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Paley-Wiener theorems with respect to the spectral parameter

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PALEY-WIENER THEOREMS WITH RESPECT TO
THE SPECTRAL PARAMETER

A Dissertation
Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in
The Department of Mathematics

by
Susanna Dann
B.S., Fachhochschule Stuttgart, Germany, 2004
M.S., Louisiana State University, 2006
August 2011
To my family
Acknowledgments

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<tr>
<td>$</td>
<td>\cdot</td>
</tr>
<tr>
<td>$(\cdot, \cdot)$</td>
<td>Inner-product on $\mathbb{C}^n$</td>
</tr>
<tr>
<td>$\mathbb{N}$</td>
<td>Set of natural numbers ${1, 2, \ldots}$</td>
</tr>
<tr>
<td>$\mathbb{N}_0$</td>
<td>Set of natural numbers including 0</td>
</tr>
<tr>
<td>$</td>
<td>\cdot</td>
</tr>
<tr>
<td>$D^\alpha$</td>
<td>Partial differential operator corresponding to the multi-index $\alpha$</td>
</tr>
<tr>
<td>$C^\infty(\Omega)$</td>
<td>Space of smooth functions on $\Omega$</td>
</tr>
<tr>
<td>$C^\infty(\Omega)_c$</td>
<td>Space of smooth compactly supported functions on $\Omega$</td>
</tr>
<tr>
<td>$</td>
<td>\cdot</td>
</tr>
<tr>
<td>$\mathcal{E}(\Omega)$</td>
<td>Space of smooth functions equipped with the Schwartz topology</td>
</tr>
<tr>
<td>$\text{supp}(f)$</td>
<td>Support of a function $f$</td>
</tr>
<tr>
<td>$\mathcal{D}_K(\Omega)$</td>
<td>Subspace of $\mathcal{E}(\Omega)$ of functions with support in $K$</td>
</tr>
<tr>
<td>$\mathcal{D}(\Omega)$</td>
<td>$C^\infty(\Omega)$ equipped with the Schwartz topology</td>
</tr>
<tr>
<td>$B_r(m)$</td>
<td>Open ball of radius $r &gt; 0$ centered at the point $m$</td>
</tr>
<tr>
<td>$\bar{B}_r(m)$</td>
<td>Closed ball of radius $r &gt; 0$ centered at the point $m$</td>
</tr>
<tr>
<td>$\mathcal{D}_r(\mathcal{M})$</td>
<td>Subspace of $\mathcal{D}(\mathcal{M})$ of functions supported in $\bar{B}_r(0)$</td>
</tr>
<tr>
<td>$S(\mathbb{R}^n)$</td>
<td>Space of Schwartz functions</td>
</tr>
<tr>
<td>$</td>
<td>\cdot</td>
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<tr>
<td>$\hat{f}, \mathcal{F}_{\mathbb{R}^n}(f)$</td>
<td>Fourier transform of $f$</td>
</tr>
<tr>
<td>$S^{n-1}$</td>
<td>Unit sphere in $\mathbb{R}^n$</td>
</tr>
<tr>
<td>$d\omega$</td>
<td>Surface measure on $S^{n-1}$</td>
</tr>
<tr>
<td>$\sigma_n$</td>
<td>Volume of the unit sphere with respect to the surface measure</td>
</tr>
<tr>
<td>$\mu_n$</td>
<td>Normalized measure on the sphere</td>
</tr>
<tr>
<td>$\xi(p, \omega)$</td>
<td>Hyperplane in $\mathbb{R}^n$</td>
</tr>
<tr>
<td>$\Xi$</td>
<td>Set of hyperplanes in $\mathbb{R}^n$</td>
</tr>
<tr>
<td>$\mathcal{R}f$</td>
<td>Radon transform of $f$</td>
</tr>
<tr>
<td>$\mathcal{S}(\mathbb{R} \times S^{n-1})$</td>
<td>Schwartz functions on $\mathbb{R} \times S^{n-1}$</td>
</tr>
<tr>
<td>$D_\omega$</td>
<td>Differential operator on $S^{n-1}$</td>
</tr>
<tr>
<td>$\mathcal{S}_H(\Xi)$</td>
<td>Certain class of Schwartz functions on $\Xi$</td>
</tr>
<tr>
<td>$\eta_{k,m,D_\omega}(\cdot)$</td>
<td>Seminorm on $\mathcal{S}_H(\Xi)$</td>
</tr>
<tr>
<td>$\mathcal{D}_H(\Xi)$</td>
<td>Compactly supported functions in $\mathcal{S}_H(\Xi)$</td>
</tr>
<tr>
<td>$\mathcal{D}_{H,r}(\Xi)$</td>
<td>Functions in $\mathcal{S}_H(\Xi)$ with support in $[-r, r] \times S^{n-1}$</td>
</tr>
<tr>
<td>$</td>
<td>f</td>
</tr>
<tr>
<td>$P(z, r)$</td>
<td>Polydisc with polyradius $r$ and center $z$</td>
</tr>
<tr>
<td>$\text{ch}(f)$</td>
<td>Cauchy integral of $f$</td>
</tr>
<tr>
<td>$\mathcal{P}\mathcal{W}_r(\mathbb{C}^n)$</td>
<td>Paley-Wiener space for smooth functions</td>
</tr>
<tr>
<td>$q_{N,r}(\cdot)$</td>
<td>Seminorm on $\mathcal{P}\mathcal{W}_r(\mathbb{C}^n)$</td>
</tr>
<tr>
<td>$s_{\alpha,r}(\cdot)$</td>
<td>Seminorm on $\mathcal{P}\mathcal{W}_r(\mathbb{C}^n)$</td>
</tr>
<tr>
<td>$\mathcal{P}\mathcal{W}(\mathbb{C}^n)$</td>
<td>Inductive limit of $\mathcal{P}\mathcal{W}_r(\mathbb{C}^n)$</td>
</tr>
<tr>
<td>$V'$</td>
<td>Continuous dual of a topological vector space $V$</td>
</tr>
<tr>
<td>$\text{co}(E)$</td>
<td>Convex hull of the set $E$</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>$\mathcal{O}(\Omega)$</td>
<td>Space of holomorphic functions on $\Omega$</td>
</tr>
<tr>
<td>$</td>
<td>\cdot</td>
</tr>
<tr>
<td>$\mathcal{D}_{\mathcal{R}}^{\mathcal{H}}$</td>
<td>Space of weakly Schwartz functions</td>
</tr>
<tr>
<td>$\nu_{N,u}(\cdot)$</td>
<td>Seminorm on $\mathcal{D}_{\mathcal{R}}^{\mathcal{H}}$</td>
</tr>
<tr>
<td>$\mathcal{D}_{\mathcal{R},c}$</td>
<td>Continuous functions in $\mathcal{D}_{\mathcal{R}}^{\mathcal{H}}$</td>
</tr>
<tr>
<td>$\mathcal{P}\mathcal{W}_{\mathcal{R}}^{\mathcal{H}}$</td>
<td>Space of weakly holomorphic functions of exponential growth $\leq r$</td>
</tr>
<tr>
<td>$\rho_{N,u}(\cdot)$</td>
<td>Seminorm on $\mathcal{P}\mathcal{W}_{\mathcal{R}}^{\mathcal{H}}$</td>
</tr>
<tr>
<td>$H_l$</td>
<td>Spherical harmonics of degree $l$</td>
</tr>
<tr>
<td>$d_n(l)$</td>
<td>Dimension of $H_l$</td>
</tr>
<tr>
<td>${Y_{l,i}}_{i=1}^{d_n(l)}$</td>
<td>Basis for $H_l$</td>
</tr>
<tr>
<td>$M(n,\mathbb{F})$</td>
<td>Set of square matrices of size $n$</td>
</tr>
<tr>
<td>$\text{GL}(n,\mathbb{F})$</td>
<td>General linear group</td>
</tr>
<tr>
<td>$\exp(\cdot)$</td>
<td>Exponential map for $M(n,\mathbb{F})$</td>
</tr>
<tr>
<td>$G^x$</td>
<td>Subgroup of $G$ that fixes $x$</td>
</tr>
<tr>
<td>$\text{Exp}$</td>
<td>Exponential map</td>
</tr>
<tr>
<td>$\text{Hom}_G(\rho,\tau)$</td>
<td>Set of intertwining operators between representations $\rho$ and $\tau$ of $G$</td>
</tr>
<tr>
<td>$\hat{G}$</td>
<td>Unitary dual of $G$</td>
</tr>
<tr>
<td>$\int_{\mathbb{R}^n} d\mu$</td>
<td>Direct integral</td>
</tr>
<tr>
<td>$\ell$</td>
<td>Left regular representation</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Right regular representation</td>
</tr>
<tr>
<td>$E(n)$</td>
<td>Group of rigid motions of $\mathbb{R}^n$</td>
</tr>
<tr>
<td>$\mathcal{F}_{E(n)}$</td>
<td>Fourier transform corresponding to $E(n)$</td>
</tr>
<tr>
<td>$\mathcal{S}^2_{H^n}$</td>
<td>Image of $\mathcal{S}(\mathbb{R}^n)$ under $\mathcal{F}_{E(n)}$</td>
</tr>
<tr>
<td>$</td>
<td>\cdot</td>
</tr>
<tr>
<td>$\mathcal{P}\mathcal{W}_{\mathbb{R}}^{\mathcal{H},2}$</td>
<td>Paley-Wiener space for the first description</td>
</tr>
<tr>
<td>$</td>
<td>\cdot</td>
</tr>
<tr>
<td>$\mathcal{S}^{n-1}_{\mathbb{C}}$</td>
<td>Complexified sphere</td>
</tr>
<tr>
<td>$\mathcal{O}_r$</td>
<td>Weakly holomorphic functions on $\mathbb{C}$ of exponential growth</td>
</tr>
<tr>
<td>$\pi_{r,N}(\cdot)$</td>
<td>Seminorm on $\mathcal{O}<em>r(\mathbb{C}, \mathcal{O}(S^{n-1}</em>{\mathbb{C}}))$</td>
</tr>
<tr>
<td>$\mathcal{P}\mathcal{W}_{\mathbb{C}}^{\mathcal{H},2}$</td>
<td>Paley-Wiener space for the second description</td>
</tr>
<tr>
<td>$\hat{\mathcal{R}}$</td>
<td>Radon type transform</td>
</tr>
<tr>
<td>$\lim S_n \rightarrow T_n$</td>
<td>Inductive limit of an inductive system ${S_n}$</td>
</tr>
<tr>
<td>$\lim T_n$</td>
<td>Projective limit of a projective system ${T_n}$</td>
</tr>
</tbody>
</table>
Abstract

One of the important questions related to any integral transform on a manifold \( \mathcal{M} \) or on a homogeneous space \( G/K \) is the description of the image of a given space of functions. If \( \mathcal{M} = G/K \), where \((G,K)\) is a Gelfand pair, then harmonic analysis on \( \mathcal{M} \) is closely related to the representations of \( G \) and the direct integral decomposition of \( L^2(\mathcal{M}) \) into irreducible representations of \( G \). \( \mathbb{R}^n \) can be realized as the quotient \( \mathbb{R}^n = E(n)/SO(n) \), where \( E(n) \) is the orientation preserving Euclidean motion group \( \mathbb{R}^n \rtimes SO(n) \). The pair \((E(n), SO(n))\) is a Gelfand pair. Hence this realization of \( \mathbb{R}^n \) comes with its own natural Fourier transform derived from the representation theory of \( E(n) \). The representations of \( E(n) \) that are in the support of the Plancherel measure for \( L^2(\mathbb{R}^n) \) are parameterized by \( \mathbb{R}^+ \). We describe the image of smooth compactly supported functions under the Fourier transform with respect to the spectral parameter. Then we discuss an extension of our description to projective limits of corresponding function spaces.
Chapter 1
Introduction

The aim of this thesis is to derive Paley-Wiener type theorems for the vector valued Fourier transform on \( \mathbb{R}^n \). Here we understand the term “Paley-Wiener type theorems” to mean the following problem: Given a manifold \( \mathcal{M} = G/H \), where \( G \) is a Lie group and \( H \) a closed subgroup of \( G \), and given a Fourier type transform on \( \mathcal{M} \), characterize the image of a given function space on \( \mathcal{M} \). More often than not, those are spaces of smooth compactly supported functions. Similar statements for square-integrable functions are called Plancherel theorems. The classical Paley-Wiener theorem identifies the space of smooth compactly supported functions on \( \mathbb{R}^n \) with certain classes of holomorphic functions on \( \mathbb{C}^n \) of exponential growth, where the exponent is determined by the size of the support via the usual Fourier transform on \( \mathbb{R}^n \).

Yet \( \mathbb{R}^n \) can also be represented as a homogeneous space: \( \mathbb{R}^n \cong E(n)/\text{SO}(n) \) with the orientation preserving Euclidean motion group \( E(n) = \mathbb{R}^n \rtimes \text{SO}(n) \). This realization comes with its own natural Fourier transform derived from the representation theory of \( E(n) \), see [28] and Section 4.2. One can again give a description of the space of smooth compactly supported functions, and in fact we will give two such descriptions. These descriptions are given in terms of the parameter in the decomposition of \( L^2(\mathbb{R}^n) \) into irreducible representations of \( E(n) \) as well as some homogeneity conditions.

This thesis is organized as follows. In Chapter 2 we recall the classical Paley-Wiener theorem on \( \mathbb{R}^n \), which describes the image of the space of smooth compactly supported functions, as well as Paley-Wiener theorems for square integrable functions and for distributions.

In Chapter 3 we extend the classical Paley-Wiener theorem to functions with values in a separable Hilbert space. It then reduces to the classical Paley-Wiener theorem by letting the Hilbert space to be one dimensional. Then we look at the particular case of the Hilbert space of square integrable functions and at functions that are \( \text{SO}(n) \)-finite.

In the first part of Chapter 4 we recall the definition of a Gelfand pair \( (G, K) \) and the Fourier transform on the associated commutative space \( G/K \). Then we apply this analysis to the Gelfand pair \( (E(n), \text{SO}(n)) \) and derive the corresponding Fourier transform on \( \mathbb{R}^n \) in Section 4.2. In the second part of this chapter we introduce our main results. We prove a topological analog of a Paley-Wiener type theorem due to Helgason. Then we prove that for smooth compactly supported functions the Fourier transform extends to a bigger space, namely \( \mathbb{C} \times S^{n-1}_\mathbb{C} \), where \( S^{n-1}_\mathbb{C} \) stands for the complexified sphere. We also describe the image of the Schwartz space.
In the last chapter, Chapter 5, we discuss the extension of the classical Paley-Wiener theorem to projective limits, which is a consequence of a result by Cowling. Then we extend our second description to the projective limits of the corresponding function spaces, which can also be interpreted as an extension to the inductive limits of the underlying spaces.
Chapter 2

Paley-Wiener Theorems on $\mathbb{R}^n$

The purpose of this chapter is to acquaint the reader with the long-known standard Paley-Wiener theorems on $\mathbb{R}^n$. To this end standard notation used throughout this manuscript is introduced in Section 2.1. We refer to [3] and [10] as the main sources for the aforementioned section as well as for proofs and further discussion. In Section 2.2 we review some basic theory of holomorphic functions in several complex variables. This material will be presented mostly without proofs, which can be found in [19]. Several Paley-Wiener theorems are presented in Section 2.3. The materials of that section can be found in [3], [18], [34], [35], and [38].

2.1 Preliminaries

Let $\mathbb{R}^n$ and $\mathbb{C}^n$ denote the usual n-dimensional real and complex Euclidean spaces respectively. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, the norm $|z|$ of $z$ is defined by $|z| := (|z_1|^2 + \cdots + |z_n|^2)^{1/2}$. The canonical inner-product of two vectors $x$ and $y$ on $\mathbb{R}^n$ or $\mathbb{C}^n$ is denoted by $x \cdot y$ and by $(x, y)$. The inner-product on $\mathbb{R}^n$ extends to a $\mathbb{C}$-bilinear form $(z, \xi) := \sum_{i=1}^{n} z_i \xi_i$ on $\mathbb{C}^n \times \mathbb{C}^n$. Let $\mathbb{N}$ be the set of natural numbers $\{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup 0$. For $j = 1, \ldots, n$, let $\partial_j := \partial/\partial z_j$. For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and $z \in \mathbb{C}^n$, put $|\alpha| := \alpha_1 + \cdots + \alpha_n$, $z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, and $D^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$.

Let $\mathcal{M}$ be a (smooth) manifold of dimension $n$. For an open subset $\Omega$ of $\mathcal{M}$, let $C^\infty(\Omega)$ and $C^\infty_c(\Omega)$ denote the spaces of smooth complex valued functions on $\Omega$ and smooth complex valued functions with compact support on $\Omega$, respectively. For each compact subset $K$ of $\Omega$, define a seminorm $|\cdot|_{K, \alpha}$ on $C^\infty(\Omega)$ by

$$|f|_{K, \alpha} := \max_{p \in K} |D^\alpha f(p)|,$$

with $\alpha \in \mathbb{N}_0^n$. The vector space $C^\infty(\Omega)$ equipped with the topology defined by these seminorms becomes a locally convex topological vector space and is denoted by $\mathcal{E}(\Omega)$. For each compact subset $K$ of $\Omega$, let $\mathcal{D}_K(\Omega)$ be the subspace of $\mathcal{E}(\Omega)$ consisting of functions $f$ with $\text{supp}(f) \subseteq K$, where $\text{supp}(f) := \{m \in \Omega : f(m) \neq 0\}$ denotes the support of a function $f$. The topology on $\mathcal{D}_K(\Omega)$ is the relative topology of $\mathcal{E}(\Omega)$, and hence it is locally convex. We recall the definition of an inductive limit topology and the most basic facts about it.

**Definition 2.1.** Let $X$ be a vector space and let $X_i$ be a collection of linear subspaces of $X$ each having a locally convex topology. Assume $\bigcup X_i = X$. The strongest locally convex topology on $X$ satisfying that for each $i$ the relative topology of $X$ on $X_i$ is weaker than the topology on $X_i$ is called the **inductive limit topology** on $X$. 

The inductive limit topology on $X = \bigcup X_i$ exists.

**Proposition 2.2.** Let $X = \bigcup X_i$ have the inductive limit topology and let $Y$ be a locally convex topological vector space. Then a linear transformation $T : X \to Y$ is continuous if and only if $T|_{X_i}$ is continuous for each $i$.

The Schwartz topology on $C_c^\infty(\Omega)$ is the inductive limit topology of the subspaces $D_K(\Omega)$ with $K \subseteq \Omega$. The space $C_c^\infty(\Omega)$ with the Schwartz topology is denoted by $D(\Omega)$. If $\mathcal{M}$ is a Riemannian manifold, then $B_r(m)$, respectively $\bar{B}_r(m)$, stands for an open, respectively closed, ball of radius $r > 0$ centered at the point $m \in \mathcal{M}$ and $D_r(\mathcal{M})$ stands for the subspace of functions in $D(\mathcal{M})$ supported in $B_r(0)$. We will also write $B_r$ for $B_r(0)$. The space of smooth rapidly decreasing functions on $\mathbb{R}^n$, the **Schwartz functions**, will be denoted by $S(\mathbb{R}^n)$. It is topologized by the seminorms

$$|f|_{N,\alpha} := \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^N |D^n f(x)|, \quad N \in \mathbb{N}_0 \text{ and } \alpha \in \mathbb{N}_0^n.$$ 

We normalize the Fourier transform by

$$\hat{f}(\lambda) = \mathcal{F}_{\mathbb{R}^n}(f)(\lambda) := \int_{\mathbb{R}^n} f(x)e^{-2\pi i x \lambda} \, dx.$$ 

The Fourier transform is a topological isomorphism of $S(\mathbb{R}^n)$ onto itself with the inverse given by $\mathcal{F}_{\mathbb{R}^n}^{-1}(g)(x) = \mathcal{F}_{\mathbb{R}^n}(g)(-x)$. It extends to a unitary isomorphism of order four of the Hilbert space $L^2(\mathbb{R}^n)$ with itself. This fact goes by the name of **Plancherel theorem**.

Denote by $S^{n-1}$ the unit sphere in $\mathbb{R}^n$ and by $d\omega$ the surface measure on $S^{n-1}$. We will sometimes use the normalized measure $\mu_n$, which is given by $\sigma_n \mu_n = d\omega$ with $\sigma_n := 2\pi^{n/2}/\Gamma(n/2)$ for $n \geq 2$. For $p \in \mathbb{R}$ and $\omega \in S^{n-1}$ denote by $\xi(p,\omega) = \{x \in \mathbb{R}^n : x \cdot \omega = p\}$ the hyperplane with the normal vector $\omega$ at signed distance $p$ from the origin. Denote by $\Xi$ the set of hyperplanes in $\mathbb{R}^n$. Then, as $\xi(r,\omega) = \xi(s,\sigma)$ if and only if $(r,\omega) = (\pm s, \pm \sigma)$, it follows that $\mathbb{R} \times S^{n-1} \ni (r,\omega) \mapsto \xi(r,\omega) \in \Xi$ is a double covering of $\Xi$. We identify functions on $\Xi$ with the corresponding even functions on $\mathbb{R} \times S^{n-1}$, that is with functions on $\mathbb{R} \times S^{n-1}$ such that $f(r,\omega) = f(-r,-\omega)$. The **Radon transform** $\mathcal{R}f$ of a function $f \in C_c^\infty(\mathbb{R}^n)$ is defined by

$$\mathcal{R}f(\xi) := \int_{\xi} f(x)dm(x),$$

where $dm$ is the Lebesgue measure on the hyperplane $\xi$. Then $\mathcal{R}f \in C_c^\infty(\Xi)$. Moreover, $\mathcal{R}$ is continuous from $L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R} \times S^{n-1})$ and its restriction from $S(\mathbb{R}^n)$ into $S(\mathbb{R} \times S^{n-1})$ is continuous [14], where $S(\mathbb{R} \times S^{n-1})$ is the space of smooth functions $\varphi$ on $\mathbb{R} \times S^{n-1}$ satisfying that for any $k, m \in \mathbb{N}_0$ and for any differential operator $D_\omega$ on $S^{n-1}$

$$\sup_{(r,\omega) \in \mathbb{R} \times S^{n-1}} (1 + r^2)^k |\partial^m_r (D_\omega \varphi)(r,\omega)| < \infty.$$
The Radon transform is related to the Fourier transform by the **Fourier-Slice theorem**

\[ \hat{f}(r\omega) = \mathcal{F}_\mathbb{R}(\mathcal{R}f)(r, \omega), \]

(2.1.1)

where the Fourier transform is taken in the first variable.

Denote by \( S_H(\Xi) \) the space of smooth functions \( f : \mathbb{R} \times S^{n-1} \to \mathbb{C} \) such that

1. \( f \) is even, i.e. \( f(r, \omega) = f(-r, -\omega) \);
2. \( \eta_{k,m,D}(f) := \sup_{(r, \omega) \in \mathbb{R} \times S^{n-1}} (1 + r^2)^k |\partial_p^m D_\omega f(r, \omega)| < \infty \) for all \( k, m \in \mathbb{N}_0 \) and for any \( D_\omega \) a differential operator on \( S^{n-1} \);
3. For each \( k \in \mathbb{N} \), the function \( \omega \mapsto \int_{-\infty}^{\infty} f(r, \omega) r^k \, dr \) is a homogeneous polynomial of degree \( k \).

The family \( \{ \eta_{k,m,D} \} \) defines a topology on \( S_H(\Xi) \).

**Theorem 2.3.** The Radon transform is a topological isomorphism of the space \( S(\mathbb{R}^n) \) with \( S_H(\Xi) \).

**Proof.** By Theorem 2.4 in [12] it is a bijection and by Corollary 4.8 in [14] it is continuous with a continuous inverse. \( \square \)

Let \( \mathcal{D}_H(\Xi) := C_c^{\infty}(\Xi) \cap S_H(\Xi) \) with the natural topology. For \( r > 0 \), let \( \mathcal{D}_{H,r}(\Xi) \) be the set \( \{ f \in \mathcal{D}_H(\Xi) : f(p, \omega) = 0 \text{ for } |p| > r \} \). The topology on \( \mathcal{D}_{H,r}(\Xi) \) is given by the seminorms

\[ |f|_{m,D_\omega} := \sup_{(p, \omega) \in [-r,r] \times S^{n-1}} |\partial_p^m D_\omega f(p, \omega)| < \infty, \]

where \( m \) is in \( \mathbb{N}_0 \) and \( D_\omega \) is any differential operator on \( S^{n-1} \). The topology on \( \mathcal{D}_H(\Xi) \) is the inductive limit topology of the subspaces \( \mathcal{D}_{H,r}(\Xi) \) with \( 0 < r < \infty \).

**Theorem 2.4.** The Radon transform is a topological isomorphism between the spaces \( \mathcal{D}_r(\mathbb{R}^n) \) and \( \mathcal{D}_{H,r}(\Xi) \).

**Proof.** By Theorems 2.4 and 2.6 and Corollary 2.8 in [12] it is a bijection and by Corollary 4.8 in [14] it is continuous with a continuous inverse. \( \square \)

### 2.2 Holomorphic Functions on \( \mathbb{C}^n \)

In this section we recall some fundamental definitions and facts about holomorphic functions in one and several complex variables.

Let \( r = (r_1, \ldots, r_n) \in \mathbb{R}^n \), we write \( r > 0 \), if \( r_j > 0 \) for all \( j \). Then

\[ P(z, r) := \{ \omega \in \mathbb{C}^n : |\omega_j - z_j| < r_j \text{ for } j = 1, \ldots, n \} \]

is called an (open) **polydisc** with **polyradius** \( r \) and with center \( z \).

**Definition 2.5.** Let \( \emptyset \neq U \subset \mathbb{C}^n \) be open, then a function \( f : U \to \mathbb{C} \) is called:
(a) **Complex differentiable** at $z \in U$ if there exists a complex linear map $Df(z) : \mathbb{C}^n \to \mathbb{C}$ such that for $\omega \in U$

$$f(\omega) = f(z) + Df(z)(\omega - z) + o(|\omega - z|).$$

(b) **Complex differentiable** on $U$ if it is complex differentiable at every point of $U$.

(c) **Analytic** or **holomorphic** on $U$ if for each $z \in U$ there exist $r \in \mathbb{R}^n, r > 0$ such that the polydisc $P(z, r) \subset U$ and on $P(z, r)$ we have

$$f(\omega) = \sum_{\alpha \in \mathbb{N}^n_0} a_{\alpha}(\omega - z)^{\alpha}$$

with $a_{\alpha} \in \mathbb{C}$.

**Proposition 2.6.** Let $z \in \mathbb{C}^n$ and $r \in \mathbb{R}^n, r > 0$. A continuous function $f : P(z, r) \to \mathbb{C}$ is complex differentiable if and only if for all $1 \leq j \leq n$ the function

$$P(z_j, r_j) \ni \omega \mapsto f(z_1, ..., z_{j-1}, \omega, z_{j+1}, ..., z_n) \in \mathbb{C}$$

is complex differentiable.

Let $Q$ be the cube $[0, 2\pi]^n$ in $\mathbb{R}^n$.

**Definition 2.7.** Let $z \in \mathbb{C}^n$ and $r, s \in \mathbb{R}^n, r, s > 0$ with $s_j < r_j$ for all $j$. For each $j$ define $\gamma_j(\theta) = z_j + s_je^{2\pi i \theta}$ and let $\gamma(\theta_1, ..., \theta_n) := (\gamma_1(\theta_1), ..., \gamma_n(\theta_n))$. Let $f$ be a continuous function on the image $\gamma(Q) \subset P(z, r)$, then

$$\text{ch}(f)(z) := \left(\frac{1}{2\pi i}\right)^n \oint_{\gamma} \frac{f(\omega)}{(\omega_1 - z_1) \cdots (\omega_n - z_n)} d\omega_1...d\omega_n$$

$$= \int_{[0, 2\pi]^n} f(\gamma_1(\theta_1), ..., \gamma_n(\theta_n)) d\theta_1...d\theta_n$$

is called the **Cauchy integral** of $f$ over $Q$.

Note that if $f$ is holomorphic, then $f(z) = \text{ch}(f)(z)$. Moreover, the integral is independent of the choice of paths $\gamma_j$ as long as each $\gamma_j$ is homotopic to a circle.

**Theorem 2.8.** Let $\emptyset \neq U \subset \mathbb{C}^n$ be open and $f : U \to \mathbb{C}$ be continuous. Then the following are equivalent:

(a) $f$ is complex differentiable on $U$.

(b) $f$ is holomorphic on $U$.

(c) $f = \text{ch}(f)$.

The following simple lemma will be needed for the proof of the Paley-Wiener theorem.
Lemma 2.9. Let $K \subset \mathbb{R}^n$ be compact, $\emptyset \neq U \subset \mathbb{C}^n$ be open, $\mu$ be a finite measure on $K$, and let $f : K \times U \to \mathbb{C}$ be measurable and bounded on compact sets. If $U \to \mathbb{C} : z \mapsto f(x, z)$ is holomorphic for each $x \in K$, then the function

$$F(z) := \int_K f(x, z) d\mu(x)$$

is holomorphic.

Proof. We can assume $\mu(K) > 0$ and $U$ is a polydisc $P(z, r)$ with some $z \in \mathbb{C}^n$ and $r \in \mathbb{R}^n$, $r > 0$. The function $F$ is continuous by Lebesgue Dominated Convergence theorem. Now compute

$$\text{ch}(F)(z) = \int_{[0,2\pi]^n} F(\gamma_1(\theta_1), ..., \gamma_n(\theta_n)) d\theta_1...d\theta_n$$

$$= \int_{[0,2\pi]^n} \int_K f(x, \gamma_1(\theta_1), ..., \gamma_n(\theta_n)) d\mu(x) d\theta_1...d\theta_n.$$

By Fubini’s theorem and Theorem 2.8 we have

$$\text{ch}(F)(z) = \int_K \int_{[0,2\pi]^n} f(x, \gamma_1(\theta_1), ..., \gamma_n(\theta_n)) d\theta_1...d\theta_n d\mu(x)$$

$$= \int_K f(x, z) d\mu(x)$$

$$= F(z).$$

Thus by Theorem 2.8 the function $F$ is holomorphic. \qed

We will make use of the following theorems for functions in one complex variable.

Theorem 2.10. If $\phi$ is a continuous function on $\{\zeta : |\zeta - P| = r\}$, then the function $f$ given by

$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta - P| = r} \frac{\phi(\zeta)}{\zeta - z} d\zeta$$

is defined and holomorphic on $B_r(P)$.

Theorem 2.11. (Cauchy’s theorem) Let $\Omega \subset \mathbb{C}$ be an open connected set and let $F$ be holomorphic on $\Omega$. Let $\gamma$ be a closed curve that is homotopic to a point in $\Omega$. Then $\oint_{\gamma} F(z) dz = 0$.

Theorem 2.12. (Morera’s theorem) Let $f$ be continuous on an open connected set $\Omega \subset \mathbb{C}$ and suppose $\oint_{\gamma} f(z) dz = 0$ for every closed curve in $\Omega$. Then $f$ is holomorphic on $\Omega$.

Theorem 2.13. (Principle of Analytic Continuation) Let $F$ and $G$ be holomorphic on an open connected set $\Omega$. Suppose there is a sequence $z_1, z_2, ...$ of distinct points of $\Omega$ converging to $z_0 \in \Omega$ such that $F(z_j) = G(z_j)$ for all $j = 1, 2, 3, ...$. Then $F = G$ on all of $\Omega$. 

7
2.3 Paley-Wiener Theorems on \( \mathbb{R}^n \)

There are many Paley-Wiener theorems or sometimes also called Paley-Wiener-Schwartz theorems in the literature. They establish a relation between some class of holomorphic functions and harmonic analysis of compactly supported functions or distributions. The classical Paley-Wiener theorem\(^1\) characterizes the space of compactly supported smooth functions on \( \mathbb{R}^n \) by means of the Fourier transform. We will take a look at several Paley-Wiener theorems. We begin by giving some definitions.

**Definition 2.14.** Let \( r \geq 0 \). An entire function \( F \) is said to be of **exponential type** \( \leq r \) if there exists a constant \( C > 0 \) such that for all \( z \in \mathbb{C}^n \)

\[
|F(z)| \leq C e^{r|\text{Im}(z)|}.
\]

Denote by \( \mathcal{PW}_r(\mathbb{C}^n) \) the space of entire functions \( H \) on \( \mathbb{C}^n \) such that \( z^\alpha H(z) \) is of exponential type \( \leq r \) for every \( \alpha \in \mathbb{N}_0^n \). The vector space \( \mathcal{PW}_r(\mathbb{C}^n) \) is topologized by the family of seminorms:

\[
q_{N,r}(H) := \sup_{z \in \mathbb{C}^n} (1 + |z|^2) e^{-r|\text{Im}(z)|} |H(z)|
\]

with \( N \in \mathbb{N}_0 \). The same topology is given by the seminorms:

\[
s_{\alpha,r}(H) := \sup_{z \in \mathbb{C}^n} e^{-r|\text{Im}(z)|} |z^\alpha H(z)|
\]

with \( \alpha \in \mathbb{N}_0^n \). Let \( \mathcal{PW}(\mathbb{C}^n) := \bigcup_{r>0} \mathcal{PW}_r(\mathbb{C}^n) \). We give it the inductive limit topology.

**Proposition 2.15.** For \( r > 0 \), the spaces \( \mathcal{D}_r(\mathbb{R}^n) \) and \( \mathcal{PW}_r(\mathbb{C}^n) \) are Fréchet spaces, i.e. complete, metrizable, locally convex vector spaces.

**Proof.** They are locally convex spaces as their topologies are defined by seminorms. It is easy to see that both spaces are separated and hence Hausdorff. As their topologies are defined by countable collections of seminorms, they are metrizable.

\(^1\)This case is often referred to as "the Paley-Wiener theorem", "the classical Paley-Wiener theorem" or "the Paley-Wiener-Schwartz theorem", e.g. see [17], [27], [32]. This labeling is well established although possibly not the most adequate. In their original work R. Paley and N. Wiener [30] developed, among other beautiful results, a theorem which nowadays may be called a Paley-Wiener theorem for the case of square-integrable functions. They were first to observe the deep relationship between the support properties of a function and the analyticity properties of its Fourier transform. In the introduction of their above mentioned work they explore properties of the Fourier transform of functions under different assumptions: functions vanishing exponentially, functions in a strip, and functions in a half-plane, as well as analyze entire functions of exponential type. They deal with the latter case in the Theorem X on p. 13 in [30]. It is perhaps the most cited of all their "Paley-Wiener" theorems and is often stated in its original form, see, for instance, Theorem 19.3 in [34] or Theorem 7.4 in [18]. L. Schwartz was the first to generalize this observation to the case of distributions in [36]. In spite of being the most widespread in the literature, as it seems, we only found contradicting remarks often without references to published materials on the authorship of the Paley-Wiener theorem for the case of smooth functions. The earliest published version of this case appears to be the Theorem 1.7.7 on p.21 in [16] by L. Hörmander. However in the historical note about Paley-Wiener theorems in the later addition of his work [17] he does not claim the authorship of this case for himself. A zillion of people discovered Paley-Wiener type theorems for many different settings. More information on those further developments of this subject can be found in the historical notes in [32] on p. 339, [17] on p. 249, [37] on p. 131, and [13] on p. 78, 229, 493, as well as in introductions of some recent articles, e.g. [27].
Let \( \{f_m\}_{m \in \mathbb{N}} \) be a Cauchy sequence in \( D_r(\mathbb{R}^n) \). Then for every \( \alpha \in \mathbb{N}_0^n \) the sequence \( \{D^\alpha f_m\}_{m \in \mathbb{N}} \) converges uniformly to some function \( f_\alpha \). \( f_0 \) is a smooth function with support contained in \( B_r(0) \), hence \( f_0 \in D_r(\mathbb{R}^n) \), and \( D^\alpha f_0 = f_\alpha \). Moreover, let \( \epsilon > 0 \), then

\[
|D^\alpha (f_k(x) - f_l(x))| \leq |f_k - f_l|_\alpha < \epsilon
\]

for all \( x \in \mathbb{R}^n \) and \( k, l \) big enough. Letting \( l \to \infty \), we obtain

\[
|D^\alpha (f_k(x) - f_0(x))| < \epsilon
\]

for all \( x \in \mathbb{R}^n \) and \( k \) big enough. Consequently, \( f_k \) converges to \( f_0 \) in the topology of \( D_r(\mathbb{R}^n) \).

Now, let \( \{F_m\}_{m \in \mathbb{N}} \) be a Cauchy sequence in \( \mathcal{PW}_r(\mathbb{C}^n) \). Hence for any \( \alpha \in \mathbb{N}_0^n \) and \( \epsilon > 0 \), there is \( N_\alpha \in \mathbb{N} \) such that

\[
s_{\alpha,r}(F_k - F_l) < \epsilon
\]

for \( k, l \geq N_\alpha \). Since the inequality

\[
|z^\alpha (F_k(z) - F_l(z))| \leq s_{\alpha,r}(F_k(z) - F_l(z))e^{r|\text{Im}(z)|}
\]

holds for all \( z \in \mathbb{C}^n \) and for any \( \alpha \in \mathbb{N}_0^n \), the sequence \( \{z^\alpha F_m\} \) converges to a function \( F_\alpha \). Moreover, as it converges uniformly over compact sets, \( F_\alpha \) is entire and \( F_\alpha(z) = z^\alpha F_0(z) \) for every \( \alpha \in \mathbb{N}_0^n \) and \( z \in \mathbb{C}^n \). This follows by taking the limit on both sides in the equality

\[
z^\alpha F_m(z) = \frac{z^\alpha}{(2\pi i)^n} \oint_{\gamma} \frac{F_m(\xi)}{(\xi - z)} d\xi
\]

with some appropriate \( \gamma \). Moreover, for all \( z \in \mathbb{C}^n \) and \( k, l \geq N_\alpha \)

\[
e^{-r|\text{Im}(z)|} |z^\alpha (F_k(z) - F_l(z))| \leq s_{\alpha,r}(F_k - F_l) < \epsilon.
\]

Letting \( l \to \infty \), we obtain that for any \( z \in \mathbb{C}^n \) and \( k \geq N_\alpha \)

\[
e^{-r|\text{Im}(z)|} |z^\alpha (F_k(z) - F_0(z))| < \epsilon.
\]

Hence \( s_{\alpha,r}(F_k(z) - F_0(z)) < \epsilon \) for \( k \geq N_\alpha \) and \( s_{\alpha,r}(F_0) \leq s_{\alpha,r}(F_0 - F_k) + s_{\alpha,r}(F_k) < \infty \) for all \( \alpha \in \mathbb{N}_0^n \). So \( F_0 \in \mathcal{PW}_r(\mathbb{C}^n) \) and \( F_k \to F_0 \) in the topology of \( \mathcal{PW}_r(\mathbb{C}^n) \).

This shows that both spaces are complete.

\[\square\]

2.3.1 The Classical Paley-Wiener Theorem

In this subsection we give a proof of the classical Paley-Wiener theorem. We start with the following lemma.
Lemma 2.16. Let $n \geq 2$, $r > 0$, $F \in \mathcal{PW}_r(\mathbb{C}^n)$, and $\xi \in \mathbb{C}$, then the function $\tilde{F} : \mathbb{C}^{n-1} \to \mathbb{C}$ defined by

$$\tilde{F}(z) := \int_{\mathbb{R}} F(z, \xi + x)dx$$

is in $\mathcal{PW}_r(\mathbb{C}^{n-1})$ and the map $\mathcal{PW}_r(\mathbb{C}^n) \to \mathcal{PW}_r(\mathbb{C}^{n-1}) : F \mapsto \tilde{F}$ is continuous.

Proof. Clearly $\tilde{F}$ is an entire function on $\mathbb{C}^{n-1}$. Since the inequality

$$e^{-r|\text{Im}(z,\xi)|} (1 + |z|^2 + |\xi + x|^2)^{N+1}|F(z, \xi + x)| \leq q_{N+1,r}(F)$$

holds for all $z \in \mathbb{C}^{n-1}$, $x \in \mathbb{R}$, and since $|\text{Im}(z, \xi)| \leq |\text{Im}(z)| + |\text{Im}(\xi)|$, we have

$$\quad e^{-r|\text{Im}(z)|} (1 + |z|^2)N (1 + |\xi + x|^2)|F(z, \xi + x)| \leq q_{N+1,r}(F) e^{r|\text{Im}(\xi)|}.$$  

This gives

$$\quad e^{-r|\text{Im}(z)|} (1 + |z|^2)^N \int_{\mathbb{R}} |F(z, \xi + x)|dx \leq q_{N+1,r}(F) e^{r|\text{Im}(\xi)|} \int_{\mathbb{R}} \frac{dx}{(1 + |\xi + x|^2)}$$

and consequently

$$\quad q_{N,r}(\tilde{F}) \leq C e^{r|\text{Im}(\xi)|} q_{N+1,r}(F),$$

where $C := \int_{\mathbb{R}} \frac{dx}{(1 + |\xi + x|^2)} < \infty$.

This shows $\tilde{F} \in \mathcal{PW}_r(\mathbb{C}^{n-1})$ and $F \mapsto \tilde{F}$ is continuous at 0. As $F \mapsto \tilde{F}$ is linear, it is continuous.

\[ \square \]

Theorem 2.17. Let $r > 0$, $F \in \mathcal{PW}_r(\mathbb{C}^n)$, and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} F(x)dx = \int_{\mathbb{R}^n} F(x + iy)dx.$$

Proof. Let $T > 0$ and consider the curves

$$\begin{align*}
\gamma_1(t) &= t, & -T \leq t \leq T, \\
\gamma_2(t) &= T + ity_n, & 0 \leq t \leq 1, \\
\gamma_3(t) &= -t + iy_n, & -T \leq t \leq T, \\
\gamma_4(t) &= -T + i(1-t)y_n, & 0 \leq t \leq 1.
\end{align*}$$

Let $\gamma$ be a simple closed curve consisting of these four curves, i.e.

$$\gamma(t) := \begin{cases}
\gamma_1(t) & \text{for } -T \leq t \leq T, \\
\gamma_2(t-T) & \text{for } T \leq t \leq T+1, \\
\gamma_3(t-(2T+1)) & \text{for } T+1 \leq t \leq 3T+1, \\
\gamma_4(t-(3T+1)) & \text{for } 3T+1 \leq t \leq 3T+2.
\end{cases}$$

As $F$ is holomorphic in each variable, we have

$$\oint_{\gamma} F(z, \omega)d\omega = 0.$$
This equality can be rewritten as:

\[
\begin{align*}
\int_{-T}^{T} F(z,t) dt + iy_n \int_{0}^{1} F(z,T+ity_n) dt - \int_{-T}^{T} F(z,t+iy_n) dt - iy_n \int_{0}^{1} F(z,-T+ity_n) dt &= 0.
\end{align*}
\]

For fixed values of \( z \in \mathbb{C}^{n-1} \) and bounded values of \( t \) we have the estimate

\[
|F(z, \pm T + ity_n)| \leq q_{1,r}(F) \frac{e^{r\sqrt{|\text{Im}(z)|^2 + (ty_n)^2}}}{1 + |z|^2 + T^2 + y_n^2 t^2} \leq C T^{-2},
\]

with some constant \( C \). This gives

\[
\lim_{T \to \infty} \int_{0}^{1} F(z, \pm T + ity_n) dt = 0.
\]

Thus

\[
\int_{-\infty}^{\infty} F(z, t) dt = \int_{-\infty}^{\infty} F(z, t + iy_n) dt.
\]

By the above lemma the function

\[
\tilde{F}(z) = \int_{-\infty}^{\infty} F(z, t) dt = \int_{-\infty}^{\infty} F(z, t + iy_n) dt
\]

is in \( \mathcal{PW}_r(\mathbb{C}^{n-1}) \). Hence we can iterate this argument to obtain the result.

\[\square\]

**Theorem 2.18.** Let \( r > 0, F \in \mathcal{PW}_r(\mathbb{C}^n) \), and \( \alpha \in \mathbb{N}_0^n \), then \( D^\alpha F \in \mathcal{PW}_r(\mathbb{C}^n) \) and the mapping \( F \mapsto D^\alpha F \) is continuous.

**Proof.** It suffices to show \( \frac{\partial}{\partial z_1} F \in \mathcal{PW}_r(\mathbb{C}^n) \), \( F \mapsto \frac{\partial}{\partial z_1} F \) is continuous, and then argue inductively. Clearly \( \frac{\partial}{\partial z_1} F \) is entire. To show it belongs to the Paley-Wiener space, write \( z = (z_1, \xi) \) with \( z_1 \in \mathbb{C} \) and \( \xi \in \mathbb{C}^{n-1} \) and let \( \gamma \) be the curve \( \gamma(t) = z_1 + e^{it} \) for \( 0 \leq t \leq 2\pi \), then

\[
\frac{\partial}{\partial z_1} F(z_1, \xi) = \frac{1}{2\pi i} \oint_\gamma \frac{F(\omega, \xi)}{(\omega - z_1)^2} d\omega.
\]

This gives

\[
(1 + |z|^2)^N e^{-r|\text{Im}(z)|} \left| \frac{\partial}{\partial z_1} F(z) \right| = \frac{1}{2\pi} \left| \oint_\gamma \frac{(1 + |z|^2)^N e^{-r|\text{Im}(z)|} F(\omega, \xi)(1 + |(\omega, \xi)|^2)^N}{(\omega - z_1)^2 (1 + |(\omega, \xi)|^2)^N} d\omega \right| \leq \frac{1}{2\pi} \oint_\gamma \frac{(1 + |z|^2)^N e^{r} q_{N,r}(F)}{(1 + |(\omega, \xi)|^2)^N} |d\omega|.
\]
As \( z = (z_1, \xi) \) and \( \omega = z_1 + e^{it} \), we have
\[
\frac{(1 + |z|^2)^N}{(1 + |(\omega, \xi)|^2)^N} = \left( \frac{1 + |z_1| + |\xi|^2}{1 + |\omega|^2 + |\xi|^2} \right)^N = \left( \frac{1 + |z_1 + e^{it}|^2 + |\xi|^2}{1 + |\omega|^2 + |\xi|^2} \right)^N \leq \left( \frac{1 + |\omega|^2 + 2|\omega| + 1 + |\xi|^2}{1 + |\omega|^2 + |\xi|^2} \right)^N \leq \left( \frac{1 + 2|\omega|^2}{1 + |\omega|^2} \right)^N = \left( \frac{2 + 2|\omega| + |\omega|^2}{1 + |\omega|^2} \right)^N \leq 5^N.
\]
So
\[
(1 + |z|^2)^N e^{-r|\Im(z)|} \left| \frac{\partial}{\partial z_1} F(z) \right| \leq \frac{1}{2\pi} \int_{\gamma} \frac{(1 + |z|^2)^N e^r q_{N,r}(F)}{(1 + |(\omega, \xi)|^2)^N} |d\omega| \leq \frac{5^N}{2\pi} e^r q_{N,r}(F) \int_0^{2\pi} dt \leq 5^N e^r q_{N,r}(F).
\]
This shows \( \frac{\partial}{\partial z_1} F \in \mathcal{PW}_r(\mathbb{C}^n) \) and \( F \mapsto \frac{\partial}{\partial z_1} F \) is continuous at 0.

**Corollary 2.19.** Let \( r > 0 \), \( F \in \mathcal{PW}_r(\mathbb{C}^n) \), then \( F|_{\mathbb{R}^n} \) is a Schwartz function and the mapping \( F \mapsto F|_{\mathbb{R}^n} \) is continuous.

**Proof.** The above proof implies \( (1 + |x|^2)^N |D^a F(x)| \leq 5^N |x|^a e^r q_{N,r}(F) \).

For a bounded measurable function with compact support the domain of definition of its Fourier transform extends to \( \mathbb{C}^n \), denote this extension by \( \mathcal{F}_{\mathbb{R}^n}^e \).

**Theorem 2.20.** (Paley-Wiener theorem for smooth functions) The Fourier transform \( \mathcal{F}_{\mathbb{R}^n}^e \) is a linear topological isomorphism of \( \mathcal{D}_r(\mathbb{R}^n) \) onto \( \mathcal{PW}_{2\pi r}(\mathbb{C}^n) \) for any \( r > 0 \).

**Proof.** Let \( f \in \mathcal{D}_r(\mathbb{R}^n) \). Define a function \( F \) by
\[
F(z) := \mathcal{F}_{\mathbb{R}^n}^e(f)(z) = \int_{B_r(0)} f(x) e^{-2\pi i z \cdot x} dx.
\]

\(^2\)This extension is sometimes called the Fourier-Laplace transform.
Then by Lemma 2.9 $F$ is a holomorphic function on $\mathbb{C}^n$. On $\bar{B}_r(0)$ we have $|e^{-2\pi i z \cdot x}| \leq e^{2\pi r |\text{Im}(z)|}$, hence

$$|F(z)| \leq e^{2\pi r |\text{Im}(z)|} \|f\|_1 \leq \text{vol}(\bar{B}_r(0)) \|f\|_{\infty} e^{2\pi r |\text{Im}(z)|}. $$

Integration by parts gives

$$(2\pi iz)^\alpha F(z) = \int_{\bar{B}_r(0)} (-1)^{|\alpha|} f(x) D_x^\alpha e^{-2\pi i z \cdot x} \, dx \tag{5}$$

Thus $(2\pi)^{|\alpha|} |z^\alpha| |F(z)| \leq \text{vol}(\bar{B}_r(0)) |f|_\alpha e^{2\pi r |\text{Im}(z)|}$. This shows that for any $\alpha \in \mathbb{N}_0^n$, $s_{\alpha,2\pi r}(F) < \infty$, so $F \in \mathcal{PW}_{2\pi r}(\mathbb{C}^n)$, and that $\mathcal{F}_{\mathbb{R}^n}^c : \mathcal{D}_r(\mathbb{R}^n) \to \mathcal{PW}_{2\pi r}(\mathbb{C}^n)$ is continuous.

Now let $F \in \mathcal{PW}_{2\pi r}(\mathbb{C}^n)$. $F$ is Schwartz on $\mathbb{R}^n$ by Corollary 2.19, thus $f$ given by

$$f(x) : = \mathcal{F}_{\mathbb{R}^n}^{-1}(F)(x) = \int_{\mathbb{R}^n} F(y) e^{2\pi i y \cdot x} \, dy$$

is Schwartz. Note $H(z) : = F(z) e^{2\pi i z \cdot x}$ is in $\mathcal{PW}_{2\pi (r+|z|)}(\mathbb{C}^n)$. Indeed,

$$\begin{align*}
(1 + |z|^2)^N e^{-2\pi(r+|z|)|\text{Im}(z)|} |H(z)| & = (1 + |z|^2)^N e^{-2\pi r |\text{Im}(z)|} |F(z)| e^{-2\pi|x||\text{Im}(z)|} e^{-2\pi r |\text{Im}(z)|} x \\
& \leq q_{N,2\pi r}(F) e^{-2\pi|x||\text{Im}(z)|} e^{2\pi |\text{Im}(z)| |x|} \\
& = q_{N,2\pi r}(F).
\end{align*}$$

Write $z = a + ib$ with $a, b \in \mathbb{R}^n$, then by Theorem 2.17

$$\int_{\mathbb{R}^n} H(y) \, dy = \int_{\mathbb{R}^n} H(y + ib) \, dy = \int_{\mathbb{R}^n} H(y + a + ib) \, dy.$$  

Consequently, for any $z \in \mathbb{C}^n$

$$f(x) = \int_{\mathbb{R}^n} F(y) e^{2\pi i y \cdot x} \, dy = \int_{\mathbb{R}^n} F(y + z) e^{2\pi i (y + z) \cdot x} \, dy.$$  

Now take $z = s ix$ with $s > 0$, then

$$f(x) = \int F(y + s ix) e^{2\pi i (y + s ix) \cdot x} \, dy$$

$$= e^{-2\pi s|x|^2} \int F(y + s ix) e^{2\pi i y \cdot x} \, dy$$

and thus

$$|f(x)| \leq e^{-2\pi s|x|^2} \int q_{N,2\pi r}(F)(1 + |y|^2)^{-N} e^{2\pi r s|x|} \, dy$$

$$= e^{2\pi s|x|(r-|x|)} q_{N,2\pi r}(F) \int (1 + |y|^2)^{-N} \, dy.$$
Choose $N$ big enough to make $\int (1 + |y|^2)^{-N}dy < \infty$ and let $s \to \infty$. We obtain $f(x) = 0$ for $|x| > r$. That is, $\text{supp}(f) \subset \bar{B}_r(0)$ and $f \in D_r(\mathbb{R}^n)$.

Let $\alpha \in \mathbb{N}_0^n$ and pick an $N$ so that $y \mapsto (1 + |y|^2)^{-N}|y_1^{\alpha_1}...y_n^{\alpha_n}|$ is integrable, then

$$|D^\alpha f(x)| \leq (2\pi)^{|\alpha|} \int |y_1^{\alpha_1}...y_n^{\alpha_n} F(y)|dy$$

$$\leq (2\pi)^{|\alpha|} \sup_{t \in \mathbb{R}^n} (1 + |t|^2)^N |F(t)| \int |y_1^{\alpha_1}...y_n^{\alpha_n}|(1 + |y|^2)^{-N}dy$$

$$\leq C q_N(2\pi r)(F).$$

Thus the inversion $\mathcal{F}^{-1}_{2\pi r} : \mathcal{PW}_{2\pi r}(\mathbb{C}^n) \to D_r(\mathbb{R}^n)$ is continuous as well. \qed

**Remark 2.21.** The spaces $D(\mathbb{R}^n)$ and $\mathcal{PW}(\mathbb{C}^n)$ are inductive limits of $D_r(\mathbb{R}^n)$ and $\mathcal{PW}_r(\mathbb{C}^n)$, respectively. It follows by Proposition 2.2 that $\mathcal{F}_{2\pi r}$ extends to a topological isomorphism of $D(\mathbb{R}^n)$ onto $\mathcal{PW}(\mathbb{C}^n)$.

### 2.3.2 Paley-Wiener Theorem for $L^2$ Functions

Here we state and prove a result that might be called the Paley-Wiener theorem for $L^2$ functions. Then we give examples, without proofs, of other results with similar flavor.

**Theorem 2.22.** (Paley-Wiener theorem for $L^2$ functions) Let $F$ be an entire function on $\mathbb{C}$ and $r > 0$. Then the following two conditions on $F$ are equivalent:

1. $F|_{\mathbb{R}} \in L^2(\mathbb{R})$ and $|F(z)| \leq Ce^{2\pi r|\text{Im}(z)|}$ with some constant $C$.

2. There exist a function $f \in L^2(\mathbb{R})$, $f(x) = 0$ for $|x| > r$ such that $F = \mathcal{F}_{2\pi r}^c(f)$.

**Proof.** Let $f \in L^2(-r, r)$. Define $F$ by

$$F(z) := \mathcal{F}_{2\pi r}^c(f)(z) = \int_{-r}^{r} f(t)e^{-2\pi itz}dt. \quad (2.3.1)$$

Write $z = x + iy$, then $|e^{-2\pi itz}| = e^{2\pi ty}$ and so

$$\left| \int_{-r}^{r} f(t)e^{-2\pi itz}dt \right| \leq \|f\|_{L^2(-r,r)} \|e^{2\pi ty}\|_{L^2_t(-r,r)} < \infty.$$

Thus the integral in (2.3.1) exists as a Lebesgue integral. The Lemma 2.9 does not apply here, so we have to work a little harder to argue $F$ is entire.

Fix $z \in \mathbb{C}$ and let $\{z_n\}$ be a sequence converging to $z$. As a Cauchy sequence it is bounded, so there is a constant $0 < b < \infty$ so that $|y_n| < b$ as well as $|y| < b$. Hence

$$|e^{-2\pi iz_n} - e^{-2\pi iz}|^2 \leq \{e^{2\pi ty} + e^{2\pi ty}\}^2 \leq 4e^{4\pi b|t|} \in L^1_t(-r, r),$$

and so by Lebesgue Dominated Convergence theorem

$$\lim_{n \to \infty} \int_{-r}^{r} |e^{-2\pi iz_n} - e^{-2\pi iz}|^2 dt = 0.$$
Applying Schwarz inequality to $|F(z_n) - F(z)|$, it follows that $F$ is continuous on $\mathbb{C}$. Let $\gamma$ be a closed path in $\mathbb{C}$, then by Fubini’s theorem

$$\oint_{\gamma} F(z) \, dz = \oint_{\gamma} \int_{-r}^{r} f(t) e^{-2\pi itz} \, dt \, dz = \int_{-r}^{r} f(t) \oint_{\gamma} e^{-2\pi itz} \, dz \, dt.$$ 

By Cauchy’s theorem $\oint_{\gamma} e^{-2\pi itz} \, dz = 0$ and hence by Morera’s theorem $F$ is entire. Moreover,

$$|F(z)| \leq \int_{-r}^{r} |f(t)| e^{2\pi \epsilon y} \, dt \leq e^{2\pi \epsilon |y|} \int_{-r}^{r} |f(t)| \, dt = \text{const. } e^{2\pi \epsilon \text{Im}(z)}.$$

Furthermore, $F|_{\mathbb{R}} \in L^2(\mathbb{R})$ by Plancherel theorem. We have shown that condition (2) implies condition (1). Now suppose $F$ is entire and satisfies the conditions in (1). Let

$$f(x) := F|_{\mathbb{R}}^{-1}(F|_{\mathbb{R}}) = \int_{\mathbb{R}} F(t) e^{2\pi itx} \, dt.$$

As $F|_{\mathbb{R}} \in L^2(\mathbb{R})$ so it $f$. All we need to show is $f(x) = 0$ for $|x| > r$. For $\epsilon > 0$ and $x$ real, let

$$g_\epsilon(x) := F(x) e^{-2\pi \epsilon |x|}.$$ 

Then $g_\epsilon$ is continuous and in $L^2(\mathbb{R})$. It is easy to see that $\|g_\epsilon - F|_{\mathbb{R}}\|_{L^2(\mathbb{R})} \to 0$ as $\epsilon \to 0$. Hence by Plancherel theorem it follows that the inverse Fourier transform of $g_\epsilon$ converges in $L^2$ to $f$ as $\epsilon \to 0$. We will show that for real $x$ with $|x| > r$

$$\lim_{\epsilon \to 0} F|_{\mathbb{R}}^{-1}(g_\epsilon)(x) = \lim_{\epsilon \to 0} \int_{\mathbb{R}} g_\epsilon(t) e^{2\pi itx} \, dt = 0. \quad (2.3.2)$$

Thus (2.3.2) will imply $f(x) = 0$ for $|x| > r$. Moreover, by the injectivity of the Fourier transform

$$F(z) = \int_{-r}^{r} f(t) e^{-2\pi itz} \, dt \quad (2.3.3)$$

for almost all real $z$ and since both sides in the above equation are entire functions, it follows that (2.3.3) holds for all complex $z$. Thus (2.3.2) will imply the theorem. To show the validity of (2.3.2) we introduce auxiliary functions $\Phi_\alpha$ below and express (2.3.2) in terms of functions $\Phi_\alpha$.

For each real $\alpha$, let $\gamma_\alpha$ be the path defined by

$$\gamma_\alpha(s) := se^{i\alpha} \text{ with } 0 \leq s \leq \infty,$$
and let $\Pi_\alpha$ be the half-plane defined by

$$\Pi_\alpha := \{ \omega \in \mathbb{C} : \text{Re}(\omega e^{i\alpha}) > r \}.$$  

For $\omega \in \Pi_\alpha$, define

$$\Phi_\alpha(\omega) := \oint_{\gamma_\alpha} F(z) e^{-2\pi\omega z} dz$$

$$= e^{i\alpha} \int_0^\infty F(se^{i\alpha}) e^{-2\pi\omega(se^{i\alpha})} ds.$$  

Since $\omega \in \Pi_\alpha$, $\text{Re}(\omega e^{i\alpha}) > r$ and so

$$|F(se^{i\alpha}) e^{-2\pi\omega(se^{i\alpha})}| \leq Ce^{2\pi rs} e^{-2\pi s \text{Re}(\omega e^{i\alpha})}$$

$$= Ce^{-2\pi s (\text{Re}(\omega e^{i\alpha}) - r)}$$

$$\in L^1_s(0, \infty).$$

Hence (2.3.4) exists as a Lebesgue integral. Let $\{\omega_n\}$ be a sequence in $\Pi_\alpha$ converging to $\omega$ and let $\delta > 0$ be so that $B_\delta(\omega) \subset \Pi_\alpha$. For $n$ big enough $\omega_ne^{i\alpha} \in B_{\frac{\delta}{2}}(\omega e^{i\alpha})$ and so $\text{Re}(\omega e^{i\alpha}) - \frac{\delta}{2} < \text{Re}(\omega_ne^{i\alpha}) < \text{Re}(\omega e^{i\alpha}) + \frac{\delta}{2}$. Hence

$$|e^{-2\pi\omega_n(se^{i\alpha})} - e^{-2\pi\omega(se^{i\alpha})}| \leq 2e^{-2\pi s (\text{Re}(\omega e^{i\alpha}) - \frac{\delta}{2})}.$$

Consequently,

$$|F(se^{i\alpha})\{e^{-2\pi\omega_n(se^{i\alpha})} - e^{-2\pi\omega(se^{i\alpha})}\}| \leq 2Ce^{-2\pi s (\text{Re}(\omega e^{i\alpha}) - \frac{\delta}{2} - r)} \in L^1_s(0, \infty).$$

By Lebesgue Dominated Convergence theorem it follows that $\Phi_\alpha$ is continuous on $\Pi_\alpha$. Let $\gamma$ be a closed path in $\Pi_\alpha$. By Fubini’s theorem

$$\oint_\gamma \Phi_\alpha(\omega) d\omega = e^{i\alpha} \int_0^\infty F(se^{i\alpha}) \oint_{\gamma} e^{-2\pi\omega(se^{i\alpha})} d\omega ds.$$  

The inner integral is 0 by Cauchy’s theorem and hence by Morera’s theorem $\Phi_\alpha$ is holomorphic on $\Pi_\alpha$. More is true for $\Phi_0$ and $\Phi_\pi$. They are holomorphic on bigger domains than $\Pi_0$ and $\Pi_\pi$:

$$\Phi_0(\omega) = \int_0^\infty F(s)e^{-2\pi\omega s} ds \quad \text{for } \text{Re}(w) > 0,$$

$$\Phi_\pi(\omega) = \int_{-\infty}^0 F(s)e^{-2\pi\omega s} ds \quad \text{for } \text{Re}(w) < 0.$$
It follows by the same argument we used at the beginning of the proof to show $F$ is holomorphic. Moreover,

$$
\Phi_0(\epsilon - it) - \Phi_\pi(-\epsilon - it) = \int_0^\infty F(s)e^{-2\pi(\epsilon - it)s}ds + \int_{-\infty}^0 F(s)e^{-2\pi(-\epsilon - it)s}ds
$$

$$
= \int_0^\infty F(s)e^{2\pi\epsilon|s|}e^{2\pi its}ds + \int_{-\infty}^0 F(s)e^{2\pi\epsilon|s|}e^{2\pi its}ds
$$

$$
= \int_{-\infty}^\infty F(s)e^{2\pi\epsilon|s|}e^{2\pi its}ds
$$

$$
= \int_\mathbb{R} g_\epsilon(s)e^{2\pi its}ds.
$$

Hence we have to show that for $|t| > r$ the difference between $\Phi_0$ and $\Phi_\pi$ tends to 0 as $\epsilon$ goes to 0. To this end we will show that under proper assumptions on $\alpha$ and $\beta$ the functions $\Phi_\alpha$ and $\Phi_\beta$ agree on the intersection of their domains of definition. In other words, they are analytic continuations of each other.

Suppose $0 < \beta - \alpha < \pi$. Put

$$
\xi := \frac{\alpha + \beta}{2} \quad \text{and} \quad \eta := \cos \frac{\beta - \alpha}{2} > 0.
$$

Let $\omega = |\omega|e^{-i\xi}$, then

$$
\text{Re}(\omega e^{i\alpha}) = |\omega| \cos \frac{\alpha - \beta}{2} = \eta |\omega| = |\omega| \cos \frac{\beta - \alpha}{2} = \text{Re}(\omega e^{i\beta}),
$$

so that $\omega \in \Pi_\alpha \cap \Pi_\beta$ as soon as $|\omega| > \frac{r}{\eta}$. Consider the integral

$$
\oint_\gamma F(z)e^{-2\pi \omega z}dz \quad (2.3.5)
$$

over the circular arc $\gamma$ given by $\gamma(t) := Re^{it}$ with $\alpha \leq t \leq \beta$ and some $R > 0$. Since $t - \xi \leq \frac{\beta - \alpha}{2}$ and $0 < \frac{\beta - \alpha}{2} < \frac{\pi}{2}$, for $z \in \gamma$

$$
\text{Re}(-\omega z) = -R|\omega|\text{Re}(e^{i(t-\xi)}) = -R|\omega| \cos(t - \xi) \leq -R|\omega| \eta.
$$

Consequently the absolute value of the integrand in (2.3.5) does not exceed

$$
Ce^{2\pi R(r - |\omega|\eta)}.
$$

So for $|\omega| > \frac{r}{\eta}$ (2.3.5) tends to 0 as $R \to \infty$. Note that by Cauchy’s theorem

$$
\oint_0^{Re^{i\beta}} F(z)e^{-2\pi \omega z}dz = \oint_0^{Re^{i\alpha}} F(z)e^{-2\pi \omega z}dz + \oint_\gamma F(z)e^{-2\pi \omega z}dz.
$$
As (2.3.5) tends to 0 as $R \to \infty$, we conclude that

$$\Phi_\alpha(\omega) = \Phi_\beta(\omega) \text{ for } \omega = |\omega|e^{-i\xi} \text{ and } |\omega| > \frac{r}{\eta}. $$

So $\Phi_\alpha$ and $\Phi_\beta$ coincide on a ray in $\mathbb{C}$, thus by Theorem 2.13 they coincide on the intersection of the half-planes on which they were originally defined. Hence for $t > r$ we have

$$\Phi_0(\epsilon - it) - \Phi_\pi(-\epsilon - it) = \Phi_\pi(\epsilon - it) - \Phi_{-\pi}(-\epsilon - it)$$

and for $t < -r$

$$\Phi_0(\epsilon - it) - \Phi_\pi(-\epsilon - it) = \Phi_{-\pi}(\epsilon - it) - \Phi_{-\pi}(-\epsilon - it).$$

Letting $\epsilon \to 0$ we obtain that the desired limit in (2.3.2) equals 0. This completes the proof. \hfill \Box

There are other Paley-Wiener theorems for $L^2$ functions. We give some examples without proofs.

Let $\Pi^+$ denote the upper half-plane: $\{z = x + iy : y > 0\}$.

**Theorem 2.23.** Let $F$ be a holomorphic function on $\Pi^+$. Then the following two conditions on $F$ are equivalent:

1. Restrictions of $F$ to horizontal lines in $\Pi^+$ form a bounded set in $L^2(\mathbb{R})$.

2. There exist a function $f \in L^2(0, \infty)$ such that $F = \mathcal{F}^c_{\mathbb{R}}(f)$.

**Theorem 2.24.** For $f \in L^2(\mathbb{R})$ the following two conditions are equivalent:

1. $f$ is the restriction to $\mathbb{R}$ of a function $F$ holomorphic on the strip $\{z = x + iy : |y| < a\}$ and satisfying

$$\sup_{-a < y < a} \int_{\mathbb{R}} |F(x + iy)|^2 \, dx = C < \infty.$$ 

2. $e^{a|\xi|} \hat{f} \in L^2(\mathbb{R})$.

### 2.3.3 Paley-Wiener Theorem for Distributions

The last case we are going to discuss is the Paley-Wiener theorem for distributions. First we need to introduce some additional notation.

For a topological vector space $V$, let $V'$ denote the continuous dual of $V$.

**Definition 2.25.** Let $\emptyset \not= \Omega \subset \mathbb{R}^n$ be an open set. The elements of $\mathcal{D}(\Omega)'$ are called distributions or generalized functions. The elements of $\mathcal{D}(\Omega)$ are called test functions.

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3This is the vector space whose elements are continuous linear functionals on $V$. 

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For a distribution $T$ and a test function $\varphi$ we introduce the following notation for the value $T(\varphi)$ of $T$ at $\varphi$:

$$T(\varphi) = \langle \varphi, T \rangle = \int \varphi(x) dT(x).$$

Let $U \subset \Omega \subset \mathbb{R}^n$ be open. We say that a distribution $T$ vanishes on $U$ if $\langle \varphi, T \rangle = 0$ for all test functions $\varphi$ with $\text{supp}(\varphi) \subset U$. Let $U_T$ be the union of all open sets $U$ on which $T$ vanishes. Then $U_T$ is open and hence $\Omega \setminus U_T$ is closed in $\Omega$. Define the support of $T$ by

$$\text{supp}(T) := \Omega \setminus U_T.$$

Recall that since $\mathcal{D}(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{E}(\mathbb{R}^n)' \hookrightarrow \mathcal{S}(\mathbb{R}^n)' \hookrightarrow \mathcal{D}(\mathbb{R}^n)'$.\(^4\)

**Proposition 2.26.** Let $T$ be a distribution, then $T \in \mathcal{E}(\Omega)'$ if and only if $\text{supp}(T)$ is compact in $\Omega$.

**Definition 2.27.** A distribution $T$ on $\mathbb{R}^n$ is a tempered distribution if $T$ has a continuous extension to $\mathcal{S}(\mathbb{R}^n)$. Since this extension is unique, we also call it $T$.

Note, since the Fourier transform is a topological isomorphism of $\mathcal{S}(\mathbb{R}^n)$ with itself, if $T$ is a tempered distribution, then $S$ defined by

$$S(\varphi) := T(\hat{\varphi})$$

is also a tempered distribution.

**Definition 2.28.** For a tempered distribution $T$ define $\mathcal{F}_{\mathbb{R}^n}(T)(\varphi) = \hat{T}(\varphi) := T(\hat{\varphi})$. We call $\hat{T}$ the Fourier transform of $T$.

The Fourier transform is a topological isomorphism of $\mathcal{S}(\mathbb{R}^n)'$ with itself.

**Theorem 2.29.** (Paley-Wiener theorem for distributions) A distribution $T \in \mathcal{D}(\mathbb{R}^n)'$ has support in $\overline{B}_r(0)$ if and only if $\hat{T}$ has an analytic continuation to an entire function on $\mathbb{C}^n$ such that for all $z \in \mathbb{C}^n$

$$|\mathcal{F}_{\mathbb{R}^n}(T)(z)| \leq C(1 + |z|^2)^N e^{2\pi r|\text{Im}(z)|}$$

with some constants $C, N \in \mathbb{N}_0$.

In conclusion we summarize the three Paley-Wiener theorems: A function or a distribution $f$ is supported in a closed ball of radius $r > 0$ centered at the origin if and only if its Fourier transform $\hat{f}$ extends to an entire function on $\mathbb{C}^n$ that is pointwise bounded by

$$|\mathcal{F}_{\mathbb{R}^n}(f)(\lambda)| \leq \begin{cases} C(1 + |\lambda|^2)^{-N} e^{2\pi r|\text{Im}(\lambda)|} & \text{for all } N \in \mathbb{N}_0 \text{ if } f \text{ is smooth,} \\ Ce^{2\pi r|\text{Im}(\lambda)|} & \text{if } f, \hat{f} \in L^2, \\ C(1 + |\lambda|^2)^N e^{2\pi r|\text{Im}(\lambda)|} & \text{for some } N \in \mathbb{N}_0 \text{ if } f \text{ is distribution.} \end{cases}$$

\(^4\)Here $\hookrightarrow$ stands for a continuous injection.
for some constant $C > 0$. We see that essentially the exponential factor $e^{2\pi r|\text{Im}\lambda|}$ in the pointwise estimate of the extended Fourier transform yields in the compact support of $f$ and the polynomial factor $(1 + |\lambda|^2)^N$ in the estimate is related to the regularity properties of $f$.

In the next chapter we will extend the classical Paley-Wiener theorem to vector valued functions.
Chapter 3
Paley-Wiener Theorems for Vector Valued Functions

In this chapter we present our first result: the extension of the classical Paley-Wiener theorem to functions with values in a separable Hilbert space. It then reduces to the classical result by choosing the Hilbert space to be one dimensional.

We begin by introducing the necessary notation and background in Section 3.1. There we first recall some standard theorems from Functional Analysis including vector valued integration. [35] and [26] are main references for this part. Then we recall the notions of holomorphic vector valued functions and holomorphic functions on a manifold, for more details on this concepts we refer to [5], [41]. In Section 3.2 we show that some particular spaces of weakly smooth and strongly smooth vector-valued functions are equal, by showing that they are isomorphic to the same Paley-Wiener space. Then we take a look at the special case of functions with values in the space of square integrable functions on the sphere.

3.1 Preliminaries

Suppose $X$ is a topological vector space whose dual $X'$ separates points in $X$. This happens in every locally convex topological vector space.

**Definition 3.1.** A set $E \subset X$ is weakly bounded if and only if every $\Lambda \in X'$ is a bounded function on $E$.

**Theorem 3.2.** In a locally convex space $X$, every weakly bounded set is bounded, and vice versa.

**Theorem 3.3.** (Banach-Steinhaus, Principle of Uniform Boundedness) Suppose $X$ and $Y$ are topological vector spaces, $\mathfrak{F}$ is a collection of continuous linear mappings from $X$ into $Y$, and $B$ is the set of all $x \in X$ whose orbits $\mathfrak{F}(x) := \{\Lambda x : \Lambda \in \mathfrak{F}\}$ are bounded in $Y$. If $B$ is of the second category in $X$, then $B = X$ and $\mathfrak{F}$ is equicontinuous.

In the following we shall rather need the following easy corollary of the Banach-Steinhaus theorem:

**Corollary 3.4.** Let $X$ be a Banach space and let $\mathfrak{F}$ be a family of bounded linear transformations from $X$ to some normed linear space $Y$. Suppose that for each $x \in X$, $\{\|Tx\|_Y : T \in \mathfrak{F}\}$ is bounded. Then $\{\|T\| : T \in \mathfrak{F}\}$ is bounded.

**Remark 3.5.** Note, in particular the above corollary implies that every weakly convergent sequence is norm bounded.

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$^1$ $X'$ denotes the continuous dual of $X$. 

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Definition 3.6. (a) Let $X$ be a vector space and $E \subset X$. Denote by $\text{co}(E)$ the intersection of all convex subsets of $X$ containing $E$. $\text{co}(E)$ is called the \textbf{convex hull} of $E$.

(b) Let $X$ be a topological vector space and $E \subset X$. The \textbf{closed convex hull} of $E$, denoted by $\overline{\text{co}}(E)$, is the closure of $\text{co}(E)$.

(c) A subset $E$ of a metric space $X$ is said to be \textbf{totally bounded} if $E$ lies in the union of finitely many open balls of radius $\epsilon$, for every $\epsilon > 0$.

Theorem 3.7. (a) If $X$ is a locally convex topological vector space and $E \subset X$ is totally bounded, then the $\text{co}(E)$ is totally bounded.

(b) If $X$ is a Fréchet space and $E \subset X$ is compact, then the $\overline{\text{co}}(E)$ is compact.

Definition 3.8. A topological vector space $X$ is called \textbf{separated} if for each $0 \neq x \in X$ there is an open neighborhood $U$ of 0 with $x \notin U$.

Proposition 3.9. A topological vector space is separated if and only if it is Hausdorff.

Proposition 3.10. Let $X$ be a locally convex topological vector space whose topology is given by seminorms $|| \cdot ||_\alpha$, $\alpha \in A$. The space $X$ is separated if and only if the assumption $||x||_\alpha = 0$ for all $\alpha \in A$ implies $x = 0$.

Theorem 3.11. A locally convex topological vector space is metrizable if and only if it is Hausdorff and its topology is given by a countable family of seminorms.

Since we will work with vector valued functions in this chapter, we briefly review vector valued integration.

Definition 3.12. Let $\mu$ be a measure on a measure space $Q$, $X$ a topological vector space on which $X'$ separates points, and let $f$ be a function from $Q$ to $X$ such that the scalar functions $\Lambda f$ are integrable with respect to $\mu$ for every $\Lambda \in X'$. Note, $\Lambda f$ is defined by

$$(\Lambda f)(q) := \Lambda(f(q)) \quad \text{for } q \in Q.$$ 

If there exists a vector $y \in X$ such that

$$\Lambda y = \int_Q (\Lambda f)(q) d\mu(q)$$

for every $\Lambda \in X'$, then we define

$$\int_Q f(q) d\mu(q) := y.$$ 

\footnote{Suppose $|| \cdot ||$ is a norm, then $||x|| = 0$ implies $x = 0$. Separatedness is an analogue of this property for a family of seminorms.}

\footnote{This characterization of an integral is called Pettis integral or Gelfand-Pettis integral and is due to I. M. Gelfand ([22]) and B. J. Pettis ([31]). In contrast to the Bochner integral, this integral is not constructed as a limit of finite sums and it therefore also called the weak integral. For a short overview on vector-valued integrals see a note by P. Garrett: [9].}
Remark 3.13. Since the dual space of $X$ separates points, it follows that such a vector $y$ is unique.

Theorem 3.14. Let $X$ be a topological vector space on which $X'$ separates points and let $\mu$ be a Borel probability measure\(^4\) on a compact Hausdorff space $Q$. If $f : Q \to X$ is continuous and if $\overline{\text{co}}(f(Q))$ is compact in $X$, then the integral

$$y = \int_Q f \, d\mu$$

exists in the sense of the above definition. Moreover, $y \in \overline{\text{co}}(f(Q))$.

In Chapter one we recalled some basic facts about holomorphic complex valued functions. Now we expand the concept of holomorphic functions from complex valued to vector valued. In this general setting there are two natural definitions available: 'weakly' and 'strongly' holomorphic.

Definition 3.15. Let $\Omega$ be an open non-empty set in $\mathbb{C}^n$ and let $X$ be a complex topological vector space.

(a) A function $f : \Omega \to X$ is said to be weakly holomorphic in $\Omega$ if $\Lambda f$ is holomorphic in the ordinary sense for every $\Lambda \in X'$.

(b) A function $f : \Omega \to X$ is said to be strongly holomorphic in $\Omega$ if

$$\lim_{\omega \to z} \frac{f(\omega) - f(z)}{|\omega - z|}$$

exists in the topology of $X$ for every $z \in \Omega$.

The continuity of the functionals $\Lambda \in X'$ makes it clear that every strongly holomorphic function is also weakly holomorphic. The converse is true when $X$ is a Fréchet space.

Theorem 3.16. Let $\Omega$ be an open set in $\mathbb{C}^n$, $X$ a complex Fréchet space, and let $f : \Omega \to X$ be weakly holomorphic. The following conclusions hold:

(a) $f$ is strongly continuous in $\Omega$.\(^5\)

(b) $f$ is strongly holomorphic in $\Omega$.

Let $\Omega$ be an open non-empty connected subset of $\mathbb{C}^n$. Denote by $\mathcal{O}(\Omega)$ the space of holomorphic functions on $\Omega$. Let the topology on this space be the topology of uniform convergence on compact sets, this means it is given by seminorms

$$|F|_K := \sup_{z \in K} |F(z)|$$

where $K$ is a non-empty compact subset of $\Omega$.

---

\(^4\)Recall that a Borel measure on a locally compact Hausdorff space $Q$ is a measure defined on the $\sigma-$algebra of all Borel sets in $Q$; this is the smallest $\sigma-$algebra containing all open subsets of $Q$. A probability measure is a positive measure of total mass 1.

\(^5\)Here 'strongly' means with respect to the topologies on $\Omega$ and $X$. 


Theorem 3.17. (Weierstrass Convergence Theorem) Let $\{F_i\}$ be a sequence of holomorphic functions on $\Omega$ that converges uniformly over compact subsets of $\Omega$ to a function $F$, then $F$ is holomorphic.

Later we will work in a more general setting of holomorphic functions on a manifold. We introduce the relevant notation here.

Definition 3.18. A topological space $\mathcal{M}$ is locally Euclidean if for every $m \in \mathcal{M}$ there is an open subset $U$ of $\mathcal{M}$ containing $m$ and an open subset $\Omega$ of $\mathbb{R}^n$ and a homeomorphism $\varphi : U \to \Omega$. We call $\varphi$ a coordinate map and the functions $x_i = \pi_i \circ \varphi$ are called the coordinate functions, where $\pi_i$ denotes the projection onto the $i^{th}$ coordinate. We call the pair $(U, \varphi)$ a system of coordinates around $m$.

Definition 3.19. A differentiable/$C^k$/smooth/analytic structure $\mathcal{A}$ on a locally Euclidean space $\mathcal{M}$ is a collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ of coordinates such that the following two conditions are satisfied:

1. $\{U_\alpha\}$ is a covering of $\mathcal{M}$, that is, $\mathcal{M} = \bigcup_{\alpha \in A} U_\alpha$.
2. For $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$ the map
   $$\varphi_\alpha \circ \varphi^{-1}_\beta : \varphi_\beta(U_\alpha \cap U_\beta) \to \mathbb{R}^n$$
   is differentiable/$C^k$/smooth/analytic.

We call the collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ an atlas. It is a maximal atlas if for an open subset $U$ of $\mathcal{M}$ and a homeomorphism $\varphi : U \to \varphi(U)$ satisfying that for every $\alpha \in A$ with $U \cap U_\alpha \neq \emptyset$ the map $\varphi \circ \varphi^{-1}_\alpha : \varphi_\alpha(U \cap U_\alpha) \to \mathbb{R}^n$ is differentiable/$C^k$/smooth/analytic implies that $(U, \alpha)$ belongs to $\mathcal{A}$.

Definition 3.20. A differentiable/$C^k$/smooth/analytic manifold is an Euclidean space $\mathcal{M}$ with a maximal differentiable/$C^k$/smooth/analytic structure.

Example 3.21. 1. Let $\Omega$ be an open non-empty subset of $\mathbb{R}^n$, then $\{(\Omega, \text{id})\}$ is an atlas for $\Omega$. Hence $\Omega$ is a manifold.

2. Let $\mathcal{M}$ and $\mathcal{N}$ be two differentiable/$C^k$/smooth/analytic manifolds of dimensions $m$ and $n$ respectively. And let the collections $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ and $\{(U_\beta, \varphi_\beta)\}_{\beta \in B}$ be the atlases for the manifolds $\mathcal{M}$ and $\mathcal{N}$ respectively. Then $\mathcal{M} \times \mathcal{N}$ is a differentiable/$C^k$/smooth/analytic manifold of dimension $m + n$ with atlas $\{(U_\alpha \times U_\beta, \varphi_\alpha \times \varphi_\beta)\}_{\alpha \in A, \beta \in B}$, where
   $$\varphi_\alpha \times \varphi_\beta : U_\alpha \times U_\beta \to \mathbb{R}^{m+n} : (a, b) \mapsto (\varphi_\alpha(a), \varphi_\beta(b)).$$

Definition 3.22. Let $\mathcal{M}, \mathcal{N}$ be smooth manifolds. A continuous map $f : \mathcal{M} \to \mathcal{N}$ is smooth if $\varphi \circ f \circ \tau^{-1}$ is smooth for each coordinate map $\tau$ on $\mathcal{M}$ and $\varphi$ on $\mathcal{N}$. 
Let $\mathcal{M}$ be a smooth manifold of dimension $d$, $(U, \varphi)$ be a system of coordinates with coordinate functions $\{x_1, \ldots, x_d\}$, and $m \in U$. For each $i \in \{1, \ldots, d\}$, define a tangent vector $(\partial/\partial x_i)|_m$ at $m$ by setting

$$\left(\frac{\partial}{\partial x_i}\right|_m (f) = \frac{\partial(f \circ \varphi^{-1})}{\partial x_i}\big|_{\varphi(m)}$$

for each function $f$ which is smooth on a neighborhood of $m$. Let $\mathcal{M}_m$ denote the linear span of $(\partial/\partial x_i)|_m$. We call $\mathcal{M}_m$ the tangent space of $\mathcal{M}$ at $m$.

Let $\psi : \mathcal{M} \to \mathcal{N}$ be smooth, and let $m \in \mathcal{M}$. The differential of $\psi$ at $m$ is a linear map

$$d\psi_m : \mathcal{M}_m \to \mathcal{N}_{\psi(m)}$$

defined as follows. Let $g$ be a smooth function on a neighborhood of $\psi(m)$. Define $d\psi_m(v)(g)$ by setting

$$d\psi_m(v)(g) = v(g \circ \psi).$$

The map $\psi$ is called non-singular at $m$ if $d\psi_m$ is non-singular, that is, if the kernel of $d\psi_m$ consists of $0$ alone.

**Definition 3.23.** Let $\psi : \mathcal{M} \to \mathcal{N}$ be smooth.

(a) $\psi$ is an immersion if $d\psi_m$ is non-singular for each $m \in \mathcal{M}$.

(b) The pair $(\mathcal{M}, \psi)$ is a submanifold of $\mathcal{N}$ if $\psi$ is a one-to-one immersion.

Let $\mathcal{M}$ be a smooth $2n$-dimensional manifold. Then $\mathcal{M}$ is a complex manifold if there is an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ such that for every $\alpha$ in the index set $\mathcal{A}$ we have $\varphi_\alpha(U_\alpha) = V_\alpha \subset \mathbb{C}^n$ and for $\alpha, \beta \in \mathcal{A}$ with $U_\alpha \cap U_\beta \neq \emptyset$ the map

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \mathbb{C}^n$$

is holomorphic. Then we say that the collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ is a complex atlas for the manifold $\mathcal{M}$.

**Definition 3.24.** If $\mathcal{M}$ is a complex manifold and $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ is a complex atlas, then $F : \mathcal{M} \to \mathbb{C}$ is holomorphic if $F$ is continuous and $F \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to \mathbb{C}$ is holomorphic for every $\alpha \in \mathcal{A}$. Denote by $\mathcal{O}(\mathcal{M})$ the space of holomorphic functions on $\mathcal{M}$. Give $\mathcal{O}(\mathcal{M})$ the topology of uniform convergence over compact subsets of $\mathcal{M}$.

**Theorem 3.25.** The space $\mathcal{O}(\mathcal{M})$ is complete.

**Proof.** Suppose $F_n \to F$ in $\mathcal{O}(\mathcal{M})$. Fix $(U_\alpha, \varphi_\alpha)$ in the complex atlas. Let $K \subset V_\alpha$ be compact, then $\varphi_\alpha^{-1}(K) \subset U_\alpha$ is compact. Hence $F_n \circ \varphi_\alpha^{-1} \to F \circ \varphi_\alpha^{-1}$ uniformly on $K$. By Weierstrass Convergence theorem the limit function $F \circ \varphi_\alpha^{-1}$ is holomorphic on $V_\alpha$. Hence $F$ is holomorphic and so $\mathcal{O}(\mathcal{M})$ is complete. \qed

**Definition 3.26.** Let $\mathcal{M}_C$ be a complex manifold. A real submanifold $\mathcal{M}$ of $\mathcal{M}_C$ is called totally real if for any function $F$ in $\mathcal{O}(\mathcal{M}_C)$ the condition $F|_{\mathcal{M}} \equiv 0$ implies $F \equiv 0$.

**Example 3.27.** $\mathbb{R}^n$ is a totally real submanifold of $\mathbb{C}^n$.  

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3.2 Paley-Wiener Type Theorems for Vector Valued Functions on $\mathbb{R}^n$

Let $\mathcal{H}$ denote a complex separable Hilbert space with a complete orthonormal set $\{e_i\}_{i \in J}$, where $J$ is a finite or a countably infinite index set. The norm in $\mathcal{H}$ is denoted by $\| \cdot \|$, and the inner-product of two elements $u, v \in \mathcal{H}$ is denoted by $\langle u, v \rangle$.

**Definition 3.28.** Let $r > 0$. The space of $\mathcal{H}$-valued functions $\varphi : \mathbb{R}^n \to \mathcal{H}$ such that for every $u \in \mathcal{H}$ the complex valued function, $x \mapsto \langle \varphi(x), u \rangle$ belongs to $\mathcal{D}_r(\mathbb{R}^n)$, is denoted by $\mathcal{D}_r^\mathcal{H} = \mathcal{D}_r(\mathbb{R}^n, \mathcal{H})$. We let the topology on $\mathcal{D}_r^\mathcal{H}$ be given by the seminorms

$$\nu_{N,u}(\varphi) := \max_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^n} |D^\alpha \langle \varphi(x), u \rangle|,$$

with $\alpha \in \mathbb{N}_0^n$, $N \in \mathbb{N}_0$, and $u \in \mathcal{H}$.

Restricting $u$ to the elements in the orthonormal basis $\{e_i\}$ gives the same topology.

**Definition 3.29.** We define another similar space of $\mathcal{H}$-valued functions on $\mathbb{R}^n$, denoted by $\mathcal{D}_r^\mathcal{H,c} = \mathcal{D}_r,c(\mathbb{R}^n, \mathcal{H})$, as the space of functions $\varphi$ in $\mathcal{D}_r^\mathcal{H}$ with an extra assumption that $\varphi : \mathbb{R}^n \to \mathcal{H}$ is continuous. We will later show that these two spaces are equal.

**Definition 3.30.** Denote by $\mathcal{PW}^\mathcal{H}_r = \mathcal{PW}_r(C^n, \mathcal{H})$ the space of weakly holomorphic functions $F : C^n \to \mathcal{H}$, which satisfy that for every $u \in \mathcal{H}$ and $N \in \mathbb{N}_0$

$$\rho_{r,N,u}(F) := \sup_{z \in \mathbb{C}^n} (1 + |z|^2)^N e^{-r|\text{Im}(z)|} |\langle F(z), u \rangle| < \infty.$$  

Let $\mathcal{PW}^\mathcal{H}_r$ be topologized by the seminorms $\rho_{r,N,u}$.

Again, it is enough to use a countable family of seminorms $\{\rho_{r,N,e_j}\}_{j \in J}$.

**Lemma 3.31.** The space $\mathcal{PW}^\mathcal{H}_r$ and its topology can be defined using the seminorms

$$\rho_{r,N}(F) := \sup_{z \in \mathbb{C}^n} (1 + |z|^2)^N e^{-r|\text{Im}(z)|} \|F(z)\|,$$

with $N \in \mathbb{N}_0$.

**Proof.** It is clear that if $\rho_{r,N}(F) < \infty$, then $\rho_{r,N,u}(F) \leq \|u\| \rho_{r,N}(F) < \infty$. For the other direction, let $E := \{ (1 + |z|^2)^N e^{-r|\text{Im}(z)|} |F(z)| : z \in \mathbb{C}^n \}$. From the assumption it follows that the set $E$ is weakly bounded. Moreover, $\mathcal{H}$ being a Hilbert space is locally convex and thus by Theorem 3.2 the set $E$ is bounded. Hence the seminorms $\rho_{r,N,u}$ can be replaced by the seminorms $\rho_{r,N}$. \qed

**Proposition 3.32.** For $r > 0$, the spaces $\mathcal{D}_r^\mathcal{H}$ and $\mathcal{PW}^\mathcal{H}_r$ are Fréchet spaces, i.e. complete, metrizable, locally convex vector spaces.
Proof. They are locally convex spaces as their topologies are defined by seminorms. We will show that both spaces are Hausdorff by arguing that they are separated. Suppose \( \nu_{N,u}(\varphi) = 0 \) for all \( N \in \mathbb{N}_0 \) and all \( u \in \mathcal{H} \). Then in particular, \( \langle \varphi(x), u \rangle = 0 \) for all \( u \) and all \( x \). Hence \( \varphi(x) = 0 \) for every \( x \). So \( \varphi \equiv 0 \). That is, \( D_r^H \) is indeed separated. Suppose \( \rho_{r,N}(F) = 0 \) for some \( N \). This implies \( \sup_{z \in \mathbb{C}^n} \|F(z)\| = 0 \), and consequently \( F \equiv 0 \). We conclude that seminorms on \( \mathcal{P}W_r^H \) are in fact norms. Hence, \( \mathcal{P}W_r^H \) is clearly separated.

As both topologies are defined by countable collections of seminorms, they are metrizable.

Let \( \{\varphi_i\}_{i \in \mathbb{N}} \) be a Cauchy sequence in \( D_r^H \). For \( u \in \mathcal{H} \) define
\[
\varphi_i^n(x) := \langle \varphi_i(x), u \rangle .
\]
Then \( \varphi_i^n \) is a Cauchy sequence in \( D_r(\mathbb{R}^n) \) and hence converges to some \( \varphi^n \in D_r(\mathbb{R}^n) \), by Proposition 2.15. Fix a vector \( x \in \mathbb{R}^n \). Since uniform convergence implies pointwise convergence, it follows that the sequence \( \{\varphi_i(x)\} \) is weakly Cauchy in \( \mathcal{H} \). Hence by an application of the Principle of Uniform Boundedness (see Corollary 3.4) this sequence is norm bounded: there is a constant \( C > 0 \) with \( \|\varphi_i(x)\| \leq C < \infty \) for any \( i \in \mathbb{N} \). We have
\[
|\varphi^n(x)| \leq |\varphi^n - \varphi_i^n(x)| + |\varphi_i^n(x)| ,
\]
choose \( N \) so that \( \sup_{x \in \mathbb{R}^n} |\varphi^n(x) - \varphi_i^n(x)| \leq \|u\| \) for \( i \geq N \), then
\[
\leq \|u\| + \|\varphi_i(x)\| \|u\| = (1 + \|\varphi_i(x)\|) \|u\| \leq (1 + C) \|u\| .
\]
This shows that the functional \( T(u) := \varphi^n(x) \) from \( \mathcal{H} \) into \( \mathbb{C} \) is bounded and hence by Riesz Representation theorem there is an element \( u_x \) in \( \mathcal{H} \) with \( T(u) = \langle u_x, u \rangle = \varphi^n(x) \). Define a vector valued function \( \varphi \) on \( \mathbb{R}^n \) by
\[
\varphi(x) := u_x ,
\]
then for every \( u \in \mathcal{H} \), \( \langle \varphi(x), u \rangle = \varphi^n(x) \in D_r(\mathbb{R}^n) \). Thus \( \varphi \in D_r^H \) and \( \{\varphi_i\} \) converges to \( \varphi \) in the topology of \( D_r^H \).

The completeness of \( \mathcal{P}W_r^H \) can be verified by a similar argument. However, in this case it can be shown more directly. Let \( \{F_i\}_{i \in \mathbb{N}} \) be a Cauchy sequence in \( \mathcal{P}W_r^H \). Then it is Cauchy and hence bounded with respect to each seminorm \( \rho_{r,N} \); that is, for each \( N \) \( \{\rho_{r,N}(F_i)\} \) is bounded by some constant, say \( M_N \). For \( u \in \mathcal{H} \) define
\[
F_i^u(z) := \langle F_i(z), u \rangle .
\]
Then \( F_i^u \) is Cauchy in \( \mathcal{P}W_r(\mathbb{C}^n) \) and hence converges to some \( F^u \in \mathcal{P}W_r(\mathbb{C}^n) \). Fix a vector \( z \in \mathbb{C}^n \), then
\[
\|F_i^u(z)\| \leq \rho_{r,0}(F_i)e^{r|\text{Im}(z)|} \leq M_0 e^{r|\text{Im}(z)|} =: M.
\]
Hence as before

\[ |F^u(z)| \leq |F^u(z) - F_i^u(z)| + |F_i^u(z)| \]
\[ \leq \|u\| + \|F_i(z)\| \|u\| \]

for \( i \) big enough,

\[ \leq (1 + M)\|u\|. \]

Thus for every \( z \in \mathbb{C}^n \) there is an element \( u_z \) in \( \mathcal{H} \) with \( F^u(z) = \langle u_z, u \rangle \). So there is a vector valued function \( F \) on \( \mathbb{C}^n \) such that for every \( u \in \mathcal{H} \), \( \langle F(z), u \rangle = F^u(z) \in \mathcal{PW}_r(\mathbb{C}^n) \). Thus \( F \in \mathcal{PW}_r^\mathcal{H} \) and \( \{F_i\} \) converges to \( F \) in the topology of \( \mathcal{PW}_r^\mathcal{H} \).

This shows, both spaces are complete. \( \square \)

**Lemma 3.33.** Let \( \varphi \in D_r^\mathcal{H} \) and \( z \in \mathbb{C}^n \), then \( x \mapsto \varphi(x)e^{-iz \cdot x} \) is weakly integrable and

\[ \left| \int_{\mathbb{R}^n} \langle \varphi(x), u \rangle e^{-iz \cdot x} \, dx \right| \leq \text{Vol}(B_r(0))\|\varphi\|_\infty\|u\|e^{r|\text{Im}z|}. \]  

(3.2.3)

**Proof.** This follows from the estimate

\[ |\langle \varphi(x), u \rangle e^{-iz \cdot x}| \leq \|\varphi(x)\|\|u\|e^{r|\text{Im}z|} \chi_{B_r(0)} \]

as the set \( \{\varphi(x)\} \) is weakly bounded by assumption, it is bounded, thus

\[ \leq \|\varphi\|_\infty\|u\|e^{r|\text{Im}z|} \chi_{B_r(0)}. \]  

(3.2.4)

\[ \square \]

### 3.2.1 The Paley-Wiener Theorem for \( D_{r,c}^\mathcal{H} \)

We define the Fourier transform of \( \varphi \in D_r^\mathcal{H} \) as the weak integral

\[ \hat{\varphi}(y) = \mathcal{F}(\varphi)(y) := \int_{\mathbb{R}^n} \varphi(x)e^{-2\pi i x \cdot y} \, dx. \]

**Theorem 3.34.** (Paley-Wiener type theorem for \( D_{r,c}^\mathcal{H} \)) If \( \varphi \in D_{r,c}^\mathcal{H} \), then \( \mathcal{F}\varphi \) extends to a weakly holomorphic function on \( \mathbb{C}^n \), denoted by \( \mathcal{F}c(\varphi) \). Moreover, \( \mathcal{F}c(\varphi) \) belongs to \( \mathcal{PW}_{2\pi r}^\mathcal{H} \). Furthermore, the Fourier transform \( \mathcal{F}c \) is a linear topological isomorphism of \( D_{r,c}^\mathcal{H} \) onto \( \mathcal{PW}_{2\pi r}^\mathcal{H} \). The inverse of \( \mathcal{F}c \) is given by the conjugate weak Fourier transform on \( \mathbb{R}^n \).

**Proof.** Let \( \varphi \) be in \( D_{r,c}(\mathbb{R}^n, \mathcal{H}) \). Then equation (3.2.4) implies that the integral \( \int \langle \varphi(x), u \rangle e^{-2\pi i z \cdot x} \, dx \) converges uniformly on every compact subset of \( \mathbb{C}^n \) and is therefore holomorphic as a function of \( z \). As \( \varphi(x) = 0 \) for \( |x| > r \) and is continuous, the image of \( \mathbb{R}^n \) under \( \varphi(\cdot)e^{-2\pi i (\cdot, \cdot)} \) is compact. Hence by Theorem 3.7 the closure of the convex hull of this set is compact in \( \mathcal{H} \) as well. Thus Theorem 3.14 applies and implies that the integral \( \int \varphi(x)e^{-2\pi i z \cdot x} \, dx \) converges to a vector in \( \mathcal{H} \), call it
\(F(z)\). This defines a mapping \(F\) from \(C^n\) into \(\mathcal{H}\). Note \(F = F^c(\varphi)\). By above we see that \(F\) is weakly holomorphic. Moreover, applying partial integration we obtain as before
\[
(2\pi)^{|\alpha|}|z^\alpha| |\langle F(z), u \rangle| \leq \text{vol}(\mathcal{B}_r) \|D_x^\alpha \langle \varphi(x), u \rangle\|_\infty e^{2\pi r|\text{Im}(z)|}.
\]

Expanding \((1 + |z|^2)^N = \sum_{|\alpha| \leq 2N} a_\alpha |z_1|^{|\alpha_1|} \cdots |z_n|^{|\alpha_n|}\), we see that
\[
(1 + |z|^2)^N e^{-2\pi r|\text{Im}(z)|} |\langle F(z), u \rangle| \leq \text{vol}(\mathcal{B}_r) \sum_{|\alpha| \leq 2N} a_\alpha (2\pi)^{-|\alpha|} \|D_x^\alpha \langle \varphi(x), u \rangle\|_\infty \nu_{2N,u}(\varphi).
\]

Thus with the constant \(C = \text{vol}(\mathcal{B}_r) \sum_{|\alpha| \leq 2N} a_\alpha (2\pi)^{-|\alpha|}\),
\[
\rho_{2\pi r,N,u}(F) \leq C \nu_{2N,u}(\varphi) < \infty.
\]

In particular, \(F \in \mathcal{PW}_{2\pi r}^n\) and \(F^c : \mathcal{D}^r_{2\pi r} \rightarrow \mathcal{PW}_{2\pi r}^n\) is continuous.

To show surjectivity, let \(F \in \mathcal{PW}_{2\pi r}^n\). Then \(F^u(z) := \langle F(z), u \rangle\) is in \(\mathcal{PW}_{2\pi r}(C^n)\), and hence \(\varphi^u := F^{-1}_{\mathbb{R}^n}(F^u)\) is in \(\mathcal{D}_{\mathbb{R}^n}\) by the classical Paley-Wiener theorem. Let \(k > n/2\), so that \(x \mapsto (1 + |x|^2)^{-k}\) is integrable. Then
\[
\int |\langle F(x), u \rangle| dx \leq \left( \rho_{2\pi r,k}(F) \int (1 + |x|^2)^{-k} dx \right) \|u\|.
\]

Hence the conjugate linear functional on \(\mathcal{H}\) into \(\mathbb{C}\) defined by
\[
T(u) := \int_{\mathbb{R}^n} \langle F(x), u \rangle e^{2\pi i x \cdot y} dx
\]
is bounded for any \(y \in \mathbb{R}^n\). Consequently, there exists an element \(g^u\) in \(\mathcal{H}\) such that
\[
T(u) = \langle g^u, u \rangle = \int \langle F(x), u \rangle e^{2\pi i x \cdot y} dx.
\]
That is, by Definition 3.12
\[
g^u = \int_{\mathbb{R}^n} F(x)e^{2\pi i x \cdot y} dx.
\]

This shows that the inverse Fourier transform of the restriction of the function \(F\) to \(\mathbb{R}^n\) exists as a weak integral. By an application of the Hahn-Banach theorem there is an element \(g\) in \(\mathcal{H}\) with \(\langle g^u, g \rangle = \|g^u\|\) and \(|\langle u, g \rangle| \leq \|u\|\) for all \(u\) in \(\mathcal{H}\).

In particular,
\[
|\langle F(x)e^{2\pi i x \cdot y}, g \rangle| \leq \|F(x)\|.
\]
Hence, we get
\[ \|g_y\| = \langle g_y, g \rangle = \int_{\mathbb{R}^n} \langle F(x), g \rangle e^{2\pi ix \cdot y} dx \leq \int_{\mathbb{R}^n} \|F(x)\| dx < \infty. \]

This shows, \( \left\| \int F(x) e^{2\pi i x \cdot y} dx \right\| \leq \int_{\mathbb{R}^n} \|F(x)\| dx \). We will use this to show that the function
\[ \varphi(y) := g_y = F^{-1}(F)(y) = \int_{\mathbb{R}^n} F(x) e^{2\pi i x \cdot y} dx \]
is continuous. Consider the difference
\[ \|\varphi(a) - \varphi(b)\| = \left\| \int_{\mathbb{R}^n} F(x) \{ e^{2\pi i x \cdot a} - e^{2\pi i x \cdot b} \} dx \right\| \leq \int_{\mathbb{R}^n} \|F(x)\| \|\{ e^{2\pi i x \cdot a} - e^{2\pi i x \cdot b} \}\| dx = \int_{\mathbb{R}^n} \|F(x)\| |\{ e^{2\pi i x \cdot a} - e^{2\pi i x \cdot b} \}| dx. \]

The function \( y \to e^{2\pi i x \cdot y} \) is continuous as a composition of two continuous functions and hence letting \( a \) approach \( b \) we get,
\[ \lim_{a \to b} \|\varphi(a) - \varphi(b)\| = 0. \]

Finally,
\[ |D^\alpha \langle \varphi(x), u \rangle| = \left| \int_{\mathbb{R}^n} \langle F(y), u \rangle (2\pi i y)^\alpha e^{2\pi i y \cdot x} dy \right| \leq (2\pi)^{|\alpha|} \int_{\mathbb{R}^n} |\langle F(y), u \rangle| |y^\alpha| dy \]
and for any \( N \)
\[ \leq (2\pi)^{|\alpha|} \rho_{2\pi r, N, u}(F) \int_{\mathbb{R}^n} |y^\alpha|(1 + |y|^2)^{-N} dy, \]
and so for \( N \) big enough it is finite. This shows that \( \varphi \in D^r_{r,c} \) and the map \( F \mapsto \varphi \) is continuous. The claim now follows as \( (\mathcal{F}^c)^{-1} \circ \mathcal{F}^c = \text{id}_{D^r_{r,c}} \) and \( \mathcal{F}^c \circ (\mathcal{F}^c)^{-1} = \text{id}_{PW_{2\pi r}}. \)

**Remark 3.35.** By examining the proof we conclude that as a consequence of part (b) of Theorem 3.7 first part of the theorem holds for Fréchet spaces.
Remark 3.36. The theorem still holds if instead of imposing the extra condition that \( \varphi : \mathbb{R}^n \to \mathcal{H} \) is continuous we require that it is smooth. The first part of the proof is clearly valid as smooth functions are continuous. For the surjectivity part we have to argue that \( \varphi \) is smooth, namely \( D^alpha \varphi(x) = \int_{\mathbb{R}^n} F(y)(2\pi iy)^\alpha e^{2\pi iy \cdot x} \, dy \).

Since \( \|F(y)\| \leq |\rho_{2\pi r,N}(F)(1 + |y|^2)^{-N} \) for any \( N \in \mathbb{N} \), it follows that for any \( N \)

\[
\int |\langle F(y), u \rangle | (2\pi iy)^\alpha |dy| \leq \left( |\rho_{2\pi r,N}(F)(2\pi)|^\alpha \int |y|^\alpha (1 + |y|^2)^{-N} \, dy \right) \|u\|.
\]

Consequently, \( T(u) := \int_{\mathbb{R}^n} \langle F(y), u \rangle (2\pi iy)^\alpha e^{2\pi iy \cdot x} \, dy \) is a bounded conjugate linear operator on \( \mathcal{H} \) and so there is an element \( g^\alpha_x \) in \( \mathcal{H} \) with

\[
T(u) = \langle g^\alpha_x, u \rangle = \int \langle F(y), u \rangle (2\pi iy)^\alpha e^{2\pi iy \cdot x} \, dy.
\]

Thus \( g^\alpha_x = \int_{\mathbb{R}^n} F(y)(2\pi iy)^\alpha e^{2\pi iy \cdot x} \, dy \). As above we have

\[
\left\| \int F(y)(2\pi iy)^\alpha e^{2\pi iy \cdot x} \, dy \right\| \leq \int_{\mathbb{R}^n} \|F(y)\| (2\pi)|^\alpha |y|^\alpha |dy|.
\]

To show that \( \varphi \) is differentiable, it is enough to show \( \frac{\partial}{\partial x_1} \varphi(x) \) exists end then argue inductively. Let \( t = (t, 0, \ldots, 0) \), we have to show

\[
\lim_{t \to 0} \left\| \frac{\varphi(x + t) - \varphi(x)}{t} - g^{(1,0,\ldots,0)} \right\| = 0.
\]

By the Mean Value theorem \( \frac{e^{2\pi iy \cdot x + t} - e^{2\pi iy \cdot x}}{t} = (2\pi iy_1) e^{2\pi iy \cdot b} \) for some \( b \in \mathbb{R}^n \) with \( b_i = x_i \) for \( i \neq 1 \) and \( b_1 \in (x_1, x_1 + t) \). Since \( \|F(y)\| (2\pi iy_1) e^{2\pi iy \cdot b} \in L^1_y(\mathbb{R}^n) \) we can apply the Lebesgue Dominated Convergence theorem in the following:

\[
\lim_{t \to 0} \left\| \frac{\varphi(x + t) - \varphi(x)}{t} - g^{(1,0,\ldots,0)} \right\| = \lim_{t \to 0} \left\| \int F(y) \left\{ \frac{e^{2\pi iy \cdot (x+t)}}{t} - \frac{e^{2\pi iy \cdot x}}{t} - (2\pi iy_1)e^{2\pi iy \cdot x} \right\} \, dy \right\|
\]

\[
\leq \lim_{t \to 0} \int \|F(y)\| \left\| \frac{e^{2\pi iy \cdot (x+t)}}{t} - \frac{e^{2\pi iy \cdot x}}{t} - (2\pi iy_1)e^{2\pi iy \cdot x} \right\| \, dy
\]

\[
= \int \|F(y)\| \lim_{t \to 0} \left\| \frac{e^{2\pi iy \cdot (x+t)}}{t} - \frac{e^{2\pi iy \cdot x}}{t} - (2\pi iy_1)e^{2\pi iy \cdot x} \right\| \, dy
\]

\[
= 0.
\]

This completes the argument.

3.2.2 The Paley-Wiener Theorem for \( D^H_r \)

In this subsection we will show that the spaces \( D^H_r \) and \( PW_{2\pi r}^H \) are isomorphic as well.
Theorem 3.37. (Paley-Wiener type theorem for $\mathcal{D}'_r$) The Fourier transform extends to a linear topological isomorphism between the spaces $\mathcal{D}'_r$ and $\mathcal{PW}^*_H$. The Fourier integral is to be understood as the weak integral.

Proof. Let $\varphi$ be in $\mathcal{D}'_r$, then $\varphi(x) = 0$ for $|x| > r$ and $\sup_{x \in \mathbb{R}^n} \|\varphi(x)\| < \infty$. For $u \in \mathcal{H}$ define $\varphi^u(x) := (\varphi(x), u)$. For $u = e_j$ we will also write $\varphi^j$ for $\varphi^{e_j}$. Observe, for any $x \in \mathbb{R}^n$

$$ |\mathcal{F}_{\mathbb{R}^n}(\varphi^u)(x)| = \left| \int_{\mathbb{R}^n} \varphi^u(y)e^{-2\pi ix \cdot y} dy \right| $$

$$ \leq \int_{B_r} |(\varphi(y), u)| dy $$

$$ \leq \int_{B_r} \|\varphi(y)\| \|u\| dy $$

$$ \leq \left( \text{vol}(B_r) \sup_{y \in \mathbb{R}^n} \|\varphi(y)\| \right) \|u\|. $$

Define an operator $T$ on $\mathcal{H}$ by $T(u) := \mathcal{F}_{\mathbb{R}^n}(\varphi^u)(x)$. By the above observation $T$ is a bounded conjugate linear operator on $\mathcal{H}$, thus there is $u_x \in \mathcal{H}$ with

$$ T(u) = (u_x, u) $$

$$ = \int_{\mathbb{R}^n} \varphi^u(y)e^{-2\pi ix \cdot y} dy $$

$$ = \int_{\mathbb{R}^n} (\varphi(y), u) e^{-2\pi ix \cdot y} dy $$

$$ = \left\langle \int_{\mathbb{R}^n} \varphi(y)e^{-2\pi ix \cdot y} dy, u \right\rangle $$

for every $u \in \mathcal{H}$. Thus $u_x = \int \varphi(y)e^{-2\pi ix \cdot y} dy$ in the sense of Definition 3.12. Define a mapping from $\mathbb{R}^n$ to $\mathcal{H}$ by

$$ F(x) := u_x. $$

As before denote by $F^u(x) := (F(x), u)$. Note $F^u(x) = \mathcal{F}_{\mathbb{R}^n}(\varphi^u)(x)$. By the classical Paley-Wiener theorem we know that $F^u$ extends to $\mathbb{C}^n$: $F^j(z) = \mathcal{F}_{\mathbb{R}^n}(\varphi^j)(z)$. Observe, for every $z \in \mathbb{C}^n$

$$ \left( \sum_{j=1}^{\infty} |F^j(z)|^2 \right)^{\frac{1}{2}} = \|\{F^j(z)\}\|_l^2 $$

$$ = \left\| \left\{ \int_{|x| \leq r} \varphi^j(x)e^{-2\pi ix \cdot z} dx \right\} \right\|_l^2 $$

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by the generalized Minkowski inequality we obtain,
\[
\leq \int_{|x| \leq r} \|\{\varphi^j(x)\}\|_2 e^{2\pi x \cdot \text{Im}(z)} dx
\]
\[
= \int_{|x| \leq r} \|\varphi(x)\| e^{2\pi x \cdot \text{Im}(z)} dx
\]
\[
\leq \text{vol}(\bar{B}_r) \sup_{x \in \mathbb{R}^n} \|\varphi(x)\| e^{2\pi r |\text{Im}(z)|}
\]
\[
< \infty.
\]
Thus the sum \(\sum_{j=1}^{\infty} F^j(z) e_j\) converges to an element in \(\mathcal{H}\), denote this element by \(F(z)\). We have extended our mapping \(F\) to \(\mathbb{C}^n\), and
\[
\langle F(z), e_i \rangle = \left\langle \sum_{j=1}^{\infty} F^j(z) e_j, e_i \right\rangle
\]
\[
= \left\langle \sum_{j=1}^{\infty} \mathcal{F}^c_{\mathbb{R}^n} (\varphi^j)(z) e_j, e_i \right\rangle
\]
\[
= \mathcal{F}^c_{\mathbb{R}^n} (\varphi^i)(z)
\]
\[
= \int_{\mathbb{R}^n} \langle \varphi(x), e_i \rangle e^{-2\pi ix \cdot z} dx
\]
This shows that
\begin{enumerate}
\item \(F(z) = \int_{\mathbb{R}^n} \varphi(x) e^{-2\pi ix \cdot z} dx\) in the sense of Definition 3.12,
\item \(\langle F(z), e_i \rangle \in \mathcal{PW}_{2\pi r}(C^n)\), and
\item \(\rho_{2\pi r, N,u}(F) \leq \text{const. } \nu_{N,u} (\varphi)\).
\end{enumerate}
Hence \(F \in \mathcal{PW}^H_{2\pi r}\) and the Fourier transform extends to a continuous map from \(\mathcal{D}^H_{r}\) into \(\mathcal{PW}^H_{2\pi r}\) mapping \(\varphi \mapsto F\).

To show surjectivity, let \(F \in \mathcal{PW}^H_{2\pi r}\). Then for any \(u \in \mathcal{H}\) the scalar valued function \(F^u(z) := \langle F(z), u \rangle\) belongs to the space \(\mathcal{PW}_{2\pi r}(\mathbb{C}^n)\) by assumption. Define \(\varphi^u := \mathcal{F}^{-1}_{\mathbb{R}^n} (F^u|_{\mathbb{R}^n})\), then \(\varphi^u \in \mathcal{D}^H_{r}(\mathbb{R}^n)\). Fix \(x \in \mathbb{R}^n\) and define an operator \(T\) on \(\mathcal{H}\) by
\[
T(u) := \varphi^u(x)
\]
and observe that $T$ is a bounded operator:

$$|T(u)| = |\varphi^u(x)|$$

$$= \left| \int_{\mathbb{R}^n} \langle F(y), u \rangle e^{2\pi iy \cdot x} dy \right|$$

$$\leq \int_{\mathbb{R}^n} \| F(y) \| \| u \| dy$$

$$\leq \left( \rho_{2\pi r,N}(F) \int_{\mathbb{R}^n} (1 + |y|^2)^{-N} dy \right) \| u \| .$$

The last inequality holds for any $N \in \mathbb{N}_0$ and for $N$ big enough the integral converges. Consequently, there is an element $u_x \in \mathcal{H}$ with

$$T(u) = \langle u_x, u \rangle$$

$$= \varphi^u(x)$$

$$= \int_{\mathbb{R}^n} F^u(y) e^{2\pi iy \cdot x} dy$$

$$= \int_{\mathbb{R}^n} \langle F(y), u \rangle e^{2\pi iy \cdot x} dy$$

$$= \left\langle \int_{\mathbb{R}^n} F(y) e^{2\pi iy \cdot x} dy, u \right\rangle .$$

Define a mapping $\varphi : \mathbb{R}^n \to \mathcal{H}$ by

$$\varphi(x) := u_x,$$

then $\varphi(x) = \int_{\mathbb{R}^n} F(y) e^{2\pi iy \cdot x} dy$ as a weak integral. As $\varphi^u$ is in $\mathcal{D}_r(\mathbb{R}^n)$ we infer that $\nu_{N,u}(\varphi) \leq \text{const.} \rho_{2\pi r,N,u}(F)$. This shows that $\varphi \in \mathcal{D}_r^H$ and the mapping $\mathcal{F}^{-1} : \mathcal{P}W_{2\pi r}^H \to \mathcal{D}_r^H$ is continuous. This proves the theorem.

**Remark 3.38.** Continuity implies weak continuity. The converse does not hold in general. The space $\mathcal{D}_r^H$ is the space of weakly smooth and so weakly continuous functions. On the other hand, for the space $\mathcal{D}_{r,c}^H$ we have an extra assumption that functions are continuous. We have shown that both spaces $\mathcal{D}_{r,c}^H$ and $\mathcal{D}_r^H$ are topologically isomorphic to the space $\mathcal{P}W_{2\pi r}^H$ and hence are equal. This presents an example where weak continuity is equivalent to continuity. In fact we have more in this case, namely: weakly smooth is equivalent to being smooth by Remark 3.36.

**Remark 3.39.** By examining the above proofs of the Paley-Wiener type theorems for $\mathcal{D}_r^H$ and $\mathcal{D}_{r,c}^H$ we notice that they can be proved without the use of the classical Paley-Wiener theorem. Instead they can be proved more directly by mimicking the proof of the classical Paley-Wiener theorem for the functions $x \mapsto \langle \varphi(x), u \rangle$ and $z \mapsto \langle F(z), u \rangle$. Hence, as pointed out at the beginning of this section, this result generalizes the classical theorem.
3.2.3 The Special Case of $\mathcal{H} = L^2(S^{n-1})$ and $\text{SO}(n)$-finite Functions

In this subsection we consider a special case of the Paley-Wiener theorem for vector valued functions by choosing the Hilbert space to be the space of square integrable functions on $S^{n-1}$. First off we recall some useful facts about this space of functions.

Denote by $\Delta := \partial_1^2 + \ldots + \partial_n^2$ the Laplacian on $\mathbb{R}^n$ and by $H_l$, with $l = 0, 1, 2, \ldots$, the space of spherical harmonics of degree $l$. This is the space of $l$-homogeneous harmonic polynomials restricted to the sphere, that is,

$$H_l := \{ p |_{S^{n-1}} : p \in \mathbb{C}[x_1, \ldots, x_n], \Delta p = 0 \text{ and } p(rx) = r^l p(x) \}.$$

The sphere is a homogeneous space: $S^{n-1} \simeq \text{SO}(n)/\text{SO}(n-1)$. The special orthogonal group acts on the sphere by the natural action. This gives an action of the group $\text{SO}(n)$ on functions defined on the sphere; namely the left-regular action:

$$g \cdot \varphi(\omega) = \varphi(g^{-1} \omega).$$

Denote the representation of $\text{SO}(n)$ on $H_l$ by $\pi_l$. Recall that each $\pi_l$ is irreducible and the space of square integrable functions on $S^{n-1}$ is isomorphic to the direct sum of spherical harmonics:

$$L^2(S^{n-1}) \simeq_{\text{SO}(n)} \bigoplus_{l=0}^{\infty} H_l.$$

$H_l$, as a linear subspace of square integrable functions on the sphere with the usual inner product $(f, g) = \int_{S^{n-1}} f(\omega) g(\omega)$, is a Hilbert space. Let $d_n(l)$ denote its dimension and let $\{Y_{l,i}\}_{i=1}^{d_n(l)}$ be an orthonormal basis of $H_l$. Then for $f \in L^2(S^{n-1})$,

$$f = \sum_{l=0}^{\infty} \sum_{i=1}^{d_n(l)} (f, Y_{l,i}) Y_{l,i}.$$

**Definition 3.40.** A square integrable function on the sphere, $f$, is called $\text{SO}(n)$-finite if the span of translates of $f$ under the elements of the special orthogonal group, $\text{span} \{ g \cdot f | g \in \text{SO}(n) \}$, is finite-dimensional. This means that $(f, Y_{l,i}) = 0$ for all but finitely many $l$.

Suppose $\varphi \in \mathcal{D}^H_r$ is $\text{SO}(n)$-finite,

$$\varphi(x) = \sum_{l=0}^{\infty} \sum_{i=1}^{d_n(l)} a_{l,i}(x) Y_{l,i},$$

with $a_{l,i}(x) \neq 0$ for finitely many $l$. As $a_{l,i}(x) = \langle \varphi(x), Y_{l,i} \rangle$, we deduce that $a_{l,i} \in \mathcal{D}_r(\mathbb{R}^n)$. Since each spherical harmonic $Y_{l,i}$ is smooth and the sum is finite, it follows that the sum converges pointwise:

$$\varphi(x)(\omega) = \sum_{l=0}^{\infty} \sum_{i=1}^{d_n(l)} a_{l,i}(x) Y_{l,i}(\omega).$$
So \( \omega \mapsto \varphi(x)(\omega) \) is a polynomial. In particular, \( \varphi \in \mathcal{D}_r(\mathbb{R}^n, C^\infty(S^{n-1})) \).

Let \( \psi \in L^2(S^{n-1}) \), then

\[
\int_{\mathbb{R}^n} \langle \varphi(x), \psi \rangle e^{-2\pi i x \cdot y} dx = \left\langle \sum_{l,i} \int_{\mathbb{R}^n} a_{l,i}(x) e^{-2\pi i x \cdot y} dx Y_{l,i}, \psi \right\rangle.
\]

Thus we see that the Fourier transform of \( \varphi \) extends to \( \mathbb{C}^n \) in the first variable and \( \langle \mathcal{F}c\varphi(z), \psi \rangle \) is in \( \mathcal{PW}_{2\pi r}(\mathbb{C}^n) \) for any \( \psi \in L^2(S^{n-1}) \). Moreover, since \( \mathcal{F}c\varphi(z) \) is a polynomial, the Fourier transform of \( \varphi \) has a holomorphic extension in the \( \omega \)-variable as well.
Chapter 4

Paley-Wiener Theorems with Respect to the Spectral Parameter

In Chapter 2 we reviewed the Paley-Wiener theorems on $\mathbb{R}^n$. There the usual Fourier transform on $\mathbb{R}^n$ was used. However $\mathbb{R}^n$ can be realized as a quotient of a Gelfand pair. This realization of $\mathbb{R}^n$ comes with its own natural Fourier transform. Then the natural question arises: Can one give a description of a given function space using this Fourier transform? In this Chapter we will give two descriptions of the space of compactly supported smooth functions with respect to this Fourier transform.

In the preliminaries we recall the notions of the Lie algebra of a linear Lie group, homogeneous spaces, representations and direct integrals. The references for this section are [38], [15], [25], [4], [3] and [6]. In Section 4.2 we discuss the Fourier transform on $\mathbb{R}^n$ with respect to the Euclidean motion group and in Section 4.3 we introduce our results: Paley-Wiener theorems with respect to the spectral parameter in the decomposition of $L^2(\mathbb{R}^n)$ into irreducible representations of the Euclidean motion group.

4.1 Preliminaries

We begin by introducing the notion of the Lie algebra $\mathfrak{g}$ of a linear Lie group $G$.

Definition 4.1. A Lie group is a group $G$ which is also an analytic manifold such that the mapping

$$G \times G \rightarrow G : (x, y) \mapsto xy^{-1}$$

is analytic.

Let $\mathbb{F}$ denote the field of real or complex numbers. Denote by $M(n, \mathbb{F})$ the set of square matrices of size $n$. As a vector space $M(n, \mathbb{F})$ is isomorphic to $\mathbb{F}^{2n}$. We write $\text{GL}(n, \mathbb{F})$ for the group of invertible matrices in $M(n, \mathbb{F})$, the general linear group, and note that

$$\text{GL}(n, \mathbb{F}) = \{A \in M(n, \mathbb{F}) : \det A \neq 0\}.$$

The operator norm on $M(n, \mathbb{F})$,

$$\|A\| := \sup\{|Ax| : x \in \mathbb{F}^n, |x| \leq 1\},$$

turns $M(n, \mathbb{F})$ into a Banach space. On every subset $S \subset M(n, \mathbb{F})$ we have the subspace topology inherited from $M(n, \mathbb{F})$. As $\det : M(n, \mathbb{F}) \rightarrow \mathbb{F}$ is a polynomial function, it follows that $\text{GL}(n, \mathbb{F})$ is an open subset of $M(n, \mathbb{F})$ and hence a manifold. The multiplication map $\text{GL}(n, \mathbb{F}) \times \text{GL}(n, \mathbb{F}) \rightarrow \text{GL}(n, \mathbb{F}) : (A, B) \mapsto AB$ is
a polynomial and the inversion map \( \text{GL}(n, \mathbb{F}) \to \text{GL}(n, \mathbb{F}) : A \mapsto A^{-1} \) is a rational map. In particular, both maps are analytic maps. Hence \( \text{GL}(n, \mathbb{F}) \) is a Lie group.

The exponential map \( \exp : M(n, \mathbb{F}) \to M(n, \mathbb{F}) \) is given by the power series

\[
\exp(X) := \sum_{j=0}^{\infty} \frac{X^j}{j!}.
\]

It converges because the majorant series \( \sum_{j=0}^{\infty} \|X\|^j \) is convergent.

**Example 4.2.**
1. If \( X = \text{diag}(\lambda_1, \ldots, \lambda_n) \), then \( e^X = \begin{pmatrix} e^{\lambda_1} & 0 \\ \vdots & \ddots \\ 0 & \cdots & e^{\lambda_n} \end{pmatrix} \).

2. If \( X = \begin{pmatrix} e^{\lambda_1} & * \\ \vdots & \ddots \\ 0 & \cdots & e^{\lambda_n} \end{pmatrix} \), then \( e^X = \begin{pmatrix} e^{\lambda_1} & * \\ \vdots & \ddots \\ 0 & \cdots & e^{\lambda_n} \end{pmatrix} \).

3. If \( X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), then \( e^{tX} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \).

4. If \( X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), then \( e^{tX} = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \).

**Proposition 4.3.** Let \( X, Y \in M(n, \mathbb{F}) \) and \( g \in \text{GL}(n, \mathbb{F}) \), then the following holds:

1. If \( X \) and \( Y \) commute, then \( e^{X+Y} = e^X e^Y \).

2. \( ge^X g^{-1} = e^{gX} g^{-1} \).

3. \( (e^X)^{tr} = e^{X^{tr}} \) and \( (e^X)^* = e^{X^*} \), where \( X^* = \overline{X}^{tr} \).

4. \( \det(e^X) = e^{\text{Tr}(X)} \). In particular, \( \det(e^X) = 1 \) iff \( \text{Tr}(X) = 0 \).

**Corollary 4.4.** If \( X \in M(n, \mathbb{F}) \), then \( e^X \in \text{GL}(n, \mathbb{F}) \) with inverse \( e^{-X} \).

**Definition 4.5.** Let \( \mathfrak{g} \) be a vector space. A **Lie bracket** on \( \mathfrak{g} \) is a bilinear map \( \left[ \cdot, \cdot \right] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \) satisfying

1. (L1) \( [x, y] = -[y, x] \) for \( x, y \in \mathfrak{g} \),

2. (L2) \( [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \) for \( x, y, z \in \mathfrak{g} \) (**Jacobi identity**).

For any Lie bracket on \( \mathfrak{g} \), the pair \( (\mathfrak{g}, \left[ \cdot, \cdot \right]) \) is called a **Lie algebra**.

If a vector subspace \( \mathfrak{h} \) of \( \mathfrak{g} \) is closed under the Lie bracket operation, then \( \mathfrak{h} \) is a Lie algebra and is called a **Lie subalgebra** of \( \mathfrak{g} \).
Example 4.6. Let $A$ be an associative algebra. Define $[\cdot, \cdot] : A \times A \to A$ by

$$[a, b] := ab - ba.$$ 

Then $(A, [\cdot, \cdot])$ is a Lie algebra. In particular, $M(n, \mathbb{F})$ is a Lie algebra. We denote this Lie algebra by $\mathfrak{gl}(n, \mathbb{F})$.

Proposition 4.7. If $G$ is a closed subgroup of $\text{GL}(n, \mathbb{F})$, then the set

$$\mathfrak{g} = \{ X \in \mathfrak{gl}(n, \mathbb{F}) : \exp(tX) \in G \text{ for all } t \in \mathbb{R} \}$$

is a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{F})$. Furthermore, there exists an open subset of $\mathfrak{g}$, $U_0$, $0 \in U_0$ and an open set $U_G \subset G$ such that $\exp : U_0 \to U_G$ is a homeomorphism.

Definition 4.8. A closed subgroup $G$ of $\text{GL}(n, \mathbb{F})$ is called a linear Lie group. The real Lie algebra defined in Proposition 4.7 is called the Lie algebra of the linear Lie group $G$.

Example 4.9. $\text{so}(n, \mathbb{F}) = \text{o}(n, \mathbb{F}) = \{ X \in \mathfrak{gl}(n, \mathbb{F}) : X^\text{tr} + X = 0 \}$.

We remark without proof and any further explanation that the Lie algebra $\mathfrak{g}$ of a Lie group $G$ is isomorphic to the tangent space of $G$ at the identity: $\mathfrak{g} \simeq T_e(G)$.

Definition 4.10. Let $G$ be a closed subgroup of $\text{GL}(n, \mathbb{F})$ and let $\mathcal{M}$ be a smooth manifold. We say that $G$ acts on $\mathcal{M}$ if there is a smooth map $m : G \times \mathcal{M} \to \mathcal{M}$ such that

(1) $m(e, x) = x$ for all $x \in \mathcal{M}$.

(2) $m(ab, x) = m(a, m(b, x))$.

We also write $g \cdot x$ for $m(g, x)$. The action is transitive if for $x, y \in \mathcal{M}$ there is an element $g \in G$ such that $g \cdot x = y$. In this case $\mathcal{M}$ is called a homogeneous space. If $G$ acts on the manifolds $\mathcal{M}$ and $\mathcal{N}$, then a smooth map $\varphi : \mathcal{M} \to \mathcal{N}$ is a $G$-map if for all $x \in \mathcal{M}$ and $g \in G$

$$\varphi(g \cdot x) = g \cdot \varphi(x).$$

We say that $\mathcal{M}$ and $\mathcal{N}$ are $G$-isomorphic if there exists a $G$-map from $\mathcal{M} \to \mathcal{N}$ which is also a diffeomorphism.

Assume $G$ acts transitively on $\mathcal{M}$. Fix a base point $x_0 \in \mathcal{M}$ and let

$$G^{x_0} := \{ g \in G : g \cdot x_0 = x_0 \}.$$ 

Then $G^{x_0}$ is a closed subgroup of $G$. Define a map from $G \to \mathcal{M}$ by $\varphi(g) = g \cdot x_0$. Then $\varphi$ is differentiable and surjective. Furthermore, it factors through $G^{x_0}$:

$$\begin{array}{ccc}
G & \xrightarrow{\varphi} & \mathcal{M} \\
\downarrow{\rho} & \cong & \downarrow{\exists \tilde{\varphi}} \\
G/G^{x_0} & \cong & \\
\end{array}$$

(4.1.1)
where \( \kappa : G \to G/G^{x_0} \) is the canonical quotient map \( g \mapsto gG^{x_0} \). We will make \( G/G^{x_0} \) into a manifold such that the action \( m(g,aG^{x_0}) = (ga)G^{x_0} \) is smooth and \( \tilde{\phi} \) is a \( G \)-isomorphism.

Let \( K \) be a closed subgroup of \( G \). Let \( G/K := \{aK : a \in G\} \) and let \( \kappa : G \to G/K \) be the canonical quotient map \( a \mapsto aK \). \( G/K \) becomes a topological space by imposing the following condition: \( U \subset G/K \) is open iff \( \kappa^{-1}(U) \subset G \) is open. Finally, \( G \) acts continuously on \( G/K \) by \( m(a,bK) = (ab)K \). Let \( \mathfrak{k} \) denote the Lie algebra of \( K \) and let \( \mathfrak{q} \subset \mathfrak{g} \) be the complementary subspace in \( \mathfrak{g} : \mathfrak{g} = \mathfrak{q} \oplus \mathfrak{k} \). Let \( U_\mathfrak{q} \subset \mathfrak{q}, U_\mathfrak{k} \subset \mathfrak{k}, \) and \( U_G \subset G \) be as in Theorem 4.7 and such that \( U_G \cap K = \exp(U_\mathfrak{k}) \).

In particular, \( \exp(U_\mathfrak{q}) \cap K = \{I\} \). Let \( U \subset \mathfrak{q} \) be a relatively compact neighborhood of zero such that \( U = -U \subset U_\mathfrak{q} \) and \( \exp(U)^2 \subset U_G \). Let \( U_{G/K} = \kappa(\exp(U)) \). As \( U_{G/K} = \kappa(\exp(U) \exp(U_t)) \), it follows that \( U_{G/K} \) is open in \( G/K \). The map

\[
\begin{align*}
\text{Exp} : \mathfrak{q} & \to G/K \\
q & \mapsto \exp(q)K,
\end{align*}
\]

restricted to \( U \) is a homeomorphism. For \( g \in G \), define \( \psi_g : gU_{G/K} \to U \) by \( g\exp(X) \mapsto X \). Then the collection \( \{(\psi_g,gU_{G/K})\}_{g \in G} \) defines an atlas for \( G/K \), which makes \( G/K \) into an analytic manifold such that the action of \( G \) on \( G/K \) is analytic.

**Theorem 4.11.** Assume that \( G \) acts transitively on \( \mathcal{M} \). Let \( x_0 \in \mathcal{M} \) and let \( G^{x_0} := \{g \in G : g \cdot x_0 = x_0\} \). Then the map

\[
\tilde{\phi} : G/G^{x_0} \to \mathcal{M}
\]

is a \( G \)-isomorphism.

**Example 4.12.** The sphere \( S^{n-1} \) is an example of a homogeneous space. Think of \( \text{SO}(n-1) \) as a subspace of \( \text{SO}(n) \):

\[
\text{SO}(n-1) \simeq \left\{ \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} : B \in \text{SO}(n-1) \right\} \subset \text{SO}(n),
\]

and let \( e_1 = (1,0,\ldots,0) \) be the base point on \( S^{n-1} \), then it is easy to see that

\[
S^{n-1} \simeq \text{SO}(n)/\text{SO}(n-1).
\]

Denote by \( I_n \) the identity matrix in \( M(n,\mathbb{R}) \) and define an involution\(^1\) of \( \text{SO}(n) \) by:

\[
\tau : \text{SO}(n) \to \text{SO}(n) : A \mapsto \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix} A \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix}.
\]

If we write \( A \) as the matrix \( \begin{pmatrix} a_{11} & y^t \\ x & B \end{pmatrix} \) with \( x,y \in \mathbb{R}^n, B \in M(n,\mathbb{R}) \), then

\[
\tau(A) = \begin{pmatrix} a_{11} & -y^t \\ -x & B \end{pmatrix}.
\]

For \( X \in \mathfrak{so}(n) \) let

\[
\dot{\tau}(X) := \frac{d}{dt}\tau(e^{tX})|_{t=0} = \frac{d}{dt}e^{\tau(tX)}|_{t=0}.
\]

\(^1\text{An involution is a function that is its own inverse.} \)
that is,

\[ \tau : \mathfrak{so}(n) \to \mathfrak{so}(n) : X \mapsto \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix} X \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix}. \]

Denote by \( \mathfrak{k} := \mathfrak{so}(n)^\tau \) and by \( \mathfrak{q} := \mathfrak{so}(n)^{-\tau} \), then

\[ \mathfrak{k} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix} : X^\text{tr} = -X \right\} \cong \mathfrak{so}(n-1) \]

and

\[ \mathfrak{q} = \left\{ \begin{pmatrix} 0 & -x^\text{tr} \\ x & 0 \end{pmatrix} : x \in \mathbb{R}^n \right\} \cong \mathbb{R}^n. \]

Hence \( \mathfrak{so}(n) = \mathfrak{k} \oplus \mathfrak{q} \). Further for \( X \in \mathfrak{q} \), define

\[ \gamma_X(t) = e^{tX} \cdot e_1 \]

\[ = (I_n + tx + \frac{t^2}{2}X^2 + \cdots + \frac{t^k}{k!}X^k + \cdots) \cdot e_1 \]

\[ = e_1 + tXe_1 + o(t). \]

Then \( d\gamma_X(0) = X \cdot e_1 = \begin{pmatrix} 0 \\ x \end{pmatrix} \). This identifies \( \mathfrak{q} \) with the tangent space of \( S^{n-1} \) at \( e_1 \):

\[ \mathfrak{q} \cong T_{e_1}S^{n-1}. \]

The exponential map is then given by:

\[ \text{Exp} : \mathfrak{q} \to S^{n-1} : X \mapsto e^X \cdot e_1. \]

Now we recall some basic representation theory. [4] and [3] are the main references for this part. In the following, we let \( G \) stand for a locally compact Hausdorff topological group, \( V \) for a topological vector space, and \( \text{GL}(V) \) for the group of continuous and continuously invertible linear transformations on \( V \).

**Definition 4.13.** A representation of \( G \) on \( V \) is a homomorphism \( \rho : G \to \text{GL}(V) \) satisfying that for each \( v \in V \)

\[ g \mapsto \pi(g)v \]

is continuous from \( G \) to \( V \). \( \rho \) is called finite dimensional if \( \text{dim}(V) < \infty \). And if \( \text{dim}(V) = 1 \), then \( \rho \) is called a character. A subspace \( W \subset V \) is invariant if \( \rho(g)W \subset W \) for any \( g \in G \). \( \rho \) is called irreducible if the only closed invariant subspaces of \( V \) are \( \{0\} \) and \( V \).

**Definition 4.14.** Let \( \rho, \tau \) be representations of \( G \) on topological vector spaces \( V_\rho, V_\tau \), respectively. A continuous linear map \( T : V_\rho \to V_\tau \) is an intertwining operator if \( T\rho(g) = \tau(g)T \) for all \( g \in G \). Denote by \( \text{Hom}_G(\rho, \tau) \) the set of intertwining operators between \( \rho \) and \( \tau \). We say that representations \( \rho \) and \( \tau \) are equivalent, denoted by \( \rho \cong \tau \), if \( \text{Hom}_G(\rho, \tau) \) contains an isomorphism.
Definition 4.15. Let \( \rho \) be a representations of \( G \) on a Hilbert space \( \mathcal{H} \). \( \rho \) is called a unitary representation if \( \rho(g) \) is unitary for each \( g \in G \). Two unitary representations \( \rho \) and \( \tau \) are unitarily equivalent if there exists a unitary isomorphism \( U \in \text{Hom}_G(\rho, \tau) \).

Example 4.16. Let \( G = \mathbb{R}^n \). For \( y \in \mathbb{R}^n \) denote by \( e_y(x) := e^{-2\pi ix \cdot y} \). For each \( y \) the map \( \mathbb{R}^n \to \mathbb{C} : x \mapsto e_y(x) \) is a character of \( \mathbb{R}^n \) acting on \( \mathbb{C} \). Define two unitary representations of \( \mathbb{R}^n \) on the \( L^2(\mathbb{R}^n) \) by

\[
L_x(f)(y) := f(y - x)
\]

and by

\[
U_x(f)(y) := e_x(y)f(y).
\]

As \( \mathcal{F}_{\mathbb{R}^n}L_xf(y) = e_x(y)\mathcal{F}_{\mathbb{R}^n}f(y) \), it follows that representations \( L_x \) and \( U_x \) are unitarily equivalent via the unitary isomorphism \( \mathcal{F}_{\mathbb{R}^n} \).

Unitary equivalence is an equivalence relation on the set of all unitary representations of \( G \). Let \( \hat{G} \) denote the set of all equivalence classes of irreducible unitary representations of \( G \). We call \( \hat{G} \) the unitary dual of \( G \).

Theorem 4.17. (Schur’s Lemma)

(a) A unitary representation \( \rho \) of \( G \) is irreducible iff \( \text{Hom}_G(\rho, \rho) \) contains only scalar multiples of the identity.

(b) Suppose \( \rho \) and \( \tau \) are irreducible unitary representations of \( G \). If \( \rho \) and \( \tau \) are equivalent, then \( \text{Hom}_G(\rho, \tau) \) is one-dimensional. Otherwise, \( \text{Hom}_G(\rho, \tau) = 0 \).

Corollary 4.18. If \( G \) is abelian, then every irreducible unitary representation of \( G \) is one-dimensional.

Corollary 4.19. Let \( G = \mathbb{R}^n \), then every irreducible unitary representation of \( G \) is of the form \( x \mapsto e^{ix \cdot y} \) for some \( y \in \mathbb{R}^n \). In particular, \( \hat{G} \simeq \mathbb{R}^n \).

Recall that for any locally compact group \( G \) there is a left (respectively right) Haar measure on \( G \), that is, a nonzero Radon measure\(^2\) \( \mu \) on \( G \) that satisfies \( \mu(gE) = \mu(E) \) (respectively \( \mu(Eg) = \mu(E) \)) for every Borel set \( E \subset G \) and every \( g \in G \). In case \( G \) is compact, there is a left Haar measure on \( G \) that is also a right Haar measure. We will assume without further notice that locally compact groups are equipped with a left Haar measure and that for compact groups the left Haar measure is also the right Haar measure.

Lastly, we briefly recall the notion of a direct integral. We follow the construction in [4], which is, along with [6], a source for further information and proofs.

Let \( (A, \mathcal{M}) \) denote a measurable space, i.e., a set equipped with a \( \sigma \)-algebra. A family \( \{\mathcal{H}_a\}_{a \in A} \) of nonzero separable Hilbert spaces indexed by \( A \) will be called a field of Hilbert spaces over \( A \), and an element of \( \prod_{a \in A} \mathcal{H}_a \) - that is, a map \( f \) on

\(^2\)The measure \( \mu \) is called a Radon measure if it is inner regular and locally finite.
A such that \( f(\alpha) \in \mathcal{H}_\alpha \) for each \( \alpha \) - will be called a vector field on \( A \). We denote the inner product and norm on \( \mathcal{H} \) by \( \langle \cdot, \cdot \rangle_\alpha \) and \( \| \cdot \|_\alpha \). A measurable field of Hilbert spaces over \( A \) is a field of Hilbert spaces \( \{ \mathcal{H}_\alpha \} \) together with a countable set \( \{ e_j \}^\infty_1 \) of vector fields with the following properties:

(i) the functions \( \alpha \mapsto \langle e_j(\alpha), e_k(\alpha) \rangle_\alpha \) are measurable for all \( j, k \),

(ii) the linear span of \( \{ e_j(\alpha) \}^\infty_1 \) is dense in \( \mathcal{H}_\alpha \) for each \( \alpha \).

**Example 4.20.** Let \( \mathcal{H} \) be a separable Hilbert space with an orthonormal basis \( \{ e_j \} \). By setting \( \mathcal{H}_\alpha = \mathcal{H} \) and \( e_j(\alpha) = e_j \) for all \( \alpha \), we obtain a measurable field of Hilbert spaces over \( A \), called a constant field.

**Example 4.21.** Suppose \( A \) is discrete (i.e., \( \mathcal{M} \) consists of all subsets of \( A \)) and \( \{ \mathcal{H}_\alpha \} \) is an arbitrary field of Hilbert spaces over \( A \). For each \( \alpha \) let \( d(\alpha) = \text{dim} \mathcal{H}_\alpha \) and let \( \{ e_j(\alpha) \}^d(\alpha)_1 \) be an orthonormal basis for \( \mathcal{H}_\alpha \). If we set \( e_j(\alpha) = 0 \) when \( j > d(\alpha) \), the vector fields \( e_j \) make \( \{ \mathcal{H}_\alpha \} \) into a measurable field.

The following proposition provides crucial information about the structure of measurable fields of Hilbert spaces.

**Proposition 4.22.** Let \( \{ \mathcal{H}_\alpha \}, \{ e_j \} \) be a measurable field of Hilbert spaces over \( A \), with \( d(\alpha) \in [1, \infty] \). Then \( \{ \alpha \in A : d(\alpha) = m \} \) is measurable for \( m = 1, \ldots, \infty \). Moreover, there is a sequence \( \{ u_k \}^\infty_1 \) of vector fields with the following properties:

(i) for each \( \alpha \), \( \{ u_k(\alpha) \}^{d(\alpha)}_1 \) is an orthonormal basis for \( \mathcal{H}_\alpha \), and if \( d(\alpha) < \infty \), then \( u_k(\alpha) = 0 \) for \( k > d(\alpha) \);

(ii) for each \( k \) there is a measurable partition of \( A \), \( A = \bigcup_{i=1}^\infty A_i^k \), such that on each \( A_i^k \), \( u_k(\alpha) \) is a finite linear combination of the \( e_j(\alpha) \)'s with coefficients depending measurably on \( \alpha \).

Given a measurable field of Hilbert spaces \( \{ \mathcal{H}_\alpha \}, \{ e_j \} \) on \( A \), a vector field \( f \) on \( A \) will be called measurable if \( \langle f(\alpha), e_j(\alpha) \rangle_\alpha \) is a measurable function on \( A \) for each \( j \).

**Proposition 4.23.** Let \( \{ u_k \} \) be as in Proposition 4.22. A vector field \( f \) on \( A \) is measurable iff \( \langle f(\alpha), u_k(\alpha) \rangle_\alpha \) is a measurable function on \( A \) for each \( k \). If \( f \) and \( g \) are measurable vector fields, then \( \langle f(\alpha), g(\alpha) \rangle_\alpha \) is a measurable function.

We are ready to define direct integrals. Suppose \( \{ \mathcal{H}_\alpha \}, \{ e_j \} \) is a measurable field of Hilbert spaces over \( A \), and suppose \( \mu \) is a measure on \( A \). The direct integral of the spaces \( \{ \mathcal{H}_\alpha \} \) with respect to \( \mu \), denoted by

\[
\int^\oplus \mathcal{H}_\alpha \, d\mu(\alpha),
\]

is the space of measurable vector fields \( f \) on \( A \) such that

\[
\| f \|^2 = \int \| f(\alpha) \|^2_\alpha \, d\mu(\alpha) < \infty.
\]
Note that the integrand is measurable by Proposition 4.23. A modification of the usual proof that $L^2(\mu)$ is complete shows that $\int^\oplus \mathcal{H}_\alpha d\mu(\alpha)$ is a Hilbert space with inner product

$$\langle f, g \rangle = \int \langle f(\alpha), g(\alpha) \rangle_\alpha d\mu(\alpha).$$

Let us see how it works for the examples discussed above.

(1) In the case of a constant field, $\mathcal{H}_\alpha = \mathcal{H}$ for all $\alpha$, $\int^\oplus \mathcal{H}_\alpha d\mu(\alpha)$ is just the space of measurable functions from $A$ to $\mathcal{H}$ that are square integrable with respect to $\mu$. We denote this space by $L^2(A, \mathcal{H}, \mu)$.

(2) If $A$ is discrete and $\mu$ is a counting measure on $A$, then $\int^\oplus \mathcal{H}_\alpha d\mu(\alpha)$ is nothing but $\bigoplus_{\alpha \in A} \mathcal{H}_\alpha$.

Note that $\int^\oplus \mathcal{H}_\alpha d\mu(\alpha)$ depends only on the equivalence class of $\mu$.

**Proposition 4.24.** Let $\{\mathcal{H}_\alpha\}, \{e_j\}$ be a measurable field of Hilbert spaces over $A$, and let $\mu$ be a measure on $A$. For $m = 1, 2, \ldots, \infty$, let $A_m = \{\alpha \in A : \dim \mathcal{H}_\alpha = m\}$. Then a choice of vector fields $\{u_j\}$ as in Proposition 4.22 defines a unitary isomorphism

$$\int^\oplus \mathcal{H}_\alpha d\mu(\alpha) \cong L^2(A_{\infty}, l^2, \mu) \oplus \bigoplus_1^\infty L^2(A_m, \mathbb{C}^m, \mu).$$

Finally note that $\int^\oplus \mathcal{H}_\alpha d\mu(\alpha)$ is independent of the choice of $\{e_j\}$ up to an isomorphism.

### 4.2 Fourier Analysis on $\mathbb{R}^n$ and the Euclidean Motion Group

One of the simplest commutative spaces is $\mathbb{R}^n$ viewed as a homogeneous space for the Euclidean motion group. It is natural to ask how the Paley-Wiener theorem extends to this setting. In this section we first recall the definition of a Gelfand pair $(G, K)$ and basic facts about the Fourier transform on the associated commutative space $G/K$. These facts are derived from the abstract Plancherel formula for the group $G$, instead of, as commonly done, from the theory of spherical functions. A more detailed discussion can be found in [42] and [40]. Then we apply the discussion to the commutative space $\mathbb{R}^n$. For some other aspects of this analysis see [28].

#### 4.2.1 Gelfand Pairs

Let $G$ be a Lie group and $K \subset G$ a compact subgroup. Denote by $\ell$ the left regular representation: $\ell(a)f(x) = f(a^{-1}x)$ and by $\rho$ the right regular representation: $\rho(a)f(x) = f(xa)$. We often identify functions on $G/K$ with right $K$-invariant functions on $G$. For $1 \leq p \leq \infty$, let

$$L^p(G/K)^K = \{f \in L^p(G) : (\forall k_1, k_2 \in K) \ell(k_1)\rho(k_2)f = f\} = \{f \in L^p(G/K) : (\forall k \in K) \ell(k)f = f\}.$$
If \( f \in L^1(G) \) and \( g \in L^p(G) \), then the convolution of \( f \) and \( g \) is the function defined by
\[
f * g(x) := \int_G f(y) g(y^{-1} x) \, dy.
\]
It is well-defined, \( f * g \in L^p(G) \) and \( \| f * g \|_p \leq \| f \|_1 \| g \|_p \). If \( g \) is also right \( K \)-invariant, then so is \( f * g \). And in case \( f \) is left \( K \)-invariant, then so is \( f * g \). It follows that \( L^1(G/K)^K \) is a Banach algebra. The pair \((G,K)\) is called a Gelfand pair if \( L^1(G/K)^K \) is abelian. In this case we call \( G/K \) a commutative space.

In case \( G/K \) is a commutative space, there exists a set \( \Lambda \subseteq \hat{G} \), such that
\[
(\ell, L^2(G/K)) \cong \int_\Lambda^ \oplus (\pi_\lambda, H_\lambda) \, d\mu(\lambda),
\]
where each \( \pi_\lambda \) is an irreducible unitary representation acting on the Hilbert space \( H_\lambda \). The important facts for us are the following: this is a multiplicity free decomposition, i.e. each representation shows up once, and \( \dim H_\lambda^K = 1 \) for almost all \( \lambda \). Here, as usually \( H_\lambda^K \) stands for the space of \( K \)-invariant vectors in \( H_\lambda \).

For details in the following arguments we refer to [42], for the case of Riemannian symmetric spaces of noncompact type see [28]. Let \( p : \Lambda \to \int_\Lambda^ \oplus H_\lambda^K \, d\mu(\lambda) \) be a measurable section such that \( \| p_\lambda \| = 1 \) for almost all \( \lambda \). For each \( \lambda \), \( p_\lambda \) is unique up to a multiplication by \( z \in \mathbb{C} \) with \( |z| = 1 \).

For \( f \in L^1(G) \) and a unitary representation \( \pi \) of \( G \), the operator valued Fourier transform of \( f \) is defined by
\[
\pi(f) := \int_G f(x) \pi(x) \, dx \in B(H_\pi),
\]
where \( B(H_\pi) \) stands for the space of bounded operators on \( H_\pi \). Furthermore, \( \| \pi(f) \| \leq \| f \|_1 \). Recall, that for Type I groups\(^3\), there exists a measure, the Plancherel measure on \( \hat{G} \), such that

1. If \( f \in C^\infty_c(G) \), then \( \pi(f) \) is a Hilbert-Schmidt operator\(^4\) and
\[
\| f \|_2^2 = \int_G \| \pi(f) \|_{HS}^2 \, d\mu(\pi).
\]

\(^3\)A unitary representation \( \rho \) of \( G \) is primary if the center of \( \text{Hom}_G(\rho, \rho) \) is trivial, i.e., consists of scalar multiples of \( \text{Id} \). By Schur’s Lemma, every irreducible representation is primary. The group \( G \) is said to be of Type I if every primary representation of \( G \) is a direct sum of copies of some irreducible representations. Every compact group, every abelian group and the Heisenberg group are of Type I.

\(^4\)A bounded linear transformation \( T \) from a Hilbert space \( \mathcal{H} \) into a Hilbert space \( \mathcal{K} \) is said to be Hilbert-Schmidt if
\[
\sum_\alpha \| Te_\alpha \|^2 < \infty
\]
for any orthonormal basis \( \{ e_\alpha \} \) of \( \mathcal{H} \). The Hilbert-Schmidt norm of \( T \) is defined by
\[
\| T \|_{HS}^2 := \sum_\alpha \| Te_\alpha \|^2.
\]
2. The operator valued Fourier transform extends to $L^2(G)$ such that (4.2.2) holds.

3. For $f \in C_c^\infty(G)$, $f(x) = \int_G \text{Tr}(\pi(x^{-1})\pi(f)) \, d\mu(x)$ pointwise and in $L^2$-sense otherwise.

The projection $pr : \mathcal{H}_\pi \to \mathcal{H}_\pi^K$ is given by

$$pr(v) = \int_K \pi(k)v \, dk.$$ 

If $f \in L^1(G/K)$, then for $k \in K$

$$\pi(f)v = \int_G f(x)\pi(x)v \, dx = \int_G f(xk^{-1})\pi(x)v \, dx = \int_G f(x)\pi(x)\pi(k)v \, dx.$$ 

As this holds for all $k \in K$, integration over $K$ gives:

**Lemma 4.25.** Let $f \in L^1(G/K)$. Then $\pi(f) = \pi(f)pr$.

It follows that $\pi(f)$ is a rank-one operator. Thus for unitary representations for which the subspace of $K$-invariant vectors is one dimensional, in particular, for a commutative space $G/K$ and for almost all $\pi_\lambda$ in the support of the Plancherel measure on $\hat{G}$, it is reasonable to define the vector valued Fourier transform of $f$ by

$$\hat{f}(\lambda) = \mathcal{F}_{G/K}(f)(\lambda) := \pi_\lambda(f)(p_\lambda),$$

where $p_\lambda \in \mathcal{H}_\lambda^K$ as above.

**Lemma 4.26.** If $f \in L^1(G)$ and $g \in C_c^\infty(G/K)$, then

$$\mathcal{F}(f * g)(\lambda) = \pi_\lambda(f)\hat{g}(\lambda).$$

**Proof.** This follows from the fact that $\pi(f * g) = \pi(f)\pi(g)$. \qed

**Theorem 4.27.** Let $f \in C_c^\infty(G/K)$. Then

$$\|f\|_2^2 = \int_{\hat{G}} \|\hat{f}(\lambda)\|_{\mathcal{H}_\lambda}^2 \, d\mu(\lambda)$$

and

$$f(x) = \int (\hat{f}(\lambda), \pi_\lambda(x)p_\lambda)_{\mathcal{H}_\lambda} \, d\mu(\lambda).$$

---

5 A finite rank operator is a bounded linear transformation from a Hilbert space $\mathcal{H}$ into a Hilbert space $K$ which has a finite dimensional range. A rank-one operator is a finite rank operator with the one dimensional range.
Hence the vector valued Fourier transform extends to a unitary isomorphism

$$L^2(G/K) = \int \oplus (\pi_\lambda, \mathcal{H}_\lambda) \, d\mu(\lambda)$$

with inverse

$$f(x) = \int (f_\lambda, \pi_\lambda(x)p_\lambda)_{\mathcal{H}_\lambda} \, d\mu(\lambda)$$

understood in the $L^2$-sense.

**Proof.** Extend $e_{1,\lambda} := p_\lambda$ to an orthonormal basis $\{e_{j,\lambda}\}_j$ of $\mathcal{H}_\lambda$. As for $j > 1$, $\pi_\lambda(f)e_{j,\lambda} = 0$, we have

$$\|\pi_\lambda(f)\|^2_{\text{HS}} = \|\pi_\lambda(f)p_\lambda\|^2_{\mathcal{H}_\lambda} = \|\hat{f}(\lambda)\|^2_{\mathcal{H}_\lambda}.$$ 

And,

$$\text{Tr}(\pi_\lambda(x^{-1})\pi_\lambda(f)) = (\pi_\lambda(x^{-1})\pi_\lambda(f)p_\lambda, p_\lambda)_{\mathcal{H}_\lambda} = (\pi_\lambda(f)p_\lambda, \pi_\lambda(x)p_\lambda)_{\mathcal{H}_\lambda}.$$ 

Hence, by the inversion formula for the operator valued Fourier transform

$$f(x) = \int (\hat{f}(\lambda), \pi_\lambda(x)p_\lambda)_{\mathcal{H}_\lambda} \, d\mu(\lambda)$$

as claimed.

Given a section $(f_\lambda) \in \int \oplus (\pi_\lambda, \mathcal{H}_\lambda) \, d\mu$, define a rank-one operator section $(T_\lambda)$ by

$$T_\lambda p_\lambda = f_\lambda \text{ and } T_\lambda|_{(\mathcal{H}_\lambda)^\perp} = 0.$$ 

Then $T_\lambda$ is a Hilbert-Schmidt operator and hence corresponds to a unique $L^2$-function

$$f(x) = \int \text{Tr}(\pi_\lambda(x^{-1})T_\lambda) \, d\mu(\lambda) = \int (f_\lambda, \pi_\lambda(x)p_\lambda)_{\mathcal{H}_\lambda} \, d\mu(\lambda)$$

and

$$\|f\|^2_2 = \int \|f_\lambda\|^2 \, d\mu(\lambda).$$

As $x \mapsto \pi_\lambda(x)p_\lambda$ is right $K$-invariant, it follows that $f \in L^2(G/K)$. Furthermore, the abstract Plancherel formula gives that $\pi_\lambda(f) = T_\lambda$ and hence $\hat{f}(\lambda) = f_\lambda$. \qed

Assume now that $f$ is left and right $K$-invariant. Then $\hat{f}(\lambda)$ is $K$-invariant and hence a multiple of $p_\lambda$: $\hat{f}(\lambda) = (\hat{f}(\lambda), p_\lambda)_{\mathcal{H}_\lambda}p_\lambda$. We have

$$(\hat{f}(\lambda), p_\lambda)_{\mathcal{H}_\lambda} = \int_G f(x)(\pi_\lambda(x)p_\lambda, p_\lambda)_{\mathcal{H}_\lambda} \, dx$$

$$= \int_{G/K} f(x)\varphi_\lambda(x) \, dx,$$
where $\varphi_\lambda(x) = (\pi_\lambda(x)p_\lambda, p_\lambda)_{H_\lambda}$ is the \textbf{spherical function} associated to $(\pi_\lambda, H_\lambda)$. Note that $\varphi_\lambda$ is independent of the choice of $p_\lambda$. Thus, in the $K$ bi-invariant case the vector valued Fourier transform reduces to the usual spherical Fourier transform on the commutative space $G/K$.

The question now is: How well does the vector valued Fourier transform on the commutative space $X$ describe the image of a given function space on $X$? Examples show that most likely there is no universal answer to this question. There is no answer so far for the Gelfand pair $(U(n \ltimes H_n, U(n)))$, where $H_n$ is the $2n + 1$-dimensional Heisenberg group. Even though some attempts have been made to address this Paley-Wiener theorem for the Heisenberg group, [7, 21, 23, 20]. The Fourier analysis for symmetric spaces of noncompact type is well understood by the work of Helgason and Gangolli, [8, 11]. On the other hand, for compact symmetric spaces $U/K$, the Paley-Wiener theorem is only known for $K$-finite functions [27, 29].

In the following, we will discuss one of the simplest cases of Gelfand pairs, the Euclidean motion group and $SO(n)$.

\subsection{4.2.2 Fourier Transform $\mathcal{F}_{\mathbb{R}^n}$ Revisited}

Every abelian locally compact group $G$, with $K$ reduced to the unit element, constitutes an example of a Gelfand pair. In particular, the pair $(\mathbb{R}^n, \{0\})$ is a Gelfand pair.

By Corollary 4.19 every irreducible unitary representation of $\mathbb{R}^n$ is given by a character of the form $e_y(x) = e^{-2\pi ix \cdot y}$. Applying Theorem 4.27 we obtain the standard Fourier transform facts. For $f \in L^2(\mathbb{R}^n)$ we have:

$$\hat{f}(\lambda) = e_\lambda(f)1 = \int_{\mathbb{R}^n} f(x)e_\lambda(x)dx = \int_{\mathbb{R}^n} f(x)e^{-2\pi ix \cdot \lambda} dx,$$

$$\|f\|_2^2 = \int_{\mathbb{R}^n} |\hat{f}(\lambda)|^2d\lambda,$$

$$f(x) = \int_{\mathbb{R}^n} (\hat{f}(\lambda), e_\lambda(x)1)d\lambda = \int_{\mathbb{R}^n} \hat{f}(\lambda)e^{2\pi i x \cdot \lambda}d\lambda.$$

The Plancherel measure on $\hat{\mathbb{R}^n}$ is the Lebesgue measure and

$$L^2(\mathbb{R}^n) \simeq \int_{\mathbb{R}^n} (e_\lambda, \mathbb{C})d\lambda.$$

\subsection{4.2.3 Representations of the Euclidean Motion Group}

Let $E(n)$ be the group of rigid motions of $\mathbb{R}^n$, the Euclidean motion group, that is, the group generated by translations and rotations. $E(n)$ is the semi-direct product

\footnote{A topological group $G$ is the \textit{semi-direct product} of two closed subgroups $N$ and $H$ if $N$ is normal in $G$ and the map $(n,h) \rightarrow nh$ from $N \times H$ to $G$ is a homeomorphism; in this case we write $G = N \rtimes H$. For further information see [4] or [15].}
of the group of rotations and the group of translations, \( E(n) = \text{SO}(n) \ltimes \mathbb{R}^n \). View elements of \( E(n) \) as diffeomorphisms of \( \mathbb{R}^n \) by \( (A, x) \cdot y = A(y) + x \).

The multiplication in \( E(n) \) is a composition of maps:

\[
(A, x)(B, y) = (AB, A(y) + x).
\]

The identity element is \((I, 0)\), where \( I \) is the identity matrix, and the inverse is \((A, x)^{-1} = (A^{-1}, -A^{-1}x)\). Let \( K = \{(A, 0) : A \in \text{SO}(n)\} \cong \text{SO}(n) \). \( K \) is the stabilizer of \( 0 \in \mathbb{R}^n \). Hence \( \mathbb{R}^n \cong E(n)/K \). Note that \( K \)-invariant functions on \( \mathbb{R}^n \) are radial functions, i.e., functions that only depend on \(|x|\).

The left regular action of \( E(n) \) on \( L^2(\mathbb{R}^n) \) is given by

\[
\ell_g f(y) = f(g^{-1} \cdot y) = f(A^{-1}(y - x)), \quad \text{with } g = (A, x).
\]

Put \( L^2(S^{n-1}) = L^2(S^{n-1}, d\mu_n) \). For \( r \in \mathbb{R} \) define a unitary representation \( \pi_r \) of \( E(n) \) on \( L^2(S^{n-1}) \) by

\[
\pi_r (A, x) \phi(\omega) := e^{-2\pi irx \cdot \omega} \phi(A^{-1}(\omega)).
\]

For \( r \neq 0 \) the representation \( \pi_r \) is irreducible, and \( \pi_r \cong \pi_s \) if and only if \( r = \pm s \). The intertwining operator is given by \([Tf](\omega) = f(-\omega)\). Note that the constant function \( p_r(\omega) := 1 \) on \( S^{n-1} \) is a \( K \)-fixed vector for \( \pi_r \).

### 4.2.4 Fourier Transform on \( \mathbb{R}^n \) with Respect to the Euclidean Motion Group: \( \mathcal{F}_{E(n)} \)

Since the Banach algebra \( L^1(\mathbb{R}^n) \) is commutative, the pair \((E(n), K)\) is a Gelfand pair. For a function \( f \) in \( L^1(\mathbb{R}^n) = L^1(E(n)/K) \) the corresponding vector valued Fourier transform, which we will denote by \( \mathcal{F}_{E(n)}(f)_r = \hat{f}_r \in L^2(S^{n-1}) \), now becomes

\[
\mathcal{F}_{E(n)}(f)_r(\omega) = [\pi_r(f) p_r](\omega)
\]

\[
= \int_{E(n)} f(g) \pi_r(g)p_r(\omega) dg
\]

\[
= \int_{E(n)} f(gk) \pi_r(g)p_r(k) p_r(\omega) dg
\]

\[
= \int_{E(n)/K} f(x) \pi_r(x) p_r(\omega) dx
\]

\[
= \int_{\mathbb{R}^n} f(x) e^{-2\pi irx \cdot \omega} dx
\]

\[
= \mathcal{F}_{\mathbb{R}^n} f(r\omega).
\]

Let \( d\tau(r) = \sigma_n r^{n-1} dr \). Then we have the following theorem:
Theorem 4.28. The Fourier transform $f \mapsto \mathcal{F}_{E(n)} f$ extends to a unitary $E(n)$-isomorphism

$$L^2(\mathbb{R}^n) \cong \int_{\mathbb{R}^+}^\oplus (\pi_r, L^2(S^{n-1})) d\tau(r)$$

$$= L^2(\mathbb{R}^+, L^2(S^{n-1}); d\tau)$$

$$\cong \{ F \in L^2(\mathbb{R}, L^2(S^{n-1}); d\tau) : F(r)(\omega) = F(-r)(-\omega) \}.$$ 

The inverse is given by

$$f(x) = \int_0^\infty (\tilde{f}_r, \pi_r(x)p_r)_{L^2(S^{n-1})} d\tau(r)$$

$$= \int_0^\infty \int_{S^{n-1}} \tilde{f}_r(\omega) e^{2\pi i x \cdot \omega} d\mu_n(\omega) d\tau(r).$$

Proof. To verify that $\mathcal{F}_{E(n)}$ is an $E(n)$-isomorphism, we compute

$$\mathcal{F}_{E(n)}(l(g)f)_r(\omega) = \int_{E(n)} l(g)f(a)\pi_r(a)p_r(\omega) da$$

$$= \int_{E(n)} f(g^{-1}a)\pi_r(a)p_r(\omega) da$$

$$= \int_{E(n)} f(a)\pi_r(ga)p_r(\omega) da$$

$$= \pi_r(g) \int_{E(n)} f(a)\pi_r(a)p_r(\omega) da$$

$$= \pi_r(g) \mathcal{F}_{E(n)}(f)_r(\omega).$$

The rest follows from Theorem 4.27.

The Fourier transform on $\mathbb{R}^n$ with respect to the Euclidean motion group, $\mathcal{F}_{E(n)}$, relates integrable functions on $\mathbb{R}^n$ to integrable functions on $\mathbb{R}^+$ with values in $L^2(S^{n-1})$.

4.3 Paley-Wiener Theorems with Respect to the Spectral Parameter

In this section we discuss the Euclidean Paley-Wiener theorem with respect to the representations of the Euclidean motion group. We will give two different descriptions of the space of smooth compactly supported functions. These descriptions will be given in terms of the spectral parameter in the decomposition of $L^2(\mathbb{R}^n)$ into irreducible representations of the Euclidean motion group as well as some homogeneity condition. For completeness we also give a description of the space of Schwartz functions.

Representations $\pi_r$ act on $L^2(S^{n-1})$ and an instance of $L^2(S^{n-1})$-valued functions is dealt with in Chapter 3. Note that the smooth vectors\(^7\) of the representation $\pi_r$.

\(^7\)Let $G$ be a Lie group and let $(\pi, V)$ be a (continuous) representation of $G$ on a topological vector space $V$. Then a vector $v$ in $V$ is a smooth vector of the representation $\pi$ if the map $G \to V : g \mapsto \pi(g)v$ is smooth.
are the smooth functions on the sphere: \( L^2(S^{n-1}) = C^\infty(S^{n-1}) \). In this section we will work with functions valued in \( C^\infty(S^{n-1}) \). We give the space \( C^\infty(S^{n-1}) \) the Schwartz topology. As the real sphere is compact, with this topology it is equal to \( \mathcal{E}(S^{n-1}) = \mathcal{S}(S^{n-1}) \).

Since \( \mathcal{S}(\mathbb{R}) \) and \( \mathcal{S}(S^{n-1}) \) are nuclear spaces,

\[
\mathcal{S}(\mathbb{R} \times S^{n-1}) = \mathcal{S}(\mathbb{R}, \mathcal{S}(S^{n-1})).^8
\]

We will also denote it simply be \( C^\infty(\mathbb{R} \times S^{n-1}) \) or \( C^\infty(\mathbb{R}, C^\infty(S^{n-1})) \) keeping in mind the Schwartz topology and that the first variable is related to the spectral decomposition. We will often identify these spaces algebraically and topologically by viewing functions \( F : \mathbb{R} \to C^\infty(S^{n-1}) \) as functions \( F : \mathbb{R} \times S^{n-1} \to \mathbb{C} \) via the mapping \( F(x, \omega) := F_x(\omega) \) and vice versa.

### 4.3.1 Description of \( \mathcal{S}(\mathbb{R}^n) \) with Respect to \( F_{E(n)} \)

Denote by \( \mathcal{S}^{\mathbb{Z}^2}(\mathbb{R}, \mathcal{S}(S^{n-1})) \) the set of smooth functions \( F \) on \( \mathbb{R} \times S^{n-1} \) which satisfy the following conditions:

1. \( F \) is even, i.e. \( F(t, \omega) = F(-t, -\omega) \),
2. \( (\frac{\partial}{\partial t})^k F(t, \omega)|_{t=0} \) is a homogeneous polynomial of degree \( k \) in \( \omega \) for any \( k \in \mathbb{N} \),
3. For \( k, l \in \mathbb{N}_0 \) and for any differential operator \( D_\omega \) on the sphere

\[
|F|_{k,l,D_\omega} := \sup_{(t,\omega) \in \mathbb{R} \times S^{n-1}} (1 + t^2)^k \left| \left( \frac{\partial}{\partial t} \right)^l D_\omega F(t, \omega) \right| < \infty.
\]

**Remark 4.29.** The space \( \mathcal{S}^{\mathbb{Z}^2}(\mathbb{R}, \mathcal{S}(S^{n-1})) \) is Fréchet with the topology given by the seminorms \( | \cdot |_{k,l,D_\omega} \). Indeed, it is locally convex. It is easy to see that it is separated and thus Hausdorff and hence metrizable. A Cauchy sequence \( \{F_m\} \) in \( \mathcal{S}^{\mathbb{Z}^2}(\mathbb{R}, \mathcal{S}(S^{n-1})) \) is Cauchy in \( \mathcal{S}(\mathbb{R}, \mathcal{S}(S^{n-1})) = \mathcal{S}(\mathbb{R} \times S^{n-1}) \), so it converges to some function \( F \) in \( \mathcal{S}(\mathbb{R} \times S^{n-1}) \). Since Schwartz convergence implies uniform and pointwise convergence of derivatives of any order, we have

\[
F(t, \omega) = \lim_{m \to \infty} F_m(t, \omega) = \lim_{m \to \infty} F_m(-t, -\omega) = F(-t, -\omega),
\]

and

\[
(\frac{\partial}{\partial t})^k F(t, c \omega)|_{t=0} = \lim_{m \to \infty} (\frac{\partial}{\partial t})^k F_m(t, c \omega)|_{t=0} = c^k \lim_{m \to \infty} (\frac{\partial}{\partial t})^k F_m(t, \omega)|_{t=0} = c^k (\frac{\partial}{\partial t})^k F(t, \omega)|_{t=0}.
\]

---

^8This notation means that this is the space of strongly Schwartz functions on \( \mathbb{R} \) with values in \( \mathcal{S}(S^{n-1}) \).
Theorem 4.30. The Fourier transform $F_{E(n)}$ is a linear topological isomorphism of $S(\mathbb{R}^n)$ onto $S^Z_H(\mathbb{R}, S(S^{n-1}))$.

Proof. Let $f \in S(\mathbb{R}^n)$ and define a function $F$ on $\mathbb{R} \times S^{n-1}$ to be the Fourier transform of $f$ with respect to the Euclidean Motion group: $F(t, \omega) := \hat{f}_t(\omega) = \mathcal{F}_{\mathbb{R}^n} f(t \omega)$. Clearly $F$ is even. By the Fourier-Slice theorem $F(t, \omega) = \mathcal{F}_{\mathbb{R}} (\mathcal{R} f)(t, \omega)$, and $\mathcal{R} f \in S_H(\Xi)$. Hence for each $k \in \mathbb{N}$, the function $\omega \mapsto \int \mathcal{R} f(t, \omega) t^k dr$ is a homogeneous polynomial of degree $k$. Observe,

$$
\left( \frac{\partial}{\partial t} \right)^k F(t, \omega) = \left( \frac{\partial}{\partial t} \right)^k \int_{\mathbb{R}} \mathcal{R} f(s, \omega) e^{-2\pi i s t} ds
$$

it is easy to see that we can interchange the order of differentiation and integration, hence

$$
\int_{\mathbb{R}} \mathcal{R} f(s, \omega)(-2\pi i s)^k e^{-2\pi i s t} ds.
$$

Consequently, $(\partial \overline{\partial})^k F(t, \omega)_{|_{t=0}} = (-2\pi i)^k \int_{\mathbb{R}} \mathcal{R} f(s, \omega) s^k ds$ and so $F$ satisfies the homogeneity condition. Since the Fourier transform is a topological isomorphism of $S(\mathbb{R})$ with itself, we have that for each $\omega$ the function $t \mapsto F(t, \omega)$ is Schwartz. That $F$ is also Schwartz in the $\omega$-variable follows by an application of the Mean Value and Lebesgue Dominated Convergence theorems, as in Remark 3.36. In particular, $F$ is a smooth function on $\mathbb{R} \times S^{n-1}$. Now

$$
t^2 \left( \frac{\partial}{\partial t} \right)^l D_\omega F(t, \omega) = t^2 \int_{\mathbb{R}} D_\omega \mathcal{R} f(s, \omega)(-2\pi i s)^l e^{-2\pi i s t} ds
$$

$$
= \int_{\mathbb{R}} D_\omega \mathcal{R} f(s, \omega)(-2\pi i s)^l (-2\pi i)^{-\frac{l}{2}} \left( \frac{\partial}{\partial s} \right)^2 e^{-2\pi i s t} ds
$$

$$
= (-2\pi i)^{-\frac{l}{2}} \int_{\mathbb{R}} s^l \left( \frac{\partial}{\partial s} \right)^2 D_\omega \mathcal{R} f(s, \omega) e^{-2\pi i s t} ds.
$$

Thus we have

$$(1 + t^2)^k \left| \left( \frac{\partial}{\partial t} \right)^l D_\omega F(t, \omega) \right| \leq \sum_{j=0}^{k} \binom{k}{j} (2\pi)^{l-2j} \int_{\mathbb{R}} |s^j| \left| \left( \frac{\partial}{\partial s} \right)^{2j} D_\omega \mathcal{R} f(s, \omega) \right| ds
$$

$$
\leq \sum_{j=0}^{k} \binom{k}{j} (2\pi)^{l-2j} \int_{\mathbb{R}} |s^j| (1 + s^2)^{-m} ds \eta_{m,2j,D_\omega}(\mathcal{R} f).
$$

We can choose $m$ big enough to make the integral converge. This shows that $F \in S^Z_H(\mathbb{R}, S(S^{n-1}))$ and the mapping $f \mapsto F$ is continuous.

To show the converse, let $F \in S^Z_H(\mathbb{R}, S(S^{n-1}))$ and let $\varphi(t, \omega) := \mathcal{F}_{\mathbb{R}}^{-1} F(t, \omega)$. Then $\varphi$ is Schwartz in both variables by similar considerations as above. Observe that $\varphi$ is even,

$$
\varphi(-t, -\omega) = \int_{\mathbb{R}} F(s, -\omega) e^{2\pi i s(-t)} ds
$$

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since $F$ is even by assumption

$$= \int_{\mathbb{R}} F(-s, \omega) e^{-2\pi ist} ds$$

making the change of variables $s \mapsto -s$ we get

$$= \int_{\mathbb{R}} F(s, \omega) e^{2\pi ist} ds$$

$$= \varphi(t, \omega).$$

Moreover,

$$= \varphi(t, \omega)$$

$$(1 + t^2)^k \left( \frac{\partial}{\partial t} \right)^t D_{\omega} \varphi(t, \omega) = \sum_{j=0}^{k} \binom{k}{j} \int_{\mathbb{R}} \left( \frac{\partial}{\partial s} \right)^{2j} D_{\omega} F(s, \omega)(2\pi is)^l(2\pi i)^{-2j} e^{2\pi ist} ds$$

$$= \sum_{j=0}^{k} \binom{k}{j} (2\pi i)^{-2j} \int_{\mathbb{R}} s^l \left( \frac{\partial}{\partial s} \right)^{2j} D_{\omega} F(s, \omega) e^{2\pi ist} ds.$$ 

Furthermore,

$$\int_{\mathbb{R}} \varphi(t, \omega) t^k dt = \int_{\mathbb{R}} \varphi(t, \omega) t^k e^{-2\pi i 0} dt$$

$$= \mathcal{F}_{\mathbb{R}}(\varphi(t, \omega) t^k)_{t=0}$$

$$= \left( i \frac{\partial}{\partial t} \right)^k \mathcal{F}_{\mathbb{R}}(\varphi(t, \omega))_{t=0}.$$

This shows that $\varphi \in S_{H}(\Xi)$ and the mapping $F \mapsto \varphi$ is continuous. Hence, by Theorem 2.3, $\mathcal{R}^{-1} \varphi \in S(\mathbb{R}^n)$ and the mapping $\varphi \mapsto \mathcal{R}^{-1} \varphi$ is continuous. Since $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{R}$ are injective, it follows that $\mathcal{F}_{E(n)}$ is injective. This proves the claim.

4.3.2 First Description of $D_r(\mathbb{R}^n)$ with Respect to $\mathcal{F}_{E(n)}$

Our first description is an analog of a variant due to Helgason, see Theorem 2.10 in [13]. We restate it here in a slightly different form.

For $r > 0$, let $\mathcal{P}\mathcal{W}_{r}^{Z;Z,H}(C \times S^{n-1})$ be the space of smooth functions $F$ on $C \times S^{n-1}$ satisfying:

1. $F$ is even, that is $F(z, \omega) = F(-z, -\omega)$.

2. For each $\omega$, the function $z \mapsto F(z, \omega)$ is a holomorphic function on $C$ with the property that for any $N \in \mathbb{N}$

$$|F(z, \omega)| \leq C_N (1 + |z|^2)^{-N} e^{r|\text{Im} z|} < \infty.$$
3. For each \( k \in \mathbb{N}^+ \), and each isotropic\(^9\) vector \( a \in \mathbb{C}^n \), the function
\[
z \mapsto z^{-k} \int_{S^{n-1}} F(z, \omega)(a, \omega) d\omega
\]
is even and holomorphic on \( \mathbb{C}^n \).

**Theorem 4.31.** The Fourier transform followed by a holomorphic extension in the spectral parameter is an injection of \( D_r(\mathbb{R}^n) \) onto \( \mathcal{PW}_{2\pi r}(\mathbb{C} \times S^{n-1}) \).

**Proof.** See [13], pages 23-28.

Note that this theorem does not contain a topological statement. In the following we prove an analogous result for vector valued functions including the topological statement.

Let \( \mathcal{PW}_{r}^{\mathbb{Z}_2,H}(\mathbb{C}, S(S^{n-1})) \) be the space of weakly\(^{10}\) holomorphic functions \( F : \mathbb{C} \mapsto \mathcal{S}(S^{n-1}) \) which satisfy

1. \( F \) is even, i.e. \( F(z, \omega) = F(-z, -\omega) \).

2. \( (\frac{\partial}{\partial z})^k F(z, \omega)|_{z=0} \) is a homogeneous polynomial of degree \( k \) in \( \omega \) for any \( k \in \mathbb{N} \).

3. For \( k \in \mathbb{N}_0 \) and for any differential operator \( D_\omega \) on the sphere
\[
|F|_{r,k,D_\omega} := \sup_{(z,\omega) \in \mathbb{C} \times S^{n-1}} (1 + |z|^2)^k e^{-r|\text{Im} z|} |D_\omega F(z, \omega)| < \infty.
\]

We let the topology on \( \mathcal{PW}_{r}^{\mathbb{Z}_2,H}(\mathbb{C}, S(S^{n-1})) \) be given by the seminorms \( |\cdot|_{r,k,D_\omega} \).

**Proposition 4.32.** \( \mathcal{PW}_{r}^{\mathbb{Z}_2,H}(\mathbb{C}, S(S^{n-1})) \) is Fréchet.

**Proof.** \( \mathcal{PW}_{r}^{\mathbb{Z}_2,H}(\mathbb{C}, S(S^{n-1})) \) is locally convex, separated and hence metrizable.

Let \( \{F_m\} \) be a Cauchy sequence in \( \mathcal{PW}_{r}^{\mathbb{Z}_2,H}(\mathbb{C}, S(S^{n-1})) \). That is, for any multi-index \( \alpha \in \mathbb{N}_0^n \), any differential operator \( D_\omega \) on the sphere, and any \( \epsilon > 0 \), there is an \( N(\alpha, D_\omega) \) such that for \( k, l \geq N(\alpha, D_\omega) \)
\[
|F_l - F_k|_{r,|\alpha|,D_\omega} < \epsilon.
\]

We have
\[
|z^\alpha D_\omega(F_l(z, \omega) - F_k(z, \omega))| \leq |F_l - F_k|_{r,|\alpha|,D_\omega} e^{r|\text{Im} z|}.
\]

We conclude that \( \{z^\alpha F_m(z, \omega)\} \) converges pointwise to some function \( F^\alpha(z, \omega) \).

Moreover, viewed as a sequence of functions in the \( z \)-variable, it converges uniformly over compact subsets of \( \mathbb{C} \). Hence \( z \mapsto F^\alpha(z, \omega) \) is holomorphic. Viewing it as a sequence of functions in the \( \omega \)-variable, gives uniform convergence of any

\(^9\)A vector \( a = (a_1, \ldots, a_n) \) in \( \mathbb{C}^n \) is called isotropic if \( a_1^2 + \cdots + a_n^2 = 0 \).

\(^{10}\)Since \( \mathcal{S}(S^{n-1}) \) is Fréchet, it is equivalent to being strongly holomorphic.
derivative in \( \omega \)-variable and so \( \omega \mapsto F^\alpha(z, \omega) \) is a Schwartz function on \( S^{n-1} \).

Observe,

\[
F^\alpha(z, \omega) = \lim_m z^\alpha F_m(z, \omega) = \lim_m \frac{z^\alpha}{2\pi i} \oint_\gamma \frac{F_m(\xi, \omega)}{\xi - z} d\xi = \lim_m \frac{z^\alpha}{2\pi i} \oint_\gamma \frac{F^0(\xi, \omega)}{\xi - z} d\xi = z^\alpha F^0(z, \omega).
\]

For any \((z, \omega) \in \mathbb{C} \times S^{n-1}\) and \(k, l \geq N(\alpha, D_\omega)\)

\[
e^{-r|\text{Im}(z)|} |z^\alpha D_\omega(F_k(z, \omega) - F_l(z, \omega))| \leq |F_l - F_k|_{r, |\alpha|, D_\omega} < \epsilon.
\]

Letting \(l \to \infty\), we get

\[
e^{-r|\text{Im}(z)|} |z^\alpha D_\omega(F_k(z, \omega) - F^0(z, \omega))| < \epsilon.
\]

This implies \(F_m \to F^0\) in the topology of \( \mathcal{PW}_{r,H}^\mathbb{C}(\mathbb{C}, \mathcal{S}(S^{n-1})) \). Next observe, \(F^0\) satisfies seminorm estimates:

\[
e^{-r|\text{Im}(z)|} |z^\alpha D_\omega F^0(z, \omega)|
\]

\[
= e^{-r|\text{Im}(z)|} |z^\alpha D_\omega(F^0(z, \omega) - F_m(z, \omega) + F_m(z, \omega))| \\
\leq e^{-r|\text{Im}(z)|} |z^\alpha D_\omega(F^0(z, \omega) - F_m(z, \omega))| + e^{-r|\text{Im}(z)|} |z^\alpha D_\omega F_m(z, \omega)| \\
< \infty,
\]

it is even:

\[
F^0(-z, -\omega) = \lim_m F_m(-z, -\omega) = \lim_m F_m(z, \omega) = F^0(-z, -\omega),
\]

and it satisfies the homogeneity condition because

\[
\lim_m \left( \frac{\partial}{\partial z} \right)^k F_m(z, \omega) = \lim_m \frac{k!}{2\pi i} \oint_\gamma \frac{F_m(\xi, \omega)}{(\xi - z)^{k+1}} d\xi = \frac{k!}{2\pi i} \oint_\gamma \frac{F^0(\xi, \omega)}{(\xi - z)^{k+1}} d\xi = \left( \frac{\partial}{\partial z} \right)^k F^0(z, \omega).
\]

To show that \(F^0 \in \mathcal{PW}_{r,H}^\mathbb{C}(\mathbb{C}, \mathcal{S}(S^{n-1}))\), it remains to verify that \(F^0 : \mathbb{C} \to \mathcal{S}(S^{n-1})\) is weakly holomorphic. Let \(\Lambda \in \mathcal{S}(S^{n-1})'\). Since each \(F_m\) is also strongly holomorphic, as \(\mathcal{S}(S^{n-1})\) is Fréchet, \(\frac{d}{dz} F_m(z)\) exists in the topology of \(\mathcal{S}(S^{n-1})\). Hence

\[
\frac{d}{dz} \Lambda(F_m)(z) = \Lambda \left( \frac{d}{dz} F_m \right)(z).
\]
Thus, $F$ hence

This shows that $\Lambda(\cdot)$.

Proof. Let $f \in D_r(\mathbb{R}^n)$, then $Rf \in D_r(\mathbb{R}, S(S^{n-1}))$. In particular, $Rf : \mathbb{R} \to S(S^{n-1})$ is smooth and so for any $\Lambda \in S(S^{n-1})$, $t \mapsto \Lambda(Rf(t))$ is in $D_r(\mathbb{R})$. Hence

$$|\Lambda(Rf(t))e^{-2\pi t\xi}| \leq \sup_{t} |\Lambda(Rf(t))| e^{2\pi r|\operatorname{Im}(\xi)|} |\chi_{[-r,r]}(t)| \in L^1(\mathbb{R}). \quad (4.3.1)$$

Observe that the first part of the Theorem 3.34 applies here, hence $F_{E(n)}f : \mathbb{C} \to S(S^{n-1})$ is weakly holomorphic. Moreover, for any differential operator $D_\omega$ on the sphere, $t \mapsto D_\omega Rf(t, \omega) \in D_r(\mathbb{R})$. Hence by the classical Paley-Wiener theorem $\xi \mapsto F_\mathbb{R}D_\omega Rf(\xi, \omega) \in \mathcal{PW}_{2\pi r}(\mathbb{C})$ and by the Lebesgue Dominated Convergence theorem $F_\mathbb{R}D_\omega Rf = D_\omega F_\mathbb{R} Rf = D_\omega F_{E(n)}f$. It follows that $F_{E(n)}f$ satisfies the seminorm estimate and the mapping $f \mapsto F_{E(n)}f$ is continuous.

The estimate in (4.3.1) also holds for $Rf$ and implies that we can differentiate under the integral sign in the following:

$$\left(\frac{\partial}{\partial \xi}\right)^k F_{E(n)}f(\xi, \omega) = \int_\mathbb{R} Rf(t, \omega) \left(\frac{\partial}{\partial \xi}\right)^k e^{-2\pi t\xi} dt$$

$$= (2\pi i)^k \int_\mathbb{R} Rf(t, \omega) t^k e^{-2\pi t\xi} dt.$$
We conclude that $F_{E(n)} f$ satisfies the homogeneity condition. As $\mathbb{R}$ is a totally real submanifold of $\mathbb{C}$, it is enough to verify that $F_{E(n)} f$ is even on $\mathbb{R} \times S^{n-1}$, which is easy. Hence $F_{E(n)} f$ is in $\mathcal{P} \mathcal{W}_{2\pi r}^{2,H}(\mathbb{C}, S(S^{n-1}))$.

To show the converse, let $F \in \mathcal{P} \mathcal{W}_{2\pi r}^{2,H}(\mathbb{C}, S(S^{n-1}))$. Then, in particular, for any $\omega \in S^{n-1}$, $\xi \mapsto F(\xi, \omega) \in \mathcal{P} \mathcal{W}_{2\pi r}^{2}(\mathbb{C})$ and so $t \mapsto F_{\mathbb{R}}^{-1}(t, \omega) \in \mathcal{D}_r(\mathbb{R})$. Since

$$|D_{\omega} F(s, \omega)| \leq |F|_{2\pi r,k,D_\omega}(1+s^2)^{-k} \in L^1(\mathbb{R})$$

for $k \geq 1$, we can interchange the order of differentiation and integration in $D_{\omega} F_{\mathbb{R}}^{-1}(t, \omega) = \int_\mathbb{R} D_{\omega} F(s, \omega) e^{2\pi ist} ds$. We conclude that $\omega \mapsto F_{\mathbb{R}}^{-1}(t, \omega)$ is smooth for any $t$. This shows that $F_{\mathbb{R}}^{-1} F \in \mathcal{D}_r(\mathbb{R} \times S^{n-1})$. It is easy to see that $F_{\mathbb{R}}^{-1} F$ is even and as $\int F_{\mathbb{R}}^{-1}(t, \omega) t^k dt = \left( i \frac{\partial}{\partial t} \right)^k F(t, \omega)|_{t=0}$, it satisfies the homogeneity condition as well. Moreover, $|s^k D_{\omega} F(s, \omega)| \leq |F|_{2\pi r,N+k,D_\omega}(1+s^2)^N$, implies

$$\left| \left( \frac{\partial}{\partial t} \right)^k D_{\omega} F_{\mathbb{R}}^{-1}(t, \omega) \right| = \left| \int D_{\omega} F(s, \omega) \left( \frac{\partial}{\partial t} \right)^k e^{2\pi ist} ds \right|$$

$$= (2\pi)^k \int (1+s^2)^N ds \ |F|_{2\pi r,N+k,D_\omega} \ < \infty.$$ 

Hence $F_{\mathbb{R}}^{-1} F \in \mathcal{D}_{H,r}(\mathbb{C})$ and the mapping $F \mapsto F_{\mathbb{R}}^{-1} F$ is continuous. We conclude that $F_{G}^{-1} F \in \mathcal{D}_r(\mathbb{R}^n)$ and the mapping $F \mapsto F_{G}^{-1} F$ is continuous. As $F_{E(n)}$ is injective, the theorem follows. \hfill \Box

**Remark 4.34.** This shows that the two Paley-Wiener spaces are isomorphic to each other:

$$\mathcal{P} \mathcal{W}_{\mathbb{R}}^{2,H}(\mathbb{C} \times S^{n-1}) \simeq \mathcal{P} \mathcal{W}_{\mathbb{C}}^{2,2,H}(\mathbb{C}, S(S^{n-1})).$$

### 4.3.3 Second Description of $\mathcal{D}_r(\mathbb{R}^n)$ with Respect to $F_{E(n)}$

First we extend the concept of polar coordinates to $\mathbb{C}^n$. Recall that $S^{n-1}$ is a homogeneous space $\text{SO}(n)/\text{SO}(n-1)$. Since the complexification of $\mathfrak{so}(n)$ is $\mathfrak{so}(n, \mathbb{C})$, we let $\text{SO}(n, \mathbb{C})$ be the complexification of $\text{SO}(n)$. We let $\mathfrak{so}(n, \mathbb{C})/\text{SO}(n-1, \mathbb{C})$ be the complexification of $S^{n-1}$ and we denote it by $S^{n-1}_\mathbb{C}$. The complex dimension of $S^{n-1}_\mathbb{C}$ is $n-1$.

**Proposition 4.35.** $S^{n-1}_\mathbb{C} \simeq \{ z \in \mathbb{C}^n : z_1^2 + \ldots + z_n^2 = 1 \}$.  

---

12 $\mathfrak{so}(n)$ is a real vector space and by the complexification of $\mathfrak{so}(n)$ we mean its complexification as a vector space. Recall, if $V$ is a real vector space, then its complexification is the complex vector space $V_\mathbb{C} = V + iV = V \otimes \mathbb{C}$, which has $V$ as a real vector subspace.

13 Recall that $\mathfrak{so}(n,F) = \{ X \in \mathfrak{gl}(n,F) : X^{tr} = -X \}$, and $\mathfrak{so}(n,F)$ is the complexification of $\mathfrak{so}(n,F)$, which has $\mathfrak{so}(n,F)$ as a real vector subspace.

14 Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Then a complexification of $G$ is a Lie group $G_\mathbb{C}$ with Lie algebra $\mathfrak{g}_\mathbb{C}$.

15 $\dim_{\mathbb{C}}(\text{SO}(n, \mathbb{C})/\text{SO}(n-1, \mathbb{C})) = \dim_{\mathbb{C}}(\text{SO}(n, \mathbb{C})) - \dim_{\mathbb{C}}(\text{SO}(n-1, \mathbb{C})) = \frac{n(n-1)}{2} - \frac{(n-1)(n-2)}{2} = n-1$. 

57
Hence \( \xi \)

Construct a set of vectors \( A \)

Proof. Fix a basepoint \( e_1 := (1, 0, \ldots, 0) \) in \( \{ z \in \mathbb{C}^n : z_1^2 + \ldots + z_n^2 = 1 \} \). Let \( \tilde{\omega} \in \{ z \in \mathbb{C}^n : z_1^2 + \ldots + z_n^2 = 1 \} \) and extend \( \{ \tilde{\omega} \} \) to a set of linearly independent vectors \( \{ v_1, \ldots, v_n \} \) in \( \mathbb{C}^n \) with \( v_1 = \tilde{\omega} \). Then use a modified Gram-Schmidt process\(^{16} \) to obtain a set of vectors \( \{ u_1, \ldots, u_n \} \) with \( u_1 = v_1 \) such that

\[
(u_j, u_k) = \begin{cases} 0 & \text{for } j \neq k \\ 1 & \text{for } j = k. \end{cases}
\]

Let \( A \) be a matrix whose column vectors are \( u_j \)'s: \( A = (u_1, \ldots, u_n) \). Then \( A \in \mathbf{O}(n, \mathbb{C}) \). Moreover, \( \det(AA^\text{tr}) = \det(I) = 1 \). As \( \det(A^\text{tr}) = \det A \) and \( \det(AA^\text{tr}) = \det(A) \det(A^\text{tr}) \), it follows that \( (\det A)^2 = 1 \). Hence \( \det A = \pm 1 \). In case \( \det A = -1 \) and \( n > 2 \), permute the last two columns of the matrix \( A \). For \( n = 2 \), multiply the last column, \( v_2 \), by \(-1 \). This will ensure that \( \det A = 1 \). Thus \( A \in \mathbf{SO}(n, \mathbb{C}) \) and \( Ae_1 = \tilde{\omega} \). This shows that \( \mathbf{SO}(n, \mathbb{C}) \) acts transitively on \( \{ z \in \mathbb{C}^n : z_1^2 + \ldots + z_n^2 = 1 \} \).

Suppose \( Ae_1 = e_1 \) for some \( A \in \mathbf{SO}(n, \mathbb{C}) \). Then \( A \) has the form

\[
\begin{pmatrix} 1 & \xi^\text{tr} \\ 0 & B \end{pmatrix},
\]

We have

\[
I = A^\text{tr} A = \begin{pmatrix} 1 & 0 \\ \xi^\text{tr} & B^\text{tr} \end{pmatrix} \begin{pmatrix} 1 & \xi^\text{tr} \\ 0 & B \end{pmatrix} = \begin{pmatrix} 1 & \xi \xi^\text{tr} B^\text{tr} B \end{pmatrix}.
\]

Hence \( \xi = 0 \) and \( A \) is equal to \( \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \). Consequently, \( B^\text{tr} B = I \) and so \( B^\text{tr} = B^{-1} \). Further, \( \det A = \det B = 1 \), so \( B \in \mathbf{SO}(n-1, \mathbb{C}) \). This shows that the stabilizer of the base point \( e_1 \) is isomorphic to \( \mathbf{SO}(n-1, \mathbb{C}) \). Hence \( \{ z \in \mathbb{C}^n : z_1^2 + \ldots + z_n^2 = 1 \} \sim \mathbf{SO}(n, \mathbb{C})/\mathbf{SO}(n-1, \mathbb{C}) \).

Let \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \), then the map \( \mathbb{C}^* \times S_{\mathbb{C}}^{n-1} \to \mathbb{C}^n \setminus \{ z : \sum_{j=1}^n z_j^2 = 0 \} : (z, \tilde{\omega}) \mapsto z\tilde{\omega} \) is a holomorphic two-to-one map.\(^{17} \) Note that the Lebesgue measure of the set \( \{ z \in \mathbb{C}^n : \sum_{i=1}^n z_i^2 = 0 \} \) is 0.

Denote by \( \mathcal{O}(S_{\mathbb{C}}^{n-1}) \) the space of holomorphic functions on the complex manifold \( S_{\mathbb{C}}^{n-1} \) with the topology of uniform convergence over compact sets. For \( r > 0 \), let \( \mathcal{O}_r(\mathbb{C}, \mathcal{O}(S_{\mathbb{C}}^{n-1})) \) be the space of weakly holomorphic functions \( F : \mathbb{C} \to \mathcal{O}(S_{\mathbb{C}}^{n-1}) \)

---

\(^{16}\)Let \( \{ v_1, \ldots, v_n \} \) be a set of linearly independent vectors in \( \mathbb{C}^n \) with \( v_1 \in \{ z \in \mathbb{C}^n : z_1^2 + \ldots + z_n^2 = 1 \} \). Construct a set of vectors \( \{ u_1, \ldots, u_n \} \) by the following procedure

\[
\begin{align*}
u_1 &= v_1 \\
v_2 - (v_2, u_1)u_1 &= u_2 = \frac{\omega_2}{\sqrt{(\omega_2, \overline{\omega_2})}} \\
v_3 - (v_3, u_1)u_1 - (v_3, u_2)u_2 &= u_3 = \frac{\omega_3}{\sqrt{(\omega_3, \overline{\omega_3})}} \\
&\vdots \\
v_n - \sum_{j=1}^{n-1} (v_n, u_j)u_j &= u_n = \frac{\omega_n}{\sqrt{(\omega_n, \overline{\omega_n})}}.
\end{align*}
\]

Then \( \{ u_j \} \) is a set of vectors in \( \mathbb{C}^n \) with \( (u_j, \overline{u}_i) = 0 \) for \( j \neq k \), and \( (u_j, \overline{u}_j) = 1 \).

\(^{17}\)Let \( (z, \omega) \in \mathbb{C}^* \times S_{\mathbb{C}}^{n-1} \), then \( \sum (z, \omega)^2 = \sum (\overline{z}, \omega) = z^2 + \sum \omega_j^2 = z^2 \neq 0 \). Next suppose \( (z, \omega), (\xi, \nu) \in \mathbb{C}^* \times S_{\mathbb{C}}^{n-1} \) with \( z\omega = \xi\nu \), then \( z^2 = \xi \nu \). So \( \xi = \pm \nu \). If \( \xi = \nu \), then \( v = \omega \). And if \( \xi = -\nu \), then \( v = -\omega \). Finally, let \( z \in \mathbb{C}^n \) with \( \sum z_j^2 = 0 \). Set \( \xi := (z, \overline{z})^{1/2} \) and \( \omega := z/(z, \overline{z})^{1/2} \). Then \( \xi \in \mathbb{C}^* \) and \( \omega \in S_{\mathbb{C}}^{n-1} \).
such that for all \( N \in \mathbb{N}_0 \)

\[
\pi_{r,N}(F) := \sup_{z \in \mathbb{C}, \omega \in S^{n-1}_C} (1 + \text{Im}(z\omega))^N e^{-r\text{Im}(z\omega)} |F(z)(\omega)| < \infty. \tag{4.3.2}
\]

Since \( \mathcal{O}(S^{n-1}_C) \) is Fréchet\(^{18}\), a function in this space is also strongly holomorphic\(^{19}\).

As spaces of holomorphic functions are nuclear, we can identify holomorphic functions \( F : \mathbb{C} \to \mathcal{O}(S^{n-1}_C) \) with holomorphic functions on the product manifold \( \mathbb{C} \times S^{n-1}_C \).\(^{20}\)

Denoted by \( \widehat{\mathcal{P}W}_{r}^{Z,2,H} = \widehat{\mathcal{P}W}_{r}^{Z,2,H}(\mathbb{C}, \mathcal{O}(S^{n-1}_C)) \) the set of even functions \( F \) in \( \mathcal{O}_{r}(\mathbb{C}, \mathcal{O}(S^{n-1}_C)) \) satisfying that for all \( \lambda \in \mathbb{C} \) and \( \omega \in S^{n-1}_C \),

\[
F(\lambda, \omega) = F(0, \omega) + \sum_{m=1}^{\infty} \frac{a_m(\omega)}{m!} \lambda^m,
\]

where each \( a_m \) is a homogeneous polynomial in \( \omega_1, \ldots, \omega_n \) of degree \( m \).

**Proposition 4.36.** \( \widehat{\mathcal{P}W}_{r}^{Z,2,H}(\mathbb{C}, \mathcal{O}(S^{n-1}_C)) \) is a Fréchet space.

**Proof.** It is locally convex as its topology is defined by seminorms. Each seminorm is a norm, so it is Hausdorff and hence metrizable. Let \( \{F_m\}_{m \in \mathbb{N}} \) be a Cauchy sequence in \( \widehat{\mathcal{P}W}_{r}^{Z,2,H} \). Then for every \((z, \omega) \in \mathbb{C} \times S^{n-1}_C \),

\[
|(z\omega)^\alpha (F_k(z, \omega) - F_l(z, \omega))| \leq \pi_{r,|\alpha|}(F_k - F_l)e^{r|\text{Im}(z\omega)|}.
\]

Hence the sequence \( \{(z\omega)^\alpha F_m\} \) converges uniformly over compact subsets of \( \mathbb{C} \times S^{n-1}_C \) to a holomorphic function \( F^\alpha \), and

\[
F^\alpha(z, \omega) = \lim_m (z\omega)^\alpha F_m(z, \omega)
= \lim_m \frac{(z\omega)^\alpha}{(2\pi i)^n} \oint_{\gamma} \frac{F_m(\xi, \eta)}{(\xi, \eta) - (z, \omega)} d(\xi, \eta)
= (z\omega)^\alpha F^0(z, \omega).
\]

\(^{18}\)In Theorem 3.25 it was shown that it is complete. Since each seminorm is actually a norm, it is Hausdorff. Embedding \( S^{n-1}_C \) into \( \mathbb{C}^n \) we conclude that the manifold \( S^{n-1}_C \) is separable and hence can be covered by countably many compact sets. The corresponding seminorms define the same topology. Hence by Theorem 3.11 it is metrizable.

\(^{19}\)See Theorem 3.16

\(^{20}\)This can also be easily seen without invoking the notion of nuclear spaces. Let \( \mathcal{M} \) and \( \mathcal{N} \) be complex manifolds. Then \( \mathcal{O}(\mathcal{M}, \mathcal{O}(\mathcal{N})) = \mathcal{O}(\mathcal{M} \times \mathcal{N}) \) with the topology of uniform convergence over compacts. Let \( F \in \mathcal{O}(\mathcal{M} \times \mathcal{N}) \) and let \( L \subset \mathcal{M} \) and \( H \subset \mathcal{N} \) be compact subsets, then \( K := L \times H \) is a compact subset of \( \mathcal{M} \times \mathcal{N} \) and

\[
\sup_{z \in L, \xi \in H} F(z, \xi) = \sup_{(z, \xi) \in K} F(z, \xi) < \infty.
\]

So \( F \in \mathcal{O}(\mathcal{M}, \mathcal{O}(\mathcal{N})) \). On the other hand, let \( G \in \mathcal{O}(\mathcal{M}, \mathcal{O}(\mathcal{N})) \) and let \( K \subset \mathcal{M} \times \mathcal{N} \), then there are compact subsets \( L \subset \mathcal{M} \) and \( H \subset \mathcal{N} \) with \( K \subset L \times H \) and

\[
\sup_{(z, \xi) \in K} G(z)(\xi) \leq \sup_{z \in L} \sup_{\xi \in H} G(z)(\xi) < \infty.
\]

So \( G \in \mathcal{O}(\mathcal{M} \times \mathcal{N}) \).
Moreover, for \((z, \bar{\omega}) \in \mathbb{C} \times S_{\mathbb{C}}^{n-1}\) and \(k, l\) big enough, we have
\[
e^{-r|\text{Im}(z\bar{\omega})|} |(z\bar{\omega})^\alpha (F_k(z, \bar{\omega}) - F_l(z, \bar{\omega}))| \leq \pi_{\alpha}(F_k - F_l) < \epsilon.
\]

Letting \(l \to \infty\), we obtain, for any multi-index \(\alpha\) and \(k\) big enough,
\[
e^{-r|\text{Im}(z\bar{\omega})|} |(z\bar{\omega})^\alpha (F_k(z, \bar{\omega}) - F^0(z, \bar{\omega}))| < \epsilon.
\]

Hence \(F_k \to F^0\) in the topology of \(\overline{\mathcal{PW}}_{r}^{S_{\mathbb{C}}^{n-1}}\), and for any \(\alpha \in \mathbb{N}^n\),
\[
e^{-r|\text{Im}(z\bar{\omega})|} |(z\bar{\omega})^\alpha F^0(z, \bar{\omega})|
\leq e^{-r|\text{Im}(z\bar{\omega})|} \left(|(z\bar{\omega})^\alpha (F^0(z, \bar{\omega}) - F_k(z, \bar{\omega}))| + |(z\bar{\omega})^\alpha F_k(z, \bar{\omega})|\right)
< \infty.
\]

This shows that \(F^0 \in \mathcal{O}_r(\mathbb{C}, \mathcal{O}(S_{\mathbb{C}}^{n-1}))\). Furthermore, for \((z, \omega) \in \mathbb{C} \times S^{n-1}\), we have
\[
\lim_{m} \left( \frac{\partial}{\partial z} \right)^k F_m(z, \omega) = \lim_{m} \frac{k!}{2\pi i} \oint_{\gamma} \frac{F_m(\xi, \omega)}{(\xi - z)^{k+1}} d\xi
= \frac{k!}{2\pi i} \oint_{\gamma} \frac{F^0(\xi, \omega)}{(\xi - z)^{k+1}} d\xi
= \left( \frac{\partial}{\partial z} \right)^k F^0(z, \omega).
\]

This implies that \(F^0\) satisfies the homogeneity condition. Finally, since uniform convergence implies pointwise convergence, \(F^0\) is even. This completes the proof.

\[\square\]

**Proposition 4.37.** \(S^{n-1}\) is a totally real submanifold of \(S_{\mathbb{C}}^{n-1}\).

**Proof.** Let \(F\) be in \(\mathcal{O}(S_{\mathbb{C}}^{n-1})\) such that \(F|_{S^{n-1}} \equiv 0\). Recall that
\[
\text{Exp} : \mathfrak{q}_{\mathbb{C}} \to S_{\mathbb{C}}^{n-1} : X \mapsto e^X \text{SO}(n - 1, \mathbb{C})
\]
is a local diffeomorphism, which means that there is an open subset of \(\mathfrak{q}_{\mathbb{C}}\) containing \(0\), \(U\), and an open subset of \(S_{\mathbb{C}}^{n-1}\) containing the basepoint \(e\text{SO}(n - 1, \mathbb{C})\), \(V\), such that
\[
\text{Exp} : U \to V
\]
is a diffeomorphism. Hence the pair \((V, \text{Exp}^{-1})\) is a coordinate chart around \(e\text{SO}(n - 1, \mathbb{C})\). Thus by definition \(F \circ \text{Exp} : U \to \mathbb{C}\) is holomorphic. Let \(W = U \cap \mathfrak{q}\), then \(0 \in W\). So \(W\) is an open subset of \(\mathfrak{q}\) and \(\text{Exp}(W)\) is an open subset of \(S^{n-1}\). Hence by assumption \(F \circ \text{Exp}(W) \equiv 0\). Since \(\mathfrak{q} \simeq \mathbb{R}^n\) is a totally real submanifold of \(\mathfrak{q}_{\mathbb{C}} \simeq \mathbb{C}^n\), we obtain that \(F \circ \text{Exp}(U) = 0\). This shows that \(F\) vanishes on an open subset of \(S_{\mathbb{C}}^{n-1} \subset \mathbb{C}^n\): \(F(V) = 0\). Hence \(F \equiv 0\) on \(S_{\mathbb{C}}^{n-1}\).

\[\square\]

Our aim is to prove the following:
**Theorem 4.38. (Euclidean Paley-Wiener type theorem)** Let \( f \in D_r(\mathbb{R}^n) \). Then \( \mathcal{F}_{E(n)} f \) extends to an even holomorphic function on \( \mathbb{C} \times S^{n-1}_c \), denote this extension by \( \mathcal{F}^e_{E(n)} f \). Moreover, \( \mathcal{F}^e_{E(n)} f \in \overline{\mathcal{P}W}_{2\pi r}^{Z_2, H} \) and the map

\[
D_r(\mathbb{R}^n) \to \overline{\mathcal{P}W}_{2\pi r}^{Z_2, H} : f \mapsto \mathcal{F}^e_{E(n)} f
\]

is a topological isomorphism.

For clarity of the exposition we will prove this result in several steps. We remark that by Theorem 2.4 it suffices to prove the following: for \( \varphi \in D_{H,r}(\Xi) \) the Fourier transform \( \mathcal{F}_\mathbb{R}(\varphi)(r, \omega) \) extends to an even holomorphic function on \( \mathbb{C} \times S^{n-1}_c \), this extension belongs to \( \overline{\mathcal{P}W}_{2\pi r}^{Z_2, H} \), and \( \mathcal{F}^e_{\mathbb{R}} \) defines a linear topological isomorphism \( D_{H,r}(\Xi) \simeq \overline{\mathcal{P}W}_{2\pi r}^{Z_2, H} \).

**Lemma 4.39.** Let \( F \) be in \( \mathcal{O}_r(\mathbb{C}, \mathcal{O}(S^{n-1}_c)) \). Then for any \( \omega \in S^{n-1}, t \mapsto F(t, \omega) \) is in \( \mathcal{S}(\mathbb{R}) \).

**Proof.** Let \( N \in \mathbb{N}, z \in \mathbb{C} \) and \( \gamma(t) = z + e^{it} \) with \(-\pi \leq t \leq \pi\). Then

\[
(1 + |z|^2)^Ne^{-r|\text{Im}(z)|} \left| \frac{d}{dz} F(z, \omega) \right| \leq \frac{1}{2\pi} \oint_{\gamma} (1 + |z|^2)^Ne^{-r|\text{Im}(z)|} |F(\xi, \omega)| |d\xi|.
\]

Subdivide the path \( \gamma \) into two paths \( \gamma_1 \) and \( \gamma_2 \) by letting \( \gamma_1 \) be \( \gamma \) for \(-\pi \leq t \leq 0 \), and \( \gamma_2 \) be \( \gamma \) for \(0 \leq t \leq \pi\). On \( \gamma_1 \) we have \( |\text{Im}(\xi)| \leq |\text{Im}(z)| \) and hence \( e^{-r|\text{Im}(\xi)|} \leq e^{-r|\text{Im}(z)|} \). Consequently,

\[
\oint_{\gamma_1} (1 + |z|^2)^Ne^{-r|\text{Im}(z)|}|F(\xi, \omega)| |d\xi| \\
\leq \oint_{\gamma_1} (1 + |z|^2)^N e^{-r|\text{Im}(\xi)|} |F(\xi, \omega)| |d\xi| \\
\leq q_{N,r}(F) \oint_{\gamma_1} (1 + |z|^2)^N |d\xi|.
\]

On \( \gamma_2 \) we have \( |\text{Im}(\xi)| - |\text{Im}(z)| \leq 1 \) and so

\[
\oint_{\gamma_2} (1 + |z|^2)^Ne^{-r|\text{Im}(z)|}e^{r|\text{Im}(\xi)|}e^{-r|\text{Im}(\xi)|}|F(\xi, \omega)| |d\xi| \\
\leq q_{N,r}(F) \oint_{\gamma_2} (1 + |z|^2)^N e^{r(|\text{Im}(\xi)| - |\text{Im}(z)|)} |d\xi| \\
\leq q_{N,r}(F) e^r \oint_{\gamma_2} (1 + |z|^2)^N |d\xi|.
\]
Moreover,

\[
\frac{1 + |z|^2}{1 + |\xi|^2} = \frac{1 + |\xi - e^{it}|^2}{1 + |\xi|^2} \leq \frac{2 + 2|\xi| + |\xi|^2}{1 + |\xi|^2} \leq 1 + \frac{1 + 2|\xi|}{1 + |\xi|^2} \leq 4.
\]

Using the above estimate along with the estimates on \( \gamma_1 \) and \( \gamma_2 \) we obtain

\[
(1 + |z|^2)^Ne^{-r|\text{Im}(z)|}\left|\frac{d}{dz}F(z, \omega)\right| \leq \frac{1}{2\pi} q_{N,r}(F) e^r \oint_{\gamma} \left( \frac{1 + |z|^2}{1 + |\xi|^2} \right)^N |d\xi| \leq 4^N e^r q_{N,r}(F).
\]

We conclude that

\[
\sup_{z \in \mathbb{C}} (1 + |z|^2)^Ne^{-r|\text{Im}(z)|}\left|\left(\frac{d}{dz}\right)^k F(z, \omega)\right| \leq k! 4^N e^r q_{N,r}(F).
\]

Restricting to the real line yields

\[
\sup_{t \in \mathbb{R}} (1 + t^2)^N \left|\left(\frac{d}{dt}\right)^k F(t, \omega)\right| \leq k! 4^N e^r q_{N,r}(F).
\]

Lemma 4.40. Let \( F \) be in the space \( \mathcal{O}_r(\mathbb{C}, \mathcal{O}(S_{n-1}^n)) \) and let \( \omega \) be a point on the real sphere \( S^{n-1} \). Then for any \( y \in \mathbb{R} \),

\[
\int_{\mathbb{R}} F(t, \omega)dt = \int_{\mathbb{R}} F(t + iy, \omega)dt.
\]

Proof. Let \( T > 0 \) and consider the following four curves

\[
\begin{align*}
\gamma_1(t) &= t & -T \leq t \leq T, \\
\gamma_2(t) &= T + ity_n & 0 \leq t \leq 1, \\
\gamma_3(t) &= -t + i y_n & -T \leq t \leq T, \\
\gamma_4(t) &= -T + i(1 - t)y_n & 0 \leq t \leq 1.
\end{align*}
\]
Define $\gamma$ to be the sum of these curves:

$$
\gamma(t) := \begin{cases} 
\gamma_1(t) & \text{for } -T \leq t \leq T, \\
\gamma_2(t - T) & \text{for } T \leq t \leq T + 1, \\
\gamma_3(t - (2T + 1)) & \text{for } T + 1 \leq t \leq 3T + 1, \\
\gamma_4(t - (3T + 1)) & \text{for } 3T + 1 \leq t \leq 3T + 2.
\end{cases}
$$

As $z \mapsto F(z, \omega)$ is holomorphic, we have:

$$\int_{\gamma} F(\xi, \omega) d\xi = 0.$$

We can rewrite the above integral over $\gamma$ as a sum of integrals over $\gamma_i$'s:

$$\int_{-T}^{T} F(z, t) dt + iy_n \int_{0}^{1} F(z, T + ity_n) dt - \int_{-T}^{T} F(z, t + i\omega) dt - iy_n \int_{1}^{0} F(z, -T + ity_n) dt = 0.$$

Since $|z\omega| = |z| |\omega| = |z|$ and $|\text{Im}(z\omega)| = |\text{Im}(z)|$, we have

$$|F(\pm T + iyt, \omega)| \leq \pi_{r,1}(F) \frac{e^{r|yt|}}{1 + T^2 + (yt)^2} \leq \pi_{r,1}(F) \frac{e^{r|yt|}}{T^2}.$$ 

Since $y$ is fixed, this yields

$$\lim_{T \to \infty} \int_{0}^{1} F(\pm T + iyt, \omega) dt = 0.$$

And we obtain what we need:

$$\int_{-\infty}^{\infty} F(t, \omega) dt = \int_{-\infty}^{\infty} F(t + iy, \omega) dt = 0.$$

\[\square\]

**Lemma 4.41.** Let $F \in \mathcal{O}_r(\mathbb{C}, \mathcal{O}(S_n^{n-1}))$ and $t \in \mathbb{R}$. Define $H(\xi, \bar{\omega}) := F(\xi, \bar{\omega}) e^{2\pi i t}$. Then $H \in \mathcal{O}_{r + 2\pi t}(\mathbb{C}, \mathcal{O}(S_n^{n-1}))$.

**Proof.** Observe that $|\text{Im}(\bar{\omega})|^2 = |\text{Im}(\xi)|^2 + |\xi|^2 |\text{Im}(\bar{\omega})|^2$. To see this, write $\xi = x + iy$ and write $\bar{\omega}_j = a_j + ib_j$. Then $\sum a_j^2 = \sum (a_j^2 + b_j^2)$ and so $\sum a_jb_j = 0$ and $\sum (a_j^2 - b_j^2) = 1$. And $\xi\bar{\omega} = ((x + iy)(a_1 + ib_1), \ldots) = (xa_1 - yb_1 + i(ya_1 + xb_1), \ldots)$, so

$$|\text{Im}(\bar{\omega})|^2 = \sum (y^2 a_j^2 + 2xy a_j b_j + x^2 b_j^2) = y^2 \sum a_j^2 + 2xy \sum a_j b_j + x^2 \sum b_j^2 = y^2 \sum (a_j^2 - b_j^2) + (x^2 + y^2) \sum b_j^2 = |\text{Im}(\xi)|^2 + |\xi|^2 |\text{Im}(\bar{\omega})|^2.$$
Hence $|\operatorname{Im}(\xi)| \leq |\operatorname{Im}(\xi \tilde{\omega})|$. We compute

$$(1 + |\xi \tilde{\omega}|^2)^N |H(\xi, \tilde{\omega})| e^{-(r+|2\pi t|)|\operatorname{Im}(\xi \tilde{\omega})|}$$

$$= (1 + |\xi \tilde{\omega}|^2)^N |F(\xi, \tilde{\omega})| e^{2\pi \xi t} e^{-(r+|2\pi t|)|\operatorname{Im}(\xi \tilde{\omega})|}$$

$$= (1 + |\xi \tilde{\omega}|^2)^N |F(\xi, \tilde{\omega})| e^{-r|\operatorname{Im}(\xi \tilde{\omega})|} e^{-2\pi t|\operatorname{Im}(\xi)|} e^{-|2\pi t| |\operatorname{Im}(\xi \tilde{\omega})|}$$

$$\leq \pi r,N(F) e^{2\pi r |\operatorname{Im}(\xi)|} e^{-|2\pi t| |\operatorname{Im}(\xi \tilde{\omega})|}$$

$$= \pi r,N(F).$$

We now complete the proof of the Theorem 4.38:

**Proof.** Let $\varphi \in D_{H,r}(\Xi)$ and fix an $\omega \in S^{n-1}$. Define $F$ to be the Fourier transform of $\varphi$ in the first variable. For $\xi \in \mathbb{C}$ and $t \in [-r, r]$ we have the estimate $|e^{-2\pi i t \xi}| \leq e^{2\pi r |\operatorname{Im}(\xi)|}$. Hence

$$|\varphi(t, \omega) e^{-2\pi i t \xi}| \leq \sup_{t \in \mathbb{R}} |\varphi(t, \omega)| e^{2\pi r |\operatorname{Im}(\xi)|} \chi_{[-r,r]}(t) \in L^1_t(\mathbb{R}).$$

This shows that $F$ is well-defined on $\mathbb{C} \times S^{n-1}$. Moreover, $\xi \mapsto F_R(\varphi)(\xi, \omega)$ converges uniformly on compact subsets of $\mathbb{C}$ and hence is holomorphic and we can differentiate inside the integral:

$$\frac{d}{d\xi} F_R(\varphi)(\xi, \omega) = \int_{\mathbb{R}} \varphi(t, \omega) \frac{d}{d\xi} e^{-2\pi i t \xi} dt.$$ 

Since $\varphi \in D_{H,r}(\Xi)$, there is a function $f \in D_r(\mathbb{R}^n)$ such that $Rf = \varphi$.

$$F(t, \omega) = F_R(\varphi)(t, \omega)$$

$$= \int_{-\infty}^{\infty} \varphi(s, \omega) e^{-2\pi i s t} ds$$

$$= \int_{-\infty}^{\infty} Rf(s, \omega) e^{-2\pi i s t} ds$$

$$= \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \omega} dx$$

$$= F_{R^n}(f)(t \omega).$$

The penultimate equality holds by the Fourier-Slice theorem. And by the classical Paley-Wiener theorem $F_{R^n}(f)$ has a holomorphic extension to $\mathbb{C}^n$. It follows that $F$ extends as a holomorphic function on a bigger domain $\mathbb{C} \times S^{n-1}_C$:

$$\mathbb{C} \times S^{n-1}_C \longrightarrow \mathbb{C}^n \longrightarrow \mathbb{C}$$

$$(z, \tilde{\omega}) \longmapsto z \tilde{\omega} \longmapsto F_{R^n}(f)(z \tilde{\omega}) =: F(z, \tilde{\omega}),$$

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and that we have the estimate
\[
\sup_{(z,\tilde{\omega})\in \mathbb{C}\times S_{C}^{n-1}} (1 + |z\tilde{\omega}|^2)^N e^{-2\pi r|\text{Im}(z\tilde{\omega})|} |F(z,\tilde{\omega})| < \infty. \tag{4.3.3}
\]

We have holomorphically extended \(F_{\mathbb{R}}(\varphi)\) in two different ways to two different domains, namely to \(\mathbb{C}\times S^{n-1}\) and to \(\mathbb{C}\times S_{C}^{n-1}\). Let us verify that these two extensions agree on their common domain, namely on \(\mathbb{C}\times S^{n-1}\):

\[
F_{\mathbb{R}}(f)(z\omega) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i(z\omega,x)} dx
= \int_{\mathbb{R}} \int_{(\omega,x)=t} f(x) e^{-2\pi iz(x\omega)} dm(x) dt
= \int_{\mathbb{R}} \int_{(\omega,x)=t} f(x) dm(x) e^{-2\pi izt} dt
= \int_{\mathbb{R}} \mathcal{R} f(t,\omega) e^{-2\pi izt} dt
= \int_{\mathbb{R}} \varphi(t,\omega) e^{-2\pi izt} dt
= F_{\mathbb{R}} \varphi(z,\omega) dt.
\]

Since \(S^{n-1}\) is a totally real submanifold of \(S_{C}^{n-1}\), to show that \(F(-\xi, -\tilde{\omega}) = F(\xi, \tilde{\omega})\) on \(\mathbb{C}\times S_{C}^{n-1}\), it is enough to verify it on \(\mathbb{R}\times S^{n-1}\).

\[
F(-t, -\omega) = \int_{\mathbb{R}} \varphi(s, -\omega) e^{2\pi ist} ds
\]

making a change of variables, we get
\[
= \int_{\mathbb{R}} \varphi(-s, -\omega) e^{-2\pi ist} dr
\]

and using that \(\varphi\) is even
\[
= \int_{\mathbb{R}} \varphi(s, \omega) e^{-2\pi ist} dr
= F(t, \omega).
\]

Let \(a_k(\omega) := \left(\frac{d}{d\xi}\right)^k F(\xi, \omega)|_{\xi=0}\). As we can differentiate inside the integral,

\[
a_k(\omega) = (-2\pi i)^k \int_{\mathbb{R}} \varphi(t, \omega) t^k dt.
\]

Thus for \(k \in \mathbb{N}^+\), \(a_k\) is a homogeneous polynomial in \(\omega_1, \ldots, \omega_n\) of degree \(k\). Furthermore, for \((\xi, \omega) \in \mathbb{C} \times S^{n-1}\) we have

\[
F(\xi, \omega) = F(0, \omega) + \sum_{m=1}^{\infty} \frac{\left(\frac{d}{d\xi}\right)^m F(\xi, \omega)|_{\xi=0}}{m!} \xi^m = F(0, \omega) + \sum_{m=1}^{\infty} \frac{a_m(\omega)}{m!} \xi^m.
\]
This shows that $F \in \mathcal{P}\mathcal{W}_{2\pi r}^{Z_2,H}$. The mapping $\varphi \mapsto F$ is injective and as the Radon transform is a linear topological isomorphism between the spaces $\mathcal{D}_r(\mathbb{R}^n)$ and $\mathcal{D}_{H,r}(\Xi)$, it is also continuous.

To show the converse, let $F$ be in $\mathcal{P}\mathcal{W}_{2\pi r}^{Z_2,H}$. By Lemma 4.39 for any $\omega \in S^{n-1}$, the function $t \mapsto F(t,\omega)$ is Schwartz. We will use the same letter for this restriction of $F$ to the real line. Since the Fourier transform is a topological isomorphism of the Schwartz space with itself, the inverse Fourier transform of $F$ in the first variable is a Schwartz function in the first variable, call it $\varphi := \mathcal{F}^{-1}_r(F)$.

By Lemma 4.41 $F(\xi,\omega)e^{-2\pi i\alpha\xi}$ is in $\mathcal{O}_{r+|2\pi s|}(\mathcal{C},\mathcal{O}(S_{2\pi s}^{n-1}))$ and by Lemma 4.40

$$\int_{\mathbb{R}} F(s,\omega)e^{-2\pi its}ds = \int_{\mathbb{R}} F(s+i\eta,\omega)e^{-2\pi it(s+i\eta)}ds,$$

for any $\eta \in \mathbb{R}$, that is,

$$\varphi(t,\omega) = \int_{\mathbb{R}} F(s+i\eta,\omega)e^{-2\pi it(s+i\eta)}ds = e^{-2\pi t\eta} \int_{\mathbb{R}} F(s+i\eta,\omega)e^{-2\pi its}ds.$$

Use the seminorm inequality $|F(s+i\eta,\omega)| \leq \pi_{r,1}(F)e^{2\pi r|\eta|}(1+s^2+\eta^2)^{-1}$ to estimate

$$|\varphi(t,\omega)| \leq e^{-2\pi t\eta} \int_{\mathbb{R}} |F(s+i\eta,\omega)|ds \leq \pi_{r,1}(F)e^{2\pi (r|\eta|-t\eta)} \int_{\mathbb{R}} \frac{ds}{1+s^2} < \infty.$$

For $t > r$, choose $\eta > 0$, then $r|\eta| - t\eta < 0$. For $t < -r$, choose $\eta < 0$, then again $r|\eta| - t\eta < 0$. In both cases, taking the limit as $\eta \to \infty$ yields $\varphi(t,\omega) = 0$ for $|t| > r$. This shows that $\text{supp}(\varphi) \subseteq [-r,r] \times S^{n-1}$.

Next we show that $\omega \mapsto \varphi(t,\omega)$ is smooth for any $t \in \mathbb{R}$. Fix $\omega \in S^{n-1}$ and let $(U,\phi)$ be a local coordinate chart around $\omega$. We have to show that $\phi(\omega) \mapsto (\varphi(t) \circ \phi^{-1})(\phi(\omega))$ is smooth. Suppose we have shown that for any multi-index $\alpha \in \mathbb{N}^n$

$$|D^\alpha_{\phi(\omega)}(F(t) \circ \phi^{-1})(\phi(\omega))| \in L^1_1(\mathbb{R}),$$

then by the Lebesgue Dominated Convergence theorem

$$D^\alpha_{\phi(\omega)}(\varphi(t) \circ \phi^{-1})(\phi(\omega)) = \int_{\mathbb{R}} D^\alpha_{\phi(\omega)}(F(t) \circ \phi^{-1})(\phi(\omega)) e^{2\pi ist}ds$$

and we are done.

Fix $t \in \mathbb{R}$. As $F$ is in $\mathcal{P}\mathcal{W}_{2\pi r}^{Z_2,H}$, $F(t) \in \mathcal{O}(S_{n-1}^{n-1})$ and so $F(t) \circ \phi^{-1} : \phi(U) \to \mathbb{C}$ is a holomorphic function on $\phi(U) \subset \mathbb{C}^n$. Let $\gamma$ be a circular path around $\phi_j(\omega)$ of

\[^{22}\text{See Theorem 2.4}\]
radius \( \delta > 0: \gamma(s) = \phi_j(\omega) + \delta e^{i\omega}, \) then

\[
\frac{\partial}{\partial [\phi_j(\omega)]}(F(t) \circ \phi^{-1})(\phi_1(\omega), \ldots, \phi_n(\omega))
= \frac{1}{2\pi i} \oint_{\gamma} \frac{F(t) \circ \phi^{-1}(\phi_1(\omega), \ldots, \phi_{j-1}(\omega), \xi, \phi_{j+1}(\omega), \ldots, \phi_n(\omega))}{(\xi - \phi_j(\omega))^2} \, d\xi.
\]

For convenience, let \( z \) denote the vector \((\phi_1(\omega), \ldots, \phi_{j-1}(\omega), \xi, \phi_{j+1}(\omega), \ldots, \phi_n(\omega))\). Each \( z \) is contained in a closed ball of radius \( \delta \) around \( \phi(\omega) \). Since \( \phi^{-1} \) is continuous, \( \phi^{-1}(B_\delta(\phi(\omega))) \) is a compact subset of \( \mathbb{S}^{m-1}_\xi \). Hence there is a constant \( \tilde{c}_j \) so that \( |\text{Im}(\phi^{-1}(z))| \leq \tilde{c}_j \). We can write \( \tilde{c}_j = \frac{\tilde{c}_j}{\delta} \), denote \( \frac{\tilde{c}_j}{\delta} \) by \( c_j \). As \( \text{Im}(t\tilde{\omega}) = t\text{Im}(\tilde{\omega}) \) and \( |t\tilde{\omega}|^2 = t^2|\tilde{\omega}|^2 \geq t^2 \), we obtain

\[
|(F(t) \circ \phi^{-1})(z)| = |F(t)(\phi^{-1}(z))|
\leq \pi r,N(F) e^{2\pi r|\text{Im}(t\phi^{-1}(z))|} \frac{(1 + |t\phi^{-1}(z)|^2)^N}{(1 + t^2)^N}
\leq \pi r,N(F) e^{2\pi r|c_j\delta}} \frac{(1 + t^2)^N}{(1 + t^2)^N}.
\]

Let \( \delta \leq \frac{1}{1 + |t|} \), then

\[
\left|\frac{\partial}{\partial [\phi_j(\omega)]}(F(t) \circ \phi^{-1})(\phi(\omega))\right|
\leq \frac{1}{2\pi} \oint_{\gamma} \frac{|F(t)(\phi^{-1}(z))|}{|\xi - \phi_j(\omega)|^2} \, |d\xi|
\leq \frac{1}{2\pi \delta^2} \pi r,N(F) e^{2\pi r|c_j\delta}} \frac{(1 + t^2)^N}{(1 + t^2)^N} \oint_{\gamma} |d\xi|
\leq \frac{1}{\delta} \pi r,N(F) e^{2\pi r c_j}} \frac{(1 + t^2)^N}{(1 + t^2)^N}.
\]

For higher order derivatives we have,

\[
\left(\frac{\partial}{\partial [\phi_j(\omega)]}\right)^k (F(t) \circ \phi^{-1})(\phi(\omega)) = \frac{k!}{2\pi i} \oint_{\gamma} \frac{F(t)(\phi^{-1}(z))}{(\xi - \phi_j(\omega))^{k+1}} \, d\xi,
\]

and consequently we have the estimate

\[
\left|\left(\frac{\partial}{\partial [\phi_j(\omega)]}\right)^k (F(t) \circ \phi^{-1})(\phi(\omega))\right|
\leq k! \left(\frac{1}{\delta} \pi r,N(F) e^{2\pi r c_j}} \frac{(1 + t^2)^N}{(1 + t^2)^N} \right).
\]

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For each $j = 1, \ldots, n$ we have the corresponding constants $c_j$, set $c := \max \{c_j\}$. Hence for a differential operator $D^\alpha_{\phi(\omega)}$ we have

$$|D^\alpha_{\phi(\omega)}(F(t) \circ \phi^{-1})(\phi(\omega))| \leq \sum_{j=1}^{n} \alpha_j! \left( \frac{1}{\delta} \right)^{\alpha_j} \pi_{r,N}(F) e^{2\pi r c_j} \frac{1}{(1 + t^2)^N}$$

$$\leq n \alpha! \left( \frac{1}{\delta} \right)^{\alpha} \pi_{r,N}(F) e^{2\pi r c} \frac{1}{(1 + t^2)^N}.$$ 

We see that by choosing $N \geq 1$ we ensure that the desired integrability condition is satisfied. We have shown that $\phi$ is a Schwartz function on $\mathbb{R} \times S^{n-1}$.

As $t \mapsto F(t, \omega)$ is Schwartz and the Schwartz space is invariant under multiplication by polynomials, we can differentiate inside the integral in

$$(\frac{d}{dt})^k \varphi(t, \omega) = \int_{\mathbb{R}} F(s, \omega) \left( \frac{d}{dt} \right)^k e^{2\pi i st} ds.$$ 

We have as well, using the estimate from above, that

$$|D^\alpha_{\omega} F(s, \omega)(2\pi i s)^k| \leq (2\pi)^k n \alpha! \left( \frac{1}{\delta} \right)^{\alpha} e^{2\pi r c} \frac{s^k}{(1 + s^2)^N} \pi_{r,N}(F),$$

which is integrable in $s$ for big enough $N$. Consequently,

$$(\frac{d}{dt})^k D^\alpha_{\omega} \varphi(t, \omega) = \int_{\mathbb{R}} D^\alpha_{\omega} F(s, \omega) \left( \frac{d}{dt} \right)^k e^{2\pi i st} ds,$$

and we have the estimate,

$$\left| \left( \frac{d}{dt} \right)^k D^\alpha_{\omega} \varphi(t, \omega) \right| \leq \frac{(2\pi)^k n \alpha! e^{2\pi r c}}{\delta^{\alpha}} \int_{\mathbb{R}} \frac{s^k}{(1 + s^2)^N} ds \pi_{r,N}(F).$$

This shows that $|\varphi|_{k,D_{\omega}} < \infty$ and that the mapping $F \mapsto \varphi$ is continuous.

By assumption, for any $k \in \mathbb{N}$, $k > 0$, $\left( \frac{d}{d\xi} \right)^k F(\xi, \omega)|_{\xi=0}$ is a homogeneous polynomial of degree $k$ in $\omega_1, \ldots, \omega_n$. Since

$$\int_{\mathbb{R}} \varphi(t, \omega) t^k dt = \left( i \frac{d}{dt} \right)^k F(t, \omega)|_{t=0},$$

$\varphi$ satisfies the homogeneity condition. That $\varphi$ is even follows again be a change of variables. We have shown, $\varphi$ is in $D_{H,r}(\Xi)$. \hfill $\Box$

### 4.3.4 Remarks

Let $S$ be the set of isotropic vectors in $\mathbb{C}^n$. It was shown above that the map $\mathbb{C}^* \times S^{n-1}_{\mathbb{C}} \to \mathbb{C}^n \setminus S: (z, \tilde{\omega}) \mapsto z\tilde{\omega}$ is a holomorphic two-to-one map. Note that $S$
is closed\textsuperscript{23} and $\mathbb{C}^n \setminus S$ is dense in $\mathbb{C}^n$.\textsuperscript{24} Let $F$ be in $\mathcal{O}(\mathbb{C}^n \setminus S)$. Then $F$ can be extended to an entire function on $\mathbb{C}^n$. To see this, let $z \in S$. There is a sequence $z^k$ in $\mathbb{C}^n \setminus S$ converging to $z$ and $\{F(z^k)\}$ is a Cauchy sequence in $\mathbb{C}$. As $\{z^k\}$ is bounded and $F$ is uniformly bounded over compacts, $|\lim F(z^k)| < \infty$. Define $F(z) := \lim F(z^k)$. Then $F$ is well-defined\textsuperscript{25} and continuous on $\mathbb{C}^n$. Consequently, by Theorem 2.10, $F$ is entire.

Let $F$ be in $\mathcal{PW}_{Z_{2,H}}(\mathbb{C}, \mathcal{O}(S_{\mathbb{C}}^{n-1}))$, then, as observed in the above paragraph, $F$ extends to a holomorphic function on $\mathbb{C}^n$, denote this extension by $\text{Ext}(F)$. Then $\text{Ext}(F)$ is in $\mathcal{PW}_r(\mathbb{C}^n)$. It is clear that this mapping is injective and continuous.

Moreover, we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{D}_H(\Xi) & \xrightarrow{F_{\tilde{R}}} & \mathcal{PW}_{2\pi r}(\mathbb{C}, \mathcal{O}(S_{\mathbb{C}}^{n-1})) \\
\mathcal{D}_r(\mathbb{R}^n) & \xrightarrow{F_{\tilde{R}}^c} & \mathcal{PW}_{2\pi r}(\mathbb{C}^n)
\end{array}
\]

Since the Fourier transforms $F_{\tilde{R}}^c$ and $F_{\tilde{R}}^c$, as well as the inverse Radon transform $R^{-1}$, are linear topological isomorphisms between the function spaces indicated in the diagram, it follows that the extension map, $\text{Ext}$, is a linear topological isomorphism as well.

In the next chapter we shall make use of the inverse of this extension mapping, which we denote by $\tilde{R}$. We will refer to it as a Radon type transform. Let us re-draw the above diagram as follows:

\[
\begin{array}{ccc}
\mathcal{D}_r(\mathbb{R}^n) & \xrightarrow{R} & \mathcal{D}_H(\Xi) \\
\mathcal{PW}_{2\pi r}(\mathbb{C}^n) & \xrightarrow{\tilde{R}} & \mathcal{PW}_{Z_{2,H}}(\mathbb{C}, \mathcal{O}(S_{\mathbb{C}}^{n-1}))
\end{array}
\]

For a function $F$ in $\mathcal{PW}_{2\pi r}(\mathbb{C}^n)$ there is a unique function $f$ in $\mathcal{D}_r(\mathbb{R}^n)$ whose Fourier-Laplace transform is $F$. Then following the arrows in the above diagram we obtain for $\tilde{R}F$ the following:

\[
\tilde{R}F(z, \tilde{\omega}) = F_{\tilde{R}}^c(Rf)(z, \tilde{\omega}) = F_{\tilde{R}}^c f(z \tilde{\omega}) = F(z \tilde{\omega}).
\]

\textsuperscript{23}Observe that $\xi = x + iy \in S$ iff $\sum x_j = \sum y_j$ and $\sum x_jy_j = 0$. Let $\{z^k\}$ be a Cauchy sequence in $S$. Then it is Cauchy in $\mathbb{C}^n$ and hence converges to some $z \in \mathbb{C}^n$. Write $z^k$ as $a^k + ib^k$ and $z$ as $a + ib$. Then the reals sequences $\{a^k_j\}$, $\{b^k_j\}$ converge to $a_j$, $b_j$, respectively, and $\sum a^k_j = \lim \sum (a^k_j)^2 = \lim \sum (b^k_j)^2 = \sum b^2_j$ as well as $\sum a_jb_j = \lim \sum a^k_j b^k_j = 0$.

\textsuperscript{24}Let $z = (z_1, \ldots, z_n) \in S$. Define $z^k := (z_1 + \frac{1}{k}, \ldots, z_n + \frac{1}{k})$, then $\{z^k \to z\}$ as $k \to \infty$ and $\sum(z^k)_j^2 = \sum(z_j + \frac{1}{k})^2 \neq 0$. So $\{z^k\}$ is a sequence in $\mathbb{C}^n \setminus S$.

\textsuperscript{25}Suppose there is another sequence $\{\xi^m\}$ in $\mathbb{C}^n \setminus S$ converging to $z$, different from $z^k$, and $\lim F(\xi^m) = \eta \neq v = \lim F(z^k)$. Then there exists an $N \in \mathbb{N}$ with $|F(\xi^m) - F(z^l)| \geq \frac{d(\eta, v)}{2}$ for $m, l > N$. But $|z^m - z^l| \to 0$ as $m, l \to \infty$ contradicting the continuity of $F$. 

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Let us make another remark. Pick a function $F$ in $\hat{\mathcal{PW}}_{2, H}(\mathbb{C}, \mathcal{O}(S^{n-1}_C))$ and consider its restriction to the real sphere. Clearly $F|_{S^{n-1}}$ is in $\mathcal{PW}_{2, H}(\mathbb{C}, \mathcal{S}(S^{n-1}))$ and this restriction map is injective. By the two Theorems 4.33 and 4.38, it is also surjective. We note that results of this flavor have been obtained before in [2]. There a local Paley-Wiener theorem is considered, and the authors give necessary and sufficient conditions for a function, which restricts analytically to the sphere $S^{n-1}$ with $n = 2, 3$, to be in a classical Paley-Wiener space, $\mathcal{PW}_r(\mathbb{C}^n)$, for some $r > 0$. 
Chapter 5

Extension of the Euclidean Paley-Wiener Theorem to Projective Limits

In this chapter we extend our second description of the Fourier transform image of $\mathcal{D}(\mathbb{R}^n)$ to the projective limits of the spaces in consideration. In Section 5.1 we recall the definition of an injective and projective limit. In the following Section 5.2 we discuss the extension of the classical Paley-Wiener theorem to the projective limits, which is a consequence of a result by Cowling. And in the last Section 5.3 we then present the extension of our description to the projective limits of the corresponding function spaces, which can also be interpreted as an extension to the inductive limits of the underlying spaces.

5.1 Inductive and Projective Limits

Following the approach in [24], which is the main reference for the first part of this section, we define inductive and projective limits by their universal mapping properties. Then we look at these two limits in the category of vector spaces.

In the following, the index set may be uncountable.

Definition 5.1. Let $\mathcal{C}$ be a category, $(A, \leq)$ a directed set, and let $\{S_\alpha\}_{\alpha \in A}$ be a family of objects in $\mathcal{C}$. Suppose that for each $\alpha \leq \beta$ there is a morphism $\phi_\alpha^\beta : S_\alpha \to S_\beta$, such that

(a) $\phi_\alpha^\alpha$ is the identity morphism for every $\alpha \in A$,

(b) $\phi_\gamma^\beta = \phi_\gamma^\delta \circ \phi_\delta^\beta$ for any $\alpha \leq \beta \leq \gamma$.

Then the pair $\{(S_\alpha)_{\alpha \in A}, \{\phi_\alpha^\beta\}_{\beta \geq \alpha}\}$ is called an inductive or a direct system in $\mathcal{C}$ indexed by $A$.

Definition 5.2. Let $\{(S_\alpha)_{\alpha \in A}, \{\phi_\alpha^\beta\}_{\beta \geq \alpha}\}$ be an inductive system in $\mathcal{C}$ and let $T$ be an object in $\mathcal{C}$. Fix an index $\delta \in A$ and a family of morphisms in $\mathcal{C}$, $\{f_\alpha\}_{\alpha \in A, \alpha \geq \delta}$, where $f_\alpha : S_\alpha \to T$. Then the family $\{f_\alpha\}_{\alpha \in A, \alpha \geq \delta}$ is called compatible if for any

\[1\] A category $\mathcal{C}$ consists of the following three entities:

- A class $\text{ob}(\mathcal{C})$, whose elements are called objects.

- A class $\text{hom}(\mathcal{C})$, whose elements are called morphisms or maps or arrows. Each morphism $\phi$ has a unique source object $a$ and a target object $b$. The expression $\text{hom}_\mathcal{C}(a,b)$ denotes all morphisms from $a$ to $b$.

- An associative binary operation $\circ$, called composition of morphisms, such that for any three objects $a$, $b$, and $c$, we have $\text{hom}(a,b) \times \text{hom}(b,c) \to \text{hom}(a,c) : (f,g) \mapsto g \circ f$, and such that for any object $a$, there exists a morphism $\text{id}_a : a \to a$ called the identity morphism for $a$, satisfying that for every morphism $f : b \to c$, we have $\text{id}_a \circ f = f = f \circ \text{id}_c$.

\[2\] A directed set is a nonempty set $A$ together with a reflexive and transitive binary relation $\leq$, with the additional property that every pair of elements has an upper bound: For any $a, b \in A$ there exists an element $c$ in $A$ such that $a \leq c$ and $b \leq c$. 

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\( \alpha \) and \( \beta \) satisfying \( \delta \leq \alpha \leq \beta \) we have

\[ f_\beta \circ \phi_\beta^{\alpha} = f_\alpha. \]

**Definition 5.3.** Let \( \{S_\alpha\}_{\alpha \in A}, \{\phi_\beta^{\alpha}\}_{\beta \geq \alpha} \) be an inductive system in \( C \), \( S \) an object in \( C \), and \( \{\phi_\alpha\}_{\alpha \in A} \) a family of morphisms in \( C \) such that whenever \( \alpha \leq \beta \)

\[ \phi_\beta \circ \phi_\beta^{\alpha} = \phi_\alpha. \]

Suppose that for every object \( T \) in \( C \) and for every compatible family of morphisms \( \{f_\alpha\}_{\alpha \in A, \alpha \geq \delta} \) in \( C \) from \( S_\alpha \) to \( T \), there is a unique morphism \( f \) from \( S \) to \( T \) in \( C \) satisfying for all \( \alpha \in A \) with \( \alpha \geq \delta \)

\[ f \circ \phi_\alpha = f_\alpha. \]

Then the pair \( (S, \{\phi_\alpha\}_{\alpha \in A}) \) is called the **inductive limit of the inductive system** \( (\{S_\alpha\}_{\alpha \in A}, \{\phi_\beta^{\alpha}\}_{\beta \geq \alpha}) \). The inductive limit is also called the **direct limit** or the **injective limit**.

The inductive limit of \( (\{S_\alpha\}_{\alpha \in A}, \{\phi_\beta^{\alpha}\}_{\beta \geq \alpha}) \) is usually denoted by

\[ (S, \{\phi_\alpha\}_{\alpha \in A}) = \lim_{\rightarrow} (\{S_\alpha\}_{\alpha \in A}, \{\phi_\beta^{\alpha}\}_{\beta \geq \alpha}) \text{ or simply by } S = \lim_{\rightarrow} S_\alpha. \]

The morphism \( f \) is called the **inductive** or the **direct limit of** \( \{f_\alpha\}_{\alpha \in A} \), denoted by \( f = \lim_{\rightarrow} f_\alpha. \)

This definition is elucidated by the following commutative diagram. There the horizontal arrows designate the inductive system and the solid arrows indicate its inductive limit, the dotted arrows show a compatible family of morphisms and the dashed arrow denotes the unique morphism, the inductive limit of the compatible family \( \{f_\alpha\} \).

The concepts of a **projective system** and the **projective** or **inverse limit** are dual to the concepts of an inductive system and the inductive limit: they are
obtained by turning around all arrows in the definition of an inductive system, compatible family of morphisms and the inductive limit. The following commutative diagram illustrates the concepts of a projective system and the projective limit.

Here the horizontal arrows designate the projective system and the solid arrows indicate its projective limit, the dotted arrows show a compatible family of morphisms and the dashed arrow denotes the projective limit of the compatible family \( \{g_\alpha\} \), which is the unique morphism whose existence is guaranteed by the universal mapping property of the projective limit.

In the category of vector spaces the inductive limit \((V, \{\phi_\alpha\}_{\alpha \in A})\), of an inductive system \((\{V_\alpha\}_{\alpha \in A}, \{\phi_\alpha^\beta\}_{\beta \geq \alpha})\) is constructed as follows. An element of the vector space \( V \) is an equivalence class \([v]\) of sets \(\{(v_\alpha, \alpha)\}\) with \(v_\alpha \in V_\alpha\). Two elements \((v_\alpha, \alpha), (v_\beta, \beta)\) are equivalent: \((v_\alpha, \alpha) \sim (v_\beta, \beta)\), if there is a \(\gamma \in A\), \(\gamma \geq \alpha, \beta\) such that

\[
\phi_\gamma^\alpha(v_\alpha) = \phi_\gamma^\beta(v_\beta).
\]

In other words, the equivalence relation is defined by the eventual behavior and the inductive limit \( V \) is the disjoint union of \( V_\alpha \)'s modulo the equivalence relation:

\[
V = \lim_{\longrightarrow} V_\alpha = \coprod V_\alpha / \sim.
\]

The vector space operations on \( V \) are then given by

\[
[v] + [w] = [v_\gamma] + [w_\gamma] = [v_\gamma + w_\gamma] \text{ and } \lambda[v] = \lambda[v_\gamma] = [\lambda v_\gamma].
\]

And the linear maps \( \phi_\alpha : V_\alpha \rightarrow V \) are defined by

\[
\phi_\alpha(v_\alpha) = [v_\gamma], \text{ with } v_\gamma = \begin{cases} 
\phi_\gamma^\alpha(v_\alpha) & \text{for } \gamma \geq \alpha, \\
0_{V_\gamma} & \text{otherwise}.
\end{cases}
\]
Example 5.4. Let $k, l \in \mathbb{N}$, $k \geq l$ and denote by $i_k^l$ the inclusion map from $\mathbb{R}^l$ into $\mathbb{R}^k$:

$$i_k^l : \mathbb{R}^l \hookrightarrow \mathbb{R}^k : x \mapsto (x, 0, \ldots, 0).$$

Then the pair $((\mathbb{R}^l)_{l \in \mathbb{N}}, \{i_k^l\}_{k \geq l})$ is an inductive system. The equivalence class for a vector $x \in \mathbb{R}^l$ with $x_l \neq 0$ is

$$[x] = \{(x, l), ((x, 0), l + 1), ((x, 0, 0), l + 2), \ldots \}$$

and $\mathbb{R}^\infty := \lim \leftarrow \mathbb{R}^l = \prod_{l=1}^{\infty} \mathbb{R}^l / \sim \cong \{(x_1, x_2, \ldots) : \text{all but finitely many } x_i = 0\}$.

The projective limit $(V, \{\psi_\alpha\}_{\alpha \in A})$ of a projective system $((V_\alpha)_{\alpha \in A}, \{\psi_\beta^\alpha\}_{\beta \leq \alpha})$, in the category of vector spaces, is a certain subgroup of the direct product of $V_\alpha$'s:

$$V = \lim \leftarrow V_\alpha = \left\{\vec{v} \in \prod_{\alpha \in A} V_\alpha : \psi_\beta^\alpha(v_\alpha) = v_\beta \text{ for all } \beta \leq \alpha \right\}.$$ 

The linear maps $\psi_\alpha : V \rightarrow V_\alpha$ are the natural projections which pick out the $\alpha$'s component of the direct product for each $\alpha \in A$:

$$\psi_\alpha(\vec{v}) = v_\alpha.$$ 

The vector space operations on $V$ are defined componentwise.

Example 5.5. Let $k, l \in \mathbb{N}$, $l \geq k$ and denote by $\text{pr}_k^l$ the projection onto the first $k$ coordinates from $\mathbb{R}^l$ onto $\mathbb{R}^k$:

$$\text{pr}_k^l : \mathbb{R}^l \rightarrow \mathbb{R}^k : (x_1, \ldots, x_k, x_{k+1}, \ldots, x_l) \mapsto (x_1, \ldots, x_k).$$

Then the pair $((\mathbb{R}^l)_{l \in \mathbb{N}}, \{\text{pr}_k^l\}_{k \leq l})$ is a projective system. Let $\vec{x}$ be an element of $\lim \leftarrow \mathbb{R}^l$, then $\vec{x} = (x^1, x^2, \ldots)$ with $x^k \in \mathbb{R}^k$ and $\text{pr}_k^l(x^l) = x^k$ for all $k \leq l$. Thus the first $k$ coordinates of $x^{k+1}$ are equivalent to $x^k$. Consequently, $(x^1, x^2, \ldots)$, where $x^k = (x^1_k, x^2_k, \ldots, x^k_k) \in \mathbb{R}^k$, represents $\vec{x}$. So the projective limit of $\mathbb{R}^k$'s is isomorphic to the set of all infinite real sequences:

$$\mathbb{R}^\infty := \lim \leftarrow \mathbb{R}^l \cong \{(x_1, x_2, \ldots)\}.$$ 

The inductive limit in the category of topological vector spaces was discussed at the beginning of this exposition and the spaces $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{PW}(\mathbb{C}^n)$ are examples of a (strict) inductive limit of topological vector spaces. For the projective limit of topological vector spaces see [33] and [39].

---

3See Definition 2.1.
5.2 Extension of the Classical Paley-Wiener Theorem to Projective Limits

The extension of the classical Paley-Wiener theorem to the projective limits of the corresponding spaces of functions is a consequence of a result by Cowling.

Let $k, l \in \mathbb{N}$, $k \geq l$. View $\mathbb{R}^l$ as a subset of $\mathbb{R}^k$ via the inclusion map $i_k^l$:

$$i_k^l(\mathbb{R}^l) = \{(x_1, \ldots, x_l, 0, \ldots, 0) : x_j \in \mathbb{R}\} \subseteq \mathbb{R}^k.$$

Write $\mathbb{R}^k$ as a direct sum of $\mathbb{R}^l$ and $((\mathbb{R}^l)^\perp)$:

$$\mathbb{R}^k = \mathbb{R}^l \oplus (\mathbb{R}^l)^\perp.$$

For $f \in \mathcal{D}_r(\mathbb{R}^k)$, define a map $C^k_l : \mathcal{D}_r(\mathbb{R}^k) \rightarrow \mathcal{D}_r(\mathbb{R}^l)$ by

$$C^k_l(f)(x) := \int_{(\mathbb{R}^l)^\perp} f(x, y) dy.$$

It is clear that $C^k_l$ maps into $\mathcal{D}_r(\mathbb{R}^l)$ and it was shown by Cowling in [1] that $C^k_l$ is a surjection. We have

$$\mathcal{D}_r(\mathbb{R}^k) \xrightarrow{C^k_l} \mathcal{D}_r(\mathbb{R}^l) \xrightarrow{\mathcal{F}_{\mathbb{R}^l}} \mathcal{P}\mathcal{W}_{2\pi r}(\mathbb{C}^k) \xrightarrow{\mathcal{F}_{\mathbb{R}^l}} \mathcal{P}\mathcal{W}_{2\pi r}(\mathbb{C}^l).$$

We want a map between the classical Paley-Wiener spaces that will make this diagram commute. This will be a simple restriction, which we denote by $R^k_l$. To see this, let $F$ be in $\mathcal{P}\mathcal{W}_{2\pi r}(\mathbb{C}^k)$, then there is a unique function $f$ in $\mathcal{D}_r(\mathbb{R}^k)$ whose Fourier-Laplace transform is $F$. Let $\xi \in \mathbb{C}^l$ and let $\tilde{\xi} = (\xi, 0, \ldots, 0) \in \mathbb{C}^k$, then

$$R^k_l(F)(\xi) = \mathcal{F}_{\mathbb{R}^l}(C^k_l(f))(\xi)$$

$$= \int \int f(x, y) dy e^{-2\pi i x \cdot \xi} dx$$

$$= \int \int f(x, y) e^{-2\pi i (x, y) \cdot (\xi, 0, \ldots, 0)} dy dx$$

$$= \int f(s) e^{-2\pi i s \cdot \tilde{\xi}} ds$$

$$= F(\tilde{\xi}) = F(\xi, 0, \ldots, 0).$$

Thus we completed the above diagram to this commutative diagram:

$$\mathcal{D}_r(\mathbb{R}^k) \xrightarrow{C^k_l} \mathcal{D}_r(\mathbb{R}^l) \xrightarrow{\mathcal{F}_{\mathbb{R}^l}} \mathcal{P}\mathcal{W}_{2\pi r}(\mathbb{C}^k) \xrightarrow{\mathcal{F}_{\mathbb{R}^l}} \mathcal{P}\mathcal{W}_{2\pi r}(\mathbb{C}^l).$$  (5.2.1)
Since \( F^c_{\mathbb{R}^k} \) is bijective, it follows that \( F^c_{\mathbb{R}^k} \circ C_k^\infty \) is surjective. We conclude that the restriction map \( R_k^\infty \) is surjective as well.

The pairs \( \{ D_r(\mathbb{R}^k) \}_{k \in \mathbb{N}}, \{ C_k^\infty \}_{i \leq k} \) and \( \{ PW_{2\pi r}(\mathbb{C}^k) \}_{k \in \mathbb{N}}, \{ R_k^\infty \}_{i \leq k} \) are projective systems in the category of vector spaces. Denote the maps in the projective limit of \( D_r(\mathbb{R}^k) \) by \( C_k^\infty \) and denote the maps in the projective limit of \( PW_{2\pi r}(\mathbb{C}^k) \) by \( R_k^\infty \). Note that both families of maps \( \{ C_k^\infty \} \) and \( \{ R_k^\infty \} \) are surjective. We have

\[
\lim D_r(\mathbb{R}^k)
\]

\[
\begin{array}{c}
D_r(\mathbb{R}^k) \quad \leftarrow \quad D_r(\mathbb{R}^{k+1}) \quad \leftarrow \quad \ldots
\end{array}
\]

\[
\begin{array}{c}
PW_{2\pi r}(\mathbb{C}^k) \quad \leftarrow \quad PW_{2\pi r}(\mathbb{C}^{k+1}) \quad \leftarrow \quad \ldots
\end{array}
\]

Observe that the maps \( F^c_{\mathbb{R}^k} \circ C_k^\infty : \lim D_r(\mathbb{R}^k) \rightarrow PW_{2\pi r}(\mathbb{C}^k) \) form a compatible family of linear maps with respect to the projective system \( \{ PW_{2\pi r}(\mathbb{C}^k) \}, \{ R_k^\infty \} \). Thus by the universal mapping property of the projective limit there is a linear map from \( \lim D_r(\mathbb{R}^k) \) to \( PW_{2\pi r}(\mathbb{C}^k) \), call it \( F_{\infty} \), such that \( R_k^\infty \circ F_{\infty} = F^c_{\mathbb{R}^k} \circ C_k^\infty \). Since the maps \( F^c_{\mathbb{R}^k} \circ C_k^\infty \) and \( R_k^\infty \) are surjective, it follows that the map \( F_{\infty} \) is surjective. Hence for any \( \tilde{F} \in \lim PW_{2\pi r}(\mathbb{C}^k) \) there is an element \( \tilde{f} \in \lim D_r(\mathbb{R}^k) \) with \( F_{\infty}(\tilde{f}) = \tilde{F} \), where \( \tilde{f} = (f^1, f^2, \ldots) \) with \( f^k \in D_r(\mathbb{R}^k) \) and \( \tilde{F} = (F^1, F^2, \ldots) \) with \( F^k \in PW_{2\pi r}(\mathbb{C}^k) \). Moreover,

\[
F^c_{\mathbb{R}^k}(f^k) = F^c_{\mathbb{R}^k} \circ C_k^\infty(\tilde{f}) = R_k^\infty \circ F_{\infty}(\tilde{f}) = F^k
\]

implies that

\[
F_{\infty}(\tilde{f}) = (F_{\mathbb{R}^1}(f^1), F_{\mathbb{R}^2}(f^2), \ldots).
\]

On the other hand, the family of maps \( F^{-1}_{\mathbb{R}^k} \circ R_k^\infty : \lim PW_{2\pi r}(\mathbb{C}^k) \rightarrow D_r(\mathbb{R}^k) \) is compatible with respect to the projective family \( \{ D_r(\mathbb{R}^k) \}_{k \in \mathbb{N}}, \{ C_k^\infty \}_{i \leq k} \). By the universal mapping property of the projective limit there is a linear map from

\[4\text{Let } f^k \in D_r(\mathbb{R}^k). \text{ For any } i \geq k, \text{ there is a function } f^i \in D_r(\mathbb{R}^i) \text{ with } C^i_m(f^i) = f^k. \text{ And for any } m \leq k, \text{ define } f^m = C^m_m(f^k). \text{ Then } \tilde{f} := (f^1, f^2, \ldots) \text{ belongs to } \lim D_r(\mathbb{R}^k) \text{ and } C_k^\infty(\tilde{f}) = f^k. \text{ Similar argument shows that the maps } R_k^\infty \text{ are surjective as well.}
\[
\lim P W_{2\pi r}(C^k) \rightarrow \lim D_r(\mathbb{R}^k), \text{ call it } F^{-1}, \text{ satisfying } C^\infty_k \circ F^{-1} = F^{-1}_\mathbb{R}^k \circ R^\infty_k. \text{ As the maps } F^{-1}_\mathbb{R}^k \circ R^\infty_k \text{ and } C^\infty_k \text{ are surjective, we conclude that the map } F^{-1} \text{ is surjective. That is, for any } \vec{f} \in \lim D_r(\mathbb{R}^k), \text{ there is an element } \vec{F} \in \lim P W_{2\pi r}(C^k) \text{ with } F^{-1}(\vec{F}) = \vec{f}. \text{ From }
\]
\[
f^k = C^\infty_k \circ F^{-1}(\vec{F}) = F^{-1}_\mathbb{R}^k \circ R^\infty_k(\vec{F}) = F^{-1}_\mathbb{R}^k(F^k)
\]

it follows that
\[
F^{-1}(\vec{F}) = (F^{-1}_\mathbb{R}^1(F^1), F^{-1}_\mathbb{R}^2(F^2), \ldots).
\]

We see that \( F^{-1} \circ F^\infty = id_{\lim D_r(\mathbb{R}^k)} \) and \( F^\infty \circ F^{-1} = id_{\lim P W_{2\pi r}(C^k)} \). This shows that \( F^\infty \) and \( F^{-1} \) are inverses of each other. We obtain the following commutative diagram
\[
\begin{array}{ccc}
D_r(\mathbb{R}^k) & \xrightarrow{C^\infty_k} & \lim D_r(\mathbb{R}^k) \\
\downarrow F^{-1}_\mathbb{R}^k & & \downarrow F^\infty \\
\lim P W_{2\pi r}(C^k) & \xleftarrow{R^\infty_k} & \lim P W_{2\pi r}(C^k)
\end{array}
\]

where the vertical maps are isomorphisms. We interpret this diagram as an extension of the classical Paley-Wiener theorem to the projective limits of the function spaces \( D_r(\mathbb{R}^k) \) and \( P W_{2\pi r}(C^k) \).

### 5.3 Extension of the Euclidean Paley-Wiener Theorem to Projective Limits

In this section we extend our second description to the projective limits of the corresponding function spaces. To do this, we first extend the commutative diagram (5.2.1) to a commutative cube by extending each side in (5.2.1) to a commutative diagram using the Radon and Radon type transforms \( R \) and \( \tilde{R} \).

We indicate with the subscript \( k \) by \( R_k \) the Radon transform on \( \mathbb{R}^k \) and with the superscript \( k \) by \( \Xi^k \) we indicate the \( k \)-dimensional product manifold \( \mathbb{R}_0^+ \times S^{k-1} \). We have the diagram
\[
\begin{array}{ccc}
D_r(\mathbb{R}^k) & \xrightarrow{C^k} & D_r(\mathbb{R}^l) \\
\downarrow R_k & & \downarrow R_l \\
D_{H,r}(\Xi^k) & & D_{H,r}(\Xi^l)
\end{array}
\]

and wish to extend it to a commutative diagram. The map between the spaces \( D_{H,r}(\Xi^k) \) and \( D_{H,r}(\Xi^l) \) that makes this diagram commute is a restriction map, which we denote by \( r^k_l \). To see this, let \( \varphi \in D_{H,r}(\Xi^k) \). Then there is a unique function \( f \in D_r(\mathbb{R}^k) \) whose Radon transform is \( \varphi \). Let \( p \in \mathbb{R}_0^+ \) and \( \omega \in S^{l-1} \). We
extend the hyperplane \( \xi(p, \omega) \) in \( \mathbb{R}^l \) to a hyperplane \( \tilde{\xi}(p, \tilde{\omega}) \) in \( \mathbb{R}^k \) by \( \tilde{\xi}(p, \tilde{\omega}) = \xi(p, \omega) + (\mathbb{R}^l)^{\perp} \).

The distance of \( \tilde{\xi} \) from the origin is \( p \) and the direction of \( \tilde{\xi} \) is \( \tilde{\omega} = (\omega, 0, \ldots, 0) \).

We compute

\[
\begin{align*}
\varphi^k(p, \omega) &= \varphi^l(R_k f)(p, \omega) \\
&= R_l(C^k_l(f))(p, \omega) \\
&= \int_{\xi(p, \omega)} \int_{(\mathbb{R}^l)^{\perp}} f(x, y) dy \, dm(x) \\
&= \int_{\tilde{\xi}(p, \tilde{\omega})} f(s) \, dm(s) \\
&= R_k(f)(p, \tilde{\omega}) \\
&= \varphi(p, \tilde{\omega}) \\
&= \varphi(p, (\omega, 0, \ldots, 0)).
\end{align*}
\]

We completed the above diagram to the commutative diagram

\[
\begin{array}{ccc}
\mathcal{D}_r(\mathbb{R}^k) & \xrightarrow{C^k_l} & \mathcal{D}_r(\mathbb{R}^l) \\
\mathcal{D}_{H,r}(\Xi^k) & \xrightarrow{r^k_l} & \mathcal{D}_{H,r}(\Xi^l)
\end{array}
\]

As the Radon transform is an isomorphism, we deduce that the restriction map \( r^k_l \) is surjective.

It remains to extend the diagram

\[
\begin{array}{ccc}
\mathcal{PW}_{2\pi r}(\mathbb{C}^k) & \xrightarrow{R^k_l} & \mathcal{PW}_{2\pi r}(\mathbb{C}^l) \\
\mathcal{PW}_{2\pi r}^{Z_2,H}(\mathbb{C}, \mathcal{O}(S_{\mathbb{C}}^{k-1})) & \xrightarrow{\mathcal{R}_k} & \mathcal{PW}_{2\pi r}^{Z_2,H}(\mathbb{C}, \mathcal{O}(S_{\mathbb{C}}^{l-1})) \\
\mathcal{PW}_{2\pi r}^{Z_2,H}(\mathbb{C}, \mathcal{O}(S_{\mathbb{C}}^{k-1})) & \xrightarrow{\mathcal{R}_l} & \mathcal{PW}_{2\pi r}^{Z_2,H}(\mathbb{C}, \mathcal{O}(S_{\mathbb{C}}^{l-1}))
\end{array}
\]

to a commutative diagram. Here as well the needed map between the spaces \( \mathcal{PW}_{2\pi r}^{Z_2,H}(\mathbb{C}, \mathcal{O}(S_{\mathbb{C}}^{k-1})) \) and \( \mathcal{PW}_{2\pi r}^{Z_2,H}(\mathbb{C}, \mathcal{O}(S_{\mathbb{C}}^{l-1})) \) works our to be a restriction map,
which we denote by $\tilde{R}^k_l$. Let $\tilde{F} \in \tilde{\mathcal{P}}\mathcal{W}_{2\pi r}^{\mathbb{Z}_2,H}(\mathbb{C}, \mathcal{O}(S^{k-1}_C))$, there is an $F \in \mathcal{P}\mathcal{W}_{2\pi r}(C^k)$ with $\tilde{F}(\xi, \tilde{\omega}) = F(\xi \tilde{\omega})$. Let $\xi \in C$ and $\tilde{\omega} \in S^{l-1}_C$, then

$$\tilde{R}_l^k(\tilde{F})(\xi, \tilde{\omega}) = \tilde{R}_l^k(F)(\xi, \tilde{\omega}) = \tilde{R}_l^k(F)(\xi \tilde{\omega}) = F(\xi \tilde{\omega}, 0, \ldots, 0) = F(\xi(\tilde{\omega}, 0, \ldots, 0)) = \tilde{F}(\xi, (\tilde{\omega}, 0, \ldots, 0)).$$

And we obtain

$$\mathcal{P}\mathcal{W}_{2\pi r}(C^k) \xrightarrow{R^k_l} \mathcal{P}\mathcal{W}_{2\pi r}(C^l) \xrightarrow{\check{R}_l} \mathcal{P}\mathcal{W}_{2\pi r}(C^k) \check{R}_l$$

(5.3.2)

Again, as the map $\check{R}_k$ is a bijection, it follows that the restriction map $\check{R}_l^k$ is surjective.

Putting the two commutative diagrams we just worked out (5.3.1) and (5.3.2) together with the commutative diagrams (5.2.1) and (4.3.4), we obtain the following commutative cube

The pairs $\{(\mathcal{D}_{H,r}(\Xi^k))_{k \in \mathbb{N}}, \{r^k_l\}_{l \leq k}\}$ and $\{(\mathcal{P}\mathcal{W}_{2\pi r}^{\mathbb{Z}_2,H}(\mathbb{C}, \mathcal{O}(S^{k-1}_C)))_{k \in \mathbb{N}}, \{\check{R}^k_l\}_{l \leq k}\}$ are projective systems. We denote the maps in the projective limit of the spaces $\mathcal{P}\mathcal{W}_{2\pi r}^{\mathbb{Z}_2,H}(\mathbb{C}, \mathcal{O}(S^{k-1}_C))$ by $\check{R}^\infty_k$ and the maps in the projective limit of the spaces $\mathcal{D}_{H,r}(\Xi^k)$ we denote by $r^\infty_k$. As a consequence of the maps $r^k_l$ and $\check{R}^k_l$ being surjective, families of maps $\{r^\infty_k\}$ and $\{\check{R}^\infty_k\}$ are surjective. By an analogous argument
as in the extension of the classical Paley-Wiener theorem to projective limits, we obtain the commutative diagram:

\[
\begin{array}{c}
D_{H,r}(\Xi^k) \xrightarrow{r^\infty_k} \lim_{\xi} D_{H,r}(\Xi^k) \\
\downarrow F^\infty_k \\
\mathcal{P}W_{2\pi r}(\mathbb{C}, \mathcal{O}(S^{k-1}_C)) \xleftarrow{\tilde{R}^\infty_k} \lim_{\xi} \mathcal{P}W_{2\pi r}(\mathbb{C}, \mathcal{O}(S^{k-1}_C)) \\
\end{array}
\]

with the vertical maps being isomorphisms. The mapping \( \tilde{F}_\infty \) has the following form, let \( \vec{\varphi} = (\varphi^1, \varphi^2, \ldots) \) be in \( \lim_{\xi} D_{H,r}(\Xi^k) \), then

\[
\tilde{F}_\infty(\vec{\varphi}) = (F^c_\mathbb{R} \varphi^1, F^c_\mathbb{R} \varphi^2, \ldots).
\]

Similarly, the two remaining maps in our commutative cube, \( \mathcal{R}_k \) and \( \tilde{\mathcal{R}}_k \), extend to mappings between the projective limits of the corresponding function spaces, call this extensions \( \mathcal{R}_\infty \) and \( \tilde{\mathcal{R}}_\infty \). For a \( \vec{f} = (f^1, f^2, \ldots) \in \lim_{\xi} D_r(\mathbb{R}^k) \)

\[
\mathcal{R}_\infty(\vec{f}) = (\mathcal{R}_1 f^1, \mathcal{R}_2 f^2, \ldots),
\]

and for a \( \vec{F} = (F^1, F^2, \ldots) \) in \( \lim_{\xi} \mathcal{P}W_{2\pi r}(\mathbb{C}^k) \)

\[
\tilde{\mathcal{R}}_\infty(\vec{F}) = (\tilde{\mathcal{R}}_1 F^1, \tilde{\mathcal{R}}_2 F^2, \ldots).
\]

Finally, as

\[
\tilde{F}_\infty(\mathcal{R}_\infty(\vec{f})) = \tilde{F}_\infty(\mathcal{R}_1 f^1, \mathcal{R}_2 f^2, \ldots)
\]

\[
= (F^c_\mathbb{R}(\mathcal{R}_1 f^1), F^c_\mathbb{R}(\mathcal{R}_2 f^2), \ldots)
\]

\[
= (\tilde{\mathcal{R}}_1(F^c_\mathbb{R} f^1), \tilde{\mathcal{R}}_2(F^c_\mathbb{R} f^2), \ldots)
\]

\[
= \tilde{\mathcal{R}}_\infty(F^c_\mathbb{R} f^1, F^c_\mathbb{R} f^2, \ldots)
\]

we obtain the commutative diagram between the projective limits:

\[
\begin{array}{c}
\lim_{\xi} D_r(\mathbb{R}^k) \xrightarrow{\mathcal{R}_\infty} \lim_{\xi} D_{H,r}(\Xi^k) \\
\downarrow F^\infty_k \\
\lim_{\xi} \mathcal{P}W_{2\pi r}(\mathbb{C}^k) \xrightarrow{\tilde{\mathcal{R}}_\infty} \lim_{\xi} \mathcal{P}W_{2\pi r}(\mathbb{C}, \mathcal{O}(S^{k-1}_C)) \\
\end{array}
\]

where all the maps are bijections. The vertical arrows have a nice interpretation as infinite dimensional Radon and Radon type transforms and horizontal arrows as
infinite dimensional Fourier transforms. We interpret the composition of the maps $\mathcal{R}_\infty$ and $\tilde{\mathcal{F}}_\infty$ as the extension of our second description to the projective limits:

$$\tilde{\mathcal{F}}_\infty \circ \mathcal{R}_\infty : \lim_{\leftarrow} D_r(\mathbb{R}^k) \rightarrow \lim_{\leftarrow} \mathcal{PW}_{2\pi r}^{Z_2,H}(\mathbb{C}, \mathcal{O}(S^{k-1})).$$

Because of the simple nature of the maps $R^k_l, r^k_l$, and $\tilde{R}^k_l$, the projective limits of the function spaces $\mathcal{PW}_{2\pi r}(\mathbb{C}^k), \mathcal{D}_{H,r}(\Xi^k)$, and $\mathcal{PW}_{2\pi r}^{Z_2,H}(\mathbb{C}, \mathcal{O}(S^{k-1}))$ allow for an interpretation as spaces of functions on the inductive limits of the underlying spaces $\mathbb{C}^k, \Xi^k, \text{and } \mathbb{C} \times S^{k-1}$. Indeed, let $\varphi = (\varphi^1, \varphi^2, \ldots) \in \lim_{\leftarrow} \mathcal{D}_{H,r}(\Xi^k)$. The inductive limit of the spaces $\Xi^k$ can be viewed as

$$\lim_{\rightarrow} \Xi^k = \lim_{\rightarrow} (\mathbb{R}_0^+ \times S^{k-1})$$

$$= (\bigsqcup \mathbb{R}_0^+ \times S^{k-1}) / \sim$$

$$= \mathbb{R}_0^+ \times (\bigsqcup S^{k-1}) / \sim$$

$$= \mathbb{R}_0^+ \times \lim_{\leftarrow} S^{k-1}.$$

Let $[\xi] \in \lim_{\rightarrow} \Xi^k$, then $[\xi] = (p, [\omega])$ with $p \in \mathbb{R}_0^+$ and $[\omega] \in \lim_{\leftarrow} S^{k-1}$. For some $l \in \mathbb{N}$, there is $\omega^l \in S^{l-1}$ with $[\omega] = [\omega^l]$. Define

$$\tilde{\varphi}([\xi]) := \varphi^l(p, \omega^l).$$

Suppose $m > l$, then $\varphi^m(p, (\omega^l, 0, \ldots, 0)) = r^m_l(\varphi^m(p, (\omega^l, 0, \ldots, 0)) = \varphi^l(p, \omega^l).$ This shows that $\tilde{\varphi}([\xi])$ is well-defined.

Similarly, let $\tilde{F} = (F^1, F^2, \ldots) \in \lim_{\leftarrow} \mathcal{PW}_{2\pi r}^{Z_2,H}(\mathbb{C} \times S^{k-1})$. Again, $\lim_{\rightarrow} \mathbb{C} \times S^{k-1} = \mathbb{C} \times \lim_{\leftarrow} S^{k-1}$. For $(\xi, [\omega]) = (\xi, [\omega^l]) \in \lim_{\rightarrow} \mathbb{C} \times S^{k-1}$, define

$$\tilde{F}(\xi, [\omega]) := F^l(\xi, [\omega^l]).$$

It is well-defined. Hence, in contrast to the extension of the classical Paley-Wiener theorem, we can also interpret the extension of the Euclidean Paley-Wiener theorem as an extension to the inductive limits of the underlying spaces.
References


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