

12-24-2004

## Quantum zeno and anti-zeno effects: An exact model

T. C. Dorlas  
*Dublin Institute for Advanced Studies*

R. F. O'Connell  
*Dublin Institute for Advanced Studies*

Follow this and additional works at: [https://digitalcommons.lsu.edu/physics\\_astronomy\\_pubs](https://digitalcommons.lsu.edu/physics_astronomy_pubs)

---

### Recommended Citation

Dorlas, T., & O'Connell, R. (2004). Quantum zeno and anti-zeno effects: An exact model. *Proceedings of SPIE - The International Society for Optical Engineering*, 5436, 194-201. <https://doi.org/10.1117/12.543312>

This Conference Proceeding is brought to you for free and open access by the Department of Physics & Astronomy at LSU Digital Commons. It has been accepted for inclusion in Faculty Publications by an authorized administrator of LSU Digital Commons. For more information, please contact [ir@lsu.edu](mailto:ir@lsu.edu).

# Quantum Zeno and anti-Zeno Effects: An Exact Model

T. C. Dorlas and R. F. O'Connell<sup>†</sup>

*School of Theoretical Physics, Dublin Institute for Advanced Studies,*

*10 Burlington Road, Dublin 4, Ireland*

(Dated: December 23, 2018)

## Abstract

Recent studies suggest that both the quantum Zeno (increase of the natural lifetime of an unstable quantum state by repeated measurements) and anti-Zeno (decrease of the natural lifetime) effects can be made manifest in the same system by simply changing the dissipative decay rate associated with the environment. We present an exact calculation confirming this expectation.

PACS number(s): 03.65.Xp, 03.65.Yz, 05.30.-d, 05.40.Jc

arXiv:quant-ph/0501098v1 18 Jan 2005

The quantum Zeno effect (QZE) predicts that the lifetime of an excited state increases by repeated measurements. It has been a subject of interest for many years [1, 2] and recent reviews appear in [3] and [4]. More recently, it has been pointed out that a decrease in lifetime, referred to as the Inverse or anti-Zeno effect (AZE) can also occur [5, 6, 7]. Whereas there are claims that the QZE has been observed, to our knowledge there has been no experimental verification yet of the AZE. On the other hand, the detailed calculations of Pascazio and Facchi [5] and Kofman and Kurizki [6] lend strong credence to the possible existence of the AZE. Since the calculations of [5] and [6] by their nature required various assumptions (such as the Weisskopf-Wigner approximation), we consider it desirable to present an exact calculation which should also delineate in a clear-cut manner the nature and magnitude of the external environment that is necessary to achieve the transition from QZE to AZE.

The system we analyze is the decay of a free particle that is placed initially in a Gaussian state:

$$\psi(x, 0) = (2\pi\sigma^2)^{-1/4} \exp\left\{-\frac{x^2}{4\sigma^2}\right\}, \quad (1)$$

where  $\sigma^2$  is the variance. The particle is regarded as part of a larger system of a particle coupled to a reservoir and the complete system is initially in equilibrium at temperature  $T$ . This was the scenario considered by Ford et al. [8] who used distribution functions defined in accordance with the quantum theory of measurement to obtain exact results for the spreading of the wave packet and for the probability at time  $t$  given by

$$P(x, t) = \frac{1}{\sqrt{2\pi w^2(t)}} \exp\left\{-\frac{x^2}{2w^2(t)}\right\}. \quad (2)$$

Here

$$\begin{aligned} w^2(t) &= \sigma^2 - \frac{[x(0), x(t)]^2}{4\sigma^2} + s(t) \\ &\equiv \sigma^2 + \sigma_q^2 + s(t), \end{aligned} \quad (3)$$

where  $\sigma^2$  is the initial variance,  $[x(0), x(t)]$  is the commutator,

$$s(t) = \langle \{x(t) - x(0)\}^2 \rangle, \quad (4)$$

is the mean square displacement and  $\sigma_q^2$  is the contribution to the spreading due to temperature-independent quantum effects. A measure of the decay rate  $R(t)$  is simply given by the ratio of the probabilities at times  $t$  and 0. However, this ratio is clearly dependent on  $x$  so, from henceforth, we take  $x = 0$  (corresponding to the maximum of the wave-packet) and write

$$\begin{aligned} R(t) &= \frac{P(0, t)}{P(0, 0)} \\ &= \left\{ \frac{\sigma^2}{w^2(t)} \right\}^{1/2}. \end{aligned} \quad (5)$$

Hence, our calculation reduces to an evaluation of the width of the wave-packet at time  $t$ .

The quantities appearing in (3) and (4) are evaluated by use of the quantum Langevin equation [9], which is a Heisenberg equation of motion for  $x(t)$ , the dynamical variable corresponding to the coordinate of a Brownian particle interacting with a linear passive heat bath. For the case of a free particle, this equation for the stationary process has the well known form,

$$m\ddot{x} + \int_{-\infty}^t dt' \mu(t-t') \dot{x}(t') = F(t), \quad (6)$$

where  $\mu(t)$  is the memory function and  $F(t)$  is a fluctuating operator force with mean zero. The solution of the quantum Langevin equation (6) can be written

$$x(t) = \int_{-\infty}^t dt' G(t-t') F(t'), \quad (7)$$

where  $G(t)$ , the Green function, can in turn be written

$$G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \alpha(\omega + i0^+) e^{-i\omega t}, \quad (8)$$

in which  $\alpha(z)$  (the Fourier transform of the Green function) is the response function. For the free particle the response function has the general form

$$\alpha(z) = \frac{1}{-mz^2 - iz\tilde{\mu}(z)}, \quad (9)$$

in which  $\tilde{\mu}(z)$  is the Fourier transform of the memory function,

$$\tilde{\mu}(z) = \int_0^\infty dt \mu(t) e^{izt}, \quad \text{Im}\{z\} > 0. \quad (10)$$

Using these results, we find that [9, 10] the mean square displacement is given by the formula

$$s(t) = \frac{2\hbar}{\pi} \int_0^\infty d\omega \text{Im} \left\{ \alpha(\omega + i0^+) \right\} \coth \frac{\hbar\omega}{2kT} (1 - \cos \omega t), \quad (11)$$

while the commutator, which is temperature independent, is given by the formula

$$[x(0), x(t)] = \frac{2i\hbar}{\pi} \int_0^\infty d\omega \text{Im} \left\{ \alpha(\omega + i0^+) \right\} \sin \omega t. \quad (12)$$

These expressions are valid for arbitrary temperature and arbitrary dissipation. (Indeed, with the appropriate expression for the response function, they are valid in the presence of an external oscillator potential.) Here, we confine our attention to the case of zero temperature and Ohmic dissipation, where  $\tilde{\mu}(z) = m\gamma$ . It then follows that

$$s(t) = \frac{2\hbar\gamma}{\pi m} t^2 \text{I}(\gamma t), \quad (13)$$

where

$$\text{I}(\gamma t) = \int_0^\infty dy \frac{(1 - \cos y)}{y [y^2 + (\gamma t)^2]}. \quad (14)$$

In addition, the commutator is given by

$$[x(0), x(t)] = i\hbar G(t) = \frac{i\hbar}{m\gamma} (1 - e^{-\gamma t}), \quad (15)$$

so that

$$\sigma_q^2 = \frac{\hbar^2}{m^2\gamma^2} \frac{(1 - e^{-\gamma t})^2}{4\sigma^2}. \quad (16)$$

Hence, we now have all the tools at our disposal in order to carry out an exact calculation of  $P(x, t)$  and hence the rate  $R(t, \gamma)$ , where we have added the argument  $\gamma$  to  $R$  in order to emphasize the fact that this dependence will be the crucial element in our calculation.

Combining the various results given above, we may write explicitly

$$R(t, \gamma) = \left\{ \left[ \sigma^2 + \left( \frac{\hbar^2}{4m^2\sigma^2} \right) t^2 \left\{ \frac{(1 - e^{-\gamma t})}{\gamma t} \right\}^2 + \frac{2\hbar t}{\pi m} (\gamma t) \text{I}(\gamma t) \right] / \sigma^2 \right\}^{-1/2}. \quad (17)$$

Our goal will be to calculate  $\{R(\tau, \gamma)\}^n$ , the rate corresponding to  $n$  measurements on the system, where  $\tau = t/n$ , and then compare it to  $R(t, \gamma)$ . As we shall see, the result depends crucially on the value of  $\gamma t$ .

The only quantity left requiring explicit evaluation is  $I(\gamma t)$  given by (14). In order to obtain more physical insight into the nature of the results obtained, we will first evaluate  $R$  analytically for both small and large values of  $\gamma t$ , which we will demonstrate correspond to the QZE and AZE, respectively. However, in order to determine for what value of  $\gamma t$  the transition between the two regimes occur, it will be necessary to carry out a numerical evaluation of  $I(\gamma)$ . First, we turn to the analytic calculation.

(a)  $\gamma t \ll 1$

Then, from (16),

$$\sigma_q^2 \approx \frac{\hbar^2}{4m^2\sigma^2} t^2, \quad (18)$$

which corresponds to the usual dynamical wave packet spreading in the absence of a dissipative environment. In addition, (13) reduces to [11]

$$s(t) = \frac{\hbar\gamma}{\pi m} t^2 \left\{ -\log(\gamma t) + \frac{3}{2} - \gamma_E \right\}, \quad (19)$$

where  $\gamma_E = 0.577$  is Euler's constant.

Using these results in (3) leads to

$$w^2(t) = \sigma^2 + \langle v^2 \rangle t^2, \quad (20)$$

where

$$\langle v^2 \rangle = \frac{\hbar^2}{4m^2\sigma^2} + \frac{\hbar\gamma}{\pi m} \{-\log(\gamma t) + 0.92\}. \quad (21)$$

Since for most reasonable scenarios  $\langle v^2 \rangle t^2 \ll \sigma^2$ , we may expand  $w^2(t)$  in a power series in  $t^2$  to get

$$\begin{aligned} R(t) &= \left\{ \frac{\sigma^2}{w^2(t)} \right\}^{1/2} \approx 1 - \frac{1}{2\sigma^2} \{\sigma_q^2 + s(t)\} \\ &\approx \left( 1 - \frac{\langle v^2 \rangle t^2}{2\sigma^2} + \dots \right). \end{aligned} \quad (22)$$

Apart from the weak  $\log(\gamma t)$  dependence of  $\langle v^2 \rangle$ , as manifest in (21), we note the ubiquitous short-time  $t^2$  behaviour of  $R(t)$ , which is characteristic of the QZE. To remind the reader, we consider  $n$  instantaneous measurements at intervals  $\tau = t/n$ . Thus, for large  $n$

$$\begin{aligned} [R(\tau)]^n &= \left(1 - \frac{\langle v^2 \rangle t^2}{n} \frac{1}{n}\right)^n \\ &= \exp\left(-\langle v^2 \rangle t \tau\right) \\ &\rightarrow 1 \text{ as } \tau \rightarrow 0. \end{aligned} \tag{23}$$

In other words, for small  $\gamma t$ , the decay rate becomes frozen as  $n \rightarrow \infty$  ( $\tau \rightarrow 0$ ), which is the extreme manifestation of the QZE. Moreover, for  $\gamma \rightarrow 0$ , we see from (21) that the second term on the right-side goes to zero. Thus, (23) still holds but now  $\langle v^2 \rangle$  is completely independent of both  $\gamma$  and  $t$ . The conclusion is that in the absence of a heat bath, the QZE always holds.

(b)  $\gamma t \gg 1$

Then, from (16),

$$\sigma_q^2 \approx \frac{\hbar^2}{4m^2\sigma^2} \frac{1}{\gamma^2}, \tag{24}$$

and (13) reduces to [11]

$$s(t) = \frac{2\hbar}{\pi m \gamma} \{\log(\gamma t) + \gamma_E\}. \tag{25}$$

Hence

$$w^2(t) = \sigma^2 + \frac{\hbar^2}{4m^2\sigma^2} \frac{1}{\gamma^2} + \frac{2\hbar}{\pi m \gamma} [\log(\gamma t) + \gamma_E]. \tag{26}$$

Since for most reasonable scenarios,  $\sigma^2 \gg s(t) \gg \sigma_q^2$  for any  $t$ , we may write

$$\begin{aligned} R(t) &\approx 1 - \frac{s(t)}{2\sigma^2} \\ &= 1 - \frac{\hbar}{\pi m \sigma^2 \gamma} [\log(\gamma t) + \gamma_E]. \end{aligned} \tag{27}$$

Because of the weak log dependence on  $t$ , it is clear that, for fixed  $\gamma$ ,  $R\left(\frac{t}{n}\right)$  is just slightly larger than  $R(t)$ , keeping in mind that one must be careful to ensure that  $\gamma \tau \gg 1$  based on our initial assumption. Thus, it is readily apparent that

$$\begin{aligned}
[R(\tau)]^n &< \left\{ 1 - \frac{\hbar}{\pi m \sigma^2 \gamma} \left[ \log \left( \frac{\gamma t}{n} \right) + \gamma_E \right] \right\}^n \\
&< R(t).
\end{aligned}
\tag{28}$$

Thus, we see that  $[R(\tau)]^n$  is always less than  $R(t)$  and decreases with increasing  $n$  (decreasing  $\tau$ ), which is the extreme manifestation of the AZE. Moreover, for  $n$  sufficiently small, we may write

$$[R(\tau)]^n \approx \exp \left\{ -\frac{\hbar n}{\pi m \sigma^2 \gamma} \left[ \log \left( \frac{\gamma t}{n} \right) + \gamma_E \right] \right\},
\tag{29}$$

the restriction on  $n$  being to ensure that the magnitude of the argument of the exponential is  $\ll 1$ .

The conclusion is that the QZE is characterized by small  $\gamma t$  values whereas the AZE is characterized by large  $\gamma t$  values. As a check on the analytic results give for  $I(\gamma t)$  in (14), we carried out a numerical evaluation of the integral and obtained excellent agreement.

In order to obtain the value of  $\gamma t$  for which the transition occurs, as well as delineating more accurately the analytic results obtained above, we now turn to a numerical evaluation of  $R(t, \gamma)$  given in (17). Thus, in Figs 1 and 2, we plot  $\left\{ R \left( \frac{t}{20} \right) \right\}^{20}$ , corresponding to 20 measurements, and compare it to  $R(t)$ , for  $t$  values ranging from 0-5 and 0-400, respectively, and taking  $\gamma = 0.1$ . We note that  $\left\{ R \left( \frac{t}{20} \right) \right\}^{20}$  is initially larger than  $R(t)$ , corresponding to the QZE but it becomes smaller (corresponding to the AZE) for  $\gamma t$  values larger than the transition  $\gamma t$  value of ? ? .

In conclusion, we have presented an exact calculation of the decay rate of a free particle that is placed initially in a Gaussian state and which is coupled to a reservoir so that the complete system is initially in equilibrium at zero temperature. The results obtained demonstrate that repeated measurements made on the system lead to a QZE effect scenario for small  $\gamma t$  values while evolving into an AZE effect scenario for large  $\gamma t$  values, confirming similar results obtained in [5] and [6].

### Acknowledgments

RFO'C would like to thank the School of Theoretical Physics, Dublin Institute for Advanced Studies, for their hospitality while this work was being carried out.



- 
- [†] Permanent address: Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803-4001.
- [1] A. Beskow, J. Nilsson: *Arkiv für Fysik* **34**, 561 (1967); L. A. Khalfin: *JETP Letters* **8**, 65 (1968).
- [2] B. Misra, E.C.G. Sudarshan: *J. Math. Phys.* **18**, 756 (1977).
- [3] D. Home and M.A.B. Whitaker, *Ann. Phys. (N.Y.)* **258**, 237 (1997).
- [4] P. Facchi and S. Pascazio in "Irreversible Quantum Dynamics", ed. by R. Benatti and R. F. Floreanini, *Lecture Notes in Physics*, Vol. 622, p. 141 (Springer, Berlin, 2003).
- [5] S. Pascazio and P. Facchi, *Acta Phys. Slovaca* **49**, 557 (1999); P. Facchi and S. Pascazio, *Phys. Rev. A* **62**, 023804 (2000); in *Progress in Optics*, edited by E. Wolf (Elsevier, Amsterdam, 2001), Vol. 41.
- [6] A.G. Kofman and G. Kurizki, *Acta Phys. Slovaca* **49**, 541 (1999); *Nature (London)* **405**, 546 (2000).
- [7] A. Luis and L.L. Sánchez-Soto, *Phys. Rev. A* **57**, 781 (1998); K. Thun and J. Peřina, *Phys. Lett. A* **249**, 363 (1998); J. Řeháček *et al.*, *Phys. Rev. A* **62**, 013804 (2000); M. Lewenstein and K. Rzażewski, *Phys. Rev.* **61**, 022105 (2000).
- [8] G. W. Ford, J.T. Lewis and R.F. O'Connell, *Phys. Rev. A* **64**, 032101 (2001).
- [9] G. W. Ford, J.T. Lewis and R. F. O'Connell, *Phys. Rev. A* **37**, 4419 (1988).
- [10] G. W. Ford and R. F. O'Connell, *J. Stat. Phys.* **57**, 803 (1989).
- [11] G. W. Ford and R. F. O'Connell, *J. Optics B, Special Issue on Quantum Computing*, **5**, S609 (2003).

This figure "rate1.jpg" is available in "jpg" format from:

<http://arxiv.org/ps/quant-ph/0501098v1>

This figure "rate2.jpg" is available in "jpg" format from:

<http://arxiv.org/ps/quant-ph/0501098v1>