V50,V51,V52,
& V53,V70,V71,V72,V90,V91,V110,V16,V35,V54,V73,V92,V111,V130,
& Z2,Z4,Z6,Z8,Z10,Z12,Z14,
& ND,NT
DATA ZERO/3.83171D00,7.01559D00,10.17347D00,13.32369D00,
& 16.47063D00,19.61586D00,22.76008D00/
C INITIALIZE
DO 5 J = 1,8
C(J) = 0.0D00
D(J) = 0.0D00
5 COLD(J) = 0.0D00
C GAS DENSITY (AIR AT STP) IN LBM/FT**3
DENG = .076
IERR = 0
NCASE = 1
C READ INPUT
READ(1,*) RAD,DELR,DEL,DELS,UCL
READ(1,*) EPS,NITER,CFO
READ(1,*) FLUID!,FLUID2.DENL,VISC,STO
READ(1,*) FMUL,XX,NREAD
XA = 1.D00 + XX
DENL = 62.24D00**DENL
VISC = 2.42D00**VISC
ETA = 9660**VISC
C READ IN INITIAL GUESS
DO 30 JJ=1,7
30 READ(1,J,C(J))
C CONVERT STO FROM DYNE/CM TO LBM/SEC**2
ST = ST0/453.5D00
IF(ST .EQ. 0.D00) ST = .00000001D00
LAM = 1728*ST/(DENL - DENG)
DIAM = 2*RAD
C STREAM FUNCTION APPROXIMATION (NT) AND SHAPE APPROXIMATION (ND)
35 READ(1,*,END=1100) NT,ND
NT = NT + 1
NT1 = NT + 1
NTM1 = NT - 1
40 CONTINUE
EOTVOS = 386.4D00*DIAM*DIAM/LAM
C GET INITIAL GUESS FOR BUBBLE VELOCITY (IN/SEC)
FR = .345
IF(EOTVOS .LT. 80.) FR = .345*(1-DEXP((3.37 - EOTVOS)/10.))
C(NT) = FR*DSQRT(386.4*RAD)

C PRINT OUTPUT

CALL CL3270
PRINT 13
13 FORMAT(' *********************************************************'
1       ', ' *********************************************************'/)
PRINT 11,FLUID1,FLUID2
11 FORMAT('/30X,2A8/)
PRINT 16,DENL,VISC,STO
16 FORMAT(FLUID DENSITY = ',F6.2,' LBM/CF *  ,5X, '  FLUID VISCOSITY'
1       , ' = ',F8.2,' LBM/FT-HR',/,' SURFACE TENSION = ',F6.2,
1       , ' DYNE/CM',/)
PRINT 15,DIAM,EOTVOS
15 FORMAT( DIAMETER = ',F6.2,' INCHES '
1       ,2X,'  EOTVOS NO. = ',F8.2,/)  
PRINT 14,NTM1,ND
14 FORMAT(IX,'N = ',13,' ND = ',13)

XNP = 3600**2*32.2DOO*DENL*DENL*(RAD/12)**3/VISC**2
XNP = DSQRT(XNP)
FMULL = 3.61/XNP**(.38)

45 REFF = RAD - DEL*FMULL
DO 42 I = 1,NTM1
42 D(I) = ZERO(I)/REFF

C CALCULATE COEFFICIENTS BY NEWTON-RAPHSON

ITER = 1
CF = CFO
SS = 1.
50 DO 60 I = 1,NT
60 COLD(I) = C(I)
     IF(SS .LE. .02) CF = 1.D00
     IF(SS .GT. .02) CF = CFO
     USOLD = C(NT)
     DO 1000 11=1,NT1
        IF(II .EQ. 1) GO TO 100
410 FUN(JJ+NT)  = FUN(JJ)
        GO TO 1000
100 CALL DERIV
111 FORMAT(6(2X,E12.4))
110 CALL FUNC
     IF(II .CT. 1) GO TO 500
     DO 410 JJ = 1,NT
410 FUN(JJ+NT) = FUN(JJ)
     GO TO 1000

C CALCULATE JACOBIAN
500 DO 510 JJ=1,NT
510 DELFUN = FUN(JJ) - FUN(JJ+NT)
ARRAY(JJ,II-1) = DELFUN/XXX
510 CONTINUE
   C(II-1) = COLD(II-1)
1000 CONTINUE
C SOLVE FOR NEW VALUES
DO 520 I = 1,NT
   520 ARRAY(I,NT1) = FUN(NT + I)
   CALL SIMEQ(ARRAY,NT,NT1,NT,DUMMY,IERR)
   IF(IERR .EQ. 1) GO TO 1100
   DO 400 J = 1,NT
      C(J) = CF*DUMMY(J) + COLD(J)
   400 CONTINUE
   C(NT) = DABS(C(NT))
   CCCCCCCCCC
   C CHECK FOR CONVERGENCE
   CCCCCCCCCC
   ITER = ITER + 1
   IF(ITER .GE. NITER) GO TO 1100
   SS = 0.0
   DO 600 J = 1,NT
      SS = SS + ((C(J) - COLD(J))/C(J))**2
   600 CONTINUE
   RC = -1/(RAD**2)
   SS = DSQRT(SS)/CF
   USFPS = C(NT)/12
   C PRINT 21, C(NT),SS
   21 FORMAT(4X,E15.6,4X,E12.4)
   IF(SS .GT. EPS) GO TO 50
   CCCCCCCCCC
   C CALCULATE FILM THICKNESS
   CCCCCCCCCC
   VAVG = 0.000
   IF(DEL .EQ. 0.000) GO TO 650
   625 DELO = DEL
   EP = DEL/RAD
   A1 = -.5D00*ETA*RAD*RAD*(1.000-DENG/DENL)*(1.000-EP)**2
   A2 = -(A1*DLOG(RAD) + .25*ETA*RAD*RAD)
   VAVG = .5D00*A1*(RAD*RAD-1.000) - (RAD - DEL)*
         (RAD - DEL)**2
   VAVG = VAVG + A2*(RAD*RAD - (RAD - DEL)**2)
   VAVG = VAVG + .125D00*ETA*(RAD**4 - (RAD - DEL)**4)
   VAVG = - VAVG/(RAD**2)
   E1 = C(NT)*(1.000 - EP)**2 - UCL
   DEL = RAD/E1/(VAVG*(2.000 - EP))
   IF(DABS(1- DEL/DEL) .LE. .0001D00) GO TO 640
   DEL = .5D00*(DEL + DELO)
   GO TO 625
   C CHECK IF SOLUTION CONVERGED
   640 IF(DABS(1- DEL/DEL) .LE. .0001D00) GO TO 650
   DEL1 = DEL
   GO TO 45
   CCCCCCCCCC
   C PRINT OUTPUT WHEN CONVERGED
   CCCCCCCCCC
CALL DERIV

VSLIP = USFPS - UCL/12
RC = -1/(RAD*Z2)
RCEFF = RC*RAD/REFF
FR = (C(NT) - UCL)/DSQRT(386.4D00*RAD)
FR = FR/1.4142D00
FREFF = (C(NT) - UCL)/DSQRT(386.4*REFF)
FREFF = FREFF/1.4142D00
VFILM = VAVG/12
REFILM = 4*3600*(DEL/12)*VFILM*DENL/VISC
VISNO = 3600*3600*14.5D00*32.2D00*DENL*DENL/(VISC*VISC)
VISNO = VISNO*(DIAM/12)**3
VISNO = VISNO**.3333D00
EO = EOTVOS
EOEFF = EOTVOS*(REFF/RAD)**2
C WRITE(2,22)EO,FR,RC,EOEFF,FREFF,RCEFF
WRITE(2,22)EO,FR

22 FORMAT(6F8.3)

PRINT 31,UCL
31 FORMAT(/' CENTERLINE VEL =',F8.3,' IN/SEC')
PRINT 4, VSLIP,FR,DEL,REFILM,VFILM,VISNO,XNP,FMULL,EOEFF,FREFF,1
1 RCEFF,RC
4 FORMAT( ' SLIP VELOCITY =',F8.3,' FT/SEC'/ ' FROUDE NUMBER ='
1 ',F8.3,/' FILM THICKNESS = ',F7.6,' INCHES',
2 ' ',F8.3,/' FILM RE = ',F7.1,
3 ' ',F8.3,/' FILM VELOCITY = ',F8.3,' FT/SEC',
4 ' ',F7.6,/' LIQ VISC NUMBER = ',F8.3,
5 ' ',F8.3,/' LIQ PROP NUMBER = ',F8.3,
6 ' ',F8.3,/' FILM MULTIPLIER = ',F8.3,
7 ' ',EOEFF = ',F8.3,
8 ' ',FREFF NUMBER = ',F8.3,
9 ' ',RCEFF NUMBER = ',F8.3,
10 ' ',NORMALIZED RADIUS OF CURVATURE AT NOSE =',F8.3//)
PRINT 7
7 FORMAT( ' COEFFICIENTS IN BESSEL SERIES FOLLOW:'//
DO 700 J = 1,NTM1
PRINT 3,J,C(J)
3 FORMAT(5X,I5,3X,E17.9)
700 CONTINUE
C PRINT VALUES OF DERIVATIVES
PRINT 17, Z2,Z4,Z6,Z8,Z10,Z12,Z14
17 FORMAT(' ',Z2 = ',E17.9,' Z4 = ',E17.9,' Z6 = ',E17.9,
1 ',Z8 = ',E17.9,' Z10 = ',E17.9,' Z12 = ',E17.9,
2 ',Z14 = ',E17.9//)

IF(NREAD .EQ. 1) GO TO 35
IF(DELR .NE. 0.0D00) GO TO 825
IF(DELS .EQ. 0.0D00) GO TO 1100
STO = STO + DELS
ST = ST0/453.5D00
LAM = 1728*ST/(DENL - DENG)
GO TO 40
825 DO 860 J = 1,NT
IF(NCASE .GE. 1) CNEW(J) = C(J)
C IF(NCASE .GE. 2) CNEW(J) = 2*C(J) - C1(J)  
C1(J) = C(J)  
C(J) = CNEW(J)
860 CONTINUE
   NCASE = NCASE + 1  
   DIAM = DIAM - 2*DELR  
   IF(DIAM .LE. 0.0) GO TO 1100  
   RAD = DIAM/2  
   GO TO 40
1100 STOP
END

SUBROUTINE SIMEQ(A,M,MP,N,STORE,IERR)
C
C SUBROUTINE SIMEQ SOLVES ANY NUMBER OF SIMULTANEOUS EQUATIONS.
C COEFFICIENTS ARE STORED IN THE CALLING MATRIX A (WHICH IS
C DESTROYED DURING THE SOLUTION PROCESS). VALUES OF THE N UNKNOWNS
C TO BE SOLVED FOR ARE STORED IN THE VECTOR STORE, IN THEIR ORDER OF
C APPEARANCE IN THE EQUATIONS. IF SOLUTION IS NOT POSSIBLE AN ERROR
C MESSAGE WILL BE PRINTED AND IERR WILL BE RETURNED AS 1. IF SOLU-
C TION IS SUCCESSFUL IERR WILL BE RETURNED AS 0.
C
C M AND MP ARE THE MAXIMUM DIMENSIONS OF THE ARRAY A
C N IS THE NUMBER OF EQUATIONS
C
IMPLICIT REAL *8(A-H,O-Z)
DIMENSION STORE(8),A(3,9)
DO 10 I = 1,8
10 STORE(I) = 0.0D0
IERR=0
IX=N+1
DO 110 MX=1,N
   DO 102 I=MX,N
      DIV=A(I,MX)
      IF(DIV)103,102,103
102 CONTINUE
   PRINT 1
   1 FORMAT(/9X,'SOLUTION OF SIMULTANEOUS EQUATIONS NOT POSSIBLE'/)
      IERR=1
      GO TO 114
103 DO 105 J=MX,IX
   Z=A(MX,J)
   A(MX,J)=A(I,J)/DIV
   IF(I-MX)104,105,104
104 A(I,J)=Z
105 CONTINUE
   IF(N-1)114,112,106
106 KY=MX+1
   IF(KY.GT.N) GO TO 110
   DO 109 L=KY,N
      AMULT=A(L,MX)
      IF(MX-N)107,109,109
DO 108 J=MX,IX
108 A(L,J)=A(L,J)-AMULT*A(MX,J)
109 CONTINUE
110 CONTINUE
KU=N-1
DO 111 J=1,KU
L=N-J
LX=L+1
DO 111 MX=LX,N
111 A(L,N+1)=A(L,N+1)-A(MX,N+1)*A(L,MX)
112 DO 113 I=1,N
113 STORE(I)=A(I,N+1)
114 CONTINUE
RETURN
END
SUBROUTINE DERIV
IMPLICIT REAL *8(A-H,O-Z)
DOUBLE PRECISION LAM
COMMON FUN,C,COLD,D,B,DUMMY,ST,UCL,RAD,LAM,
& U01,U02,U03,U04,U05,U06,U20,U21,U22,U23,U24,U25,
& U40,U41,U42,U43,U44,U60,U61,U62,U63,U80,U81,U82,U100,
& U101,U120,U07,U26,U45,U64,U83,U102,U121,U140,
& V10,V11,V12,V13,V14,V15,V30,V31,V32,V33,V34,V50,V51,V52,
& V53,V70,V71,V72,V90,V91,V110,V16,V35,V54,V73,V92,V111,V130,
& Z2,Z4,Z6,Z8,Z10,Z12,Z14,
& ND,NT
DIMENSION D(8),B(17),C(8),COLD(8),FUN(16),DUMMY(8)
Z2 = 0.D00
Z4 = 0.D00
Z6 = 0.D00
Z8 = 0.D00
Z10 = 0.D00
Z12 = 0.D00
Z14 = 0.D00
DO 100 I = 1,17
100 B(I) = 0.0D00
NB = 2*NT + 1
NB = 17
NTM1 = NT - 1
DO 200 J = 1,NTM1
DO 200 I = 1,NB
B(I) = B(I) + C(J)*D(J)**I
200 CONTINUE
C CALCULATE DERIVATIVES
AL = 1.D00 - UCL/C(NT)
U01 = - AL*B(2)
U02 = AL*B(3)
U03 = - AL*B(4)
U04 = AL*B(5)
U05 = - AL*B(6)
U06 = AL*B(7)
$U_{07} = - \text{AL*B(8)}$
$U_{20} = - \text{AL*B(3)/2}$
$U_{21} = \text{AL*B(4)/2}$
$U_{22} = - \text{AL*B(5)/2}$
$U_{23} = \text{AL*B(6)/2}$
$U_{24} = - \text{AL*B(7)/2}$
$U_{25} = \text{AL*B(8)/2}$
$U_{26} = - \text{AL*B(9)/2}$
$U_{40} = 3\text{AL*B(5)/8}$
$U_{41} = - 3\text{AL*B(6)/8}$
$U_{42} = 3\text{AL*B(7)/8}$
$U_{43} = - 3\text{AL*B(8)/8}$
$U_{44} = 3\text{AL*B(9)/8}$
$U_{45} = - 3\text{AL*B(10)/8}$
$U_{60} = - 5\text{AL*B(7)/16}$
$U_{61} = 5\text{AL*B(8)/16}$
$U_{62} = - 5\text{AL*B(9)/16}$
$U_{63} = 5\text{AL*B(10)/16}$
$U_{64} = - 5\text{AL*B(11)/16}$
$U_{80} = 35\text{AL*B(9)/128}$
$U_{81} = - 35\text{AL*B(10)/128}$
$U_{82} = 35\text{AL*B(11)/128}$
$U_{83} = - 35\text{AL*B(12)/128}$
$U_{100} = - 63\text{AL*B(11)/256}$
$U_{101} = 63\text{AL*B(12)/256}$
$U_{102} = - 63\text{AL*B(13)/256}$
$U_{120} = 231\text{AL*B(13)/1024}$
$U_{121} = - 231\text{AL*B(14)/1024}$
$U_{140} = - 429\text{AL*B(15)/2048}$
$V_{10} = \text{AL*B(2)/2}$
$V_{11} = - \text{AL*B(3)/2}$
$V_{12} = \text{AL*B(4)/2}$
$V_{13} = - \text{AL*B(5)/2}$
$V_{14} = \text{AL*B(6)/2}$
$V_{15} = - \text{AL*B(7)/2}$
$V_{16} = \text{AL*B(8)/2}$
$V_{30} = - 3\text{AL*B(4)/8}$
$V_{31} = 3\text{AL*B(5)/8}$
$V_{32} = - 3\text{AL*B(6)/8}$
$V_{33} = 3\text{AL*B(7)/8}$
$V_{34} = - 3\text{AL*B(8)/8}$
$V_{35} = 3\text{AL*B(9)/8}$
$V_{50} = 5\text{AL*B(6)/16}$
$V_{51} = - 5\text{AL*B(7)/16}$
$V_{52} = 5\text{AL*B(8)/16}$
$V_{53} = - 5\text{AL*B(9)/16}$
$V_{54} = 5\text{AL*B(10)/16}$
$V_{70} = - 35\text{AL*B(8)/128}$
$V_{71} = 35\text{AL*B(9)/128}$
$V_{72} = - 35\text{AL*B(10)/128}$
$V_{73} = 35\text{AL*B(11)/128}$
$V_{90} = 63\text{AL*B(10)/256}$
$V_{91} = - 63\text{AL*B(11)/256}$
$V_{92} = 63\text{AL*B(12)/256}$

$2*UCL/(C(NT)^{*}RAD^{*2})$
\[ V_{110} = -\frac{231 \cdot AL \cdot B(12)}{1024} \]
\[ V_{111} = \frac{231 \cdot AL \cdot B(13)}{1024} \]
\[ V_{130} = \frac{429 \cdot AL \cdot B(14)}{2048} \]

**C** Calculate derivatives of bubble shape at nose

\[ Z_2 = \frac{U_{20}}{2 \cdot V_{10} - U_{01}} \]

If \( ND \cdot EQ. \ 2 \) RETURN

\[ Z_4 = \frac{(U_{40} + (6 \cdot U_{21} - 4 \cdot V_{30}) \cdot Z_2 + (3 \cdot U_{02} - 12 \cdot V_{11}) \cdot Z_2 \cdot Z_2)}{4 \cdot V_{10} - U_{01}} \]

If \( ND \cdot EQ. \ 4 \) RETURN

\[ Z_6 = \frac{(U_{60} + (15 \cdot U_{41} - 6 \cdot V_{50}) \cdot Z_2 + (15 \cdot U_{02} - 20 \cdot V_{31}) \cdot Z_2 \cdot Z_2)}{6 \cdot V_{10} - U_{01}} \]

If \( ND \cdot EQ. \ 6 \) RETURN

\[ Z_8 = \frac{(U_{80} + (28 \cdot U_{21} - 56 \cdot V_{30}) \cdot Z_6 + (28 \cdot U_{02} - 22 \cdot V_{11}) \cdot Z_2 \cdot Z_6)}{8 \cdot V_{10} - U_{01}} \]

If \( ND \cdot EQ. \ 8 \) RETURN

\[ Z_{10} = \frac{(945 \cdot Z_2^5 \cdot (U_{05} - 10 \cdot V_{14}) + 15 \cdot V_{33} \cdot Z_2^4 \cdot (-8 \cdot V_{53} + 3 \cdot U_{24}) + 3150 \cdot Z_2^3 \cdot Z_4 \cdot (U_{03} - 10 \cdot V_{13}) + 1350 \cdot Z_2^2 \cdot Z_6 \cdot (U_{04} - 12 \cdot V_{12}) + 120 \cdot Z_2 \cdot Z_8 \cdot (-10 \cdot V_{11} + U_{02}) + 5 \cdot U_{01})}{10 \cdot V_{10} - U_{01}} \]

If \( ND \cdot EQ. \ 10 \) RETURN

\[ Z_{12} = \frac{10395 \cdot Z_2^5 \cdot (U_{06} - 12 \cdot V_{15}) + 20790 \cdot Z_2^4 \cdot (-10 \cdot V_{34} + 3 \cdot U_{25}) + 1 \cdot 51975 \cdot Z_2^3 \cdot Z_4 \cdot (U_{05} - 12 \cdot V_{14}) + 10395 \cdot Z_2^2 \cdot Z_6 \cdot (-8 \cdot V_{53} + 5 \cdot U_{14}) + 2 \cdot 6930 \cdot Z_2 \cdot Z_8 \cdot (-10 \cdot V_{32} + 3 \cdot U_{23}) + 3 \cdot 1980 \cdot Z_2^3 \cdot (U_{04} - 12 \cdot V_{13}) + 4 \cdot 20790 \cdot Z_2^2 \cdot Z_4 \cdot (-8 \cdot V_{52} + 5 \cdot U_{43}) + 13860 \cdot Z_2 \cdot Z_6 \cdot (-10 \cdot V_{31} + 3 \cdot U_{23}) + 5 \cdot 1485 \cdot Z_2^2 \cdot Z_8 \cdot (U_{03} - 12 \cdot V_{12}) + 165 \cdot Z_2 \cdot Z_8 \cdot (-4 \cdot V_{91} + 9 \cdot U_{82})}{10 \cdot V_{10} - U_{01}} \]
Z12 = Z12 +
6 34650*Z2*Z4*2*(-10*V32+3*U23)+13860*Z2*Z4*Z6*(U03-12*V12)+
7 1980*Z2*Z4*(-6*V71+7*U62)+2772*Z2*Z6*(-8*V51+5*U42)+990*Z2
8 *Z8*(-10*V31+3*U22)+662*Z2*Z10*(-12*V11+U02)+6*Z2*(-2*V110+
9 11*U101)+5775*Z4*3*(U03-12*V12)+3465*Z4*2*(-8*V51+5*U42)
Z12 = Z12 +
1 4620*Z4*Z6*(-10*V31+3*U22)+695*Z4*Z8*(-12*V11+U02)+55*Z4*(-
2 4*V90+9*U81)+462*Z6*Z2*(-12*V11+U02)+132*Z6*(-6*V70+7*U61)
3 +99*Z8*(-8*V50+5*U41)+22*Z10*(3*U21-10*V30)+U120
Z12 = Z12/(12*V10 - U01)
IF(ND .EQ. 12) RETURN

500 RETURN

END

SUBROUTINE FUNC
IMPLICIT REAL *(A-H,O-Z)
DOUBLE PRECISION LAM,LAMM
COMMON FUN, C, COLD,D,B,DUMMY,ST,UCL,RAD,LAM,
& U01,U02,U03,U04,U05,U06,U20,U21,U22,U23,U24,U25,
& U40,U41,U42,U43,U44,U60,U61,U62,U63,U80,U81,U82,U100,
& U101,U120,U07,U26,U45,U64,U83,U102,U121,U140,
& V10,V11,V12,V13,V14,V15,V30,V31,V32,V33,V34,V50,V51,V52,
& V53,V70,V71,V72,V90,V91,V110,V16,V35,V73,V92,V111,V130,
& Z2,Z4,Z6,Z8,Z10,Z12,Z14,
& ND,NT
DIMENSION D(8),B(17),C(8),COLD(8),FUN(16),DUMMY(8)
Z16 = 0.000
LAMM = LAM/C(NT)**2
G = 386.4000/C(NT)**2
CC T = (U**2 + V**2)/2
T02=U01**2
T04=4*U03**2+3*U02**2
T05=5*(U04*U01+2*U03*U02)
T06=6*U05*U01+15*U04*U02+10*U03**2
T07=7*(U06*U01+3*U05*U02+5*U04*U03)
T20=V10**2
T21=U20*U01+2*V11*V10
T22=U20*U02+2*U01*U21+2*V12*V10+2*V11**2
T23=U20*U03+3*U02*U21+3*U01*U22+2*V13*V10+6*V12*V11
T24=U20*U04+4*U03*U21+6*U02*U22+4*U01*U23+2*V14*V10+8*V13
1 V11+6*V12**2
T25=U20*U05+5*U04*U21+10*U03*U22+10*U02*U23+5*U01*U24+2*V15
1 V10+10*V14*V11+20*V13*V12
T26=U20*U06+6*U05*U21+15*U04*U22+20*U03*U23+15*U02*U24+6*U01
T40=3*U20**2+4*V30*V10
T41=6*U20+U21+U01*U40+4*V31*V10+4*V30*V11
T42=6*U20+U22+U02*U40+2*U01*U41+4*V32*V10+8*V31*V11+4*V30*1
1 V12+6*U21**2
T43=6*U20+U23+U03*U40+3*U02*U41+3*U01*U42+4*V33*V10+12*V32
1 V11+12*V31*V12+4*V30*V13+18*U22*U21
T44=6*U20+U24+U04*U40+4*U03*U41+6*U02*U42+4*U01*U43+4*V34*
1 V10+16*V33*V11+24*V32*V12+16*V31*V13+4*V30*V14+24*U23*U21 +
2 18*U22**2
T45=6*U20+U25+U05*U40+5*U04*U41+10*U03*U42+10*U02*U43+5*U01*
1 U44+4*V35*V10+20*V34*V11+40*V33*V12+40*V32*V13+40*V31*V14+
2 4*V30*V15+30*U24*U21+60*U23*U22
T60=15*U20*U40+6*V50*V10+10*V30*V11
T61=15*U20*U41+U01*U60+6*V51*V10+6*V50*V11+20*V31*V30+15*U40
1 *U21
T62=15*U20*U42+U02*U60+2*U01*U61+6*V52*V10+12*V51*V11+6*V50
1 V12+20*V32*V30+20*V31*V30+30*U41*U21+15*U40*U22
T63=15*U20*U43+U03*U60+6*V53*V10+3*U02*U61+3*U01*U62+18*V52*
1 V11+18*V51*V12+6*V50*V13+20*V33*V30+60*V32*V31+45*U42*U21 +
2 45*U41*U22+15*U40*U23
T64=15*U20*U44+6*V54*V10+U04*U60+4*U63*U01+4*U03*U61+24*V53*
1 V11+6*U02*U62+16*V52*V12+24*V51*V13+6*V50*V14+20*V34*V30+
2 80*V33*V31+60*U43*U21+60*V32*V32+90*U42*U22+60*U41*U23+15*
3 U40*U24
T80=28*U20*U60+56*V50*V30+35*U40**2+8*V70*V10
T81=28*U20*U61+U01*U80+56*V51*V30+56*V50*V31+28*U60*U21+70*
1 U1*U40+8*V71*V11+8*V10
T82=28*U20*U62+U02*U80+2*U01*U81+56*V52*V30+112*V51*V31+56*
1 U61+U21+56*V50*V32+28*U60*U22+70*U42*U40+70*U41**2+8*V72*
2 V10+16*V71*V11+8*V12*V70
T83=28*U20*U63+U03*U80+56*V53*V30+3*U02*U81+3*U01*U82+168*
1 \ V52*V31+84*U62*U21+168*V51*V32+84*U61*U42+210*U60*U41+8*V73*V10+24*V72*V11+8*V13*V70+
2 \ 24*V71*V12
T100=45*U20*U80+126*V50*V70+210*U60*U40+10*V90*V10+120*V30*V70+120*V31*V71+120*V32*V72+90*U81*U21+
3 \ 45*U80*U22
T120=66*U20*U100+792*V50*V70+462*U60**2+12*V110*V10+220*V90+
1 \ V30+495*U40*U80
T121=66*U20*U101+U01*U100+252*V51*V70+924*U61*U40+792*V50*V71+
1 \ +12*V111*V10+12*V110*V11+220*V91*V30+66*U100*U21+220*V90+
2 \ 495*U41*U80+495*U40*U81
T140=91*U20*U120+14*V130*V10+200*V50*V90+3003*U60*U80+364*
1 \ V100*V30+100**1*U40+1716*V70**2

\text{FUN}(1) = \text{B}(1) - 1.D00

\text{FUN}(2) = Z2*G + T20
\text{FUN}(2) = \text{FUN}(2) + \text{LAMM}\cdot F2
\text{IF}(\text{NT} = 2) \text{RETURN}

\text{FUN}(3) = Z4*G + 3\cdot T02\cdot Z2**2 + 6\cdot T21*Z2 + T40
\text{FUN}(3) = \text{FUN}(3) + \text{LAMM}\cdot F4
\text{IF}(\text{NT} = 3) \text{RETURN}

\text{FUN}(4) = Z6*G + 15\cdot T03\cdot Z2**3 + 45\cdot T22*Z2**2 + 15\cdot T41*Z2 + T60
\text{FUN}(4) = \text{FUN}(4) + \text{LAMM}\cdot F6
\text{IF}(\text{NT} = 4) \text{RETURN}
$$\text{FUN}(5) = 28*Z6*T02*Z2 + 28*Z6*T21 + 210*T03*Z4*Z2**2 + 420*T22*Z4*$$
$$1 \ *Z2 + 70*T41*Z4 + 35*Z4**2*Z02 + 105*Z2**4*T04 + 420*Z2**3*T23$$
$$2 \ + 210*Z2**2*T42 + 28*Z2*T61 + G*Z8 + T80$$

$$C$$

$$F8= 10*(5670*Z6*Z2**4 - 1512*Z6*Z2*Z4 + 99225*Z2**9 -132300*Z2**6$$
$$1 \ *Z4 + 37800*Z2**3*Z4**2 - 108*Z2**2*Z8 - 840*Z4**3 + Z10)/9$$

$$C$$

$$\text{FUN}(5) = \text{FUN}(5) + \text{LAMM}*F8$$
$$\text{IF}(\text{NT} \ . \ \text{EQ.} \ 5) \ \text{RETURN}$$

$$\text{FUN}(6) = 630*Z6*T03*Z2**2 + 1260*Z6*T22*Z2 + 210*Z6*T41 + 210*Z6*$$
$$1 \ *Z4*T02 + 1575*T03*Z4**2*Z2 + 1575*T22*Z4**2 + 3150*Z4*Z2**3*T04$$
$$2 \ + 9450*Z4*Z2**2*T23 + 3150*Z4*Z2*T42 + 210*Z4*T61 + 45*T02*Z2*Z8$$
$$3 \ + 45*T21*Z8 + 945*Z2**5*T05 + 4725*Z2**4*T24 + 3150*Z2**3*T43$$
$$4 \ + 630*Z2**2*T62 + 45*Z2*T81 + G*Z10 + T100$$

$$C$$

$$F10= 12*(-4158*Z6**2*Z2-727650*Z6*Z2**6+415800*Z6*Z2**3*Z4-$$
$$1 \ 13860*Z6*Z4**2-9823275*Z2**11+16372125*Z2**8*Z4-727650*Z2**5$$
$$2 \ *Z4**2 + 14850*Z2**4*Z8 + 693000*Z2**2*Z4**3 -165*Z2**2*Z10$$
$$3 \ - 3960*Z2*Z4*Z8 + Z12)/11$$

$$C$$

$$\text{FUN}(6) = \text{FUN}(6) + \text{LAMM}*F10$$
$$\text{IF}(\text{NT} \ . \ \text{EQ.} \ 6) \ \text{RETURN}$$

$$\text{FUN}(7) = 462*Z6**2*T02 + 13860*Z6*T03*Z4*Z2 + 13860*Z6*T22*Z4$$
$$1 \ + 13860*Z6*Z2**3*T04 + 41580*Z6*Z2**2*T23 + 13860*Z6*Z2**2*T42$$
$$2 \ + 924*Z6*T61 + 5775*T03*Z4**3 + 1485*T03*Z2**2*Z8 + 2970*T22*Z2*$$
$$3 \ Z8 + 495*T41*Z8 + 51975*Z4**2*Z2**2*T04 + 103950*Z4**2*Z2**2*T23$$
$$4 \ + 17325*Z4**2*T42 + 495*Z4*T02*Z8 + 51975*Z4*Z2**4*T05 +$$
$$5 \ 207900*Z4*Z2**3*T24 + 103950*Z4*Z2**2*T43 + 13860*Z4*Z2**2*T62 +$$
$$6 \ 495*Z4*T81 + 66*T02*Z2*Z10 + 66*T21*Z10 + 10395*Z2**6*T06 +$$
$$7 \ 62370*Z2**5*T25 + 51975*Z2**4*T44 + 13860*Z2**3*T63$$
$$8 \ + 1485*Z2**2*T82 + 66*Z2*T101 + G*Z12 + T120$$

$$C$$

$$A0 = \ 1621620.000$$
$$A1 = \ 12770257.000$$
$$A2 = \ 113513400.000$$
$$A3 = \ 1404728325.000$$
$$A4 = \ 2809456650.000$$
$$A5 = \ 1702701000.000$$
$$A6 = \ 315315000.000$$
$$A7 = \ 16216200.000$$

$$C$$

$$F12= A0*Z6**2*Z2**3 - 108108*Z6**2*Z4 + A1*Z6*Z2**8$$
$$1 \ - A2*Z6*Z2**5*Z4 + A7*Z6*Z2**2*Z4**2 - 30888*Z6*Z2$$
$$2 \ + Z8* \ A3*Z2**13 - A4*Z2**10*Z4 + A5*Z2**7*$$
$$3 \ Z4**2 - 2702700*Z2**6*Z8 - A6*Z2**4*Z4**3 + 32175*Z2**4*Z10 +$$
$$4 \ 15444000*Z2**3*Z4**2*Z12 + 9009000*Z2*Z4**4 - 8580*Z2*Z4$$
$$5 \ *Z10 - 51480*Z4**2*Z8 + Z14$$

$$F12 = 14*F12/13$$
\[\text{FUN}(7) = \text{FUN}(7) + \text{LAMM} \cdot \text{F12}\]

IF\(\text{NT} \cdot \text{EQ} \cdot 7\) RETURN

\[\text{FUN}(8) = \]
\[\begin{align*}
1 & \quad 1001 \cdot T41 \cdot Z10 + 42042 \cdot T22 \cdot Z6 \cdot Z8 + 45045 \cdot T22 \cdot Z4 \cdot Z8 \cdot 6006 \cdot T22 \cdot Z2 \cdot Z2 \\
2 & \quad Z10 + 42042 \cdot T03 \cdot Z6 \cdot Z2 \cdot Z2 + 105105 \cdot T03 \cdot Z6 \cdot Z4 \cdot Z2 \cdot Z8 + 45045 \cdot T03 \cdot Z4 \cdot Z2 \cdot Z8 \\
3 & \quad + 3003 \cdot T03 \cdot Z2 \cdot Z2 \cdot Z10 + 3003 \cdot Z6 \cdot T02 \cdot Z8 + 630630 \cdot Z6 \cdot Z4 \cdot Z2 \cdot Z8 \cdot Z2 \cdot T04 \\
4 & \quad + 1261260 \cdot Z6 \cdot Z4 \cdot Z2 \cdot T23 + 210210 \cdot Z6 \cdot Z4 \cdot T42 + 315315 \cdot Z6 \cdot Z2 \cdot Z4 \cdot Z2 \cdot T05 \\
5 & \quad + 1261260 \cdot Z6 \cdot Z2 \cdot Z3 \cdot T24 + 630630 \cdot Z6 \cdot Z2 \cdot Z3 \cdot Z4 \cdot T43 + 840840 \cdot Z6 \cdot Z2 \cdot Z6 \cdot T62 \\
\end{align*}\]

\[\text{FUN}(8) = \text{FUN}(8) + 3003 \cdot Z6 \cdot T81\]

\[\begin{align*}
1 & \quad 91 \cdot T21 \cdot Z12 + 1001 \cdot T02 \cdot Z4 \cdot Z10 + 91 \cdot T02 \cdot Z2 \cdot Z12 + 525525 \cdot Z4 \cdot Z3 \\
2 & \quad * Z2 \cdot T04 + 525525 \cdot Z4 \cdot Z3 \cdot T23 + 1576575 \cdot Z4 \cdot Z2 \cdot Z8 \cdot T05 + 4729725 \cdot Z4 \\
3 & \quad * Z2 \cdot Z2 \cdot Z3 \cdot T24 + 1576575 \cdot Z4 \cdot Z2 \cdot Z8 \cdot Z3 + 105105 \cdot Z4 \cdot Z2 \cdot T06 + 945945 \cdot Z4 \\
4 & \quad * Z2 \cdot Z2 \cdot Z3 \cdot T06 + 4729725 \cdot Z4 \cdot Z2 \cdot Z8 \cdot T25 + 315315 \cdot Z4 \cdot Z2 \cdot Z2 \cdot T44 \\
5 & \quad + 630630 \cdot Z4 \cdot Z2 \cdot Z2 \cdot T63 \\
6 & \quad + 45045 \cdot Z4 \cdot Z2 \cdot T82 + 1001 \cdot Z4 \cdot T101 + 135135 \cdot Z2 \cdot Z7 \cdot T07 \\
7 & \quad + 945945 \cdot Z2 \cdot Z2 \cdot T26 + 945945 \cdot Z2 \cdot Z2 \cdot Z5 \cdot T45 + 315315 \cdot Z2 \cdot Z2 \cdot Z4 \cdot T64 \\
\end{align*}\]

\[\text{FUN}(8) = \text{FUN}(8) + 45045 \cdot Z2 \cdot Z3 \cdot T83\]

\[\begin{align*}
1 & \quad 45045 \cdot Z2 \cdot Z3 \cdot T04 \cdot Z8 + 3003 \cdot Z2 \cdot Z2 \cdot T102 + 135135 \cdot Z2 \cdot Z2 \cdot T23 \cdot Z8 \\
2 & \quad + 91 \cdot Z2 \cdot Z121 + 45045 \cdot Z2 \cdot T42 \cdot Z8 + 3003 \cdot T61 \cdot Z8 \\
3 & \quad + G \cdot Z14 + T140 \\
\end{align*}\]
FUN(8) = FUN(8) + F14*LAMM

500 RETURN
END
APPENDIX F.2

Slug Velocity Calculation for Viscous Liquids in Tubes

THIS PROGRAM WILL CALCULATE THE FROUDE NUMBER AND SLIP VELOCITY OF A GAS SLUG IN A VERTICAL TUBE FILLED WITH A VISCOUS LIQUID

REAL NP,NU
A = 1.339

PRINT 1
1 FORMAT(' ENTER TUBE DIAMETER (CM), LIQ DENS(G/ML), VISC(CP), ST',
     ' (DYNE/CM)')
READ(5,*) DIAM,DENL,VISC,ST
DIAM = DIAM/2.54
RAD = DIAM/2.

DEN = DENL*62.4
VIS = VISC*2.42/3600
R = RAD/12
NP = (32.2*DEN**2*R**3/VIS**2)**.5
IF(NP .LT. 22.) GO TO 300
RCM = 2.5*4 - • RAD
EO = 4*DENL*981*RCM**2/ST

NU = 6.40*NP**(-.60)
EP = 0.1
100 F = (1 - NU*EP)**.5
FF = EO**4
G = 1 - 3.18/FF - 14.77/FF**2
FUN = F**G*(1-EP) - A*NP*EP**3

C CALCULATE DERIVATIVES WRT EP
DF = - .5*NU/F
DG = (12.72/(F*FF) + 118.16/(F*FF**2))*DF
EPO = EP
EP = EP - FUN/DFUN
IF(ABS(1 - EP/EPO) .GE. .0001) GO TO 100

RR = (1 - NU**EP)
EOEFF = EO*RR**2
FREEFF = .352*(1 - 3.18/EOEFF - 14.77/EOEFF**2)
PR = RR**.5*FREEFF
US = FR*SQRT(386.4*DIAM)
PRINT 2,EO,FR,US,NP
2 FORMAT(' EOTVOS NUMBER = ',F6.1,/
     ' FROUDE NUMBER = ',F6.3,/
     ' SLIP VELOCITY = ',F6.2,' FT/SEC',/
     ' LIQ PROP NUM = ',E12.4)
STOP
300 PRINT 10
10 FORMAT(' THE LIQUID PROPERTY IS LESS THAN 22. SOLUTION NOT'
1     ' VALID')
STOP
END
APPENDIX F.3
Slug Flow in Vertical 2-D
Rectangular Channel

IMPLICIT REAL *8(A-H,O-Z)
REAL*8 LEN,LAMM,LMAX
COMMON FUN(16),C(8),COLD(8),ALPHA(8),DUMMY(8),BETA,LAMM,
& U10,U11,U12,U13,U14,U15,U16,
& U30,U31,U32,U33,U34,U35,
& U50,U51,U52,U53,U54,
& U70,U71,U72,U73,
& U90,U91,U92,
& U110,U111,U130,
& W01,W02,W03,W04,W05,W06,W07,
& W20,W21,W22,W23,W24,W25,W26,
& W40,W41,W42,W43,W44,W45,
& W60,W61,W62,W63,W64,
& W80,W81,W82,W83,
& W100,W101,W102,
& W120,W121,W140,
& Z2,Z4,Z6,Z8,Z10,Z12,Z14
DIMENSION CNEW(8),Cl(8),C2(8), ARRAY(8,9)
DENG = .076D00
EO = 9999.D00
DO 10 I = 1,8
10 COLD(I) = 0.0D00
IERR = 0
NCASE = 1
PI = 3.14159265359D00
READ(1,*) LEN, DELX, LMAX, DELS
READ(1,*) NT, ND, NITER, FAC, XX, EPS, IPRINT
READ(1,*) DENL, ST0
XA = 1.D00 + XX
  C DEQ = EQUIPERIPHERAL DIAMETER FOR RECTANGULAR SLOT
  DEQ = 2*LEN/PI
  BETA = 2*PI/LEN
  NTO = NT + 1
  NT = 0
  C CONVERT DENSITY TO LBM/FT3
  DENL = DENL*62.24D00
  C CONVERT SURFACE TENSION TO LBM/SEC**2
  ST = ST0/453.2D00
  LAMM = 1728*ST/(DENL - DENG)
30 READ(1,*,END = 40) I,C(I)
  NT = NT + 1
  IF(NT .LT. NTO) GO TO 30
40 NT1 = NT + 1
  NTM1 = NT - 1
PRINT 20
45 ITER = 1
  IF(IPRINT .NE. 1) PRINT 15,LEN,NTM1,ND,DENL,ST0
15 FORMAT(/,20X,F6.2,' INCH CHANNEL',//,
1   25X,' N = ',I3,' ND = ',I3,//,
2   ' FLUID DENSITY = ',F6.2,' LBM/FT**3',//,
3   ' FLUID SURFACE TENSION = ',F6.2,' DYNE/CM',//)

50 CONTINUE
   DO 55 I = 2,NT
55   ALPHA(I) = BETA*(I - 1)
   ITER = ITER + 1
   IF(ITER .GE. NITER) GO TO 1100
   DO 60 I = 1,NT
60   COLD(I) = C(I)
   DO 1000 11=1,NT1
    JJ = 11 - 1
    IF(JJ .GT. 0) GO TO 500
   CALL DERIV
   CALL FUNC
   DO 405 J=1,NT
    FUN(J+NT) = FUN(J)
405   ARRAY(J,NT1) = - FUN(J)
   GO TO 1000
C PRINT 222, (FUN(I), I = 1,NT)
   CALL SIMEQ( ARRAY, NT,NT1, NT,DUMMY,IERR)
   IF(IERR .EQ. 1) GO TO 1100
   DO 420 I = 1,NT
420   C(I) = FAC*DUMMY(I) + COLD(I)
   C CHECK FOR CONVERGENCE
   SS = 0.0D00
   ICONV = 1
   DO 600 I = 1,NT
700   SS = SS + (FAC*DUMMY(I)/C(I))**2
   SS1 = DABS(FAC*DUMMY(I)/C(I))
   IF(SS1 .GT. EPS) ICONV = 0
600 CONTINUE
   SS = DSQRT(SS/FAC)
C PRINT 222, SS,C
222 FORMAT(8D9.3)
C IF(SS .GT. EPS) GO TO 50
C IF(ICONV .EQ. 0) GO TO 50
12 FORMAT(4E12.5)
650 US = C(1)/12
RX = -2/(LEN*Z2)

C PRINT OUTPUT WHEN CONVERGED
IF(LAMM .NE. 0.0D00) EO = 386.4D00*DEQ**2/LAMM
IF(LAMM .EQ. 0.0D00) EO = 9999.

C CALCULATE FROUDE NUMBER
FR = C(1)/DSQRT(386.4D00*DEQ)

C WRITE(2,11) LEN,US,RX,EO,FR
C WRITE(2,11) EO,FR
C WRITE(2,11) EO,RX

11 FORMAT(5F11.3)
   IF(IPRINT .EQ. 1 )  GO TO 740
   PRINT 4, US,FR,EO,RX

4 FORMAT( ' SLIP VELOCITY = ',F6.3,' FT/SEC',/,
         ' FROUDE NUMBER = ',F6.3,'/,
         ' EOTVOS NUMBER = ',F8.2,'/,
         ' NORMALIZED RC IN X DIR = ',F8.4,'/)

DO 700 I = 1,NT
   KI = I - 1
   IF(I .EQ. 1 )  GO TO 700
   IF(C(I) .EQ, 0.0D00) GO TO 700
   PRINT 3,KI,C(I)

700 CONTINUE
3 FORMAT(5X,I5,3X,E17.9)

C PRINT VALUES OF DERIVATIVES
CALL DERIV
CALL FUNC
IF(ND .GE. 2) PRINT 16, Z2
16 FORMAT(//,' Z2 = ',E17.9)
IF(ND .GE. 4) PRINT 17, Z4
17 FORMAT( ' Z4 = ',E17.9)
IF(ND .GE. 6) PRINT 18, Z6
18 FORMAT( ' Z6 = ',E17.9)
IF(ND .GE. 8) PRINT 19, Z8
19 FORMAT( ' Z8 = ',E17.9)
IF(ND .GE. 10) PRINT 22, Z10
22 FORMAT( ' Z10 = ',E17.9)
IF(ND .GE. 12) PRINT 23, Z12
23 FORMAT( ' Z12 = ',E17.9)
IF(ND .GE. 14) PRINT 24, Z14
24 FORMAT( ' Z14 = ',E17.9)
PRINT 20
20 FORMAT (/'*********************

CCCCC
DO 725 I = 2,NT
   IF(C(I) .EQ. 0.0D00) GO TO 725
   II = I - 1
   X = ALPHA(I)*C(I)
   PRINT 730, II,X

725 CONTINUE
730 FORMAT(I4,D12.5)

CCCCC
740 IF(DELX .NE. 0.0D00) GO TO 750
IF(DELS .EQ. 0.0D00) GO TO 1100
STO = STO + DELS
IF(STO .GT. 75. .OR. STO .LT. 0.1) GO TO 1100
LAMM = 1728*STO/(453.2D00*(DENL - DENG))
GO TO 45
750 LEN = LEN - DELX
IF(LEN .GE. LMAX .OR. LEN. LE. 0.0D00) GO TO 1100
DEQ = 2*LEN/PI
BETA = 2*PI/LEN
C SAVE PREVIOUS VALUES FOR NEXT GUESS
DO 860 I = 1,NT
C IF(NCASE .GE. 1)
C IF(NCASE .GE. 2)
C IF(NCASE .GE. 3)
C IF(NCASE .GE. 4)
C 1     C3(I) = C2(I)
C 2     C2(I) = C1(I)
C 3     C1(I) = C(I)
C 4     C(I) = CNEW(I)
860 CONTINUE
NCASE = NCASE + 1
GO TO 45
1100 STOP
END

SUBROUTINE SIMEQ(A,M,MP,N,STORE,IERR)
C
C SUBROUTINE SIMEQ SOLVES ANY NUMBER OF SIMULTANEOUS EQUATIONS.
C COEFFICIENTS ARE STORED IN THE CALLING MATRIX A (WHICH IS
C DESTROYED DURING THE SOLUTION PROCESS). VALUES OF THE N UNKNOWNS
C TO BE SOLVED FOR ARE STORED IN THE VECTOR STORE, IN THEIR ORDER OF
C APPEARANCE IN THE EQUATIONS. IF SOLUTION IS NOT POSSIBLE AN ERROR
C MESSAGE WILL BE PRINTED AND IERR WILL BE RETURNED AS 1. IF SOLU-
C TION IS SUCCESSFUL IERR WILL BE RETURNED AS 0.
C
C M AND MP ARE THE MAXIMUM DIMENSIONS OF THE ARRAY A
C N IS THE NUMBER OF EQUATIONS
C
IMPLICIT REAL *8(A-H,O-Z)
DIMENSION STORE(8),A(8,9)
DO 10 I = 1,8
10 STORE(I) = 0.0D00
IERR=0
IX=N+1
DO 110 MX=1,N
DO 102 I=MX,N
DIV=A(I,MX)
IF(DIV)103,102,103
102 CONTINUE
PRINT 1
1 FORMAT(/9X,'SOLUTION OF SIMULTANEOUS EQUATIONS NOT POSSIBLE/')
IERR=1
GO TO 114
103 DO 105 J=MX,IX
   Z=A(MX,J)
   A(MX,J)=A(I,J)/DIV
   IF(I-MX)104,105,104
104 A(I,J)=Z
105 CONTINUE
   IF(N-1)114,112,106
106 KY=MX+1
   IF(KY.GT.N) GO TO 110
   DO 109 L=KY,N
      AMULT=A(L,MX)
      IF(MX-N)107,109,109
108 A(L,J)=A(L,J)-AMULT*A(MX,J)
109 CONTINUE
110 CONTINUE
   KU=N-1
   DO 111 J=1,KU
      L=N-J
      LX=L+1
      DO 111 MX=LX,N
111 A(L,N+1)=A(L,N+1)-A(MX,N+1)*A(L,MX)
   112 DO 113 I=1,N
   113 STORE(I)=A(I,N+1)
114 CONTINUE
RETURN
END

SUBROUTINE DERIV
IMPLICIT REAL*8(A-H,O-Z)
REAL*8 LAMM
COMMON FUN(16),C(8),COLD(8),ALPHA(8),DUMMY(8),BETA,LAMM,
& U10,U11,U12,U13,U14,U15,U16,
& U30,U31,U32,U33,U34,U35,
& U50,U51,U52,U53,U54,
& U70,U71,U72,U73,
& U90,U91,U92,
& U110,U111,U130,
& W01,W02,W03,W04,W05,W06,W07,
& W20,W21,W22,W23,W24,W25,W26,
& W40,W41,W42,W43,W44,W45,
& W60,W61,W62,W63,W64,
& W80,W81,W82,W83,
& W100,W101,W102,
& W120,W121,W140,
& Z2,Z4,Z6,Z8,Z10,Z12,Z14,NT,ND
Z2 = 0.0D00
Z4 = 0.0D00
Z6 = 0.0D00
Z8 = 0.0D00
Z10 = 0.0D00
Z12 = 0.0D00
Z14 = 0.0D00
U10 = 0.0D00
U11 = 0.0D00
U12 = 0.0D00
U13 = 0.0D00
U14 = 0.0D00
U15 = 0.0D00
U16 = 0.0D00
U30 = 0.0D00
U31 = 0.0D00
U32 = 0.0D00
U33 = 0.0D00
U34 = 0.0D00
U35 = 0.0D00
U50 = 0.0D00
U51 = 0.0D00
U52 = 0.0D00
U53 = 0.0D00
U54 = 0.0D00
U72 = 0.0D00
U73 = 0.0D00
U90 = 0.0D00
U91 = 0.0D00
U92 = 0.0D00
U110 = 0.0D00
U111 = 0.0D00
U130 = 0.0D00
W01 = 0.0D00
W02 = 0.0D00
W03 = 0.0D00
W04 = 0.0D00
W05 = 0.0D00
W06 = 0.0D00
W07 = 0.0D00
W26 = 0.0D00
W45 = 0.0D00
W64 = 0.0D00
W83 = 0.0D00
W102 = 0.0D00
W121 = 0.0D00
W140 = 0.0D00
C C(1) = SLIP VELOCITY US
DO 100 I=2,NT
IF(C(I) .EQ. 0.0D00) GO TO 100
BB = (I-1)*BETA
B2 = BB**2
B4 = BB**4
B6 = BB**6
B8 = BB**8
B10 = BB**10
B12 = BB**12
B14 = BB**14
AL = ALPHA(I)
CI = C(I)
U10 = U10 - B2*CI
U11 = U11 + AL*B2*CI
U12 = U12 - AL**2*B2*CI
U13 = U13 + AL**3*B2*CI
U14 = U14 - AL**4*B2*CI
U15 = U15 + AL**5*B2*CI
U16 = U16 - AL**6*B2*CI
U30 = U30 + B4*CI
U31 = U31 - AL*B4*CI
U32 = U32 - AL**2*B4*CI
U33 = U33 - AL**3*B4*CI
U34 = U34 - AL**4*B4*CI
U35 = U35 - AL**5*B4*CI
U50 = U50 - B6*CI
U51 = U51 + AL*B6*CI
U52 = U52 + AL**2*B6*CI
U53 = U53 + AL**3*B6*CI
U54 = U54 + AL**4*B6*CI
U70 = U70 + B8*CI
U71 = U71 - AL*B8*CI
U72 = U72 + AL**2*B8*CI
U73 = U73 - AL**3*B8*CI
U90 = U90 - B10*CI
U91 = U91 + AL*B10*CI
U92 = U92 + AL**2*B10*CI
U110 = U110 + B12*CI
U111 = U111 - AL*B12*CI
U130 = U130 - B14*CI
W01 = W01 + AL**2*CI
W02 = W02 - AL**3*CI
W03 = W03 - AL**4*CI
W04 = W04 - AL**5*CI
W05 = W05 + AL**6*CI
W06 = W06 - AL**7*CI
W07 = W07 + AL**8*CI
W26 = W26 + AL**7*B2*CI
W45 = W45 + AL**6*B4*CI
W64 = W64 + AL**5*B6*CI
W83 = W83 + AL**4*B8*CI
W102 = W102 + AL**3*B10*CI
W121 = W121 + AL**2*B12*CI
W140 = W140 + AL*B14*CI

100 CONTINUE
W20 = U11
W21 = U12
W22 = U13
W23 = U14
W24 = U15
W25 = U16
W40 = U31
\[ W_{41} = U_{32} \]
\[ W_{42} = U_{33} \]
\[ W_{43} = U_{34} \]
\[ W_{44} = U_{35} \]
\[ W_{60} = U_{51} \]
\[ W_{61} = U_{52} \]
\[ W_{62} = U_{53} \]
\[ W_{63} = U_{54} \]
\[ W_{80} = U_{71} \]
\[ W_{81} = U_{72} \]
\[ W_{82} = U_{73} \]
\[ W_{100} = U_{91} \]
\[ W_{101} = U_{92} \]
\[ W_{120} = U_{111} \]

**Z2 =** \( \frac{W_{20}}{2U_{10} - W_{01}} \)

**IF(ND .EQ. 2) RETURN**

**Z4 =** \( \frac{W_{40} + F_1Z_2 + F_2Z_2Z_2}{F_3} \)

**IF(ND .EQ. 4) RETURN**

**Z6 =** \( \frac{W_{60} + Z_2(15W_{41} - 6U_{50}) + Z_4(15W_{21} - 60U_{31})}{6U_{10} - W_{01}} \)

**IF(ND .EQ. 6) RETURN**

**Z8 =** \( \frac{W_{80} + Z_6(28W_{21} - 56U_{30}) + Z_2Z_6(28W_{02} - 224U_{11}) + Z_4(70W_{41} - 56U_{50}) + Z_2Z_4(210W_{03} - 168U_{12})}{8U_{10} - W_{01}} \)

**IF(ND .EQ. 8) RETURN**

**Z10 =** \( \frac{945Z_2Z_4(280W_{61} - 8U_{70})}{8U_{10} - W_{01}} \)
7 (3*W21-8*U30)+W100)/(10*U10-W01)
IF(ND .EQ. 10) RETURN
Z12 = (3*W21-8*U30)+W100)/(10*U10-W01)

Z12 = 10395*Z2**6*(W06-12*U15)+W1975*Z2**4*Z4*(W05-12*U14)+10395*Z2**4*(-8*U53+5*W44)+
2 69300*Z2**3*Z4*(-10*U33+3*W24)+13860*Z2**3*Z6*(-W04+12*U13)+
3 1980*Z2**2*Z4*(-6*U72+7*W63)+51975*Z2**2*Z4**2*(-W04-12*U13)+
4 20790*Z2**2*Z4*(-8*U52+5*W43)+13860*Z2**2*Z6*(-10*U32+3*W23)+
5 1485*Z2**2*Z8*(-W03-12*U12)+165*Z2**2*(-4*U11+9*W82)

Z12 = Z12 +
6 34650*Z2*Z4**2*(-10*U32+3*W23)+13860*Z2*Z4**2*(-W03-12*U12)+
7 1980*Z2*Z4**2*(-6*U71+7*W62)+2772*Z2*Z6*(-8*U51+5*W42)+990*Z2+
8 *Z8*(-10*U31+3*W22)+66*Z2*Z10*(-12*U11+W02)+6*Z2*(-2*U110+
9 11*W101)+5775*Z4**3*(-W03-12*U12)+3465*Z4**2*(-8*U51+5*W42)

Z12 = Z12 +
1 4620*Z4*Z6*(-10*U31+3*W22)+495*Z4*Z6*(-12*U11+W02)+55*Z4+
2 (-4*U90+9*W81)+62*Z6**2*(-12*U11+W02)+132*Z6*(-6*U70+7*W61)+
3 +99*Z8*(-8*U50+5*W41)+2772*Z4**2*(-4*U33+W24)+273*Z2**2*(-4*U33+W24)+
4 429*Z8*(-8*U70+7*W61)+1001*Z10*(-2*U50+W41)+91*Z12*(-W21-4*)
5 U30)+W140

ANS2=105105*Z4**2*Z6*(-W03-14*U12)+15015*Z4**2*(-8*U71+7*W62)+
1 +210210*Z4*Z6*(-2*U51+W42)+45045*Z4*Z8*(-4*U31+2+W22)+1001*Z4+
2 *Z10*(-14*U11+W02)+91*Z4*(-14*U11+W02)+42042*Z6**2*(-4*U33+W24)+
3 *U31+3*W22)+3003*Z6**2*(-14*U11+W02)+1001*Z6*(-2*U90+3*W82)+
4 +990*Z8*(-14*U11+W02)+5775*Z4**3*(-W03-12*U12)+3465*Z4**2*(-8*U51+5*W42)+
5 91*Z4*(-4*U110+11*W101)+42042*Z6**2*(-4*U33+W24)+1261260*Z2**3*Z6*(-4*U33+W24)+

SUBROUTINE FUNC

Z14=ANS1/(14*U10-W01)
RETURN

END

SUBROUTINE FUNC

Z14=ANS2/(14*U10-W01)
RETURN

END
IMPLICIT REAL *8(A-H,O-Z)
REAL*8 LAMM, LAM
COMMON FUN(16), C(8), COLD(8), ALPHA(8), DUMMY(8), BETA, LAMM,
& U10, U11, U12, U13, U14, U15, U16,
& U30, U31, U32, U33, U34, U35,
& U50, U51, U52, U53, U54,
& U70, U71, U72, U73
& U90, U91, U92,
& U110, U111, U130,
& W01, W02, W03, W04, W05, W06, W07,
& W20, W21, W22, W23, W24, W25, W26,
& W40, W41, W42, W43, W44, W45,
& W60, W61, W62, W63, W64,
& W80, W81, W82, W83,
& W100, W101, W102,
& W120, W121, W140,
& Z2, Z4, Z6, Z8, Z10, Z12, Z14, NT, ND
Z16 = 0.0000
US = C(1)
G = 386.4D00/(US*US)
LAM = LAMM/(US*US)
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T02 = W01**2
T03 = 3*W02*W01
T04 = 4*W03*W01 + 3*W02**2
T05 = 5*(W04*W01 + 2*W03*W02)
T06 = 6*W05*W01 + 15*W04*W02 + 10*W03**2
T07 = 7*(W06*W01 + 3*W05*W02 + 5*W04*W03)
T20 = U10**2
T21 = W20*W01 + 2*U11*U10
T22 = W20*W02 + 2*W01*W21 + 2*U12*U10 + 2*U11**2
T23 = W20*W03 + 3*W02*W21 + 3*W01*W22 + 2*U13*U10 + 6*U12*U11
T24 = W20*W04 + 4*W03*W21 + 6*W02*W22 + 4*W01*W23 + 2*U14*U10 + 8*U13*U12 + 2*U11**2
T25 = W20*W05 + 5*W04*W21 + 10*W03*W22 + 10*W02*W23 + 5*W01*W24 + 2*U15
1 U10 + 14*U11 + 20*U13 + 12*U12 + 6*U11*U10
T26 = W20*W06 + 6*W05*W21 + 15*W04*W22 + 20*W03*W23 + 15*W02*W24 + 6*W01
1 *W25 + 2*U16 + U10 + 12*U15 + 30*U14 + U12 + 20*U13 + 2*U11**2
T40 = 3*W20**2 + 4*U30*U10
T41 = 6*W20*W21 + W01*W40 + 4*U31*U10 + 4*U30*U11
T42 = 6*W20*W22 + W02*W40 + 2*W01*W41 + 4*U32*U10 + 8*U31*U11 + 4*U30*U12 + 2*U12**2
T43 = 6*W20*W23 + W03*W40 + 3*W02*W41 + 3*W01*W42 + 4*U33*U10 + 12*U32 + 12*U31*U11 + 6*U30*U12 + 2*U11**2
T44 = 6*W20*W24 + W04*W40 + 4*W03*W41 + 6*W02*W42 + 4*W01*W43 + 4*U34 + 12*U33*U10 + 20*U32*U11 + 10*U31*U12 + 2*U10**2
T45 = 6*W20*W25 + W05*W40 + 5*W04*W41 + 10*W03*W42 + 10*W02*W43 + 5*W01
1 W44*4 + U35*U10 + 20*U34*U11 + 40*U33*U12 + 40*U32*U13 + 20*U31*U14 + 2*U30*U15 + 30*W24 + 16*U31 + 4*U30*U14 + 24*W23 + 2*W22**2
T61 = 15*W20*W24 + W01*W60 + 6*U51*U10 + 6*U50*U11 + 20*U31*U30 + 15*W40
1  *W21
2  +5*W41*W22+15*W40*W23
T80=28*W20*W60+56*U50*U30+35*W40**2+8*U70*U10
T81=28*W20*W61+W01*W80+56*U51*U30+56*U50*U31+28*W60*W21+70*W41*W40+70*W41*W22+70*W42*W40+70*W42*W23+15*W40--W24
T100=45*W20*W80+126*U50*U30+462*W60*W40+10*U90*U10+120*U30*U70
1  U70
T101=45*W20*W81+W01*W100+252*U51*U50+210*W61*W40+210*W60*W41+1  10*U91*U10+10*U90*U11+120*U31*U70+120*U30*U71+45*W80*W21
T102=45*W20*W82+W02*W100+2*W01*W101+252*U52*U50+210*W62*W40+1  252*U51*2*2+420*W61*W41+420*W60*W42+10*U92*U10+20*U91*U11+2  120*U32*U70+10*U90*U12+240*U31*U71+120*U30*U72+90*W81*W21+3  45*W80*W22
T120=66*W20*W100+792*U50*U70+462*W60*W40+2+12*U110*U10+220*U90*
1  U30+495*W40*W80
T121=66*W20*W101+W01*W120+792*U51*U70+924*W61*W60+792*U50*U71
1  +12*U111*U10+12*U110*U11+220*U91*U30+66*W100*W21+220*U90*
2  U31+495*W41*W80+495*W50*W81
T140=91*W20*W120+14*U130*U10+2002*U50*U90+3003*W60*W80+364*
1  U110*U30+1001*W100*W40+1716*U70+2
CCCCCCCCCCCCCCCCCCCCC AXIAL VELOCITY = 0 CCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
SUM = 0.000
DO 5 I = 2,NT
IF(C(I) .NE. 0.000)
1SUM = SUM + ALPHA(I)*C(I)
5 CONTINUE
FUN(1) = SUM + 1.000
CCCCCCCCCCCCCCCCCCCCCCCCC Z2 CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
FUN(2)= U10*U10 + G*Z2 + LAM*(Z4 - 3*Z2**3 )
IF(NF .EQ. 2) RETURN
CCCCCCCCCCCCCCCCCCCCCCCCC Z4 CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
FUN(3) =4*U10*U30 + 3*W20*W20 + (12*U10*U11 + 6*W01*W20
1  + 3*W01*W01*Z2) + G*Z4
2  + LAM*(Z6 - 30*Z2*Z2*Z4 + 45*Z2**5)
IF(NF .EQ. 3) RETURN
CCCCCCCCCCCCCCCCCCCCCCCCCCC Z6 CCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
FUN(4) = (6*U10*U50 + 10*U30*U30 + 15*W20*W40) + 15*(2*U10*U11
1  + W20*W01 + W01*W01*Z2)*Z4 + 45*(2*U10*U12 + 2*U11*U11
2  + W20*W02
3  + 2*W01*W21)*Z2*Z2 + 45*W01*W02*Z2**3 + 15*(4*U10*U31
4  + 4*U30*U11 + 6*W20*W21 + W40*W01)*Z2 + C*Z6
5  + LAM*Z8 - 210*Z4**2*Z2 + 1575*Z4*Z2**4
6  - 1575*Z2**7 - 63*Z2**2*Z6)
IF(NT .EQ. 4) RETURN

FUN(5) = (8*U10*U70 + 56*U30*U50 + 28*W20*W60 + 35*W40*W40)
1  + (56*U10*U11 + 28*W01*W20)*Z6 + 28*W01*W01*Z2*Z6
2  + (280*U10*U11 + 280*W20*W21 + 70*W40*W01)*Z4
3  + (168*U10*U51 + 560*U30*U31 + 168*U11*U50 + 420*W20*W41
4  + 420*W40*W01 + 35*W20*W02)*Z2**3 + G*Z8
5  + LAM*(Z8 - 840*Z4**3 + 37800*Z4**2*Z2**3
6  - 132300*Z4*Z2**6 - 108*Z2**2*Z8 + 99225*Z2**9 + 5670*Z2**4*Z6)
IF(NT .EQ. 5) RETURN

FUN(6) = 630*Z6*T03*Z2*Z2 + 1260*Z6*T22*Z2 + 1260*Z6*T23*Z2 + 1260*Z6*T24*Z2
1  + 9450*Z4*Z2**2*T23 + 3150*Z4*Z2**2*T04 + 17325*Z4*Z2**2*T24
2  + 14850*Z2*Z8 + 495*T02*Z2*Z8 + 2970*T22*Z2*Z8 + 495*T03*Z2*Z8
3  + 103950*Z4*Z2**2*T24 + 207900*Z4*Z2**2*T25 + 51975*Z2**4*T26
4  + 3960*Z2*Z8 + 3960*Z2*Z8

C
A0 = 9823275.D00
A1 = 7276500.D00
A2 = 16372125.D00
A3 = 727650.D00
A4 = 415800.D00
A5 = 693000.D00
F6 = Z12 - 4158*Z6**2*Z2 - A3*Z6*Z2**6 + A4*Z6*Z2**3*Z4
1  - 13860*Z6*Z4**2 - A0*Z2**11 + A2*Z2**5*Z4**2
2  + 14850*Z2*Z8 + A5*Z2*Z8 + A5*Z2*Z8 - 165*Z2*Z8 + A5*Z2*Z8
3  - 3960*Z2*Z8

C
FUN(6) = FUN(6) + LAM*F6
IF(NT .EQ. 6) RETURN

FUN(7) = 462*Z6**2*T02 + 13860*Z6*T03*Z2*Z2 + 1260*Z6*T22*Z2 + 210*Z6*T41 + 210*Z6*
1  + 13860*Z6*T23*Z2 + 13860*Z6*T24*Z2
2  + 9450*Z4*Z2**2*Z2 + 14850*Z4*Z2**2*Z2 + 2970*T22*Z2*Z8 + 103950*Z4*Z2**2*Z2
3  + 495*T04*Z2*Z8 + 51975*Z4*Z2**2*Z2**2*T04 + 103950*Z4*Z2**2*Z2**2*T04
4  + 17325*Z4*Z2**2*T24 + 495*Z4*Z2**2*T24 + 51975*Z4*Z2**2*T24 + 495*Z4*Z2**2*T24
6  + 495*Z4*Z2**2*T24 + 66*T22*Z2*Z10 + 66*T21*Z10 + 103950*Z2**6*T06 +
7  + 62370*Z2**6*T25 + 51975*Z2**6*T26 + 13860*Z2**3*T63
8  + 1485*Z2**2*Z10 + 66*Z2*Z10 + G*Z12 + T120

C
\[ \text{F7} = A0 \times Z6^{**2} \times Z2^{**3} - 108108 \times Z6^{**2} \times Z4 + A1 \times Z6 \times Z2^{**8} \\
1 - A2 \times Z6 \times Z2^{**5} \times Z4 + A7 \times Z6 \times Z2^{**2} \times Z4^{**2} = 30888 \times Z6 \times Z2 \times Z8 \\
2 + A3 \times Z2^{**13} - A4 \times Z2^{**10} \times Z4 + A5 \times Z2^{**7} \times Z4^{**2} \\
3 - A10 \times Z2^{**6} \times Z8 - A6 \times Z2^{**4} \times Z4^{**4} + 32175 \times Z2 \times Z4^{**3} \\
4 + A8 \times Z2^{**3} \times Z4^{**2} \times Z12 + A9 \times Z2 \times Z4^{**4} - 8580 \times Z2 \times Z4 \times Z10 \\
5 - 51480 \times Z4^{**2} \times Z8 + Z14 \]

\[ \text{FUN(7)} = \text{FUN(7)} + \text{LAM*F7} \]

\[ \text{IF(NT .EQ. 7) RETURN} \]

\[ \text{CC}ccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc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2 + A16*Z2**6*T26 + A16*Z2**5*T45 + A11*Z2**4*T64

FUNCTION(8) = FUNCTION(8) + A5*Z2**3*T83

1 + A5*Z2**3*T04*Z8 + A8*Z2**2*T102 + A17*Z2**2*T23*Z8
2 + A13*Z2*T121 + A5*Z2*T42*Z8 + A8*T61*Z8
3 + C*Z14 + T140

C

A1 = 595945350.00
A2 = 170270100.00
A3 = 29499294825.00
A4 = 35756721000.00
A5 = 9932422500.00
A6 = 16216200.00
A7 = 378378000.00
A8 = 27392202375.00
A9 = 63951387875.00
A10 = 491654913750.00
A11 = 638512875.00
A12 = 13905391500.00
A13 = 567567000.00
A14 = 11036025000.00
A15 = 81081000.00
A16 = 630630000.00
A17 = 4504500.00
A18 = 7882875.00
A19 = 1081080.00
A20 = 378378.00
A21 = 90090.00
A22 = 61425.00
A23 = 77220.00
A24 = 150150.00
A25 = 16380.00
A26 = 315.00

C

1 - A3*Z6**2*Z2**10 + A4*Z6**7*Z4
2 - A5*Z6**2*Z2**4*Z4**2 + A6*Z6**2*Z2**3*Z8 + A7*Z6**2*Z2**2*Z4**3

FUNCTION(8) = FUNCTION(8) + LAM*F8

RETURN

END
APPENDIX F.4
Slug Flow in Vertical 3-D Rectangular Channel

IMPLICIT REAL *8(A-H,O-Z)
DOUBLE PRECISION LEN,LAMM,WID
COMMON FUN(30),C(6,6),COLD(6,6),ST,
& ALPHA(6,6),DUMMY(15),BETA,GAMMA,LAMM,LEN,WID,
& U1,U2,U3,U4,U5,U6,U7,U8,U9,UA,UB,UC,UD,UE,UF,UG,UI,UJ,
& V1,V2,V3,V4,V5,V6,V7,V8,V9,VA,VB,VC,VD,VE,VF,VG,VI,VJ,
& W1,W2,W3,W4,W5,W6,W7,W8,W9,WA,WC,WD,WE,WF,WH,WI,WJ,
& WK,WL,WM,WN,WQ,WR,WS,WU,WV,WW,UK,VK,WX,WY,
& Z20,Z02,Z22,Z40,Z04,Z60,Z06,Z24,Z42,Z80,Z08,Z44,Z26,Z62,
& Z100,Z82,264,Z46,Z28,Z10,
& NTERM,NA,ND
DIMENSION CNEW(6,6),C1(6,6),C2(6,6)
DENG = .076D00
EO = 9999.D00
do 5  I  = 1,6
  do 5 J = 1,6
  C(I,J) = 0.D00
  5  COLD(I, J) = 0.D00
ierr = 0
ncase = 1
nterm = 0
pi = 3.141592654d00
xA = 1.00001d00
xx = .00001d00
READ(l,*) LEN,WID,DELX,DELS
READ(l,*) I  PRINT
READ(l,*) EPS,NT,ND,NITER,FAC
READ(l,*)DENL,STO
NTERM0 = (NT+1)•(NT + 2) /  2
C CONVERT DENSITY TO LBM/FT3
DENL = DENL*62.24
C CONVERT SURFACE TENSION TO LBM/SEC**2
ST = STO/453.2
LAMM = 1728*ST/((DENL - DENG)
BETA = 2*pi/LEN
GAMMA = 2*pi/WID
30 READ(l,*,END = 40) I,J,C(I+1,J+1)
  I = I + 1
  J = J + 1
  IF(I .GT. NA) NA = I
  IF(J .GT. NA) NA = J
  ALPHA(I,J) = DSQRT(((I-1)*BETA)**2 + ((J-1)*GAMMA)**2)
  NTERM = NTERM + 1
  IF(NTERM .LT. NTERM0) GO TO 30
40 NTERM1 = NTERM + 1
PRINT 20
45 ITER = 1
PRINT 15,LEN,WID,NT,ND,DENL,ST0
15 FORMAT(/,20X,F6.2,' BY ',F6.2,' CHANNEL'/,
1   25X, ' N = ',13,' ND = ',13,//,
2   ' FLUID DENSITY = ',F6.2,' LBM/FT**3',/,,
3   ' FLUID SURFACE TENSION = ',F6.2,' DYNE/CM',//)

50 DO 60 I = 1,6
   DO 60 J = 1,6
60   COLD(I,J) = C(I,J)
   MM = 0
   DO 1000 I1=1,NTERM1
      IF(I1 .EQ. 1) GO TO 100
   DO 70 I = 1,6
   DO 70 J = 1,6
      IF(C(I, J) .EQ. 0.0) GO TO 70
      MMM = 10*I + J
      IF(MMM .LE. MM) GO TO 70
      C(I,J) = XA*COLD(I,J)
      XXX = XX*COLD(I, J)
      MM = MMM
   GO TO 100
70 CONTINUE
100 CONTINUE
   CALL DERIV
   CALL FUNC
   IF(I1 .GT. 1) GO TO 500
   DO 405 J=1,NTERM
405   FUN(J+NTERM) = FUN(J)
   GO TO 1000
C CALCULATE JACOBIAN
500 CALL JACOB(I1,XXX,IERR)
   C(I,J) = COLD(I,J)
1000 CONTINUE
C SOLVE FOR NEW VALUES
   XXX = 100000.000
   CALL JACOB(I1,XXX,IERR)
   IF(IERR .EQ. 1) GO TO 1100
   II = 1
   DO 420 I = 1,6
   DO 420 J = 1,6
      IF(C(I, J) .EQ. 0.000) GO TO 420
      C(I,J) = FAC*DUMMY(II) + COLD(I,J)
   II = II+1
420 CONTINUE
   C(1,1) = DABS(C(1,1))
   US = C(1,1)/12
   RX = -2/(LEN*Z20)
   RY = -2/(WID*Z02)
C CHECK FOR CONVERGENCE
   SS = 0.0000
   ITER = ITER + 1
   IF(ITER .GE. NITER) GO TO 1100
   DO 600 I = 1,6
   DO 600 J = 1,6
      IF(C(I,J) .NE. 0.0)
D00 CONTINUE
SS = DSQRT(SS/FAC)
12 FORMAT(4E12.5)
   IF(SS .GT. EPS) GO TO 50
C PRINT OUTPUT WHEN CONVERGED
C EO NUMBER BASED ON EQ. PERIPHERAL DIAMETER
DEQ = 2*(LEN + WID)/PI
   IF(LAMM .NE. 0.0000) EO = 386.4D00*DEQ**2/LAMM
C CALCULATE FROUDE NUMBER BASED ON EQUIVALENT PERIPHERAL DIAMETER
FR = C(1,1)/DSQRT(386.4D00*DEQ)
WRITE(2,11) LEN,WID,US,RX,RY,EO,FR
11 FORMAT(7F9.3)
   PRINT 4, US,FR,EO,RX,RY
4 FORMAT(' SLIP VELOCITY      = ',F6.3, ' FT/SEC',/,
   ' FROUDE NUMBER       = ',F6.3,/, 
   ' EOTVOS NUMBER       = ',F8.2,/, 
   ' NORMALIZED RC IN X DIR = ',F8.4,/, 
   ' NORMALIZED RC IN Y DIR = ',F8.4,//)
   IF(IPRINT .NE. 1) GO TO 710
DO 700 I = 1,6
K1 = I - 1
DO 700 J = 1,6
   IF(I .EQ. 1 .AND. J .EQ. 1) GO TO 700
   IF(C(I,J) .EQ. 0.0000) GO TO 700
K2 = J - 1
   PRINT 3,K1,K2,C(I,J)
CONTINUE
3 FORMAT(5X,I5,I5,3(3X,E17.9))
C PRINT VALUES OF DERIVATIVES
710 CALL DERIV
   CALL FUNC
   IF(IPRINT .NE. 1) GO TO 720
   IF(ND .GE. 2) PRINT 16, Z20,Z02
16 FORMAT(//' Z20 = ',E17.9,' Z02 = ',E17.9)
   IF(ND .GE. 4) PRINT 17, Z40,Z04,Z22
17 FORMAT(' Z40 = ',E17.9,' Z04 = ',E17.9,' Z22 = ',E17.9)
   IF(ND .GE. 6) PRINT 18, Z60,Z06,Z42,Z24
18 FORMAT(' Z60 = ',E17.9,' Z06 = ',E17.9,' Z42 = ',E17.9,
  1 ' Z24 = ',E17.9)
   IF(ND .GE. 8) PRINT 19, Z80,Z08,Z44,Z62,Z26
19 FORMAT(' Z80 = ',E17.9,' Z08 = ',E17.9,' Z44 = ',E17.9,
  1 ' Z62 = ',E17.9,' Z26 = ',E17.9)
   PRINT 20
20 FORMAT(//'******************************************************************',
  1 '******************************************************************'/)
CCCCCC
Y = 0.
   DO 111 I = 1,6
   DO 111 J = 1,6
      K1 = I - 1
      J1 = J - 1
      X = C(I,J)*ALPHA(I,J)
      Y = Y + X
111 CONTINUE
IF(X .NE. 0.0D00) PRINT 112, II, JJ, X
112 FORMAT(I4, I4, F8.4)
CONTINUE
PRINT 112, II, JJ, Y

IF(SS .GT. EPS) GO TO 50
IF(DELX .NE. 0.0D00) GO TO 750
IF(DELS .EQ. 0.0D00) GO TO 1100
STO = STO + DELS
LAMM = 1728*STO/(453.2D00*(DENL - DENG))
GO TO 45
750 WID = WID - DELX
GAMMA = 2*PI/WID
DO 800 I = 1, 6
DO 800 J = 1, 6
IF(C(I, J) .NE. 0.0D00)
1ALPHA(I, J) = DSQRT(((I-1)*BETA)**2 + ((J-l)*GAMMA)**2)
800 CONTINUE
C SAVE PREVIOUS VALUES FOR NEXT GUESS
DO 860 I = 1, 6
DO 860 J = 1, 6
IF(NCASE .EQ. 1) CNEW(I,J) = C(I,J)
IF(NCASE .GE. 2) CNEW(I,J) = 2*C(I,J) - C1(I,J)
IF(NCASE .GE. 3) CNEW(I,J) = 3*(C(I,J) - C1(I,J)) + C2(I,J)
IF(NCASE .GE. 4) CNEW(I,J) = 4*(C(I,J) + C2(I,J) -6*C1(I,J)
C 1 - C3(I,J)
C C3(I,J) = C2(I,J)
C C2(I,J) = C1(I,J)
C1(I,J) = C(I,J)
C(I,J) = CNEW(I,J)
860 CONTINUE
NCASE = NCASE + 1
IF(NCASE .GE. 3) FAC = 1.0D0
GO TO 45
1100 STOP
END

SUBROUTINE SIMEQ(A, M, MP, N, STORE, IERR)

C

SUBROUTINE SIMEQ SOLVES ANY NUMBER OF SIMULTANEOUS EQUATIONS.
COEFFICIENTS ARE STORED IN THE CALLING MATRIX A (WHICH IS
DESTROYED DURING THE SOLUTION PROCESS). VALUES OF THE N UNKNOWNS
TO BE SOLVED FOR ARE STORED IN THE VECTOR STORE, IN THEIR ORDER OF
APPEARANCE IN THE EQUATIONS. IF SOLUTION IS NOT POSSIBLE AN ERROR
MESSAGE WILL BE PRINTED AND IERR WILL BE RETURNED AS 1. IF SOLU-
TION IS SUCCESSFUL IERR WILL BE RETURNED AS 0.
M AND MP ARE THE MAXIMUM DIMENSIONS OF THE ARRAY A
N IS THE NUMBER OF EQUATIONS
IMPLICIT REAL *8(A-H, O-Z)
DIMENSION STORE(15), A(M, MP)
DO 10 I = 1,15
10 STORE(I) = 0.0
IERR=0
IX=N+1
DO 110 MX=1,N
DO 102 I=MX,N
DIV=A(I,MX)
IF(DIV)103,102,103
102 CONTINUE
PRINT 1
1 FORMAT(/9X,'SOLUTION OF SIMULTANEOUS EQUATIONS NOT POSSIBLE'/)
IERR=1
GO TO 114
103 DO 105 J=MX,IX
Z=A(MX,J)
A(MX,J)=A(I,J)/DIV
IF(I-MX)104,105,104
104 A(I,J)=Z
105 CONTINUE
IF(N-1)114,112,106
106 KY=MX+1
IF(KY.GT.N) GO TO 110
DO 109 L=KY,N
AMULT=A(L,MX)
IF(MX-N)107,109,109
107 DO 108 J=MX,IX
108 A(L,J)=A(L,J)-AMULT*A(MX,J)
109 CONTINUE
110 CONTINUE
KU=N-1
DO 111 J=1,KU
L=N-J
LX=L+1
DO 111 MX=LX,N
111 A(L,N+1)=A(L,N+1)-A(MX,N+1)*A(L,MX)
112 DO 113 1=1,N
113 STORE(I)=A(I,N+1)
114 CONTINUE
RETURN
END

SUBROUTINE DERIV
IMPLICIT REAL *8(A-H,O-Z)
DOUBLE PRECISION LAMM,LEN,WID
COMMON FUN(30),C(6,6),COLD(6,6),ST,
& ALPHA(6,6),DUMMY(15),BETA,GAMMA,LAMM,LEN,WID,
& U1,U2,U3,U4,U5,U6,U7,U8,U9,UB,UC,UD,UE,UF,UG,UH,UI,UF,
& V1,V2,V3,V4,V5,V6,V7,V8,V9,VA,VB,VC,VD,VE,VF,VG,WH,VI,VJ,
& W1,W2,W3,W4,W5,W6,W7,W8,W9,WA,WC,WD,WE,WF,WC,WH,WI,WJ,
& WK,WL,WM,WN,WP,WQ,WR,WS,WT,WU,WV,WW,UK,VK,WX,WF,
WM = 0.D00
WN = 0.D00
WO = 0.D00
WP = 0.D00
WQ = 0.D00
WR = 0.D00
WS = 0.D00
WT = 0.D00
WU = 0.D00
WV = 0.D00
WW = 0.D00
WX = 0.D00
WY = 0.D00

DO 100 I = 1,6
BB = (I-1)*BETA
B2 = BB**2
B4 = BB**4
B6 = BB**6
B8 = BB**8

DO 100 J = 1,6
IF(I .EQ. 1 .AND. J .EQ. 1) GO TO 100
C
IF(C(I,J) .EQ. 0.D00) GO TO 100
AL = ALPHA(I,J)
CIJ = C(I,J)
GG = (J-1)*GAMMA
G2 = GG**2
G4 = GG**4
G6 = GG**6
G8 = GG**8
U1 = U1 - B2*CIJ
U2 = U2 + AL*B2*CIJ
U3 = U3 + B4*CIJ
U4 = U4 - AL*B4*CIJ
U5 = U5 - B6*CIJ
U6 = U6 + AL*B6*CIJ
U7 = U7 - AL*AL*B2*CIJ
U8 = U8 + B2*G2*CIJ
U9 = U9 - AL*B2*G2*CIJ
UB = UB + AL*AL*B4*CIJ
UC = UC + AL**3*B2*CIJ
UD = UD + B8*CIJ
UE = UE - B2*G4*CIJ
UF = UF + B4*G4*CIJ
UJ = UJ + B6*G2*CIJ
UK = UK + B2*G6*CIJ
V1 = V1 - G2*CIJ
V2 = V2 + AL*G2*CIJ
V3 = V3 + G4*CIJ
V4 = V4 - AL*G4*CIJ
V5 = V5 - G6*CIJ
V6 = V6 + AL*G6*CIJ
V7 = V7 - AL*AL*G2*CIJ
VD = VD + G8*CIJ
W1 = W1 + AL**2*CIJ
W8 = W8 - AL**3*CIJ
W9 = W9 + AL**4*CIJ
WD = WD + AL**2*G4*CIJ
WF = WF + AL**3*G2*CIJ
WG = WG + AL**2*G2*B2*CIJ
WI = WI + AL*B4*G2*CIJ
WJ = WJ + AL*B2*G4*CIJ
WK = WK - AL*B8*CIJ
WL = WL - AL*C8*CIJ
WM = WM - AL**2*B6*CIJ
WN = WN - AL**2*G6*CIJ
WO = WO - AL**5*CIJ
WP = WP - AL**4*B2*CIJ
WQ = WQ - AL**4*G2*CIJ
WR = WR - AL**3*B4*CIJ
WS = WS - AL**3*G4*CIJ
WT = WT - AL*B4*G4*CIJ
WU = WU - AL**2*B2*G4*CIJ
WV = WV - AL**3*B2*G2*CIJ
WW = WW - AL**2*B4*G2*CIJ
WX = WX - AL*B6*G2*CIJ
WY = WY - AL*G6*B2*CIJ
100 CONTINUE
UG = WJ
UH = WG
UI = WI
W2 = U2
W3 = U7
W4 = U4
W5 = UB
W6 = U6
W7 = UC
WA = V2
WB = V7
WC = V4
WE = V6
WH = U9
V8 = U8
V9 = U9
VA = UA
VB = WD
VC = WF
VE = UE
VF = UF
VG = WJ
VH = WG
VI = WI
VJ = UJ
VK = UK
ND = 2
Z20 = W2/(2*U1 - W1)
Z02 = WA/(2*V1 - W1)
IF(ND .EQ. 2) RETURN
Z22 = -(WH + 3*W8*Z20*Z02 + Z20*(WB - 2*U8))
& + Z02*(W3 - 2*V8))/(3*W1)
Z40 = (W4 + (6*W3 - 4*U3)*Z20 + (3*W8 - 12*U2)*Z20*Z20)/
& (4*U1 - W1)
Z04 = (WC + (6*WB - 4*V3)*Z02 + (3*W8 - 12*V2)*Z02*Z02)/
& (4*V1 - W1)
IF(ND .EQ. 4) RETURN

Z60 = (W6 + Z20*Z40*(15*W8 - 6*U5) + Z40*(15*W8 - 20*U3))
& + Z20*Z40*(15*W8 - 90*U2) + Z20*Z20*(45*W7 - 60*U4)
& + Z02*Z04*(15*W8 - 90*U7))/(6*U1 - W1)
Z06 = (WE + Z02*Z04*(15*W8 - 6*V5) + Z04*(15*WB - 20*V3)
& + Z02*Z04*(15*W8 - 90*V2) + Z02*Z02*(45*WF - 60*V4)
& + Z02*Z03*(15*W9 - 90*V7))/(6*V1 - W1)
Z42 = (WI + Z02*(W5 - 2*VA) + Z20*(6*WG - 4*UA)
& + Z20*Z02*(6*W7 - 4*U4 - 12*V9) + Z20*Z40*Z20*(6*W7 - 4*V2 - 2*U2) + Z20*Z02*Z20*(3*W9 - 12*V7 - 6*U7)
& + Z20*Z02*(3*W9 - 12*V7 - 6*U7)))/6*V1 - W1)
Z24 = (WJ + Z20*(WD - 2*UE) + Z02*(6*WG - 4*VE)
& + Z02*Z20*(6*WF - 4*V4 - 12*U9) + Z20*Z40*(6*WF - 4*V4 - 24*U8) + Z04*Z04*(W8 - 4*V2 - 2*U2)
& + Z02*Z02*(6*WF - 4*V4 - 12*V2) + Z02*Z02*Z04*(3*W9 - 12*V7 - 6*U7)
& + Z02*Z02*(3*W7 - 12*V9))/(4*V1 + 2*U1 - W1)
IF(ND .EQ. 6) RETURN

Z80 = (WK + Z60*(28*W3 - 56*U3) + Z20*Z60*(28*W8 - 224
& *U4) + Z40*(35*W8 - 280*U2) + Z20*Z20*Z40*(210*
& W9 - 1680*U7) + Z20*Z20*(210*WR - 168*U6) + Z20*Z20
& *Z20*(420*WF - 840*U8) + Z20**4*(105*WO - 840*UC)
& + Z20*(28*WM - 8*U0))/(8*U1 - W1)
Z08 = (WL + Z06*(28*WB - 56*V3) + Z02*Z06*(28*W8 - 224
& *V4) + Z04*Z04*(15*W8 - 56*U5) + Z02*Z04*(420*WF - 840*
& V9) + Z04*Z04*(35*W8 - 280*V2) + Z02*Z02*Z04*(210*
& V9 - 1680*V7) + Z02*Z02*(210*WS - 168*V6) + Z02*Z02
& *Z02*(420*WQ - 840*V8) + Z02**4*(105*WO - 840*VC)
& + Z02*(28*WQ - 8*VD))/(8*V1 - W1)
Z44 = WT + Z42*(6*WB - 4*V3 - 24*U8) + Z42*Z02
& *(6*W8 - 24*V2 - 24*U2) + Z40*(WD - 4*UE) + Z40*Z04*
& (W8 - 4*V2 - 4*U2) + Z02*Z40*(420*WF - 840*V4 - 24*U9)
& + Z02*Z02*Z40*(3*W9 - 12*V7 - 12*U7) + Z22*Z22*
& (18*W8 - 72*V2 - 72*U2) + Z20*Z24*(6*W8 - 24*V2 - 24
& *U2) + Z20*Z22*(36*WF - 24*V4 - 144*U9) + Z20*Z20
& *Z22*(36*W9 - 144*V7 - 144*U7) + Z20*Z20*(3*WS - 12*UG)
& + Z20*Z20*Z02*(18*WQ - 12*V8 - 72*UH) + Z20*Z20*Z04*
& (3*WS - 12*V7 - 12*U7)
Z44 = Z44 + Z20*Z20*Z02*Z02*(9*WO - 36*VC - 36*UC) +
& Z24*(6*W3 - 24*V8 - 4*U3) +
& Z22*(36*WG - 24*VE - 24*UA) + Z22*Z02*(36*W7 -
& 144*V9 - 24*U4) + Z20*(6*WU - 4*UF) + Z20*Z02*(36*WV
& - 24*VC - 24*UI) + Z20*Z04*(6*W7 - 24*V9 - 4*U4) +
& Z20*Z02*Z02*(18*WP - 72*VH - 12*UB) + Z02*(6*WW - 4*
& VP) + Z02*Z02*(3*WR - 12*VI) + Z04*(W5 - 4*VA)
Z44 = Z44/(4*U1 + 4*V1 - W1)
Z62 = Z42*(30*V8 + 20*U3 - 15*W3) + Z42*Z20*(30*V2+
& 90*U2 - 15*W8) + Z22*(30*VA + 6*U5 - 15*W5) + Z22*
& Z22*(180*V9 + 120*U4 - 90*W7) + Z02*(2*VJ - WM) +
& (30*VI + 6*U6 - 15*WR) + Z02*Z40*(30*V9 + 20*U4 - 15*
& W7) + Z20*Z20*Z02*(90*VH + 90*U7 - 15*W9) + Z02*Z22
& *(30*V2 + 90*U2 - 15*W8) + Z60*(6*W7 - 24*V9 - 4*U4) +
& Z22*(90*V9 + 20*U4 - 15*W7) + Z02*Z20*Z20*(90*
& U7 + 270*V7 - 45*W9) + Z02*(2*VJ - WM) + Z02*Z20*
& (30*UC + 90*VC - 15*WO) + Z06*(6*W8 - W3) + Z04*(20*
& UA - 15*WG) + Z20*Z40*(90*U9 - 15*WF)
Z62 = Z62 + Z20*Z20*(60*UI - 45*WV) + Z20*(6*UJ -
& 15*WW) + Z20**3*(90*UH - 15*WQ) - WX
Z62 = Z62/(W1 - 6*U1 - 2*V1)
Z26 = Z24*(30*U8 + 20*V3 - 15*WB) + Z24*Z02*(30*U2 +
& 90*V2 - 15*W8) + Z22*(30*UE + 6*V5 - 15*WD) + Z02*
& Z22*(180*U9 + 120*V4 - 90*WF) + Z22*Z02*Z02*(90*
& U7 + 270*V7 - 45*W9) + Z20*(2*UK - WN) + Z02*Z20*
& (30*UC + 6*V6 - 15*WS) + Z20*Z04*(30*U9 + 20*V4 - 15*
& WF) + Z02*Z02*Z20*(90*UH + 60*WB - 45*U8) + Z04*Z22
& *(30*U2 + 90*V2 - 15*W8) + Z60*(6*V9 - 45*W9) + Z02*Z20
& *Z20*(30*U7 + 90*V7 - 15*W9) + Z20*Z02**3*
& (30*UC + 90*VC - 15*WO) + Z06*(6*V8 - W3) + Z04*(20*
& VE - 15*WC) + Z02*Z04*(90*V9 - 15*WF)
Z26 = Z26 + Z02*Z02*(60*VC - 45*WV) + Z02*(6*VK -
& 15*WW) + Z26/(W1 - 6*V1 - 2*U1)
RETURN
END

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SUBROUTINE FUNC
IMPLICIT REAL *8(A-H,O-Z)
DOUBLE PRECISION LAMM,LAM,LEN,WID
COMMON FUN(30),C(6,6),COLD(6,6),ST,
& ALPHA(6,6),DUMMY(15),BETA,GAMMA,LAMM,LEN,WID,
& U1,U2,U3,U4,U5,U6,U7,U8,U9,UA,UB,UC,UD,UE,UF,UG,UI,UY,
& VI,V2,V3,V4,V5,V6,V7,V8,V9,VA,VB,VC,VD,VE,VF, VG,VI,VJ,
& WI,W2,W3,W4,W5,W6,W7,W8,W9,WA,WB,WC,WD,WE,WF, WG,WH, WI,WJ,
& WK,WL,WM,WN,WO,WP,WQ,WR,WS,WT,WW,UK,VK, WX,WY,
& Z20,Z02,Z22,Z40,Z04,Z60,Z06,Z24,Z42,Z80,Z08,Z44,Z26,Z62,
& Z100,Z82,Z64,Z46,Z28,Z10,
& NTERM,NA,ND
US = C(1,1)
G = 386.4D00/(US*US)
LAM = LAMM/(US*US)

CCCCCCCCCCCCCCCCCCCCCCCCCCCCC AXIAL VELOCITY = 0 CCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
SUM = 0.0D00
DO 10 I = 1, 6
DO 10 J = 1, 6
IF(I .EQ. 1 .AND. J .EQ. 1) GO TO 10
SUM = SUM - ALPHA(I,J)*C(I,J)
10 CONTINUE
FUN(1) = SUM - 1.000
FUN(2) = U1*U1 + G*Z20 + LAM*(Z40 + Z22 - 3*Z20**3 - Z20**2*Z20)
FUN(3) = V1*V1 + G*Z20 + LAM*(Z20**3 - 3*Z20**2*Z20)
IF(INTERM .EQ. 3) RETURN
FUN(4) = 2*(U1*U8 + V1*V8) + W2*WA + (2*U1*U2)
1 + W1*W2)*Z20 + (2*V1*V2 + WA*W1)*Z20 + G*Z22
2 + LAM*(Z22 - Z02**2*Z20 - 3*Z02**3 - Z02**2*Z20)
3 + 9*Z20*Z20**3 + 9*Z20**2*Z20 - 9*Z20*Z20**3
4 - 9*Z02*Z02*Z22 - 12*Z22*Z20*Z20)
FUN(5) = 4*U1*U3 + 3*W2*W2 + (12*U1*U2)
1 + W2*W3)*Z20 + G*Z40 + LAM*(Z60 + Z42 - 4*Z20*Z02*Z40
2 - 6*Z20*Z20*Z22 - 30*Z20*Z20*Z40 + 9*Z02*Z20**2
3 + 45*Z20**2)
FUN(6) = 4*V1*V3 + 3*WA*WA + (12*V1*V2)
1 + WA*W4 + WA*W2* + W1*W1*Z02)*Z04 + 45*(2*V1*V7 + 2*V2*V2 + WA*W8
2 + 2*W1*W3)*Z20*Z20 + 45*W1*W8*Z20**3 + 15*(4*V1*U4
3 + 4*V3*V2 + 6*WA*W8 + W4*W1)*Z20 + G*Z60
4 + LAM*(-10*Z40**2*Z20 - 210**2*Z20 - 60*Z40*Z22 + 180*Z40*Z02*
5 Z20**2 + 1575*Z40*Z20**4 + 4*Z22*Z20**4 + 6*Z02*
6 Z20**2 - 1575*Z20**2 + 63*Z20**2 + 15*Z02**2 + Z42*Z80 + Z60)
FUN(7) = (6*U1*U5 + 10*U3*U3 + 15*W2*W4) + 15*(2*U1*U2)
1 + W2*W1 + W1*W1*Z20)*Z40 + 45*(2*U1*U7 + 2*U2*U2 + W2*W8
2 + 2*W1*W3)*Z20*Z20 + 45*W1*W8*Z20**3 + 15*(4*U1*U4
3 + 4*U3*U2 + 6*WA*W3 + W4*W1)*Z20 + G*Z60
4 + LAM*(-10*Z40**2*Z20 - 210**2*Z20 - 60*Z40*Z22 + 180*Z40*Z02*
5 Z20**2 + 1575*Z40*Z20**4 + 4*Z22*Z20**4 + 6*Z02*
6 Z20**2 - 1575*Z20**2 + 63*Z20**2 + 15*Z02**2 + Z42*Z80 + Z60)
FUN(8) = (6*V1*V5 + 10*V3*V3 + 15*WA*WC) + 15*(2*V1*V2)
1 + WA*W1 + W1*W1*Z20)*Z04 + 45*(2*V1*V7 + 2*V2*V2 + WA*W8
2 + 2*W1*W3)*Z20*Z20 + 45*W1*W8*Z20**3 + 15*(4*V1*V4
3 + 4*V3*V2 + 6*WA*W8 + W4*W1)*Z20 + G*Z60
4 + LAM*(-10*Z40**2*Z20 - 210**2*Z20 - 60*Z40*Z22 + 180*Z40*Z02*
5 Z20**2 + 1575*Z40*Z20**4 + 4*Z22*Z20**4 + 6*Z02*
6 Z20**2 - 1575*Z20**2 + 63*Z20**2 + 15*Z02**2 + Z42*Z80 + Z60)
C  IF(INTERM .EQ. 10) LAM = 0.000
FUN(9) = 6*(2*U1*U9 + 2*U8*U2 + 2*V1*V9 + 2*V8*V2 + WA*W3
1 + W2*W2 + WH*W1)*Z20 + (4*U1*U4 + 4*U3*U2 + 6*W2*W3
2 + W4*W1)*Z20 + (4*V1*U4 + 4*V3*V2 + 6*W1*W3)
3 + Z22
4 + 6*W1*W1**2*Z20 + 3*(2*V1*V7 + 2*V2*V2 + 2*W1*W8
5 + WA*W8)*Z20 + Z20
6 + (2*V1*V2 + WA*W1)*Z40 + W1*W1*Z02*Z40 + (4*U1*UA
7 + 4*U3*U8  
8 + 2*V1*VA + 6*V8*V8 + W4*WA + 6*W2*WH) + 6*(2*U1*U7  
9 + 2*U2*U2 + W2*W8 + 2*W1*W3)*Z20*Z02 + G*Z42  
FUN(9) = FUN(9) + LAM*  
1 (9*Z04*Z20**4-4*Z04*Z20*Z40-66*Z22**2*Z20-54*Z22**2*Z02+  
2 225*Z22*Z20**4+288*Z22*Z20**3*Z02+162*Z22*(Z20**2)*Z02**2-60*  
3 Z22*Z04-32*Z20**2-Z20*Z40-225*(Z20**5)*Z02**2-135*Z22*Z02**3  
4 +90*(Z20**2)*Z02**2*Z40-30*Z20**2*Z24-6*Z20**2*Z24+36*Z20*Z02**3  
5 *Z40-22*Z20*Z02*Z42-Z02**2*Z60-9*Z02**2*Z42+Z44+Z62)  

FUN(10) = 6*(2*V1*V9 + 2*V8*V2 + 2*U1*U9 + 2*U8*U2 + W2*WB  
1 + WA*W3 + WH*W1)*Z02 + (4*V1*V4 + 4*V3*V2 + WC*W1)*Z20 + 9  
2 *W1*W8 *Z0 2 *Z0 2 *Z2 0 + 6*(2*V1*V2 + WA*W1  
3 )*Z22  
4 + 6*W1*W1*Z02*Z22 + 3*(2*U1*U7 + 2*U2*U2 + W1*W3  
5 + W2*W8)*Z02*Z02  
6 + (2*U1*U2 + W2*W1)*Z04 + W1*W1*Z20*Z04 + (4*V1*VE  
7 + 4*V3*V8  
8 + 2*U1*UE + 6*U8*U8 + WC*W2 + 6*WA*WH) + 6*(2*V1*V7  
9 + 2*V2*V2 + WA*W8 + 2*W1*WB)*Z02*Z20 + G*Z42  

FUN(11) = FUN(11) + LAM*  
1 (9*Z40*Z02**4-4*Z40*Z02*Z40-66*Z22**2*Z20-54*Z22**2*Z02+  
2 225*Z22*Z20**4+288*Z22*Z20**3*Z02+162*Z22*(Z20**2)*Z02**2-60*  
3 Z22*Z04-32*Z20**2-Z20*Z40-225*(Z20**5)*Z02**2-135*Z22*Z02**3  
4 +90*(Z20**2)*Z02**2*Z40-30*Z20**2*Z24-6*Z20**2*Z24+36*Z02*Z20**3  
5 *Z40-22*Z20*Z02*Z42-Z02**2*Z60-9*Z02**2*Z42+Z44+Z62)  

IF(NTERM .EQ. 10) RETURN  

FUN(12) = (8*U1*UD + 56*U3*U5 + 28*W2*W6 + 35*W4*W4  
1 + (56*U1*U2 + 28*W1*W2)*Z60 + 28*W1*W1*Z02*Z60  
2 + (280*U1*U4 + 280*U3*U2 + 420*W2*W3 + 70*W4*W1)*Z40  
3 + (168*U1*U6 + 560*U3*U4 + 168*U2*U5 + 420*W2*W5  
4 + 420*W4*W3 + 28*W6*W1)*Z20 + (840*U1*U7 + 840*U2*U2  
6 + 35*W1*W1*Z40 + (840*U1*UC + 2520*U2*U7  
7 + 1260*W1*W7 + 1260*W8*W3 + 420*W9*W2)*Z02**3  
8 + (840*U1*UB + 1680*U2*U4 + 840*U3*U7 + 1260*W2*W7  
9 + 1260*W3*W3 + 210*W4*W8 + 420*W5*W1)*Z20*Z20  
10 + (420*W1*W9 + 315*W3*W8)*Z20*Z40 + G*Z80  
F80 = -840*Z40**2-280*Z40**2*Z22+2520*Z40**2*Z02*Z20**2+37800*  
1 Z40**2*Z02**3+5040*Z40*Z22*Z20**3-12600*Z40*Z02*Z20**5-Z40**4  
2 +Z02*Z40-13200*Z20**6-1512*Z40*Z20*Z60-280*Z40*Z20*Z42-  
3 6300*Z22*Z20**6+168*Z22*Z20*Z60+11025*Z02*Z20**8+8504*Z02*Z20**3  
4 *Z60-8*Z02*Z20**2+99225*Z02**9+5670*Z20**4*Z60+630*Z20**2  
5 4*Z42-108*Z20**2-28*Z20**2*Z62+82*Z100  

FUN(11) = FUN(11) + LAM*F80  

IF(NTERM .EQ. 15)LAM = 0.0000
\[ \begin{align*}
6 + 35W1W1Z04Z04 + (840V1V1VC + 2520V2V7) & \\
7 + 1260W1WF + 1260W8WB + 420W9WA & \times Z02^2*3 \\
8 + (840V1V1WB + 1680V2V4 + 840V3V7 + 1260W8WF & \\
9 + 1260WBWB + 210WCW8 + 420WDWD1W1 & \times Z02Z02 \\
1 + (420W1W1W9 + 315W8WBWB1W2Z02^2*4 & \times GZ02 \\
F08 = -840Z04Z04^2*3-280Z04Z04^2*2Z22Z22+2520Z04Z02Z02^2*2+37800 & \\
1 Z04Z04Z02Z02^2*3+5040Z04Z22Z02Z02^2*3-12600Z04Z20Z02^2*5-56Z04^2 & \\
2 \times Z20Z06-132300Z04Z02Z02^2*6-1512Z04Z02Z06-280Z04Z02Z22Z4 & \\
3 6300Z22Z02Z02^2*6-168Z22Z02Z06+11025Z20Z02Z08^2+504Z20Z20Z02^2*3 & \\
4 \times Z068Z20Z02Z08+9925Z20Z02Z09^2+5670Z20Z02^4Z06+630Z02 & \\
5 4ZZV4-108Z02Z02Z08-28Z20Z22Z20Z22 + Z28Z010 & \\
FUN(12) = FUN(12) + LAM^*F08 & \\
\end{align*} \]
4 *Z24-30*Z20**2*Z44-6*Z20**2*Z26-60*Z20*Z42*Z04+Z46+Z64
FUN(13) = FUN(13) + LAM*=F44

C011 = 4*V1*V2 + 2*WA*W1 + 2*W1*W1*Z02
G031 = 2*(4*V1*V4 + 4*V3*V2 + 6*WA*WB + W1*WC) + 2*W1*W1*
1 Z04 + 6*(2*V1*V7 + 2*V2*V2 +WA*WB + 2*W1*WB)*Z02
G211 = 2*(2*V1*V9 + 2*V8*V2 + 2*U1*U9 + 2*U2*U8 + WA*W8 +
1 W1*WH + W3*WA) + W1*W1*Z22 + (2*U1*U7 + Z04 + Z02*Z02 + 2*W1*WB)
2 W2*W8 + 2*W1*W3)*Z02)
G012 = 2*(2*V1*V7 + 2*V2*V2 + 2*W1*WB + W8*WA + 3*W1*W8*
1 Z02)
G051 = 2*(6*V1*V6 + 20*V3*V4 + 6*V2*V5 + 15*WA*WB + 15*WC*WB
2 + WE*W1) + 2*W1*W1*Z06 + 20*(2*V1*V7 + 2*V2*V2 + WA*WB
4 + 2*W1*WB)*Z04 + 10*(4*V1*VB + 4*V3*V7 + 8*V2*V4
5 + WC*WB* 6*WA*WF + 2*W1*WD + 6*WB*WB)*Z02
G032 = 2*(4*V1*VB + 8*V2*V4 + 4*V3*V7 + 6*WA*WF + 6*WB
1 *WB + WC*WB + 2*W1*WD) + 6*W1*W1*Z04 + 6*(2*V1*VC
2 + 6*V2*V7 + WA*W9 + 3*WB*WB + 3*W1*WF)*Z02
G231 = 2*(4*V8*V4 + 4*V1*VG + 4*VE*V2 + 2*U1*UG + 12*U8*U9
1 + 2*U2*UE + 6*WA*WG + W2*WD + 6*WH*WB + W1*WJ + W3*WC +
2 4*V3*V9) + 2*W1*W1*Z24 + 6*(2*V1*V7 + 2*V2*V2 + 2*W1*WB
3 + WA*WB)*Z22 + 2*(2*U1*U7 + 2*U2*U2 + 2*W1*W3
4 + W2*W8)*Z04 + 6*(2*V1*VH + 2*V8*V7 + 4*V2*V9 + 2*U1*U7 +
5 UH + 2*U8*U7 + 4*U2*U9 + WA*W7 + W2*WF + WH*WB + 2*W1*WG
F 26 = -2.10*Z04*2*Z22+90*Z04**2*Z20**3+630*Z04**2*Z20**2*Z02-10
   *Z04**2*Z40-390*Z04**2*Z22+1620*Z04**2*Z20**2*Z02+5400*Z04*
   +3*Z22**2*Z02**2+6300*Z04**2*Z22**3-2700*Z04**2*Z20**3*Z02**3-
   3*8785*Z04**2*Z20**2*Z02**4-130*Z04**2*Z20**8*Z04**2*Z20**3*Z40-60*
   +4*Z04**2*Z22**3-420*Z04**2*Z22**3+2430*Z02**2*Z20**2*Z02**2
   +3*3510*Z22**2*Z02**3-6075*Z22**2*Z02**2*Z02**4-13500*Z22**2*Z02**3
   +6*5-60*Z22**2*Z20**2*Z06-270*Z22**2*Z24-11025*Z22**2*Z02**6-126*Z22**2
   +7*Z02**2*Z06-480*Z22**2*Z20**2*Z24+4725*Z20**3*Z02**3+54*Z20**3*Z02**3
   +8*11025*Z22**2*Z02**7+189*Z20**2*Z02**2*Z20**2*Z02**2+405*Z20**2*Z02**2
   +9*Z24-Z20**2*Z08-9*Z20**2*Z26+1170*Z20**2*Z24-32*Z20**2*Z02
   +1*Z26-225*Z02**2*Z40+135*Z02**2*Z42+1575*Z02**4*Z24-15*Z02**2*
   +2*Z44-63*Z02**2*Z26-6*Z02*Z06*Z40+Z28+Z46
FUN(15) = FUN(15) + LAM*F26
RETURN
IF(XXX .GT. 99999.D00) GO TO 500
IF(NTERM .EQ. 3) GO TO 100
IF(NTERM .EQ. 6) GO TO 200
IF(NTERM .EQ. 10) GO TO 300
IF(NTERM .EQ. 15) GO TO 400
100 DO 110 J=1,NTERM
110 A3(J,II-1) = (FUN(J) - FUN(J+NTERM))/XXX
RETURN
200 DO 210 J=1,NTERM
210 A6(J,II-1) = (FUN(J) - FUN(J+NTERM))/XXX
RETURN
300 DO 310 J=1,NTERM
310 A10(J,II-1) = (FUN(J) - FUN(J+NTERM))/XXX
RETURN
400 DO 410 J=1,NTERM
410 A15(J,II-1) = (FUN(J) - FUN(J+NTERM))/XXX
RETURN
500 IF(NTERM .EQ. 3) GO TO 1100
IF(NTERM .EQ. 6) GO TO 1200
IF(NTERM .EQ. 10) GO TO 1300
IF(NTERM .EQ. 15) GO TO 1400
1100 DO 1110 I = 1,NTERM
1110 A3(I,NTERM1) = - FUN(NTERM + I)
CALL SIMEQ(A3,NTERM,NTERM1,NTERM,DUMMY,IERR)
RETURN
1200 DO 1210 I = 1,NTERM
1210 A6(I,NTERM1) = - FUN(NTERM + I)
CALL SIMEQ(A6,NTERM,NTERM1,NTERM,DUMMY,IERR)
RETURN
1300 DO 1310 I = 1,NTERM
1310 A10(I,NTERM1) = - FUN(NTERM + I)
CALL SIMEQ(A10,NTERM,NTERM1,NTERM,DUMMY,IERR)
RETURN
1400 DO 1410 I = 1,NTERM
1410 A15(I,NTERM1) = - FUN(NTERM + I)
CALL SIMEQ(A15,NTERM,NTERM1,NTERM,DUMMY,IERR)
RETURN
END
APPENDIX F.5

Bubble Shape in Vertical Tube

THIS PROGRAM SOLVES FOR THE BUBBLE SHAPE FOR A GAS SLUG IN A CIRCULAR PIPE.

IMPLICIT REAL *8(A-H,O-Z)
REAL ZSR,RSR
DIMENSION D(6),ZERO(6),C(6)
data zero,eps/3.83171d00,7.01559d00,10.17347d00,13.32369d00,16.47063d00,19.61586d00,.0001d00/
10 FORMAT(2F12.4)

READ IN INPUT
DIAM = TUBE INNER DIAMETER
FILM = LAMINAR FILM THICKNESS
FMUL = FILM MULTIPLIER
RC =
NR = NUMBER OF RADIAL
C = COEFFICIENTS OF BESSEL SERIES
MESH POINTS EXPANSION

READ(1,*),DIAM,FILM,FMUL,RC,NR
READ(1,*),C
RAD = DABS(DIAM)/2
DELR = RAD/NR
IF(DIAM .LT. 0.0) GO TO 500
REFF = RAD - FILM*FMUL
DO 42 I = 1,6
42 D(I) = ZERO(I)/REFF

SOLVE FOR BUBBLE SHAPE

RSR = 0.
ZSR = 0.
WRITE (2,10)RSR,ZSR
ZS = 0.
DO 100 J = 1,NR
RS = J*DELR
IF(RS .GE. (REFF - EPS)) GO TO 1000
50 ZSOLD = ZS
F = 0.
DF = 0.
DO 60 I = 1,6
IF(C(I) .EQ. 0.) GO TO 65
XX = C(I)*BJ1(D(I)*RS)*DEXP(-D(I)*ZSOLD)
Y = -D(I)*XX
F = F + XX
60 DF = DF + Y
65 F = 2*F - RS
DF = 2*DF
ZS = ZSOLD - F/DF
IF(DABS(1 - ZSOLD/ZS) .LT. EPS) GO TO 50
ZSR = ZS/RAD
RSR = RS/RAD
WRITE (2,10)RSR,ZSR
100 CONTINUE
C SPHERICAL CAP
500 DELR = RAD*RC/NR
   NR1 = NR + 1
   DO 600 J = 1, NR1
      RS = (J-1)*DELR/RAD
      IF(RS .GT. RC) GO TO 1000
      ZS = RC - DSQRT(RC*RC - RS*RS)
      ZSR = - ZS
      RSR = RS
      WRITE (2,10)RSR,ZSR
   600 CONTINUE
1000 STOP
END

FUNCTION BJ1(X)
C THIS FUNCTION EVALUATES THE ORDINARY BESSEL FUNCTION OF THE 1ST KIND
C OF ORDER ONE. THE ALGORITHM HEREIN IS TAKEN FROM "HANDBOOK OF
C MATHEMATICAL FUNCTIONS WITH FORMULAS, GRAPHS AND MATHEMATICAL TABLES",
C NATIONAL BUREAU OF STANDARDS APPLIED MATHEMATICS SERIES 55, JUNE 1964,
C PAGES 369-370.
IMPLICIT REAL *8(A-H,O-Z)
IF(X .GT. 3.) GO TO 100
IF(X .GE. -3.) GO TO 200
BJ1 = -100.
RETURN
200 Y = X/3.D00
BJ1 = .5 - .56249985D00*Y*Y + .21093573D00*Y**4
   1 - .03954289D00*Y**6 + .01659667D00*Y**8
   2 - .0031761D00*Y**10 + .0001109D00*Y**12
BJ1 = BJ1*X
RETURN
100 Y = 3.D00/X
F1 = .79788456D00 + .00000156D00*Y*Y + .01659667D00*Y*Y
   1 + .00017105D00*Y**3 - .00249511D00*Y**4
   2 + .00113653D00*Y**5 - .00020033D00*Y**6
THT1 = X - 2.35619449D00 + .12499612D00*Y
   1 + .000565D00*Y*Y - .00637879D00*Y**3
   1 + .00074348D00*Y**4 + .00079824D00*Y**5
   2 - .0029166D00*Y**6
BJ1 = F1*DCOS(THT1)/DSQRT(X)
RETURN
END
APPENDIX F.6

Bubble Shape in Vertical Rectangular Slot

This program solves for the bubble shape of a gas slug in a rectangular slot using Runge-Kutta order 4. Viscous effects are assumed negligible.

IMPLICIT REAL *8(A-H,O-Z)
DOUBLE PRECISION K1X,K2X,K3X,K4X,LEN
DOUBLE PRECISION K1Y,K2Y,K3Y,K4Y
COMMON C(6,6),ALPHA(6,6),BETA,GAMMA,NA1,NA

Boundary conditions

NTERM = 0
NA = 1
PI = 3.141592654D00
DO 10 I = 1,6
   DO 10 J = 1,6
      C(I,J) = 0.0
10   ALPHA(I,J) = 0.

Read in input

LEN = LENGTH   WID = WIDTH   N = NUMBER OF X (AND Y) MESH POINTS
C = ARRAY OF COEFFICIENTS IN THE DOUBLE SERIES EXPANSION

READ(1,*) LEN,WID,N
READ(1,*) NTERM0
BETA = 2*PI/LEN
GAMMA = 2*PI/WID
30 READ(1,*,END = 40) I, J, C(I+1,J+1)
   I = I + 1
   J = J + 1
   IF(I .GT. NA) NA = I
   IF(J .GT. NA) NA = J
   ALPHA(I,J) = DSQRT(((I-1)*BETA)**2 + ((J-1)*GAMMA)**2)
   NTERM = NTERM + 1
   IF(NTERM .LT. NTERM0) GO TO 30

NA1 = NA + 1
IAX = 0
50 BX = LEN/3
BY = WID/3
XNOR = LEN/2
YNOR = WID/2
HX = LEN/N
HY = WID/N
XIX = 0.
XIY = 0.
WIX = 0.
WIY = 0.
XIN = XIX/XNOR
WINX= WIX/XNOR
WINY= WIY/YNOR
100 WRITE(2,1) XIN,WINX,WINY
1 FORMAT(3F12.6)

C CALCULATE NEXT X,Z

K1X = HX*F(XIX, WIX, 0)
K2X = HX*F(XIX + HX/2, WIX + K1X/2, 0)
K3X = HX*F(XIX + HX/2, WIX + K2X/2, 0)
K4X = HX*F(XIX + HX, WIX + K3X, 0)

C CALCULATE NEXT VALUE OF X,Z

XIX = XIX + HX
WIX = WIX + (K1X + 2*K2X + 2*K3X + K4X)/6

C CALCULATE NEXT Y,Z

K1Y = HY*F(XIY, WIY, 1)
K2Y = HY*F(XIY + HY/2, WIY + K1Y/2, 1)
K3Y = HY*F(XIY + HY/2, WIY + K2Y/2, 1)
K4Y = HY*F(XIY + HY, WIY + K3Y, 1)

C CALCULATE NEXT VALUE OF Y,Z

XIY = XIY + HY
WIY = WIY + (K1Y + 2*K2Y + 2*K3Y + K4Y)/6

C QUIT IF END POINT REACHED

IF(XIX .LT. BX) GO TO 100

200 STOP
END

FUNCTION F(X,Z,IAX)
IMPLICIT REAL *8(A-H,O-Z)
COMMON C(6,6),ALPHA(6,6),BETA,GAMMA,NA1,NA
IF(X .EQ. 0.D00) GO TO 400
IF(IAX .EQ. 1 )  CO TO 200
C CALCULATE AXIAL VELOCITIES FOR X-DIRECTION

VZ = 0.
VX = 0.
DO 100 I=1,NA
NAI = NA1 - I
BI = (I-1)*BETA
DO 100 J = 1,NAI
IF(I .EQ. 1 .AND. J .EQ. 1) GO TO 100
IF(C(I,J) .EQ. 0.0D00) GO TO 100
CIJ = C(I,J)
AIJ = ALPHA(I,J)
EX = DEXP(-AIJ*Z)
VZ = VZ - AIJ*CIJ*DCOS(BI*X)*EX
VX = VX - BI*CIJ*DSIN(BI*X)*EX
100 CONTINUE
VZ = VZ - 1
C CALCULATE DERIVATIVE
F = VZ/VX
RETURN
C CALCULATE AXIAL VELOCITIES FOR Y-DIRECTION
200 VZ = 0.
  VY = 0.
  Y = X
  DO 300 I=1,NA
    NAI = NA1 - I
    DO 300 J = 1,NAI
      IF(I .EQ. 1 .AND. J .EQ. 1) GO TO 300
      IF(C(I,J) .EQ. 0.0D00) GO TO 300
      CIJ = C(I,J)
      AIJ = ALPHA(I,J)
      GJ = (J-1)*GAMMA
      EX = DEXP(-AIJ*Z)
      VZ = VZ - AIJ*CIJ*DCOS(GJ*Y)*EX
      VY = VY - GJ*CIJ*DSIN(GJ*Y)*EX
300 CONTINUE
VZ = VZ - 1
C CALCULATE DERIVATIVE
F = VZ/VY
RETURN
400 F = 0.
RETURN
END
Henry Valma Nickens is the son of Mr. and Mrs. Milburn T. Nickens, Sr. and was born October 3, 1947 in Prarieville, Louisiana. I was graduated from Dutchtown High School in 1965 and received the Bachelor of Science in Physics from Louisiana State University in 1969.

From 1969 to 1973, I served in the United States Air Force as an instructor in the Air Traffic Control Radar Maintenance School in Biloxi, Mississippi. During this time, I received the Master of Science in Mathematics from the University of Southern Mississippi.

From 1973 to 1978, I was employed with Westinghouse Electric Corporation in Pittsburgh, Pennsylvania as a Nuclear Engineer at the Bettis Atomic Power Laboratory. During this time, I received the Master of Nuclear Engineering Degree from the Carnegie Institute of Technology.

From 1978 to 1981, I was enrolled as a Doctoral Candidate in the Department of Mechanical Engineering at Louisiana State University. In 1981 I accepted employment with Amoco Production Research in Tulsa, Oklahoma, while completing the doctoral dissertation.

In 1972, I met and married Judith Ann Goddard. We have two daughters, Shannon (10) and Lindsay (7) and reside in Tulsa, Oklahoma.
EXAMINATION AND THESIS REPORT

Candidate: Henry Valma Nickens

Major Field: Mechanical Engineering

Title of Thesis: The Velocity and Shape of Gas Slugs Rising in Vertical Tubes and Rectangular Slots

Approved:

[Signatures]

Major Professor and Chairman
Dean of the Graduate School

EXAMINING COMMITTEE:

[Signatures]

Date of Examination:

November 21, 1984