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# Differential geometry in cartesian closed categories of smooth spaces

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DIFFERENTIAL GEOMETRY IN CARTESIAN CLOSED CATEGORIES OF SMOOTH SPACES

A Dissertation

Submitted to the Graduate Faculty of the  
Louisiana State University and  
Agricultural and Mechanical College  
in partial fulfillment of the  
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in

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by

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I dedicate this dissertation to my parents.

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# Abstract

The main categories of study in this thesis are the categories of diffeological and Frölicher spaces. They form concrete cartesian closed categories. In Chapter 1 we provide relevant background from category theory and differentiation theory in locally convex spaces. In Chapter 2 we define a class of categories whose objects are sets with a structure determined by functions into the set. Frölicher's  $M$ -spaces, Chen's differentiable spaces and Souriau's diffeological spaces fall into this class of categories. We prove cartesian closedness of the two main categories, and show that they have all limits and colimits. We exhibit an adjunction between the categories of Frölicher and diffeological spaces. In Chapter 3 we define a tangent functor for the two main categories. We define a condition under which the tangent spaces to a Frölicher space are vector spaces. Frölicher groups satisfy this condition, and under a technical assumption on the tangent space at identity, we can define a Lie bracket for Frölicher groups.

# Chapter 1

## Introduction

This thesis is concerned with two cartesian closed categories  $\mathcal{C}$  which generalize the category  $\mathbf{Mfd}$  of smooth finite-dimensional manifolds. Here we say that  $\mathcal{C}$  generalizes  $\mathbf{Mfd}$  if there is a full and faithful functor from  $\mathbf{Mfd}$  to  $\mathcal{C}$ . The two categories generalizing  $\mathbf{Mfd}$  are the category  $\mathcal{F}$  of Frölicher spaces and the category  $\mathcal{D}$  of diffeological spaces. Our thesis has two objectives, which are respectively addressed in Chapters 2 and 3. First we will compare the categories of Frölicher and diffeological spaces. To our knowledge they have only been treated separately in the literature; we propose a general framework of ‘spaces with structure determined by functions’, of which both diffeological and Frölicher spaces, as well as Frölicher’s  $M$ -spaces and Chen’s differentiable spaces, are examples. Furthermore we exhibit an adjunction from  $\mathcal{F}$  to  $\mathcal{D}$ . Our second objective is to generalize notions from differential geometry to a more general setting. This is done in Chapter 3, where we show that the tangent functor for finite-dimensional manifolds can be extended to both  $\mathcal{F}$  and  $\mathcal{D}$ . Vector fields and differential forms can be defined in  $\mathcal{F}$  and  $\mathcal{D}$ , and we go some way toward defining a Lie functor in  $\mathcal{F}$ .

We would like to sketch some of the history of diffeological and Frölicher spaces. This is not meant to be a complete account. Our main purpose here is to mention some of the sources we have studied in the course of our research.

Let us start with the theory of convenient vector spaces, which are closely related to Frölicher spaces. In [FB66], Frölicher and Bucher develop a theory of differentiation for functions  $f : V \rightarrow W$ , where  $V$  and  $W$  are non-normable topological vector spaces. One problem in working with non-normable vector spaces is that

there is no topology on the space  $L(V, W)$  of continuous linear maps making the evaluation

$$eval : L(V, W) \times V \rightarrow W$$

continuous. Frölicher and Bucher use pseudo-topologies instead and arrive at a concept of smooth maps for which they can show that there is a linear homeomorphism

$$C^\infty(U, C^\infty(V, W)) \cong C^\infty(U \times V, W).$$

This is the exponential law, which plays a central role in the definition of cartesian closedness. In joint work, Kriegl, Frölicher developed a theory of convenient vector spaces, see their monograph [FK88]. A convenient vector space can be characterized in various equivalent ways, in particular by a completeness condition (in our thesis, we use the term ‘Mackey complete locally convex space’ rather than convenient vector space). There is also a characterization in terms of smooth curves. A locally convex vector space  $V$  is convenient if a function  $c : \mathbb{R} \rightarrow V$  is smooth if and only if for all continuous linear functionals  $\Lambda : V \rightarrow \mathbb{R}$ , the composition  $\Lambda \circ c$  is smooth. It is this type of condition which underlies the definition of  $M$ -spaces. In [Frö86], Frölicher defines a category associated with an arbitrary collection  $M$  of functions. An  $M$ -structure on a set  $X$  is determined by curves into the set and functions on the set, such that the compositions of a curve with a function is always an element of  $M$ . Frölicher determines conditions on  $M$  under which the resulting category is cartesian closed. For example  $M = C^\infty(\mathbb{R}, \mathbb{R})$  yields a cartesian closed category, whose objects Frölicher calls smooth spaces, but which later became known as Frölicher spaces. Frölicher spaces have since been studied by Ntumba and Batubenge [NB05], who investigate groups in the Frölicher category, and Cherenack



and Bentley [BC05], who investigate Frölicher structures on singular spaces. Many results and references can be found in [KM97], Chapter V.23.

Before turning to diffeological spaces, let me mention the closely related ‘differentiable spaces’ as defined by Chen in [Che73]. For Chen, a differentiable space is a Hausdorff topological space  $X$  together with a collection of continuous maps  $\alpha : C \rightarrow X$ . The maps  $\alpha$  are called plots, and the collection of plots satisfies certain axioms. The domains  $C$  of the plots are convex subset of some  $\mathbb{R}^n$ . Chen’s main object of study are spaces of smooth paths and loops in manifolds. In his later paper [Che86], he drops the condition that the plots be continuous.

Souriau defines diffeologies on groups in [Sou80]. Diffeologies consist of families of maps  $\alpha : U \rightarrow X$  (called ‘plaques’) which satisfy similar axioms to those of Chen’s plots. The main difference is that the domains  $U$  are *open* subsets of  $\mathbb{R}^n$ . Souriau uses diffeological groups as a tool in his work on a formalization of quantum mechanics, and that he defines diffeologies in order to equip diffeomorphism groups with a smooth structure. In [Sou85] he drops the restriction on groups and defines general diffeological spaces. He defines homotopy groups and differential forms for diffeological spaces, and discusses the irrational torus as an example. There is further research on the irrational torus by Souriau’s doctoral students Donato and Iglesias-Zemmour, together with Lachaud [DI85],[IL90]. Iglesias-Zemmour is still actively working on diffeological spaces, in particular in relation with symplectic geometry, and he is writing on a book on the topic which is available on his web page. Hector [Hec95] defines tangent spaces and singular homology for diffeological spaces, and investigates diffeological groups joint with Macías-Virgós in [HVM02]. See also Leslie’s article [Les03] on diffeological groups associated to Kac-Moody Lie algebras. There are various generalizations of diffeological spaces. Losik [Los94] considers categories of local structures on sets, and Giordano [Gio04] defines the

cartesian closure of a category. Diffeological spaces arise as cartesian closure of a category of open subsets of  $\mathbb{R}^n$  and smooth maps. Kock and Reyes [KR04] define a topos of which the category of diffeological spaces is a full subcategory.

Let us briefly outline the content of this thesis. In the remaining sections of the current chapter, we review the definitions and results from category theory and the theory of locally convex vector spaces needed in later chapters. The categories of interest are concrete categories, and we define this notion. We think of objects of concrete categories as sets with an underlying structure, and we define an order relation on the collection of structures on a fixed set. For example the category of topological spaces is concrete, and the relation is the well known relation of ‘finer’ for topologies on a given set. We define cartesian closedness only in the context of concrete categories, thus avoiding the use of adjunctions and natural transformations.

In Chapter 2 we introduce the cartesian closed categories we are concerned with in this thesis. These are the categories  $\mathcal{D}$  of diffeological spaces and  $\mathcal{F}$  of Frölicher spaces, which arise as a special case of the more general categories of  $M$ -spaces if  $M = C^\infty(\mathbb{R}, \mathbb{R})$ . As already mentioned, both categories are concrete. Moreover, in both categories the structure of an object is given by a collection of functions in the underlying set. In the case of diffeological spaces, the functions are called plots, and their domains are open subsets of  $\mathbb{R}^n$  for arbitrary  $n$ . In the case of  $M$ -spaces, the functions are called curves, and their domain is a fixed but arbitrary set  $A$ , which in the case of Frölicher spaces is the set  $\mathbb{R}$ .

Therefore, in Section 2.1 we define a general notion of structures determined by functions, which subsumes  $M$ -spaces and Frölicher spaces, but also Chen’s differentiable spaces. One could use this general notion to define analogues of diffeological

spaces for which the domains of plots are open in more general vector spaces than  $\mathbb{R}^n$ , or using analytic or holomorphic rather than smooth functions.

In Section 2.2 we prove properties of the category of diffeological spaces, including existence of limits and colimits and cartesian closedness. We then show that there is a full and faithful functor from the category **Mfd** of smooth finite-dimensional manifolds into  $\mathcal{D}$ . In this sense, diffeological spaces can be said to generalize smooth manifolds. Sections 2.3 and 2.4 are parallel to Section 2.2; we prove existence of limits, colimits and Hom-objects in the category  $\mathcal{K}_M$  of  $M$ -spaces and give Frölicher's condition for cartesian closedness of  $\mathcal{K}_M$ . In Section 2.4. we specialize to Frölicher spaces to show that  $\mathcal{F}$  is cartesian closed and there is a full and faithful functor **Mfd**  $\rightarrow$   $\mathcal{F}$ . In the last section of Chapter 2 we define two functors **F** :  $\mathcal{D} \rightarrow \mathcal{F}$  and **D** :  $\mathcal{F} \rightarrow \mathcal{D}$  and show that they form an adjunction from  $\mathcal{F}$  to  $\mathcal{D}$ .

In Chapter 3 we study differential geometry in the categories  $\mathcal{F}$  and  $\mathcal{D}$ . We describe the construction of tangent functors for both categories. The construction in  $\mathcal{F}$  is easier, however the tangent space at a point is not necessarily a vector space. Under the condition that the underlying space is of  $L$ -type we can show that the tangent space is always a vector space. This applies in particular to all Frölicher groups. In Section 3.1.6 we consider the subspace of  $\mathbb{R}^2$  consisting of the union of the coordinate axes. We show that the tangent space at the singular point  $(0, 0)$  is not a vector space. In 3.1.7 we show that the tangent functors on  $\mathcal{F}$  and  $\mathcal{D}$  extend the classical tangent functor.

In Section 3.2 we define vector fields on a Frölicher space  $X$  and show that they give rise to derivations of the algebra  $F$  of smooth functions on  $X$ . We also define differential forms on diffeological and Frölicher spaces.

Our goal in Section 3.3 is to define a Lie bracket on the tangent space  $\mathfrak{g} = T_e G$  to the identity of a Frölicher group  $G$ . The idea is to identify elements of  $\mathfrak{g}$  with derivations of the space  $F$  of smooth functions on  $G$ . The space of derivations is a Lie algebra. If  $\xi_v$  and  $\xi_w$  are derivations associated with elements  $v, w \in \mathfrak{g}$ , the main problem is then to find an element in  $\mathfrak{g}$  which yields the derivation given by  $[\xi_v, \xi_w]$ . The construction involves the second derivative of the commutator map  $K(a, b) = aba^{-1}b^{-1}$  of  $G$ . In Section 3.3.2 we discuss the group structure on  $TG$  and its semidirect product decomposition. In Section 3.3.3 we describe the Lie bracket of derivations associated with elements of  $\mathfrak{g}$ . Finally, in Section 3.3.4 we describe a decomposition of the second iterated tangent group  $T^2G$ . Here, the tangent space  $T_0\mathfrak{g}$  occurs as a factor, and under the additional hypothesis that  $T_0\mathfrak{g}$  is isomorphic to  $\mathfrak{g}$ , we can define the Lie bracket of two elements of  $\mathfrak{g}$ .

## 1.1 Foundational Material

### 1.1.1 Category Theory

The present thesis is concerned with various categories which share some basic properties. Most importantly, objects are always sets equipped with an additional structure. Furthermore, the categories under consideration are closed under limits and colimits. Lastly, the Hom-sets carry a natural structure, making them objects in the respective categories. In the present section we introduce some basic constructions from category theory, and fix notation. We assume familiarity with the definitions of category, subcategory and functor, and refer to [Mac71] for details.

**Definition 1.1.** *We let  $\mathbf{Set}$  denote the category of sets and set maps, and  $\mathbf{Mfd}$  the category of smooth finite dimensional manifolds and smooth maps. Manifolds will always assumed to be paracompact and second countable. For brevity we will*

just say manifold for smooth finite-dimensional manifold. Let  $\text{Map}(X, Y)$  denote the collection of set maps from  $X$  to  $Y$ , and as usual we write  $C^\infty(M, N)$  for  $\text{Hom}_{\text{Mfd}}(M, N)$ .

**Definition 1.2.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is full if for each pair  $(X, Y)$  of objects of  $\mathcal{C}$  and every morphism  $g : FX \rightarrow FY$  there is a morphism  $f : X \rightarrow Y$  such that  $F(f) = g$ . The functor is faithful if  $F(f_1) = F(f_2)$  implies that  $f_1 = f_2$  for morphisms  $f_i : X \rightarrow Y$  in  $\mathcal{C}$ . We say that a subcategory  $\mathcal{D}$  of  $\mathcal{C}$  is full if the corresponding inclusion functor is full.

Limits and Colimits

**Definition 1.3.** A directed graph  $\mathcal{G} = (V, E)$  consists of a class  $V$  of vertices and  $E$  of edges and, if  $E$  is non-empty, two functions  $h, t : E \rightarrow V$ . Every edge  $\alpha$  has a head  $h(\alpha)$  and a tail  $t(\alpha)$ , and we write  $i \xrightarrow{\alpha} j$  if  $\alpha$  has tail  $i$  and head  $j$ .

Here are some important examples.

**Example 1.4.** If  $\mathcal{C}$  is a category, the underlying graph has the objects of  $\mathcal{C}$  as vertices and the morphisms  $f : X \rightarrow Y$  as directed edges.

**Example 1.5.** If  $E$  is the empty set, we call  $\mathcal{G} = (V, E)$  the discrete graph with vertices  $V$ .

We recall that a set  $X$  with a binary relation  $\leq$  is called a *pre-ordered set* if  $\leq$  is reflexive and transitive. If the relation  $\leq$  is also antisymmetric, then  $(X, \leq)$  is called a *partially ordered set* or *poset*. If  $(X, \leq)$  is a pre-ordered set such that any two elements in  $X$  have an upper bound, then  $(X, \leq)$  is called a *directed set*.

**Example 1.6.** To each pre-ordered set  $V$  we can define a directed graph  $(V, E)$  by letting  $E$  consist of one arrow  $x \rightarrow y$  for each pair  $x, y \in V$  with  $x \leq y$ . An important basic example is  $V = \mathbb{N}$  with its natural order relation.

We next define the limit and colimit of an arbitrary *diagram* in a category.

**Definition 1.7.** *If  $\mathcal{G} = (V, E)$  is a graph and  $\mathcal{C}$  is a category with objects  $O$  and morphisms  $M$ , then a diagram  $J$  of shape  $\mathcal{G}$  in  $\mathcal{C}$  is given by maps  $J_V : V \rightarrow O$  and  $J_E : E \rightarrow M$  such that  $J_E(\alpha) : J_V(i) \rightarrow J_V(j)$  if  $\alpha$  is an edge from  $i$  to  $j$ . In other words, a diagram  $J$  is a morphism of directed graphs from  $\mathcal{G}$  to the graph underlying  $\mathcal{C}$ . Usually we write  $X_i$  and  $f_\alpha$  for the object and morphism associated to vertex  $i$  and edge  $\alpha$ , respectively.*

**Definition 1.8 (Limit).** *The limit of a diagram  $J : \mathcal{G} \rightarrow \mathcal{C}$  is an object  $X$  together with morphisms  $a_i : X \rightarrow X_i$  such that for each  $f : X_i \rightarrow X_j$  in the diagram, we have  $a_j = f \circ a_i$ . Furthermore  $X$  has the following universal property: If  $X'$  is another object with morphisms  $b_i : X' \rightarrow X_i$  satisfying  $b_j = f \circ b_i$ , then there is a unique morphism  $\varphi : X' \rightarrow X$  such that  $b_i = a_i \circ \varphi$  for all vertices  $i$ .*

The dual notion is that of a colimit. It is obtained from the previous definition by ‘reversing arrows’.

**Definition 1.9 (Colimit).** *The colimit of a diagram  $J : \mathcal{G} \rightarrow \mathcal{C}$  is an object  $X$  together with morphisms  $\alpha_i : X_i \rightarrow X$  such that for each  $f : X_i \rightarrow X_j$  in the diagram, we have  $\alpha_i = \alpha_j \circ f$ . The object  $X$  has the following universal property: If  $X'$  is another object with morphisms  $\beta_i : X_i \rightarrow X'$  satisfying  $\beta_i = \beta_j \circ f$ , then there is a unique morphism  $\varphi : X \rightarrow X'$  such that  $\beta_i = \varphi \circ \alpha_i$ .*

*If the graph  $\mathcal{G}$  comes from a directed set as in Example 1.6, then the colimit of a diagram of shape  $\mathcal{G}$  is called a filtered colimit.*

**Remark 1.10.** *Limits and colimits are unique up to isomorphism. From now on we will speak of the limit or colimit, keeping in mind that there might be several isomorphic limits or colimits.*

**Example 1.11.** If  $\mathcal{G} = (V, E)$  is small in the sense that  $V$  and  $E$  are sets, then all limits and colimits of diagrams of shape  $\mathcal{G}$  in **Set** exist. A proof of this fact can be found for example in [Mac71], Chapters III.3. for colimits and V.1. for limits.

Here we give an example of a filtered colimit.

**Example 1.12.** Consider the graph associated to the directed set  $\mathbb{N}$ . Define a diagram of shape  $\mathbb{N}$  in the category of topological spaces as follows: To each vertex  $n$  we associate  $\mathbb{S}^n$ , the  $n$ -sphere, and to an arrow  $n \mapsto n+1$  the embedding  $\mathbb{S}^n \rightarrow \mathbb{S}^{n+1}$  as equator. Then the colimit in the category of topological spaces is the infinite sphere  $\mathbb{S}^{(\infty)}$ . As a topological space,  $\mathbb{S}^{(\infty)}$  has the following description. It consists of the union of all  $\mathbb{S}^n$ , where we regard  $\mathbb{S}^n$  as a subset of  $\mathbb{S}^m$  if  $n \leq m$ . A subset  $U \subset \mathbb{S}^{(\infty)}$  is open if and only if its intersection with all  $\mathbb{S}^n$  is open.

**Example 1.13.** Let  $\mathcal{G}$  be the discrete graph on a set  $I$ . Then a diagram of shape  $\mathcal{G}$  is just a collection  $X_i$  of objects indexed by  $I$ . Let us consider the category of vector spaces over  $\mathbb{R}$ . Then the limit of such a diagram is given by the direct product  $\prod_{i \in I} V_i$  of the vector spaces, together with the projections  $\pi_i$  onto the  $i$ -th coordinate. The colimit is given by the direct sum  $\bigoplus_{i \in I} V_i$  together with the injections  $\iota_i : V_i \rightarrow \bigoplus_{i \in I} V_i$ .

**Definition 1.14.** More generally, let  $\mathcal{C}$  be any category, and  $\{X_i \mid i \in I\}$  a family of objects. If the limit of the discrete diagram with vertices  $X_i$  exists, we denote it by  $\prod_{i \in I} X_i$  and call it the product of the  $X_i$ . If the colimit exists, we denote it by  $\coprod_{i \in I} X_i$  and call it the coproduct of the  $X_i$ .

**Lemma 1.15.** Suppose that in  $\mathcal{C}$ , the product of a family of  $X_i$  exists. Then for every object  $Y$  in  $\mathcal{C}$  there is a bijection of sets

$$\varphi : \text{Hom}_{\mathcal{C}} \left( Y, \prod_i X_i \right) \rightarrow \prod_i \text{Hom}_{\mathcal{C}}(Y, X_i).$$

*Proof.* The product  $X = \prod_i X_i$  comes with projections  $\pi_i : X \rightarrow X_i$ , and we can easily define  $\varphi(f)$  by  $\varphi(f)_i = \pi_i \circ f$ . To construct an inverse  $\psi$  to  $\varphi$ , we will use the universal property. If  $(f_i)_{i \in I} \in \prod_i \text{Hom}_{\mathcal{C}}(Y, X_i)$ , then by the universal property, there is a unique morphism  $f : Y \rightarrow X$  such that  $\pi_i \circ f = f_i$ . Let  $\psi((f_i)_{i \in I}) = f$ . By construction we have  $\varphi \circ \psi = \text{id}$ . The fact that  $\psi \circ \varphi = \text{id}$  follows from the uniqueness assertion of the universal property.  $\square$

**Definition 1.16.** A terminal object of a category  $\mathcal{C}$  is an object  $t$  such that for every other object  $X$ , there is a unique morphism  $X \rightarrow t$ . An object  $i$  in  $\mathcal{C}$  is called initial if for every object  $X$ , there is a unique morphism  $i \rightarrow X$ .

**Example 1.17.** In **Set**, every one-point set is terminal, and the empty set is initial. In the category of groups and homomorphisms, the one-element group  $\{e\}$  is both initial and terminal.

**Remark 1.18.** If the empty graph  $\emptyset$  is defined to be the graph with no vertices and no edges, then the limit of a diagram of shape  $\emptyset$ , if it exists, is a terminal object. The colimit is an initial object.

We will work only with very special categories  $\mathcal{C}$  whose objects are sets with some additional structure. This can be made precise as follows.

**Definition 1.19.** A category  $\mathcal{C}$  is concrete if there is a faithful functor  $F : \mathcal{C} \rightarrow \mathbf{Set}$ . This functor is called a forgetful functor or grounding functor.

**Remark 1.20.** It is customary to denote  $X$  and  $FX$  by the same letter if  $\mathcal{C}$  is a concrete category. For example, if  $(X, \mathcal{P})$  is a topological space, we use  $X$  to denote both the set and the topological space, unless we consider several topologies on the same set.



If  $X$  is an object in a concrete category  $\mathcal{C}$ , then by an element  $x$  of  $X$  we mean an element  $x \in F(X)$  of the underlying set. We write  $x \in X$  if  $x$  is an element of the object  $X$ . If  $f : X \rightarrow Y$  is a morphism in  $\mathcal{C}$  and  $x$  an element of  $X$ , let us write  $f(x)$  for the element of  $Y$  given by  $F(f)(x) \in F(Y)$ . Suppose that  $X$  and  $Y$  are objects in  $\mathcal{C}$  and  $f : FX \rightarrow FY$  is a set map. Let us say that  $f$  is a morphism from  $X$  to  $Y$  if there is a morphism  $g : X \rightarrow Y$  with  $f = F(g)$ . Since  $F$  is faithful,  $g$  is then uniquely determined by  $f$ .

The categories under consideration in this thesis not only have limits and colimits. The limits and colimits actually arise from putting a structure on the corresponding limits and colimits of underlying sets. This is formally expressed by saying that the grounding functor *creates* limits and colimits.

**Definition 1.21.** Let  $\mathcal{C}$  be a concrete category with grounding functor  $F$ . If  $J : \mathcal{G} \rightarrow \mathcal{C}$  is a diagram, we can compose with  $F$  to get a diagram  $F \circ J$  in **Set**. Now we say that  $F$  creates the limit (colimit) of the diagram  $J$  if there is a limit (colimit) of  $J$ , given by an object  $X$  and morphisms  $f_i$ , and if furthermore the limit (colimit) of  $F \circ J$  in **Set** is given by the set  $FX$  together with the morphisms  $Ff_i$ .

**Definition 1.22.** We say that a concrete category  $\mathcal{C}$  has all finite products if for every finite collection  $X_1, \dots, X_n$  of objects of  $\mathcal{C}$ , the product  $X_1 \times \dots \times X_n$  exists and is created by  $F$  and if  $\mathcal{C}$  has a terminal object.

Now we make more precise what we mean by a *structure* on a set, in the context of a concrete category.

**Definition 1.23.** If  $\mathcal{C}$  is a concrete category with grounding functor  $F$ , then a  $\mathcal{C}$ -structure on a set  $X$  is an object  $I$  in  $\mathcal{C}$  such that  $FI = X$ . If  $I$  and  $J$  are structures on  $X$ , we say that  $I$  is coarser than  $J$  or  $J$  is finer than  $I$  if the identity  $\text{id}_X$  is a morphism  $J \rightarrow I$ . We write  $I \preceq J$  if  $J$  is finer than  $I$ . The finest structure

on a set  $X$  is called the discrete structure, and the coarsest structure is called the indiscrete structure.

It is immediate that  $\preceq$  is a reflexive and transitive relation. However, it is not necessarily antisymmetric. If  $J$  is finer than  $I$  and  $I$  is finer than  $J$ , in general one can only say that  $I$  and  $J$  are isomorphic.

**Example 1.24.** *The category  $\mathbf{Top}$  of topological spaces, with the ordinary grounding functor, is a concrete category. A  $\mathbf{Top}$ -structure on a set  $X$  is simply a topology on  $X$ . The relation  $\preceq$  is the usual notion of finer in topology, since if  $\mathcal{O}$  and  $\mathcal{O}'$  are topologies on a set  $X$ , the identity  $\text{id}_X$  gives rise to a continuous map  $(X, \mathcal{O}) \rightarrow (X, \mathcal{O}')$  if and only if  $\mathcal{O}' \subset \mathcal{O}$ . In fact, in this example  $\preceq$  is antisymmetric (not just up to isomorphism), so that the topologies on a given set  $X$  form a partially ordered set.*

**Example 1.25.** *Let  $\mathcal{G}$  be the category of groups. If  $G$  and  $H$  are two groups with the same underlying set  $X$ , then  $H \preceq G$  if the identity on  $X$  gives rise to a homomorphism of groups. But a bijective homomorphism is automatically an isomorphism in this category. Hence  $G = H$ , and the relation  $\preceq$  is trivial.*

#### Initial and Final Structures

A main property shared by the categories in the present thesis is the existence of all final and initial structures.

**Definition 1.26.** *Let  $\mathcal{C}$  be a concrete category with grounding functor  $F : \mathcal{C} \rightarrow \mathbf{Set}$ , let  $\{X_i \mid i \in I\}$  be a family of objects and  $X$  be any set. If  $f_i : X \rightarrow FX_i$  is a family of set maps, then we say that the set  $X$  admits an initial structure with respect to the maps  $f_i$  if there is a unique object  $U$  in  $\mathcal{C}$  such that*

$$i) \quad FU = X$$

ii) The maps  $f_i$  define morphisms  $f_i : U \rightarrow X_i$

iii) If  $V$  is another object with properties i) and ii), then the identity  $\text{id}_X$  is a morphism  $V \rightarrow U$ , that is, the structure  $V$  is finer than the structure  $U$ .

If  $U$  and  $V$  are initial structures on  $X$ , then by iii), the identity on  $X$  gives rise to an isomorphism between  $U$  and  $V$ . Therefore we will speak of *the* initial structure on a set, if it exists.

**Example 1.27.** *In the category of topological spaces, the product topology on  $X \times Y$  is the initial structure with respect to the projections onto  $X$  and  $Y$ .*

Dually, we define final structures.

**Definition 1.28.** *Let  $\mathcal{C}$  be a concrete category with grounding functor  $F : \mathcal{C} \rightarrow \mathbf{Set}$ , let  $\{X_i \mid i \in I\}$  be a family of objects and  $X$  be any set. If  $g_i : FX_i \rightarrow X$  is a family of set maps, then we say that  $X$  admits a final structure with respect to the maps  $g_i$  if there is a unique object  $U$  in  $\mathcal{C}$  such that*

i)  $FU = X$

ii) The maps  $g_i$  define morphisms  $g_i : X_i \rightarrow U$

iii)  $U$  is the finest structure on  $X$  satisfying i) and ii).

As in the case of initial structures, final structures are unique up to isomorphism. An example of a final structure in the topological category is given by the quotient topology, which is the finest topology for which the quotient map is continuous.

We will see later on that in the categories of interest, the grounding functor will create limits and colimits because we can equip objects with initial and final structures with respect to the maps  $f_i : U \rightarrow X_i$  and  $g_i : X_i \rightarrow X$  that arise in the definition of limit and colimit.

## Cartesian Closedness

Suppose that  $X \times Y$  is the cartesian product of two sets. If  $f : X \times Y \rightarrow Z$  is a set map, then for every element  $x \in X$  we get a map  $y \mapsto f(x, y)$  from  $Y$  to  $Z$ . This assignment can be thought of as a map from  $X$  to  $\text{Map}(Y, Z)$ . Let us call this map  $\bar{f}$ . Conversely, if  $f \in \text{Map}(X, \text{Map}(Y, Z))$ , we can define a map  $\tilde{f}$  on  $X \times Y$  by letting  $\tilde{f}(x, y) = f(x)(y)$ . It is now clear that  $\bar{\tilde{f}}(x)(y) = \tilde{f}(x, y) = f(x)(y)$  for all  $x \in X$  and  $y \in Y$ , which shows that  $\bar{\tilde{f}} = f$ . Similarly,  $\tilde{\bar{f}} = f$  for every map  $f : X \times Y \rightarrow Z$ . We have thus established a bijection

$$\text{Map}(X \times Y, Z) \cong \text{Map}(X, \text{Map}(Y, Z)).$$

This bijection is called an *exponential law*, because if we write  $\text{Map}(X, Y)$  as  $Y^X$ , the bijection takes on the form

$$Z^{X \times Y} \cong (Z^Y)^X.$$

Now roughly speaking, a cartesian closed category is a category in which a suitable form of the exponential law holds. We will not need the most general definition, which can be found in [Mac71], but rather give a simplified definition sufficient for our purpose. We will comment on the relation to the more general definition in Remark 1.31

**Definition 1.29.** *Let  $X, Y$  and  $Z$  be sets. We define maps*

$$\Psi : \text{Map}(X \times Y, Z) \rightarrow \text{Map}(X, \text{Map}(Y, Z))$$

$$\Phi : \text{Map}(X, \text{Map}(Y, Z)) \rightarrow \text{Map}(X \times Y, Z),$$

$$\text{eval} : \text{Map}(X, Y) \times X \rightarrow Y$$

by  $\Psi(f)(x)(y) = f(x, y)$ ,  $\Phi(f)(x, y) = f(x)(y)$  and  $\text{eval}(f, x) = f(x)$ . We abbreviate

$$\bar{f} := \Psi(f) \quad \text{and} \quad \tilde{f} := \Phi(f),$$

and call *eval* the evaluation map.

**Definition 1.30.** A concrete category  $\mathcal{C}$  with grounding functor  $F$  is said to have Hom-objects if for each pair  $(X, Y)$  of objects in  $\mathcal{C}$  there is an object  $Y^X$ , and the following conditions are satisfied:

- $F(Y^X) = \text{Hom}_{\mathcal{C}}(X, Y)$ .
- If  $\varphi : Y \rightarrow Z$  is a morphism, then there is a morphism  $\varphi_* : Y^X \rightarrow Z^X$  given by  $\varphi_*(f) = \varphi \circ f$ .
- If  $\psi : Z \rightarrow Y$  is a morphism, then there is a morphism  $\psi^* : X^Y \rightarrow X^Z$  given by  $\psi^*(f) = f \circ \psi$ .

Note that  $\varphi_*(f)$  has to be understood in the sense of Remark 1.20. Therefore,  $\varphi_*(f) = \varphi \circ f$  is equivalent to  $F(\varphi_*(f)) = F(\varphi) \circ F(f)$ . From now on, if a category  $\mathcal{C}$  has Hom-objects, we will use  $Y^X$  and  $\text{Hom}_{\mathcal{C}}(X, Y)$  interchangeably.

A concrete category  $\mathcal{C}$  with grounding functor  $F$  is cartesian closed if it has all finite products, Hom-objects and if for all objects  $X, Y$  and  $Z$ , there is a bijection  $\Psi : \text{Hom}_{\mathcal{C}}(X \times Y, Z) \rightarrow \text{Hom}_{\mathcal{C}}(X, Z^Y)$  given by  $\Psi(f)(x)(y) = f(x, y)$ . We denote the inverse of  $\Psi$  by  $\Phi$ .

**Remark 1.31.** Mac Lane defines a cartesian closed category to be a category with all finite products, and with a specified right adjoint  $R : X \mapsto X^Y$  to the functor  $L : X \mapsto X \times Y$ . In fact, the cartesian product defines a bifunctor  $(X, Y) \mapsto X \times Y$ , and it follows from a general theorem ([Mac71, IV.7. Theorem 3]) that  $(X, Y) \mapsto X^Y$  is also a bifunctor. For convenience, we made this part of the definition of Hom-object.

Every adjunction can be uniquely described by different data, as Mac Lane shows in [Mac71, IV.1. Theorem 2]. One of these descriptions corresponds to our defi-

inition of cartesian closedness in terms of a bijection between Hom-sets. Another description given by Mac Lane is in terms of a natural transformation  $\varepsilon : LR \rightarrow \text{id}$ , which has the property that  $\varepsilon_X : Y^X \times X \rightarrow Y$  is universal from  $L$  to  $Y$ . In the category of sets,  $LR(X) = X^Y \times Y$ , and the natural transformation is simply

$$\text{eval} : X^Y \times Y \rightarrow Y.$$

Being universal from  $L$  to  $Y$  means that if any map  $f : L(Z) = Z \times X \rightarrow Y$  is given, then there is a unique map  $g : Z \rightarrow Y^X$  such that  $\text{eval} \circ F(g) = f$ . The map  $g$  is given by  $g(z) = f(z, \cdot) : x \mapsto f(z, x)$ . This motivates the following lemma, which characterizes cartesian closedness in terms of the evaluation map.

**Lemma 1.32.** *For a concrete category  $\mathcal{C}$  with grounding functor  $F$ , all finite products and Hom-objects, the following are equivalent:*

i)  $\mathcal{C}$  is cartesian closed.

ii) For objects  $Z$  and  $Y$ , the set map  $\text{eval} : \text{Map}(Y, Z) \times Y \rightarrow Z$  gives rise to a morphism  $\text{eval} : Z^Y \times Y \rightarrow Z$  in  $\mathcal{C}$  with the following universal property:

Whenever  $f : X \times Y \rightarrow Z$  is a morphism, then there is a unique morphism

$g : X \rightarrow Z^Y$  such that  $\text{eval} \circ (g, \text{id}_Y) = f$ .

*Proof.* First, assume that  $\mathcal{C}$  is cartesian closed. Then there is an object  $Y^X$  for each pair of objects  $(X, Y)$  in  $\mathcal{C}$ , and a bijection  $\Phi : \text{Hom}_{\mathcal{C}}(Y^X, Y^X) \rightarrow \text{Hom}_{\mathcal{C}}(Y^X \times X, Y)$ . Under this bijection, the identity  $\text{id}$  on  $Y^X$  goes to  $\Phi(\text{id})(f, x) = \text{id}(f)(x) = f(x)$ . Hence  $\text{eval}$  is a morphism from  $X^Y \times X$  to  $Y$ . It remains to prove the universal property. But if  $f \in \text{Hom}_{\mathcal{C}}(X \times Y, Z)$ , then  $g = \Phi^{-1}(f)$  is the desired morphism which satisfies  $\text{eval}(g(x), y) = g(x)(y) = f(x, y)$ .

Now conversely, assume that  $\mathcal{C}$  is a concrete category with finite products and Hom-objects. If  $\text{eval}$  gives rise to a morphism, we need to show that there is a

bijection  $\Psi'$  as in the definition of cartesian closedness. For  $f \in \text{Hom}_{\mathcal{C}}(X \times Y, Z)$  we define  $\Psi'(f)$  to be the morphism  $g$  whose existence is guaranteed by condition ii). Then  $\text{eval}(g(x), y) = g(x)(y) = f(x, y)$  by ii), which shows that  $\Psi'$  comes from the corresponding map  $\Psi$  defined on the underlying sets. More precisely:  $F(\Psi'(f)(x)) = \Psi(F(f))(x) \in \text{Map}(F(Y), F(Z))$ . Since  $F$  is faithful and  $\Psi$  is injective, so is  $\Psi'$ . It remains to show that  $\Psi'$  is onto. To this end, let  $f \in (Z^Y)^X$ . Then  $\tilde{f}(x, y) = f(x)(y) = (\text{eval} \circ \rho)(x, y)$  where  $\rho(x, y) = (f(x), y)$ . By assumption on  $f$ , the map  $x \mapsto f(x)$  is a morphism, hence  $\rho$  is a morphism, and so is  $\tilde{f}$ . Clearly  $\Psi'(\tilde{f}) = f$ , which shows that  $\Psi'$  is onto.  $\square$

It is true in general that, if  $\mathcal{C}$  is cartesian closed, then the bijection  $\Psi$  is also an isomorphism in  $\mathcal{C}$ . This is exercise IV.6.3 in [Mac71]. We give the proof for the case of a concrete category.

**Lemma 1.33.** *If  $\mathcal{C}$  is concrete and cartesian closed, then  $\Phi$  yields an isomorphism of the objects  $Z^{X \times Y}$  and  $(Z^Y)^X$  in  $\mathcal{C}$ .*

*Proof.* We will make use of characterization ii) of cartesian closedness in Lemma 1.32. If we compose

$$(Z^Y)^X \times X \times Y \xrightarrow{(\text{eval}, \text{id}_Y)} Z^Y \times Y \xrightarrow{\text{eval}} Z,$$

the universal property gives us a morphism  $\bar{\Phi} : (Z^Y)^X \rightarrow Z^{X \times Y}$ . Also by the universal property,  $\bar{\Phi}(f)(x, y) = f(x)(y)$ , which shows that  $\bar{\Phi}$  corresponds to the previously defined map  $\Phi$  on underlying sets. It remains to show that the inverse of  $\bar{\Phi}$  is also a morphism. To this end, consider the object  $U = Z^{X \times Y} \times Y$ . Evaluation yields a morphism  $U \times X \rightarrow Z$ , so there is a unique morphism  $\varphi : U \rightarrow Z^X$  by Lemma 1.32, which maps  $(f, y) \in U$  to the morphism  $x \mapsto f(x, y)$ . Now we apply

the lemma again, this time to the morphism  $\varphi : U = Z^{X \times Y} \times Y \rightarrow Z^X$ , and we get a morphism  $Z^{X \times Y} \rightarrow (Z^X)^Y$  which is inverse to  $\Phi$ .  $\square$

**Corollary 1.34.** *In a concrete cartesian closed category the following set maps are morphisms:*

- *eval,  $\Phi$  and  $\Psi$  as defined above.*
- *ins :  $X \rightarrow \text{Hom}(Y, X \times Y)$ ,  $x \mapsto (y \mapsto (x, y))$*
- *comp :  $\text{Hom}(X, Y) \times \text{Hom}(Z, X) \rightarrow \text{Hom}(Z, Y)$ ,  $(f, g) \mapsto f \circ g$*
- 

$$\begin{aligned} \text{Hom}(-, -) : \text{Hom}(X, Y) \times \text{Hom}(U, V) &\rightarrow \text{Hom}(U^Y, V^X) \\ (f, g) &\mapsto (h \mapsto g \circ h \circ f) \end{aligned}$$

*Proof.* See [KM97, Corollary I.3.13].  $\square$

## 1.1.2 Locally Convex Vector Spaces

Throughout this thesis, vector space means vector space over  $\mathbb{R}$ . A *functional* on a vector space  $V$  is a linear map  $V \rightarrow \mathbb{R}$ . We denote by  $V^*$  the vector space of functionals on  $V$  (the algebraic dual of  $V$ ), and if  $V$  is a topological vector space, we denote by  $V'$  the vector space of continuous functionals (the continuous or topological dual of  $V$ ).

**Definition 1.35.** *By a locally convex space we mean a Hausdorff topological vector space in which every zero-neighborhood contains a convex neighborhood. In other words,  $0$  has a neighborhood basis of convex sets.*

**Definition 1.36.** *A subset  $A$  of a locally convex space is:*

- *Balanced if  $\lambda x \in A$  whenever  $x \in A$  and  $|\lambda| \leq 1$ .*



- Absolutely convex if it is convex and balanced.
- Bounded if for every zero-neighborhood  $U$  there is a constant  $C > 0$  such that for all  $\lambda$  with  $|\lambda| \geq C$ , one has  $A \subset \lambda U$ .

A sequence  $\{x_n\}$  in a locally convex space  $V$  is called a Cauchy sequence if for each zero-neighborhood  $U$  there is an  $N$  such that  $x_n - x_m \in U$  whenever  $n, m \geq N$ . The sequence is a Mackey-Cauchy sequence if there is a bounded absolutely convex subset  $B$  of  $V$  and a sequence  $\{\mu_n\}$  in  $\mathbb{R}$  such that  $\lim_{n \rightarrow \infty} \mu_n = 0$  and  $x_n - x_m \in \mu_N B$  whenever  $n, m \geq N$ . We call the locally convex space  $V$  complete or Mackey-complete if every Cauchy sequence or every Mackey-Cauchy sequence converges, respectively.

**Remark 1.37.** Let  $\{x_n\}$  be a Mackey-Cauchy sequence and  $B, \mu_n$  as in the definition, so that  $x_n - x_m \in \mu_N B$  for  $n, m$  large enough. Then, since  $B$  is bounded, for every zero-neighborhood  $U$  we can choose  $N$  such that  $B \subset \frac{1}{\mu_N} U$ , or  $\mu_N B \subset U$ . Hence  $x_n - x_m \in U$  for large  $n$  and  $m$ . This shows that Mackey-Cauchy sequences are Cauchy sequences and that every complete locally convex space is also Mackey complete. Now suppose  $(V, \|\cdot\|)$  is a normed vector space and  $\{x_n\}$  is a Cauchy sequence. We let  $B = B_1(0)$  be the closed unit ball, which is bounded and absolutely convex. If we let  $\mu_N = \inf\{\varepsilon \mid (\forall n, m \geq N) : x_n - x_m \in \varepsilon B\}$ , then  $\mu_N$  is a zero-sequence, and  $x_n - x_m \in \mu_N B$  for  $n, m \geq N$ . This shows that for normable locally convex spaces, completeness and Mackey completeness are the same.

**Remark 1.38.** Kriegl and Michor [KM97] call Mackey complete locally convex spaces convenient vector spaces and develop a theory of convenient manifolds, which are modeled on convenient vector spaces.

**Definition 1.39.** *A complete metrizable locally convex space is called Fréchet space, and a complete normable locally convex space is called Banach space.*

The following characterizations of metrizable and normable locally convex spaces are well known. See for example [RR73].

**Theorem 1.40.** *A locally convex space  $V$  is metrizable if and only if there is a countable neighborhood basis of convex sets. A locally convex space is normable if and only if it contains a bounded zero-neighborhood.*

Examples

**Example 1.41.** *If  $[a, b] \subset \mathbb{R}$  is a compact interval, define*

$$\|f\|_{[a,b]} = \sup_{x \in [a,b]} |f(x)|$$

*for continuous functions  $f \in C([a, b])$ . Then  $\|\cdot\|_{[a,b]}$  is a norm which turns  $C([a, b])$  into a Banach space.*

**Example 1.42.** *Let  $C^\infty(\mathbb{R}, \mathbb{R})$  be the vector space of indefinitely differentiable functions on  $\mathbb{R}$ . We define a family of seminorms by*

$$|f|_{n,m} = \|f^{(n)}\|_{[-m,m]}.$$

*The sets*

$$B_{n,m,k} = \{f \mid |f|_{n,m} < 1/k\}$$

*form a countable basis of convex zero-neighborhoods for a topology on  $C^\infty(\mathbb{R}, \mathbb{R})$ .*

*One can show that the vector space is complete with this topology, hence it is a Fréchet space.*

Next we give an example of a colimit of locally convex spaces.

**Example 1.43.** *Let  $V = C_c^\infty(\mathbb{R}, \mathbb{R})$  denote the vector space of smooth functions with compact support. For each compact subset  $K \subset \mathbb{R}$ , consider the subspace*

$C_K^\infty(\mathbb{R}, \mathbb{R}) \subset V$  of functions with support in  $K$ . If we order the set  $I$  of compact subsets of  $\mathbb{R}$  by inclusion, then  $I$  is a directed set since the union of two compact sets is compact. There is a topology on  $V$  which makes  $V$  the colimit of the corresponding diagram in the category of locally convex spaces. It can be shown (see [RR73] Chapter V.7. Supplement 3) that one gets the same topology on  $V$  if one considers the diagram of shape  $\mathbb{N}$  whose vertices are given by  $V_n = C_{[-n, n]}^\infty(\mathbb{R}, \mathbb{R})$ .

Duality

**Definition 1.44.** A pair  $(V, V')$  of real vector spaces, together with a bilinear map  $b(\cdot, \cdot) : V \times V' \rightarrow \mathbb{R}$  is called a dual pair if

(D1) For each  $v \neq 0$  in  $V$  the linear map  $v' \mapsto b(v, v')$  on  $V'$  is not the zero map.

(D2) For each  $v' \neq 0$  in  $V'$  the linear map  $v \mapsto b(v, v')$  on  $V$  is not the zero map.

In view of (D2), if  $(V, V')$  is a dual pair, then we can identify  $V'$  with a subspace of the algebraic dual of  $V$ . A topology on  $E$  which makes  $E$  a locally convex space, such that the continuous dual of  $E$  agrees with  $E'$  under this identification, is called a topology of the dual pair  $(V, V')$ .

Here are three examples:

- If  $(V, V')$  is a dual pair, then so is  $(V', V)$ .
- If  $V^*$  is the algebraic dual of a vector space  $V$ , then  $(V, V^*)$  is a dual pair.
- If  $V$  is a locally convex space with continuous dual  $V'$ , then  $(V, V')$  is a dual pair. This is a consequence of the Hahn-Banach theorem.

**Theorem 1.45.** For all dual topologies of  $(V, V')$ , the space  $V$  has the same bounded sets.

*Proof.* See [RR73] Chapter IV Theorem 1. □

### 1.1.3 Differentiation Theory

Good references for calculus in locally convex spaces are Milnor [Mil84], who considers complete locally convex spaces, and Glöckner [Glö02] for the general case. See also Neeb [Nee06] for a survey of Lie groups modeled on locally convex spaces.

**Definition 1.46.** *If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a map, we denote by  $d_x f : \mathbb{R}^n \rightarrow \mathbb{R}$  the total derivative of  $f$  at  $x$ . We write  $d_x f(h)$  for the directional derivative of  $f$  at  $x$  in direction  $h \in \mathbb{R}^n$ . If  $e_i \in \mathbb{R}^n$  is the  $i$ -th standard unit vector, then the  $i$ -th partial derivative of  $f$  is given by  $d_x f(e_i)$ . It is denoted  $\frac{\partial f}{\partial x_i}(x)$  or simply  $\partial_i f(x)$ .*

The following theorem implies that the smooth functions on  $\mathbb{R}^n$  can be detected by composing with all smooth curves.

**Theorem 1.47** (Boman's Theorem). *Let  $f$  be a function from  $\mathbb{R}^n$  to  $\mathbb{R}$  and assume that the composition  $f \circ c$  belongs to  $C^\infty(\mathbb{R}, \mathbb{R})$  for every smooth  $c \in C^\infty(\mathbb{R}, \mathbb{R}^n)$ . Then  $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ .*

*Proof.* See [Bom67]. □

#### Differentiation in Locally Convex Spaces

**Definition 1.48.** *Let  $V$  and  $W$  be topological vector spaces,  $U \subset V$  open and  $f : U \rightarrow W$  continuous. If  $x \in U$  and  $h \in V$ , we define*

$$d_x f(h) = \lim_{t \rightarrow 0} t^{-1}(f(x + th) - f(x))$$

*whenever the limit exists. The vector  $d_x f(h)$  is then called directional derivative or Gâteaux-derivative of  $f$  at  $x$  in direction  $h$ . If for some  $x \in U$  the limit exists for all  $h \in V$ , then we say that  $f$  is Gâteaux-differentiable at  $x$ .*

*We say that  $f$  is differentiable in the sense of Michal-Bastiani or  $f$  is of class  $C_{MB}^1$  on  $U$  if  $f$  is Gâteaux-differentiable at every  $x \in U$  and if*

$$df : U \times V \rightarrow W$$

is a continuous map.

**Remark 1.49.** We follow Neeb [Nee06] in using the notation  $C_{MB}^1$ . Milnor [Mil84] and Gloeckner [Glö02] simply use the symbol  $C^1$ .

**Definition 1.50.** If  $f : U \rightarrow W$  is of class  $C_{MB}^1$ , then for each  $h \in V$  we can consider the map

$$D_h f : x \mapsto d_x f(h)$$

from  $U$  to  $W$ . We now say that  $f$  is of class  $C_{MB}^2$  if  $D_h f$  is of class  $C_{MB}^1$  for all  $h \in V$  and if furthermore the map

$$d^2 f : U \times V \times V \rightarrow W$$

defined by  $d^2 f(x, h_1, h_2) = d_x(D_{h_1} f)(h_2)$  is continuous. Inductively we define  $d^n f(x; h_1, \dots, h_n) = d_x(D_{h_{n-1}} \circ \dots \circ D_{h_1}(f))(h_n)$  and say that  $f$  is of class  $C_{MB}^n$  if the resulting map

$$d^n f : U \times V^n \rightarrow W$$

is continuous. Finally we say that  $f$  is of class  $C_{MB}^\infty$  if it is of class  $C_{MB}^n$  for all  $n \in \mathbb{N}$ .

**Remark 1.51.** If  $V$  and  $W$  are normed spaces, there is the stronger concept of Fréchet derivative. We say that  $f : V \rightarrow W$  is Fréchet differentiable at  $x \in V$  if there is a bounded linear operator  $A_x : V \rightarrow W$  such that

$$\lim_{\|h\| \rightarrow 0} \frac{\|f(x+h) - f(x) - A_x h\|}{\|h\|} = 0.$$

The function  $f$  is Fréchet differentiable on an open set  $U$  if it is Fréchet differentiable at each point  $x \in U$ , and if the map  $x \mapsto A_x$  is a continuous map from  $U$  into the normed space of bounded linear operators  $L(V, W)$ . One can then define higher Fréchet differentiability, and it turns out that if  $f$  is of class  $C_{MB}^{n+1}$ , then

it is  $n$ -times Fréchet differentiable (see Warning 3.6. in [Mil84]). Therefore, both definitions of differentiability lead to the same ‘ $C^\infty$ -maps’.

If  $V$  and  $W$  are locally convex, one can prove some of the important theorems from calculus. Let us here just mention linearity of  $d_x f$  and the chain rule.

**Lemma 1.52.** *Suppose that  $V, W$  and  $Z$  are locally convex spaces, and  $U \subset V$  is open. If  $f : U \rightarrow W$  is of class  $C_{MB}^1$ , then for any  $x \in U$ , the map  $d_x f$  is linear and continuous. If  $U' \subset W$  is open,  $f$  maps  $U$  into  $U'$  and  $g : U' \rightarrow Z$  is of class  $C_{MB}^1$ , then  $g \circ f$  is of class  $C_{MB}^1$  and  $d_x(g \circ f) = d_{f(x)}g \circ d_x f$ .*

*Proof.* See [Glö02] Lemma 1.9 and Proposition 1.12. □

**Remark 1.53.** *If we consider curves  $c : \mathbb{R} \rightarrow V$  with values in a locally convex space, then the definition of differentiability simplifies somewhat. Suppose that  $c$  is of class  $C_{MB}^1$ . Since by the above lemma  $dc(x, h)$  is linear in the second argument, we have  $d_x c(h) = h d_x c(1)$ , and we define  $c'(x) = d_x c(1)$ . We observe that  $c$  is of class  $C_{MB}^1$  if and only if  $c'(x)$  exists for all  $x \in \mathbb{R}$ , and if  $c' : \mathbb{R} \rightarrow V$  is continuous. Iteratively,  $c$  is of class  $C_{MB}^n$  when its  $n$ -th derivative  $c^{(n)} = (c^{(n-1)})'$  exists and is continuous; we then have*

$$d^n c(x; h_1, \dots, h_n) = h_1 \dots h_n c^{(n)}(x).$$

*See also [KM97], Section I.1. for a detailed discussion of curves into locally convex spaces.*

**Lemma 1.54.** *A continuous linear map  $L : V \rightarrow W$  between locally convex spaces is smooth.*

*Proof.* By linearity of  $L$  we get  $t^{-1}(L(x + th) - L(x)) = L(h)$ . Hence  $dL(x, h) = L(h)$ , which is continuous on  $V \times V$ . Furthermore, it is constant in the first argument. Therefore all higher directional derivatives vanish, and  $L$  is smooth. □

To see that local convexity is an important assumption, we give an example of a differentiable curve into a non-locally convex space, whose derivative is 0 but which is not constant. Hence, the Fundamental Theorem of Calculus fails in general.

**Example 1.55.** *The spaces  $V = L^p[0, 1]$  with  $0 < p < 1$  are examples of topological vector spaces which are not locally convex. They have a metric defined by  $d(f, g) = |f - g|$ , where*

$$|f| = \int_0^1 |f(x)|^p dx.$$

*Let  $\chi_{[s,t]}$  denote the characteristic function of the interval  $[s, t] \subset \mathbb{R}$ , and let  $c : \mathbb{R} \rightarrow V$  be the curve defined by  $c(t) = \chi_{[0,t]}$ . Then*

$$|h^{-1}(c(t+h) - c(t))| = \int_0^1 |h^{-1}\chi_{[t,t+h]}(x)|^p dx = \frac{h}{h^p}$$

*which goes to 0 as  $h \rightarrow 0$ , since  $p < 1$ . This shows that  $c$  is differentiable with constant derivative  $c'(t) = 0$ .*

The following example, due to Glöckner, gives a map which is not smooth in the sense just defined, but will turn out to be smooth in the category of Frölicher spaces.

**Example 1.56.** *Recall that  $V = C_c^\infty(\mathbb{R}, \mathbb{R})$  carries a topology which makes it the colimit of its subspaces  $V_n = C_{[-n,n]}^\infty(\mathbb{R}, \mathbb{R})$  (Example 1.43). Now Glöckner defines a map  $f : V \rightarrow V$  by  $f(\gamma) = \gamma \circ \gamma - \gamma(0)$ . He then shows that  $f$  is not continuous in the direct limit topology on  $V$ , but its restriction to  $V_n$  is smooth for all  $n$ . See [Glö06] for details.*

We conclude the introductory chapter by quoting some theorems from [KM97]. The first one gives a characterization of Mackey complete locally convex spaces.

**Theorem 1.57.** *For a locally convex space  $V$ , the following are equivalent:*

- i)  $V$  is Mackey complete.*

ii) A map  $c : \mathbb{R} \rightarrow V$  is smooth if and only if for each continuous functional  $\Lambda \in V'$ , the composition  $\Lambda \circ c$  is smooth.

*Proof.* See [KM97], Theorem I.2.14. □

The next theorem will later be used to prove cartesian closedness of a certain category.

**Theorem 1.58.** *For a set map  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ , the following are equivalent:*

i) *For all smooth curves  $c : \mathbb{R} \rightarrow \mathbb{R}^2$ , the composite  $f \circ c$  is smooth.*

ii) *For every  $x \in \mathbb{R}$ , the map  $y \mapsto f(x, y)$  is smooth. Therefore  $\bar{f}$  is a map from  $\mathbb{R}$  to  $C^\infty(\mathbb{R}, \mathbb{R})$ . If  $C^\infty(\mathbb{R}, \mathbb{R})$  is considered as a Fréchet space as in Example 1.42, then  $\bar{f}$  is a smooth curve.*

*Proof.* See [KM97, I.3.2.]. □

The last result states that the set of smooth curves into a locally convex space depends only on the bounded sets.

**Theorem 1.59.** *Let  $V$  be a vector space with locally convex topologies  $\mathcal{T}$  and  $\mathcal{S}$ . If  $(V, \mathcal{T})$  and  $(V, \mathcal{S})$  have the same bounded sets, then the same curves  $c : \mathbb{R} \rightarrow V$  are smooth for both topologies. In particular, all topologies of a dual pair  $(V, V')$  yield the same smooth curves.*

*Proof.* The first statement can be found in [KM97], Corollary I.1.8. and the comments thereafter. The latter statement now follows, since by Theorem 1.45 all topologies of a dual pair yield the same bounded sets. □



# Chapter 2

## Smooth Spaces

In this chapter we will introduce the main categories considered in this thesis. These are the categories of diffeological spaces and that of  $M$ -spaces, as well as Frölicher spaces which are a special case of  $M$ -spaces. All of these categories have in common that they are concrete categories, and that their structures in the sense of Chapter 1 are determined by functions into the underlying set. Therefore, in the first section we discuss the abstract idea of spaces whose structure is determined by functions. In the following sections we then define diffeological spaces,  $M$ -spaces and Frölicher spaces and discuss some of their category theoretical properties. In the last section we show that there is an adjoint pair of functors between the categories of Frölicher and diffeological spaces, which allows us to carry over some definitions from one to the other category, such as  $L$ -type and differential forms.

### 2.1 Structures Determined by Functions

Let us define a general type of category, whose objects are pairs  $(X, C)$  consisting of a set  $X$  and a collection  $C$  of maps into  $X$ .

**Definition 2.1.** *Let  $\mathcal{Q}$  denote any nonempty collection of sets. We say that a category  $\mathcal{C}$  is a  $\mathcal{Q}$ -category if it has the following properties: Its objects are pairs  $(X, C_X)$ , where  $C_X$  is a collection of set maps into  $X$ , whose domains are elements of  $\mathcal{Q}$ . A morphism  $f : (X, C_X) \rightarrow (Y, C_Y)$  is given by any set map  $f : X \rightarrow Y$  for which  $f \circ c \in C_Y$  for every  $c \in C_X$ .*

**Remark 2.2.** Any  $\mathcal{Q}$ -category is concrete, when we define the grounding functor to map a space  $(X, C)$  to the underlying set  $X$ , and a morphism to the corresponding set map. As noted in Chapter 1, we often identify  $(X, C)$  and  $X$  for notational convenience. If  $C$  and  $C'$  are two structures on the same set  $X$ , then  $C$  is finer than  $C'$ , or  $C' \preceq C$ , if and only if  $C \subset C'$ . In particular, the relation  $\preceq$  is antisymmetric and hence a partial order on the set of structures on  $X$ .

Let us generalize the definition of smooth map.

**Definition 2.3.** Let  $V$  and  $W$  be locally convex spaces, and let  $A \subset V$  be an arbitrary subset. We say that a map  $f : A \rightarrow W$  is smooth if for every point  $x \in A$ , there is an open neighborhood  $U$  of  $x$  in  $V$  and a smooth map  $\tilde{f} : U \rightarrow W$  whose restriction to  $U \cap A$  agrees with  $f$ .

We now describe a notion of differentiable space due to Chen [Che86] (see [Che73] for a slightly different, earlier notion). Differential spaces form a  $\mathcal{Q}$ -category.

**Example 2.4.** Let  $\mathcal{Q}$  be the collection of all convex subsets of all  $\mathbb{R}^n$ ,  $n \in \mathbb{N}$ , with non-empty interior. Chen defines a differentiable space or  $C^\infty$ -space to be a set  $X$  together with a family  $C$  of set maps, called plots, which satisfy:

- Every plot is a set map  $\alpha : U \rightarrow X$  where  $U \in \mathcal{Q}$ .
- If  $\alpha : U \rightarrow X$  is a plot and  $V \in \mathcal{Q}$ , then for every smooth map  $h : V \rightarrow U$ , the map  $\alpha \circ h$  is also a plot.
- Every constant map from a convex set to  $X$  is a plot.
- Let  $\{U_i\}$  be an open convex covering of a convex set  $U$ , and let  $\alpha : U \rightarrow X$  be such that its restriction to each  $U_i$  is a plot. Then  $\alpha$  is a plot.

By definition, a morphism of  $\mathcal{Q}$ -spaces is a set map  $f : X \rightarrow Y$  such that  $f \circ \alpha$  is a plot for  $Y$  whenever  $\alpha$  is a plot for  $X$ .

In the following example, the structure consists of functions into the set, as well as functions from the set into the real numbers. However, the structure is still determined by the functions into the set, making the category a  $\mathcal{Q}$ -category.

**Example 2.5.** Let  $V$  be a locally convex space, and let  $\mathcal{Q}$  be any collection of subsets of  $V$ . We define a  $\mathcal{Q}$ -differentiable space to be a triple  $(X, C, F)$  where  $X$  is a set,  $C$  a collection of maps into  $X$ , called curves, and  $F$  a collection of maps  $f : X \rightarrow \mathbb{R}$ , which we call functions. The following conditions are required to hold:

- Every curve  $c \in C$  is of the form  $c : U \rightarrow X$  with  $U \in \mathcal{Q}$ .
- $C = \{c : A \rightarrow X \mid A \in \mathcal{Q}, f \circ c \text{ is smooth for all } f \in F\}$ .
- $F = \{f : X \rightarrow \mathbb{R} \mid f \circ c \text{ is smooth for all } c \in C\}$ .

Here ‘smooth’ is meant in the sense of Definition 2.3. Note that by definition, if  $(X, C, F)$  is a  $\mathcal{Q}$ -differentiable space, then  $F$  is uniquely determined by  $C$ . Furthermore, if we define morphisms to be set maps  $f : X \rightarrow Y$  such that  $f \circ c$  is a curve into  $Y$  for every curve  $c$  into  $X$ , then it is clear that  $\mathcal{Q}$ -differentiable spaces form a  $\mathcal{Q}$ -category.

The following observation follows directly from the definition of initial and final structures.

**Remark 2.6.** Let  $\mathcal{C}$  be a  $\mathcal{Q}$ -category with grounding functor  $F$ , for some set  $\mathcal{Q}$ . Let  $X$  be a set and  $\{X_i\}$  a family of objects of  $\mathcal{C}$ . If  $f_i : X \rightarrow FX_i$  and  $g_i : FX_i \rightarrow X$  are families of set maps, consider the following subsets of the poset of structures on  $X$  : Let  $\mathcal{A}$  be the set of structures for which all  $f_i$  are morphisms, and let  $\mathcal{B}$

be the set of structures for which all  $g_i$  are morphisms. The set  $X$  has an initial structure with respect to the family  $\{f_i\}$  if and only if  $\mathcal{A}$  has a coarsest element. Dually,  $X$  has a final structure with respect to the family  $\{g_i\}$  if and only if  $\mathcal{B}$  contains a finest element. This follows directly from Definitions 1.28 and 1.26.

## 2.2 Diffeological Spaces

Souriau describes the axioms of a diffeology in [Sou80], where he restricts his attention to groups. The more general definition follows in [Sou85]. See [HMV02] and [Lau06] for a survey and further references.

The definition of a diffeological space is very similar to that of Chen's differentiable spaces described in Example 2.4. The only difference is that Souriau uses *open* subsets of  $\mathbb{R}^n$ , rather than convex subsets, as domains for the plots. We now give the definition of a diffeology.

**Definition 2.7.** *Let  $X$  be a set and for each  $n \in \mathbb{N}$  let  $\mathcal{P}^n(X)$  be a collection of set maps  $\alpha : U_\alpha \rightarrow X$ , where  $U_\alpha \subset \mathbb{R}^n$  is an open set. Let  $\mathcal{P}(X) = \cup_n \mathcal{P}^n(X)$  denote the collection of all those maps. We say that  $\mathcal{P}(X)$  is a diffeology on  $X$  and that  $(X, \mathcal{P}(X))$  is a diffeological space if the following conditions are satisfied.*

(D1) *Every constant map  $c : U \rightarrow X$  is in  $\mathcal{P}(X)$ .*

(D2) *Given a family of maps  $\alpha_i : U_i \rightarrow X$  in  $\mathcal{P}^n(X)$  such that  $\alpha_i$  and  $\alpha_j$  agree on  $U_i \cap U_j$ , then the unique map  $\alpha : \cup_i U_i \rightarrow X$  extending the  $\alpha_i$  is again in  $\mathcal{P}^n(X)$ .*

(D3) *If  $\alpha : U_\alpha \rightarrow X$  is in  $\mathcal{P}^n(X)$  and  $h : V \rightarrow U_\alpha$  is smooth, where  $V \subset \mathbb{R}^m$  is open, then  $\alpha \circ h \in \mathcal{P}^m(X)$ .*

*Elements of  $\mathcal{P}(X)$  and  $\mathcal{P}^n(X)$  are called plots and  $n$ -plots respectively.*

**Remark 2.8.** *Axiom (D2) implies that being a plot is a local property. That is, if  $\alpha : U \rightarrow X$  is a map into a diffeological space and for every  $x \in U$ , the restriction of  $\alpha$  to some neighborhood of  $x$  is a plot, then  $\alpha$  is a plot.*

**Example 2.9.** *Every set has two extreme diffeologies, consisting of all maps and all locally constant maps, respectively. They are called indiscrete and discrete diffeology, respectively. Let  $X$  be a smooth manifold, not necessarily finite dimensional. Then the smooth maps  $\alpha : U \rightarrow X$  for  $U \subset \mathbb{R}^n$  open form a diffeology on  $X$ . We call this the manifold diffeology. Let us briefly check if the three axioms of a diffeology are satisfied for smooth maps into a manifold. (D1) Clearly, constant maps are smooth. (D2) Let  $\alpha$  be the smallest extension of the  $\alpha_i$ . Then given  $x \in U_\alpha$ , there is an  $i \in I$  such that  $x \in U_i$  and hence  $\alpha|_{U_i} = \alpha_i$  is smooth. Now smoothness is a local condition, so  $\alpha$  is smooth on all of  $U_\alpha$ . (D3) Compositions of smooth maps are smooth.*

**Remark 2.10** ( $\mathcal{D}$ -topology). *A diffeological space can be equipped with the initial topology with respect to all of its plots. Let us call this the  $\mathcal{D}$ -topology. It can be shown [Lau06] that the  $\mathcal{D}$ -topology is always locally arc-connected. If  $X$  carries the discrete and indiscrete diffeology, this topology is discrete or indiscrete, respectively. If  $X$  is a manifold with its manifold diffeology, then the  $\mathcal{D}$ -topology coincides with the topology of  $X$  as a manifold. The  $\mathcal{D}$ -topology is not necessarily Hausdorff. For a simple example, take the real line with a double point. That is, let  $X = (\mathbb{R} - 0) \cup \{a, b\}$  and say that  $\alpha : U \rightarrow X$  is smooth if and only if it is smooth as a map into  $X - b \cong \mathbb{R}$  or into  $X - a \cong \mathbb{R}$ .*

We will now define morphisms between diffeological spaces.

**Definition 2.11.** *If  $(X, \mathcal{P}(X))$  and  $(Y, \mathcal{P}(Y))$  are diffeological spaces and  $f : X \rightarrow Y$  a set map, then we say that  $f$  is smooth if  $f \circ \alpha \in \mathcal{P}(Y)$  for every  $\alpha \in \mathcal{P}(X)$ .*

It is now easily verified that diffeological spaces and smooth maps form a category, which we denote  $\mathcal{D}$ . We write  $\text{Hom}_{\mathcal{D}}(X, Y)$  for the set of smooth maps from  $X$  to  $Y$ .

Clearly  $\mathcal{D}$  is a  $\mathcal{Q}$ -category, where  $\mathcal{Q}$  is the set of all open subsets of all  $\mathbb{R}^n$ .

### 2.2.1 Generating Families and Dimension

Intersections of diffeologies are again diffeologies, and therefore it is easy to define the diffeology generated by a given family of functions. We characterize the plots of a diffeology in terms of a generating family, and in the following subsection we use generating families to define initial and final diffeologies.

**Lemma 2.12.** *Given a family  $\{\mathcal{P}_i \mid i \in I\}$  of diffeologies on a set  $X$ , the intersection  $\cap_i \mathcal{P}_i$  is again a diffeology.*

*Proof.* We have to verify the three axioms (D1)-(D3) for  $\mathcal{P} := \cap_i \mathcal{P}_i$ . By definition, each of the diffeologies  $\mathcal{P}_i$  contains all constant maps, thus the same is true for their intersection. Every compatible family of  $n$ -plots in  $\mathcal{P}$  is also compatible in each of the  $\mathcal{P}_i$ . Therefore the smallest common extension is element of each  $\mathcal{P}_i$ , hence also of  $\mathcal{P}$ . Similarly, if  $\alpha \in \mathcal{P}$  and  $h$  are composable, then  $\alpha \circ h \in \mathcal{P}_i$  for all  $i \in I$ , and therefore also  $\alpha \circ h \in \mathcal{P}$ .  $\square$

**Corollary 2.13.** *Let  $\mathcal{F}$  be a collection of maps into  $X$ , whose domains are open subsets of some  $\mathbb{R}^n$ . Then there is a unique smallest diffeology on  $X$  containing  $\mathcal{F}$ . We denote this diffeology by  $\langle \mathcal{F} \rangle$ .*

*Proof.* Note that the indiscrete diffeology contains  $\mathcal{F}$ . So the collection of all diffeologies on  $X$  which contain  $\mathcal{F}$  is nonempty, and we can take its intersection.  $\square$

**Remark 2.14.** *If  $X$  is a set, the diffeologies on  $X$  are partially ordered by the relation  $\preceq$ . Every collection of diffeologies on  $X$  has an infimum, given by the*

intersection, and a supremum, given by the diffeology generated by the union. This means that the poset  $(\mathcal{A}, \preceq)$  of diffeologies on  $X$  is a complete lattice.

**Definition 2.15.** If  $\mathcal{P}(X) = \langle \mathcal{F} \rangle$ , then  $\mathcal{F}$  is called a generating family for the diffeology. For a given generating family  $\mathcal{F}$ , let  $n_{\mathcal{F}}$  denote the supremum

$$n_{\mathcal{F}} := \sup\{\dim(U_{\alpha}) \mid \alpha \in \mathcal{F}\}.$$

Then the dimension of a diffeological space  $(X, \mathcal{P}(X))$  is the infimum

$$\dim(X, \mathcal{P}(X)) := \inf\{n_{\mathcal{F}} \mid \mathcal{F} \text{ generates } \mathcal{P}(X)\}.$$

**Lemma 2.16.** Let  $X$  be a set and  $\mathcal{F}$  be a family of maps into  $X$ , whose domains are open subsets of some  $\mathbb{R}^n$ . Then a map  $\alpha : U_{\alpha} \rightarrow X$  is in  $\langle \mathcal{F} \rangle$  if and only if it satisfies:

(G) For each point  $x \in U_{\alpha}$  there is an open neighborhood  $V \subset U_{\alpha}$  of  $x$  such that the restriction of  $\alpha$  to  $V$  is either constant or of the form  $f \circ h$  for some  $f \in \mathcal{F}$  and a smooth map  $h : V \rightarrow U_f$ .

*Proof.* Let  $\mathcal{P}$  denote the collection of maps  $\alpha$  satisfying (G). Clearly,  $\mathcal{P}$  contains  $\mathcal{F}$ . We will first show that  $\mathcal{P}$  is a diffeology. Constant maps certainly satisfy (G), therefore (D1) holds. Now take a compatible family  $\alpha_i \in \mathcal{P}$  with smallest extension  $\alpha$ . Each  $x \in U_{\alpha}$  is contained in some  $\text{dom}(\alpha_i) = U_i$  and therefore has a neighborhood in  $U_i$  on which  $\alpha$  is constant or of the form  $f \circ g$  with smooth  $g$  and  $f \in \mathcal{F}$ . Therefore  $\alpha$  is in  $\mathcal{P}$ , which shows (D2). To verify (D3), let  $\alpha$  and  $h : U \rightarrow U_{\alpha}$  be composable, where  $\alpha$  is in  $\mathcal{P}$  and  $h$  is smooth. Let  $x \in U$  and  $y = h(x)$ . Then there is a neighborhood  $V$  of  $y$  such that either

- $\alpha|_V$  is constant. Then  $\alpha \circ h|_{h^{-1}(V)}$  is also constant.

or

- $\alpha|_V$  is of the form  $f \circ g$  for smooth  $g$  and  $f \in \mathcal{P}$ . Then

$$\alpha \circ h|_{h^{-1}(V)} = f \circ g \circ h.$$

As  $h^{-1}(V)$  is an open neighborhood of  $x$ , in either case we have that  $\alpha \circ h \in \mathcal{P}$ . This proves (D3). So  $\mathcal{P}$  is a diffeology containing  $\mathcal{F}$ . If we can show  $\mathcal{P} \subset \langle \mathcal{F} \rangle$ , we can conclude equality as  $\langle \mathcal{F} \rangle$  is minimal containing  $\mathcal{F}$ . Let  $\alpha \in \mathcal{P}$ . Then each point  $x \in U_\alpha$  has a neighborhood  $V_x$  such that  $\alpha_x := \alpha|_{V_x}$  is either constant or of the form  $f \circ g$  with smooth  $g$  and  $f \in \mathcal{F}$ . So in either case,  $\alpha_x$  is in  $\langle \mathcal{F} \rangle$ , and the collection of the  $\alpha_x$  is a compatible family with smallest extension  $\alpha$ . Thus  $\alpha \in \langle \mathcal{F} \rangle$ , which completes the proof.  $\square$

**Lemma 2.17.** *If the diffeology  $\mathcal{P}(X)$  is generated by a family  $\mathcal{F}$  of functions, then a map  $f : X \rightarrow Y$  is smooth if  $f \circ \alpha \in \mathcal{P}(Y)$  for all  $\alpha \in \mathcal{F}$ .*

*Proof.* Let  $\alpha$  be a plot for  $X$ , and  $x \in U_\alpha$ . By Lemma 2.16 we can choose an open neighborhood  $U_x$  of  $x$  in  $U_\alpha$  such that  $\alpha|_{U_x}$  is either constant or of the form  $\beta \circ h$  for some  $\beta \in \mathcal{F}$  and a smooth map  $h$ . In the first case  $f \circ \alpha|_{U_x}$  is constant and thus in  $\mathcal{P}(Y)$ . In the second case,  $f \circ \alpha|_{U_x} = f \circ \beta \circ h$  which is in  $\mathcal{P}(Y)$  since by assumption,  $f \circ \beta \in \mathcal{P}(Y)$ . The open sets  $U_x$  cover  $U_\alpha$ , so we use axiom (D2) to conclude that  $f \circ \alpha \in \mathcal{P}(Y)$ , hence  $f$  is smooth.  $\square$

## 2.2.2 Initial and Final Structures

We now define initial and final structures in the diffeological category, and show that the grounding functor creates all limits and colimits.

**Lemma 2.18.** *If  $X$  is a set,  $(Y_i, \mathcal{P}(Y_i))$  a family of diffeological spaces and  $f_i : X \rightarrow Y_i$  a family of set maps, then the initial diffeology on  $X$  with respect to the  $f_i$  is given by*

$$\mathcal{P}(X) = \{\alpha : U_\alpha \rightarrow X \mid f_i \circ \alpha \in \mathcal{P}(Y_i) \text{ for all } i\}.$$



If  $g_i : Y_i \rightarrow X$  is a family of set maps, then the final diffeology on  $X$  with respect to the  $g_i$  is the diffeology generated by

$$\mathcal{F} = \{f_i \circ \alpha \mid i \in I \text{ and } \alpha \in \mathcal{P}(Y_i)\}$$

*Proof.* This follows from Remark 2.6, since the diffeologies are the coarsest and finest making all  $f_i$  and all  $g_i$  smooth, respectively.  $\square$

**Corollary 2.19.** *Let  $X, Y$  and  $X_i$  be diffeological spaces. Suppose that  $X$  carries the initial diffeology with respect to maps  $f_i : X \rightarrow X_i$ , and  $\varphi : Y \rightarrow X$  is a set map such that all  $f_i \circ \varphi$  are smooth. Then  $\varphi$  is smooth. If  $X$  carries the final diffeology with respect to maps  $g_i : X_i \rightarrow X$ , and if  $\psi : X \rightarrow Y$  is a map such that all  $\psi \circ g_i$  are smooth, then  $\psi$  is smooth.*

*Proof.* Let  $\alpha$  be a plot for  $Y$ . Then  $f_i \circ \varphi \circ \alpha$  is smooth by assumption on  $\varphi$ , hence  $\varphi \circ \alpha$  is plot for the initial diffeology on  $X$ , by definition of the initial diffeology. Thus  $\varphi$  is smooth.

Now to check that  $\psi$  is smooth, we only need to compose with an element of the generating family  $\mathcal{F}$  for the final diffeology. We get  $\psi \circ g_i \circ \alpha$ , which is smooth by assumption on  $\psi$ .  $\square$

**Theorem 2.20.** *In the category  $\mathcal{D}$  of diffeological spaces, all limits and colimits exist and are created by the grounding functor  $F : \mathcal{D} \rightarrow \mathbf{Set}$ .*

*Proof.* Fix a diagram  $J : \mathcal{G} \rightarrow \mathcal{D}$ . In  $\mathbf{Set}$ , all limits and colimits exist. Let  $X$  be the limit of  $F \circ J : \mathcal{G} \rightarrow \mathbf{Set}$ , which comes with a set map  $a_i : X \rightarrow FX_i$  for each vertex of the diagram. We equip  $X$  with the initial diffeology with respect to the  $a_i$ , and claim that the resulting space is the limit of  $J$  in  $\mathcal{D}$ . By definition, the  $a_i$  yield smooth maps  $X \rightarrow X_i$ , and for edges  $f : X_i \rightarrow X_j$  in the diagram, one has  $a_j = f \circ a_i$ . It remains to show that  $X$  satisfies the universal property, so let  $X'$

be another diffeological space and  $b_i : X' \rightarrow X_i$  smooth maps with  $b_j = f \circ b_i$ . We need to construct a smooth map  $\varphi : X' \rightarrow X$  with  $b_i = a_i \circ \varphi$  for all  $i$ . Clearly, there is a set map  $\varphi : FX' \rightarrow FX$  with that property, since  $FX$  is the limit of  $F \circ J$ . But since  $a_i \circ \varphi = b_i$  is smooth and  $X$  carries the initial diffeology with respect to the  $a_i$ , it follows from Corollary 2.19 that  $\varphi$  is smooth. The proof for colimits is analogous, and we omit the details.  $\square$

As in the theory of topological spaces, we can define embeddings and quotient maps.

**Definition 2.21.** *In Lemma 2.18, suppose that the family consists of only one space  $(Y, \mathcal{P}(Y))$ . Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  be set maps. Then the initial diffeology with respect to  $f$  consists of all maps  $\alpha : U_\alpha \rightarrow X$  such that  $f \circ \alpha \in \mathcal{P}(Y)$ . Let us denote this diffeology  $\mathcal{P}^f(X)$ . Similarly, let us write  $\mathcal{P}_g(X)$  for the final diffeology with respect to  $g$ , which is generated by the maps  $g \circ \alpha$  for  $\alpha \in \mathcal{P}(Y)$ .*

*Now suppose that  $X$  carries a diffeology  $\mathcal{P}(X)$  and  $f$  and  $g$  are morphisms. If  $f$  is surjective and  $\mathcal{P}^f(X) = \mathcal{P}(X)$  we call  $f$  a quotient map. If  $g$  is injective and  $\mathcal{P}_g(X) = \mathcal{P}(X)$  we call  $g$  an embedding.*

**Example 2.22** (Subspaces and Quotients). *Given a diffeological space  $(X, \mathcal{P}(X))$  and a subset  $A \subset X$ , we obtain a diffeology on  $A$  by just taking plots  $\alpha \in \mathcal{P}(X)$  which map into  $A$ . If  $\iota : A \rightarrow X$  is the inclusion map, we recognize this diffeology as  $\mathcal{P}_\iota(A)$ . Thus the inclusion map is an embedding.*

*Given a diffeological space  $(X, \mathcal{P}(X))$  and an equivalence relation  $\sim$  on  $X$ , we equip  $X/\sim$  with the final diffeology with respect to the projection  $\pi$ . This diffeology makes  $\pi$  a quotient map.*

**Example 2.23** (Direct Products and Sums). *Given a family  $\{X_i \mid i \in I\}$  of diffeological spaces, we can form disjoint union and direct product of the sets  $X_i$  in the*

category of sets and maps. We equip the direct product

$$\prod_i X_i$$

with the initial diffeology with respect to the projections  $\pi_i$ , and the disjoint union

$$\coprod_i X_i$$

with the final diffeology with respect to the injections  $\iota_i$ . Then the projections are quotient maps and the injections are embeddings. Note that for two spaces  $X, Y$  the product diffeology is given by plots of the form  $x \mapsto (\alpha_1(x), \alpha_2(x))$  where  $\alpha_1 \in \mathcal{P}(X)$  and  $\alpha_2 \in \mathcal{P}(Y)$  are plots with the same domain.

### 2.2.3 Hom-objects and Cartesian Closedness

In this subsection we show that  $\mathcal{D}$  has Hom-objects, and that it is a cartesian closed category. Let us first define a diffeology on the Hom-sets.

**Definition 2.24.** *Let  $X$  and  $Y$  be diffeological spaces, and let  $U \subset \mathbb{R}^n$  be open. Then  $U$  carries the manifold diffeology, and we equip  $U \times X$  with the product diffeology. Now we say that  $\alpha : U \rightarrow \text{Hom}_{\mathcal{D}}(X, Y)$  is smooth if  $\tilde{\alpha} : U \times X \rightarrow Y$  is smooth.*

**Lemma 2.25.** *Let  $X$  and  $Y$  be diffeological spaces. The smooth maps  $\alpha : U \rightarrow \text{Hom}_{\mathcal{D}}(X, Y)$  form a diffeology, making  $\text{Hom}_{\mathcal{D}}(X, Y)$  a Hom-object.*

*Proof.* The axioms for a diffeology are immediately verified. Let us just give the details for (D2): If  $\alpha_i$  is a family of smooth maps such that  $\alpha_i$  and  $\alpha_j$  agree on  $U_i \cap U_j$ , we need to show that their unique extension  $\alpha$  to  $U = \cup U_i$  is smooth. Let  $\beta = (\beta_1, \beta_2) : U_\beta \rightarrow U \times X$  be a plot. The sets  $V_i = \beta_1^{-1}(U_i)$  form an open cover of  $U_\beta$ , and the restriction of  $\tilde{\alpha} \circ \beta$  to  $V_i$  is given by  $\tilde{\alpha}_i \circ \beta$ , hence smooth. This shows that  $\alpha \circ \beta$ , and hence  $\alpha$ , is smooth.

Now we show that  $\text{Hom}_{\mathcal{D}}(X, Y)$  is a Hom-object. By definition, the underlying set is the Hom-set, so only two things remain to check. First, let  $\varphi : Y \rightarrow Z$  be smooth. We have to show that  $f \mapsto \varphi \circ f$  is a smooth map from  $\text{Hom}(X, Y)$  to  $\text{Hom}(X, Z)$ . So let  $\alpha$  be a plot for  $\text{Hom}(X, Y)$ . Then  $(\varphi_* \circ \alpha)(u)(x) = (\varphi \circ \alpha(u))(x) = \varphi(\tilde{\alpha}(u, x))$ , hence

$$\widetilde{\varphi_* \circ \alpha} = \varphi \circ \tilde{\alpha},$$

which is smooth. Lastly, let  $\psi : Z \rightarrow X$ , and we show that  $f \mapsto f \circ \psi$  defines a smooth map from  $\text{Hom}(X, Y)$  to  $\text{Hom}(Z, Y)$ . Let  $\alpha$  be a plot for  $\text{Hom}(X, Y)$ . One computes  $\widetilde{\psi^* \circ \alpha} = \tilde{\alpha} \circ (\text{id}, \psi)$ , which is smooth.  $\square$

Now we can use Lemma 1.32 to show that  $\mathcal{D}$  is cartesian closed.

**Theorem 2.26.** *The category of diffeological spaces is cartesian closed.*

*Proof.* By Lemma 1.32 it suffices to show that the evaluation map gives rise to a morphism, and that for every morphism  $f : X \times Y \rightarrow Z$ , the set map  $g : X \rightarrow \text{Map}(Y, Z)$  sending  $x$  to  $y \mapsto f(x, y)$  actually yields a morphism in  $\text{Hom}(X, \text{Hom}(Y, Z))$ . We start by showing that the evaluation map is smooth. Let  $\alpha = (\alpha_1, \alpha_2) : U \rightarrow \text{Hom}(Y, Z) \times Y$  be a plot. Then the composition  $eval \circ \alpha$  maps  $u \mapsto \alpha_1(u)(\alpha_2(u)) = \tilde{\alpha}_1(u, \alpha_2(u))$ , which is smooth because  $\tilde{\alpha}_1$  is smooth by definition of the diffeology of  $\text{Hom}(Y, Z)$ . This proves that  $eval$  is smooth.

Now let  $f : X \times Y \rightarrow Z$  be a smooth map. Then for each  $x \in X$ , the map  $y \mapsto f(x, y)$  is smooth, because  $y \mapsto (x, y)$  is smooth. It remains to show that the map  $g$  defined by  $g(x)(y) = f(x, y)$  is smooth, so let  $\alpha$  be a plot for  $X$ . We get

$$\widetilde{g \circ \alpha}(u, y) = f(\alpha(u), y),$$

which is a smooth map into  $Z$ , and this completes the proof.  $\square$

## 2.2.4 Manifolds as Diffeological Spaces

An important fact about diffeological spaces is that they generalize manifolds in the sense of the following lemma.

**Lemma 2.27.** *There is a full and faithful functor from  $\mathbf{Mfd}$  to  $\mathcal{D}$ , which assigns to each manifold its manifold diffeology.*

*Proof.* Fix a pair of manifolds  $M, N$  and let  $\varphi \in C^\infty(M, N)$ . By the chain rule, if  $\alpha : U \rightarrow M$  is smooth, then so is  $\varphi \circ \alpha$ . This shows that  $\varphi$  yields a morphism  $\varphi \in \text{Hom}_{\mathcal{D}}(M, N)$  if  $M$  and  $N$  carry the manifold diffeology. Thus we get a functor  $H : \mathbf{Mfd} \rightarrow \mathcal{D}$ . This functor is clearly faithful, and it remains to show that it is full. To this end, let  $f \in \text{Hom}_{\mathcal{D}}(M, N)$ . We have to show that  $f$  is smooth as a map between manifolds, so let  $\varphi : U \rightarrow \mathbb{R}^n$  be a chart in a given atlas  $\mathcal{A}$  of  $M$ . By choice of  $f$ , the map  $f \circ \varphi^{-1}$  is a plot for  $N$ , and therefore a smooth map. This is true for every chart in  $\mathcal{A}$ , which shows that  $f \in C^\infty(M, N)$ .  $\square$

**Remark 2.28.** *Note that open subsets  $U \subset \mathbb{R}^n$  are manifolds, and hence can be equipped with the manifold diffeology. Now if  $X$  is a diffeological space, then it is easy to see that  $\alpha : U \rightarrow X$  is a plot if and only if  $\alpha \in \text{Hom}_{\mathcal{D}}(U, X)$ .*

**Lemma 2.29.** *Let  $M$  be an  $n$ -dimensional manifold with atlas  $\mathcal{A}$  and the diffeology  $\mathcal{P}(M)$  as described above. Then the family  $\mathcal{A}^{-1} = \{\varphi^{-1} \mid \varphi \in \mathcal{A}\}$  generates  $\mathcal{P}(M)$ .*

*Proof.* Clearly  $\langle \mathcal{A}^{-1} \rangle \subset \mathcal{P}(M)$  as the inverse charts  $\varphi^{-1}$  are smooth. For the other inclusion, we use the characterization (G) of a generated diffeology from Lemma 2.16. Let  $\alpha : U_\alpha \rightarrow M$  be a smooth map,  $x \in U_\alpha$  and let  $\varphi : U \rightarrow \mathbb{R}^n$  be a chart about  $\alpha(x)$ . Then  $V = \alpha^{-1}(U)$  is an open neighborhood of  $x$  in  $U_\alpha$ . The map  $h := \varphi \circ \alpha|_V$  is smooth and we can write  $\alpha|_V = \varphi^{-1} \circ h$ , where  $\varphi^{-1} \in \mathcal{A}^{-1}$  and  $h$  is smooth. So by Lemma 2.16 it follows that  $\alpha \in \langle \mathcal{A}^{-1} \rangle$ .  $\square$

**Corollary 2.30.** *Given an  $n$ -dimensional manifold, its dimension as a diffeological space (see Definition 2.15) is also  $n$ .*

*Proof.* Lemma 2.29 immediately yields that the diffeological dimension is at most  $n$ . Now if it were strictly less than  $n$ , the diffeology would be generated by its collection of  $n-1$ -plots. But then by Lemma 2.16, every smooth map into  $M$  would locally be constant or factor through an open subset of  $\mathbb{R}^{n-1}$ . This is certainly not true for local diffeomorphisms, e.g. inverse maps of coordinate charts.  $\square$

The next theorem will characterize the diffeological spaces which are manifolds.

**Theorem 2.31.** *Let  $(X, \mathcal{P}(X))$  be a diffeological space, equipped with the  $\mathcal{D}$ -topology. The following are equivalent.*

1. *There is a smooth atlas  $\mathcal{A}$  making  $X$  a  $n$ -dimensional manifold, such that  $\mathcal{P}(X)$  is the corresponding manifold diffeology.*
2. *The  $\mathcal{D}$ -topology is Hausdorff and paracompact. There is an open cover  $\{U_i\}_{i \in I}$  of  $X$  and an  $n \in \mathbb{N}$  such that for each  $i \in I$  there is a diffeomorphism  $\varphi_i$  between  $U_i$  and some open set  $V_i \subset \mathbb{R}^n$ . Here  $U_i$  is equipped with the subset diffeology, and  $V_i$  carries the manifold diffeology.*

*Proof.* First assume that (1) holds, and let  $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ . The open sets  $U_i$  cover  $X$ , and we claim that the maps  $\varphi_i$  are isomorphisms in  $\mathcal{D}$ . But this is clear, since they are isomorphisms in the category **Mfd** of manifolds, and there is a full and faithful functor **Mfd**  $\rightarrow$   $\mathcal{D}$ . Now we assume that (2) holds. Clearly, the pairs  $(U_i, \varphi_i)$  form an atlas, since all compositions  $\varphi_i \circ \varphi_j^{-1}$  are smooth. Since  $\varphi_i^{-1} \in \mathcal{P}(X)$ , the corresponding manifold diffeology is contained in  $\mathcal{P}(X)$ . To show equality, let  $\alpha$  be any plot. But then  $\alpha$  is also smooth as a map between manifolds, because for each  $i \in I$ , the map  $\varphi_i^{-1} \circ \alpha$  is smooth.  $\square$

## 2.2.5 Spaces of $L$ -type and Diffeological Groups

We define a certain natural condition on diffeological spaces, which is satisfied by manifolds and diffeological groups. The definition is due to Leslie [Les03].

**Definition 2.32.** *Let  $(X, \mathcal{P})$  be a diffeological space and  $x \in X$ . Given two plots  $\alpha$  and  $\beta$  at  $x$ , we write  $\beta \leq \alpha$  if  $\beta$  factors through  $\alpha$ . More precisely,*

$$\beta \leq \alpha \quad \text{if there is a smooth map } h : U_\beta \rightarrow U_\alpha \quad \text{such that } \beta = \alpha \circ h.$$

*We say that the diffeological space is of lattice type or of  $L$ -type if for each point  $x \in X$  and all plots  $\alpha, \beta \in \mathcal{P}_x$  there is a plot  $\gamma$  and neighborhoods  $V \subset U_\alpha$  and  $W \subset U_\beta$  of 0 such that  $\alpha|_V \leq \gamma$  and  $\beta|_W \leq \gamma$ .*

**Example 2.33.** *Let  $X \subset \mathbb{R}^2$  be the union of the coordinate axes, equipped with the subset diffeology. Then  $X$  is not of  $L$ -type. For example, let  $\alpha : x \mapsto (x, 0)$  and  $\beta : x \mapsto (0, x)$  be plots centered at  $(0, 0)$ . Suppose there is a plot  $\gamma$  such that  $\alpha|_V = \gamma \circ h_1$  and  $\beta|_W = \gamma \circ h_2$  for some neighborhoods  $V, W$  of 0 in  $\mathbb{R}$  and smooth maps  $h_1, h_2$ . With  $u = d_0 h_1(1)$  and  $v = d_0 h_2(1)$ , the chain rule would give us  $d_0 \gamma(u) = (1, 0)$  and  $d_0 \gamma(v) = (0, 1)$ . Then  $d_0 \gamma(w) = (1, 1)$  for  $w = u + v$ . Let  $\rho : \mathbb{R} \rightarrow \mathbb{R}^2$  be the restriction of  $\gamma$  to the line through  $w$ , such that  $\rho'(0) = d_0 \gamma(w) = (1, 1)$ . Hence both components of  $\rho$  have non-zero derivative at 0. Using elementary calculus, one can deduce that there is a small  $s \in \mathbb{R}$  such that both components of  $\rho$  are non-zero, which is a contradiction since  $\rho(s)$  is contained in  $X$ .*

It suffices to check the  $L$ -type condition for a generating family.

**Lemma 2.34.** *Suppose the diffeology on  $X$  is generated by a family  $\mathcal{F}$ . Then  $X$  is of  $L$ -type if for any two elements  $\alpha, \beta \in \mathcal{F}$  with  $\alpha(u) = \beta(v) = x$ , there are open neighborhoods  $U, V$  of  $u$  and  $v$  and a plot  $\gamma$  such that  $\alpha|_U = \gamma \circ h_1$  and  $\beta|_V = \gamma \circ h_2$  for smooth maps  $h_1$  and  $h_2$ .*

*Proof.* Let  $\alpha, \beta \in \mathcal{P}_x$  where  $\mathcal{P} = \langle \mathcal{F} \rangle$ . Then by Lemma 2.16 we know that there are neighborhoods  $U, V$  of 0 in  $U_\alpha$  and  $U_\beta$  respectively, on which  $\alpha$  and  $\beta$  are either constant or of the form  $f \circ h$  for  $f \in \mathcal{F}$  and smooth  $h$ . We distinguish two cases. First, assume that at least one of the restrictions is constant; without loss of generality let  $\alpha|_U$  be constant. It is then constant  $x$ , since  $\alpha$  is centered at  $x$ . We then let  $\gamma = \beta|_V$ . Let  $h_1 : U \rightarrow V$  be the constant map with value 0, and let  $h_2 = \text{id} : V \rightarrow V$ . Then we have  $\beta|_V = \gamma \circ h_2$  by definition, and  $\alpha|_U = \gamma \circ h_1$  because the right hand side is constant  $\beta(0) = x$ .

The second case is  $\alpha|_U = f_\alpha \circ h_\alpha$  and  $\beta|_V = f_\beta \circ h_\beta$ . Now we use that the condition for  $L$ -type diffeologies is satisfied by  $\mathcal{F}$ , hence there are open neighborhoods  $U_1, V_1$  of 0 in  $U$  and  $V$  respectively, such that  $f_\alpha|_{U_1} = \gamma \circ k_\alpha$  and  $f_\beta|_{V_1} = \gamma \circ k_\beta$  for a plot  $\gamma \in \mathcal{F}$  and smooth maps  $k_\alpha, k_\beta$ . If we let  $U_2 = k_\alpha^{-1}(U_1)$  and  $V_2 = k_\beta^{-1}(V_1)$  then

$$\alpha|_{U_2} = f_\alpha \circ h_\alpha|_{U_2} = \gamma \circ (h_\alpha \circ k_\alpha|_{U_2})$$

and similarly

$$\beta|_{V_2} = f_\beta \circ h_\beta|_{V_2} = \gamma \circ (h_\beta \circ k_\beta|_{V_2})$$

which concludes the proof.  $\square$

**Corollary 2.35.** *Let  $X$  be a diffeological space of  $L$ -type and  $\sim$  any equivalence relation on  $X$ . Then  $X/\sim$  is of  $L$ -type.*

*Proof.* The quotient diffeology is generated by  $\mathcal{F} = \{\pi \circ \alpha \mid \alpha \in \mathcal{P}(X)\}$ , and if  $\alpha, \beta$  factor through  $\gamma$ , it is clear that  $\pi \circ \alpha$  and  $\pi \circ \beta$  factor through  $\pi \circ \gamma$ .  $\square$

In the two following lemmas we show that manifolds and diffeological groups are of  $L$ -type.

**Lemma 2.36.** *Manifolds, when equipped with the standard diffeology, are of  $L$ -type.*



*Proof.* We use Lemma 2.34 with  $\mathcal{F} = \{\varphi^{-1} \mid \varphi \in \mathcal{A}\}$  for some atlas  $\mathcal{A}$  as a generating family. Suppose  $\alpha = \varphi^{-1}$  and  $\beta = \psi^{-1}$  for two charts  $\varphi, \psi$  at  $x$ . So  $\alpha(u) = x = \beta(v)$  for  $u \in U_\alpha$  and  $v \in U_\beta$  respectively. The intersection  $U$  of the domains of  $\varphi$  and  $\psi$  is then an open neighborhood of  $x$  in  $M$ . Let  $U_1 = \varphi(U) \subset U_\alpha, U_2 = \psi(U) \subset U_\beta$ . Further, let  $h_1 : U_\alpha \rightarrow U_1$  be the restriction and  $h_2 = \psi \circ \varphi^{-1}$ . Then  $\alpha|_{U_1} = \alpha \circ h_1$  and  $\beta|_{U_2} = \alpha \circ h_2$ .  $\square$

**Lemma 2.37.** *Diffeological groups are of L-type.*

*Proof.* Given a diffeological group  $(G, \mathcal{P})$ , let  $\alpha, \beta \in \mathcal{P}_g$  be plots centered at  $g$ . If  $\lambda_{g^{-1}}$  denotes left multiplication by  $g^{-1}$ , then  $\lambda_{g^{-1}} : G \rightarrow G$  is a smooth map. Therefore  $\alpha' = \lambda_{g^{-1}} \circ \alpha$  is a plot centered at the identity. Similarly, we define  $\beta' = \lambda_{g^{-1}} \circ \beta$ . Now we use smoothness of multiplication to see that

$$\gamma' : U_\alpha \times U_\beta \rightarrow G, \quad (u, v) \mapsto \alpha'(u)\beta'(v)$$

is a plot centered at the identity. We let  $\gamma(u, v) = g\gamma'(u, v)$  and claim that both  $\alpha$  and  $\beta$  factor through  $\gamma$ . To see this, let

$$f : U_\alpha \rightarrow U_\alpha \times U_\beta, \quad u \mapsto (u, 0)$$

and

$$g : U_\beta \rightarrow U_\alpha \times U_\beta, \quad v \mapsto (0, v).$$

Then  $(\gamma \circ f)(u) = g\gamma'(u, 0) = g\alpha'(u)\beta'(0) = \alpha(u)$  and similarly  $(\gamma \circ g)(v) = \beta(v)$ .  $\square$

**Corollary 2.38.** *Homogeneous spaces  $G/H$ , where  $G$  is a diffeological group and  $H$  any subgroup, are of L-type.*

*Proof.* This follows immediately since diffeological groups are of L-type and this property is preserved under forming quotients.  $\square$

As examples of diffeological groups which are not Lie groups, we now discuss the groups of mappings  $\text{Diff}(M)$  and  $C^\infty(M, G)$ .

**Example 2.39.** *Let  $\text{Diff}(X) \subset \text{Hom}_{\mathcal{D}}(X, X)$  carry the subset diffeology. Composition  $\circ : \text{Hom}_{\mathcal{D}}(X, X) \times \text{Hom}_{\mathcal{D}}(X, X) \rightarrow \text{Hom}_{\mathcal{D}}(X, X)$  is smooth by Corollary 1.34 since  $\mathcal{D}$  is cartesian closed. Since restriction and corestriction of a smooth map are smooth, it follows that the group multiplication in  $\text{Diff}(X)$  is smooth.*

*Now suppose that  $M$  is a smooth manifold modeled on a Banach space, and equip  $M$  with the manifold diffeology. Let  $U \subset \mathbb{R}^n$  be open. Glöckner and Neeb show the following in their upcoming book [GN08]: If  $f : \rightarrow \text{Diff}(M)$  is a function such that  $\tilde{f} : U \times M \rightarrow M$  is smooth, then the function  $(u, m) \mapsto f^{-1}(u)(m)$  is also smooth. This implies that inversion  $i : \text{Diff}(M) \rightarrow \text{Diff}(M)$  is a morphism in  $\mathcal{D}$ , hence  $\text{Diff}(M)$  is a diffeological group.*

**Lemma 2.40.** *If  $G$  is a diffeological group and  $X$  any diffeological space, then  $\text{Hom}_{\mathcal{D}}(X, G)$  is a diffeological group under pointwise multiplication and inversion.*

*Proof.* Let  $H = \text{Hom}_{\mathcal{D}}(X, G)$ . We have to show that multiplication and inversion are smooth. Let  $(\alpha, \beta)$  be a plot for  $H \times H$ . After composing with the multiplication map, we get the map  $\gamma : U \rightarrow H$  given by  $\gamma(u)(x) = \alpha(u)(x)\beta(u)(x)$ . In other words,  $\tilde{\gamma} = \text{mult} \circ (\tilde{\alpha}, \tilde{\beta})$  where  $\text{mult}$  is multiplication in  $G$ . Now by hypothesis,  $\text{mult}$ ,  $\tilde{\alpha}$  and  $\tilde{\beta}$  are smooth, hence  $\gamma$  is a plot for  $H$ . Now let  $\alpha$  be a plot for  $H$ , and compose with the inversion. Then we get a plot  $\delta$  given by  $\delta(u)(x) = \alpha(u)(x)^{-1}$ . Thus  $\tilde{\delta} = \text{inv} \circ \tilde{\alpha}$ , and we see that  $\delta$  is a plot for  $H$ .  $\square$

Consequently, for all manifolds  $M$  and Lie groups  $G$ , the group  $C^\infty(M, G)$  is a diffeological group.

## 2.3 $M$ -spaces

We now turn to a class of categories which was investigated by Frölicher in [Frö86].

They are special cases of  $\mathcal{Q}$ -categories, where  $\mathcal{Q}$  consists of a single set  $A$ .

**Definition 2.41.** *Given two sets  $A, B$  and a subset  $M \subset \text{Map}(A, B)$  we define an  $M$ -structure on the set  $X$  to be a pair  $(C, F)$  of sets  $C \subset \text{Map}(A, X)$  and  $F \subset \text{Map}(X, B)$  which satisfies*

$$F = \{f : X \rightarrow B \mid \forall c \in C : c \circ f \in M\}$$

and

$$C = \{c : A \rightarrow X \mid \forall f \in F : c \circ f \in M\}.$$

If  $(C, F)$  is an  $M$ -structure on  $X$ , we call the triple  $(X, C, F)$  an  $M$ -space. The elements of  $C$  are the curves into  $X$  and the elements of  $F$  the functions on  $X$ .

Given two  $M$ -spaces  $(X, C_X, F_X)$  and  $(Y, C_Y, F_Y)$ , we say that a set map  $\varphi : X \rightarrow Y$  is a morphism of  $M$ -spaces if one has  $\varphi \circ c \in C_Y$  for all  $c \in C_X$ . We get a category  $\mathcal{K}_M$  of  $M$ -spaces and  $M$ -morphisms. Let us denote the collection of  $M$ -morphisms from  $X$  to  $Y$  by  $\text{Hom}_M(X, Y)$ .

Note that, if  $(C, F)$  is an  $M$ -structure, then  $C$  and  $F$  determine one another uniquely. As a consequence, one can use functions rather than curves to check whether a set map is a morphism.

**Lemma 2.42.** *Given two  $M$ -spaces  $(X, C_X, F_X)$  and  $(Y, C_Y, F_Y)$ , for a set map  $\varphi : X \rightarrow Y$  the following are equivalent:*

*i)  $\varphi$  is an  $M$ -morphism.*

*ii) For all pairs  $(f, c) \in F_Y \times C_X$ , the map  $f \circ \varphi \circ c$  is in  $M$ .*

iii) For all functions  $f \in F_Y$ , the map  $f \circ \varphi$  is in  $F_X$ .

*Proof.* The implications  $i) \Rightarrow ii) \Rightarrow iii) \Rightarrow i)$  all follow directly from the definition of an  $M$ -structure.  $\square$

Examples

**Example 2.43.** Suppose that  $X = \{\text{pt}\}$  is a one-point set. Note that we have not excluded the case of a structure  $(X, C, F)$  with  $C = \emptyset$ . Therefore, there are two cases:  $C$  is empty or  $C$  contains the constant map  $c : A \rightarrow X$ . In the first case,  $F$  consists of all maps  $f : X \rightarrow B$ , whereas in the second case  $F$  contains only maps  $f : X \rightarrow B$  such that the constant map  $f \circ c$  is in  $M$ . The first structure is the finest structure on  $X$ , the second one is the coarsest structure on  $X$ , making  $X$  the final object in  $\mathcal{K}_M$ .

Note that if  $M$  contains all constant maps from  $A$  to  $B$ , then  $C$  cannot be empty, since it contains at least all constant maps. This will be the case in all of the following examples.

**Definition 2.44.** A bornology on a set  $X$  is a collection of subsets (the bounded subsets) which contains all singletons, and is closed under taking subsets and finite unions. Examples are metric spaces  $(X, d)$ , where a subset  $A \subset X$  is bounded if  $\sup_{x, y \in A} d(x, y) < \infty$ , and topological vector spaces  $E$ , where we say that  $A \subset E$  is bounded if for every neighborhood  $U$  of  $0$ , there is a  $\lambda > 0$  such that  $\lambda A \subset U$ .

**Example 2.45.** Let  $A = \mathbb{N}$ ,  $B = \mathbb{R}$  and  $M = l^\infty$  be the set of all bounded sequences of real numbers. Every set  $X$  with bornology has a canonical  $l^\infty$ -structure, where  $C$  consists of all sequences in  $X$  whose image is a bounded subset of  $X$ . This yields a natural  $\mathcal{K}_{l^\infty}$ -structure on metric spaces and on topological vector spaces.

**Example 2.46.** One can let  $M$  be the continuous functions  $C(\mathbb{R}, \mathbb{R})$ , or holomorphic functions  $\mathbb{C} \rightarrow \mathbb{C}$ . One can also take  $M$  to be  $k$ -times differentiable functions

whose derivatives of order  $k$  are Lipschitz continuous. This might be useful since the general version of Boman's theorem (Theorem 1.47, [Bom67]) is stated for such classes of functions.

Here we define the category of Frölicher spaces, which will be discussed in more detail in the following sections.

**Definition 2.47.** *If  $M = C^\infty(\mathbb{R}, \mathbb{R})$ , the resulting category  $\mathcal{K}_M$  is called the category of Frölicher spaces. We will denote this category by  $\mathcal{F}$ . Morphisms are called Frölicher maps or, if there is no danger of confusion, smooth maps, and we write  $\text{Hom}_{\mathcal{F}}(X, Y)$  for the smooth maps from  $X$  to  $Y$ .*

For  $M = l^\infty$  and  $C^\infty(\mathbb{R}, \mathbb{R})$  we will show below that the corresponding category  $\mathcal{K}_M$  is cartesian closed.

### 2.3.1 Generating Families

As for diffeological spaces, one can generate  $M$ -structures using arbitrary sets of curves or functions. To check whether a set map is a morphism, it suffices to use generating families.

**Definition 2.48.** *If  $X$  is a set and  $\tilde{F} \subset \text{Map}(X, B)$ , let*

$$C = \left\{ c : A \rightarrow X \mid \forall f \in \tilde{F} : c \circ f \in M \right\}.$$

*Then  $C$  is the set of curves of a unique  $M$ -structure  $(C, F)$  on  $X$ , the  $M$ -structure generated by  $\tilde{F}$ . Note that in general  $\tilde{F} \subset F$  is a proper inclusion. Similarly one can use any set  $\tilde{C} \subset \text{Map}(A, X)$  to define the  $M$ -structure generated by  $\tilde{C}$ .*

**Lemma 2.49.** *Let  $(X, F_X, C_X)$  and  $(Y, F_Y, C_Y)$  be  $M$ -spaces. Suppose that  $(F_X, C_X)$  is generated by  $\tilde{C} \subset C_X$  and that  $(F_Y, C_Y)$  is generated by  $\tilde{F} \subset F_Y$ . Let  $\varphi : X \rightarrow Y$  be a set map. The following are equivalent.*

*i)  $\varphi$  is an  $M$ -morphism*

ii)  $\varphi \circ c \in C_Y$  for all  $c \in \tilde{C}$

iii)  $f \circ \varphi \in F_X$  for all  $f \in \tilde{F}$ .

*Proof.* Clearly i) implies ii) and iii). Suppose ii) holds. Then  $f \circ \varphi \circ c \in M$  for all  $c \in \tilde{C}$  and  $f \in F_Y$ . Hence  $f \circ \varphi \in F_X$  for all  $f \in F_Y$ , so  $\varphi$  is an  $M$ -morphism. Similarly if iii) holds, then  $f \circ \varphi \circ c \in M$  for all  $f \in \tilde{F}$  and  $c \in C_X$ , hence  $\varphi \circ c \in C_Y$  for all  $c \in C_X$ , so again  $\varphi$  is an  $M$ -morphism.  $\square$

**Lemma 2.50.** *If  $A$  and  $B$  are sets, and  $M \subset \text{Map}(A, B)$ , then  $A$  and  $B$  carry  $M$ -structures of the form  $(C_A, M)$  and  $(M, F_B)$ , respectively.*

*Proof.* The set  $M$  consists of functions from  $A$  to  $B$  and therefore generates an  $M$ -structure  $(C_A, F_A)$ . We have  $M \subset F_A$ , and we need to show that  $F_A \subset M$ . Note that  $\text{id}_A \in C_A$ , hence if  $f \in F_A$ , then  $f \circ \text{id} = f \in M$ . This proves  $F_A = M$ . Similarly,  $M$  generates an  $M$ -structure  $(C_B, F_B)$  on  $B$  and one sees easily that  $C_B = M$ .  $\square$

**Corollary 2.51.** *Let  $A$  and  $B$  carry the  $M$ -structures with functions  $F_A = M$  and curves  $C_B = M$ , respectively. Then for every  $M$ -space  $(X, C_X, F_X)$  we have  $C_X = \text{Hom}_M(A, X)$  and  $F_X = \text{Hom}_M(X, B)$ .*

*Proof.* By definition of  $M$ -structures,  $c \in C_X$  if and only if  $f \circ c \in M$  for all  $f \in F_X$  which by Lemma 2.42 is equivalent to  $c \in \text{Hom}_M(A, X)$ . A similar argument shows that  $f \in F_X$  if and only if  $f \in \text{Hom}_M(X, B)$ .  $\square$

**Remark 2.52.** *The ‘finer’-relation for  $M$ -structures is given by  $(C_1, F_1) \preceq (C_2, F_2)$  if and only if  $C_2 \subset C_1$  or equivalently if and only if  $F_1 \subset F_2$ . Consider a family  $(C_i, F_i)$  of  $M$ -structures on a fixed set  $X$ . As in the case of diffeologies (see Remark 2.14), the family has an infimum and a supremum. The infimum is the  $M$ -structure*

generated by  $\cup_i C_i$ , and has functions  $F_{\text{inf}} = \cap_i F_i$ . Similarly, the supremum has curves  $C_{\text{sup}} = \cap_i C_i$ . Question: Is the lattice of  $M$ -structures on  $X$  modular or distributive? The lattice of topologies on a fixed set  $X$  is not modular if  $|X| \geq 3$  (see [Ste66]).

### 2.3.2 Initial and Final Structures

Let  $X$  be any set and  $\{(X_i, C_i, F_i) \mid i \in I\}$  a family of  $M$ -spaces. Suppose that for each  $i \in I$  there is a set map  $f_i : X \rightarrow X_i$ . There are two natural ways to generate an  $M$ -structure on  $X$ , using  $\tilde{F} = \{h \circ f_i \mid i \in I, h \in F_i\}$  or  $\tilde{C} = \{c : A \rightarrow X \mid (\forall i \in I) : f_i \circ c \in C_i\}$ . By definition, the structure generated by  $\tilde{F}$  has curves

$$\begin{aligned} C &= \{c : A \rightarrow X \mid (\forall i \in I)(\forall h \in F_i) : h \circ f_i \circ c \in M\} \\ &= \{c : A \rightarrow X \mid (\forall i \in I) : f_i \circ c \in C_i\} \\ &= \tilde{C}, \end{aligned}$$

which shows that  $\tilde{F}$  and  $\tilde{C}$  generate the same structure, and that the curves of that structure are given by  $\tilde{C}$ .

Similarly, if we are given a family  $\{(X_i, C_i, F_i) \mid i \in I\}$  of  $M$ -spaces and set maps  $g_i : X_i \rightarrow X$ , let  $\tilde{F} = \{f : X \rightarrow B \mid (\forall i \in I) : f \circ g_i \in F_i\}$  and  $\tilde{C} = \{g_i \circ c \mid i \in I, c \in C_i\}$ . It turns out that both generate the same  $M$ -structure on  $X$  with  $F = \tilde{F}$ .

It is clear that these structures on  $X$  are the coarsest and finest with respect to which all  $f_i$  and all  $g_i$  are morphisms, respectively. This discussion proves the following result:

**Lemma 2.53.** *Suppose that  $\{(X_i, C_i, F_i) \mid i \in I\}$  is a family of  $M$ -spaces and there are given set maps  $f_i : X \rightarrow X_i$  and  $g_i : X_i \rightarrow X$ . The unique  $M$ -structure on  $X$*

having

$$C = \{c : A \rightarrow X \mid (\forall i \in I) : f_i \circ c \in C_i\}$$

as curves is the initial  $M$ -structure on  $X$  with respect to the maps  $f_i$ , and the unique  $M$ -structure on  $X$  having

$$F = \{f : X \rightarrow B \mid (\forall i \in I) : f \circ f_i \in F_i\}$$

as functions is the final  $M$ -structure on  $X$  with respect to the maps  $f_i$ .

*Proof.* See discussion preceding this lemma. □

**Lemma 2.54.** *Let  $X$  carry the final  $M$ -structure with respect to maps  $\{f_i : X_i \rightarrow X \mid i \in I\}$  and let  $Y$  carry the initial  $M$ -structure with respect to maps  $\{g_j : Y \rightarrow Y_j \mid j \in J\}$ . A set map  $\varphi : X \rightarrow Y$  is an  $M$ -morphism if either of the following is true:*

i)  $\varphi \circ f_i$  is an  $M$ -morphism for all  $i \in I$ .

ii)  $g_j \circ \varphi$  is an  $M$ -morphism for all  $j \in J$ .

*Proof.* Let us assume that i) holds. Let  $f \in F_Y$ . Then by assumption,  $f \circ \varphi \circ f_i$  is a function on  $X_i$  for all  $i \in I$ . Thus by definition of the final  $M$ -structure,  $f \circ \varphi$  is a function on  $X$ . This holds for all  $f \in F_Y$ , hence  $\varphi$  is a  $M$ -morphism. Similarly, assume that ii) holds and let  $c \in C_X$  be a curve. Then  $g_j \circ \varphi \circ c$  is a curve in  $Y_j$  for all  $j \in J$ , and by definition of initial structures, this implies that  $\varphi \circ c$  is a curve in  $Y$ . This is true for all curves  $c$ , thus  $\varphi$  is a  $M$ -morphism. □

Now it follows, as in the case of diffeological spaces, that all limits and colimits in  $\mathcal{K}_M$  exist.

**Theorem 2.55.** *In the category  $\mathcal{K}_M$  of  $M$ -spaces, all limits and colimits exist and are created by the grounding functor  $F : \mathcal{K}_M \rightarrow \mathbf{Set}$ .*



*Proof.* The proof is analogous to the proof of the corresponding theorem for diffeological spaces. Given a diagram  $J : \mathcal{G} \rightarrow \mathcal{K}_M$ , we equip the limit  $X$  of  $F \circ J$  in **Set** with the initial  $M$ -structure with respect to the maps  $a_i : X \rightarrow FX_i$  associated with the limit. Now it remains to show that  $X$  satisfies the universal property; but if  $X'$  is an  $M$ -space, and the  $b_i : X' \rightarrow X_i$  are  $M$ -morphisms such that  $b_j = f \circ b_i$  for every edge  $f : X_i \rightarrow X_j$  in the diagram, the universal property of  $X$  as a limit in **Set** gives us a set map  $\varphi : X' \rightarrow X$  with  $a_i \circ \varphi = b_i$ . The  $b_i$ , hence the  $a_i \circ \varphi$  are morphisms, and thus by Lemma 2.54,  $\varphi$  is a morphism. This proves the universal property. A similar argument shows that all colimits exist and are created by  $F$ .  $\square$

We conclude the subsection with some examples.

**Example 2.56** (Subspaces). *Let  $(X, C_X, F_X)$  be an  $M$ -space and  $Y \subset X$  a subset. The initial structure on  $Y$  with respect to the inclusion  $\iota : Y \rightarrow X$  can be generated by*

$$\tilde{F} = \{f \circ \iota \mid f \in F_X\} = \{f|_Y \mid f \in F_X\},$$

*and we have seen above that the curves for the resulting  $M$ -structure  $(C_Y, F_Y)$  on  $Y$  are given by  $C = \{c : A \rightarrow Y \mid \iota \circ c \in C_X\}$ . It is easily seen that the restriction to  $Y$  of a morphism on  $X$  is again a morphism. Also, if  $f : Z \rightarrow X$  is a morphism and  $Y = f(Z)$ , then the corestriction  $f : Z \rightarrow Y$  is a morphism if  $Y$  carries the subspace structure.*

*Let us show that in general  $\tilde{F} \neq F_Y$ , or in other words, not every function on  $Y$  is the restriction of a function on  $X$ . Let  $M = C(\mathbb{R}, \mathbb{R})$  and  $X = \mathbb{R}$ , with  $M$ -structure given by  $C_X = F_X = C(\mathbb{R}, \mathbb{R})$ . Now consider  $Y = (0, 1) \subset \mathbb{R}$  with the subspace structure. Clearly  $f(x) = 1/x$  is in  $F_Y$ , since for every continuous*

$c : \mathbb{R} \rightarrow (0, 1)$ , the composition  $f \circ c$  is continuous. However,  $f \notin \tilde{F}$  since  $f$  is not the restriction of a continuous function on  $\mathbb{R}$  to  $(0, 1)$ .

**Example 2.57** (Quotient Spaces). *If  $(X, C, F)$  is an  $M$ -space and  $\sim$  is an equivalence relation on  $X$ , we can form the quotient  $\tilde{X} = X / \sim$ . The quotient  $M$ -structure on  $\tilde{X}$  is the final  $M$ -structure with respect to the quotient map  $\pi : X \rightarrow \tilde{X}$ . A function  $f : \tilde{X} \rightarrow B$  is in  $F_{\tilde{X}}$  if and only if  $f \circ \pi \in F_X$ . More generally, a map  $\varphi : \tilde{X} \rightarrow Y$  is an  $M$ -morphism if and only if  $\varphi \circ \pi$  is an  $M$ -morphism. An example of a quotient construction is given by the irrational torus, see Example 2.82 below.*

**Example 2.58** (Direct Product). *Let  $I = \mathcal{G}$  be a discrete graph and  $J : I \rightarrow \mathcal{K}_M$  any functor. Then the underlying set of the limit is the cartesian product  $X = \prod_{i \in I} X_i$  together with the projections  $\pi_i : X \rightarrow X_i$ , and a map  $c : A \rightarrow X$  is a curve if and only if its components  $\pi_i \circ c$  are curves into the  $X_i$ . More generally it is true that a map  $\varphi : Y \rightarrow X$  is an  $M$ -morphism if and only if its components  $\pi_i \circ \varphi$  are  $M$ -morphisms.*

**Example 2.59** (Initial and Terminal Object). *The colimit of the empty graph, which is the initial object of  $\mathcal{K}_M$ , is given by the empty set with its unique  $M$ -structure. The limit of the empty graph, which is the terminal object of  $\mathcal{K}_M$ , is given by a one-element set  $X = \{\text{pt}\}$  with the indiscrete  $M$ -structure.*

### 2.3.3 Hom-objects and Cartesian Closedness

In [Frö86], Frölicher gave a necessary and sufficient condition on  $M$  for  $\mathcal{K}_M$  to be cartesian closed. We will prove his result using Lemma 1.32, so first let us show that  $\mathcal{K}_M$  has Hom-objects.

**Definition 2.60.** As usual, for a map  $c : A \rightarrow M$  we let  $\tilde{c}$  denote the map from  $A \times A$  to  $B$  given by  $\tilde{c}(x, y) = c(x)(y)$ . Let

$$\tilde{C} = \{c : A \rightarrow M \mid (\forall c_1, c_2 \in C_A) : \tilde{c} \circ (c_1, c_2) \in M\}.$$

Let  $F_{A \times A}$  denote the functions for the product structure on  $A \times A$ . Then  $c \in \tilde{C}$  if and only if  $\tilde{c} \in F_{A \times A}$ . The set  $\tilde{C}$  generates an  $M$ -structure on  $M$ . Let us denote this  $M$ -structure by  $(C_M, F_M)$ .

Now we can define an  $M$ -structure on Hom-sets.

**Definition 2.61.** Let  $(X, C_X, F_X)$  and  $(Y, C_Y, F_Y)$  be  $M$ -spaces. For each pair  $(f, c) \in F_Y \times C_X$  we define a map

$$(f, c) : \text{Hom}_M(X, Y) \rightarrow M$$

by  $\varphi \mapsto f \circ \varphi \circ c$ . Now we equip the set  $\text{Hom}_M(X, Y)$  with the initial structure with respect to all such maps  $(f, c)$ .

**Lemma 2.62.** The concrete category  $\mathcal{K}_M$  has Hom-objects.

*Proof.* Let  $X, Y$  and  $Z$  be  $M$ -spaces, and let  $\varphi : Y \rightarrow Z$  be a morphism. We need to show that  $\varphi_* : \text{Hom}(X, Y) \rightarrow \text{Hom}(X, Z)$  is a morphism. We will do that by composing with  $(f, c) \in F_Z \times C_X$ . We get

$$((f, c) \circ \varphi_*)(g) = (f, c)(\varphi \circ g) = f \circ \varphi \circ g \circ c = (f \circ \varphi, c)(g),$$

which is a morphism  $\text{Hom}(X, Y) \rightarrow M$  since  $f \circ \varphi \in F_Y$ . By Lemma 2.54 this suffices to conclude that  $\varphi_*$  is a morphism.

Now let  $\psi : Z \rightarrow X$  be an  $M$ -morphism. We show that  $\psi^* : \text{Hom}(X, Y) \rightarrow \text{Hom}(Z, Y)$  is a morphism by letting  $(f, c) \in F_Y \times C_Z$  and composing:

$$((f, c) \circ \psi^*)(g) = (f, c)(g \circ \psi) = (f \circ g \circ \psi \circ c) = (f, \psi \circ c)(g),$$

which implies by Lemma 2.54 that  $\psi^*$  is an  $M$ -morphism.  $\square$

Now we give a proof of Frölicher's result on cartesian closedness of  $\mathcal{K}_M$ .

**Theorem 2.63** ([Frö86]). *Given sets  $A, B$  and  $M \subset \text{Map}(A, B)$ , assume  $M$  contains all constant maps. Let  $M, \tilde{C}$  and  $C_M$  be as in Definition 2.60. The category  $\mathcal{K}_M$  is cartesian closed if and only if  $C_M = \tilde{C}$ . In other words,  $\mathcal{K}_M$  is cartesian closed if and only if  $\text{Hom}_M(A, M) \cong \text{Hom}_M(A \times A, B)$ .*

*Proof.* First assume that  $\mathcal{K}_M$  is cartesian closed. Then  $c \in \tilde{C}$  if and only if  $\tilde{c} : A \times A \rightarrow B$  is a morphism. By cartesian closedness this is the case if and only if  $c : A \rightarrow M$  is a morphism. Now  $\text{Hom}_M(A, M) = C_M$  by Corollary 2.51, hence  $C_M = \tilde{C}$ .

Conversely assume that  $C_M = \tilde{C}$ . By Lemma 1.32, we need to show that  $eval : \text{Hom}_M(X, Y) \times X \rightarrow Y$  is a morphism. Take a function  $f$  on  $Y$  and a curve  $c = (c_1, c_2)$  for  $\text{Hom}_M(X, Y) \times X$ . Then  $f \circ eval \circ c = f \circ \tilde{c}_1 \circ (\text{id}_X, c_2)$ . It remains to show that  $\tilde{c}_1 : A \times X \rightarrow Y$  is a morphism. We will show that, more generally,

$$\varphi \in \text{Hom}_M(X, \text{Hom}_M(Y, Z)) \Rightarrow \tilde{\varphi} \in \text{Hom}_M(X \times Y, Z). \quad (2.1)$$

To see this, let  $\varphi \in \text{Hom}_M(X, \text{Hom}_M(Y, Z))$ , let  $c$  be a curve for  $X$  and let  $(f, d) \in F_Z \times C_Y$ . Then by definition of the  $M$ -structure on Hom-sets, the composition  $(f, d) \circ \varphi \circ c$  is an element of  $C_M$ . Now by hypothesis,  $C_M = \tilde{C}$ , hence

$$(x, y) \mapsto f(\varphi(c(x))(d(y))) = f(\tilde{\varphi}(c(x), d(y)))$$

is smooth for all choices of  $f, c$  and  $d$ . This implies that  $\tilde{\varphi}$  is a morphism.

Finally we need to prove the universal property for  $eval$ . That is, for each  $f : X \times Y \rightarrow Z$  we need to show that the corresponding map  $g : X \rightarrow \text{Map}(Y, Z)$  takes values in  $\text{Hom}_M(Y, Z)$ , and is an  $M$ -morphism if  $\text{Hom}_M(Y, Z)$  carries its Hom-object structure. First note that, since  $M$  contains all constant functions by assumption, the map  $y \mapsto (x, y)$  is a morphism from  $Y$  to  $X \times Y$ , and therefore

$y \mapsto f(x, y)$  is in  $\text{Hom}_M(Y, Z)$ . Now to show that  $g$  is a morphism, let  $(f, c) \in F_Z \times C_Y$  and  $d \in C_X$  and compose to get a map  $\rho = (f, c) \circ g \circ d : A \rightarrow M$ . One computes  $\tilde{\rho}(a, b) = f(\tilde{g}(d(a), c(b)))$ , which is a morphism because  $\tilde{g} = f$  is a morphism. Hence  $\rho$ , and therefore also  $g$ , is a morphism.  $\square$

**Example 2.64.** *If  $M = l^\infty$ , then the category  $\mathcal{K}_M$  is cartesian closed. To see this, note that  $C_A = \text{Map}(\mathbb{N}, \mathbb{N})$ . Then  $\tilde{C}$  consists of all  $c : \mathbb{N} \rightarrow l^\infty$  such that  $\tilde{c} : \mathbb{N}^2 \rightarrow \mathbb{R}$  is bounded, or in other words, bounded maps into  $(l^\infty, \|\cdot\|_\infty)$ . The set  $C_M$  consists of all bounded maps  $\mathbb{N} \rightarrow l^\infty$  and thus agrees with  $\tilde{C}$ .*

The next lemma describes  $\text{Hom}_M(X, Y)$  as an object of  $\mathcal{K}_M$  if one of  $X, Y$  is discrete.

**Lemma 2.65.** *Let  $M$  be such that  $\mathcal{K}_M$  is cartesian closed, and let  $J$  be a discrete object in  $\mathcal{K}_M$ . Then for every other object  $X$ , the object  $\text{Hom}_M(J, X)$  is isomorphic to the direct product  $\prod_{j \in J} X$ . On the other hand,  $\text{Hom}_M(X, J)$  is isomorphic to  $J$ .*

*Proof.* Since evaluation is a morphism, each  $j \in J$  yields a morphism  $a_j : \text{Hom}_M(J, X) \rightarrow X$ ,  $\varphi \mapsto \varphi(j)$ . To show that  $\text{Hom}_M(J, X)$  is the product of copies of  $X$  indexed by  $J$ , it remains to prove the universal property. So let  $Y$  be an object together with a morphism  $b_j : Y \rightarrow X$  for each  $j \in J$ . Then we define a set map  $\Phi : Y \rightarrow \text{Hom}_M(J, X)$  by  $\Phi(y)(j) = b_j(y)$ . By cartesian closedness, we can use  $\tilde{\Phi}(y, j) = b_j(y)$  to check whether  $\Phi$  is a morphism. Now  $J$  is discrete, so that curves are constant, and each curve  $c : A \rightarrow Y \times J$  is of the form  $a \mapsto (c(a), j)$  for some fixed  $j \in J$ . Therefore,  $\tilde{\Phi} \circ c : a \mapsto b_j(c(a))$ , which is a morphism by assumption. Hence  $\Phi$  is a morphism, and  $\text{Hom}_M(J, X)$  satisfies the required universal property.

Since  $J$  is discrete,  $\text{Hom}_M(X, J)$  contains only constant maps and is, as a set, isomorphic to  $J$ . It remains to show that  $\text{Hom}_M(X, J)$  is discrete as an  $M$ -space.

But if  $c : A \rightarrow \text{Hom}_M(X, J)$  is a curve, by cartesian closedness the map  $\tilde{c} : A \times X \rightarrow J$  has to be a morphism and hence constant. But then  $c$  itself is constant, which completes the proof.  $\square$

## 2.4 Frölicher Spaces

The category  $\mathcal{F}$  of Frölicher spaces is defined to be the category  $\mathcal{K}_M$  for  $M = C^\infty(\mathbb{R}, \mathbb{R})$ . We will see below that there is a full and faithful functor  $\mathbf{Mfd} \rightarrow \mathcal{F}$ , that Frölicher spaces are related to diffeological spaces via an adjunction, and that it is possible to generalize certain concepts from classical differential geometry to the category  $\mathcal{F}$ . First, let us use Theorem 2.63 to show that  $\mathcal{F}$  is cartesian closed.

**Lemma 2.66.** *The category  $\mathcal{F}$  of Frölicher spaces is a cartesian closed concrete category.*

*Proof.* For brevity, let us write  $M = C^\infty(\mathbb{R}, \mathbb{R})$  throughout the proof. We want to use Theorem 2.63, so we have to compute  $\tilde{C}$  as in Definition 2.60 and show that  $\tilde{C}$  is the set of curves of an  $M$ -structure on  $M$ . First note that the set  $C_{\mathbb{R}}$  of curves for the standard  $M$ -structure on  $\mathbb{R}$  is simply  $M$ . Recall that

$$\tilde{C} = \{c : \mathbb{R} \rightarrow M \mid (\forall c_1, c_2 \in C_{\mathbb{R}}) : \tilde{c} \circ (c_1, c_2) \in M\}.$$

Since a map  $(c_1, c_2)$  with smooth  $c_i$  is the same as a smooth map from  $\mathbb{R}$  to  $\mathbb{R}^2$ , we conclude that  $\tilde{C}$  consists of maps  $c : \mathbb{R} \rightarrow M$  for which  $\tilde{c}$  is smooth along smooth curves into  $\mathbb{R}^2$ . By Theorem 1.58, this is equivalent to  $c$  being a smooth map into  $M$ , when  $M$  is considered as a Fréchet space with the topology of uniform convergence of all derivatives (see Example 1.42).

We noted in Remark 1.37 that complete locally convex spaces are Mackey complete, hence  $M$  is Mackey complete and by Theorem 1.57, a function  $c : \mathbb{R} \rightarrow M$

is smooth if and only if its composition with all continuous linear functionals on  $M$  is smooth. But this implies that  $\tilde{C}$  is the set of functions for an  $M$ -structure on  $M$ , namely the structure generated by the set  $\tilde{F} = C^\infty(\mathbb{R}, \mathbb{R})'$  of continuous linear functionals. This is what we had to show.  $\square$

**Definition 2.67.** *If  $M$  is a manifold, then the Frölicher structure on  $M$  generated by the set  $\tilde{C} = C^\infty(\mathbb{R}, M)$  of smooth curves into  $M$  is called the manifold Frölicher structure on  $M$ .*

**Lemma 2.68.** *There is a full and faithful functor  $\mathbf{Mfd} \rightarrow \mathcal{F}$ , which assigns to each manifold its manifold Frölicher structure.*

*Proof.* Let  $M$  and  $N$  be smooth manifolds and  $\varphi : M \rightarrow N$  a smooth map. Clearly, the composition of  $\varphi$  with smooth curves  $c : \mathbb{R} \rightarrow M$  is smooth, so that  $\varphi$  defines a morphism in  $\mathcal{F}$  if  $M$  and  $N$  are equipped with their manifold Frölicher structure. Thus we have a functor  $\mathbf{Mfd} \rightarrow \mathcal{F}$ . Let  $H$  denote this functor. For every pair  $M, N$  of smooth manifolds and every pair of smooth maps  $\varphi, \psi : M \rightarrow N$ , the equality  $H(\varphi) = H(\psi)$  implies  $\varphi = \psi$ . This shows that  $H$  is faithful. To see that  $H$  is full, let  $f \in \text{Hom}_{\mathcal{F}}(M, N)$ . We need to show that  $f$  is smooth, which is a local condition and can be checked in local coordinates. Thus we can assume that  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , and by assumption  $f$  is smooth along smooth curves. It follows from Boman's Theorem 1.47 that  $f$  is smooth.  $\square$

**Corollary 2.69.** *The manifold Frölicher structure on a manifold  $M$  is given by  $(C^\infty(\mathbb{R}, M), C^\infty(M, \mathbb{R}))$ .*

We give an example of a Frölicher space which is the colimit of manifolds. Then we show that the map from Example 1.56 is a morphism in  $\mathcal{F}$ .

**Example 2.70.** Let  $\mathcal{G} = \mathbb{N}$  with directed edges  $n \rightarrow n + 1$ . Let  $J(n) = \mathbb{S}^n$  be the  $n$ -sphere with its manifold Frölicher structure. To each edge  $n \rightarrow n + 1$  associate the embedding of  $\mathbb{S}^n$  into  $\mathbb{S}^{n+1}$  as equator. This yields a diagram  $J$  of shape  $\mathbb{N}$ . The colimit  $\mathbb{S}^{(\infty)}$  of  $J$  has as underlying set the set of finite sequences  $(x_1, \dots, x_k)$  of real numbers which satisfy  $x_1^2 + \dots + x_k^2 = 1$ . A function  $f : \mathbb{S}^{(\infty)} \rightarrow \mathbb{R}$  is smooth if and only if for all inclusions  $i : \mathbb{S}^n \subset \mathbb{S}^{(\infty)}$ , the composition  $f \circ i : \mathbb{S}^n \rightarrow \mathbb{R}$  is smooth in the usual sense.

**Example 2.71.** Let  $V = C_c^\infty(\mathbb{R}, \mathbb{R})$  with the topology described in Example 1.56. Recall that the map

$$f : V \rightarrow V, f(\gamma) = \gamma \circ \gamma - \gamma(0)$$

is not smooth as a map between locally convex spaces. However, if  $V \subset \text{Hom}_{\mathcal{F}}(\mathbb{R}, \mathbb{R})$  carries the subspace Frölicher structure, then  $f$  is easily seen to be a morphism in  $\mathcal{F}$ : If  $c : \mathbb{R} \rightarrow V$  is a curve, then

$$\widetilde{(f \circ c)}(s, t) = \tilde{c}(s, \tilde{c}(s, t)) - \tilde{c}(s, 0)$$

which is smooth because  $\tilde{c}$  is smooth.

The following lemma is a consequence of Boman's Theorem.

**Lemma 2.72.** Given Frölicher spaces  $X_1, \dots, X_n$ , let  $X = \prod_{i=1}^n X_i$ . If the  $c_i : \mathbb{R} \rightarrow X_i$  are smooth curves, let

$$c(t_1, \dots, t_n) := (c_1(t_1), \dots, c_n(t_n)).$$

Then  $c : \mathbb{R}^n \rightarrow X$  is smooth, where  $\mathbb{R}^n$  carries the standard Frölicher structure.

*Proof.* Let  $d : \mathbb{R} \rightarrow \mathbb{R}^n$  be a smooth curve. Then its components  $d_i$  are smooth, and hence the components of  $c \circ d$ , which are given by  $c_i \circ d_i$ , are in  $C_{X_i}$ . This shows that  $c \circ d \in C_X$ . Now let  $f$  be in  $F_X$ . Then by what we just said, the function



$f \circ c : \mathbb{R}^n \rightarrow \mathbb{R}$  is smooth along curves. Hence by Boman's theorem, it is smooth.

This is true for all  $f \in F_X$ , hence  $c$  is smooth.  $\square$

**Remark 2.73.** *Let us note that each Frölicher space carries a canonical topology, the initial topology with respect to the curves  $c : \mathbb{R} \rightarrow X$ . This topology is called  $c^\infty$ -topology in [KM97].*

## 2.4.1 Frölicher Vector Spaces

**Definition 2.74.** *Let  $V$  be a Frölicher space and a vector space over  $\mathbb{R}$ . Equip  $\mathbb{R}$  with the standard Frölicher structure of a smooth manifold, and equip  $\mathbb{R} \times V$  and  $V \times V$  with the product structure. Then we say that  $V$  is a Frölicher vector space if scalar multiplication and vector addition are smooth maps  $\mathbb{R} \times V \rightarrow V$  and  $V \times V \rightarrow V$ , respectively.*

**Lemma 2.75.** *Let  $(V, C_V, F_V)$  be a vector space and a Frölicher space, and let  $(X, C_X, F_X)$  be a Frölicher space. Then  $\text{Hom}_{\mathcal{F}}(X, V)$  is a Frölicher vector space if and only if  $V$  is a Frölicher vector space.*

*Proof.* First assume that  $V$  is a Frölicher vector space. Let  $(c_1, c_2)$  be a curve for  $\text{Hom}_{\mathcal{F}}(X, V) \times \text{Hom}_{\mathcal{F}}(X, V)$ . If  $\oplus$  denotes addition in  $\text{Hom}_{\mathcal{F}}(X, V)$ , let  $d = \oplus \circ (c_1, c_2)$ . Then  $\tilde{d}(s, x) = c_1(s)(x) + c_2(s)(x) = \tilde{c}_1(s, x) + \tilde{c}_2(s, x)$  where  $+$  is addition in  $V$ . Since addition in  $V$  is smooth, we conclude that  $\tilde{d}$  is a curve  $\mathbb{R} \times X \rightarrow V$ , which shows that  $\oplus$  is smooth. Similar arguments show that scalar multiplication is smooth. Now suppose that  $\text{Hom}_{\mathcal{F}}(X, V)$  is a Frölicher vector space, and let  $(c_1, c_2)$  be a curve into  $V \times V$ . Let  $f$  and  $g$  be the curves into  $\text{Hom}_{\mathcal{F}}(X, V)$  which map  $s$  to the constant function  $f(s) : x \mapsto c_1(s)$  and  $g(s) : x \mapsto c_2(s)$ , respectively. By cartesian closedness,  $f$  and  $g$  are smooth. Thus by assumption, so is the sum  $f \oplus g$ . This shows that  $c + d$  is smooth. Scalar multiplication is treated similarly.  $\square$

**Example 2.76.** *The smooth functions  $F = \text{Hom}_{\mathcal{F}}(X, \mathbb{R})$  on a Frölicher space form a Frölicher vector space. If  $V$  is any Frölicher vector space, consider  $X = \mathbb{R}$  with the discrete, standard or indiscrete Frölicher structure, respectively. Then we get Frölicher vector spaces  $\text{Map}(\mathbb{R}, V)$ ,  $C^\infty(\mathbb{R}, V)$  and  $V$ , respectively.*

**Example 2.77.** *If  $V$  is a Frölicher vector space and  $H$  a linear subspace of  $V$ , then the quotient vector space is again a Frölicher vector space. Let us check smoothness of vector addition*

$$\oplus : V/H \times V/H \rightarrow V/H$$

*By definition of product and quotient structure, we need to check smoothness of*

$$\oplus \circ (\pi, \pi) : V \times V \rightarrow V/H,$$

*which is given by  $(v, w) \mapsto v + w + H$ , thus  $\oplus \circ (\pi, \pi) = \pi \circ +$  where  $+$  denotes vector addition in  $V$ . But the latter map is smooth, hence  $\oplus$  is smooth. Smoothness of scalar multiplication is proven similarly.*

## 2.5 An Adjunction from $\mathcal{D}$ to $\mathcal{F}$

In this section we compare diffeological and Frölicher spaces using a pair of functors between the corresponding categories. As an example, we discuss the ‘irrational torus’, which is a quotient  $T_\alpha = \mathbb{R}/(\mathbb{Z} + \alpha\mathbb{Z})$  for irrational  $\alpha \in \mathbb{R}$ , equipped with the quotient diffeology. This space has been studied by Donato, Iglesias-Zemmour and Lachaud in [DI85] and [IL90]. Here we only use it to show that  $\mathcal{D}$  and  $\mathcal{F}$  are not isomorphic.

**Definition 2.78.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be concrete categories with grounding functors  $F$ , so that for each pair  $(X, Y)$  of objects, we can identify  $\text{Hom}(X, Y)$  with a subset of*

$\text{Map}(FX, FY)$ . Given functors  $U : \mathcal{A} \rightarrow \mathcal{B}$  and  $V : \mathcal{B} \rightarrow \mathcal{A}$ , we say that  $(U, V)$  is an adjunction from  $\mathcal{A}$  to  $\mathcal{B}$  if for each pair of objects  $X \in \mathcal{A}$  and  $Y \in \mathcal{B}$ , we have

$$\text{Hom}_{\mathcal{B}}(UX, Y) = \text{Hom}_{\mathcal{A}}(X, VY)$$

as subsets of  $\text{Map}(FX, FY)$ . We say that  $V$  is a right adjoint for  $U$ , and  $U$  is a left adjoint for  $V$ .

**Definition 2.79.** Given a Frölicher space  $(X, C, F)$ , there is a natural diffeology on  $X$ , given by all functions  $\alpha : U_\alpha \rightarrow X$  which are smooth if we equip  $U_\alpha \subset \mathbb{R}^n$  with its standard Frölicher structure. It is clear that a smooth map  $f : X \rightarrow Y$  of Frölicher spaces becomes a smooth map of diffeological spaces if we equip both  $X$  and  $Y$  with the natural diffeology. Thus we get a functor  $\mathbf{D} : \mathcal{F} \rightarrow \mathcal{D}$ .

**Lemma 2.80.** Let  $(X, \mathcal{P}(X))$  be a diffeological space. If  $(C, F)$  is the Frölicher structure on  $X$  generated by  $\text{Hom}_{\mathcal{D}}(\mathbb{R}, X)$ , then  $F = \text{Hom}_{\mathcal{D}}(X, \mathbb{R})$ . Let us denote this Frölicher space by  $\mathbf{F}(X, \mathcal{P}(X))$ . If  $\varphi : (X, \mathcal{P}(X)) \rightarrow (Y, \mathcal{P}(Y))$  is a smooth map between diffeological spaces, then  $\varphi$  is also a morphism between the corresponding Frölicher spaces  $\mathbf{F}(X, \mathcal{P}(X))$  and  $\mathbf{F}(Y, \mathcal{P}(Y))$ .

*Proof.* Let  $(C, F)$  denote the Frölicher structure on  $X$  generated by  $\text{Hom}_{\mathcal{D}}(\mathbb{R}, X)$ . It is clear that  $\text{Hom}_{\mathcal{D}}(X, \mathbb{R}) \subset F$ , and we need to show equality. Let  $f \in F$  and  $\alpha \in \mathcal{P}(X)$ . By Boman's Theorem, in order to show that  $f \circ \alpha$  is smooth it suffices to show that  $f \circ \alpha$  is smooth along smooth curves. Now if  $c : \mathbb{R} \rightarrow U$  is a smooth curve, then  $\alpha \circ c \in \text{Hom}_{\mathcal{D}}(\mathbb{R}, X)$ , hence  $f \circ \alpha \circ c$  is smooth by assumption on  $f$ . This holds true for all smooth curves  $c$ , and therefore  $f \circ \alpha$  is a smooth map. This shows that  $f \in \text{Hom}_{\mathcal{D}}(X, \mathbb{R})$ .

Now let  $(Y, \mathcal{P}(Y))$  be a second diffeological space, and  $\varphi \in \text{Hom}_{\mathcal{D}}(X, Y)$ . Since  $f \circ \varphi \in \text{Hom}_{\mathcal{D}}(X, \mathbb{R})$  for every  $f \in \text{Hom}_{\mathcal{D}}(Y, \mathbb{R})$ , the map  $\varphi$  gives rise to a morphism  $\varphi : \mathbf{F}X \rightarrow \mathbf{F}Y$  of Frölicher spaces.  $\square$

**Definition 2.81.** *Let  $\mathbf{F} : \mathcal{D} \rightarrow \mathcal{F}$  be the functor defined by above lemma.*

**Example 2.82.** *Let  $T = \mathbb{R}/A$  be an irrational torus with the quotient diffeology. For example let  $A = \mathbb{Z} + \mathbb{Z}\sqrt{2}$ , but all we will use is that  $A$  is dense in  $\mathbb{R}$ . We claim that the quotient diffeology is not discrete, so there are maps  $\alpha : U_\alpha \rightarrow T$  which are not plots. However,  $\mathbf{F}T$  is discrete, that is  $C = \text{Map}(T)$  and  $F$  contains only the constant functions. First, pick a number not in  $\mathbb{Z} + \mathbb{Z}\sqrt{2}$ , say  $1/2$ . Then define  $\alpha$  by  $\alpha(0) = 0$  and  $\alpha(x) = 1/2$  for  $x \neq 0$ . If  $\alpha$  were a plot, then locally at 0 we could write  $\alpha = \pi \circ h$  for some smooth map  $h$ . Then  $h$  can not be constant, and by density of  $A$  we can find  $s \neq 0$  close to 0 in  $\mathbb{R}$  with  $h(s) \in A$ . For this  $s$  we have  $\alpha(s) = \pi(h(s)) = 0$ , which is a contradiction. Now we go on to show that  $\mathbf{F}T$  is discrete by proving that the only smooth functions  $\mathbf{F}T \rightarrow \mathbb{R}$  are the constant ones. Otherwise we could find  $f \in F$  and  $\bar{x}, \bar{y} \in T$  with  $f(\bar{x}) \neq f(\bar{y})$ . Let  $c$  be a non-constant one-plot with  $c(0) = \bar{x}$ . Then  $c$  is also a curve, and hence  $f \circ c$  is smooth. Now write  $c = \pi \circ h$  around 0, where  $h(0) = x$  with  $x$  being a representative of  $\bar{x}$ . By density of  $A$ , we can pick a representative  $y$  for  $\bar{y}$  arbitrarily close to  $x$ . Since  $c$  is not constant, neither is  $h$ . But then  $f \circ \pi \circ h$  attains the value  $f(\bar{y})$  arbitrarily close to 0 and can thus not be continuous, a contradiction.*

**Remark 2.83.** *Above example shows that  $\mathbf{D}\mathbf{F}$  is not the identity. However, its restriction to the subcategory of smooth manifolds is the identity.*

**Lemma 2.84.** *The functors  $\mathbf{F}$  and  $\mathbf{D}$  form an adjunction  $(\mathbf{F}, \mathbf{D})$  from  $\mathcal{D}$  to  $\mathcal{F}$ . Furthermore,  $\mathbf{F}\mathbf{D}$  is the identity functor of  $\mathcal{F}$ .*

*Proof.* Let  $(X, \mathcal{P})$  be a diffeological space and  $(Y, F, C)$  a Frölicher space. Our goal is to show

$$\mathrm{Hom}_{\mathcal{F}}(\mathbf{F}X, Y) = \mathrm{Hom}_{\mathcal{D}}(X, \mathbf{D}Y).$$

First let  $\varphi \in \mathrm{Hom}_{\mathcal{D}}(X, \mathbf{D}Y)$ . To show that  $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\mathbf{F}X, Y)$ , we can show that  $f \circ \varphi$  is a smooth function on  $\mathbf{F}X$  for all  $f \in F$ . By Lemma 2.80, the smooth functions on  $\mathbf{F}X$  are just the ones in the diffeological sense. So we compose  $f \circ \varphi$  with a plot to get  $f \circ \varphi \circ \alpha$ . The map  $\varphi \circ \alpha$  is smooth in the Frölicher sense by definition of  $\mathbf{D}Y$  and by choice of  $\varphi$ . Thus  $f \circ \varphi \circ \alpha$  is smooth, which is what we needed to show. Now choose  $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\mathbf{F}X, Y)$ . Given a plot  $\alpha$  for  $X$  we need to show that  $\varphi \circ \alpha$  is smooth in the Frölicher sense. Take a curve  $c$  into  $U_{\alpha}$ . Then  $\alpha \circ c \in \mathrm{Hom}(\mathbb{R}, X)$  is a smooth curve in  $\mathbf{F}X$  and hence  $\varphi \circ \alpha \circ c \in C$ . This holds for all smooth  $c$ , hence  $\varphi \circ \alpha$  is smooth in the Frölicher sense.

To see  $\mathbf{F}\mathbf{D} = \mathrm{id}$ , recall from Lemma 2.80 that given a Frölicher space  $(X, C, F)$ , the smooth functions on  $\mathbf{F}\mathbf{D}(X)$  are the same as those on  $\mathbf{D}(X)$ . But these are exactly the functions  $f$  on  $X$  for which  $f \circ \alpha$  is smooth for all functions  $\alpha : U_{\alpha} \rightarrow X$  which are smooth in the Frölicher sense. These are exactly the functions  $f \in F$ . Thus  $X$  and  $\mathbf{F}\mathbf{D}(X)$  carry the same Frölicher structure.  $\square$

**Remark 2.85.** *In the proof of the following corollary we use the fact that  $\mathbf{D}(U \times X) = U \times \mathbf{D}X$  for open  $U \subset \mathbb{R}^n$ . This is easily seen as follows:  $\alpha : V \rightarrow \mathbf{D}(U \times X)$  is a plot if and only if  $\alpha = (\alpha_1, \alpha_2) : V \rightarrow U \times X$  is a morphism in  $\mathcal{F}$ , which in turn is true if and only if  $\alpha : V \rightarrow U \times \mathbf{D}X$  is a plot.*

**Corollary 2.86.** *Given a Frölicher space  $(X, C, F)$  and a diffeological space  $(Y, \mathcal{P})$ ,*

- (a) *the Frölicher spaces  $\mathbf{F}\mathrm{Hom}_{\mathcal{D}}(\mathbf{D}X, Y)$  and  $\mathrm{Hom}_{\mathcal{F}}(X, \mathbf{F}Y)$  are isomorphic,*
- (b) *the diffeological spaces  $\mathbf{D}\mathrm{Hom}_{\mathcal{F}}(X, \mathbf{F}Y)$  and  $\mathrm{Hom}_{\mathcal{D}}(\mathbf{D}X, Y)$  are isomorphic.*

*Proof.* Both in (a) and in (b), the underlying sets are equal, and we need to show that the smooth structures are the same, too. To prove (a), note that  $C_X = \text{Hom}_{\mathcal{F}}(\mathbb{R}, X)$  for any Frölicher space  $X$ . We use above adjointness and cartesian closedness to get

$$\begin{aligned}
\text{Hom}_{\mathcal{F}}(\mathbb{R}, \mathbf{F}\text{Hom}_{\mathcal{D}}(\mathbf{D}X, Y)) &= \text{Hom}_{\mathcal{D}}(\mathbb{R}, \text{Hom}_{\mathcal{D}}(\mathbf{D}X, Y)) \\
&= \text{Hom}_{\mathcal{D}}(\mathbb{R} \times \mathbf{D}X, Y) \\
&= \text{Hom}_{\mathcal{F}}(\mathbb{R} \times X, \mathbf{F}Y) \\
&= \text{Hom}_{\mathcal{F}}(\mathbb{R}, \text{Hom}_{\mathcal{F}}(X, \mathbf{F}Y)).
\end{aligned}$$

The proof of (b) is completely analogous, using that plots are exactly the smooth maps from open subsets in  $\mathbb{R}^n$ . We get

$$\begin{aligned}
\text{Hom}_{\mathcal{D}}(U, \mathbf{D}\text{Hom}_{\mathcal{F}}(X, \mathbf{F}Y)) &= \text{Hom}_{\mathcal{F}}(U, \text{Hom}_{\mathcal{F}}(X, \mathbf{F}Y)) \\
&= \text{Hom}_{\mathcal{F}}(U \times X, \mathbf{F}Y) \\
&= \text{Hom}_{\mathcal{D}}(U \times \mathbf{D}X, Y) \\
&= \text{Hom}_{\mathcal{D}}(U, \text{Hom}_{\mathcal{D}}(\mathbf{D}X, Y)).
\end{aligned}$$

□

The following corollary says that the subcategory of Frölicher spaces  $X$  for which  $\mathbf{D}\mathbf{F}(X) = X$  is closed under forming Hom-objects.

**Corollary 2.87.** *Whenever  $\mathbf{D}\mathbf{F}(X) = X$  we have*

$$\mathbf{D}\mathbf{F}\text{Hom}_{\mathcal{D}}(X, Y) = \text{Hom}_{\mathcal{D}}(X, Y).$$

*In particular for smooth manifolds  $M$  and  $N$  this yields  $\mathbf{D}\mathbf{F}(C^\infty(M, N)) = C^\infty(M, N)$ .*

*Proof.* Using above corollary and  $\mathbf{DF}(X) = X$  we get the following chain of equalities:

$$\begin{aligned}
\mathbf{DF}\mathrm{Hom}_{\mathcal{D}}(X, Y) &= \mathbf{DF}\mathrm{Hom}_{\mathcal{D}}(\mathbf{DF}X, Y) \\
&= \mathbf{D}\mathrm{Hom}_{\mathcal{F}}(\mathbf{F}X, \mathbf{F}Y) \\
&= \mathrm{Hom}_{\mathcal{D}}(\mathbf{DF}X, Y) \\
&= \mathrm{Hom}_{\mathcal{D}}(X, Y)
\end{aligned}$$

□

In the remainder of the present subsection we discuss an example is used as a model space in the theory of manifolds with corners.

Example

Let  $X = [0, \infty)^n$  denote the subset of  $\mathbb{R}^n$  of vectors with non-negative coordinates. In [KM97], Chapter V.24, Kriegl and Michor discuss subsets of Mackey complete vector spaces, and it follows from their Proposition V.24.10 that there is a linear map

$$E : \mathrm{Hom}_{\mathcal{F}}(X, \mathbb{R}) \rightarrow C^\infty(\mathbb{R}^n, \mathbb{R})$$

which is an ‘extension operator’: The restriction of  $E(f)$  to  $X$  equals  $f$ . The proof is by induction on  $n$  and makes use of cartesian closedness of  $\mathcal{F}$ .

Note that if  $f$  can be extended to a smooth function on a neighborhood  $U$  of  $X$ , then  $f$  is smooth in the sense of Definition 2.3. So if one is only interested in extending elements  $f \in \mathrm{Hom}_{\mathcal{F}}(X, \mathbb{R})$  to smooth functions, not necessarily in a linear way, there is a more direct approach which does not use cartesian closedness. We will sketch a proof of the following lemma.

**Lemma 2.88.** Equip  $X = [0, \infty)^n$  with the diffeology generated by the map  $\alpha : \mathbb{R}^n \rightarrow X$  given by squaring all coordinates, that is

$$\alpha(x_1, \dots, x_n) = (x_1^2, \dots, x_n^2).$$

Then  $f \in \text{Hom}_{\mathcal{D}}(X, \mathbb{R})$  if and only if  $f$  can be extended to a smooth function  $F$  on  $\mathbb{R}^n$ .

This statement is a priori stronger than the result proved by Kriegl and Michor in that we use a diffeology with fewer plots than the subdiffeology. But as a corollary, since the diffeologies yield the same set  $\text{Hom}_{\mathcal{D}}(X, \mathbb{R})$ , they actually agree.

Let us now discuss the lemma. Let  $X$  and  $\alpha$  be as above, and equip  $X$  with the final diffeology with respect to  $\alpha$ . We want to describe the corresponding Frölicher space  $\mathcal{F}X = (X, C, F)$ , and we already know by Lemma 2.80 that  $F = \text{Hom}_{\mathcal{D}}(X, \mathbb{R})$ . If  $f \in F$ , then the composition  $g = f \circ \alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is smooth and even in all coordinates, by which we mean that if  $y$  is obtained from  $x \in \mathbb{R}^n$  by changing the sign of any coordinates, then  $g(y) = g(x)$ . Let  $U = (0, \infty)^n \subset X$  be the interior of  $X$ . The plot  $\alpha$  can be restricted to a diffeomorphism from  $U$  onto itself, its inverse given by taking the square roots of the entries of a vector  $x \in U$ . From this we can conclude that the restriction of  $f$  to  $U$  must be smooth.

We claim that all derivatives of  $f$  have a limit as  $x \in U$  approaches the boundary  $\partial X$ . Let  $x \in \partial X$  be a boundary point. This means that  $i \geq 1$  coordinates of  $x$  are 0. The point  $x$  has a neighborhood in  $X$  of the form  $[0, \varepsilon)^i \times V$  where  $V \subset \mathbb{R}^{n-i}$  is open. We can compute the  $k$ -th Taylor approximation of  $g$  at  $x$ . For a multiindex  $\alpha \in \mathbb{N}^n$ , let  $D^\alpha g(x) = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} g(x)$  denote the corresponding derivative of  $g$ . Recall that a differentiable even function has derivative zero at 0; this implies that  $D^\alpha g(x) = 0$  whenever  $\alpha_i$  is odd and  $x_i = 0$  for some index  $i$ . By  $o(\|x - y\|^k)$  we



denote a function  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  which satisfies

$$\lim_{x \rightarrow y, x \neq y} \frac{\varphi(x - y)}{\|x - y\|^k} = 0.$$

With this notation, the  $k$ -th order Taylor approximation of  $g$  can be written as

$$g(y) = \sum'_{|\alpha| \leq k} \frac{D^\alpha g(x)}{\alpha!} (y - x)^\alpha + o(\|y - x\|^k)$$

where the sum  $\sum'$  is over all multiindices  $\alpha$  for which  $\alpha_i$  is *even* whenever  $x_i = 0$ .

Now  $f(y) = g(\beta(y))$ , whence

$$\begin{aligned} f(y) &= \sum'_{|\alpha| \leq k} \frac{D^\alpha g(x)}{\alpha!} (\beta(y) - x)^\alpha + o(\|\beta(y) - x\|^k) \\ &= \Psi_k(x, y) + o(\|\beta(y) - x\|^k). \end{aligned} \quad (2.2)$$

We claim that  $\Psi_k$  is a smooth function for  $y$  near  $x$ . To see this, recall that for  $i \in J$ , we have that  $\alpha_i = 2\gamma_i$  is even and  $x_i = 0$ , so that  $(\sqrt{y_i} - x_i)^{\alpha_i} = (\sqrt{y_i})^{2\gamma_i} = y_i^{\gamma_i}$  for  $y_i \geq 0$ . If  $i \notin J$ , then  $(\sqrt{y_i} - x_i)^{\alpha_i}$  is also smooth for  $y_i$  near  $x_i$  since  $x_i > 0$ . Furthermore, this shows that  $\Psi_k$  allows a smooth extension to a neighborhood of  $x$  by

$$\Psi_k(x, y) = \sum'_{|\alpha| \leq k} \frac{D^\alpha g(x)}{\alpha!} \prod_{i \in J} y_i^{\gamma_i} \prod_{i \notin J} (\sqrt{y_i} - x)^{\alpha_i}.$$

We claim that there are continuous functions  $f_\alpha$  on  $X$  for each multiindex  $\alpha$ , such that the restriction of  $f_\alpha$  to  $U$  agrees with the derivative  $D^\alpha f$ . This will follow from the next lemma.

**Lemma 2.89.** *Suppose that  $\varphi$  is  $o(\|x - y\|^k)$ , and let  $\beta$  and  $\alpha$  be defined as above.*

*Then*

$$\lim_{\substack{x \rightarrow y \\ x \neq y \in X}} \frac{\varphi(\beta(x) - \beta(y))}{\|x - y\|^l} = 0$$

*if  $l \leq k/2$ .*

*Proof.* First write

$$\frac{\varphi(\beta(x) - \beta(y))}{\|x - y\|^l} = \frac{\varphi(\beta(x) - \beta(y))}{\|\beta(x) - \beta(y)\|^k} \cdot \frac{\|\beta(x) - \beta(y)\|^k}{\|x - y\|^l}$$

and note that the first factor on the right hand side tends to zero as  $x \rightarrow y$ . We need to show that the second factor is bounded if  $l \leq k/2$ . Note that  $|a - b| \leq |a| + |b|$  for real numbers  $a$  and  $b$ . We apply this to the components  $\sqrt{x_i}$  and  $\sqrt{y_i}$  of  $\beta(x)$  and  $\beta(y)$  to get an inequality

$$(\sqrt{x_i} - \sqrt{y_i})^2 \leq |\sqrt{x_i} - \sqrt{y_i}| |\sqrt{x_i} + \sqrt{y_i}| = |x_i - y_i|.$$

Let  $\|\cdot\|_1$  denote the standard  $l^1$ -norm on  $\mathbb{R}^n$ . The inequality then implies

$$\|\beta(x) - \beta(y)\|^k \leq \|x - y\|_1^{k/2}.$$

Recall that  $\|\cdot\|_1$  and  $\|\cdot\|$  are equivalent (in fact,  $\|x\|_1 \leq \sqrt{n}\|x\|$  for  $x \in \mathbb{R}^n$ ), to see that

$$\frac{\|\beta(x) - \beta(y)\|^k}{\|x - y\|^l} \leq C\|x - y\|^{k/2-l}$$

for some constant  $C$ . This expression is bounded if  $l \leq k/2$ .  $\square$

**Corollary 2.90.** *Let  $f \in \text{Hom}_{\mathcal{F}}(X, \mathbb{R})$ . Each derivative  $D^\alpha f$  is smooth on the interior  $U$  of  $X$  and admits a unique continuous extension to the boundary  $\partial X$ .*

*Proof.* It is clear that  $D^\alpha f$  is smooth on  $U$  because  $f$  is smooth on  $U$ . The second statement follows from the Taylor approximation  $f(y) = \Psi_k(x, y) + o(\|\beta(y) - x\|^k)$  at a boundary point  $x$ . Since  $o(\|\beta(y) - x\|^k)$  vanishes of order  $k/2$ , it follows that the derivative of  $f$  at  $x$  exists and agrees with the derivative of  $\Psi_k$ . Since  $U$  is dense in  $X$ , it follows from a theorem in general topology that the extension of the derivative of  $f$  is unique and continuous.  $\square$

The following theorem is due to Seeley.

**Theorem 2.91.** *Let  $U \subset \mathbb{R}^n$  denote the open half-space described by  $x_1 > 0$ , and let*

$$D_+ = \{f \in C^\infty(U), f \text{ and all its limits have continuous derivatives as } x_1 \rightarrow 0.\}$$

*Equip  $D_+$  with the topology of uniform convergence of all derivatives on compact subsets of the closure of  $U$  in  $\mathbb{R}^n$ , and  $C^\infty(\mathbb{R}^n)$  with the corresponding topology, as described in Example 1.42. Then there is a continuous linear extension operator  $E : D_+ \rightarrow C^\infty(\mathbb{R}^n)$ .*

*Proof.* See [See64]. □

We claim that this theorem can be used in our more general situation. To illustrate this, let us consider the case of  $n = 2$ , when  $X$  is the first quadrant in the plane. We can restrict our function  $f : X \rightarrow \mathbb{R}$  to vertical lines, and get functions  $f_x : y \mapsto f(x, y)$ . Furthermore, let  $f_V$  denote the function  $f$ , restricted to the subset  $V = \{(x, y) \in X \mid x > 0\}$  of  $X$ . Note that  $V$  is diffeomorphic to a closed half-space, and therefore Seeley's Theorem can be applied to the functions  $f_x$  and to  $f_V$ , and furthermore if  $E$  is Seeley's extension operator, then by construction of  $E$  we have that the restriction of  $E(f_V)$  to the vertical line through  $(x, 0)$  agrees with  $E(f_x)$ . Now  $f$  is a continuous function on  $X$ , from which one can deduce that  $f_x$  converges to  $f_0$  uniformly on compact sets. It follows that  $E(f_x)$  converges to  $E(f_0)$ . Consequently, there is a continuous function on the closed half-space  $x \geq 0$  which extends  $f$ . The same argument goes through with  $f$  replaced by  $D^\alpha f$ , which yields a function  $E(f)$  on the half-space  $x \geq 0$  which is smooth on the interior (since there it is given by  $E(f_V)$ ) and whose derivatives have a continuous limit as  $(x, y)$  approaches the boundary. Hence we can apply Seeley's theorem once more to get a smooth extension of  $f$  to all of  $\mathbb{R}^2$ .

# Chapter 3

## Differential Geometry of Diffeological and Frölicher Spaces

In this chapter we generalize notions from differential geometry to the categories  $\mathcal{D}$  and  $\mathcal{F}$ . We start with the tangent space and tangent bundle for diffeological spaces. Then we do the same for Frölicher spaces, and show that for Frölicher spaces of  $L$ -type, the tangent spaces are vector spaces. We discuss the coordinate cross as an example which is not of  $L$ -type, and we show that the tangent space at the singular point is not a vector space. As a second example, we discuss vector spaces and show that if  $V$  is a Fréchet space with its natural Frölicher structure, then  $T_0V \cong V$ . We conclude Section 3.1 by showing that the tangent functors we defined extend the classical tangent functor on **Mfd**.

In Section 3.2 we define vector fields and derivations associated to vector fields. We briefly discuss differential forms.

The final section is devoted to groups in the Frölicher category. We show that the tangent bundle  $TG$  to a group can be trivialized, and that elements of  $\mathfrak{g} = T_eG$  give rise to derivations of the algebra of smooth functions on  $G$ . If  $T_0\mathfrak{g} \cong \mathfrak{g}$ , we can use the group commutator map of  $G$  to define an element  $[v, w]$  associated to each pair of elements  $v, w \in \mathfrak{g}$ . If  $\xi_v$  is the derivation associated to  $v \in \mathfrak{g}$ , we now hope to be able to show that  $[\xi_v, \xi_w] = \xi_{[v, w]}$ . This would then make  $(\mathfrak{g}, [\cdot, \cdot])$  into a Lie algebra.

## 3.1 Tangent Functors

Tangent spaces and a tangent functor for diffeological spaces were defined by Hector in [Hec95], see also [Lau06]. For Frölicher spaces, the definition of a tangent space is more straightforward, see [Frö86]. In this section we will define the diffeological and Frölicher tangent functors and show that they extend the classical tangent functor.

### 3.1.1 Tangent Spaces for Diffeological Spaces

Given a diffeological space  $X$  and a point  $x \in X$ , we construct a vector space  $T_x X$  and a linear map  $j_\alpha : T_0 U_\alpha \rightarrow T_x X$  for each plot  $\alpha$  centered at  $x$ . The maps  $j_\alpha$  will be interpreted as *differentials* of the plots. The construction is motivated by the idea that those differentials satisfy a chain rule for plots of the form  $\alpha \circ h$ . Every choice of a plot  $\alpha$  and a smooth map  $h$  with  $h(0) = 0$  gives a reparametrization as in the following Figure 3.1.

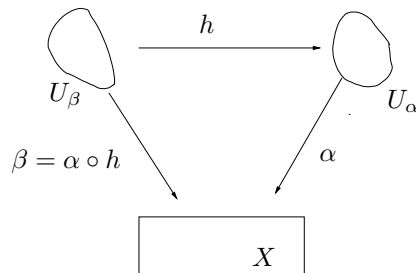


FIGURE 3.1. Reparametrization

By ‘chain rule’ we mean that the corresponding diagram of differentials should commute:

$$\begin{array}{ccc}
 T_0 U_\beta & \xrightarrow{d_0 h} & T_0 U_\alpha \\
 j_\beta \searrow & & \swarrow j_\alpha \\
 & T_x X &
 \end{array}$$

So we require

$$j_{\alpha \circ h} = j_\alpha \circ d_0 h$$

which is the chain rule for the composition  $\alpha \circ h$ . This motivates the definition of the tangent space.

**Definition 3.1** (Tangent Space). *Let  $E_\alpha$  denote the tangent space  $T_0 U_\alpha$ . Let*

$$E_x := \bigoplus_{\alpha \in \mathcal{P}_x} E_\alpha$$

which comes with injections

$$\iota_\alpha : E_\alpha \rightarrow E_x$$

Often we will identify  $v \in E_\alpha$  with its image under  $\iota_\alpha$ . Define a linear subspace of  $E_x$  as follows:

$$\hat{E}_x := \langle \iota_\beta(v) - (\iota_\alpha \circ d_0 h)(v) \mid \beta = \alpha \circ h \text{ and } v \in T_0 U_\beta \rangle,$$

where  $\langle \rangle$  denotes the linear span in  $E_x$ . Then the quotient space

$$T_x X := E_x / \hat{E}_x,$$

is the tangent space to  $X$  at  $x$ . If  $\pi : E_x \rightarrow T_x X$  is the linear projection, we get a family  $j_\alpha := \pi \circ \iota_\alpha : E_\alpha \rightarrow T_x X$  of linear maps indexed by  $\mathcal{P}_x$ .

By construction, the tangent space together with the family  $(j_\alpha)_{\alpha \in \mathcal{P}_x}$  is a colimit in the category of vector spaces. Thus we have a universal property.

**Lemma 3.2** (Universal Property). *Assume that  $F$  is a vector space together with a family of linear maps  $(k_\alpha)_{\alpha \in \mathcal{P}_x}$  such that  $k_\alpha : E_\alpha \rightarrow F$  and all the triangles*

$$\begin{array}{ccc} E_\beta & \xrightarrow{d_0 h} & E_\alpha \\ & \searrow k_{\alpha \circ h} & \swarrow k_\alpha \\ & & F \end{array} \tag{3.1}$$

commute. Then there is a unique linear map  $k : T_x X \rightarrow F$  such that for each  $\alpha \in \mathcal{P}_x$  the triangle

$$\begin{array}{ccc}
 & E_\alpha & \\
 j_\alpha \swarrow & & \searrow k_\alpha \\
 T_x X & \xrightarrow{k} & F
 \end{array} \tag{3.2}$$

commutes.

*Proof.* First note that by the universal property for the direct sum  $E_x$ , the linear map

$$\hat{k} := \bigoplus_{\alpha \in \mathcal{P}_x} k_\alpha$$

is the unique map such that

$$k_\alpha = \hat{k} \circ j_\alpha$$

for all  $\alpha \in \mathcal{P}_x$ . Using diagram (3.1) we see that  $\hat{k}$  vanishes on the subspace  $\hat{E}_x$ . Thus  $\hat{k}$  factors through  $T_x X$  :

$$\begin{array}{ccc}
 E_x & \xrightarrow{\hat{k}} & F \\
 \pi \downarrow & \nearrow k & \\
 E_x / \hat{E}_x & & 
 \end{array}$$

which yields existence of a map  $k$  making the diagrams (3.2) commute. To see that  $k$  is unique, suppose that  $k'$  is another map making all diagrams (3.2) commute. Note that the vector space  $T_x X$  is generated by elements of the form  $j_\alpha(v)$  with  $v \in E_\alpha$ . Commutativity of diagram (3.2) yields

$$k'(j_\alpha(v)) = k(v).$$

So  $k'$  and  $k$  agree on a set of vectors spanning  $T_x X$ , which proves  $k' = k$ .  $\square$

### 3.1.2 Tangent Bundle for Diffeological Spaces

In this section we construct a new diffeological space, the *tangent bundle* of  $X$ . Just as for the classical tangent bundle, the underlying set will be the union of the

tangent spaces to all points of  $X$ . We equip this set with the diffeology generated by the differentials of plots.

**Definition 3.3.** Let  $TX$  be the disjoint union of all tangent spaces to  $X$ ,

$$TX := \bigcup_{x \in X} T_x X.$$

Next we define the differential  $d\alpha : TU_\alpha \rightarrow TX$  of a plot by translating  $U_\alpha$ .

**Definition 3.4.** Let  $\alpha$  be any plot. Given a point  $u \in U_\alpha$ , let

$$h : U_\alpha \rightarrow U_\alpha - u$$

$$y \mapsto y - u.$$

Then  $\beta = \alpha \circ h^{-1}$  is a plot centered at  $\alpha(u)$ , and we can define

$$d_u \alpha := j_\beta \circ d_u h : T_u U_\alpha \rightarrow T_{\alpha(u)} X.$$

This yields a map

$$d\alpha : TU_\alpha \rightarrow TX$$

$$v \mapsto d_u \alpha(v) \quad \text{if } v \in T_u U_\alpha.$$

For  $u = 0$ , the map  $h$  is the identity map and we have  $j_\alpha = d_0 \alpha$ .

Now we are ready to define a diffeology on  $TX$ .

**Definition 3.5.** Since  $TU_\alpha \cong U_\alpha \times \mathbb{R}^n$  we can regard  $TU_\alpha$  as an open subset of  $\mathbb{R}^{2n}$ . Therefore we can use the maps  $d\alpha$ ,  $\alpha \in \mathcal{P}(X)$  to generate a diffeology  $\mathcal{P}(TX)$  on the set  $TX$ . The diffeological space  $(TX, \mathcal{P}(TX))$  is the tangent bundle to the diffeological space  $(X, \mathcal{P})$ .

This is a natural definition and is similar to the manifold structure on the tangent bundle to a manifold. We will show that the projection map and scalar multiplication on  $TX$  are smooth if  $TX$  is equipped with this diffeology.



**Lemma 3.6.** *The bundle projection  $\pi : TX \rightarrow X$  is a smooth map.*

*Proof.* By Lemma 2.17 it suffices to verify smoothness on a generating family, so we have to show that  $\pi \circ d\alpha \in \mathcal{P}(X)$  for all plots  $\alpha \in \mathcal{P}(X)$ . But this follows immediately from the definitions, since  $\pi \circ d\alpha = \alpha$ .  $\square$

**Lemma 3.7.** *The scalar multiplication*

$$\begin{aligned} m : \mathbb{R} \times TX &\rightarrow TX \\ (r, v) &\mapsto rv \end{aligned}$$

*is smooth.*

*Proof.* The diffeology of  $\mathbb{R} \times TX$  is generated by  $(\alpha, d\beta)$ , where  $\beta : U \rightarrow X$  is a plot and  $\alpha : TU \rightarrow \mathbb{R}$  is a smooth map. So let  $(\alpha, d\beta)$  be a plot in the generating family. Then observe that

$$m \circ (\alpha, d\beta)(u) = \alpha(u)d\beta(u) = d\beta(\alpha(u)u)$$

because of the linearity of each  $d_x\beta$ . Now define the map  $\gamma : TU \rightarrow TU$  by  $\gamma(u) = \alpha(u)u$ . The map  $\alpha$  is smooth, and so is the scalar multiplication on  $TU$ , thus the map  $\gamma$  is a smooth map, and we see that  $m \circ (\alpha, d\beta) = d\beta \circ \gamma$  is a plot for  $TX$ . This implies that  $m$  is a smooth map.  $\square$

We have defined a map  $T$  on the objects of  $\mathcal{D}$ . In the next subsection we will extend  $T$  to a functor on  $\mathcal{D}$ .

### 3.1.3 Tangent Functor for Diffeological Spaces

Let us consider two spaces  $(X, \mathcal{P}(X))$  and  $(Y, \mathcal{P}(Y))$  and a smooth map  $f \in \text{Hom}_{\mathcal{D}}(X, Y)$ . We will define a smooth map  $Tf : TX \rightarrow TY$  by defining its restriction  $T_x f$  to each fiber  $T_x X$ . So let  $x \in X$  be any point and let  $y = f(x)$ . We will now use the Universal Property (Lemma 3.2) for  $T_x X$  to define the map  $T_x f : T_x X \rightarrow T_y Y$ .

**Definition 3.8.** Consider the family

$$\mathcal{F} = \{f \circ \alpha \mid \alpha \in \mathcal{P}_x(X)\} \subset \mathcal{P}_y(Y).$$

If  $h$  is smooth with  $h(0) = 0$ , then the map  $f \circ \alpha \circ h$  is a plot for  $Y$ , centered at  $y$ .

In particular we have that

$$j_{f \circ \alpha \circ h} = j_{f \circ \alpha} \circ d_0 h,$$

so the family  $\mathcal{F}$  makes diagram (3.1) commute. Hence by Lemma 3.2 there is a unique linear map

$$T_x f : T_x X \rightarrow T_y Y$$

such that for every  $\alpha \in \mathcal{P}_x(X)$  we have

$$d_0(f \circ \alpha) = T_x f \circ d_0 \alpha.$$

**Remark 3.9.** Note that we use different notation in order to distinguish differentials of plots  $d\alpha : TU_\alpha \rightarrow TX$  and of smooth maps  $Tf : TX \rightarrow TY$ , because these two concepts are defined in a different way. However, we can regard  $U_\alpha$  as a manifold and thus as a diffeological space, and in Subsection 3.1.7 below we will prove the equality  $d\alpha = T\alpha$  for plots.

Uniqueness of the differential  $T_x$  yields the following chain rule.

**Lemma 3.10** (Chain Rule). Given smooth maps  $f$  and  $g$ , we have

$$T_x(g \circ f) = T_{f(x)}g \circ T_x f.$$

*Proof.* Looking at

$$\begin{array}{ccccc} & & E_\alpha & & \\ & \swarrow & \downarrow & \searrow & \\ T_x X & \xrightarrow{T_x f} & T_y Y & \xrightarrow{T_y g} & T_z Z \end{array}$$

we see that the map  $T_{f(x)}g \circ T_xg$  satisfies

$$j_{g \circ f \circ \alpha} = T_{f(x)}g \circ T_xg \circ j_\alpha.$$

But by Definition 3.19,  $T_x(g \circ f)$  is the unique linear map with that property. Thus we have the equality

$$T_x(g \circ f) = T_{f(x)}g \circ T_xg.$$

□

**Lemma 3.11** (Chain Rule for Plots). *Let  $\alpha \in \mathcal{P}(X)$  and  $f \in \text{Hom}_{\mathcal{D}}(X, Y)$ . Then*

$$d(f \circ \alpha) = Tf \circ d\alpha.$$

*Proof.* Given  $u \in U_\alpha$ , by definition of differential for plots we have

$$d_u(f \circ \alpha) = d_0(f \circ \alpha \circ h^{-1}) \circ d_uh$$

and

$$T_{\alpha(u)}f \circ d_u\alpha = T_{\alpha(u)}f \circ d_0(\alpha \circ h^{-1}) \circ d_uh.$$

We claim that the left hand sides are equal, so let us show that the right hand sides are equal. Let us write  $\delta := \alpha \circ h^{-1} \in \mathcal{P}(X)$ . Then by Definition 3.19 we have that  $d_0(f \circ \delta) = T_{\alpha(u)}f \circ d_0\delta$ , which implies equality of the right hand sides. □

**Lemma 3.12.** *The map  $Tf : TX \rightarrow TY$  is smooth.*

*Proof.* By Lemma 2.17 it suffices to show

$$Tf \circ d\alpha \in \mathcal{P}(TY)$$

for all  $\alpha \in \mathcal{P}(X)$ . But this follows immediately from the chain rule for plots, since  $Tf \circ d\alpha = d(f \circ \alpha) \in \mathcal{P}(TY)$ . □

**Corollary 3.13.** *The assignments  $X \mapsto TX$  and  $f \mapsto Tf$  define a functor*

$$T : \mathcal{D} \rightarrow \mathcal{D}.$$

In Subsection 3.1.7 we will show that this functor extends the classical tangent functor for smooth manifolds.

**Lemma 3.14.** *If  $X$  is a diffeological space of  $L$ -type, then  $TX$  is also of  $L$ -type.*

*Proof.* We use Lemma 2.34, which allows us to check the  $L$ -type condition on a generating family. The diffeology on  $TX$  is generated by the maps  $d\alpha$ , where  $\alpha$  is a plot for  $X$ . Given  $d\alpha$  and  $d\beta$ , let  $\gamma$  be a plot through which  $\alpha$  and  $\beta$  factor. Then it follows from the chain rule that  $d\alpha$  and  $d\beta$  factor through  $d\gamma$ .  $\square$

### 3.1.4 Tangent Functor for Frölicher Spaces

For Frölicher spaces, the connection between smooth curves and smooth functions allows a more direct definition of tangent spaces than in the diffeological case.

**Definition 3.15.** *Given a Frölicher space  $(X, C, F)$  and  $x \in X$ , let  $C_x$  be the set of curves  $c \in C$  for which  $c(0) = x$ . We define equivalence relations  $\sim$  and  $\sim_x$  on  $C_x$  and  $F$  respectively by*

$$c_1 \sim c_2 \iff \forall f \in F : (f \circ c_1)'(0) = (f \circ c_2)'(0)$$

and similarly

$$f_1 \sim_x f_2 \iff \forall c \in C_x : (f_1 \circ c)'(0) = (f_2 \circ c)'(0).$$

Then the tangent space to  $X$  at  $x$  is  $T_x X = C_x / \sim$  and the cotangent space to  $X$  at  $x$  is  $T^x X = F / \sim_x$ .

**Remark 3.16.** *Recall that the vector space  $F$  of functions on  $X$  is a Frölicher vector space (see Example 2.75). The cotangent space  $T^x X$  is the quotient of  $F$  by*

the vector subspace

$$\{f \in F \mid (f \circ c)'(0) = 0 \text{ for all } c \in C_x\}.$$

By Example 2.77,  $T^x X$  is also a Frölicher vector space. Let us denote the cotangent vector represented by  $f \in F$  as  $[f]_x$ , or simply by  $[f]$  if it is clear that  $[f] \in T^x X$ . Note that the tangent space  $T_x X$  is not a vector space in general, as we illustrate in Section 3.1.6 below.

**Definition 3.17.** If a tangent vector  $v$  is represented by a curve  $c$ , we write  $v = [c]$ . If  $s \in \mathbb{R}$  and  $v = [c]$ , let  $sv$  be the tangent vector represented by  $d(t) = c(st)$ . The unique vector represented by the constant curve with value  $x$  is called the zero vector in  $T_x X$  and denoted  $0_x$  or simply  $0$ .

**Definition 3.18.** Let

$$b : T_x X \times T^x X \rightarrow \mathbb{R}$$

$$b([c], [f]) = (f \circ c)'(0).$$

Note that the map  $b$  is well defined by definition of the equivalence relations  $\sim$  and  $\sim_x$ . We say that  $T_x X$  has vector addition if for each pair  $v, w \in T_x X$  there is a third vector  $u \in T_x X$  such that  $b(u, \xi) = b(v, \xi) + b(w, \xi)$  for all  $\xi \in T^x X$ . In this case we write  $u = v + w$ , and we will see in Corollary 3.29 that if  $T_x X$  has vector addition, it is in fact a vector space.

**Definition 3.19** (Differential). Given a Frölicher space  $(X, C, F)$ , let  $TX$  be the disjoint union of all tangent spaces  $T_x X$ . If  $\varphi : X \rightarrow Y$  is a smooth map of Frölicher spaces, define a map  $T\varphi : TX \rightarrow TY$  as follows. If  $v = [c] \in T_x X$ , let

$$T\varphi(v) = [\varphi \circ c] \in T_{\varphi(x)} Y.$$

We need to show that this definition does not depend on the representative  $c$  of  $[c]$ , so suppose that  $[c] = [d]$ . Let  $f \in F_Y$ . Then  $f \circ \varphi \in F_X$ , so that  $(f \circ \varphi \circ c)'(0) =$

$(f \circ \varphi \circ d)'(0)$  by assumption. But then  $[\varphi \circ c] = [\varphi \circ d]$ , so  $T\varphi$  is well defined. The map  $T\varphi$  is called the differential of  $\varphi$ , and its restriction  $T_x\varphi$  to the tangent space  $T_xX$  at  $x$  is called the differential of  $\varphi$  at  $x$ .

**Remark 3.20** (Chain Rule). *The chain rule for Frölicher spaces is trivial. Given smooth maps  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ , we see that  $T(g \circ f)([c]) = [g \circ f \circ c] = Tg(Tf([c]))$ .*

**Definition 3.21** (Tangent Bundle). *We equip the set  $TX$  with the initial Frölicher structure with respect to all differentials  $Tf : TX \rightarrow T\mathbb{R}$  of smooth functions  $f \in F_X$ . Here  $T\mathbb{R} \cong \mathbb{R}^2$  carries the manifold Frölicher structure. The resulting Frölicher space  $(TX, C_{TX}, F_{TX})$  is called the tangent bundle of  $X$ . If a vector  $v \in TX$  is contained in  $T_xX$ , then the point  $x$  is called the base point of  $v$ , and we say that  $v$  is a tangent vector at  $x$ . The map  $\pi : TX \rightarrow X$  which sends a vector to its base point is called the projection of the tangent bundle. From now on we assume that the tangent spaces  $T_xX$  carry the subspace structure induced from  $TX$ .*

**Remark 3.22.** *Let us describe the curves for  $TX$  as well as those for the subset structure of  $T_xX$ . Given a curve  $c : \mathbb{R} \rightarrow TX$ , let  $c_s(t)$  be the curve in  $X$  representing the vector  $c(s)$ . By definition,  $c$  is smooth if and only if for all functions  $f$  on  $X$ , the composition  $Tf \circ c$  is smooth into  $T\mathbb{R} \cong \mathbb{R}^2$ . We compute*

$$Tf \circ c(s) = ((f \circ \pi \circ c)(s), (f \circ c_s)'(0))$$

*which is smooth if and only if its two components are smooth. The curves for the subset structure of  $T_xX$  are exactly those for which  $\pi \circ c$  is constant with value  $x$ .*

*Now let us write  $\gamma_c(s, t) = c_s(t)$ . Then  $\gamma_c$  satisfies*

- i) For each  $s \in \mathbb{R}$ , the map  $t \mapsto \gamma_c(s, t)$  is a curve into  $X$ .*
- ii)  $s \mapsto \gamma_c(s, 0)$  is a curve into  $X$ .*

iii) The map  $\partial_2(f \circ \gamma_c)(s, 0)$  is smooth in  $s$  for all smooth functions  $f$  on  $X$ .

By definition, elements of  $T^2X = T(TX)$  are given by curves in  $TX$ , so above remark relates elements of  $T^2X$  with maps  $\gamma : \mathbb{R}^2 \rightarrow TX$ . This can be generalized as follows.

**Lemma 3.23.** *A map  $\gamma : \mathbb{R}^2 \rightarrow TX$  gives rise to a curve  $c : \mathbb{R} \rightarrow TX$  via  $c(s) = [t \mapsto \gamma(s, t)]$  if and only if  $\gamma$  satisfies i) - iii) above.*

*Proof.* We have seen that if  $c$  is a curve into  $TX$  and  $c(s)$  is represented by  $t \mapsto \gamma(s, t)$ , then  $\gamma$  satisfies i)-iii). Now conversely suppose that  $\gamma$  satisfying i)-iii) is given. By condition i) we can define  $c(s) = [t \mapsto \gamma(s, t)]$ . It remains to show that the map  $c$  thus defined is a curve into  $TX$ . To see this, we compose with  $Tf$  for some  $f \in F_X$  and get

$$Tf(c(s)) = (\pi(c(s)), (f \circ c(s))'(0)) = (\gamma(s, 0), \partial_2(f \circ \gamma)(s, 0)),$$

which is smooth by conditions ii) and iii). □

**Definition 3.24.** *Every element  $v \in T^nX$  determines a map  $\gamma : \mathbb{R}^n \rightarrow X$  which can be defined iteratively as follows. The vector  $v$  is represented by a curve  $\gamma_1 : \mathbb{R} \rightarrow T^{n-1}X$ , and then  $\gamma_1(s)$  is again represented by a curve  $t \mapsto \gamma_2(s, t) \in T^{n-2}X$ . Iterate this to get maps  $\gamma_j : \mathbb{R}^j \rightarrow T^{n-j}X$  and finally  $\gamma = \gamma_n : \mathbb{R}^n \rightarrow X$ . These maps are not uniquely determined. For example if  $\rho : \mathbb{R} \rightarrow \mathbb{R}$  is a reparametrization with  $\rho'(0) = 1$ , then  $c : \mathbb{R} \rightarrow X$  and  $c \circ \rho$  yield the same tangent vector.*

**Remark 3.25.** *We do not know whether a condition analogous to iii) in Remark 3.22 above characterizes the  $\gamma$  which can occur as representatives of vectors  $v \in T^nX$ . This is because for  $n \geq 2$ , we do not know whether the Frölicher structure of  $T^nX$  is generated by the maps  $T^n f$  for  $f \in F_X$ .*

**Lemma 3.26.** *The map  $b$  is separately smooth. It is linear in the second argument, and for all  $v \in T_x X, \alpha \in \mathbb{R}$  and  $\xi \in T_x X$  we have  $b(0, \xi) = 0$  and  $b(\alpha v, \xi) = \alpha b(v, \xi)$ .*

*Proof.* First fix  $[f] \in T_x X$ . Let  $c : \mathbb{R} \rightarrow T_x X$  be a curve and let  $c(s)$  be represented by a curve  $c_s$  into  $X$ . Then  $s \mapsto b(c(s), [f]) = (f \circ c_s)'(0)$  is smooth by Remark 3.22. Now let us fix  $[c] \in T_x X$ . Since  $T_x X$  carries the quotient Frölicher structure, to check that  $\varphi : [f] \mapsto b([c], [f])$  is smooth it suffices to verify that  $\varphi \circ \pi : F \rightarrow \mathbb{R}$  is smooth. Let  $d : \mathbb{R} \rightarrow F$  be a curve. Then  $(\varphi \circ \pi \circ d)(s) = (d(s) \circ c)'(0)$ . Note that we can write the latter expression as the derivative with respect to the second argument of the function

$$\gamma : (s, t) \mapsto \tilde{d}(s, c(t)).$$

The function  $\gamma$  is smooth, so  $(\varphi \circ \pi \circ d)(s) = \partial_2 \gamma(s, 0)$  is a smooth function of  $s$ . This completes the proof of separate smoothness of  $b$ .

Linearity of  $b$  in the second argument is clear, since

$$((f + \alpha g) \circ c)'(0) = (f \circ c)'(0) + \alpha(g \circ c)'(0).$$

Now let  $v = [c]$  and  $d(t) = c(\alpha t)$ . By the chain rule,  $(f \circ d)'(0) = \alpha(f \circ c)'(0)$ , which shows  $b(\alpha v, \xi) = \alpha b(v, \xi)$  as well as  $b(0, \xi) = 0$ .  $\square$

**Lemma 3.27.** *If the Frölicher structure on the product  $T_x X \times T_x X$  is generated by curves  $(c_1, c_2)$ , where  $c_2$  is in the generating family  $\{\pi \circ c \mid c \in C_F\}$  for the Frölicher structure on  $T_x X$ , then  $b$  is smooth.*

*Proof.* By assumption, it suffices to check whether  $b \circ (c_1, \pi \circ c_2)$  is smooth, where  $c_1$  is a curve into  $T_x X$  and  $c_2$  is a curve into  $F_X$ . If  $c_1(s)$  is represented by  $t \mapsto \gamma(s, t)$ ,



we compute

$$\begin{aligned} b(c_1(s), c_2(s)) &= \left. \frac{\partial}{\partial t} \right|_{t=0} c_2(s)(\gamma(s, t)) \\ &= \left. \frac{\partial}{\partial t} \right|_{t=0} \tilde{c}_2(s, \gamma(s, t)). \end{aligned}$$

The map  $\tilde{c}_2$  is smooth by cartesian closedness. The map  $\Gamma : (s, t) \mapsto (s, \gamma(s, t))$  represents a smooth curve in  $T(\mathbb{R} \times X)$  because its components define smooth functions into  $T\mathbb{R}$  and  $TX$ , respectively (see Lemma 3.35 below). Thus, by condition iii) in Remark 3.22, the function  $s \mapsto \left. \frac{\partial}{\partial t} \right|_{t=0} \tilde{c}_2(s, \gamma(s, t))$  is smooth.  $\square$

**Remark 3.28.** *We do not know whether in general, if a set  $\tilde{C}_i$  of curves generates the Frölicher structure on  $X_i$ , then the product Frölicher structure on  $X_1 \times X_2$  is generated by the curves  $(c_1, c_2)$  with  $c_i \in \tilde{C}_i$ .*

*This is true if  $X_i = M_i$  are manifolds, and we define the generating set of curves as follows: Let  $\mathcal{A}_i$  be a maximal atlas on  $M_i$ . Let  $\tilde{C}_i$  consist of smooth functions  $c : \mathbb{R} \rightarrow M_i$  whose image is contained in a chart for the atlas  $\mathcal{A}_i$ . Then  $f : M_1 \times M_2 \rightarrow \mathbb{R}$  is smooth if and only if  $f \circ (c_1, c_2)$  is smooth for all  $(c_1, c_2) \in \tilde{C}_1 \times \tilde{C}_2$ .*

**Corollary 3.29.** *If  $T_x X$  has vector addition, then  $v \mapsto b(v, \cdot)$  is an injective linear map from  $T_x X$  into the smooth dual  $(T^x X)'_s$  of the cotangent space at  $x$ . Furthermore,  $b$  is a pairing of vector spaces making  $(T_x X, T^x X)$  a dual pair (Definition 1.44).*

*Proof.* If  $T_x X$  has vector addition, then  $b$  is additive in the first argument by definition of the sum in  $T_x X$ . Since  $b$  is separately smooth, the map  $b_v : \xi \mapsto b(v, \xi)$  is smooth for every  $v \in T_x X$ . Therefore,  $v \mapsto b_v$  defines a linear map from  $T_x X$  to the smooth dual of  $T^x X$ . The fact that  $b$  is a proper pairing follows directly from the definition of the equivalence relations defining  $T_x X$  and  $T^x X$ .  $\square$

**Lemma 3.30.** *Given a Frölicher space  $(X, C, F)$ , the projection  $\pi : TX \rightarrow X$  is smooth. If  $\varphi : X \rightarrow Y$  is a smooth map of Frölicher spaces, then  $T\varphi : TX \rightarrow TY$  is smooth. The maps  $T_x\varphi : T_xX \rightarrow T_{\varphi(x)}Y$  are smooth, and if  $T_xX$  and  $T_{\varphi(x)}Y$  are vector spaces, they are also linear.*

*Proof.* Let  $f \in F$  be a smooth function on  $X$ . If  $\pi' : T\mathbb{R} \rightarrow \mathbb{R}$  is the projection of the tangent bundle of  $\mathbb{R}$ , then  $f \circ \pi = \pi' \circ Tf$ . This shows that  $f \circ \pi$  is smooth for all  $f \in F$ , hence  $\pi$  is smooth. The chain rule implies smoothness of  $Tf$  as follows: Since  $TY$  carries the initial Frölicher structure with respect to the  $Tf, f \in F_Y$ , we need to look at compositions  $T\varphi \circ Tf = T(\varphi \circ f)$  which are clearly smooth functions  $TX \rightarrow T\mathbb{R}$ . Hence  $T\varphi$  is smooth. It follows that the restrictions  $T_x\varphi$  are smooth (see Example 2.56). Now let us show that  $T_x\varphi$  is linear. If  $v = [c]$  and  $\alpha$  is a scalar, then  $T_x\varphi(\alpha v)$  is represented by the curve  $s \mapsto \varphi(c(\alpha s))$ , which also represents  $\alpha T_x\varphi(v)$ . Now suppose the vectors  $v = [c_1]$  and  $w = [c_2]$  have a sum  $[c_3] = u = v + w$ . Let  $f$  be a function on  $Y$ . Then  $f \circ \varphi$  is a function on  $X$ , and by Definition 3.18 we have that

$$\begin{aligned} (f \circ \varphi \circ c_3)'(0) &= b([c_3], [\varphi \circ c_3]) = \\ &= b([c_1], [\varphi \circ c_3]) + b([c_2], [\varphi \circ c_3]) = (f \circ \varphi \circ c_1)'(0) + (f \circ \varphi \circ c_2)'(0), \end{aligned}$$

hence  $T_x\varphi(u) = T_x\varphi(v) + T_x\varphi(w)$ . □

**Corollary 3.31** (Tangent Functor). *The assignments  $X \mapsto TX$  and  $\varphi \mapsto T\varphi$  define a functor  $T : \mathcal{F} \rightarrow \mathcal{F}$ .*

**Definition 3.32.** *Let  $X_1, \dots, X_n$  be Frölicher spaces and  $X = \prod_{i=1}^n X_i$ . If  $f : X \rightarrow \mathbb{R}$  is smooth, let us define  $\partial^i f : TX \rightarrow T\mathbb{R}$  for  $i = 1, \dots, n$  as follows. If  $v \in TX$ , let  $\pi(v) = (x_1, \dots, x_n)$  be the base point of  $v$ , and let  $\varphi_i : X_i \rightarrow \mathbb{R}$  be*

given by  $x \mapsto f(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n)$ . Let  $v_i = T\pi_i(v)$  where  $\pi_i : X \rightarrow X_i$  is the projection onto  $X_i$ . Then we define  $\partial^i f(v) = T\varphi_i(v_i)$ .

**Lemma 3.33.** *If  $X = \prod_{i=1}^n X_i$  is a product of Frölicher spaces, then the maps  $\partial^i f : TX \rightarrow \mathbb{R}$  are smooth.*

*Proof.* Let  $c$  be a curve into  $TX$ , which means that  $c(s) \in TX$  can be represented by a curve  $\gamma_s : t \mapsto (c_s^1(t), \dots, c_s^n(t))$ . The map  $\gamma(s, t) = \gamma_s(t)$  satisfies conditions i)-iii) in Remark 3.22. We need to show that  $\partial^i f \circ c : \mathbb{R} \rightarrow T\mathbb{R}$  is smooth. Consider the function  $\rho : (s, t) \mapsto (c_s^1(0), \dots, c_s^i(t), \dots, c_s^n(0))$ . This function satisfies conditions i)-iii) and defines a curve  $d$  in  $TX$ . Furthermore,  $\partial^i f(c(s)) = Tf(d(s))$ . This proves that  $\partial^i f \circ c$  is smooth, hence  $\partial^i f$  is smooth.  $\square$

**Lemma 3.34.** *Let  $X_1, \dots, X_n$  be Frölicher spaces, and let  $X = \prod_{i=1}^n X_i$ . If  $f : X \rightarrow \mathbb{R}$  is smooth and  $v \in TX$ , then  $Tf(v) = \sum_{i=1}^n \partial^i f(v)$ .*

*Proof.* Let  $v \in T_x X$  be represented by a curve  $c = (c_1, \dots, c_n)$  with  $c(0) = x$ , and let  $\rho(s_1, \dots, s_n) = (c_1(s_1), \dots, c_n(s_n))$ . Then  $\rho : \mathbb{R}^n \rightarrow X$  is smooth by Lemma 2.72. If  $\Delta : \mathbb{R} \rightarrow \mathbb{R}^n$  is the diagonal map  $\Delta(s) = (s, \dots, s)$ , we can write  $c = \rho \circ \Delta$ . If  $f : X \rightarrow \mathbb{R}$  is smooth, the chain rule then implies

$$(f \circ c)'(0) = \sum_{i=1}^n \frac{\partial}{\partial x_i} (f \circ \rho)(0, \dots, 0).$$

Now we need to show that the vector  $\partial^i f(v) \in T\mathbb{R}$  coincides with

$$\frac{\partial}{\partial x_i} (f \circ \rho)(0, \dots, 0) \in T_{f(x)}\mathbb{R}.$$

Let us write  $c(0) = x = (x_1, \dots, x_n)$ . By definition, the  $i$ -th partial derivative of  $f \circ \rho$  is given by differentiating the function  $\varphi_i(x) = f(c_1(0), \dots, c_i(x), \dots, c_n(0)) = f(x_1, \dots, c_i(x), \dots, x_n)$  with respect to  $x$ . Now observe that if  $v$  is represented by

$c$ , the vector  $v_i = \pi_i(v) \in TX_i$  is represented by the curve  $c_i(x)$ . This shows that

$$\partial^i f(v) = \frac{\partial}{\partial x_i}(f \circ \rho)(0, \dots, 0) \in T_{f(x)}\mathbb{R}.$$

□

**Lemma 3.35.** *If  $\{X_i \mid i \in I\}$  is a family of Frölicher spaces and  $X = \prod_{i \in I} X_i$  their direct product, then*

$$P = (T\pi_i)_{i \in I} : TX \rightarrow \prod_{i \in I} TX_i$$

$$[(c_i)_{i \in I}] \mapsto ([c_i])_{i \in I}$$

*is a smooth surjection. If  $I$  is finite, then  $P$  is a diffeomorphism.*

*Proof.* The map  $P$  is smooth because its components  $T\pi_i$  are smooth. It is surjective, since if  $v \in \prod_{i \in I} TX_i$  has components  $v_i = [c_i]$ , then the curve  $c : \mathbb{R} \rightarrow X$  with components  $c_i$  satisfies  $P([c]) = v$ .

Now suppose that  $I$  is finite, and let us show that  $P$  is injective. Let  $v = [c]$  and  $w = [d]$  be tangent vectors to  $X$  with  $P([c]) = P([d])$ . This means that for every index  $i$ , the components  $v_i = [c_i]$  and  $w_i = [d_i]$  agree. Then for every smooth function on  $X$ , we have  $\partial^i f(v) = \partial^i f(w)$ , and hence by Lemma 3.34,

$$Tf(v) = \sum_i \partial^i f(v) = \sum_i \partial^i f(w) = Tf(w).$$

Since this is true for every smooth function  $f$  on  $X$ , we can conclude that  $v = w$ .

Lastly we have to show that if  $I$  is finite, then  $P^{-1}$  is smooth. Let  $c : \mathbb{R} \rightarrow \prod_i TX_i$  be a smooth curve with components  $c_i : \mathbb{R} \rightarrow TX_i$ . If  $f$  is a smooth function on  $X$ , we use Lemma 3.34 to get

$$(Tf \circ P^{-1} \circ c)(s) = \sum_i \partial^i f(v).$$

The terms on the left hand side are smooth by Lemma 3.33, which completes the proof. □

### 3.1.5 $L$ -type

The  $L$ -condition ensures that all tangent spaces are vector spaces. The condition is satisfied, for example, by all manifolds and all groups.

**Definition 3.36.** *A Frölicher space  $(X, C, F)$  is of  $L$ -type if  $\mathbf{D}X$  is a diffeological space of  $L$ -type (see Subsection 2.2.5).*

**Example 3.37.** *If  $M$  is a manifold and  $G$  a Frölicher group, then  $\mathbf{D}M$  carries the manifold diffeology, and  $\mathbf{D}G$  is a diffeological group. Since manifolds and diffeological groups are of  $L$ -type, this implies that manifolds and Frölicher groups are Frölicher spaces of  $L$ -type.*

**Lemma 3.38.** *If  $X$  is of  $L$ -type at  $x$ , then  $T_x X$  is a vector space.*

*Proof.* Suppose that  $X$  is of  $L$ -type at  $x \in X$ . Take two tangent vectors at  $x$  represented by curves  $c_1$  and  $c_2$ . By the  $L$ -condition, there is a smooth function  $\varphi : U \rightarrow X$ , where  $U \subset \mathbb{R}^n$  is an open neighborhood of 0, and there are smooth maps  $h_i, i = 1, 2$  with  $h_i(0) = 0$  and such that locally,  $c_i = \varphi \circ h_i$ . Let  $v = h'_1(0) + h'_2(0) \in T_0 U$  and choose a curve  $\tilde{c}_3 : \mathbb{R} \rightarrow U$  such that  $\tilde{c}'_3(0) = v$ . Let  $c_3 = \varphi \circ \tilde{c}_3$ . The following short computation shows that  $c_3$  represents the sum of the vectors represented by  $c_1$  and  $c_2$ .

$$\begin{aligned}
 b([c_3], [f]) &= (f \circ c_3)'(0) \\
 &= (f \circ \varphi \circ \tilde{c}_3)'(0) \\
 &= d_0(f \circ \varphi)(h'_1(0) + h'_2(0)) \quad (\text{chain rule}) \\
 &= (f \circ \varphi \circ h_1)'(0) + (f \circ \varphi \circ h_2)'(0) \\
 &= b([c_1], [f]) + b([c_2], [f]).
 \end{aligned}$$

□

For Frölicher groups and manifolds, the structure of the tangent spaces can be described more explicitly.

**Lemma 3.39.** *If  $G$  is a Frölicher group and  $[c]$  and  $[d]$  are vectors in  $T_gG$ , then their sum is represented by the curve  $s \mapsto c(s)g^{-1}d(s)$ . The vector  $-[c]$  is represented by  $s \mapsto gc(s)^{-1}g$ .*

*If  $m$  and  $i$  are multiplication and inversion in  $G$ , then  $T_e m(v, w) = v + w$  and  $T_e i(v) = -v$ .*

*Proof.* Let us first treat the case of  $[c], [d] \in T_eG$ , and let us keep the notation from the proof of Lemma 3.38. We know that  $c$  and  $d$  factor through  $\varphi(s, t) = c(s)d(t)$  via  $h_1(x) = (x, 0)$  and  $h_2(x) = (x, 0)$ . Hence  $v = h_1'(0) + h_2'(0) = (1, 1)$ . The curve  $\tilde{c}_3$  can simply be chosen as  $\tilde{c}_3(s) = (s, s)$ . Hence by Lemma 3.38, the curve  $\varphi \circ \tilde{c}_3 : s \mapsto c(s)d(s)$  represents  $[c] + [d]$ . In particular,  $[c^{-1}] + [c]$  is represented by the constant curve, and therefore  $[c^{-1}] = -[c]$ .

Now we reduce the general case to the case of curves through  $e$ . Let  $c$  and  $d$  be curves through  $g \in G$ . By the case treated above,  $\gamma(s) = g^{-1}c(s)g^{-1}d(s)$  represents the sum of  $[g^{-1}c]$  and  $[g^{-1}d]$ . We claim that  $[g\gamma] = [c] + [d]$ . To this end, let  $f \in F$ , and let  $\lambda_g$  be left multiplication on  $G$  by  $g$ . Thus,  $f \circ g\gamma = f \circ \lambda_g \circ \gamma$ , and with  $h = f \circ \lambda_g$ ,

$$(h \circ \gamma)'(0) = (h \circ g^{-1}c)'(0) + (h \circ g^{-1}d)'(0) = b([f], [c]) + b([f], [d]).$$

This shows  $[g\gamma] = [c] + [d]$ . If  $g^{-1}c(s)g^{-1}d(s)$  represents the constant curve, one can easily solve for  $d(s)$  to get  $gc(s)^{-1}g$ , which consequently represents  $-[c]$ .

The last statement of the lemma is a consequence of the fact that  $T_e m([c], [d])$  and  $T_e i([c])$  are represented by  $c(s)d(s)$  and  $c(s)^{-1}$ , respectively. But we have just seen that these curves represent  $[c] + [d]$  and  $-[c]$ .  $\square$

Our next goal is to show that scalar multiplication and addition of tangent vectors are smooth maps.

**Definition 3.40.** *Let  $X$  be a Frölicher space and let*

$$m : \mathbb{R} \times TX \rightarrow TX, \quad (r, v) \mapsto rv$$

*be scalar multiplication on  $TX$ . If  $X$  is of  $L$ -type, let*

$$a : \text{Dom}(a) \rightarrow TX, \quad (v, w) \mapsto v + w$$

*be the vector addition, defined on  $\text{Dom}(a) = \{(v, w) \in TX \times TX \mid \pi(v) = \pi(w)\}$ .*

**Corollary 3.41.** *Scalar multiplication is smooth, and if  $X$  is of  $L$ -type and we equip  $\text{Dom}(a)$  with its subspace structure, then vector addition is smooth.*

*Proof.* Since  $TX$  carries the final structure with respect to the maps  $Tf$ , we compose  $m$  and  $a$  with  $Tf$ . By Lemma 3.30,  $Tf$  is multiplicative and additive whenever the sum of two vectors is defined. This yields  $Tf \circ a = a \circ (Tf \times Tf)|_{\text{Dom}(a)}$ , where the  $a$  on the right side denotes vector addition in  $T\mathbb{R}$ . Similarly,  $Tf \circ m = m \circ (\text{id}_{\mathbb{R}}, Tf)$  where the  $m$  on the right hand side is scalar multiplication in  $T\mathbb{R}$ . Since in  $T\mathbb{R}$ , scalar multiplication and vector addition are smooth, this shows that the corresponding maps in  $TX$  are also smooth.  $\square$

### 3.1.6 Examples

In this subsection we give two examples of tangent spaces. First, we consider the coordinate cross as a subspace of  $\mathbb{R}^2$ , and compute its tangent space at the point where the axes intersect. Then we consider tangent spaces of locally convex spaces.

Coordinate Cross

Let  $X = \{(x, 0), (0, y) \mid x, y \in \mathbb{R}\} \subset \mathbb{R}^2$ , equipped with the subspace Frölicher structure  $(C, F)$ . Then  $C$  consists of smooth maps  $c : \mathbb{R} \rightarrow \mathbb{R}^2$  whose image lies in  $X$ .

**Definition 3.42.** A function  $f \in C^\infty(\mathbb{R}, \mathbb{R})$  is said to be  $n$ -flat at  $x \in \mathbb{R}$  if the first  $n$  derivatives of  $f$  at  $x$  vanish. It is said to be  $\infty$ -flat at  $x$  if it is  $n$ -flat at  $x$  for all  $n$ .

For example,

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is  $\infty$ -flat at 0.

**Lemma 3.43.** Let  $(X, C, F)$  be the coordinate cross in  $\mathbb{R}^2$ , equipped with the subspace structure. Then the curves and functions can be described as follows.

- i) If  $c = (c_1, c_2) \in C$  and  $c(x) = (0, 0)$ , then at least one of  $c_1, c_2$  is  $\infty$ -flat at  $x$ .
- ii)  $F$  can be identified with pairs of smooth functions  $(f, g)$  for which  $f(0) = g(0)$ .

*Proof.* Suppose that  $c(x) = (0, 0)$  and  $c_1$  is not  $\infty$ -flat at  $x$ . Let  $n$  be the smallest number such that  $c_1^{(n)}(x) \neq 0$ . Then we can use Taylor approximation to write  $c_1(y) = \alpha(y-x)^n + \varphi(y)$ , where  $\alpha = c_1^{(n)}(x)$  and  $\varphi$  vanishes at  $x$  of order  $n+1$ . We will show that  $x$  is an isolated zero of  $c_1$ , which implies that  $c_2$  has to be  $\infty$ -flat at  $x$ . We can write

$$c_1(y) = (y-x)^n \left( \alpha + (y-x) \frac{\varphi(y)}{(y-x)^{n+1}} \right),$$

and since  $\alpha \neq 0$  and  $\frac{\varphi(y)}{(y-x)^{n+1}}$  is bounded, this shows that for  $y$  close enough to  $x$  we have  $c_1(y) \neq 0$ . But whenever  $c_1(y) \neq 0$ , we have  $c_2(y) = 0$  since  $c(y) \in X$ . This shows that  $c_2$  is constant 0 in a small neighborhood around  $x$ , and hence  $\infty$ -flat at  $x$ .

Now let us turn to statement ii). If  $\varphi : X \rightarrow \mathbb{R}$  is a smooth function, then its restriction to the axes has to be smooth, since the restrictions can be described as



composition of  $\varphi$  with the smooth curves  $x \mapsto (x, 0)$  and  $x \mapsto (0, x)$ , respectively. Now we need to show that conversely, if  $f, g \in C^\infty(\mathbb{R}, \mathbb{R})$  are given smooth functions with  $f(0) = g(0)$ , then the corresponding map  $\varphi$  on  $X$  defined by  $\varphi(x, 0) = f(x)$  and  $\varphi(0, x) = g(x)$ , is smooth. First, consider the map  $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$  given by  $\psi(x, y) = f(x) + g(y) - f(0)$ . This map is certainly smooth, since  $f, g$  and addition in  $\mathbb{R}^2$  are smooth. But the restriction of  $\psi$  to  $X$  is exactly  $\varphi$ , which shows that  $\varphi$  is smooth.  $\square$

From the proof we get the following corollary.

**Corollary 3.44.** *If  $c = (c_1, c_2)$  is a curve into  $X$  with  $c(0) = (0, 0)$ , and  $\varphi$  is a smooth function on  $X$  whose restriction to the axes is given by  $f_1$  and  $f_2$  respectively, then  $(\varphi \circ c)'(0) = (f_1 \circ c_1)'(0) + (f_2 \circ c_2)'(0)$ , and at least one of the summands is zero.*

*Proof.* It follows from the proof of above lemma that, if  $c_i$  is not flat, then  $\varphi \circ c$  is given by  $f_i \circ c_i$  in a small neighborhood of 0. Hence in that case,  $(\varphi \circ c)'(0) = (f_i \circ c_i)'(0)$ . Now suppose that both  $c_1$  and  $c_2$  are  $\infty$ -flat. Then both  $f_i \circ c_i$  for  $i = 1, 2$  are  $\infty$ -flat by the chain rule. Therefore  $\frac{\varphi(c(x))}{|x|^n}$  is bounded for all  $n$ , which implies that  $(\varphi \circ c)'(0) = 0$ .  $\square$

This corollary shows that for  $v = [c] \in T_0X$ , the vector  $(c'_1(0), c'_2(0))$  lies in  $X$ . It is independent of the representative  $c$ , since if  $[c] = [d]$ , we can use the smooth functions on  $X$  given by  $(0, \text{id})$  and  $(\text{id}, 0)$  to deduce that  $c'_i(0) = d'_i(0)$ .

**Definition 3.45.** *Let  $\iota : X \rightarrow \mathbb{R}^2$  be the inclusion, and  $\pi_i : \mathbb{R}^2 \rightarrow \mathbb{R}$  the projections onto the axes. Then we define a smooth map  $F : T_0X \rightarrow X$  by*

$$F([c]) = (T_0(\pi_1 \circ \iota)([c]), T_0(\pi_2 \circ \iota)([c])) = (c'_1(0), c'_2(0)).$$

Our goal now is to show that  $F$  is in fact a diffeomorphism. Since the components of  $F$  are differentials, they are additive whenever the sum of two elements  $v, w \in T_0X$  is defined.

**Lemma 3.46.** *The sum of  $v, w \in T_0X$  is defined if and only if  $F(v)$  and  $F(w)$  lie in the same axis.*

*Proof.* It is clearly necessary that  $F(v)$  and  $F(w)$  lie in the same axis, because otherwise  $F(v) + F(w) = F(v + w)$  does not lie in  $X$ . Assume now that both  $F(v) = (c'_1(0), c'_2(0))$  and  $F(w) = (d'_1(0), d'_2(0))$  lie in the same axis, say the first, and also assume that both  $v, w$  are non-zero. Then both  $c'_1(0)$  and  $d'_1(0)$  are nonzero, and there is a neighborhood  $U$  of 0 in  $\mathbb{R}$  on which  $c_2$  and  $d_2$  vanish. Let  $V \subset U$  be a smaller neighborhood of 0, and let  $\rho$  be a function with support in  $U$ , which is constant 1 on  $V$ . Consider the curve  $\rho(c_1 + d_1, c_2)$ . By construction of  $\rho$ , this is a curve into  $X$  which represents  $v + w$ .  $\square$

**Theorem 3.47.** *The function  $F : T_0X \rightarrow X$  defines a diffeomorphism of Frölicher spaces which respects addition and scalar multiplication.*

*Proof.* It only remains to show that the inverse of  $F$  is smooth. If  $(a, b) \in X$ , then  $F^{-1}(a, b)$  can be represented by the curve  $t \mapsto (ta, tb)$ . To check that  $F^{-1}$  is smooth, let  $c = (c_1, c_2)$  be a curve into  $X$ , so that  $F^{-1}(c(s)) = [t \mapsto (tc_1(s), tc_2(s))]$ . This is a curve into  $T_0X$  if  $\gamma(s, t) = (tc_1(s), tc_2(s))$  satisfies i)-iii) in Remark 3.22. Conditions i) and ii) are clearly satisfied. Let  $\varphi$  be a smooth function on  $X$ , given by  $(f, g)$ . By Corollary 3.44 and the chain rule,

$$\frac{\partial}{\partial t} \Big|_{t=0} \varphi(\gamma(s, t)) = f'(0)c_1(s) + g'(0)c_2(s)$$

which is a smooth function in  $s$ . Therefore, iii) holds and  $F^{-1}$  is smooth.  $\square$

**Lemma 3.48.** *There is an isomorphism of vector spaces  $T^0X \rightarrow \mathbb{R}^2$  given by  $[(f_1, f_2)] \mapsto (f_1'(0), f_2'(0))$ , where we identify smooth functions on  $X$  with pairs  $(f, g)$  as described above.*

*Proof.* If  $\varphi : X \rightarrow \mathbb{R}$  is given by a pair  $(f_1, f_2)$ , we map  $[\varphi] \in T^0X$  to  $(f_1'(0), f_2'(0))$ . This map is linear and surjective, and we have to show that it is well-defined and injective. Using  $\iota_1(x) = (x, 0)$  and  $\iota_2(x) = (0, x)$ , it is immediate that if  $(f_1, f_2)$  and  $(g_1, g_2)$  represent the same element of  $T^0X$ , then  $f_i'(0) = g_i'(0)$  for  $i = 1, 2$ . This shows well-definedness. Now suppose that  $[(f_1, f_2)] \neq [(g_1, g_2)]$ . Then there is a curve  $c$  through 0 and an index  $i \in 1, 2$  such that  $(f_i \circ c_i)'(0) \neq (g_i \circ c_i)'(0)$ . Using the chain rule and  $c(0) = 0$  we get  $c_i'(0)f_i'(0) \neq c_i'(0)g_i'(0)$ , which is only possible if  $c_i'(0)$  is non-zero, in which case we can divide and get  $f_i'(0) \neq g_i'(0)$ , proving injectivity.  $\square$

## Vector Spaces

In classical differential geometry, the tangent space to a manifold is naturally isomorphic to the vector space on which the manifold is modeled. In particular, the tangent space to any point of a vector space is isomorphic to that vector space. Here we ask under which circumstances a similar result holds for Frölicher tangent spaces to Frölicher vector spaces.

**Definition 3.49.** *If  $V$  is a Frölicher vector space and  $v \in V$ , the straight line  $t \mapsto tv$  defines a smooth curve  $c_v$  through 0. We define a map  $\Xi : V \rightarrow T_0V$  by  $\Xi(v) = [c_v]$ .*

**Lemma 3.50.** *For a Frölicher vector space  $V$ , the map  $\Xi : V \rightarrow T_0V$  is smooth. If  $\Xi$  is injective and the Frölicher structure on  $V$  is generated by the smooth linear functionals on  $V$ , then  $\Xi$  is a diffeomorphism onto its image.*

*Proof.* Let  $c$  be a curve in  $V$ . Then  $(\Xi \circ c)(s) = [t \mapsto tc(s)]$ , and  $\gamma(t, s) = tc(s)$  is smooth in  $s$  and  $t$  since  $V$  is a Frölicher vector space. Now let  $f$  be a smooth function on  $V$ , and compute

$$Tf \circ \Xi \circ c(s) = (f(0), \partial_1 \gamma(0, s))$$

which is smooth in  $s$ . Hence  $\Xi$  is smooth.

Now suppose that  $\Xi$  is injective and that the smooth linear functionals generate the Frölicher structure on  $V$ . Take a curve  $c : \mathbb{R} \rightarrow T_0V$  whose image lies in  $\Xi(V)$ . Then each vector  $c(s)$  can be represented by a curve  $t \mapsto tv_s$  for some  $v_s \in V$ . We have  $\Xi^{-1}(c(s)) = v_s$ , so we need to show that  $s \mapsto v_s$  is smooth. To this end, note that  $c$  yields an element of  $T^2V$ , which is represented by  $\gamma(s, t) = tv_s$ . Now we use iii) in Remark 3.22 with a smooth linear functional  $f = \Lambda$ . We get that  $(\Lambda \circ \gamma)(s, t) = t\Lambda(v_s)$ , and hence  $\partial_2(\Lambda \circ \gamma)(s, 0) = v_s$ , is smooth in  $s$  for every such  $\Lambda$ . Since by assumption the smooth linear functionals generate the structure, the map  $s \mapsto v_s$  is smooth.  $\square$

**Corollary 3.51.** *If  $V$  is a locally convex space, and we equip  $V$  with the Frölicher structure generated by  $V'$ , then  $\Xi : V \rightarrow T_0V$  is injective and a diffeomorphism onto its image.*

*Proof.* By definition, the smooth linear functionals contain  $V'$ , and they generate the Frölicher structure. If  $c_v(t) = tv$  with  $v \neq 0$ , there is a functional  $\Lambda$  with  $(\Lambda \circ c_v)(t) = t\Lambda(v) \neq 0$ , which implies  $[c_v] \neq 0$  and  $\Xi$  is injective. Now it follows from Lemma 3.50 that  $\Xi$  is a diffeomorphism onto its image.  $\square$

Now let us describe a class of locally convex spaces for which  $T_0V \cong V$ . This condition is of importance in the Lie theory we develop in Section 3.3.

**Lemma 3.52.** *Let  $V$  be a Mackey complete locally convex space. Then the Frölicher structure on  $V$  generated by the continuous dual  $V'$  has  $C^\infty(\mathbb{R}, V)$  as set of curves. If furthermore the topology of  $V$  is the initial topology with respect to  $C^\infty(\mathbb{R}, V)$ , then the set of functions is  $C^\infty(V, \mathbb{R})$ , and  $T_0V$  and  $V$  are isomorphic Frölicher vector spaces.*

*Proof.* The first statement follows directly from the characterization of Mackey completeness in Theorem 1.57. Now suppose that the topology of  $V$  is initial with respect to  $C^\infty(\mathbb{R}, V)$ . Let  $F$  be the functions on  $V$ ; it follows from the chain rule that  $C^\infty(V, \mathbb{R}) \subset F$ . We need to show that each  $f \in F$  is smooth in the sense of Definition 1.48. Recall that

$$df(x)(h) = \lim_{t \rightarrow 0} t^{-1}(f(x + th) - f(x)).$$

This is equal to the derivative  $(f \circ c)'(0)$  where  $c(t) = x + th$  is a smooth curve in  $V$ . This proves existence of  $df$ . It is easy to check that  $df : V \times V \rightarrow \mathbb{R}$  is smooth in the Frölicher sense. This implies that its composition with all curves into  $V \times V$  is smooth, and hence by assumption on the topology on  $V$ , the map  $df$  is also continuous. Since  $df$  is smooth in the Frölicher sense, the argument can be repeated to show that all  $d^n f$  exist and are continuous, hence  $f \in C^\infty(V, \mathbb{R})$ .

Lastly, to show that  $\Xi$  is an isomorphism, it suffices to show that it is onto. So let  $[c] \in T_0V$ , and let  $v = c'(0)$ . We will show that  $[c] = \Xi(v)$ . Let  $f \in F = C^\infty(V, \mathbb{R})$  and let  $d(t) = tv$  so that  $\Xi(v) = [d]$ . It now follows from the chain rule that  $(f \circ c)'(0) = d_0 f(c'(0)) = d_0 f(v) = (f \circ d)'(0)$ .  $\square$

**Corollary 3.53.** *If  $V$  is a Fréchet space, then  $(V, C^\infty(\mathbb{R}, V), C^\infty(V, \mathbb{R}))$  is a Frölicher vector space and  $T_0V \cong V$ .*

*Proof.* By [KM97] Theorem I.4.11, the topology on a Fréchet space  $V$  is equal to the initial topology with respect to the elements of  $C^\infty(\mathbb{R}, V)$ . Hence Lemma 3.52 applies.  $\square$

### 3.1.7 Extension of the Classical Tangent Functor

As we have seen in Lemmas 2.27 and 2.68, there are full and faithful functors from the category **Mfd** of manifolds into the categories  $\mathcal{D}$  and  $\mathcal{F}$ . The goal of the present subsection is to show that the diffeological and Frölicher tangent functors, when restricted to the subcategory **Mfd**, coincide with the classical tangent functor. This statement for the diffeological tangent functor was proven by Hector [Hec95, Ch.4.1].

#### Diffeological Tangent Functor

Until we prove equality, let us write  $\tilde{T}$  for the diffeological tangent functor and  $\tilde{d}$  for the differential of plots from Definition 3.4. We have to show the following:

- The vector spaces  $\tilde{T}_x M$  and  $T_x M$  are isomorphic. This implies that the underlying sets of the tangent bundles are equal.
- The diffeology  $\mathcal{P}(\tilde{T}M)$  defined in 3.5 is the same as the manifold diffeology consisting of all smooth maps into  $TM$ .
- $\tilde{T}_x f = T_x f$  for smooth maps  $f \in C^\infty(M, N) = \text{Hom}_{\mathcal{D}}(M, N)$ .

It then follows that  $\tilde{T}f$  is smooth in the classical sense, because

$$C^\infty(TM, TN) = \text{Hom}_{\mathcal{D}}(\tilde{T}M, \tilde{T}N).$$

We start by proving the first point:

**Theorem 3.54.** *Let  $M$  be a smooth manifold. For every point  $x$  on  $M$ , the vector spaces  $\tilde{T}_x M$  and  $T_x M$  are isomorphic.*

*Proof.* For each plot  $\alpha$  centered at  $x$  let

$$k_\alpha := d_0\alpha : E_\alpha \rightarrow T_x M,$$

where  $d_0$  denotes the classical differential. If  $\beta = \alpha \circ h$  it follows from the classical chain rule that the triangle (3.1) commutes. So by Lemma 3.2 there is a unique linear map  $k : \tilde{T}_x M \rightarrow T_x M$  such that  $d_0\alpha = k \circ j_\alpha$  for each  $\alpha \in \mathcal{P}_x(M)$ . We claim that  $k$  is an isomorphism. Let  $(\psi, U)$  be a chart of  $M$  about  $x$  such that  $\psi(x) = 0$ . Then  $\varphi := \psi^{-1}$  is a plot centered at  $x$ , and  $d_0\varphi$  is a bijection. As  $d_0\varphi = k \circ j_\varphi$ , the map  $k$  is necessarily surjective. It remains to show that  $k$  is injective. In order to make the notation more readable we will identify  $v_\alpha \in E_\alpha$  with its image under  $\iota_\alpha$  in  $E_x$ , and we will write  $[x]$  for the class of  $x \in E_x$  in  $T_x X$ . So suppose

$$\begin{aligned} 0 = k \left( \left[ \sum_{\alpha \in \mathcal{P}_x} c_\alpha v_\alpha \right] \right) &= \sum_{\alpha} c_\alpha (k \circ j_\alpha)(v_\alpha) \\ &= \sum_{\alpha} c_\alpha d_0\alpha(v_\alpha) \end{aligned}$$

for some element  $\sum_{\alpha} c_\alpha v_\alpha \in E_x$ , where  $v_\alpha \in E_\alpha$ . Choose a chart  $\psi$  as above, such that its inverse  $\varphi$  is a plot centered at  $x$ . Then

$$d_x\psi : T_x M \rightarrow E_\varphi$$

is an isomorphism, and therefore

$$\begin{aligned} 0 &= d_x\psi \left( \sum_{\alpha} c_\alpha d_0\alpha(v_\alpha) \right) \\ &= \sum_{\alpha} c_\alpha d_0(\psi \circ \alpha)(v_\alpha). \end{aligned}$$

The vector  $v_\alpha - d_0(\psi \circ \alpha)(v_\alpha)$  lies in  $\hat{E}_x$  as  $\alpha = \varphi \circ (\psi \circ \alpha)$ . So

$$\begin{aligned} \sum_{\alpha} c_\alpha v_\alpha &= \sum_{\alpha} c_\alpha v_\alpha - \sum_{\alpha} c_\alpha d_0(\psi \circ \alpha)(v_\alpha) \\ &= \sum_{\alpha} c_\alpha (v_\alpha - d_0(\psi \circ \alpha)(v_\alpha)) \end{aligned}$$

represents the zero vector in  $E_x/\hat{E}_x$ , and therefore  $k$  is injective.  $\square$

To show the second point, we will first prove that the differential of plots as defined in 3.4 agrees with the classical differential.

**Corollary 3.55.** *For a plot  $\alpha$ , the maps  $\tilde{d}\alpha$  and  $d\alpha$  agree.*

*Proof.* We use the map  $k$  from Theorem 3.54 above to identify diffeological and classical tangent space. So the equality

$$k \circ j_\alpha = d_0\alpha,$$

shows that  $j_\alpha = \tilde{d}_0\alpha$  is to be identified with  $d_0\alpha$ . Now for  $u \neq 0$  the map  $\tilde{d}_u\alpha$  is defined as

$$j_\beta \circ d_u h$$

where  $\beta = \alpha \circ h^{-1}$ . Using  $j_\beta = d_0\beta$  we get

$$\tilde{d}_u\alpha = d_0\beta \circ d_u h = d_u(\beta \circ h) = d_u\alpha.$$

□

**Corollary 3.56.** *The manifold diffeology on  $TM$  and the tangent diffeology from Definition 3.5 are equal.*

*Proof.* Recall from Definition 3.5 that the tangent diffeology on  $TM$  is generated by the differentials  $d\alpha$  of charts, which are smooth maps by Corollary 3.55. So the tangent diffeology is contained in the manifold diffeology, which consists of *all* smooth maps into  $TM$ .

On the other hand, by Lemma 2.29 the manifold diffeology is generated by  $\mathcal{A}^{-1}$  where  $\mathcal{A}$  is an atlas for the manifold. The tangent bundle has an atlas given by

$$T\mathcal{A} := \{d\varphi \mid \varphi \in \mathcal{A}\},$$



so the inverse charts are of the form  $d(\varphi^{-1})$ . Now the smooth maps  $\varphi^{-1}$  are plots for  $M$ . Therefore the maps  $d(\varphi^{-1})$  are contained in the generating set for the tangent diffeology. Hence the manifold diffeology is contained in the tangent diffeology. This implies equality of the two diffeologies.  $\square$

**Corollary 3.57.** *Given a smooth map  $f : M \rightarrow N$  between smooth manifolds  $M$  and  $N$ , the differentials  $\tilde{T}_x f$  and  $T_x f$  agree.*

*Proof.* We have established the equality  $j_\alpha = d_0 \alpha$  for plots centered at  $x$ . Thus  $j_{f \circ \alpha} = d_0(f \circ \alpha) = T_x f \circ d_0 \alpha$  by the classical chain rule. But by construction, the diffeological differential  $\tilde{T}_x f$  is the unique linear map satisfying this equality for all plots  $\alpha \in \mathcal{P}_x X$ . Thus  $\tilde{T}_x f = T_x f$ .  $\square$

Frölicher Tangent Functor

The Frölicher tangent functor also extends the classical tangent functor. In order to prove this, let us denote the Frölicher tangent functor by  $\tilde{T}$  and the classical one by  $T$ . Furthermore, fix a smooth  $n$ -dimensional manifold  $M$  with charts  $(U_i, \varphi_i)$  indexed by a set  $I$ . Then  $TM$  can be described as set of equivalence classes  $[x, i, a]$  where  $x \in M, i \in I$  and  $a \in \mathbb{R}^n$ . Triples  $(x, i, a)$  and  $(y, j, b)$  are equivalent if  $x = y$  and  $d_{\varphi_i(x)}(\varphi_j \circ \varphi_i^{-1})(a) = b$ . The manifold structure of  $TM$  is given by the charts  $T\varphi_i : TU_i \rightarrow U_i \times \mathbb{R}^n, [x, i, a] \mapsto (\varphi_i(x), a)$ .

We define a map  $\Phi : \tilde{T}M \rightarrow TM$  as follows. If  $[c] \in \tilde{T}M$ , let  $x = c(0)$  and choose  $i \in I$  such that  $x \in U_i$ . Now let  $a = (\varphi_i \circ c)'(0)$  and set  $\Phi([c]) = [x, i, a]$ .

**Theorem 3.58.** *The map  $\Phi : \tilde{T}M \rightarrow TM$  is well defined and a diffeomorphism of Frölicher spaces.*

*Proof.* Suppose that  $[c] = [d]$ . Then  $c(0) = d(0)$ , and we can pick  $i \in I$  such that  $\Phi([c]) = [x, i, a]$  and  $\Phi([d]) = [x, i, b]$ . It remains to show that  $a = b$ . We can use a

smooth bump function  $\rho$  which is supported in  $U_i$  and is constant 1 on a smaller neighborhood of  $x$ , to define  $f = \rho\varphi_i$ . Since  $[c] = [d]$ , we get  $a = (f \circ c)'(0) = (f \circ d)'(0) = b$ . This proves well-definedness of  $\Phi$ .

Next we check smoothness of  $\Phi$ . Let  $c : \mathbb{R} \rightarrow \tilde{T}M$  be a curve. Then  $\pi \circ c : \mathbb{R} \rightarrow M$  is smooth, and we can choose a neighborhood  $U$  of  $0 \in \mathbb{R}$  which is mapped into a chart  $U_i$ . Compose  $\Phi \circ c : U \rightarrow TM$  with  $T\varphi_i$  to get the map

$$s \mapsto (\varphi_i \circ \pi \circ c)(s), (\varphi_i \circ c_s)'(0),$$

where  $c_s$  represents the vector  $c(s) \in \tilde{T}M$ . This map is smooth by Remark 3.22.

Let us construct an inverse map to  $\Phi$ . If  $U_i \times \mathbb{R}^n$  is a chart for  $TM$ , we map  $(x, a) \in U_i \times \mathbb{R}^n$  to the vector in  $\tilde{T}M$  represented by the curve  $[t \mapsto \varphi_i^{-1}(x + ta)]$ . This map is easily seen to be inverse to  $\Phi$ . To see that it is smooth, recall that  $\tilde{T}M$  carries the initial Frölicher structure with respect to the maps  $Tf$  for all smooth functions  $f$  on  $M$ . If  $v \in T_x X$ , we can represent  $v$  by a curve  $t \mapsto \gamma(t)$  with  $\gamma(0) = x$ . By construction,  $\Phi^{-1}(v) = [c]$ , and  $(Tf \circ \Phi^{-1})(v) = d_x f(v)$  is the classical differential. But  $df$  is smooth, which shows that  $\Phi^{-1}$  is smooth.  $\square$

## 3.2 Vector Fields and Differential Forms

Vector fields on a diffeological or Frölicher space  $X$  can be simply defined as sections of  $TX \rightarrow X$ . Differential forms for diffeological spaces were defined by Souriau in [Sou85], the same definition works also for Frölicher spaces.

### 3.2.1 Vector Fields and Derivations

**Definition 3.59.** *Let  $X$  be a Frölicher space with tangent bundle  $\pi : TX \rightarrow X$ . Then a smooth vector field on  $X$  is a smooth map  $\xi : X \rightarrow TX$  such that  $\pi \circ \xi = \text{id}_X$ . We denote the set of vector fields on  $X$  by  $\mathcal{V}(X)$ , it carries the subspace*

Frölicher structure induced from  $\text{Hom}_{\mathcal{F}}(X, TX)$ . If all tangent spaces  $T_x X$  are vector spaces, then  $\mathcal{V}(X)$  is a vector space under pointwise addition and scalar multiplication:  $(\xi + \alpha\eta)(x) = \xi(x) + \alpha\eta(x)$  for  $\xi, \eta \in \mathcal{V}(X)$  and  $\alpha \in \mathbb{R}$ .

**Definition 3.60.** Let  $(X, C, F)$  be a Frölicher space. Note that  $F$  is an algebra under pointwise multiplication of functions. A smooth derivation of  $F$  is a smooth linear map  $D : F \rightarrow F$  such that

$$D(fg) = fD(g) + gD(f).$$

We let  $\text{Der}(F)$  be the set of smooth derivations of  $F$ .

**Remark 3.61.** We can add and multiply by scalars in  $\text{Der}(F)$ , since  $F$  is a Frölicher vector space. Thus  $\text{Der}(F)$  is a vector space.

We make  $\text{Der}(F)$  into a Lie algebra if we define  $[D, H] = D \circ H - H \circ D$  for derivations  $D$  and  $H$ . Bilinearity is clear, and the Jacobi identity is easy to verify, so it suffices to show that  $[D, H]$  is again a derivation:

$$\begin{aligned} (D \circ H)(fg) &= D(fH(g)) + gH(f) \\ &= f(D \circ H)(g) + D(f)H(g) + g(D \circ H)(f) + D(g)H(f). \end{aligned}$$

Now subtract  $(H \circ D)(fg)(x)$  from this expression. The second and fourth terms cancel out, and one sees that  $[D, H]$  is indeed a derivation.

**Remark 3.62.** We now show that each vector field on  $X$  gives rise to a derivation of  $F_X$ . As is common, if  $\xi : X \rightarrow TX$  is a vector field, we will denote the corresponding derivation by the same symbol  $\xi : F \rightarrow F$ .

**Lemma 3.63.** Let  $(X, C, F)$  be a Frölicher space and  $\xi \in \mathcal{V}(X)$ . Let  $\xi(x)$  be represented by a curve  $\xi_x \in C$ . For  $f \in F$  we define a function  $\xi(f) : X \rightarrow \mathbb{R}$  by

$$\xi(f)(x) = (f \circ \xi_x)'(0).$$

Then  $\xi(f) \in F$  and the induced map  $\xi : F \rightarrow F$  is a smooth derivation.

*Proof.* Let  $c \in C$ . Then  $s \mapsto [\xi_{c(s)}] = \xi(c(s))$  is a smooth curve in  $TX$ , hence by Remark 3.22 the expression

$$(\xi(f) \circ c)(s) = (f \circ \xi_{c(s)})'(0)$$

is smooth in  $s$ . This shows that  $\xi(f) \in F$ . Now let us show that  $\xi : F \rightarrow F$  is smooth. Let  $c : \mathbb{R} \rightarrow F$  be a curve. Then

$$\begin{aligned} \widetilde{\xi \circ c} : \mathbb{R} \times X &\rightarrow \mathbb{R} \\ (s, x) &\mapsto \xi(c(s))(x) \\ &= (c(s) \circ \xi_x)'(0). \end{aligned}$$

If we let  $\gamma(s, t, x) = \tilde{c}(s, \xi_x(t))$ , then the last expression can be written as  $\partial_2 \gamma(s, 0, x)$ . By assumption,  $\tilde{c}$  is smooth on  $\mathbb{R} \times X$ , and we can use Remark 3.22 to conclude that  $\partial_2 \gamma(s, 0, x)$  is smooth on  $\mathbb{R} \times X$ . This shows that  $\xi : F \rightarrow F$  is smooth. It is clear that  $\xi$  is linear. Now  $\xi(fg)(x)$  is the derivative of the function

$$s \mapsto f(\xi_x(s))g(\xi_x(s)),$$

at 0, so the standard product rule yields

$$\begin{aligned} \xi(fg)(x) &= f(\xi_x(0))\xi(g)(x) + g(\xi_x(0))\xi(f)(x) \\ &= f(x)\xi(g)(x) + g(x)\xi(f)(x), \end{aligned}$$

because  $\xi_x(0) = x$ . This shows that  $\xi$  is a smooth derivation.  $\square$

**Lemma 3.64.** *The map  $\mathcal{V}(X) \rightarrow \text{Der}(F_X)$  defined in above lemma is linear and injective.*

*Proof.* Recall that vector fields form a vector space under pointwise addition and scalar multiplication. Let  $\xi, \eta \in \mathcal{V}(X)$  and let  $\alpha \in \mathbb{R}$  be a scalar. For  $x \in X$ , let  $\xi_x$

and  $\eta_x$  be curves representing  $\xi(x)$  and  $\eta(x)$ . Then

$$\begin{aligned}
(\xi + \alpha\eta)(f)(x) &= (f \circ (\xi_x + \alpha\eta_x))'(0) \\
&= b(\xi(x) + \alpha\eta(x), [f]) \\
&= b(\xi(x), [f]) + \alpha b(\eta(x), [f]) \\
&= \xi(f)(x) + \alpha\eta(f)(x)
\end{aligned}$$

using the bilinearity of  $b(\cdot, \cdot)$ . To prove injectivity, suppose that  $\xi(x) \neq \eta(x)$  for some  $x \in X$ . Then the curves  $\xi_x$  and  $\eta_x$  represent different tangent vectors, hence there is a function  $f$  on  $X$  for which  $(f \circ \xi_x)'(0) \neq (f \circ \eta_x)'(0)$ .  $\square$

**Remark 3.65.** Next we compute  $[\xi, \eta]$  for vector fields  $\xi$  and  $\eta$  on a Frölicher space  $X$ . If  $\xi(x) \in T_x X$  is represented by a curve  $\xi_x : \mathbb{R} \rightarrow X$ , we let  $\tilde{\xi}(s, x) := \xi_x(s)$ . Similarly we define  $\tilde{\eta}(s, x)$ . Then  $\eta(f)(x) = (f \circ \tilde{\eta}(\cdot, x))'(0)$ . Hence we can compute  $\xi \circ \eta$  as follows.

$$\begin{aligned}
(\xi \circ \eta)(f)(x) &= \frac{d}{ds}(\eta(f) \circ \tilde{\xi}(s, x))(0) \\
&= \frac{\partial^2}{\partial s \partial t} [(f \circ \tilde{\eta})(t, \tilde{\xi}(s, x))] \Big|_{(0,0)}.
\end{aligned}$$

From this we subtract  $(\eta \circ \xi)(f)(x)$  in order to obtain

$$[\xi, \eta](f)(x) = \frac{\partial^2}{\partial s \partial t} [(f \circ \tilde{\eta})(t, \tilde{\xi}(s, x)) - (f \circ \tilde{\xi})(t, \tilde{\eta}(s, x))] \Big|_{(0,0)}.$$

### 3.2.2 Differential Forms

Recall that in classical differential geometry, a differential  $p$ -form  $\omega$  on a manifold  $M$  is given by a smooth section of the  $p$ -th exterior product of the cotangent bundle  $T^*M$ . In other words, to each point  $x$  on  $M$ , a differential form assigns an alternating  $p$ -linear map

$$\omega_x : T_x M \times \cdots \times T_x M \rightarrow \mathbb{R},$$

such that  $\omega_x$  depends smoothly on  $x$  in an appropriate sense. If  $h : N \rightarrow M$  is a smooth map between manifolds and  $\omega$  is a  $p$ -form on  $M$ , then the pull-back  $h^*\omega$  is defined as  $(h^*\omega)_x(v_1, \dots, v_p) = \omega_{h(x)}(d_x h(v_1), \dots, d_x h(v_p))$ , where the  $v_i$  are in  $T_x N$ .

The definition of differential form for diffeological spaces was given by Souriau in [Sou85]. His definition also works for Frölicher spaces.

**Definition 3.66.** *Let  $X$  be a diffeological or a Frölicher space. Let  $\mathcal{Q}$  denote the collection of all open subsets of all  $\mathbb{R}^n$ . Then each  $U \in \mathcal{Q}$  carries the manifold diffeological or Frölicher structure and one can speak of smooth maps  $U \rightarrow X$ . A differential form  $\omega$  of degree  $p$  on  $X$  is an assignment to every  $U \in \mathcal{Q}$  and every smooth  $\alpha : U \rightarrow X$  of a classical  $p$  form  $\omega(\alpha) \in \Omega^p(U)$ .*

*The following compatibility condition is required to hold: If  $h : V \rightarrow U$  and  $\alpha : U \rightarrow X$  are smooth, then  $\omega(\alpha \circ h) = h^*\omega(\alpha)$ .*

In [Lau06] we show that for manifolds, the differential forms just defined can be identified with classical differential forms.

**Remark 3.67.** *The de Rham differential  $d : \Omega^p(M) \rightarrow \Omega^{p+1}(M)$  for manifolds satisfies  $d \circ \varphi^* = \varphi^* \circ d$  whenever  $\varphi : M \rightarrow N$  is a smooth map. This makes it possible to define the de Rham differential for diffeological and Frölicher spaces. Also,  $d \circ d = 0$  remains true for this extension, so that there is a de Rham cohomology theory for  $\mathcal{D}$  and  $\mathcal{F}$ .*

**Remark 3.68.** *One can also define the value  $\omega_x$  of a differential form at a point  $x$ . Given differential forms  $\omega$  and  $\eta$  and a point  $x \in X$ , we say that  $\omega$  and  $\eta$  take the same value at  $x$  and write  $\omega \sim_x \eta$  if for all smooth  $\alpha : U \rightarrow X$  we have  $\omega(\alpha)_0 = \eta(\alpha)_0$ . The equivalence class of a form  $\omega$  under  $\sim_x$  is the value of  $\omega$  at  $x$ , denoted by  $\omega_x$ .*

### 3.3 Frölicher Groups and Their Lie Algebra

The main goal of this final section is to define a Lie bracket on  $\mathfrak{g} = T_e G$ , where  $G$  is a Frölicher group. We can achieve this under the hypothesis that  $T_0 \mathfrak{g} \cong \mathfrak{g}$  as vector spaces. Our approach is to take the second derivative of the commutator map of  $G$ , which has been done by Bertram [Ber06] in a different context.

#### 3.3.1 Semidirect Product Groups

**Definition 3.69.** *If  $G$  is a Frölicher group, let  $\text{Aut}(G)$  be the group of automorphisms of  $G$  which are smooth with smooth inverse. We equip  $\text{Aut}(G) \subset \text{Hom}_{\mathcal{F}}(G, G)$  with the subset structure.*

**Remark 3.70.** *The group multiplication in  $\text{Aut}(G)$  is given by composition of functions, which is smooth by Corollary 1.34. However we can not show that inversion is smooth in general, so that we do not know whether  $\text{Aut}(G)$  is a Frölicher group.*

**Definition 3.71** (Semidirect Product). *Let  $N$  and  $H$  be Frölicher groups and  $\alpha : H \rightarrow \text{Aut}(N)$  be a smooth homomorphism. Then we define a multiplication on  $H \times N$  via*

$$(h_1, n_1)(h_2, n_2) = (h_1 h_2, \alpha(h_2)^{-1}(n_1) n_2).$$

*This defines a group structure on  $H \times N$  with identity element  $(e, e)$  and inverse  $(h, n)^{-1} = (h^{-1}, \alpha(h)(n^{-1}))$ . This group is called the semidirect product of  $H$  and  $N$  and denoted  $H_\alpha \times N$ . We will simply write  $H \times N$  if the homomorphism  $\alpha$  is understood.*

**Lemma 3.72.** *Let  $N$  and  $H$  be Frölicher groups and  $\alpha : H \rightarrow \text{Aut}(N)$  be a smooth homomorphism. Then  $H \times N$  is a Frölicher group if we equip the underlying set  $H \times N$  with the product Frölicher structure.*

*Proof.* By cartesian closedness, the map  $\tilde{\alpha} : H \times N \rightarrow N$  is smooth, and we can write multiplication and inversion in  $H \times N$  as

$$(h_1, n_1)(h_2, n_2) = (h_1 h_2, \tilde{\alpha}(h_2^{-1}, n_1) n_2)$$

$$(h, n)^{-1} = (h^{-1}, \tilde{\alpha}(h, n^{-1})).$$

This shows that multiplication and inversion are smooth maps. □

### 3.3.2 Group Structure on $TG$

The goal of this subsection is to show that for a Frölicher group  $G$ , the tangent bundle  $TG$  is isomorphic to a semidirect product of Frölicher groups.

**Lemma 3.73.** *If  $G$  is a Frölicher group with multiplication  $m$  and inversion  $i$ , then  $TG$  is a Frölicher group with multiplication  $Tm$ , inversion  $Ti$  and identity element  $0 \in T_e G$ .*

*Proof.* The maps  $Tm$  and  $Ti$  are smooth. Furthermore, we know that  $T(G \times G) = TG \times TG$  and  $(X \times Y) \times Z = X \times (Y \times Z)$ . If  $c(s) = e$  is the constant curve through  $e$  and if  $v = [d]$ , then  $Tm(0, v)$  is represented by  $s \mapsto c(s)d(s) = d(s)$ , hence  $Tm(0, v) = v$ . Similarly,  $Tm(v, 0) = v$ . Now it remains to check the various group axioms. Let us show that  $Tm(v, Ti(v)) = 0$ .

$$\begin{aligned} Tm(v, Ti(v)) &= Tm \circ (\text{id}, Ti)(v) \\ &= T(m \circ (\text{id}, i))(v) \quad (\text{chain rule}) \\ &= 0 \end{aligned}$$

The last equality holds because  $m \circ (\text{id}, i)$  is the constant function with value  $e$ . Similarly, one uses the chain rule and the group axioms of  $G$  to check the remaining group axioms for  $TG$ . □



**Lemma 3.74.** *The vector space  $\mathfrak{g} = T_e G$  and the zero section  $\{0 \in T_g G \mid g \in G\}$  are subgroups of  $TG$ . The zero section isomorphic to  $G$ .*

*Proof.* By Lemma 3.39,  $(\mathfrak{g}, +)$  is a subgroup of  $TG$ . Now let  $z : G \rightarrow TG$  be the zero section, that is,  $z(g) \in T_g G$  is the zero vector for each  $g \in G$ . Let  $\tilde{G} = z(G)$  be the image of the zero section, equipped with the subspace structure. We claim that  $\tilde{G}$  and  $G$  are isomorphic. To show that  $z$  is smooth, let  $f \in F_G$  and compose  $Tf$  with  $z$  to get

$$Tf(z(g)) = Tf(0) = (f(g), 0) \in T\mathbb{R}.$$

Both components are smooth, hence so is  $Tf \circ z$ . So the corestriction  $z : G \rightarrow \tilde{G}$  is smooth, and it remains to show that its inverse, say  $\gamma$ , is smooth. Let  $c : \mathbb{R} \rightarrow \tilde{G}$  be a curve. Then  $c(s) = (c_1(s), [c_s])$ , and  $\gamma \circ c(s) = c_1(s)$  which is a curve in  $G$  by Remark 3.22. This shows  $G \cong \tilde{G}$ .  $\square$

**Definition 3.75.** *The map  $Tm$  can be restricted to maps  $TG \times G \rightarrow TG$  and  $G \times TG \rightarrow TG$ . These maps define smooth left and right actions of  $G$  on  $TG$ , respectively. We denote these actions by  $\lambda_g$  and  $\rho_g$ , respectively. The adjoint action of  $G$  on  $\mathfrak{g}$  is defined as*

$$\text{Ad}(g) : \mathfrak{g} \rightarrow \mathfrak{g}, \quad \text{Ad}(g)(v) = \lambda_g(\rho_{g^{-1}}(v)) = gvg^{-1}$$

where we regard  $G$  and  $\mathfrak{g}$  as subgroups of  $TG$ .

**Lemma 3.76.** *The map  $\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g})$  is a smooth homomorphism.*

*Proof.* The map  $\widetilde{\text{Ad}} : G \times \mathfrak{g} \rightarrow \mathfrak{g}$  is a restriction of the smooth map

$$TG \times TG \rightarrow TG, \quad (g, h) \mapsto ghg^{-1},$$

so by cartesian closedness, the map  $\text{Ad}$  is smooth. Clearly  $\text{Ad}(e)$  is the identity morphism, and

$$\text{Ad}(gh)(v) = (gh)v(gh)^{-1} = g(hvh^{-1})g^{-1} = \text{Ad}(g)(\text{Ad}(h)(v))$$

shows that  $\text{Ad}$  is a group homomorphism.  $\square$

**Corollary 3.77.** *The semidirect product  $G_{\text{Ad}} \ltimes \mathfrak{g}$  is a Frölicher group.*

*Proof.* Combine Lemmas 3.72 and 3.76.  $\square$

**Lemma 3.78.** *The Frölicher groups  $TG$  and  $G \ltimes \mathfrak{g}$  are isomorphic.*

*Proof.* As a set,  $TG$  is the disjoint union of the vector spaces  $T_gG$  for  $g \in G$ . Since the  $T_gG$  are isomorphic to  $\mathfrak{g}$ , we get a bijection

$$\Phi : TG \rightarrow G \times \mathfrak{g}, \quad v \mapsto (\pi(v), \pi(v)^{-1}v).$$

We need to show that  $\Phi$  is a group homomorphism and a diffeomorphism. Let  $v = [c]$  and  $w = [d]$  be vectors in  $TG$ . Then their product is represented by the curve  $s \mapsto c(s)d(s)$ . Clearly,  $\pi(vw) = c(0)d(0) = \pi(v)\pi(w)$ . Let  $g = \pi(v)$  and  $h = \pi(w)$ . In  $TG$  we can compute

$$\begin{aligned} (gh)^{-1}vw &= h^{-1}g^{-1}vw \\ &= h^{-1}g^{-1}vhh^{-1}w \\ &= \text{Ad}(h^{-1})(g^{-1}v) + h^{-1}w, \end{aligned}$$

using that multiplication when restricted to  $\mathfrak{g} \in TG$  is just vector addition. We conclude that  $\Phi(vw) = \Phi(v)\Phi(w)$ . It is clear that  $\Phi$  is smooth since both components are smooth. The inverse is given by  $(g, v) \mapsto gv$ , which is also smooth, hence  $\Phi$  is a diffeomorphism.  $\square$

**Definition 3.79.** *The isomorphism  $\Phi$  in Lemma 3.78 is called the left trivialization of  $TG$ .*

### 3.3.3 Invariant Vector Fields and Derivations

**Definition 3.80.** A vector field  $\xi$  on a Frölicher group  $(G, C, F)$  is called left invariant if  $\xi(g) = \lambda_g(\xi(e))$  for all  $g \in G$ . A derivation  $D$  of  $F$  is called left invariant if  $D(f \circ \lambda_g) = D(f) \circ \lambda_g$  for all  $g \in G$  and  $f \in F$ . We denote the sets of invariant vector fields and derivations by  $\mathcal{V}_l(G)$  and  $\text{Der}_l(F)$ , respectively.

**Remark 3.81.** Let  $\xi$  and  $\eta$  be left invariant derivations of the ring of functions  $F$  of a Frölicher group  $(G, C, F)$ . If

$$\xi(f)(e) = \eta(f)(e)$$

for all  $f \in F$ , then  $\xi = \eta$ .

**Lemma 3.82.** The Frölicher vector spaces  $\mathfrak{g}$  and  $\mathcal{V}_l(G)$  are isomorphic.  $\text{Der}_l(F_G)$  is a Lie algebra and there is a injective linear map  $\mathfrak{g} \cong \mathcal{V}_l \rightarrow \text{Der}_l$ .

*Proof.* Let  $\Phi : \mathfrak{g} \rightarrow \mathcal{V}_l(G) \subset \text{Hom}_{\mathcal{F}}(G, TG)$  be given by  $v \mapsto (g \mapsto gv)$ . Then  $\tilde{\Phi}(g, v) = gv$  is multiplication in  $TG$ , which is smooth, and therefore  $\Phi$  is smooth. The inverse map is evaluation at  $e$ , which is also smooth. Thus,  $\Phi$  is an isomorphism. Next, let  $\xi$  and  $\eta$  be invariant derivations. Then  $\xi(\eta(f \circ \lambda_g)) = \xi(\eta(f)) \circ \lambda_g$ , and it follows that  $[\xi, \eta]$  is also invariant. Hence  $\text{Der}_l(F_G)$  is a Lie algebra. Lastly we need to show that the linear map  $\mathcal{V}(G) \rightarrow \text{Der}(F_G)$  from Lemma 3.64 maps invariant vector fields to invariant derivations. So let  $\xi$  be an invariant vector field. Then

$$\xi(f \circ \lambda_g)(h) = (f \circ \lambda_g \xi_h)'(0) = (f \circ \xi_{gh})'(0) = \xi(f)(gh).$$

□

The main goal of the next subsection is to show that the image of  $\mathfrak{g}$  in  $\text{Der}_l$  is a Lie subalgebra. We will need the following lemma.

**Lemma 3.83.** *Let  $(G, C, F)$  be a Frölicher group. If  $\xi$  and  $\eta$  are the invariant derivations of  $F$  determined by elements  $[c]$  and  $[d]$  of  $\mathfrak{g}$ , then*

$$[\xi, \eta](f)(e) = \frac{\partial^2}{\partial t \partial s} [f(c(t)d(s)) - f(d(t)c(s))] \Big|_{(0,0)}.$$

*Proof.* We first compute  $(\xi \circ \eta)(f)(e)$ . By definition in Lemma 3.63 of the derivation associated to a vector field,  $\xi(\eta(f))(e)$  is given by

$$(\eta(f) \circ \xi_e)'(0).$$

Here  $\xi_e$  is a curve representing the tangent vector  $\xi(e)$ , which is  $[c]$ . Therefore  $(\eta(f) \circ \xi_e)(t) = \eta(f)(c(t))$ , which again by definition equals  $(f \circ \eta_{c(t)})'(0)$ , so that we arrive at

$$\xi(\eta(f))(e) = \frac{\partial}{\partial t \partial s} f(\eta_{c(t)}(s)) \Big|_{(0,0)}.$$

Now  $\eta_{c(t)}$  is a curve representing  $\eta(c(t)) = c(t)[d] \in T_{c(t)}G$ , by definition of the invariant vector field given by the vector  $[d] \in \mathfrak{g}$ . Therefore  $\eta(c(t))$  can be represented by the curve  $s \mapsto c(t)d(s)$ , and finally we have

$$\xi(\eta(f))(e) = \frac{\partial}{\partial t \partial s} f(c(t)d(s)) \Big|_{(0,0)}.$$

Subtracting  $\eta(\xi(f))(e)$  yields the desired result. □

### 3.3.4 Higher Tangent Groups and Lie Bracket

Recall from Definition 3.24 that, if  $X$  is a Frölicher space, then elements of higher tangent bundles  $T^n X$  can be represented by set maps from  $\mathbb{R}^n$  into  $X$ . Let us introduce some more notation. If  $c$  is a curve in  $T^{k-1}X$ , let  $\pi_k(c)$  be the corresponding element of  $T_{c(0)}T^{k-1}X \subset T^k X$ . Define  $T^0 X = X$ . Then  $\pi_1(c) = [c]$ , using our previous notation for the vector represented by  $c$ . Now let us assume that  $v \in T^n X$  is given, and can be written as  $\pi_n(c)$  for some curve into  $T^{n-1}X$ . That means that

each  $c(s)$  can be written as  $\pi_{n-1}(c(s, \cdot))$  where  $t \mapsto c(s, t)$  is a curve in  $T^{n-2}X$ . We can iterate this and write

$$v = \pi_n(\pi_{n-1} \dots (\pi_1(\gamma)) \dots),$$

where  $\gamma : \mathbb{R}^n \rightarrow X$  is the map representing  $v$ . We will also frequently write

$$\pi_2\pi_1\gamma = [s \mapsto [t \mapsto \gamma(s, t)]]$$

for elements of  $T^2X$ .

**Lemma 3.84.** *Let  $\varphi : X \rightarrow Y$  be a smooth map between Frölicher spaces  $X$  and  $Y$ , and let  $v \in T^nX$  be represented by  $\gamma : \mathbb{R}^n \rightarrow X$ . Then*

$$T^{n-1}\varphi \circ \pi_n = \pi_n \circ T^{n-1}\varphi$$

and consequently  $T^n\varphi(v)$  is represented by  $\varphi \circ \gamma : \mathbb{R}^n \rightarrow Y$ .

*Proof.* By definition,  $T\varphi(\pi_1(c)) = T\varphi([c]) = [\varphi \circ c] = \pi_1(\varphi \circ c)$ . The general result follows similarly.  $\square$

**Remark 3.85.** *If  $G$  is a Frölicher group, then so is  $TG$ . This process can be iterated, and we get Frölicher groups  $T^nG$ . In the case of Lie groups, these iterated tangent groups have been described in great detail by Bertram and Didry ([Ber06], [Did06]). We describe here the group  $T^2G$  in some detail, since this group is needed in our definition of the Lie bracket. If we agree to use left trivialization as in Definition 3.79, there are still two different ways to trivialize  $T^2G$ . If for any Frölicher group  $H$  we denote the left trivialization by  $\Phi_H : TH \rightarrow H \times T_eH$ , then we can either use*

$$T\Phi_G : TTG \rightarrow T(G \times T_eG)$$

or

$$\Phi_{TG} : TTG \rightarrow TG \times T_e(TG).$$

Both yield an isomorphism of  $T^2G$  with  $G \times \mathfrak{g} \times \mathfrak{g} \times T_0\mathfrak{g}$ . We discuss these isomorphisms in detail. Throughout, let  $v \in T^2G$  be represented by  $\gamma : \mathbb{R}^2 \rightarrow G$ , that is

$$v = \pi_2\pi_1(\gamma) = [s \mapsto [t \mapsto \gamma(s, t)]],$$

and let  $g = \gamma(0, 0)$ .

**Trivialization using  $T\Phi_G$ .** By definition of differentials and of  $\Phi_G$ , we have

$$\begin{aligned} T\Phi_G(v) &= [s \mapsto \Phi_G([t \mapsto \gamma(s, t)])] \\ &= [s \mapsto (\gamma(s, 0), [t \mapsto \gamma(s, 0)^{-1}\gamma(s, t)])] \\ &\cong ([s \mapsto \gamma(s, 0)], [s \mapsto [t \mapsto \gamma(s, 0)^{-1}\gamma(s, t)]] \in TG \times T\mathfrak{g}. \end{aligned}$$

Now we use  $(\Phi_G, \Phi_{\mathfrak{g}}) : TG \times T\mathfrak{g} \rightarrow G \times \mathfrak{g} \times \mathfrak{g} \times T_0\mathfrak{g}$ . Let us treat the components separately. The first component becomes

$$\Phi_G([s \mapsto \gamma(s, 0)]) = (g, [s \mapsto g^{-1}\gamma(s, 0)]),$$

and the second component becomes

$$([t \mapsto g^{-1}\gamma(0, t)], [s \mapsto [t \mapsto \gamma(0, t)^{-1}g\gamma(s, 0)^{-1}\gamma(s, t)]]).$$

**Trivialization using  $\Phi_{TG}$**  We start with

$$\begin{aligned} \Phi_{TG}(\pi_2(\pi_1(\gamma))) &= ([t \mapsto \gamma(0, t)], Tm(Ti[t \mapsto \gamma(0, t)], [s \mapsto [t \mapsto \gamma(s, t)]]) \\ &= ([t \mapsto \gamma(0, t)], [s \mapsto [t \mapsto \gamma(0, t)^{-1}\gamma(s, t)]]). \end{aligned}$$

The first component is mapped by  $\Phi_G$  to

$$(g, [t \mapsto g^{-1}\gamma(0, t)]).$$

We apply  $T_e\Phi_G$  to the second component and use  $T_e(G \times \mathfrak{g}) \cong T_eG \times T_0\mathfrak{g}$  to get

$$\begin{aligned} & [s \mapsto \Phi_G([t \mapsto \gamma(0, t)^{-1}\gamma(s, t)])] \\ &= [s \mapsto (g^{-1}\gamma(s, 0), [t \mapsto \gamma(s, 0)^{-1}g\gamma(0, t)^{-1}\gamma(s, t)])] \\ &\cong ([s \mapsto g^{-1}\gamma(s, 0)], [s \mapsto [t \mapsto \gamma(s, 0)^{-1}g\gamma(0, t)^{-1}\gamma(s, t)]]). \end{aligned}$$

This shows that the two trivializations are indeed different, for example, the second and third component are interchanged. Let us also describe the inverse mappings of the two trivializations. Throughout the following, let  $(g, [c], [d], \pi_2\pi_1(\rho)) \in G \times \mathfrak{g} \times \mathfrak{g} \times T_0\mathfrak{g}$ .

**First trivialization** If  $(g, [c], [d], \pi_2\pi_1(\rho))$  corresponds to

$$\begin{aligned} & (\gamma(0, 0), [s \mapsto \gamma(0, 0)^{-1}\gamma(s, 0)], [t \mapsto \gamma(0, 0)^{-1}\gamma(0, t)], \\ & [s \mapsto [t \mapsto \gamma(0, t)^{-1}\gamma(0, 0)\gamma(s, 0)^{-1}\gamma(s, t)]]), \end{aligned}$$

then it is easy to solve for  $\gamma$ , and we get

$$\gamma(s, t) = gc(s)d(t)\rho(s, t).$$

**Second trivialization** If  $(g, [c], [d], \pi_2\pi_1(\rho))$  corresponds to

$$\begin{aligned} & (\gamma(0, 0), [t \mapsto \gamma(0, 0)^{-1}\gamma(0, t)], [s \mapsto \gamma(0, 0)^{-1}\gamma(s, 0)], \\ & [s \mapsto [t \mapsto \gamma(s, 0)^{-1}\gamma(0, 0)\gamma(0, t)^{-1}\gamma(s, t)]]), \end{aligned}$$

we solve for  $\gamma$ , and again we get

$$\gamma(s, t) = gc(s)d(t)\rho(s, t).$$

**Definition 3.86.** Let

$$\Phi_G : T^2G \rightarrow G \times \mathfrak{g} \times \mathfrak{g} \times T_0\mathfrak{g}$$

denote the isomorphisms induced by  $T\Phi_G$ , as described in Remark 3.85 above.

Recall that

$$\Phi_G(\pi_2\pi_1\gamma) = (g, \pi_1(g^{-1}\gamma(s, 0)), \pi_1(g^{-1}\gamma(0, t)), \pi_2\pi_1(\gamma(0, t)^{-1}g\gamma(s, 0)^{-1}\gamma(s, t)))$$

and

$$\Phi_G^{-1}(g, [c], [d], \pi_2\pi_1(\rho)) = \pi_2\pi_1(gc(s)d(t)\rho(s, t)).$$

Following Bertram and Didry, we write

$$G \times \mathfrak{g} \times \mathfrak{g} \times T_0\mathfrak{g} = \mathfrak{g}_{00} \times \mathfrak{g}_{10} \times \mathfrak{g}_{01} \times \mathfrak{g}_{11}$$

and let  $\pi_\alpha$  and  $\iota_\alpha$  for

$$\alpha \in \{(00), (10), (01), (11)\}$$

be the corresponding projections and inclusions .

**Corollary 3.87.** *It follows that  $\iota_{10}([c])$  is represented by  $\gamma(s, t) = c(s)$ , and  $\iota_{01}([d])$  is represented by  $\gamma(s, t) = d(t)$ .*

Before we proceed to define the Lie bracket, let us describe the derivations corresponding to elements of  $T_0\mathfrak{g}$  in case  $T_0\mathfrak{g} \cong \mathfrak{g}$ .

**Lemma 3.88.** *Suppose that the canonical map  $\Xi : \mathfrak{g} \rightarrow T_0\mathfrak{g}$  from Definition 3.49 is an isomorphism. If  $c : \mathbb{R} \rightarrow \mathfrak{g}$  is a smooth curve representing an element  $v \in T_0\mathfrak{g}$ , let  $w = [d] = \Phi^{-1}(v)$ . Then*

$$\partial_1\partial_2(f \circ \gamma_c)(0, 0) = (f \circ d)'(0)$$

for all  $f \in F_G$ .

*Proof.* By definition, the vector  $v = \Phi(w)$  is represented by the straight line  $t \mapsto tw$ . Now by Definition 3.17,  $tw$  is represented by  $s \mapsto d(st)$ , hence  $\gamma_d(s, t) = d(st)$



represents  $v$  in  $T_0\mathfrak{g}$ . For now, let  $\bar{\gamma}_c(s) = [t \mapsto \gamma_c(s, t)]$  and  $\bar{\gamma}_d(s) = [t \mapsto d(st)]$ . Then by definition of  $T_0\mathfrak{g}$ , for all smooth functions  $\varphi$  on  $\mathfrak{g}$  the following holds:

$$(\varphi \circ \bar{\gamma}_d)'(0) = (\varphi \circ \bar{\gamma}_c)'(0). \quad (3.3)$$

For each  $f \in F_G$  we get a smooth function on  $\mathfrak{g}$  by restricting  $Tf$  and then taking the second component of  $Tf$  as a map to  $T\mathbb{R} \cong \mathbb{R}^2$ . Now let us compute the left hand and the right hand side of equation (3.3) for  $\varphi = \pi_2 \circ Tf$ . We have

$$\pi_2(Tf(\bar{\gamma}_c(s))) = (f \circ \gamma(s, \cdot))'(0) = \partial_2(f \circ \gamma_c)(s, 0),$$

so that the right hand side equals

$$\partial_1 \partial_2(f \circ \gamma_c)(0, 0).$$

The right hand side is computed similarly. Now we know that  $\gamma_d(s, t) = d(st)$ , hence  $\partial_2(f \circ \gamma_d)(s, t) = s(f \circ d)'(st)$  by the chain rule. We use the product rule to get

$$\partial_1 \partial_2(f \circ \gamma_d)(s, t) = (f \circ d)'(st) + s^2(f \circ d)''(st).$$

We plug in  $(s, t) = (0, 0)$ , and equation (3.3) becomes

$$\partial_1 \partial_2(f \circ \gamma_c)(0, 0) = (f \circ d)'(0).$$

□

We will now define the Lie bracket on  $\mathfrak{g}$  by differentiating the commutator map  $K(a, b) = aba^{-1}b^{-1}$  of  $G$ . Recall that  $\Phi_G : T^2G \rightarrow G \times \mathfrak{g} \times \mathfrak{g} \times T_0\mathfrak{g}$  is an isomorphism. We will abuse notation and let  $TK$  denote the map induced by  $TK$  on the trivialization of  $T^2G$ . So we will write  $T^2K$  rather than  $\Phi_G \circ T^2K \circ \Phi_G^{-1}$ .

**Lemma 3.89.** *Let  $v, w \in \mathfrak{g}$  be given by curves  $c$  and  $d$ , respectively. Then*

$$\pi_{11}T^2K(\iota_{10}(v), \iota_{01}(w))$$

is represented by the map  $\gamma : \mathbb{R}^2 \rightarrow G$  given by

$$\gamma(s, t) = c(s)d(t)c(s)^{-1}d(t)^{-1}.$$

*Proof.* We have seen that in our chosen trivialization,  $a = \iota_{10}(v)$  and  $b = \iota_{01}(w)$  are represented by  $\rho(s, t) = c(s)$  and  $\eta(s, t) = d(t)$ , respectively. Now by Lemma 3.84, the element  $T^2K(a, b) \in T^2G$  is represented by

$$\gamma(s, t) = K(\rho(s, t), \eta(s, t)) = c(s)d(t)c(s)^{-1}d(t)^{-1}.$$

Note that  $\gamma(s, 0) = \gamma(0, t) = e$  for all  $s, t \in \mathbb{R}$ , so if we use Definition 3.86 to compute the 11-part we simply get that  $\pi_{11}(T^2K(a, b)) \in T_0\mathfrak{g}$  is represented by

$$\begin{aligned} \gamma_{11}(s, t) &= \gamma(s, 0)^{-1}\gamma(s, t)\gamma(0, t)^{-1}\gamma(0, 0) \\ &= \gamma(s, t) \end{aligned}$$

which completes the proof. □

We now use the commutator map to define the Lie bracket on  $\mathfrak{g}$ .

**Definition 3.90.** *Let  $G$  be a Frölicher group for which the canonical map  $\Xi : \mathfrak{g} \rightarrow T_0\mathfrak{g}$  (Definition 3.49) is an isomorphism. Then we will say that  $G$  is a Frölicher group with Lie algebra, and we define*

$$[v, w] = \Xi^{-1}\pi_{11}T^2K(\iota_{01}(v), \iota_{10}(w)) \in \mathfrak{g}.$$

**Theorem 3.91.** *Let  $(G, C, F)$  be a Frölicher group with Lie algebra  $\mathfrak{g}$ , let  $[\cdot, \cdot]$  the bracket operation on  $\mathfrak{g}$ , and for  $w \in \mathfrak{g}$  let  $\xi_w \in \mathcal{V}_l(G)$  denote the corresponding invariant derivation. Then*

$$\xi_{[u, v]} = [\xi_u, \xi_v] \in \text{Der}_l(F),$$

and  $(\mathfrak{g}, [\cdot, \cdot])$  is a Lie algebra.

*Proof.* By Lemma 3.82, the map  $w \mapsto \xi_w$  is linear and injective, so we can identify  $\mathfrak{g}$  with a vector subspace of the Lie algebra  $\text{Der}_l(F)$  of invariant derivations of  $F$ . If we can show that  $\xi_{[v,w]}(f)(e) = [\xi_v, \xi_w](f)(e)$  for all  $f$ , it follows that  $\xi_{[v,w]} = [\xi_v, \xi_w]$ . This would imply that  $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  is a Lie bracket.

In what follows, let  $\partial_s \partial_t \gamma(s, t)$  be short for  $\frac{\partial^2}{\partial s \partial t} \gamma(s, t)|_{(0,0)}$ , and similarly  $\partial_t \partial_s \gamma(s, t)$ , whenever  $\gamma : \mathbb{R}^2 \rightarrow \mathbb{R}$ . Note that if  $\gamma$  is smooth, then  $\partial_s \partial_t \gamma(s, t) = \partial_t \partial_s \gamma(s, t)$ .

As we have seen in Lemma 3.83,

$$[\xi_v, \xi_w](f)(e) = \partial_s \partial_t [f(c(s)d(t)) - f(d(t)c(s))].$$

Using Lemmas 3.88 and 3.89 we get

$$\xi_{[v,w]}(f)(e) = \partial_s \partial_t f(c(s)d(t)c(s)^{-1}d(t)^{-1}).$$

To finish the proof we show that the right hand sides of these equations are equal. We will use Lemma 3.39 several times. The functions being differentiated are both smooth functions in two variables, and therefore we can interchange the order of differentiation. First, fix  $s$  and note that the negative of the vector  $[t \mapsto d(t)c(s)] \in T_{c(s)}G$  is given by  $[t \mapsto d(t)^{-1}c(s)]$ . Hence by Lemma 3.39,

$$\begin{aligned} [\xi_v, \xi_w](f)(e) &= \partial_s \partial_t [f(c(s)d(t)) - f(d(t)c(s))] \\ &= \partial_s \partial_t [f(c(s)d(t)) + f(d(t)^{-1}c(s))] \\ &= \partial_s \partial_t f(c(s)d(t)c(s)^{-1}d(t)^{-1}c(s)). \end{aligned}$$

The last equality comes from the formula for addition of  $[t \mapsto c(s)d(t)]$  and  $[t \mapsto d(t)^{-1}c(s)]$ . We change the order of differentiation and view the argument of  $f$  in the last expression as a curve through  $e$  with parameter  $s$ . We get

$$\begin{aligned} \partial_t \partial_s f(c(s)d(t)c(s)^{-1}d(t)^{-1}c(s)) &= \partial_t \partial_s [f(c(s)d(t)c(s)^{-1}d(t)^{-1}) + f(c(s))] \\ &= \partial_s \partial_t [f(c(s)d(t)c(s)^{-1}d(t)^{-1}) + f(c(s))] \\ &= \partial_s \partial_t f(c(s)d(t)c(s)^{-1}d(t)^{-1}). \end{aligned}$$

This proves  $[\xi_v, \xi_w] = \xi_{[v,w]}$ , and therefore  $(\mathfrak{g}, [\cdot, \cdot])$  is a Lie algebra.  $\square$

**Remark 3.92.** Given  $[c], [d] \in \mathfrak{g}$ , consider the curve  $\varphi(t) = c(t)d(t)c(t)^{-1}d(t)^{-1}$ .

This curve represents the zero vector, but in classical Lie theory it can be reparametrized and  $\psi(t) = \varphi(\sqrt{t})$  is a curve which represents the Lie bracket of  $[c]$  and  $[d]$ . One can ask whether there is a smooth reparametrization which represents the Lie bracket. The answer is negative, as the following example shows. Let  $G$  be the  $ax + b$ -group

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{R}, a > 0 \right\}$$

with Lie algebra

$$\mathfrak{g} = \left\{ \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix} \mid u, v \in \mathbb{R} \right\}$$

and exponential function

$$\exp\left(t \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} e^{tu} & \frac{v}{u}(e^{tu} - 1) \\ 0 & 1 \end{pmatrix}.$$

Let us take two non-commuting elements  $x = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}$ . Then

$$\exp(tx) \exp(ty) \exp(-tx) \exp(-ty) = \begin{pmatrix} 1 & t(e^{2t} - 1) \\ 0 & 1 \end{pmatrix}.$$

Let  $\gamma(t) = t(e^{2t} - 1)$ . Then  $\gamma(\sqrt{t})$  has right derivative 2 at  $t = 0$ . However, if  $\rho$  is a smooth reparametrization with  $\rho(0) = 0$ , then

$$(\gamma \circ \rho)' = \rho'(e^{2\rho} - 1) + 2\rho\rho'e^{2\rho},$$

and this shows that  $(\gamma \circ \rho)'(0) = 0$ .

**Theorem 3.93.** *Let  $G$  and  $H$  be Frölicher groups with  $T_0\mathfrak{g} \cong \mathfrak{g}$  and  $T_0\mathfrak{h} \cong \mathfrak{h}$ . If  $\alpha : G \rightarrow H$  is a homomorphism of Frölicher groups, then  $T_e\alpha : \mathfrak{g} \rightarrow \mathfrak{h}$  is a homomorphism of Lie algebras.*

*Proof.* Let  $u = [c]$  and  $v = [d]$  be given elements of  $\mathfrak{g}$ . Then

$$[u, v] = \Xi^{-1}(w),$$

where  $w \in T_0\mathfrak{g}$  is given by the map  $\gamma(s, t) = c(s)d(t)c(s)^{-1}d(t)^{-1} = K(c(s), d(t))$ . Then by Lemma 3.84,  $T^2\alpha(w)$  is represented by  $\alpha \circ \gamma$ , and since  $\alpha$  is a homomorphism we have that  $(\alpha \circ \gamma)(s, t) = K(\alpha(c(s)), \alpha(d(t)))$ . This proves that  $\Xi^{-1}(T^2\alpha(w)) = [T_e\alpha u, T_e\alpha v]$ , and it remains to show that the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{T_e\alpha} & \mathfrak{h} \\ \downarrow \Xi & & \downarrow \Xi \\ T_0\mathfrak{g} & \xrightarrow{T^2\alpha} & T_0\mathfrak{h} \end{array}$$

commutes. To this end, recall that  $\Xi([c])$  is given by the curve  $s \mapsto s[c]$  in  $\mathfrak{g}$  which, by definition of multiplication, can be represented by  $\gamma(s, t) = c(st)$ . Hence, using Lemma 3.84,  $T^2\alpha(\Xi([c]))$  is represented by  $(s, t) \mapsto \alpha(c(st))$ . It is easily seen that  $T_e\alpha(\Xi([c]))$  is represented by the same map, hence  $T^2\alpha \circ \Xi = \Xi \circ T_e\alpha$ .  $\square$

# References

- [BC05] H. L. Bentley and Paul Cherenack. Boman's theorem: strengthened and applied. *Quaest. Math.*, 28(2):145–178, 2005.
- [Ber06] Wolfgang Bertram. Differential geometry, lie groups and symmetric spaces over general base fields and rings. Preprint arXiv:math/0502168v3, 2006. Accepted for Publication in the Memoirs of the AMS.
- [Bom67] Jan Boman. Differentiability of a function and of its compositions with functions of one variable. *Math. Scand.*, 20:249–268, 1967.
- [Che73] Kuo-tsai Chen. Iterated integrals of differential forms and loop space homology. *Ann. of Math. (2)*, 97:217–246, 1973.
- [Che86] Kuo Tsai Chen. On differentiable spaces. In *Categories in continuum physics (Buffalo, N.Y., 1982)*, volume 1174 of *Lecture Notes in Math.*, pages 38–42. Springer, Berlin, 1986.
- [DI85] Paul Donato and Patrick Iglésias. Exemples de groupes difféologiques: flots irrationnels sur le tore. *C. R. Acad. Sci. Paris Sér. I Math.*, 301(4):127–130, 1985.
- [Did06] Manon Didry. *Structures Algébriques sur les Espaces Symétriques*. PhD thesis, Université Henri Poincaré - Nancy I, 2006.
- [FB66] A. Frölicher and W. Bucher. *Calculus in vector spaces without norm*. Lecture Notes in Mathematics, No. 30. Springer-Verlag, Berlin, 1966.
- [FK88] Alfred Frölicher and Andreas Kriegl. *Linear spaces and differentiation theory*. Pure and Applied Mathematics (New York). John Wiley & Sons Ltd., Chichester, 1988. A Wiley-Interscience Publication.
- [Frö86] Alfred Frölicher. Cartesian closed categories and analysis of smooth maps. In *Categories in continuum physics (Buffalo, N.Y., 1982)*, volume 1174 of *Lecture Notes in Math.*, pages 43–51. Springer, Berlin, 1986.
- [Gio04] P. Giordano. Infinitesimal differential geometry. *Acta Math. Univ. Comenian. (N.S.)*, 73(2):235–278, 2004.
- [Glö02] Helge Glöckner. Infinite-dimensional Lie groups without completeness restrictions. In *Geometry and analysis on finite- and infinite-dimensional Lie groups (Będlewo, 2000)*, volume 55 of *Banach Center Publ.*, pages 43–59. Polish Acad. Sci., Warsaw, 2002.

- [Glö06] Helge Glöckner. Discontinuous non-linear mappings on locally convex direct limits. *Publ. Math. Debrecen*, 68(1-2):1–13, 2006.
- [GN08] Helge Glöckner and Karl-Hermann Neeb. *Infinite-dimensional Lie groups*. 2008. In Preparation.
- [Hec95] G. Hector. Géométrie et topologie des espaces difféologiques. In *Analysis and geometry in foliated manifolds (Santiago de Compostela, 1994)*, pages 55–80. World Sci. Publ., River Edge, NJ, 1995.
- [HMV02] Gilbert Hector and Enrique Macías-Virgós. Diffeological groups. In *Recent advances in Lie theory (Vigo, 2000)*, volume 25 of *Res. Exp. Math.*, pages 247–260. Heldermann, Lemgo, 2002.
- [IL90] P. Iglésias and G. Lachaud. Espaces différentiables singuliers et corps de nombres algébriques. *Ann. Inst. Fourier (Grenoble)*, 40(3):723–737, 1990.
- [KM97] Andreas Kriegl and Peter W. Michor. *The convenient setting of global analysis*, volume 53 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.
- [KR04] Anders Kock and Gonzalo E. Reyes. Categorical distribution theory; heat equation. Preprint arXiv:math/0407242v1, 2004.
- [Lau06] Martin Laubinger. Diffeological spaces. *Proyecciones*, 25(2):151–178, 2006.
- [Les03] Joshua Leslie. On a diffeological group realization of certain generalized symmetrizable Kac-Moody Lie algebras. *J. Lie Theory*, 13(2):427–442, 2003.
- [Los94] M. V. Losik. Categorical differential geometry. *Cahiers Topologie Géom. Différentielle Catég.*, 35(4):274–290, 1994.
- [Mac71] Saunders MacLane. *Categories for the working mathematician*. Springer-Verlag, New York, 1971. Graduate Texts in Mathematics, Vol. 5.
- [Mil84] J. Milnor. Remarks on infinite-dimensional Lie groups. In *Relativity, groups and topology, II (Les Houches, 1983)*, pages 1007–1057. North-Holland, Amsterdam, 1984.
- [NB05] P. P. Ntumba and T. A. Batubenge. On the way to Frölicher Lie groups. *Quaest. Math.*, 28(1):73–93, 2005.
- [Nee06] Karl-Hermann Neeb. Towards a Lie theory of locally convex groups. *Jpn. J. Math.*, 1(2):291–468, 2006.

- [RR73] A. P. Robertson and Wendy Robertson. *Topological vector spaces*. Cambridge University Press, London, second edition, 1973. Cambridge Tracts in Mathematics and Mathematical Physics, No. 53.
- [See64] R. T. Seeley. Extension of  $C^\infty$  functions defined in a half space. *Proc. Amer. Math. Soc.*, 15:625–626, 1964.
- [Sou80] J.-M. Souriau. Groupes différentiels. In *Differential geometrical methods in mathematical physics (Proc. Conf., Aix-en-Provence/Salamanca, 1979)*, volume 836 of *Lecture Notes in Math.*, pages 91–128. Springer, Berlin, 1980.
- [Sou85] Jean-Marie Souriau. Un algorithme générateur de structures quantiques. *Astérisque*, (Numero Hors Serie):341–399, 1985. The mathematical heritage of Élie Cartan (Lyon, 1984).
- [Ste66] A. K. Steiner. Complementation in the lattice of  $T_1$ -topologies. *Proc. Amer. Math. Soc.*, 17:884–886, 1966.



# Vita

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