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Quantum measurement and decoherence

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Abstract

Distribution functions defined in accord with the quantum theory of measurement are combined with results obtained from the quantum Langevin equation to discuss decoherence in quantum Brownian motion. Closed form expressions for wave packet spreading and the attenuation of coherence of a pair of wave packets are obtained. The results are exact within the context of linear passive dissipation. It is shown that, contrary to widely accepted current belief, decoherence can occur at high temperature in the absence of dissipation. Expressions for the decoherence time with and without dissipation are obtained that differ from those appearing in earlier discussions.

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The problem of decoherence in quantum systems has been of considerable recent interest. A sampling of earlier theoretical work appears in Refs. [1–5], but for a detailed survey we refer to a recent book devoted wholly to the subject [6]. In an introduction to the contents of this book, Joos surveys the current situation and, in discussing the mechanism of decoherence, states that "irreversible coupling to the environment seems to have become widely accepted (and even quite popular) during the last decade, not least through the various contributions by Woljciech Zurek and his collaboratorsIn this present work we come to quite the opposite conclusion and in fact show that, in the high temperature case considered by Zurek [4] and others, decoherence occurs in a characteristic time τ_d [see Eq. (21) below] that is independent of the Ohmic decay rate γ [see Eq. (19) below] which characterizes the strength of the coupling to the environment. Furthermore, the formulation we use is exact and enables us to show explicitly that previous estimates of the decoherence time arise from an inconsistent application of long-time asymptotic formulas to obtain a short-time result. Apart from its relevance to the question of classical-quantum correspondence and the foundations of quantum mechanics, this work, especially that part dealing with entangled states, is clearly relevant to the host of experiments on decoherence [7], quantum teleportation [8] and quantum information and computation [9,10].

Much of the discussion of decoherence has been in terms of the simple problem of a particle moving in one dimension that is placed in an initial superposition state (Schrödinger "cat" state) corresponding to two widely separated wave packets. Decoherence is said to occur when the long-time interference pattern is destroyed. The key questions asked are, first, under what conditions does decoherence occur and, second, what is the decoherence time. Previous discussions of the problem have either used the Feynman-Vernon influence functional technique [1] or master equation techniques [2–4] and have been confined to the case of Ohmic dissipation. In either case it is assumed that the initial state is decoupled from the environment. Here we use a different formulation in terms of quantum distribution functions introduced some time ago by Ford and Lewis [11]. With this formulation we are able to obtain exact closed form expressions for the spreading of a wave packet and

for the attenuation of interference in the two wave packet problem, which are valid for all temperatures and for a very general dissipative environment. In particular, we avoid the assumption that the particle is initially decoupled from the environment; the particle is in equilibrium with the environment at the time it is put into the initial state by a measurement. An important conclusion is that decoherence can occur for $\gamma \to 0$ (absence of dissipation). A further feature of the Ford-Lewis formulation is that the density matrix for the entire system is employed, i.e., one does not trace over the environment to obtain a reduced density matrix as previous investigations have done.

In the formulation of Ford and Lewis, we regard the particle as part of a larger system of particle coupled to a reservoir. Initially (or in the distant past) the complete system is in equilibrium at temperature T, described by the normalized density matrix

$$\rho_0 = \frac{e^{-H/kT}}{\mathrm{Tr}\{e^{-H/kT}\}}, \tag{1}$$

where H is the *system* Hamiltonian. To a measurement of x one associates a measuring function $\alpha(x_1)$ (x_1 is a c-number) such that $\|\alpha(x - x_1)\Phi\|^2 \, dx_1$ is the probability that if the system is in a normalized quantum state Φ the instrument will read between x_1 and $x_1 + dx_1$. An example is the "Gaussian slit" [12] for which

$$\alpha(x_1) = (2\pi\sigma_1^2)^{-1/4} \exp\{-\frac{x_1^2}{4\sigma_1^2}\}, \tag{2}$$

where σ_1 is the experimental width. It follows that if x is measured at time t_1, the probability that the instrument will read between x_1 and dx_1 is $W(x_1, t_1)dx_1$, where

$$W(x_1, t_1) = \mathrm{Tr}\{\alpha[x(t_1) - x_1]\rho_0\alpha[x(t_1) - x_1]^\dagger\}. \tag{3}$$

Here $x(t)$ is the Heisenberg operator at time t,

$$x(t) = e^{iHt/\hbar} x e^{-iHt/\hbar}. \tag{4}$$

In the same way, if x is measured at time t_1 and again at a later time t_2 the probability that the first instrument will read in range dx_1 and the second in dx_2 is $W(x_1, t_1; x_2, t_2)dx_1 dx_2$. Using an obvious shorthand notation,

$$W(1,2) = \mathrm{Tr}\{\alpha(2)\alpha(1)\rho_0\alpha(1)^\dagger\alpha(2)^\dagger\}. \tag{5}$$

Here in $\alpha(j) = \alpha[x(t_j) - x_j]$ the index j is meant to indicate not only the time t_j and instrumental position x_j, but also the instrumental parameters such as a width σ_j. In this way one can go on to define higher order distribution functions.

The distribution functions can be expressed in terms of the corresponding characteristic functions,

$$\begin{aligned} W(1) &= \int_{-\infty}^{\infty} \frac{dk_1}{2\pi}\xi(1)e^{-ik_1x_1}, \\ W(1,2) &= \int_{-\infty}^{\infty} \frac{dk_1}{2\pi}\int_{-\infty}^{\infty} \frac{dk_2}{2\pi}\xi(1,2)e^{-i(k_1x_1+k_2x_2)}. \end{aligned} \tag{6}$$

Now the key formulas needed from Ford and Lewis [11] is that for quantum Brownian motion these characteristic functions are given by the general formulas

$$\begin{aligned} \xi(1) &= \exp\{-\frac{1}{2}k_1^2\left\langle x(t_1)^2\right\rangle\} \\ &\quad\times\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}\tilde{\alpha}(q_1-\frac{k_1}{2})^*\tilde{\alpha}(q_1+\frac{k_1}{2}), \\ \xi(1,2) &= \exp\{-\frac{1}{4}\sum_{j=1}^{2}\sum_{l=1}^{2}\left\langle x(t_j)x(t_l)+x(t_l)x(t_j)\right\rangle k_jk_l\} \\ &\quad\times\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}\int_{-\infty}^{\infty}\frac{dq_2}{2\pi}\prod_{j=1}^{2}\tilde{\alpha}(q_j-\frac{k_j}{2})^*\tilde{\alpha}(q_j+\frac{k_j}{2}) \\ &\quad\times\exp\{q_1k_2[x(t_1),x(t_2)]\}. \end{aligned} \tag{7}$$

Here $\tilde{\alpha}$ is the Fourier transform of the function α describing the measurement,

$$\tilde{\alpha}(1) = \int_{-\infty}^{\infty} dx_1\alpha(x_1)e^{-iq_1x_1}. \tag{8}$$

We should remark that in the derivation of these formulas it was necessary to assume that the commutator $[x(t_1), x(t_2)]$ is a c-number. This is the case for quantum Brownian motion [13].

We first apply these formulas to obtain an expression for the spreading of a wave packet. That is, at time t_1 a measurement is made with an associated function of the form (2) and then at a later time t_2 a second measurement of the same form is made (with index $1 \to 2$). The integrals are all standard Gaussian integrals, and we obtain the results

$$W(1) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{x_1^2}{2\sigma^2}\},$$
$$W(1,2) = \frac{\exp\{-\frac{1}{2(1-\rho^2)}(\frac{x_1^2}{\sigma^2} - 2\frac{\rho x_1 x_2}{\sigma\tau} + \frac{x_2^2}{\tau^2})\}}{2\pi\sigma\tau(1-\rho^2)^{1/2}}, \tag{9}$$

where (note the misprint in the Eq. (7.18) of [11])

$$\begin{aligned} \sigma^2 &= \sigma_1^2 + \left\langle x^2 \right\rangle, \\ \tau^2 &= \sigma_2^2 - \frac{[x(t_1), x(t_2)]^2}{4\sigma_1^2} + \left\langle x^2 \right\rangle, \\ 2\sigma\rho\tau &= -s(t_2 - t_1) + 2\left\langle x^2 \right\rangle. \end{aligned} \tag{10}$$

In this last, $s(t_2 - t_1)$ is the mean square displacement

$$s(t) = \left\langle \{x(t_1) - x(t_1 + t)\}^2 \right\rangle. \tag{11}$$

The distribution $W(1,2)$ is a Gaussian quadratic form, with mean square width given by

$$\begin{aligned} w^2(t) &\equiv \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 (x_1 - x_2)^2 W(1,2) \\ &= \sigma_1^2 - \frac{[x(t_1), x(t_1 + t)]^2}{4\sigma_1^2} + s(t) + \sigma_2^2. \end{aligned} \tag{12}$$

This is an exact general formula for the spreading of a Gaussian wave packet, expressed in terms of the mean square displacement and the nonequal-time commutator. For the special case when the second measurement is made with infinite precision ($\sigma_2 = 0$) and for a free particle without dissipation and at zero temperature ($s(t) = 0$, $[x(t_1), x(t_1 + t)] = i\hbar t/m$) this reduces to the familiar formula of elementary quantum mechanics [15].

In the case of an unbound (free) particle the mean square displacement $\langle x^2 \rangle$ diverges. We can obtain a simple expression in this limit if we introduce the conditional probability

$$P(x_2 - x_1, t_2 - t_1) = \lim_{\langle x^2 \rangle \to \infty} \frac{W(1,2)}{W(1)}. \tag{13}$$

Using the expressions (9) we find

$$P(x,t) = \frac{\exp\{-\frac{x^2}{2w^2(t)}\}}{\sqrt{2\pi w^2(t)}}. \tag{14}$$

Thus, the conditional probability is a normal distribution with variance $w^2(t)$. Here we should refer to the work of Hakim and Ambegaokar [14], who use path integral methods for the special case of a free particle interacting with an Ohmic bath to obtain an equivalent expression for wave packet spreading. (note the misprint in their expression (38) for the width)

Next, we consider the case where the initial measurement forms two widely separated wave packets, which corresponds to the measurement function

$$\alpha(1) = \frac{\exp\{-\frac{(x_1-d/2)^2}{4\sigma_1^2}\} + \exp\{-\frac{(x_1+d/2)^2}{4\sigma_1^2}\}}{[8\pi\sigma_1^2(1+e^{-d^2/8\sigma_1^2})^2]^{1/4}}, \tag{15}$$

where d is the separation of the wave packets, the width of each being σ_1 and x_1 being the center of the wave packet pair. The second measurement is then made with a single slit instrument corresponding to a function of the form (2) (with index $1 \to 2$). Again, the integrals are all standard Gaussian integrals, and we obtain results for $W(1)$ and $W(1,2)$. Again, there is a considerable simplification if we introduce the conditional probability (13). We find

$$\begin{aligned} P(x,t) = & \frac{1}{\sqrt{2\pi w^2}(1+e^{-d^2/8\sigma_1^2})} \\ & \times \left(\exp\{-\frac{x^2 + (\sigma_1^2 + s + \sigma_2^2)\frac{d^2}{4\sigma_1^2}}{2w^2}\} \cos\frac{xd[x(t_1), x(t_1+t)]}{4i\sigma_1^2 w^2} \right. \\ & \left. + \frac{1}{2}\exp\{-\frac{(x-\frac{d}{2})^2}{2w^2}\} + \frac{1}{2}\exp\{-\frac{(x+\frac{d}{2})^2}{2w^2}\} \right). \end{aligned} \tag{16}$$

This conditional probability is the sum of three contributions, corresponding to the three terms within the parentheses. The second and third clearly correspond to the sum of probabilities of the form (14) from two single slits, while the first term (that involving the cosine) is an interference term. It is of interest to study the ratio, $a(t_2 - t_1)$ of the amplitude of the interference term to twice the geometric mean of the other two terms, which we will refer to as the attenuation factor. We find

$$a(t) = \exp\{-\frac{(s(t)+\sigma_2^2)d^2}{8\sigma_1^2 w^2(t)}\}. \tag{17}$$

In general, the interest is in the case where the second measurement is made with infinite precision, so in the following discussion we set σ_2 equal to zero.

For quantum Brownian motion, the mean square displacement and the commutator are given by the formulas

$$\begin{aligned} s(t) &= \frac{2\hbar}{\pi}\int_0^\infty d\omega \mathrm{Im}\{\alpha(\omega + i0^+)\}\coth\frac{\hbar\omega}{2kT}(1-\cos\omega t), \\ [x(t_1), x(t_1+t)] &= \frac{2i\hbar}{\pi}\int_0^\infty d\omega \mathrm{Im}\{\alpha(\omega + i0^+)\}\sin\omega t. \end{aligned} \tag{18}$$

where α is the response function. In the so-called Ohmic case, where the mean motion is $m\langle\ddot{x}\rangle + m\gamma\langle\dot{x}\rangle = 0$,

$$\mathrm{Im}\{\alpha(\omega + i0^+)\} = \frac{\gamma}{m\omega(\omega^2+\gamma^2)}. \tag{19}$$

Consider first the case of vanishingly small dissipation ($\gamma \to 0$). Then, setting $\sigma_2^2 = 0$ and putting $s = \frac{kT}{m}t^2$ and $[x(t_1), x(t_1+t)] = i\frac{\hbar}{m}t$ in (12) and (17) we see that

$$a(t) = \exp\{-\frac{d^2}{8\sigma_1^2 + 2\bar{\lambda}^2 + 8\frac{m\sigma_1^4}{kTt^2}}\} \qquad \text{no dissipation,} \tag{20}$$

where $\bar{\lambda} = \hbar/\sqrt{mkT}$ is the mean thermal de Broglie wavelength. Now the interest is always in the case where the wave packets are widely separated, $d \gg \sigma_1$. From this expression we see that for long time, the attenuation factor will be small (i.e., there will be decoherence) if the temperature is sufficiently high that the mean de Broglie wavelength is small compared with the spacing, $d \gg \bar{\lambda}$. The characteristic time for decoherence to occur will then be

$$\tau_d = \frac{\sigma_1^2}{\bar{v}d}, \tag{21}$$

where $\bar{v} = \sqrt{kT/m}$ is the mean thermal velocity. This decoherence time is the time for a particle travelling with the mean thermal velocity to traverse the slit width multiplied by the ratio of the slit width to the slit spacing. Thus we see that we can have *decoherence without dissipation* (in the sense that τ_d is independent of γ, which characterizes the strength of the coupling to the environment).

Next we consider the case of Ohmic dissipation at high temperature, where by high temperature we mean $kT \gg \hbar\gamma$. Then, using (19) we see from the formulas (18) that

$$s(t) = \frac{2kT}{m\gamma}(t - \frac{1 - e^{-\gamma t}}{\gamma}),$$
$$[x(t_1), x(t_1 + t)] = \frac{i\hbar}{m\gamma}(1 - e^{-\gamma t}). \tag{22}$$

For short times ($\gamma t \ll 1$) these reduce to the those for the case of vanishingly small dissipation, for which the decoherence time is given by (21). Thus if $\gamma\tau_d = \gamma\sigma_1^2/\bar{v}d$ is small (and this will generally be the case at high temperature) the decoherence time will be the same as for the case of vanishingly small dissipation, given by (21).

These exact results are strikingly different from those obtained by previous investigators [2–5]. It appears that the disagreement arises from the fact that others have implicitly used a long time ($\tau_d \gg \gamma^{-1}$) approximation to obtain characteristic decay times. To see how this comes about, we evaluate the high temperature formulas (22) for very long times ($\gamma t \gg 1$). Putting the result in (17) and setting $\sigma_1^2 = 0$ we find $a(t) \sim \exp\{-d^2\gamma t/\bar{\lambda}^2\}$. This is an exponential decaying with characteristic time $\gamma^{-1}\bar{\lambda}^2/d^2$, which is exactly twice the expression for the decoherence time obtained by previous authors [3–5]. But we see that it is inconsistent with the assumption of long time used to obtain it. At short times, as we have seen, we recover the estimate (21). One possible reason why others have obtain inconsistent results may be due to the fact that they assumed that the system and its environment are initially decoupled whereas, by contrast, in the formulation we use the particle state is entangled with the environment (i.e., in equilibrium) at the time it is put into the initial state by a measurement. Putting this point in another way, previous discussions have been in terms of the reduced density matrix but, as pointed out by Ambegaokar [16], on such short time scales the time evolution does not operate on the reduced density matrix alone."

In conclusion we have seen that the simple and general formulation of quantum measurement given in [11] provides a powerful method for discussing quantum stochastic systems. The formalism is in terms of quantum distribution functions and, when combined with results obtained from the quantum Langevin equation, has enabled us to obtain exact explicit

expressions for wave packet spreading and the coherence attenuation factor. In discussing the latter we have seen that decoherence occurs at high temperature with or without dissipation. In either case the decoherence time is the same, given by (21). At zero temperature, decoherence occurs only in the presence of dissipation.

Note added in proof:

In order to understand more clearly the origin of our (no dissipation) result (20), we recently showed that it may be derived in a simple manner solely within the framework of elementary quantum mechanics and equilibrium statistical mechanics [17]. In addition, we have recently obtained an explicit general solution of the exact master equation [18]. When applied to the situation considered in previous discussions, namely a particle at temperature zero suddenly coupled to a bath at high temperature, we are led to an expression for the decoherence time differing by a factor of 6 from the conventional result. We see therefore that the conventional result corresponds to a particle that is "warming up" over a time of order γ^{-1}; our result corresponds to a particle that is initially at the same temperature as the bath.

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