Equations of parametric surfaces with base points via syzygies

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EQUATIONS OF PARAMETRIC SURFACES WITH BASE POINTS VIA SYZYGIES

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Abstract

Suppose S is a parametrized surface in complex projective 3-space $\mathbf{P}^3$ given as the image of $\phi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$. The implicitization problem is to compute an implicit equation $F = 0$ of S using the parametrization ϕ. An algorithm using syzygies exists for computing F if ϕ has no base points, i.e. ϕ is everywhere defined. This work is an extension of this algorithm to the case of a surface with multiple base points of total multiplicity k.

We accomplish this in three chapters. In Chapter 2, we develop the concept and properties of Castelnuovo-Mumford regularity in biprojective spaces. In Chapter 3, we give a criterion for regularity in biprojective spaces. These results are applied to the implicitization problem in Chapter 4.

1. Introduction

1.1 Motivation for the Implicitization Problems

We start with a simple example: The curve in Figure 1.1 is given by the following

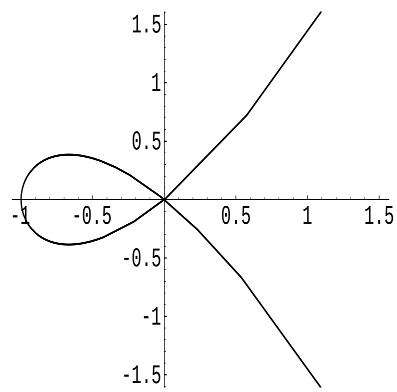

FIGURE 1.1. Curve

parametric equations:

$$\begin{aligned} x &= t^2 - 1, \\ y &= t(t^2 - 1). \end{aligned}$$

The implicitization problem is to convert the parametrization into a defining equation for the curve, which we find is:

$$y^2 - x^2 - x^3 = 0.$$

Parametric surfaces are widely used in computer aided design projects since it is easy to describe the points of the surface by means of the parameter values. Given the parametric equations, the computer can evaluate for different parameter values and then plot the points. But it is hard to decide whether a point is on the surface which is parametrically presented. For example the graph in

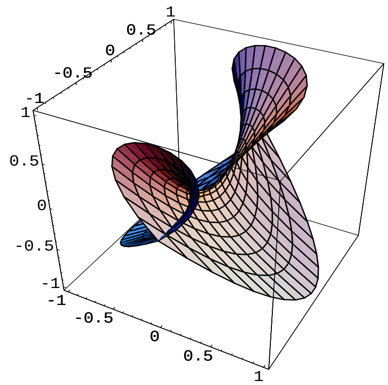

FIGURE 1.2. Surface

Figure 1.2 is plotted by using the parametric representation:

$$\begin{aligned} x &= t(u^2 - t^2), \\ y &= u, \\ z &= u^2 - t^2 \end{aligned}$$

To answer the following question, it is useful to have an implicit representation of a variety.

Question: *Is the point* (x_0, y_0, z_0) *on the above surface?*

Answer:

To decide this question from the knowledge of the parametric equations, we need to solve the equations

$$\begin{aligned} x_0 &= t(u^2 - t^2), \\ y_0 &= u, \\ z_0 &= u^2 - t^2. \end{aligned}$$

for t, u, if possible.

Trying to solve these equations leads to

$$x_0^2 - y_0^2 z_0^2 + z_0^3 = 0,$$

as a criterion for the solveability of the parametric equations. That is the implicit equation $x^2 - y^2z^2 + z^3 = 0$, an easily checked criterion for deciding if a point (x_0, y_0, z_0) is on the parametrized surface. For example, $(1, 2 - 1)$ is not on the surface since

$$1^2 - 2^2(-1)^2 + (-1)^3 = 1 - 4 - 1 = -1 \neq 0,$$

while $(10, 3, 5)$ is on the surface since

$$10^2 - 3^2 5^2 + 5^3 = 0.$$

To describe the set of points which are common to two different parametrically presented surfaces is a difficult problem using the parametric descriptions. If the surfaces are described by means of external, i.e. implicit equations, then to find the set of common points of two surfaces reduces to the problem of finding the common solutions of two explicitly given polynomial equations. This is a problem which can be handled relatively easily. For example, let's consider the following question:

Question: *What is the intersection of the parametric surfaces S_1 and S_2?*

The surface S_1 (Figure 1.3) is given by the parametric representation: Figure 1.3

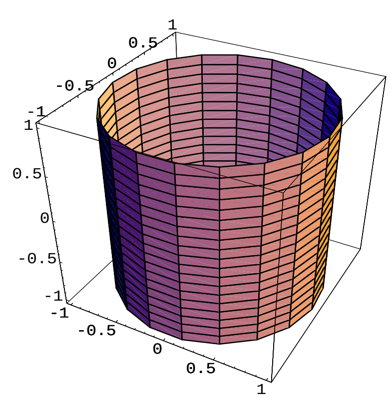

FIGURE 1.3. Surface S_1

is plotted by using the parametric representation given by:

$$x = \frac{1-u^2}{1+u^2},$$
$$y = \frac{2u}{1+u^2},$$
$$z = v;$$

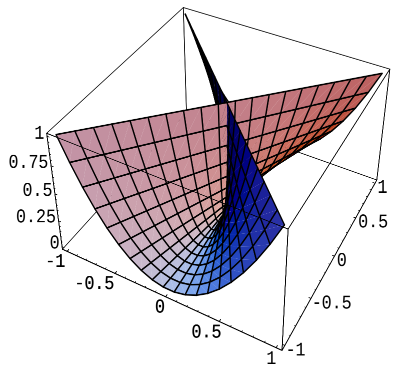

FIGURE 1.4. Surface S_2

While S_2 (Figure 1.4) is given by:

$$x = uv,$$
$$y = v,$$
$$z = u^2.$$

Figure 1.5 is the picture of the intersection of the above two surfaces: It is not easy to describe the intersection if the surfaces are represented parametrically. If we use implicit equations to describe the surface, then finding intersections is just to find the solutions of the two polynomials:

$$x^2 + y^2 = 1$$
$$y^2 z - x^2 = 0$$

The solution set is:

$$\{(\pm\sqrt{1-y^2}, y, \frac{1-y^2}{y^2}) : 0 < |y| \leq 1\}.$$

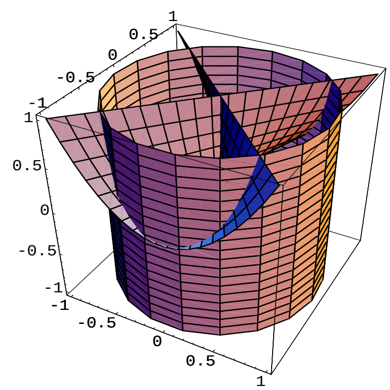

FIGURE 1.5. $S_1 \cap S_2$

Thus there is a need for being able to go back and forth between a parametric and an implicit description of a surface. This is, in essence, the implicitization problem. *Describe algorithms to produce an implicit equation of a surface for which one knows a parametric description.*

We will work over the field $\mathbb{C}$, since it is algebraically closed and the commutative algebra needed is developed over $\mathbb{C}$. Suppose S is a parametrized surface in complex projective 3-space $\mathbf{P}^3$ given as the image of

$$\phi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$$

where

$$\phi = (a(s,u,t,v) : b(s,u,t,v) : c(s,u,t,v) : d(s,u,t,v)).$$

where $a, b, c, d \in R = \mathbb{C}[s,u,t,v]$ are bihomogeneous polynomials of bidegree (m, n). The map ϕ is known as a parametrization of the surface. $\mathrm{Im}(\phi)$ is a constructible subset in $\mathbf{P}^3$. If ϕ is a generically one-to-one map, then $\mathrm{Im}(\phi)$ has dimension 2, that is a surface S in $\mathbf{P}^3$. The closure of $\mathrm{Im}(\phi)$ is an algebraic subset of dimension 2, and it can be expressed by an equation. The implicitization problem is to compute an implicit equation $F = 0$ of the closure of $\mathrm{Im}(\phi)$ using the parametrization ϕ.

1.2 Three Major Techniques

For any parametrization, we can find the implicit equation via elimination of the parameters. In practice, there are three methods used: resultants, Gröbner bases, and syzygies.

Resultant computations are based on the methods developed by Macaulay [28], Cayley [7], Bezout, Dixon [14]. The resultant can tell us whether two polynomials have a common factor. To find an implicit equation via resultants is to eliminate a subset of variables from a set of polynomials.

For example, we can rewrite the parametric equations of Figure 1.1 as the following equations:

$$\begin{aligned} t^2 - (1+x) &= 0 \\ t^3 - t - y &= 0. \end{aligned}$$

Then the *Sylvester Resultant* of the above equation with respect to t can be written as a determinant of 5×5 matrix N,

$$N = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ -(1+x) & 0 & 1 & -1 & 0 \\ 0 & -(1+x) & 0 & -y & -1 \\ 0 & 0 & -(1+x) & 0 & -y \end{bmatrix}$$

And the implicit equation for the curve is $|N| = -x^2 - x^3 + y^2 = 0$.

It is not an easy task to compute the implicit equations via resultants. It often involves an extraneous factor and requires a polynomial division to eliminate the extraneous factor.

Gröbner bases were proposed by Buchberger [5] for efficient computation in polynomial rings. Many problems about ideals in polynomial rings can be attacked by

Gröbner bases. Gröbner bases can be used to find solutions to a set of polynomials, compute projections of their variety into lower dimensional spaces and test polynomials for ideal membership. With Gröbner bases, we can compute the equations satisfied by given elements of an affine or homogeneous coordinate ring. Geometrically, this is the computation of the closure of the image of an affine or projective variety under a morphism. This method requires an ordering of the monomials in the polynomial ring. The algorithm gives a bases of the ideal generated by the parametric equations. This method will produce the implicit equation without any extraneous factor [3]. Let's use the example of the equations of Figure 1.1 again. We will take the lex order in the ring $K[x, y, t]$ by the variable ordering $t > x > y$, where K is an algebraically closed field. Using Mathematica, we find the Gröbner bases of the ideal

$$\tilde{I} = \langle x - (t^2 - 1), y - t(t^2 - 1) \rangle$$

is:

$$\{x^2 + x^3 - y^2, -x - x^2 + ty, tx - y, 1 - t^2 + x\}$$

The implicit equation is:

$$y^2 - x^2 - x^3 = 0.$$

The first polynomial in the Gröbner bases eliminates the variable t, since t is the largest in the monomial order. This polynomial is the implicit equation of our curve. However, in practice, Gröbner bases calculations require more time and memory than resultant calculations (see [27] and [42]). Resultants are still the preferred choice to compute the implicit equations.

Syzygy techniques have been developed recently as a tool for finding implicit equations. The first introduction of syzygy-like techniques in the implicitization problem was the the use of moving lines to produce implicit equations for curves by Sederberg and Chen [36]. For curves, the goal is to find the implicit equation of a parametrized curve in the projective plane $\mathbf{P}^2$ given by

$$\phi : \mathbf{P}^1 \to \mathbf{P}^2$$

$$\phi = (a(s,u) : b(s,u) : c(s,u)),$$

where a, b, c are homogeneous polynomials in the polynomial ring $\mathbb{C}[s,u]$ of degree n, and $\gcd(a,b,c) = 1$. A *moving line* in $\mathbf{P}^2$ is a linear form

$$A(s,u)x_1 + B(s,u)x_2 + C(s,u)x_3,$$

where $A, B, C \in \mathbb{C}[s,u]$ are homogeneous of the same degree. We say that a moving line has degree k if A, B, C are homogeneous of degree k. If

$$A(s,u)a(s,u) + B(s,u)b(s,u) + C(s,u)c(s,u) = 0, \quad \forall (s:u) \in \mathbf{P}^1,$$

then we say *the moving line follows the parametrization* ϕ. In the terminology of commutative algebra, we say that (A, B, C) is a *syzygy* of (a, b, c), and we write this as $(A, B, C) \in \mathrm{Syz}(a, b, c)$, where $\mathrm{Syz}(a, b, c)$ is the *syzygy module* of (a, b, c) over the ring $\mathbb{C}[s,u]$. We let $\mathrm{Syz}(a, b, c)_k$ denote the set of syzygies (A, B, C) with A, B, C homogeneous of degree k. $\mathrm{Syz}(a, b, c)_k$ is a vector space over $\mathbb{C}$ which consists of the moving lines of degree k. The number of linearly independent moving lines of degree k is the dimension of the kernel of the following map:

$$R_k^3 \xrightarrow{(a,b,c)} R_{n+k}$$

$$(A, B, C) \to Aa + Bb + Cc$$

where $R = \mathbb{C}[s,u]$ and R_k denotes the homogeneous forms of degree k. If $k = n-1$, we have n linearly independent moving lines of degree $n-1$ of the form

$$A_i x_1 + B_i x_2 + C_i x_3 = \sum_{j=0}^{n-1} L_{ij}(x_1, x_2, x_3) s^j u^{n-1-j},$$

where $(A_i, B_i, C_i) \in \mathrm{Syz}(a, b, c)_{n-1}$ are homogeneous polynomials in s, u [9]. We can use these n moving lines to construct an $n \times n$ matrix, where the columns of the matrix are indexed by the monomial bases of R_{n-1}, the rows of the matrix are indexed by the linearly independent moving lines $A_i x_1 + b_i x_2 + C_i x_3$, and the entries of the matrix are the coefficients $L_{ij}(x_1, x_2, x_3)$ of $s^j t^{n-1-j}$ in the moving lines.

The following result is proved in [12]:

Theorem 1.2.1. *The implicit equation of ϕ is $F = 0$, where*

$$F^h = \det(L_{ij}(x_1, x_2, x_3))$$

and h is the generic degree of $\phi : \mathbf{P}^1 \to \mathbf{P}^2$.

This theorem uses smaller determinants than the classical methods to find the implicit equation of the parametrized curve. Shortly after that, Cox, Goldman and Zhang extended these ideas to show the validity of implicitization by moving quadrics for rational surfaces with no base points. No base points means that the parametric equation is defined for all values of the projective parameter. In case there is no base point and the parametrization is given by

$$\phi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$$

$$\phi = (a(s, u, t, v) : b(s, u, t, v) : c(s, u, t, v), d(s, u, t, v)).$$

where $a, b, c, d \in R = \mathbb{C}[s, u, t, v]$ are bihomogeneous polynomials of bidegree (m, n) and $\gcd(a, b, c, d) = 1$, the goal is to compute the implicit equation $F = 0$ of $\mathrm{Im}(\phi)$. In the paper [10], the concepts of moving planes and moving quadrics are introduced. A *moving plane* is a linear form

$$Ax_1 + Bx_2 + Cx_3 + Dx_4 \in R[x_1, x_2, x_3, x_4],$$

and a *moving plane follows* the parametrization ϕ if

$$A(p)a(p) + B(p)b(p) + C(p)c(p) + D(p)d(p) = 0, \quad \forall p = (s:u;t:v) \in \mathbf{P}^1 \times \mathbf{P}^1.$$

This is equivalent to saying that $(A, B, C, D) \in \mathrm{Syz}(a, b, c, d)$, where $\mathrm{Syz}(a, b, c, d)$ is the syzygy module of (a, b, c, d) over the ring $R = \mathbb{C}[s, u, t, v]$. Similarly, a *moving quadric* is a quadric form

$$Ax_1^2 + Bx_1x_2 + \cdots + Jx_4^2 \in R[x_1, x_2, x_3, x_4],$$

and the *moving quadric follows* the parametrization ϕ if

$$A(p)a(p)^2 + B(p)b(p)^2 + \cdots + J(p)d(p)^2 = 0, \quad \forall p = (s:u;t:v) \in \mathbf{P}^1 \times \mathbf{P}^1.$$

This is equivalent to saying that $(A, B, \cdots, J) \in \mathrm{Syz}(a^2, ab, \cdots, d^2)$. In analogy with moving lines, the moving planes and moving quadrics of bidegree (k, l) are denoted by $\mathrm{Syz}(a, b, c, d)_{k,l}$, $\mathrm{Syz}(a^2, ab, \cdots, d^2)_{k,l}$ respectively. Let $R_{k,l}$ denote the bihomogeneous forms of bidegree (k, l) in s, u, t, v, and consider the map:

$$MP : R^4_{m-1,n-1} \xrightarrow{(a,b,c,d)} R_{2m-1,2n-1}$$

$$(A, B, C, D) \rightarrow Aa + Bb + Cc + Dd.$$

This map can be represented by a $mn \times mn$ matrix and

$$\dim(\mathrm{Syz}(a, b, c, d)_{m-1,n-1}) = \dim \ker(MP).$$

Also, there is a map:

$$MQ : R^{10}_{m-1,n-1} \xrightarrow{(a^2,ab,\cdots,d^2)} R_{3m-1,3n-1}$$

$$(A, B, \cdots, J) \rightarrow Aa^2 + Bab + \cdots + Jd^2.$$

This map can be represented by a $9mn \times 10mn$ matrix and

$$\dim(\mathrm{Syz}(a^2, ab, \cdots, d^2)_{m-1,n-1}) = \dim \ker(MQ).$$

If we construct an $mn \times mn$ matrix M whose columns are indexed by the monomial bases of $R_{m-1,n-1}$, the rows are indexed by linearly independent moving planes and moving quadrics, and the entries of the matrix are the coefficients of the moving planes and moving quadrics with respect to the monomial bases of $R_{m-1,n-1}$.

The result of [9], [10] says:

Theorem 1.2.2. *If $\phi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$ has no base points and is generically one-to-one, then $\dim(\mathrm{Syz}(a, b, c, d)_{m-1,n-1}) = 0$, $\dim(\mathrm{Syz}(a^2, ab, \cdots, d^2)_{m-1,n-1}) = mn$, and the implicit equation of the surface $S \subset \mathbf{P}^3$ parametrized by ϕ is*

$$F = \det(M),$$

where M is the matrix described above.

Current research is directed to the case where base points are present. Cox and Schenck [11] gave a nice theorem about the syzygies when base points are present. Recently, Cox, Busé, and D'Andrea [6] produced an algorithm for finding an implicit equation of a rational surface for the parametrization $\phi : \mathbf{P}^2 \to \mathbf{P}^3$ with base points present. Cox [9] also gave conjectures about the algorithm for finding an implicit equation of a rational surface for the parametrization $\phi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$ with base points present. My research is concerned with the problem of finding implicit equations of rational surfaces for the parametrization $\phi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$ with base points via syzygies.

1.3 Questions When Base Points Appear

Let's still consider the case of a parametrization

$$\phi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$$

$$\phi = (a(s, u, t, v) : b(s, u, t, v) : c(s, u, t, v) : d(s, u, t, v),$$

where $a, b, c, d \in R = \mathbb{C}[s, u, t, v]$ are bihomogeneous polynomials of bidegree (m, n), $\gcd(a, b, c, d) = 1$. Let $I = \langle a, b, c, d \rangle$. The *base points* are the common zeros of a, b, c, d, i.e. $\mathbb{V}(I)$, the variety of the ideal I. We assume $\mathbb{V}(I)$ is a finite subset of $\mathbf{P}^1 \times \mathbf{P}^1$.

Three questions arise when base points are present:

Question 1: *What is* $\dim \mathrm{Syz}(I)_{m-1,n-1}$*?*

Question 2: *What is* $\dim \mathrm{Syz}(I^2)_{m-1,n-1}$*?*

Question 3: *Will* $|M| = 0$ *define the image of* ϕ*?* M *is constructed as described before Theorem 1.2.2 in Section 1.2.*

The goal of this dissertation is to answer these questions. We will prove the following theorems.

Definition 1.3.1. The base points are *local complete intersection* (LCI) if for every point $p \in \mathbb{V}(I)$, I_p is a complete intersection ideal. This means that I_p is generated by 2 elements of $\mathcal{O}_p$.

Theorem 1.3.2.

$$\dim(\mathrm{Syz}(I)_{m-1,n-1}) = \sum_{p \in V(I)} e(I, p) = k$$

where $e(I, p)$ *denotes the multiplicity of the base point, provided*

1. *there are finitely many base points and the base points are LCI.*

2. $k = \dim(R/I)_{2m-1,2n-1} \leq mn$*.*

Definition 1.3.3. For a bigraded ideal $I \subset R = \mathbb{C}[x_0, \cdots, x_m, y_0, \cdots, y_n]$ where bidegree of x_i is $(1, 0)$ and bidegree of y_i is $(0, 1)$. The *saturation* of I is defined to be

$$I^{\mathrm{sat}} = \{f \in R : (x_i y_j)^k f \in I, 0 \leq i \leq m, 0 \leq j \leq n, \text{for some } k\}.$$

I^{sat} is the largest ideal containing I such that locally I^{sat} defines the same ideal as I, that is $\mathbb{V}(I) = \mathbb{V}(I^{\text{sat}})$ as a set and $\tilde{I} = \tilde{I}^{\text{sat}}$ as sheaves in $\mathbf{P}^m \times \mathbf{P}^n$ defined by I and I^{sat}.

Theorem 1.3.4. *In addition to the conditions in Theorem 1.3.2, assume* $d \in \text{sat}(a, b, c)$ *and* $\dim(\text{Syz}(a, b, c)_{m-1,n-1}) = 0$. *Then*

$$\dim(\text{Syz}(I^2)_{m-1,n-1}) = mn + 3k.$$

Theorem 1.3.5. *Under the conditions of Theorem 1.3.2 and Theorem 1.3.4,* $|M| = 0$ *is the implicit equation of the image of* ϕ.

1.4 Dissertation Overview

The proofs of Theorem 1.3.2, Theorem 1.3.4, and Theorem 1.3.5 will require the development of some background material. Some of these topics to be developed in Chapter 2, Chapter 3, Chapter 4 are as follows:

Regularity for Biprojective Space: Suppose J is a homogeneous ideal in a graded ring A. The regularity of J, denoted by $\text{reg}(J)$, is the smallest integer such that $J_k = A_k$ for all $k \geq \text{reg}(J)$. In general, regularity of a module is computed from the minimal free resolution of the module. Since we are working in the case of $\mathbf{P}^1 \times \mathbf{P}^1$, we need to extend these concepts to the bigraded modules in biprojective spaces. In simple terms, if I is a bihomogeneous ideal in bigraded ring R, then $\text{reg}(I) = (p, p')$ where p, p' are the smallest integers such that $I_{k,k'} = R_{k,k'}$ for all $k \geq p$ and $k' \geq p'$. The current definitions and properties about regularity only apply to graded modules. We will develop definitions and properties of (p, p')-regularity for coherent sheaves on $\mathbf{P}^m \times \mathbf{P}^n$ similar to Castelnuovo-Mumford's regularity [31] for coherent sheaves on $\mathbf{P}^m$. We will also give a definition of weak (p, p')-regularity for a bigraded module similar to the work of Johnston, Katz [26] and Ooishi [33]. However, in attempting to generalize a theorem such as [1, Definition 3.2] for free resolutions of bigraded modules, the conditions for weak regularity are inadequate. Therefore, we define a new concept, called strong regularity, and we will prove a relationship between strong regularity of a bigraded module and the minimal free resolution of the module. This work was done in collaboration with Dr. J. William Hoffman and is included in Chapter 2.

Regularity and Saturation: In Chapter 3, we define what it means for a bigraded ideal to be weakly (p, p')-saturated. This extends some of Bayer and Stillman's [2] results concerning saturation and regularity to the situation of a bigraded ideal. We will discuss some equivalence conditions between weakly (p, p')-saturated and weakly (p, p')-regular, and give a criterion for an bigraded ideal to be weakly (p, p')-regular. With the properties of weak saturation and weak regularity, we are able to provide a formula to compute the weak regularity of the saturation of a power of a bigraded ideal. This is similar to the results for a graded ideal indicated in Chandler's paper [8]. With the tools of regularity and saturation we will give a relation between a bigraded ideal I in a bigraded ring R being (p, p')-regular and the dimension of $\dim(R/I)_{p,p'}$. This work is a modified version of Cox, Busé, and D'Andrea [6]. Recently, Cox and Schenck [11] have shown that the module of syzygies vanishing at $\mathbb{V}(I)$ is generated by the Koszul syzygies if and only if $\mathbb{V}(I)$ is a local complete intersection. We will extend these theorems to the bigraded case.

Implicitization: In Chapter 4, we give the main result of the dissertation. We will prove that the algorithm conjectured by Cox [9] for producing the implicit equation of a rational surface from a parametrization $\phi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$ with base points via syzygies is valid. The key to the syzygy method is finding a square matrix M, in which the rows are formed by the coefficients of certain moving planes and moving quadrics. The degree of the determinant of the matrix M is the degree of the surface. The conditions for finding the appropriate number of moving planes and moving quadrics of a certain degree are based on properties of the regularity and the saturation of the bigraded ideal $I = \langle a, b, c, d \rangle$. In order to prove that the method will produce the implicit equation, we will also need to show that the determinant of the matrix M does not vanish identically. The syzygy algorithm for finding implicit equations of parametric surfaces with base points uses smaller determinants than resultants.

2. Castelnuovo-Mumford Regularity in Biprojective Spaces

2.1 Introduction

In chapter 14 of [30] Mumford introduced the concept of *regularity* for a coherent sheaf $\mathcal{F}$ on projective space $\mathbf{P}^n$: $\mathcal{F}$ is p-regular if, for all $i \geq 1$ we have vanishing for the twists

$$\mathrm{H}^i(\mathbf{P}^n, \mathcal{F}(k)) = 0, \quad \text{for all } k + i = p.$$

This in turn implies the stronger condition of vanishing for $k + i \geq p$. Regularity was investigated later by several people, notably Bayer and Mumford [1], Bayer and Stillman [2], Eisenbud and Goto [16], and Ooishi [33]. Let $R = K[x_0, ..., x_n]$ be the polynomial algebra in $n+1$ variables over a field K, graded in the usual way. If M is a finitely generated graded R-module, then the local cohomology groups $H^i_{\mathbf{m}}(M)$ with respect to the ideal $\mathbf{m} = (x_0, ..., x_n)$ are graded in a natural way and we say that M is p-regular if

$$H^i_{\mathbf{m}}(M)_k = 0 \quad \text{for all } k + i \geq p + 1.$$

If $\mathcal{F}$ is the coherent sheaf on $\mathbf{P}^n$ associated with M in the usual way, we have

$$H^{i+1}_{\mathbf{m}}(M)_k = \mathrm{H}^i(\mathbf{P}^n, \mathcal{F}(k)) \quad \text{for all } i \geq 1,$$

which shows the compatibility of these definitions (see [41, Lemma 1.8]). An important result in this theory is:

Theorem 2.1.1. *(see [1, Definition 3.2]) Suppose K is a field and $I \subset R$ is a graded ideal. Then I is p-regular if and only if the minimal free graded resolution of I has the form*

$$0 \longrightarrow \bigoplus_{\alpha=1}^{r_s} Re_{\alpha,s} \longrightarrow \cdots \longrightarrow \bigoplus_{\alpha=1}^{r_1} Re_{\alpha,1} \longrightarrow \bigoplus_{\alpha=1}^{r_0} Re_{\alpha,0} \longrightarrow I \longrightarrow 0$$

where $\deg(e_{\alpha,i}) \le p + i$ *for all* $i \ge 0$.

The conditions of p-regularity can be derived quasi-axiomatically from the following considerations. One seeks a condition of the form:

$$H^i_{\mathbf{m}}(M)_k = 0 \text{ all } i \ge 0, \text{ all } k \in C_i(p) \Longrightarrow R_s M_p = M_{p+s} \text{ all } s \ge 0, \tag{2.1}$$

for certain regions $C_i(p) \subset \mathbb{Z}$. One postulates:

1. For each i, the region $C_i(p)$ is independent of the number $n+1$ of variables.

2. If M is p-regular in the sense of the left-hand side of (2.1), then for a generic linear form $x \in R_1$, $\bar{M} = M/xM$ is p-regular over $\bar{R} = R/xR$.

First, when $n + 1 = 0$, that is, we are considering graded K-modules, since $\mathbf{m} = (0)$, we have $H^0_{\mathbf{m}}(M) = M$, and since $R_s = 0$ for $s \ge 1$, property (2.1) forces $M_k = H^0_{\mathbf{m}}(M)_k = 0$ for $k \ge p+1$ in this case, so we set $C_0 = \{k : k \ge p+1\}$. By principle 1., this must hold for all n. Assuming that $\mathbf{m} \notin \operatorname{Ass}(M)$ where $\operatorname{Ass}(M)$ denotes the associated primes for M, and K is infinite, then x may be chosen so that we have an exact sequence

$$0 \longrightarrow xM = M(-1) \longrightarrow M \longrightarrow \bar{M} \longrightarrow 0$$

which gives rise to the long exact sequence in cohomology. We have

$$H^0_{\mathbf{m}}(M)_k \longrightarrow H^0_{\bar{\mathbf{m}}}\left(\bar{M}\right)_k \longrightarrow H^1_{\mathbf{m}}(M(-1))_k = H^1_{\mathbf{m}}(M)_{k-1}.$$

In order that we have $H^0_{\bar{\mathbf{m}}}\left(\bar{M}\right)_k = 0$ for $k \ge p+1$, as is demanded by principle 2., we must have $H^1_{\mathbf{m}}(M)_k = 0$ for $k \ge p$. In a similar way, we obtain the vanishing region for $H^2_{\mathbf{m}}(M)_k$ from that of $H^1_{\mathbf{m}}(M)_k$, etc., and we find that they are exactly the conditions of p-regularity given. Of course, one deduces property (2.1) from the definition of p-regularity, by induction on the number of variables $n+1$, by a reversal of the above steps.

The other essential feature of p-regularity is that

3. R is 0-regular.

Note: $H^i(\mathbf{P}^n, \mathcal{O}(k)) = 0$ for all $i \geq 1$ and $i + k \geq 0$. This follows from Serre's calculations of the cohomology of the invertible sheaves $\mathcal{O}(k)$ on $\mathbf{P}^n$ ([38]), as reinterpreted by Grothendieck in the language of local cohomology (combine [19, Prop. (2.1.5)] with [20, Exp. II, Prop. 5]).

Our definition of regularity for bigraded modules follows this pattern. Let $R = K[x, y] = K[x_0, ..., x_m, y_0, ..., y_n]$, which is bigraded in the usual way. Let $\mathbf{m} = (xy) = (x_i y_j)$ be the irrelevant ideal. We seek regions $C_i(p, p') \subset \mathbb{Z}^2$ with the property that

$$H^i_{\mathbf{m}}(M)_{k,k'} = 0, \forall i \geq 0, \forall (k, k') \in C_i(p, p') \tag{2.2}$$

$$\Downarrow$$

$$R_{s,s'} M_{p,p'} = M_{p+s,p'+s'}, \forall s, s' \geq 0$$

One postulates the analogs of 1. and 2. above. For 2. we need regularity for both M/xM and M/yM for generic $x \in R_{1,0}$ and $y \in R_{0,1}$. This leads to regions called $Reg_{i-1}(p, p')$ (the shift $i \to i-1$ is explained later). We are able to prove analogs in this setting of many of the classical results of regularity for graded modules (see Theorem 2.3.4 and Proposition 2.3.5). Actually, we first do a separate treatment for sheaves, the way Mumford did (Propositions 2.2.7 and 2.2.8). However, in attempting to generalize Theorem 2.1.1 to a structure theorem for free resolutions for bigraded modules, the conditions we have proposed are seen to be inadequate. Therefore, we define a new concept of *strong* regularity and prove that it does indeed give the structure theorem that we want (Theorem 2.4.10). This involves vanishing conditions on $H^*_I(M)$ for each of the three ideals $I = (x), (y), (x, y)$. The previous notion of regularity is now called *weak* regularity. We show that strong regularity implies weak regularity, and that R itself is strongly $(0,0)$-regular. As far as we can determine, there is no simple vanishing condition for $H^*_{(xy)}(M)$ alone that implies the structure theorem that we want.

In the last section we write down a free resolution that permits computation of $H^i_{\mathbf{m}}(M)$. We will also give an example to show that weakly regular does not imply strongly regular.

2.2 Regularity for Coherent Sheaves

First, we will give the definition and some properties of regularity of a coherent sheaf similar to the treatment of Mumford [30, Ch. 14]. Let K be a field, and $R = K[x_0, \cdots, x_m, y_0, \cdots, y_n]$ be the polynomial ring, bigraded with variables x having bidegree $(1,0)$ and variables y having bidegree $(0,1)$. We let

$$\mathbf{m} = R_+ = \bigoplus_{a>0,b>0} R_{a,b},$$

be the *irrelevant ideal*. Some of the general theory of graded and multigraded algebras used here can be found in [17], [18].

Let $X = \mathbf{P}^m \times \mathbf{P}^n$, which when regarded as a scheme, is $\mathrm{Proj}\,(R)$, where by definition, this is the set of bigraded prime ideals $\mathbf{p}$ that do not contain the irrelevant ideal $\mathbf{m}$. There are projections p_1 and p_2 of X onto its two factors. If $\mathcal{F}_1$ is a sheaf of $\mathcal{O}_{\mathbf{P}^m}$-modules, and $\mathcal{F}_2$ is a sheaf of $\mathcal{O}_{\mathbf{P}^n}$-modules, we denote

$$\mathcal{F}_1 \boxtimes \mathcal{F}_2 = p_1^*\mathcal{F}_1 \otimes p_2^*\mathcal{F}_2, \text{ an } \mathcal{O}_X\text{-module.}$$

As in the usual case of projective space there is a functor $M \to \tilde{M}$ from bigraded R-modules to quasi-coherent sheaves on X, and every quasi-coherent sheaf $\mathcal{F}$ arises this way, in a nonunique fashion. In fact, if

$$M = \bigoplus_{(a,b)\in\mathbb{Z}^2} \mathrm{H}^0(X, \mathcal{F}(a,b))$$

then $\mathcal{F} \cong \tilde{M}$. Here, for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, we denote

$$\mathcal{F}(a,b) = \mathcal{F} \otimes \mathcal{O}_X(a,b)$$

where $\mathcal{O}_X(a,b) = \mathcal{O}_{\mathbf{P}^m}(a) \boxtimes \mathcal{O}_{\mathbf{P}^n}(b)$ is the invertible sheaf associated to the graded R-module $R(a,b)$. Recall that if M is any graded R-module, $M(a,b)$ is the graded module with degrees shifted via $M(a,b)_{d,e} = M_{d+a,e+b}$. If Z is a scheme, tensor products involving $\mathcal{O}_Z$-modules will always be relative to $\mathcal{O}_Z$ unless otherwise stated.

When $m \geq 1$, and $n \geq 1$, the Picard group $\mathrm{Pic}(X)$ is isomorphic with $\mathbb{Z}^2$ with (a,b) corresponding to $\mathcal{O}_X(a,b)$. Interpreting the Picard group as the group of divisor-classes, $\mathcal{O}_X(a,b)$ corresponds to the divisor $aL_1 + bL_2$, where $L_1 = H_1 \times \mathbf{P}^n$, $H_1 \subset \mathbf{P}^m$ being any hyperplane, and $L_2 = \mathbf{P}^m \times H_2$, $H_2 \subset \mathbf{P}^n$ being any hyperplane.

Note the special case: if m or n is 0, the biprojective space reduces to a projective space. Except in the case where both are 0, the Picard group $\mathrm{Pic}(X)$ is isomorphic with $\mathbb{Z}$. If both are 0, the space reduces to a point, and its Picard group is trivial. Even in these degenerate cases we still use notations such as $\mathcal{F}(a,b)$, where one or other twisting by a or b might be trivial.

Definition 2.2.1. For each integer $i > 0$, let

$$\begin{aligned} St_i &= \{(r,s) \in \mathbb{Z}^2 : r+s = -i-1,\ r<0,\ s<0\} \\ &= \{(-i,-1),\,(-i+1,-2),\,\ldots,\,(-2,-i+1),\,(-1,-i)\}. \end{aligned}$$

For $i \leq 0$, let

$$\begin{aligned} St_i &= \{(r,s) \in \mathbb{Z}^2 : r+s = -i,\ r \geq 0,\ s \geq 0\} \\ &= \{(-i,0),\,(-i-1,1),\,\ldots,\,(1,-i-1),\,(0,-i)\}. \end{aligned}$$

For each $(p,p') \in \mathbb{Z}^2$ let $St_i(p,p') = (p,p') + St_i$.

For $i \geq 0$, let $Reg_i(p,p') = \mathbb{Z}_+^2 + St_i(p,p')$ where $\mathbb{Z}_+ = \{n \in \mathbb{Z} : n \geq 0\}$.

For $i = -1$, let $Reg_{-1}(p,p') = \mathbb{Z}_+^2 + (p+1, p'+1)$.

Let $Reg'_{-1}(p,p') = (p+1,p') + \mathbb{Z}_+^2$.

Let $Reg''_{-1}(p,p') = (p,p'+1) + \mathbb{Z}_+^2$.

For $i \geq 0$, define $DReg_i(p,p') = \mathbb{Z}_-^2 + St_{-i}(p,p')$ where $\mathbb{Z}_- = \{n \in \mathbb{Z} : n \leq 0\}$.

Note that, for all $i \geq -1$,

$$Reg_i(p,p') \ = \ \mathbb{Z}^2 \cap \{(x,y) \in \mathbb{R}^2 \mid x \geq p-i, \ y \geq p'-i, \ x+y \geq p+p'-i-1\}$$

and, for all $i \geq 0$,

$$DReg_i(p,p') = -Reg_{i+1}(-p+1, -p'+1)$$

Remark 2.2.2. For $i \geq 0$, and for all p, p', we have

1. $(k,k') \in St_i(p,p') \Rightarrow (k-1,k'), (k,k'-1) \in St_{i+1}(p,p')$.

2. $St_i(p,p') \in Reg_i(p,p')$.

3. $(k,k') \in Reg_i(p,p') \Rightarrow (k-1,k'), (k,k'-1) \in Reg_{i+1}(p,p')$.

4. $Reg_i(q,q') \subset Reg_i(p,p')$, if $q \geq p, q' \geq p'$.

5. $(k,k') \in Reg'_{-1}(p,p') \Longrightarrow (k-1,k') \in Reg_0(p,p')$.

6. $(k,k') \in Reg''_{-1}(p,p') \Longrightarrow (k,k'-1) \in Reg_0(p,p')$.

Figure 1. and Figure 2. are pictures for $Reg_i(p,p')$ and $DReg_i(p,p')$:

Using these notations, we make the following definition.

Definition 2.2.3. Let $\mathcal{F}$ be a coherent sheaf on X. We will say that $\mathcal{F}$ is (p,p')-*regular* if, for all $i \geq 1$,

$$H^i(X, \mathcal{F}(k,k')) = 0$$

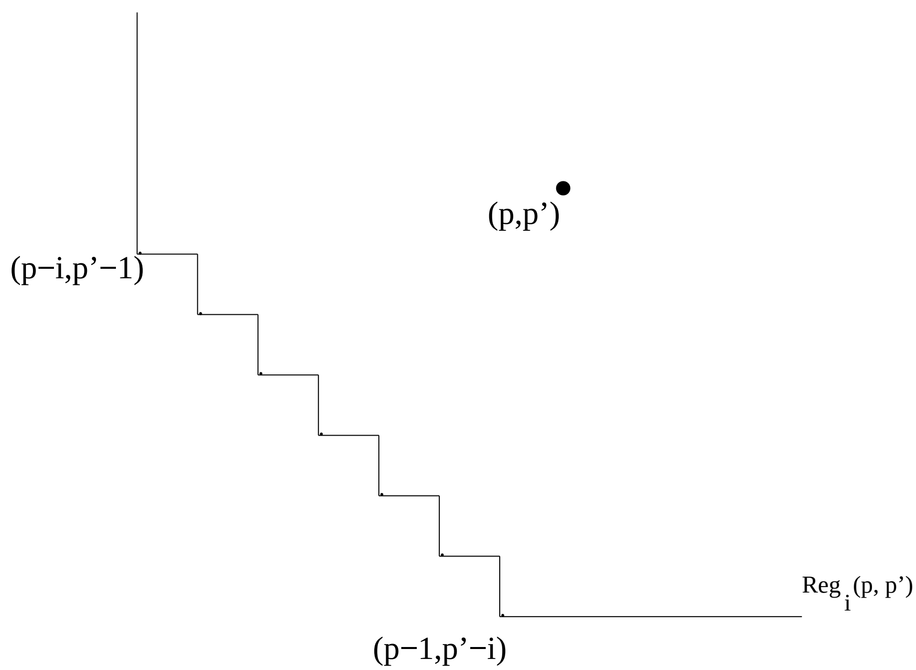

FIGURE 2.1. $Reg_i(p,p')$

whenever $(k,k') \in St_i(p,p')$.

Remark 2.2.4. If $n = 0$, $\mathbf{P}^m \times \mathbf{P}^0 \cong \mathbf{P}^m$, so every coherent sheaf on $\mathbf{P}^m \times \mathbf{P}^0$ is naturally identified with a sheaf on $\mathbf{P}^m$. The sheaf $\mathcal{F}(p,p')$ is independent of p'. Under this identification, $\mathcal{F}$ is (p,p')-regular on $\mathbf{P}^m \times \mathbf{P}^0$ in the sense of Definition 2.2.3, if and only if $\mathcal{F}$ is p-regular on $\mathbf{P}^m$ in the sense of Mumford.

Proof. First, we will show that (p,p')-regular implies p-regular.

In this case, $\mathcal{F}(k,k') \cong \mathcal{F}(k)$. $\mathcal{F}$ is (p,p')-regular means that for all $i \geq 1$,

$$H^i(\mathbf{P}^m \times \mathbf{P}^0, \mathcal{F}(k,k')) = H^i(\mathbf{P}^m, \mathcal{F}(k)) = 0,$$

where $p - i \leq k \leq p - 1$. Since $k + i \geq p$, according to [30, p. 100], $\mathcal{F}$ is p-regular.

Second, we will show that $\mathcal{F}$ is p-regular implies (p,p')-regular.

If $\mathcal{F}$ is p-regular, then $H^i(\mathbf{P}^m, \mathcal{F}(k)) = 0$ whenever $k + i \geq p$, this implies

$$H^i(\mathbf{P}^m \times \mathbf{P}^0, \mathcal{F}(k,k')) = 0$$

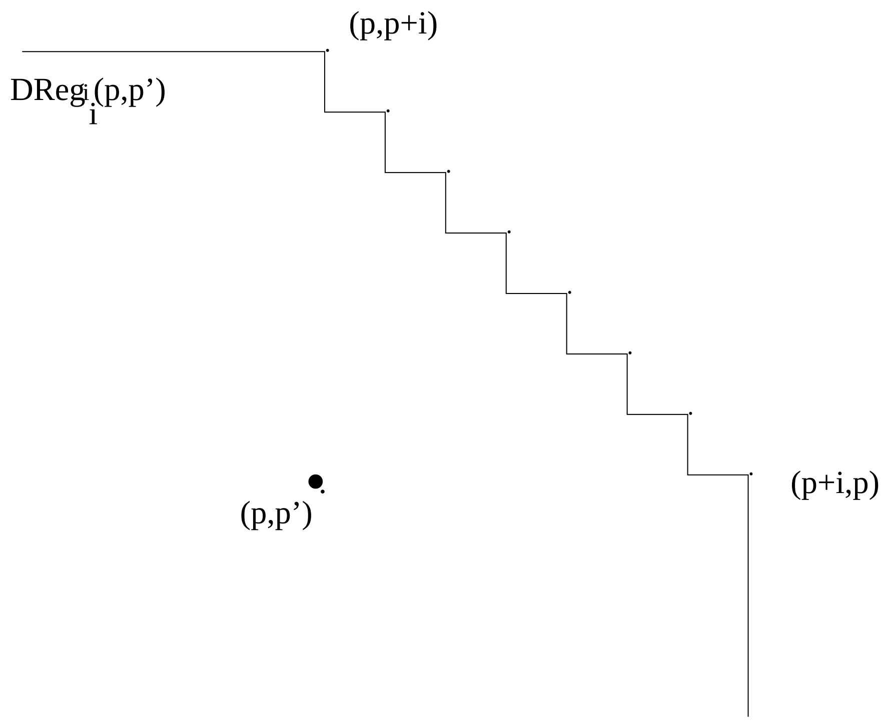

FIGURE 2.2. $DReg_i(p,p')$

for any $k' \in \mathbb{Z}$. In particular, $H^i(\mathbf{P}^m \times \mathbf{P}^0, \mathcal{F}(k,k')) = 0$ for all $(k,k') \in St_i(p,p')$. Therefore, $\mathcal{F}$ is (p,p')-regular. □

Proposition 2.2.5. *$\mathcal{O}_X$ is $(0,0)$-regular.*

Proof. If m or $n = 0$, by the previous remark, $\mathcal{O}_X$ is $(0,0)$-regular $\Leftrightarrow$ $\mathcal{O}_X$ is 0-regular. But $\mathcal{O}_{\mathbf{P}^m}$ is 0-regular since

$$\mathrm{H}^a\left(\mathbf{P}^m,\, \mathcal{O}_{\mathbf{P}^m}(k)\right) = 0, \text{ if } a \geq 1 \text{ and } a+k \geq 0 \tag{2.3}$$

$$\mathrm{H}^0\left(\mathbf{P}^m,\, \mathcal{O}_{\mathbf{P}^m}(k)\right) = 0, \text{ if } k \leq -1. \tag{2.4}$$

These formulas are a consequence of Serre's results on the cohomology of projective space. [21]

If m and $n \geq 1$, we can apply the Künneth formula [35],

$$\mathrm{H}^i\left(X,\, \mathcal{O}_{\mathbf{P}^m}(k) \boxtimes \mathcal{O}_{\mathbf{P}^n}(k')\right) = \bigoplus_{a+b=i} \mathrm{H}^a\left(\mathbf{P}^m,\, \mathcal{O}_{\mathbf{P}^m}(k)\right) \otimes \mathrm{H}^b\left(\mathbf{P}^n,\, \mathcal{O}_{\mathbf{P}^n}(k')\right).$$

We will show that $\mathrm{H}^a(\mathbf{P}^m, \mathcal{O}_{\mathbf{P}^m}(k)) = 0$ or $\mathrm{H}^b(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(k')) = 0$ whenever $a+b = i$ and $(k,k') \in St_i(0,0)$. If $(k,k') \in St_i(0,0)$, then $k = -i+l$ and $k' = -1-l$ where $0 \le l \le i-1$. If $a = 0$ or $b = 0$, we are done by Equation (2.4), since $k, k' < 0$. If $a > 0$, and $b > 0$, we only need to show $a - i + l \ge 0$ or $b - 1 - l \ge 0$. Suppose both $a - i + l \le -1$ and $b - 1 - l \le -1$. Since $a + b = i$,

$$-1 = (a - i + l) + (b - 1 - l) \le -2.$$

This contradiction shows that either $a - i + 1 \ge 0$ or $b - 1 - l \ge 0$, and the proof is completed by Equation (2.4). □

Lemma 2.2.6. *Assume that K is infinite, and that $m \ge 1$. Let $\mathcal{F}$ be a coherent sheaf on X. Let L_1 be a hyperplane defined by $\sum_{i=0}^m a_i x_i = 0$, and let $\mathcal{F}_{L_1} = \mathcal{F} \otimes \mathcal{O}_{L_1}$ denote the sheaf $\mathcal{F}$ restricted to L_1. If $\mathcal{F}$ is (p,p')-regular, then $\mathcal{F}_{L_1}$ is (p,p')-regular for a generic L_1. The similar statement is true for hyperplanes L_2 defined by a form $\sum_{i=0}^n b_i y_i = 0$ assuming $n \ge 1$.*

Proof. Given $\mathcal{F}$, choose a hyperplane L_1, where L_1 is defined by an equation of the form $f = \sum_{i=0}^m a_i x_i = 0$, such that L_1 does not contain any of points of the finite set of associated primes $A(\mathcal{F})$ (for the definition of this, see [30, p.40]). Note that this is possible: $A(\mathcal{F})$ is finite, and because K is infinite, we can find a linear form missing the p_1-projections of the associated primes. Tensor the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-1,0) \xrightarrow{f} \mathcal{O}_X \longrightarrow \mathcal{O}_{L_1} \longrightarrow 0$$

with $\mathcal{F}(k,k')$. For all $x \in X$, multiplication by f is injective in $\mathcal{F}_x$, since by construction, f is a unit at all associated primes of $\mathcal{F}_x$. Therefore the resulting sequence is exact:

$$0 \longrightarrow \mathcal{F}(k-1,k') \xrightarrow{f} \mathcal{F}(k,k') \longrightarrow \mathcal{F} \otimes \mathcal{O}_{L_1}(k,k') = \mathcal{F}_{L_1}(k,k') \longrightarrow 0 \tag{2.5}$$

This gives an exact cohomology sequence:

$$\cdots \longrightarrow H^i(\mathcal{F}(k,k')) \longrightarrow H^i(\mathcal{F}_{L_1}(k,k')) \longrightarrow H^{i+1}(\mathcal{F}(k-1,k')) \longrightarrow \cdots$$

Note, $H^i(\mathcal{F}) = H^i(X,\mathcal{F})$. If $(k,k') \in St_i(p,p')$, then $(k-1,k') \in St_{i+1}(p,p')$ by Remark 2.2.2, and the first and the last groups vanish when $i \geq 1$, since we are assuming that $\mathcal{F}$ is (p,p')-regular. This forces the second group to vanish, thus proving that $\mathcal{F}_{L_1}$ is (p,p')-regular. □

Proposition 2.2.7. *If $\mathcal{F}$ is a (p,p')-regular coherent sheaf on $X = \mathbf{P}^m \times \mathbf{P}^n$, then for all $i \geq 1$,*

$$H^i(X,\mathcal{F}(k,k')) = 0 \tag{2.6}$$

whenever $(k,k') \in Reg_i(p,p')$. That is, $\mathcal{F}$ is (q,q')-regular for $q \geq p,\ q' \geq p'$.

Proof. We will prove (2.6) by double induction on (m,n). If $m = 0$ or $n = 0$, by Remark 2.2.4 (p,p')-regularity reduces to ordinary p-regularity or p'-regularity for projective space, and (2.6) is true by Mumford's result [30]. So assume $m \geq 1$ and $n \geq 1$. Every element of $Reg_i(p,p')$ is of the form $(k+r,k'+s)$ for some $(k,k') \in St_i(p,p')$, and $(r,s) \geq (0,0)$. Now we will do double induction on the pair (r,s). The case $(r,s) = (0,0)$ is true by assumption of (p,p')-regularity for $\mathcal{F}$. Choose a hyperplane L_1 as in Lemma 2.2.6 such that $\mathcal{F}_{L_1}$ is (p,p')-regular. Consider the cohomology exact sequence attached to (2.5) with (k,k') replaced by $(k+r+1,k'+s)$:

$$H^i(\mathcal{F}(k+r,k'+s)) \longrightarrow H^i(\mathcal{F}(k+r+1,k'+s)) \longrightarrow H^i(\mathcal{F}_{L_1}(k+r+1,k'+s))$$

Since $\mathcal{F}_{L_1}$ is (p,p')-regular, and since L_1 is a biprojective space of lower dimension, the induction hypothesis says that the right-hand term is 0. The left-hand side also vanishes, by the induction hypothesis on (r,s). Hence the middle term vanishes, as required. A symmetric argument shows vanishing for $(k+r,k'+s+1)$. □

Proposition 2.2.8. *If $\mathcal{F}$ is a (p,p')-regular coherent sheaf on X, then $H^0(X,\mathcal{F}(k,k'))$ is spanned by*

$$H^0(X,\mathcal{F}(k-1,k'))\otimes H^0(X,\mathcal{O}(1,0)),$$

if $k>p, k'\geq p'$; and it is spanned by

$$H^0(X,\mathcal{F}(k,k'-1))\otimes H^0(X,\mathcal{O}(0,1)),$$

if $k\geq p, k'>p'$.

Proof. We use induction on $\dim(X)$: for $\dim(X)=0$, the result is true. By Lemma 2.2.6, we know that $\mathcal{F}_{L_1}$ is (p,p')-regular. Consider the following diagram:

$$\begin{array}{ccc}
H^0(\mathcal{F}(k-1,k'))\otimes H^0(\mathcal{O}_X(1,0)) & \xrightarrow{\sigma} & H^0(\mathcal{F}_{L_1}(k-1,k'))\otimes H^0(\mathcal{O}_{L_1}(1,0)) \\
\big\downarrow{\scriptstyle\mu} & & \big\downarrow{\scriptstyle\tau} \\
H^0(\mathcal{F}(k,k')) & \xrightarrow{\nu} & H^0(\mathcal{F}_{L_1}(k,k')) \\
\big\uparrow{\scriptstyle\alpha} & & \\
H^0(\mathcal{F}(k-1,k')) & &
\end{array}$$

If $k>p$ and $k'\geq p'$, σ is surjective because $\mathcal{F}$ is (p,p')-regular, and $H^1(\mathcal{F}(k-2,k'))=0$. τ is surjective by induction hypothesis. ν is also surjective, since $H^1(\mathcal{F}(k-1,k))=0$.

Let $t\in H^0(\mathcal{F}(k,k'))$, we have $\nu(t)=\tau(s)=\tau\sigma(s')$ for some

$$s\in H^0(\mathcal{F}_{L_1}(k-1,k'))\otimes H^0(\mathcal{O}_{L_1}(1,0)),\ \text{and}\ s'\in H^0(\mathcal{F}(k-1,k'))\otimes H^0(\mathcal{O}_Q(1,0)).$$

We have $\nu(\mu(s'))=\tau(\sigma(s'))=\nu(t)$, and $t-\mu(s')\in\ker(\nu)$. Since the last row of the diagram is exact in the middle, so we have $t'\in H^0(\mathcal{F}(k-1,k'))$ such that $\alpha(t')=t-\mu(s')$. This says that $H^0(\mathcal{F}(k,k'))$ is spanned by the image of μ and the image of α. But the image of α is in $H^0(\mathcal{F}(k-1,k'))\otimes H^0(\mathcal{O}(1,0))$, because the map

α is the multiplication by f, and $f \in H^0(\mathcal{O}(1,0))$. This means that $H^0(\mathcal{F}(k,k'))$ is spanned by

$$H^0(\mathcal{F}(k-1,k')) \otimes H^0(\mathcal{O}(1,0))$$

By symmetry, we can show that if $k \geq p, k' > p'$, $H^0(\mathcal{F}(k,k'))$ is spanned by

$$H^0(\mathcal{F}(k,k'-1)) \otimes H^0(\mathcal{O}(0,1)).$$

□

2.3 Weak Regularity for Bigraded Modules

We will give the definition and some properties of regularity for a bigraded module similar to Ooishi [33] and Johnston and Katz [26]. Let A be a noetherian ring, and let now $R = \oplus_{a,b\geq 0} R_{a,b}$ be any bigraded ring over A, with $R_{0,0} = A$. We assume that it is finitely generated by homogeneous elements of bidegrees $(1,0)$ and $(0,1)$. Such a ring will be called a *bihomogeneous A-algebra*. Previously we considered only the case of a polynomial ring in two sets of variables over a field. Let $\mathbf{m} = R_+ = \oplus_{a>0,b>0} R_{a,b}$ be the irrelevant ideal; it is a bigraded R-module. There is a scheme $X = \mathrm{Proj}\,(R)$, whose points are the bihomogeneous prime ideals $\mathbf{p}$ of R that do not contain the irrelevant ideal. We also have a functor $M \to \tilde{M}$ from bigraded modules to quasicoherent $\mathcal{O}_X$-modules with similar properties to those discussed in section 2.2. Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module. If we set $M = \oplus_{a,b\in\mathbb{Z}} H^0(X, \mathcal{F}(a,b))$, then we have $\mathcal{F} = \tilde{M}$.

If R is a bigraded A-algebra, then it defines a graded A-algebra

$$R^\sharp_n = \bigoplus_{i+j=n} R_{i,j}$$

and similarly we have a graded $R^\sharp$-module $M^\sharp$ associated to a bigraded R-module M.

Let $M = \oplus_{a,b\in\mathbb{Z}} M_{a,b}$ be a bigraded R-module. The local cohomology groups $H^i_{\mathbf{m}}(M)$ are bigraded R-modules, and let $H^i_{\mathbf{m}}(M)_{a,b}$ denote the (a,b) part. The general theory of local cohomology is found in [20]. Note that, if $J \subset A$ is an ideal in a ring, and $V(J) \subset C = \mathrm{Spec}(A)$ is the corresponding closed subset, then

$$H^*_J(M) = H^*_{V(J)}\left(C, \tilde{M}\right)$$

where $\tilde{M}$ is the quasi-coherent sheaf on C associated with the A-module M.

We have

$$H^i_{\mathbf{m}}(M)^\sharp = H^i_{\mathbf{m}^\sharp}\left(M^\sharp\right), \text{ ie., } H^i_{\mathbf{m}^\sharp}\left(M^\sharp\right)_n = \bigoplus_{k+k'=n} H^i_{\mathbf{m}}(M)_{k,k'}$$

Generally we omit the $\sharp$ from $\mathbf{m}$ and M, as it is clear in context that we are referring to the graded, as opposed to the bigraded structure.

We recall the following fact [41, Lemma 1.8]: Let R be any ring, $I \subset R$ an ideal and M an R-module. If $\mathrm{Supp}(M) \subset V(I)$ then

$$H^0_I(M) = M, \text{ and } H^i_I(M) = 0 \text{ for } i \geq 1.$$

Also, if R is Noetherian and M is finitely generated,

$$\mathrm{Ass}(M) \subset \mathrm{Supp}(M),$$

and both have the same minimal elements where $\mathrm{Ass}(M)$ denotes the associated primes of M, $\mathrm{Supp}(M)$ denotes the support of M. $\mathrm{Ass}(M)$ is finite.

We allow the case where $R_{a,b} = 0$ for all $a > 0$, or $R_{a,b} = 0$ for all $b > 0$. For then $\mathbf{m} = 0$, and thus for all R-modules M,

$$H^0_{\mathbf{m}}(M) = M, \text{ and } H^i_{\mathbf{m}}(M) = 0 \text{ for } i \geq 1.$$

since $V(\mathbf{m}) = \mathrm{Spec}(R)$, so $\mathrm{Supp}(M) \subset V(\mathbf{m})$ always holds. This extreme case plays an important role in the proofs of the main theorems about regularity, which are by induction on the number of variables.

Definition 2.3.1. We say that a bigraded R-module M is *weakly (p,p')-regular*, if for all $i \geq 0$,

$$H^i_{\mathbf{m}}(M)_{k,k'} = 0 \text{ for all } (k,k') \in Reg_{i-1}(p,p')$$

The connection with the previous concept of regularity for coherent sheaves is established by the following:

Proposition 2.3.2. *(see [24]) Let $X = \mathrm{Proj}(R)$. For any finitely generated bigraded R-module M we have an exact sequence of bigraded R-modules*

$$0 \longrightarrow H^0_{\mathbf{m}}(M) \longrightarrow M \longrightarrow \bigoplus_{(a,b)\in\mathbb{Z}^2} H^0(X, \mathcal{M}(a,b)) \longrightarrow H^1_{\mathbf{m}}(M) \longrightarrow 0$$

and an isomorphism of bigraded R-modules

$$H^{i+1}_{\mathbf{m}}(M) = \bigoplus_{(a,b)\in\mathbb{Z}^2} H^i(X, \mathcal{M}(a,b)), \quad \forall i \geq 1$$

Corollary 2.3.3. *Let $\tilde{M}$ be the sheaf on X associated to the bigraded R-module M. If M is weakly (p,p')-regular, then $\tilde{M}$ is (p,p')-regular in the sense of definition 2.2.3. This explains the shift in index from i to $i-1$ in the definition of weak regularity for modules.*

The main result for weak regularity is the following:

Theorem 2.3.4. *Let R be a bihomogeneous A-algebra, M a finitely generated bigraded R-module. Fix (p,p').*

1. *Suppose that $H^i_{\mathbf{m}}(M)_{k,k'} = 0$ for all $i \geq 1$ and all $(k,k') \in St_{i-1}(p,p')$, then*

$$H^i_{\mathbf{m}}(M)_{k,k'} = 0 \text{ for all } i \geq 1 \text{ and all } (k,k') \in Reg_{i-1}(p,p')$$

2. *Moreover,*

   a. *if $H^0_{\mathbf{m}}(M)_{k,k'} = 0$ for $(k,k') \in Reg'_{-1}(p,p')$, then we have $R_{d,0}M_{k,k'} = M_{d+k,k'}$ for every $d \geq 0, k \geq p, k' \geq p'$;*

b. *if* $H^0_{\mathbf{m}}(M)_{k,k'} = 0$ *for* $(k,k') \in Reg''_{-1}(p,p')$*, then we have* $R_{0,d'}M_{k,k'} = M_{k,k'+d'}$ *for every* $d' \geq 0, k \geq p, k' \geq p'$.

3. *if* M *is weakly* (p,p')*-regular, and if* $H^0_{\mathbf{m}}(M)_{k,k'} = 0$ *for* $(k,k') \in Reg^{'}_{-1}(p,p') \cup Reg^{''}_{-1}(p,p')$*, then* $R_{d,d'}M_{k,k'} = M_{k+d,k'+d'}$ *for all* $d, d' \geq 0$, $k \geq p, k' \geq p'$.

Proof. First, by the same argument as in Ooishi [33, Theorem 2], we may reduce to the case where A is a local ring with infinite residue field, and assume that $R = A[x_0, \cdots, x_m, y_0, \cdots, y_n]$, with irrelevant ideal $\mathbf{m}$ generated by the $x_i y_j$. We will prove the claim by induction on (m,n). If either $m = -1$ or $n = -1$ (i.e., either x or y variables are missing), or if

$$\mathrm{Ass}_+(M) = \{\mathbf{p} \in \mathrm{Ass}(M) : \mathbf{p} \not\supseteq \mathbf{m}\} = \emptyset$$

the claim is true: in the first case the irrelevant ideal $\mathbf{m} = 0$, so that the remark before the statement of Proposition 2.3.2 applies; in the second case, we have $\mathrm{Supp}(M) \subset V(\mathbf{m})$. In either case, $H^0_{\mathbf{m}}(M) = M$ and $H^i_{\mathbf{m}}(M) = 0$ for every $i \geq 1$.

Suppose that both $m \geq 0$ and $n \geq 0$, and $\mathrm{Ass}_+(M) = \{\mathbf{p}_1, \cdots, \mathbf{p}_r\}$. By our assumptions, A is a um-ring in the terminology of [34], and by theorem 2.3 of that paper we conclude that if we had an equality of A-modules

$$R_{1,0} = \max(A)R_{1,0} \cup (\mathbf{p}_1 \cap R_{1,0}) \cup \cdots \cup (\mathbf{p}_r \cap R_{1,0})$$

then $R_{1,0}$ would have to be equal to one of the terms in the union. It clearly is not the first term. If, say $R_{1,0} = \mathbf{p}_1 \cap R_{1,0}$ we would have

$$\mathbf{m} \subset (x_0, ..., x_m) = R \cdot R_{1,0} \subset \mathbf{p}_1$$

which is contrary to the fact that $\mathbf{p}_1$ does not contain $\mathbf{m}$. Thus we can find an element

$$x \in R_{1,0} - \max(A)R_{1,0} \cup (\mathbf{p_1} \cap R_{1,0}) \cup \cdots \cup (\mathbf{p_r} \cap R_{1,0})$$

which we can take as part of a free basis of $R_{1,0}$. Since $x \notin \max(A)R_{1,0}$, the image of x in $R/\max(A)R_{1,0}$ is non-zero. A non-zero element in a vector space can be extended to a basis, then by Nakayama's Lemma x is a free basis. By change of coordinate, we may assume that $x = x_m$.

(1.) Consider the following exact sequence:

$$0 \longrightarrow M_1 \longrightarrow M \xrightarrow{x} xM(1,0) \longrightarrow 0.$$

This implies:

$$H^i_{\mathbf{m}}(M_1) \longrightarrow H^i_{\mathbf{m}}(M) \longrightarrow H^i_{\mathbf{m}}(xM(1,0)) \longrightarrow H^{i+1}_{\mathbf{m}}(M_1). \tag{2.7}$$

Since x was chosen not to belong to any of the $\mathbf{p}_i$, $\mathrm{Supp}(M_1) \subset V(\mathbf{m})$, and so by the remarks above, the first and last terms above vanish when $i \geq 1$, and so $H^i_{\mathbf{m}}(M) \cong H^i_{\mathbf{m}}(xM(1,0))$ for every $i \geq 1$. Set $\bar{R} = R/xR = A[x_0, \cdots, x_{m-1}, y_0, \cdots, y_n]$, $\bar{\mathbf{m}} = \bar{R}_+$ and $\bar{M} = M/xM$. From

$$0 \longrightarrow xM \longrightarrow M \longrightarrow \bar{M} \longrightarrow 0,$$

we have the exact sequence:

$$H^i_{\mathbf{m}}(M)_{k,k'} \longrightarrow H^i_{\bar{\mathbf{m}}}(\bar{M})_{k,k'} \longrightarrow H^{i+1}_{\mathbf{m}}(xM)_{k,k'} = H^{i+1}_{\mathbf{m}}(M)_{k-1,k'}. \tag{2.8}$$

If $(k,k') \in St_{i-1}(p,p')$, then the first term is 0, by our assumption on M.

Now assume that $i \geq 2$. Then, $(k-1,k') \in St_i(p,p')$ by Remark 2.2.2, and so the last term above is 0, also by our assumption on M, so that $H^i_{\bar{\mathbf{m}}}(\bar{M})_{k,k'} = 0$. By induction hypothesis $H^i_{\bar{\mathbf{m}}}(\bar{M})_{k,k'} = 0$ for every $i \geq 1$ and $(k,k') \in Reg_{i-1}(p,p')$. If $i \geq 2$ and $(k,k') \in St_{i-1}(p+1,p')$, then in the exact sequence

$$H^i_{\mathbf{m}}(M)_{k-1,k'} = H^i_{\mathbf{m}}(xM)_{k,k'} \longrightarrow H^i_{\mathbf{m}}(M)_{k,k'} \longrightarrow H^i_{\bar{\mathbf{m}}}(\bar{M})_{k,k'} \tag{2.9}$$

the first and last terms are 0 because $(k-1,k') \in St_{i-1}(p,p')$ and $(k,k') \in Reg_{i-1}(p,p')$, so $H^i_{\mathbf{m}}(M)_{k,k'} = 0$ when $i \geq 1$ and $(k,k') \in St_{i-1}(p+1,p')$. Repeating the argument we get $H^i_{\mathbf{m}}(M)_{k,k'} = 0$ when $i \geq 1$ and $(k,k') \in St_{i-1}(p+d,p')$ for every $d \geq 0$, and by symmetry, arguing with a $y \in R_{0,1}$, we get $H^i_{\mathbf{m}}(M)_{k,k'} = 0$ when $i \geq 1$ and $(k,k') \in St_{i-1}(p+d,p'+d')$ for every $d,d' \geq 0$, which is the first claim for $i \geq 2$.

When $i = 1$, the only changes to make in the argument are the following. If $(k,k') \in St_0(p,p')$, then $(k-1,k') \in Reg_1(p,p')$, by Remark 2.2.2. But then $H^2_{\mathbf{m}}(M)_{k-1,k'} = 0$ has been established by the argument in the previous paragraph. Also, when $(k,k') \in St_0(p+1,p')$, we have $(k-1,k') \in St_0(p,p')$ and $(k,k') \in Reg_0(p,p')$, so that the first and last terms in the sequence (2.9) vanish when $i = 1$, too.

(2a.) Let $\mathrm{Ass}_+(M) = \{\mathbf{p} \in \mathrm{Ass}(M) : \mathbf{p} \not\supseteq \mathbf{m}\}$. Suppose $m,n \geq 0$ and $\mathrm{Ass}_+(M) = \{\mathbf{p}_1, \cdots, \mathbf{p}_r\}$. As before, we change coordinates so that $x = x_m \notin \mathbf{p}_i$, for any i. Set $\bar{R} = R/xR = A[x_0,\cdots,x_{m-1},y_0,\cdots,y_n]$, $\bar{\mathbf{m}} = \bar{R}_+$ and $\bar{M} = M/xM$. We claim that the induction hypothesis can be applied to $\bar{M}$. First, by the argument proving (1.), we saw that

$$H^i_{\bar{\mathbf{m}}}(\bar{M})_{k,k'} = 0 \text{ for } i \geq 1 \text{ and } (k,k') \in Reg_{i-1}(p,p').$$

From the sequence (2.8) above with $i = 0$, we see that $H^0_{\bar{\mathbf{m}}}(\bar{M})_{k,k'} = 0$ for every $(k,k') \in Reg'_{-1}(p,p')$, because the extreme terms vanish: the left-hand one because of our assumption on M, the right-hand one because $(k-1,k') \in Reg_0(p,p')$ by Remark 2.2.2 and vanishing of this term has been established above. Thus by the induction hypothesis applied to $\bar{M}$, and we have $\bar{R}_{d,0}\bar{M}_{k,k'} = \bar{M}_{d+k,k'}$, which implies $R_{d,0}M_{k,k'} + xM_{d+k-1,k'} = M_{d+k,k'}$. Reasoning by induction on $d \geq 1$, we assume that $M_{d+k-1,k'} = R_{d-1,0}M_{k,k'}$ has been established, the case $d = 1$ being trivial. Then

$$M_{d+k,k'} = R_{d,0}M_{k,k'} + xM_{d+k-1,k'} = R_{d,0}M_{k,k'} + xR_{d-1,0}M_{k,k'} = R_{d,0}M_{k,k'}.$$

This proves our claim. By symmetry, arguing with a y_n, we get the assertion $M_{k,d'+k'} = R_{0,d'}M_{k,k'}$.

(3.) This follows by repeated application of (2a) and (2b). $\square$

For bigraded ideals in the polynomial ring $R = K[x_0, \cdots, x_m, y_0, \cdots, y_n]$, we have:

Proposition 2.3.5. *Let K be a field, let $I \subset R$ be any ideal generated by bi-homogeneous polynomials, let $\mathcal{I}$ be the corresponding sheaf of ideals in $\mathcal{O}_X$. The following properties are equivalent.*

I. The ideal I is weakly (p, p')-regular in the sense of Definition 2.3.1.

II. The natural map $I_{p,p'} \to H^0(\mathcal{I}(p, p'))$ is an isomorphism and $\mathcal{I}$ is (p, p')-regular in the sense of Definition 2.2.3.

III. The natural map $I_{d,d'} \to H^0(\mathcal{I}(d, d'))$ is an isomorphism and $\mathcal{I}$ is (d, d')-regular, for all $d \geq p, d' \geq p'$.

Proof. There is no loss in generality in assuming that K is infinite, because we may tensor the whole situation by the algebraic closure of K.

(I $\Rightarrow$ II) If I is weakly (p, p')-regular in the sense of Definition 2.3.1, then we have $H^i_{\mathbf{m}}(I)_{k,k'} = 0$ for $(k, k') \in Reg_{i-1}(p, p')$ for $i \geq 1$. But for an ideal in a polynomial ring, we also have $H^0_{\mathbf{m}}(I) = 0$, since there are no 0-divisors in the ring R. By Proposition 2.3.2, $H^i(\mathcal{I}(k, k')) = H^{i+1}_{\mathbf{m}}(I)_{k,k'} = 0$ for $i \geq 1$, $(k, k') \in Reg_i(p, p')$ for $i \geq 1$; and $I_{p,p'} \cong H^0(\mathcal{I}(p, p'))$.

(II $\Rightarrow$ I) If $\mathcal{I}$ is (p, p')-regular in the sense of Definition 2.2.3, then $H^i(\mathcal{I}(k, k')) = 0$ for $(k, k') \in Reg_i(p, p')$, $i \geq 1$ by Proposition 2.2.7. I is an ideal, $H^0_{\mathbf{m}}(I) = 0$, in particular, $H^0_{\mathbf{m}}(I)_{k,k'} = 0$ for $(k, k') \in Reg_{-1}(p, p')$. Since $I_{p,p'} \cong H^0(\mathcal{I}(p, p'))$, by Proposition 2.3.2 and Proposition 2.2.8, we have $H^1_{\mathbf{m}}(I)_{k,k'} = 0$ for all $(k, k') \in$

$Reg_0(p,p')$, and $H^{i+1}_{\mathbf{m}}(I)_{k,k'} = H^i(\mathcal{I}(k,k')) = 0$ for $(k,k') \in Reg_i(p,p')$, $i \geq 1$. Therefore $H^i_{\mathbf{m}}(I)_{k,k'} = 0$ for all $(k,k') \in Reg_{i-1}(p,p')$, $i \geq 0$, i.e. I is weak (p,p')-regular in the sense of Definition 2.3.1.

(II $\Rightarrow$ III) follows from Proposition 2.2.7, and Proposition 2.2.8.

(III $\Rightarrow$ II) is obvious, we just take $d = p, d' = p'$. □

2.4 Strong Regularity for Bigraded Modules

From now on, K is a field and $R = K[x_0, \cdots, x_m, y_0, \cdots, y_n] = K[x,y]$ is a polynomial algebra, bigraded in the usual way. We will be using the ideals $(x) = (x_0, \cdots, x_m)$, $(y) = (y_0, \cdots, y_n)$, $(x,y) = (x_0, \cdots, x_m, y_0, \cdots, y_n)$, and $(xy) = \mathbf{m} = (x_i y_j)$.

In addition to the graded $K[x,y]$-module $M^\sharp$ introduced above, we need to consider the following graded modules. Fix j', and let $M^{[1]}_{j'} = \oplus_j M_{j,j'}$, which is a $K[x] = K[x_0, \cdots, x_m]$-module; fix j, and let $M^{[2]}_j = \oplus_{j'} M_{j,j'}$, which is a $K[y] = K[y_0, \cdots, y_m]$-module. Observe that

$$M = \bigoplus_{j'} M^{[1]}_{j'} = \bigoplus_{j} M^{[2]}_{j}$$

as $K[x]$-module (resp. as $K[y]$-module). Also, each $H^i_{(x)}\left(M^{[1]}_{j'}\right)$ is a graded $K[x]$-module (resp. each $H^i_{(y)}\left(M^{[2]}_{j}\right)$ is a graded $K[y]$-module), but both $H^i_{(x)}(M)$ and $H^i_{(y)}(M)$ are bigraded $K[x,y]$-modules.

$$H^i_{(x)}(M) = \bigoplus_{j'} H^i_{(x)}\left(M^{[1]}_{j'}\right)$$

$$H^i_{(y)}(M) = \bigoplus_{j} H^i_{(y)}\left(M^{[2]}_{j}\right)$$

$$H^i_{(x)}(M)_{j,j'} = H^i_{(x)}\left(M^{[1]}_{j'}\right)_j$$

$$H^i_{(y)}(M)_{j,j'} = H^i_{(y)}\left(M^{[2]}_{j}\right)_{j'}$$

Definition 2.4.1. Let M be a bigraded R-module and let $d \geq 0$.

I. M satisfies the *vanishing condition* $VC_d(p, p')$ if for all $i \geq 0$

$$\begin{aligned} H^i_{(x)}(M)_{k,k'} &= H^i_{(x)}(M^{[1]}_{k'})_k = 0,\ \forall k \geq p + d + 1 - i, \forall k'; \\ H^i_{(y)}(M)_{k,k'} &= H^i_{(y)}(M^{[2]}_{k})_{k'} = 0,\ \forall k' \geq p' + d + 1 - i, \forall k; \\ & H^i_{(x,y)}(M^\sharp)_{k+k'} = 0,\ \forall k + k' \geq p + p' + d + 1 - i. \end{aligned}$$

II. M is *strongly* (p, p')*-regular* if M satisfies $VC_0(p, p')$.

Remark 2.4.2. For all p, p', we have

1. If M satisfies $VC_0(p, p')$, then M satisfies $VC_d(p, p')$ for all $d \geq 0$.
2. If M satisfies $VC_d(p, p')$, then $M(a, b)$ satisfies $VC_d(p - a, p' - b)$.
3. For all $(\alpha, \alpha') \in DReg_d(p, p')$, if M satisfies $VC_0(\alpha, \alpha')$, then M satisfies $VC_d(p, p')$.

Proposition 2.4.3. *Let* $R = K[x_0, ..., x_m, y_0, ..., y_n]$ *be a bigraded polynomial algebra over a field* K*. Assume that* $m, n \geq 0$*. Then* R *is strongly* $(0, 0)$*-regular.*

Proof. If $R = K[z_1, ..., z_s]$ is any polynomial algebra over a field K, the relationship between local cohomology and sheaf cohomology tells us that $H^i_{(z)}(R)_k = H^{i-1}(\tilde{R}(k))$ for $i \geq 1$. Since $H^i(\tilde{R}(k)) = 0$ for $i + k \geq 1$. Therefore we have that $H^i_{(z)}(R)_k = 0$ whenever $i + k \geq 1$. This verifies the vanishing statement for $H^i_{(x,y)}(R^\sharp)_{k+k'}$ for $R = K[x, y]$. For the case $H^i_{(x)}(R^{[1]}_{k'})_k$, note that

$$R^{[1]}_{k'} = \bigoplus_{|\beta| = k'} K[x] y^\beta.$$

As each term is a free module over $K[x]$ and local cohomology commutes with direct sum, the requisite vanishing follows from Serre's result. □

Proposition 2.4.4. *If a bigraded R-module M satisfies $VC_d(p, p')$, then*

$$H^i_{(xy)}(M)_{k,k'} = 0$$

for all $(k, k') \in Reg_0(p + d + 1 - i, p' + d + 1 - i), \quad 0 \leq i \leq d + 2$*; and for all* $(k, k') \in Reg_{i-d-1}(p, p'), \quad i > d + 2.$

Proof. By the Mayer-Vietoris sequence, we have

$$H^i_{(x,y)}(M) \longrightarrow H^i_{(x)}(M) \oplus H^i_{(y)}(M) \longrightarrow H^i_{(xy)}(M) \longrightarrow H^{i+1}_{(x,y)}(M)$$

(see [21, Exercise 2.4, Ch. III, p. 212]; Note that if $Y_1 = V(x)$, $Y_2 = V(y)$, then $Y_1 \cup Y_2 = V(xy)$ and $Y_1 \cap Y_2 = V(x, y)$ as subsets of $C = \mathrm{Spec}(K[x, y])$.) Assuming that M satisfies $VC_d(p, p')$ we see that $H^i_{(xy)}(M)_{k,k'} = 0$ for all (k, k') that satisfy the inequalities:

$$k \geq p + d + 1 - i, \quad k' \geq p' + d + 1 - i, \;\; k + k' \geq p + p' + d + 1 - (i + 1).$$

If $0 \leq i \leq d + 2$ the last condition above is redundant, and so we obtain vanishing in the region described by the first two inequalities, which is just $Reg_0(p + d + 1 - i, p' + d + 1 - i)$. If $i > d + 2$, these three inequalities describe $Reg_{i-d-1}(p, p')$. □

Corollary 2.4.5. *If M is strongly* (p, p')*-regular, then it is weakly* (p, p')*-regular.*

Proof. We have $H^i_{\mathbf{m}}(M)_{k,k'} = 0$ for all $(k, k') \in Reg_{i-1}(p, p')$, according to the Proposition 2.4.4, whenever $i \geq 2$. For $i = 0, 1$, this is zero for $(k, k') \in Reg_0(p + 1 - i, p' + 1 - i)$, but these are exactly the regions $Reg_{-1}(p, p')$ and $Reg_0(p, p')$. Thus we have the conditions for weak (p, p')-regularity. □

Remark 2.4.6. If M is strongly (resp. weakly) (p, p')-regular, then $M(a, b)$ is strongly (resp. weakly) $(p - a, p' - b)$-regular.

Proposition 2.4.7. *If a finitely generated bigraded R-module M satisfies $VC_d(p,p')$, then M is generated by elements of bidegree $(k,k') \in DReg_d(p,p')$.*

Proof. Let A be a homogeneous algebra,(i.e., for a ring R, a graded ring A is called a graded R-algebra if $A_0 = R$, and A is called a homogeneous R-algebra if $A_0 = R$ and A is generated by A_1 over R.) (see introduction to [33]), with maximal ideal P. If N is a finitely generated graded module over A, then [33, Thm. 2] asserts that if $H^i_P(N)_k = 0$ for all $i + k \geq m + 1$, then N is generated by elements of degrees $\leq m$.

We first apply this to the graded module $N = M^\sharp$ over the graded ring $A = R^\sharp$. Since M satisfies $VC_d(p,p')$, we have

$$H^i_{(x,y)}(M^\sharp)_{k+k'} = 0, \ \forall k + k' \geq p + p' + d + 1 - i$$

so that by the previous remark, $M^\sharp$ can be generated by elements of degree $\leq p + p' + d$. This means that the bigraded M can be generated by bihomogeneous elements of bidegree (k,k') with $k + k' \leq p + p' + d$. Now let $A = K[x]$, and for a fixed k', regard $N = M^{[1]}_{k'}$ as an A-module. That M satisfies $VC_d(p,p')$ means here that

$$H^i_{(x)}(M^{[1]}_{k'})_k = 0, \ \forall k \geq p + d + 1 - i$$

and thus by Ooishi's result, that $M^{[1]}_{k'}$ can be generated as a $K[x]$-module by elements of degree $\leq p + d$. This being true for every k', we see that

$$R_{s,0}M_{p+d,k'} = M_{p+d+s,k'} \quad \text{for all } s \geq 0, k'.$$

Similar reasoning applied to $M^{[2]}_k$ as a $K[y]$-module leads to

$$R_{0,s}M_{k,p'+d} = M_{k,p'+d+s} \quad \text{for all } s \geq 0, k.$$

Combining this information gives that M can be generated by bihomogeneous elements of degree (k, k') where

$$k \leq p + d, \quad k' \leq p' + d \quad k + k' \leq p + p' + d$$

This is the description of the region $DReg_d(p, p')$. $\square$

In the following context M_d is a bigraded R module which satisfies $VC_d(p, p')$, note, the index d is not the degree index of the module. By Proposition 2.4.7, M_d is generated by elements of bidegree $\deg(e_{\alpha,d}) = (\alpha_d, \alpha'_d) \in DReg_d(p, p')$. We can find an exact sequence:

$$0 \longrightarrow M_{d+1} \longrightarrow \bigoplus_{\alpha=1}^{r_d} Re_{\alpha,d} \xrightarrow{\phi_d} M_d \longrightarrow 0,$$

where $M_{d+1} = \ker \phi_d$.

Proposition 2.4.8. *Let M_d be as above. If M_d satisfies $VC_d(p, p')$, then M_{d+1} satisfies $VC_{d+1}(p, p')$, and are generated by elements of bidegree in $DReg_{d+1}(p, p')$.*

Proof. For the case $i = 0$, we have an injection

$$H^0_{(x)}\left(M_{d+1}\right)_{k,k'} \subset \bigoplus_{\alpha=1}^{r_d} H^0_{(x)}(R)_{k-\alpha_d, k'-\alpha'_d} = 0,$$

so we can assume that $i \geq 1$. Consider the local cohomology sequence with $I = (x)$ of the above exact sequence:

$$H^{i-1}_I(M_d)_{k,k'} \longrightarrow H^i_I(M_{d+1})_{k,k'} \longrightarrow \bigoplus_{\alpha=1}^{r_d} H^i_I(R)_{k-\alpha_d, k'-\alpha'_d}$$

Suppose that $k + i \geq p + (d + 1) + 1$. Then the left-hand side above vanishes by assumption on M_d, because $k + (i - 1) \geq p + d + 1$. That (α_d, α'_d) belongs to $DReg_d(p, p')$ means that $\alpha_d \leq p + d$. Thus, $k - \alpha_d + i \geq 2$, and since R is $(0, 0)$-regular by Proposition 2.4.3, the last term vanishes.

By similar reasoning, we get the vanishing of $H^i_{(y)}(M_{d+1})_{k,k'}$ for $k' + i \geq p' + (d+1) + 1$, for all k.

Now look at the local cohomology sequence with $I = (x, y)$. Again we may assume that $i \geq 1$. If (k, k') satisfies $k+k'+i \geq p+p'+(d+1)+1$, our assumption on M_d shows the vanishing of the left-hand side because $k+k'+(i-1) \geq p+p'+d+1$. That (α_d, α'_d) belongs to $DReg_d(p, p')$ means that $\alpha_d + \alpha'_d \leq p + p' + d$, so that $k+k'-\alpha_d-\alpha'_d+i \geq 2$. Thus the right-hand side vanishes because R is $(0, 0)$-regular.

In all three cases we have verified vanishing in the appropriate region to satisfy $VC_{d+1}(p, p')$. □

Conversely:

Proposition 2.4.9. *Let M_{d+1} be a finitely generated bigraded R-module. If M_{d+1} satisfies $VC_{d+1}(p, p')$ and if there is an exact sequence:*

$$0 \longrightarrow M_{d+1} \longrightarrow \bigoplus_{\alpha=1}^{r_d} Re_{\alpha,d} \xrightarrow{\phi_d} M_d \longrightarrow 0,$$

where $M_{d+1} = \ker \phi_d$, and $\deg(e_{\alpha,d}) = (\alpha_d, \alpha'_d) \in DReg_d(p, p')$, then M_d satisfies $VC_d(p, p')$. Therefore M_d is generated by elements of bidegree in $DReg_d(p, p')$.

Proof. Let I be any one of the ideals (x), (y), (x, y). Look at the segment of the local cohomology sequence associated with the above exact sequence:

$$\bigoplus_{\alpha=1}^{r_d} H^i_I(R)_{k-\alpha_d,k'-\alpha'_d} \longrightarrow H^i_I(M_d)_{k,k'} \longrightarrow H^{i+1}_I(M_{d+1})_{k,k'}$$

Let $I = (x)$. and suppose $k+i \geq p+d+1$, we have $k+(i+1) \geq p+(d+1)+1$, and the last group vanishes by assumption on M_{d+1}. Also, in this region, $k - \alpha_d + i \geq 1$, and the first term vanishes by Proposition 2.4.3. Therefore, $H^i_{(x)}(M_d)_{k,k'} = 0$ if $k + i \geq p + d + 1$.

By similar reasoning, we obtain the vanishing of $H^i_{(y)}(M_d)_{k,k'}$ if $k'+i \geq p'+d+1$.

For $I = (x, y)$, suppose $k + k' + i \geq p + p' + d + 1$. We have $k + k' + (i + 1) \geq p + p' + (d + 1) + 1$, so that the last group vanishes by assumption on M_{d+1}. Also, because $\alpha_d + \alpha'_d \leq p + p' + d$, we get $k + k' - \alpha_d - \alpha'_d - i \geq 1$, last term vanishes because R is $(0, 0)$-regular. Therefore, $H^i_{(x,y)}(M)_{k,k'} = 0$ if $k+k'+i \geq p+p'+d+1$.

In all three cases we have verified vanishing in the appropriate region to satisfy $VC_d(p, p')$. □

We prove some equivalent conditions for regularity of a module similar to those of Bayer, Mumford and Stillman (see [2] and [1]). In the formulation below, R is a polynomial algebra over K in two sets of variables x and y bigraded in the usual way. We assume both variable sets are nonempty.

Theorem 2.4.10. *Let M be a finitely generated bigraded module over R. The following properties are equivalent.*

I. M is strongly (p, p')-regular in the sense of definition 2.4.1.

II. The minimal resolution of M by free bigraded $R = K[x, y]$-modules:

$$0 \longrightarrow \bigoplus_{\alpha=1}^{r_s} Re_{\alpha,s} \longrightarrow \cdots \longrightarrow \bigoplus_{\alpha=1}^{r_0} Re_{\alpha,0} \longrightarrow M \longrightarrow 0,$$

satisfies $\deg(e_{\alpha,d}) = (\alpha_d, \alpha'_d) \in DReg_d(p, p')$.

III. There exists a free resolution with the properties above.

Proof. (I ⇒ II) Let $M_0 = M$. We will inductively construct a sequence of bigraded modules M_d that satisfy $VC_d(p, p')$ and that fit into an exact sequence

$$0 \longrightarrow M_{d+1} \longrightarrow \bigoplus_{\alpha=1}^{r_d} Re_{\alpha,d} \xrightarrow{\phi_d} M_d \longrightarrow 0 \tag{2.10}$$

where $\deg(e_{\alpha,i}) = (\alpha_i, \alpha'_i) \in DReg_d(p, p')$. By Proposition 2.4.8, we know that M_{d+1} will satisfy $VC_{d+1}(p, p')$ and therefore we can find generators for it whose bidegrees are in $DReg_{d+1}(p, p')$. In other words, we may construct the above exact sequence but with d replaced by $d+1$. By Hilbert's syzygy theorem, M_d will become a free bigraded module, with generators in $DReg_d(p, p')$, and by splicing these short sequences together, we get our resolution. We can start this induction at $d = 0$, because by hypothesis, $M = M_0$ is strongly (p, p')-regular, and by Proposition 2.4.7, we know M_0 is generated by elements whose bidegrees are in $DReg_0(p, p')$.

(II $\Rightarrow$ III) is trivial.

(III $\Rightarrow$ I) Break the given resolution into short sequences as in equation (2.10) above. We will show by descending induction on d that M_d satisfies $VC_d(p, p')$. Since the last stage of this, namely M_0, is the module M itself, we will be done, since the condition $VC_0(p, p')$ is exactly strong (p, p')-regularity. The starting point of the induction is the extreme left-hand term of the resolution $M_s = \oplus_{\alpha=1}^{r_s} Re_{\alpha,s}$. Because R is $(0, 0)$-regular by Proposition 2.4.3, and because of Remark 2.4.2, we see that M_s satisfies $VC_s(p, p')$. If $d < s$ and we assume by induction that M_{d+1} satisfies $VC_{d+1}(p, p')$, from the exact sequence (2.10) and Proposition 2.4.9, we find that M_d satisfies $VC_d(p, p')$, verifying the induction step. □

Corollary 2.4.11. *Any finitely generated bigraded module over $K[x, y]$ is (p, p')-regular for some p, p'.*

Proof. Look at the minimal free bigraded resolution of M, which we know exists and is unique up to isomorphism. Whatever are the bidegrees $\deg(e_{\alpha,d})$ of the generators of the various terms in this, it is clear that by taking p and p' sufficiently large, for all d these will belong to the region $DReg_d(p, p')$. □

Remark 2.4.12. Let K be an infinite field and let $I \subset K[x_0, y_0, \cdots, y_n]$ be an ideal such that $I = x_0^m J$ where $J \subset K[y_0, \cdots, y_n]$ is a homogeneous ideal. Then I is strongly (p, p')-regular if and only if $p \geq m$ and J is p'-regular.

Proof. Suppose J is p'-regular and $p \geq m$, we would like to show that I is strongly (p, p')-regular. Let $R = K[y_0, \cdots, y_n]$, and take a minimal free resolution of J as follows:

$$0 \longrightarrow \bigoplus_{\alpha=1}^{r_s} Re_{\alpha,s} \longrightarrow \cdots \xrightarrow{d_1} \bigoplus_{\alpha=1}^{r_0} Re_{\alpha,0} \xrightarrow{d_0} R \longrightarrow R/J \longrightarrow 0. \tag{2.11}$$

Since J is p'-regular, then we have $\deg(e_{\alpha,i}) \leq p' + i$. Also, note the map d_i is represented by a matrix. Since the free resolution is minimal, then the matrix has no entry in K^* where $K^* = K \setminus \{0\}$. [40, Proposition 11.5]

We can break this exact sequence into two exact sequences:

$$0 \longrightarrow C_n \longrightarrow \cdots \longrightarrow C_1 \longrightarrow \ker(d_0) \longrightarrow 0, \tag{2.12}$$

and

$$0 \longrightarrow \ker(d_0) \longrightarrow \bigoplus_{\alpha=1}^{r_0} Re_{\alpha,0} \xrightarrow{d_0} R \longrightarrow R/J \longrightarrow 0. \tag{2.13}$$

If we let $S = K[x_0, y_0, \cdots, y_n]$, we have

$$0 \longrightarrow \ker(x_0^m d_0) \longrightarrow \bigoplus_{\alpha=1}^{r_0} Se'_{\alpha,0} \xrightarrow{d_0} S \longrightarrow S/I \longrightarrow 0. \tag{2.14}$$

If we tensor the exact sequence (2.12) with S over R, since S is flat, then we will have an exact sequence:

$$0 \longrightarrow C_n \otimes S \longrightarrow \cdots \longrightarrow C_1 \otimes S \longrightarrow \ker(d_0) \otimes S \longrightarrow 0. \tag{2.15}$$

Note, at each stage, the matrix which represents the map has no entry in K^*. Since $\ker(x_0^m d_0) = S \otimes_R \ker(d_0)$, we can piece exact sequence (2.14) and (2.15) together, we will form a free resolution of I as follow:

$$0 \longrightarrow C_n \otimes S \cdots \longrightarrow C_1 \otimes S \longrightarrow \bigoplus_{\alpha=1}^{r_0} Se'_{\alpha,0} \xrightarrow{d_0} S \longrightarrow S/I \longrightarrow 0. \tag{2.16}$$

This free resolution is minimal, since the matrix that represents the map has no entry in K^*. And we can rewrite the minimal free resolution (2.16) as follow:

$$0 \longrightarrow \bigoplus_{\alpha=1}^{r_s} Se'_{\alpha,s} \longrightarrow \cdots \xrightarrow{d'_1} \bigoplus_{\alpha=1}^{r_0} Se'_{\alpha,0} \xrightarrow{d'_0} S \longrightarrow S/I \longrightarrow 0 \tag{2.17}$$

where $d'_0 = x_0^m d_0$, and $d'_1 = d_1$, and $\deg(e'_{\alpha,i}) = (-m, \deg(e_{\alpha,i}))$. If $m \leq p$ and $\deg(e_{\alpha,i}) \leq p' + i$, by the equivalent relation of minimal free resolution and strongly (p, p')-regular, we know that I is strongly (p, p')-regular.

On the other hand, suppose I is strongly (p, p')-regular, there is a minimal free resolution of I as (2.17), where $\deg(e'_{\alpha,i}) = (a_{\alpha,i}, \deg(e_{\alpha,i}))$ and $a_{\alpha,i} \leq p$ and $\deg(e_{\alpha,i}) \leq p' + i$. Note, at each stage, the matrix representing the map has no entry in K^*. And we can split the free resolution into two exact sequences: the free resolution of I (2.14) and

$$0 \longrightarrow D_n \xrightarrow{d'_n} \cdots \longrightarrow D_1 \xrightarrow{d'_1} \ker(x_0^m d_0) \longrightarrow 0, \tag{2.18}$$

We always have a resolution of J as (2.13). Since $\ker(x_0^m d_0) = S \otimes_R \ker(d_0)$, we will have an exact sequence as follow:

$$0 \longrightarrow C_n \xrightarrow{d_n} \cdots \longrightarrow C_1 \xrightarrow{d_1} \ker(d_0) \longrightarrow 0, \tag{2.19}$$

where $d_i = d'_i$. We can piece the two exact sequences (2.19) and (2.13) together to get:

$$0 \longrightarrow C_n \xrightarrow{d_n} \cdots \longrightarrow C_1 \xrightarrow{d_1} \bigoplus_{\alpha=1}^{r_0} Re_{\alpha,0} \xrightarrow{d_0} R \longrightarrow R/J \longrightarrow 0,$$

which can be written as (2.11). Since the matrix representing d_i has no entry in K^*, the free resolution (2.11) is minimal, and $\deg(e_{\alpha,i}) \leq p' + i$. According to [1, Definition 3.2], the existence of a free resolution of this type implies that J is p'-regular. □

Example 2.4.13. This example will show that weakly regular does not imply strongly regular. Let $I = (s, u, t, v) \subset K[s, u, t, v]$. $\mathbb{V}(I) = \emptyset \subset \mathbf{P}^1 \times \mathbf{P}^1$. Let $\mathbf{m} = (st, sv, tu, tv)$ be the irrelevant ideal of $K[s, u, t, v]$. As a sheaf $\tilde{I}$, is $(0, 0)$-regular, i.e.

$$H_{\mathbf{m}}^{i+1}(I)_{k,k'} = H^i(\tilde{I}(k, k')) = 0, \quad \forall k, k' \geq 0, \quad \forall i \geq 1.$$

Also $H_{\mathbf{m}}^0(I) = 0$, and

$$H_{\mathbf{m}}^1(I)_{k,k'} = (I^{\mathbf{m}\text{sat}}/I)_{k,k'} = (R/I)_{k,k'} = 0, \quad \forall k, k' \neq 0.$$

Therefore I is weakly $(0, 1)$-regular or $(1, 0)$-regular. Also consider the free resolution of I:

$$0 \longrightarrow R(-2,-2) \xrightarrow{\phi_3} \oplus R^2(-1,-2) \oplus R^2(-2,-1) \xrightarrow{\phi_2}$$

$$R(0,-2) \oplus R^4(-1,-1) \oplus R(-2,0) \xrightarrow{\phi_1} R^2(0,-1) \oplus R^2(-1,0) \xrightarrow{\phi_0} I \longrightarrow 0$$

where ϕ_0 is represented by the 1×4 matrix $[v, t, u, s]$, ϕ_1 is represented by the 4×6 matrix:

$$\begin{bmatrix} t & u & 0 & s & 0 & 0 \\ -v & 0 & u & 0 & s & 0 \\ 0 & -v & -t & 0 & 0 & s \\ 0 & 0 & 0 & -v & -t & -u \end{bmatrix},$$

ϕ_2 is represented by the 6×4 matrix:

$$\begin{bmatrix} u & s & 0 & 0 \\ -t & 0 & s & 0 \\ v & 0 & 0 & s \\ 0 & -t & -u & 0 \\ 0 & v & 0 & -u \\ 0 & 0 & v & t \end{bmatrix},$$

and ϕ_3 is represented by the 4×1 matrix $[s, -u, t, -v]^T$. Therefore, by Theorem 2.4.10, I is strongly $(1, 1)$-regular, and cannot be either strongly $(0, 1)$ or $(1, 0)$-regular because of the degree shifts in the free resolution.

3. Regularity and Saturation in Biprojective Spaces

3.1 Saturation and Regularity

Let $R = K[x_0, \cdots, x_m, y_0, \cdots, y_n]$ be a bigraded ring, with $m, n \geq 1$. Let $X = \mathbf{P}^m \times \mathbf{P}^n$. Let $\mathbf{m} = R_+ = \oplus_{k>0,k'>0} R_{k,k'}$ be the irrelevant ideal.

Let $(x) = (x_0, \cdots, x_m)$, $(y) = (y_0, \cdots, y_n)$, $(xy) = \mathbf{m} = (x_i y_j)$, and $(x, y) = (x_0, \cdots, y_n)$.

Definition 3.1.1. If $J \subset R$ is an ideal, then

$$I^{J\text{sat}} = \{r \in R : J^\mu r \subset I, \text{ for some } \mu \in \mathbb{N}\} = \bigcup_{\mu=1}^{\infty} (I :_R J^\mu).$$

An ideal $I \subset R$ is called *J-saturated* if $I = I^{J\text{sat}}$.

Remark 3.1.2. $J_1 \subset J_2 \Rightarrow I^{J_2\text{sat}} \subset I^{J_1\text{sat}}$, and $I^{J_1\text{sat}} \cap I^{J_2\text{sat}} = I^{\langle J_1, J_2\rangle\text{sat}}$. In particular, $I^{(x)\text{sat}} \cap I^{(y)\text{sat}} = I^{(x,y)\text{sat}}$.

Lemma 3.1.3. *I is a proper J-saturated ideal if and only if J is not contained in any associated primes of R/I.*

Proof. ($\Leftarrow$) We do the contrapositive, i.e. if I is not a proper J-saturated ideal, then $J \subset \mathbf{p}$ for some $\mathbf{p} \in \text{Ass}(R/I)$ where $\text{Ass}(R/I)$ denotes the associated primes of R/I.

Let $r \in I^{J\text{sat}} \setminus I$. $r \notin I$, but $rJ^\mu \subset I$ for some $\mu \geq 1$. Choose μ minimal. Let $J = \langle a_1, \cdots, a_t\rangle$. Then $ra^\beta \in I$ for all monomials $a^\beta = a_1^{\beta_1} \cdots a_t^{\beta_t}$ where $|\beta| = \mu$.

Claim: there exists $r' \notin I$ such that $r'J \subset I$. Proof of claim: If $\mu = 1$, we take $r' = r$. If $\mu \geq 2$, then there exists a monomial a^γ with $|\gamma| = \mu - 1$ such that $r' = a^\gamma \notin I$(for otherwise μ would not be the minimal). But then $a_i r' \in I$ for all i, i.e. $r'J \subset I$.

By [37, Proposition 2, page 8], we have $J \subset \mathbf{p}$ for some $\mathbf{p} \in \text{Ass}(R/I)$.

($\Rightarrow$) Let $\mathrm{Ass}(R/I) = \{\mathbf{p}_1, \cdots, \mathbf{p}_r\}$. Suppose J is in an associated prime of R/I, say $\mathbf{p}_1$. Then $J \subset \mathbf{p}_1 = (I : r)$ for some $r \in R$. This means that $Jr \subset I$. Since I is J-saturated, we have $r \in I^{Jsat} = I$. But if $r \in I$, then we have $(I : r) = R$, which is contrary to $(I : r) = \mathbf{p}_1$. Therefore, J is not in any associate primes of R/I. □

Through this chapter we will use J to denote any one of the ideals $(x), (y), (x, y), \mathbf{m}$.

Definition 3.1.4. An ideal $I \subset R$ is called *saturated* if $I = I^{\mathbf{m}\mathrm{sat}}$, and we will write I^{sat} for $I^{\mathbf{m}\mathrm{sat}}$.

Definition 3.1.5. An ideal $I \subset R$ is called *p-saturated for* (x) (resp. p'*-saturated for* (y)) if and only if it satisfies the following conditions:

$$I_{k,k'}^{(x)\mathrm{sat}} = I_{k,k'}, \quad \forall k \geq p, \forall k', (\text{ resp. } I_{k,k'}^{(y)\mathrm{sat}} = I_{k,k'}, \quad \forall k, \forall k' \geq p').$$

Definition 3.1.6. An ideal $I \subset R$ is called *strongly* (p, p')*-saturated* if and only if it is both p-saturated for (x), and p'-saturated for (y).

Definition 3.1.7. An ideal $I \subset R$ is called *weakly* (p, p')*-saturated* if and only if it satisfies the following conditions:

$$I_{k,k'}^{\mathrm{sat}} = I_{k,k'}, \quad \forall k \geq p, \forall k' \geq p'.$$

Remark 3.1.8.

$$\begin{cases} I_{k,k'}^{(x)\mathrm{sat}} = I_{k,k'}, \quad \forall k \geq p, \forall k', \\ I_{k,k'}^{(y)\mathrm{sat}} = I_{k,k'}, \quad \forall k, \forall k' \geq p', \end{cases}$$

Remark 3.1.9. 1. We have

$$H_J^0(R/I) = \bigcup_{\mu}^{\infty} (0 :_{R/I} J^{\mu}) = \{\bar{r} \in R/I | J^{\mu} r \subset I\} = I^{J\mathrm{sat}}/I.$$

Consider the exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

which gives a long exact local cohomology sequence

$$H^0_J(I) \longrightarrow H^0_J(R) \longrightarrow H^0_J(R/I) \longrightarrow H^1_J(I) \longrightarrow H^1_J(R).$$

Since R is an integral domain, and $m, n \geq 1$, we have $\operatorname{depth}_J R \geq 2$, so $H^0_J(R) = H^1_J(R) = 0$. Thus $H^0_J(R/I) = H^1_J(I) = I^{J\text{sat}}/I$.

2. I is strongly (p, p')-saturated if and only if

$$\begin{cases} H^1_{(x)}(I)_{k,k'} = 0, & \forall k \geq p, \forall k', \\ H^1_{(y)}(I)_{k,k'} = 0, & \forall k, \forall k' \geq p'. \end{cases}$$

3. I is weakly (p, p')-saturated if and only if $H^1_{\mathbf{m}}(I)_{k,k'} = 0$ for all $k \geq p, k' \geq p'$.

4. If I is strongly (resp. weakly) (p, p')-regular, then I is strongly (resp. weakly) (p, p')-saturated. Any ideal $I \subset R$ is strongly (p, p')-saturated for some p, p', since it is strongly (p, p')-regular for some p, p'. Therefore, we have

$$\begin{cases} I^{(x)\text{sat}}_{k,k'} = I_{k,k'}, \forall k \geq p, \forall k', \\ I^{(y)\text{sat}}_{k,k'} = I_{k,k'}, \forall k, \forall k' \geq p'; \end{cases} \quad \text{and } I^{\text{sat}}_{k,k'} = I_{k,k'}, \forall k \geq p, \forall k' \geq p' \text{ for some } p, p'.$$

Remark 3.1.10. If K is infinite field, and (x) (resp. (y), resp. $\mathbf{m}$) is not contained in any associated primes of R/I, then there is a Zariski open dense subset with elements $h \in R_{1,0}$, (resp. $h' \in R_{0,1}$, resp. $h'' \in R_{1,1}$) which are not a zero divisor on R/I.

Proof. Let $\operatorname{Ass}(R/I) = \{\mathbf{p}_1, \cdots, \mathbf{p}_r\}$. Suppose every $h \in R_{1,0}$ is a zero divisor on R/I. Since the set of 0-divisors of R/I is the union of $\operatorname{Ass}(R/I)$, we have

$$R_{1,0} = (\mathbf{p}_1 \cap R_{1,0}) \cup \cdots \cup (\mathbf{p}_r \cap R_{1,0})$$

Since a vector space over the infinite field K is not the union of finitely many proper subspaces, then $R_{1,0}$ is equal to one of the terms in the union. If say $R_{1,0} = \mathbf{p}_1 \cap R_{1,0}$, we would have

$$(x) = R \cdot R_{1,0} \subset \mathbf{p}_1$$

which is contrary to the fact that $\mathbf{p}_1$ does not contain (x). Thus we can find an element

$$h \in R_{1,0} - ((\mathbf{p}_1 \cap R_{1,0}) \cup \cdots \cup (\mathbf{p}_r \cap R_{1,0})) = V.$$

Since each $\mathbf{p}_i$ is a proper closed subset of $R_{1,0}$, V is a Zariski open dense subset of $R_{1,0}$. By the similar reason, we can find $h' \in R_{0,1}$ and $h'' \in R_{1,1}$ which is not zero divisor on R/I. □

Definition 3.1.11. We that $h \in R$ is *J-generic for I* if and only if h is not a 0-divisor on $R/I^{J\text{sat}}$, provided $I^{J\text{sat}}$ is a proper ideal in R. If $I^{J\text{sat}} = R$, then every element of R is J-generic for I.

3.2 Criterion for Weak (p, p')-Regularity

Lemma 3.2.1. *Let $I \subset R$ be a proper $\mathbf{m}$-saturated ideal, let $h \in R$ be bihomogeneuous.*

1. *If h is not a zero divisor on R/I, then $(I : h) = I$.*
2. *If h is a zero divisor on R/I, then there exist d, d' such that $(I : h)_{k,k'} \neq I_{k,k'}, \forall k \geq d$ and $k' \geq d'$.*

Proof. Since the first result is from the definition, we will just show the second statement. If h is a zero-divisor on R/I, then we can choose $f \in (I : h) - I$ such that $fh \in I$, where f has bidegree (d, d'). By Lemma 3.1.3 and Remark 3.1.10, we can find $g \in R_{1,1}$ with g not a 0-divisor on R/I, then $gf \in (I : h) - I$.

Iterating this process, we can find elements in $(I : h)_{k,k'}$ which are not in $I_{k,k'}$ for all $k \geq d, k' \geq d'$. □

Definition 3.2.2. For $j > 0$ define $U_j(I)$ to be the subset

$\{(h_1, \cdots, h_j) \in R^j_{1,1} | h_1$ is **m**-generic for I, h_i is **m**-generic for $(I, h_1, \cdots, h_{i-1}), 2 \leq i \leq j\}$

of $R^j_{1,1}$. Since K is infinite, by Remark 3.1.10 the set of $h \in R_{1,1}$ which are **m**-generic for I form a non-empty Zariski open set of $R_{1,1}$. $U_j(I)$ is a non-empty open subset of $R^j_{1,1}$.

Lemma 3.2.3. *Let $I \subset R$ be an ideal, let $h \in R_{1,1}$. The following are equivalent.*

1. $(I : h)_{k,k'} = I_{k,k'} \quad \forall k \geq p, \forall k' \geq p'$,

2. *I is weakly (p, p')-saturated, and h is* **m***-generic for I.*

Proof. (1. ⇒ 2.) Let $d \geq p, d' \geq p'$. Suppose $f \in I^{\text{sat}}_{d,d'} - I_{d,d'}$. We know that $I^{\text{sat}}_{k,k'} = I_{k,k'}$ for $k, k' \gg 0$. Let $\alpha \in \mathbb{N}$ and $\alpha \geq 1$ be the minimal number such that $h^\alpha f \in I$. Then $h^{\alpha-1} f \in (I : h)_{d+\alpha-1, d'+\alpha-1} = I_{d+\alpha-1, d'+\alpha-1}$, since $d+\alpha-1 \geq p, d'+\alpha-1 \geq p'$, and $(I : h)_{k,k'} = I_{k,k'}$ when $k \geq p, k' \geq p'$. Therefore $h^{\alpha-1} f \in I$, which is contrary to the minimality of α. Therefore $I^{\text{sat}}_{k,k'} = I_{k,k'}$ for all $d \geq p, d' \geq p'$. By Definition 3.1.4, I is weakly (p, p')-saturated. By Lemma 3.2.1, h is **m**-generic for I.

(2. ⇒ 1.) If $I^{\text{sat}} = R$, and I is weakly (p, p')-saturated, then we have

$$(I : h)_{k,k'} = (I^{\text{sat}} : h)_{k,k'} = (R : h)_{k,k'} = R_{k,k'} = I^{\text{sat}}_{k,k'} = I_{k,k'}, \quad \forall k \geq p, \forall k' \geq p'.$$

If $I^{\text{sat}} \neq R$, and I is weakly (p, p')-saturated, then

$$(I : h)_{k,k'} = (I^{\text{sat}} : h)_{k,k'} = I^{\text{sat}}_{k,k'} = I_{k,k'}, \quad \forall k \geq p, \forall k' \geq p'.$$

□

Lemma 3.2.4. *Let $I \subset R$ with* $\dim R/I = 0$*, the following are equivalent:*

1. *I is weakly (p,p')-saturated.*

2. *I is weakly (p,p')-regular.*

3. *$I_{k,k'} = R_{k,k'}\ \ \forall k \geq p, \forall k' \geq p'$.*

Proof. (1. $\Leftrightarrow$ 3.) Obvious. (2. $\Rightarrow$ 1.) By Remark 3.1.9. (1. $\Rightarrow$ 2.) If $\dim R/I = 0$, then $H^i_{\mathbf{m}}(R/I) = 0$ for $i \geq 1$. Consider the following exact sequence:

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

The cohomology sequence will be:

$$H^{i-1}_{\mathbf{m}}(R/I) \longrightarrow H^i_{\mathbf{m}}(I) \longrightarrow H^i_{\mathbf{m}}(R) \longrightarrow H^i_{\mathbf{m}}(R/I).$$

If $i \geq 2$, then $H^i_{\mathbf{m}}(I) = H^i_{\mathbf{m}}(R)$. If $i = 1$, then $H^1_{\mathbf{m}}(I) = I^{\mathrm{sat}}/I$. If $i = 0$, then $H^0_{\mathbf{m}}(I) = 0$. Since R is $(0,0)$-regular, we have $H^i_{\mathbf{m}}(I)_{k,k'} = H^i_{\mathbf{m}}(R)_{k,k'} = 0$ for all $(k,k') \in \mathit{Reg}_{i-1}(0,0)$. I is weakly (p,p')-saturated, $H^1_{\mathbf{m}}(I)_{k,k'} = I^{\mathrm{sat}}_{k,k'}/I_{k,k'} = 0$ for all $(k,k') \in \mathit{Reg}_0(p,p')$. Therefore, we have the following:

$$H^i_{\mathbf{m}}(I)_{k,k'} = 0, \ \ \forall (k,k') \in \mathit{Reg}_{i-1}(p,p').$$

Thus, I is weakly (p,p')-regular. □

Lemma 3.2.5. *Let $I \subset R$ be an ideal, let $h \in R_{1,1}$ be $\mathbf{m}$-generic for I and $Q = (I : h)/I$. If I is weakly (p,p')-saturated, then* $\mathrm{Supp}(Q) \subset \mathbb{V}(\mathbf{m})$ *where* $\mathrm{Supp}(Q)$ *is the support of Q and $\mathbb{V}(\mathbf{m}) \subset \operatorname{Spec} R$.*

Proof. We prove the following 2 claims:

Claim 1: $\mathrm{Ass}(Q) \subset \mathbb{V}(\mathbf{m})$.

Proof of claim 1: Suppose not, then there exist $\mathbf{p} \in \mathrm{Ass}(Q) \setminus \mathbf{m}$, and $t \in \mathbf{m} \setminus \mathbf{p}$. $Q_{\mathbf{p}} = \{\frac{q}{s} : s \notin \mathbf{p}\}$ and

$$\frac{q}{s} \sim 0 \Leftrightarrow \exists v \in \mathbf{p} \text{ such that } vq = 0.$$

Since I is weakly (p,p')-saturated, we can take $v = t^{\alpha} \in \mathbf{m} \setminus \mathbf{p}$ for some α big enough. This is contrary to $\mathbf{p} \in \operatorname{Ass}(Q)$. Therefore $\operatorname{Ass}(Q) \subset \mathbb{V}(\mathbf{m})$.

Claim 2: $\operatorname{Supp}(Q) \subset \mathbb{V}(\mathbf{m})$.

Proof of claim 2: Let $\mathbf{P} \in \operatorname{Supp}(Q)$. Since $\operatorname{Ass}(Q)$ and $\operatorname{Supp}(Q)$ contains the same minimal elements, $\mathbf{P}$ contains a minimal element $\mathbf{p} \in \operatorname{Ass}(Q)$. Thus $\mathbf{m} \subset \mathbf{p} \subset \mathbf{P}$. Therefore $\operatorname{Supp}(Q) \subset \mathbb{V}(\mathbf{m})$. □

Lemma 3.2.6. *Let $I \subset R$ be an ideal. Suppose that $h \in R_{1,1}$ is $\mathbf{m}$-generic for I. The following are equivalent.*

1. *I is weakly (p,p')-regular.*
2. *I is weakly (p,p')-saturated, and (I,h) is weakly (p,p')-regular.*

Proof. We start with some information before we prove the equivalent relations. Suppose I is weakly (p,p')-saturated. Let $Q = (I : h)/I$, we have

$$0 \longrightarrow I \longrightarrow (I : h) \longrightarrow Q \longrightarrow 0.$$

By Lemma 3.2.5, $\operatorname{Supp}(Q) \subset \mathbb{V}(\mathbf{m})$, thus $H^i_{\mathbf{m}}(Q) = 0$ for all $i \geq 1$. $H^0_{\mathbf{m}}(Q) = Q$. We have an exact local cohomology sequence:

$$H^{i-1}_{\mathbf{m}}(Q) \longrightarrow H^i_{\mathbf{m}}(I) \longrightarrow H^i_{\mathbf{m}}((I : h)) \longrightarrow H^i_{\mathbf{m}}(Q).$$

If $i \geq 2$, then $H^i_{\mathbf{m}}(I) \cong H^i_{\mathbf{m}}(I : h)$.

If $i = 1$, then $Q \longrightarrow H^1_{\mathbf{m}}(I) \longrightarrow H^1_{\mathbf{m}}(I : h) \longrightarrow 0$, and $H^1_{\mathbf{m}}(I : h)_{k,k'} \cong H^1_{\mathbf{m}}(I)_{k,k'}$, $\forall k \geq p, \forall k' \geq p'$. If $i = 0$, then $H^0_{\mathbf{m}}(I) = H^0_{\mathbf{m}}((I : h)) = 0$. Therefore, we have the following cohomology relations:

$$\begin{cases} H^i_{\mathbf{m}}(I : h)_{k,k'} \cong H^i_{\mathbf{m}}(I)_{k,k'}, \quad \forall i \geq 2; \\ H^1_{\mathbf{m}}(I : h)_{k,k'} \cong H^1_{\mathbf{m}}(I)_{k,k'}, \quad \forall k \geq p, k' \geq p; \\ H^0_{\mathbf{m}}(I : h)_{k,k'} \cong H^0_{\mathbf{m}}(I)_{k,k'} = 0. \end{cases} \tag{3.1}$$

(1. $\Rightarrow$ 2.) Suppose I is weakly (p,p')-regular; by Remark 3.1.9, I is weakly (p,p')-saturated. We will show that (I,h) is weakly (p,p')-regular. Consider

$$0 \longrightarrow I \cap (h) \longrightarrow I \oplus (h) \longrightarrow (I,h) \longrightarrow 0.$$

Since $I \cap h = (I:h) \cdot h$, and $h \in R_{1,1}$, we have

$$0 \longrightarrow (I:h)(-1,-1) \xrightarrow{h} I \oplus (h) \longrightarrow (I,h) \longrightarrow 0.$$

This gives

$$H^i_{\mathbf{m}}(I \oplus (h))_{k,k'} \longrightarrow H^i_{\mathbf{m}}((I,h))_{k,k'} \longrightarrow H^{i+1}_{\mathbf{m}}((I:h))_{k-1,k'-1}.$$

I is weakly (p,p')-regular implies that I is weakly (p,p')-saturated. According to the cohomology relations (3.1), and I being weakly (p,p')-regular, we have

$$H^{i+1}_{\mathbf{m}}(I:h)_{k-1,k'-1} \cong H^{i+1}_{\mathbf{m}}(I)_{k-1,k'-1} = 0, \quad \forall (k,k') \in Reg_i(p-1,p'-1).$$

Since $H^i_{\mathbf{m}}(I \oplus (h))_{k,k'} \cong H^i_{\mathbf{m}}(I)_{k,k'} \oplus H^i_{\mathbf{m}}((h))_{k,k'}$, and the fact that $(h) \cong R(1,1)$ is $(1,1)$-regular, we have

$$H^i_{\mathbf{m}}(I \oplus (h))_{k,k'} = 0, \quad \forall (k,k') \in Reg_{i-1}(p,p').$$

Since $Reg_{i-1}(p,p') \subset Reg_i(p-1,p'-1)$, we must have

$$H^i_{\mathbf{m}}((I,h))_{k,k'} = 0, \quad \forall (k,k') \in Reg_{i-1}(p,p').$$

Therefore (I,h) is weakly (p,p')-regular.

(2. $\Rightarrow$ 1.) Suppose (I,h) is weakly (p,p')-regular, and I is weakly (p,p')-saturated. From

$$0 \longrightarrow (I:h)(-1,-1) \xrightarrow{h} I \oplus (h) \longrightarrow (I,h) \longrightarrow 0,$$

we have

$$H^{i-1}_{\mathbf{m}}((I,h))_{k,k'} \to H^i_{\mathbf{m}}((I:h))_{k-1,k'-1} \to H^i_{\mathbf{m}}(I \oplus (h))_{k,k'} \to H^i_{\mathbf{m}}((I,h))_{k,k'}.$$

Since (I, h) is weakly (p, p')-regular, we have

$$H^{i-1}_{\mathbf{m}}((I,h))_{k,k'} = H^{i}_{\mathbf{m}}((I,h))_{k,k'} = 0, \quad \forall (k,k') \in Reg_{i-2}(p,p').$$

Therefore we have

$$H^{i}_{\mathbf{m}}((I:h))_{k-1,k'-1} \cong H^{i}_{\mathbf{m}}(I \oplus (h))_{k,k'}, \quad \forall (k,k') \in Reg_{i-2}(p,p').$$

Since I is weakly (p, p')-saturated, we have the cohomology relations (3.1). This says that

$$H^{i}_{\mathbf{m}}(I)_{k-1,k'-1} \cong H^{i}_{\mathbf{m}}(I \oplus (h))_{k,k'}, \quad \forall (k,k') \in Reg_{i-2}(p,p').$$

Since any ideal I is (d, d')-regular for some $d, d' \gg 0$ by Remark 3.1.9, we have $H^{i}_{\mathbf{m}}(I)_{k,k'} = 0$ for $k, k' \gg 0$. Therefore,

$$H^{i}_{\mathbf{m}}(I)_{k-1,k'-1} = 0, \quad \forall (k,k') \in Reg_{i-2}(p,p').$$

But for every $(k-1, k'-1) \in Reg_{i-1}(p,p')$ we have $(k,k') \in Reg_{i-1}(p+1), (p+1) \subset Reg_{i-2}(p,p')$. Therefore, I is (p,p')-regular. □

Theorem 3.2.7. *Criterion for weak (p, p')-regularity*

Let $I \subset R$ be a bigraded ideal. The following are equivalent:

1. *I is weakly (p, p')-regular.*

2. *Let $r = \dim R/I$ where $\dim R/I$ refers to the Krull dimension of R/I, then for all $h_1, \cdots, h_r \in U_r(I)$, and all $k \geq p, k' \geq p'$,*

$$((I, h_1, \cdots, h_{i-1}) : h_i)_{k,k'} = (I, h_1, \cdots, h_{i-1})_{k,k'}, \quad i = 1, \cdots, r;$$

and

$$(I, h_1, \cdots, h_r)_{k,k'} = R_{k,k'}.$$

Note, when $r = 0$ this means that

$$I_{k,k'} = R_{k,k'}.$$

3. If $r = \dim R/I$*, there exist* $h_1, \cdots, h_r \in U_r(I)$*, such that for all* $k \geq p, k' \geq p'$*,*

$$((I, h_1, \cdots, h_{i-1}) : h_i)_{k,k'} = (I, h_1, \cdots, h_{i-1})_{k,k'}, \quad i = 1, \cdots, r;$$

and

$$(I, h_1, \cdots, h_r)_{k,k'} = R_{k,k'}.$$

Note, when $r = 0$ *this means that*

$$I_{k,k'} = R_{k,k'}.$$

Proof. (2. $\Rightarrow$ 3.) This is obvious.

(3. $\Rightarrow$ 1.) We prove this by induction on r. If $r = 0$, $I_{k,k'} = R_{k,k'}$ for all $k \geq p, k' \geq p'$. I is weakly (p, p')-regular by Lemma 3.2.4. By the induction hypothesis (I, h_1) is weakly (p, p')-regular, and $(I : h_1)_{k,k'} = I_{k,k'}$ for all $k \geq p, k' \geq p'$. By Lemma 3.2.3, we have I is weakly (p, p')-saturated. By Lemma 3.2.6, I is weakly (p, p')-regular.

(1. $\Rightarrow$ 2.) We prove this by induction on r. If $r = 0$, By Lemma 3.2.4 $I_{k,k'} = R_{k,k'}$ for all $k \geq p, k' \geq p'$. Let $(h_1, \cdots, h_r) \in U_r(I)$. By Lemma 3.2.6, (I, h_1) is weakly (p, p')-regular. By the induction hypothesis, (2.) holds. By construction, $(h_2, \cdots, h_r) \in U_{r-1}(I, h)$. Since (I, h) is weakly (p, p')-regular, it follows from the induction hypothesis for (I, h_1) that the remaining equalities holds. By Remark 3.1.9, I is weakly (p, p')-saturated, and $(I : h)_{k,k'} = I_{k,k'}$ for all $k \geq p, k' \geq p'$ by Lemma 3.2.3. ∎

3.3 Weak Regularity of a Power of an Ideal

In this section, we will prove some results similar as those in Chandler [8] about the regularity of the power of an ideal. We will compare the regularity of I with that of its kth power I^k.

Proposition 3.3.1. *Let $I \subset R$ be weak $p, p')$-regular and generated by bihomogeneous forms of bidegree (d_i, d_i') $(d_i \leq m, d_i' \leq m')$. If $\dim R/I = 0$, then I^e is weakly (l, l')-regular for some l, l' with $l \leq (e-1)m + p, l' \leq (e-1)m' + p'$.*

Proof. We prove this by induction on $e \geq 1$. It is true for $e = 1$ by assumption.

Let $I = (f_1, \cdots, f_r)$, where f_i has bidegree (d_i, d_i'), and $d_i \leq m, d_i' \leq m'$ for all i. Let M be any monomial of bidegree (k, k'), where $k \geq p$, and $k' \geq p'$. Since I is weakly (p, p')-regular, Lemma 3.2.4 shows that $M \in I$, so $M = \sum_{i=1}^{r} N_i f_i$, where the bidegree of N_i is denoted by (n_i, n_i'), where $n_i \geq k - m, n_i' \geq k' - m'$. Let N be any monomial of bidegree $((e-1)m, (e-1)m')$. We will show that $MN \in I^e$. $N_i N$ has bidegree $(n_i + (e-1)m, n_i' + (e-1)m'$. Since $n_i \geq k - m, n_i' \geq k' - m'$, we have the bidegree of $N_i N$ $(\geq (e-1)m + p - m, \geq (e-1)m' + p' - m')$. Since $k \geq p$, and $k' \geq p'$, that is the bidegree $(\geq (e-2)m + p, \geq (e-2)m' + p')$. By induction, $N_i N \in I^{e-1}$, then $MN = \sum_{i=1}^{r} N_i N f_i \in I^{e-1} I = I^e$. Thus I^e contains any monomial of bidegree (a, a') with $a \geq (e-1)m + p, a' \geq (e-1)m' + p'$. Therefore, by Lemma 3.2.4, I^e is $((e-1)m + p, (e-1)m' + p')$-regular. □

Proposition 3.3.2. *If I is a bihomogenous ideal of R with $\dim R/I \leq 1$, then I^{sat} and $\mathrm{sat}(I^e)$ are strongly $(0,0)$-regular, where $\mathrm{sat}(I^e)$ denotes the saturation of I^e.*

Proof. First, note that if $\dim R/I \leq 1$, then $\mathbb{V}(I) \subset \mathbf{P}^m \times \mathbf{P}^n = \emptyset$. This implies that $\mathbb{V}(I) \subset \mathbb{V}(\mathbf{m})$, and . Thus $\mathbf{m} = \sqrt{\mathbf{m}} \subset \sqrt{I}$. Thus $\mathbf{m}^\mu \subset I$ for some μ and $\mathbf{m}^{\mu e} \subset I^e$. Thus, $I^{\mathrm{sat}} = R$ and $\mathrm{sat}(I^e) = R$. Therefore, I^{sat} and $\mathrm{sat}(I^e)$ are strongly $(0,0)$-regular. □

Proposition 3.3.3. *Let I be a bihomogenous ideal in $R = K[x_0, ..., x_m, y_0, ..., y_n]$. Assume that*

i. $Z = \mathbb{V}(I) \subset X = \mathbf{P}^m \times \mathbf{P}^n$ *is finite.*

ii. I is weakly (p, p')-regular.

iii. I is generated by forms of bidegree $(\leq m, \leq m')$.

Then the saturation J of I^e is weakly $((e-1)m + p, (e-1)m' + p')$-regular.

Proof. The proof is by induction on e. Suppose $e = 1$. In this case, it is necessary to show J is weakly (p, p')-regular, i.e., $H^i_{\mathbf{m}}(J)_{k,k'} = 0$ for all $(k, k') \in Reg_{i-1}(p, p')$. Since J is a saturated ideal, $H^0_{\mathbf{m}}(J) = (0 :_R J) = 0$ since $J \subset R$, and $H^1_{\mathbf{m}}(J) = J^{\text{sat}}/J = 0$ by Remark 3.1.9(5). Then $H^i_{\mathbf{m}}(J)_{k,k'} = 0$ for $i = 0, 1$ and for all k, k'.

If $i \geq 2$, let $\mathcal{I}, \mathcal{J}$ be the sheafification of I, J respectively. Tensor the exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{J} \longrightarrow \mathcal{J}/\mathcal{I} \longrightarrow 0$$

with $\mathcal{O}(k, k')$ and consider the resulting cohomology sequence:

$$H^{i-1}(Z, \mathcal{J}/\mathcal{I}(k, k')) \to H^i(X, \mathcal{I}(k, k')) \to H^i(X, \mathcal{J}(k, k')) \to H^i(Z, \mathcal{J}/\mathcal{I}(k, k')).$$

Since $\dim Z = 0$, $H^i(Z, \mathcal{J}/\mathcal{I}(k, k')) = 0$ and for $i \geq 1$ for all k, k'. Since I is weakly (p, p')-regular, $H^1(\mathcal{I}(k, k')) = 0$ for $(k, k') \in Reg_1(p, p')$. Thus, we have

$$H^1(X, \mathcal{J}(k, k')) = 0, \quad \forall (k, k') \in Reg_1(p, p'), and \tag{3.2}$$

$$H^i(X, \mathcal{J}(k, k')) = H^i(X, \mathcal{I}(k, k')), \quad \forall i \geq 2.$$

Since I is weakly (p, p')-regular, we have

$$H^i(X, \mathcal{J}(k, k')) = H^i(X, \mathcal{I}(k, k')) = 0, \quad \forall i \geq 2, \quad \forall (k, k') \in Reg_i(p, p'). \tag{3.3}$$

Combining Equation (3.3) and Equation (3.2), we see that

$$H^i(X, \mathcal{J}(k, k')) = 0, \quad \forall i \geq 1, \quad (k, k') \in Reg_i(p, p').$$

Since $H_{\mathbf{m}}^{i+1}(J)_{k,k'} = H^i(\mathcal{J}(k,k'))$, for all $i \geq 1$, we have

$$H_{\mathbf{m}}^{i+1}(J)_{k,k'} = 0, \quad \forall i \geq 1, \quad (k,k') \in Reg_{i-1}(p,p').$$

Therefore, when $e = 1$, J is weakly (p,p')-regular.

Assume that $e \geq 2$. Let $\mathcal{I}$ denote the sheafification of I. The sheafification of J is $\mathcal{I}^e$, and $H^0(X, \mathcal{I}^e(k,k')) = J_{k,k'}$. Define $Z^{(d)} = \mathbb{V}(I^d)$, which has the same support as Z and is hence finite.

Since J is saturated, we have $H_{\mathbf{m}}^i(J) = 0$ for $i = 0, 1$. Let $(l,l') = ((e-1)m + p, (e-1)m' + p'))$. We must show that

$$H^i(X, \mathcal{I}^e(k,k')) = 0 \text{ for } (k,k') \in Reg_i(l,l'), \text{ all } i \geq 1.$$

Tensor the following exact sequence

$$0 \longrightarrow \mathcal{I}^e \longrightarrow \mathcal{I}^{e-1} \longrightarrow \mathcal{I}^{e-1}/\mathcal{I}^e \longrightarrow 0$$

with $\mathcal{O}(k,k')$ and consider the resulting cohomology sequence. Since the support of $\mathcal{I}^{e-1}/\mathcal{I}^e$ is contained in Z, which is 0-dimensional, $\mathrm{H}^i\left(X, \mathcal{I}^{e-1}/\mathcal{I}^e(k,k')\right) = 0$ for $i \geq 2$. Therefore, we have

$$\mathrm{H}^i\left(X, \mathcal{I}^e(k,k')\right) = \mathrm{H}^i\left(X, \mathcal{I}^{e-1}(k,k')\right) \text{ for all } i \geq 2,$$

and the latter group vanishes by induction for all

$$(k,k') \in Reg_i((e-2)m + p, (e-2)m' + p') \supset Reg_i((e-1)m + p, (e-1)m' + p').$$

Thus, we have the required vanishing for $i \geq 2$. Now look at the sequence

$$H^0(X, \mathcal{I}^{e-1}(k,k')) \xrightarrow{\phi} H^0(X, \mathcal{I}^{e-1}/\mathcal{I}^e(k,k'))$$

$$\longrightarrow H^1(X, \mathcal{I}^e(k,k')) \longrightarrow H^1(X, \mathcal{I}^{e-1}(k,k'))$$

By induction, the last term vanishes for all $(k, k') \in Reg_1(l, l')$, so that the next-to-last term will vanish there provided we show that ϕ is onto for those same (k, k').

Suppose $Z = \{p_1, \cdots, p_s\}$. Note, since the support is finite, we have

$$H^0(X, \mathcal{I}^{e-1}/\mathcal{I}^e(k, k')) = H^0(X, \mathcal{I}^{e-1}/\mathcal{I}^e) = \bigoplus_{p \in Z} (I^{e-1}\mathcal{O}_{X,p}/I^e\mathcal{O}_{X,p})(k, k')$$

We will show that for $(k, k') \in Reg_1(l, l')$ and for any

$$\left(\frac{u_1}{v_1}, ..., \frac{u_s}{v_s}\right) \in \bigoplus_i (I^{e-1}\mathcal{O}_{X,p_i}/I^e\mathcal{O}_{X,p_i})(k, k')$$

with bihomogeneous forms with $\deg u_i - \deg v_i = (k, k')$, $u_i \in I^{e-1}$ we can find a bihomogeneous $g \in \mathrm{sat}(I^{e-1})_{k,k'}$ and forms H_i with $H_i(p_i) \neq 0$, such that

$$H_i(gv_i - u_i) \in I^e \text{ for all } i. \tag{3.4}$$

This will prove the surjectivity of ϕ.

Let I be generated by bihomogenous elements $f_1, \cdots, f_r$ with bidegree $(m_i, m_i') \leq (m, m')$. We can write

$$u_i = \sum a_{ij} f_j, \text{for some } a_{ij} \in I^{e-2}_{k-m_j, k'-m_j'}$$

Note that $(\alpha, \alpha') = (k - m_j, k' - m_j') \in Reg_1((e-2)m + p, (e-2)m' + p')$, by our initial choice of (k, k'). Tensor the following exact sequence

$$0 \longrightarrow \mathcal{I}^{e-1} \longrightarrow \mathcal{I}^{e-2} \longrightarrow \mathcal{I}^{e-2}/\mathcal{I}^{e-1} \longrightarrow 0$$

with $\mathcal{O}_X(\alpha, \alpha')$. Look at the sequence

$$H^0(X, \mathcal{I}^{e-2}(\alpha, \alpha')) \xrightarrow{\psi} H^0(X, \mathcal{I}^{e-2}/\mathcal{I}^{e-1}(\alpha, \alpha'))$$

$$\longrightarrow H^1(X, \mathcal{I}^{e-1}(\alpha, \alpha')) \longrightarrow H^1(X, \mathcal{I}^{e-2}(\alpha, \alpha'))$$

Reasoning as before, we see that ψ is onto for this (α, α'). This means that for every j, and each

$$\left(\frac{a_{1j}}{v_1}, ..., \frac{a_{sj}}{v_s}\right) \in \bigoplus_i (I^{e-2}\mathcal{O}_{X,p_i}/I^{e-1}\mathcal{O}_{X,p_i})(k - m_j, k' - m'_j)$$

we can find a bihomogeneous $g_j \in \mathrm{sat}(I^{e-2})_{\alpha,\alpha'}$ and forms H_{ij} with $H_{ij}(p_i) \neq 0$, such that

$$H_{ij}(g_j v_i - a_{ij}) \in I^{e-1} \text{ for all } i. \tag{3.5}$$

We may replace each H_{ij} by $H_i = \prod_j H_{ij}$. Multiply equation (3.5) by f_j and sum the result over j and define $g = \sum g_j f_j \in \mathrm{sat}(I^{e-1})_{k,k'}$. Then we have obtained equation (3.4), as required. □

3.4 Rank and Regularity

In this section, we will prove some regularity results similar to those of Busé, Cox, and D'Andrea [6] about the regularity. In this Chapter, we only talk about weak regularity.

Lemma 3.4.1. *Let $\bar{I} \subset S = \mathbb{C}[s, t, v]$ be minimally generated by r bihomogeneous forms of bidegree (m, n), which means that the generators have the form $\sum_{j=0}^{n} a_{ij} s^m t^i v^{n-j}$. That is $\bar{I} = s^m J$ where J is generated by homogeneous generators of degree n. If $\mathbb{V}(J) = \emptyset$ in $\mathbf{P}^1$, then $\bar{I}$ is (p, p')-regular for all $p \geq m$ and $p' \geq 2n - r + 1$.*

Proof. This follows from Remark 2.4.12 and Lemma B.1 in [6]. □

Remark 3.4.2. Similarly, let $\bar{I} \subset S = \mathbb{C}[s, u, t]$ be minimally generated by r bihomogeneous forms of bidegree (m, n), which means that the generators have the form $\sum_{i=0}^{n} a_{ij} s^i u^{m-i} t^n$. That is $\bar{I} = t^n J$ where J is generated by homogeneous generators of degree m. If $\mathbb{V}(J) = \emptyset$ in $\mathbf{P}^1$, then $\bar{I}$ is (p, p')-regular for all $p \geq 2m-r+1$ and $p' \geq n$.

Lemma 3.4.3. *Let $I \subset R = \mathbb{C}[s,u,t,v]$ be minimally generated by $r \geq 4$ bihomogeneous forms of bidegree (m,n) with both $m,n \geq 1$. Assume $\mathbb{V}(I) \subset \mathbf{P}^1 \times \mathbf{P}^1$ is finite. Given $\ell \in R_{1,0}$, let I_ℓ be the image of I in the quotient ring $R/\langle \ell \rangle$. Then for a generic ℓ, I_ℓ is minimally generated by at least 2 elements.*

Proof. Let I be minimally generated by $p_1, \cdots, p_r$, where each p_i has bidegree (m,n) with $m,n \geq 1$. Let

$$Z \subset \mathbf{P}^{r-1} \times \mathbf{P}(R_{1,0}) = \mathbf{P}^{r-1} \times \mathbf{P}^1$$

be defined by

$$Z = \{([a_1, \cdots, a_r], [\ell]) \mid \ell | (a_1 p_1 + \cdots + a_r p_r)\},$$

and let $\pi_1 : Z \to \mathbf{P}^{r-1}$ and $\pi_2 : Z \to \mathbf{P}^1$ be the natural projections.

Since $\mathbb{V}(I)$ is finite, $p_1, \cdots p_r$ have no common factors. Otherwise, the common factor will give a curve in $\mathbb{V}(I)$ which contradicts the finiteness of $\mathbb{V}(I)$. Thus the linear system of divisors given by $a_1 p_1 + \cdots + a_r p_r = 0$ is reduced (see Page 130 [25]). According to Bertini's theorem (see Theorem 7.19 [25]) the general member of the linear system is irreducible. Thus $\pi_1^{-1}(\bar{a}) = \emptyset$ for a generic point $\bar{a} \in \mathbf{P}^{r-1}$. This means that $\pi_1(Z)$ is a proper subset of $\mathbf{P}^{r-1}$. Furthermore, if $\pi_1^{-1}(\bar{a}) \neq \emptyset$, then $\pi_1^{-1}(\bar{a})$ is finite since $a_1 p_1 + \cdots + a_r p_r$ is divisible by at most m linear forms. This means the map π_1 is finite to 1. Thus $\dim(Z) \leq r - 2$.

Now consider a generic $\ell \in \mathbf{P}^1$ and let $(p_i)_\ell$ denote the image of p_i in $R/\langle \ell \rangle$. We consider two cases.

Case 1: If $\pi_2(Z) \neq \mathbf{P}^1$, and if $\ell \notin \pi_2(Z)$, we have that $\pi_2^{-1}(\ell) = \emptyset$. This implies that there does not exist non zero a_i's such that $a_1 p_1 + \cdots + a_r p_r = 0$ in $R/\langle \ell \rangle$. So $(p_i)_\ell$ are linearly independent.

Case 2: If $\pi_2(Z) = \mathbf{P}^1$, then $\pi_2^{-1}(\ell) \neq \emptyset, \quad \forall \ell \in \mathbf{P}^1$. By dimension theorem in [39, Theorem 7, page 60], we know that $\text{codim}(\pi_2^{-1}(\ell)) = 1$ in Z. This implies that $\dim(\pi_2^{-1}(\ell)) \leq r-3$ for a generic ℓ.

Since $\pi_2^{-1}(\ell) =$ projective space of linear relations among $\{p_1, \cdots, p_r\}$ in $R/\langle \ell \rangle$, the space of linear relations among the $(p_i)_\ell$ has dimension $\leq r-2$. We know that the sum of the dimension of the span and the dimension of the linearly independent relations is r, this implies that at least 2 of $(p_i)_\ell$ are linearly independent for generic ℓ. □

Remark 3.4.4. The above result is true if the given generic element of ℓ is chosen from $R_{0,1}$.

Theorem 3.4.5. *Let $I \subset R = \mathbb{C}[s,u,t,v]$ be minimally generated by $r \geq 4$ bihomogeneous forms of bidegree (m,n) with both $m,n \geq 1$, and assume that $\mathbb{V}(I) \subset \mathbf{P}^1 \times \mathbf{P}^1$ is finite. If $\mathcal{I}$ is the associated sheaf on $\mathbf{P}^1 \times \mathbf{P}^1$, then:*

I. $H^2(\mathcal{I}(k,k')) = 0 \ \ \forall k, k' \geq 0.$

II. $H^1(\mathcal{I}(k,k')) = 0 \ \ \forall k \geq 2m-2$ *and* $k' \geq 2n-2.$

Proof. Proof of the first claim: let $Z = \mathbb{V}(I) \subset \mathbf{P}^1 \times \mathbf{P}^1$. Consider the following exact sequence:

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1} \to \mathcal{O}_Z \to 0.$$

Tensoring with $\mathcal{O}(k,k')$ gives a long exact cohomology sequence:

$$\to H^1(\mathcal{O}_Z(k,k')) \to H^2(\mathcal{I}(k,k')) \to H^2(\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(k,k')) \to H^2(\mathcal{O}_Z(k,k')) \to .$$

Since Z is finite, $H^i(\mathcal{O}_Z) = 0, \ \ \forall i \geq 1$. So we have

$$H^2(\mathcal{I}(k,k')) = H^2(\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(k,k')) = 0, \ \ \forall k, k' \geq 0$$

by Künneth formula [35].

To prove the second statement, choose a line $\ell \in R_{1,0}$ such that $\mathbb{V}(\ell) \cap \mathbb{V}(I) = \emptyset$ and $\bar{I} = I_\ell =$ the image of I in $R/\langle \ell \rangle$ is minimally generated by at least two elements. This is possible since $\mathbb{V}(I)$ is finite, and by Lemma 3.4.3, $\bar{I}$ is minimally generated by at least 2 elements. Then by Lemma 3.4.1, we know that $\bar{I}$ is (p, p')-regular for $p \geq m$ and $p' \geq 2n - 1$. If $\bar{\mathcal{I}}$ is the sheaf associated to $\bar{I}$, then by definition of regularity, we have

$$\bar{I}_{k,k'} \cong H^0(\bar{\mathcal{I}}(k,k')) \ \ \forall k \geq m, \quad k' \geq 2n-1,$$

$$H^1(\bar{\mathcal{I}}(k,k')) = 0 \ \ \forall k \geq m-1, \quad k' \geq 2n-2.$$

Now, we consider the following exact sequence:

$$0 \to \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(-1,0) \to \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1} \to \mathcal{O}_{L_1} \cong \mathcal{O}_{\mathbf{P}^0 \times \mathbf{P}^1} \cong \mathcal{O}_{\mathbf{P}^1} \to 0.$$

Tensoring with $\mathcal{I}(k,k')$ gives the exact sequence:

$$Tor_1^{\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}}(\mathcal{I}(k,k'), \mathcal{O}_{\mathbf{P}^1}) \to \mathcal{I}(k-1,k') \to \mathcal{I}(k,k') \to \mathcal{O}_{\mathbf{P}^1} \otimes_{\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}} \mathcal{I}(k,k') \to 0.$$

Note $\mathcal{O}_{\mathbf{P}^1} \otimes_{\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}} \mathcal{I}(k,k') \cong \bar{\mathcal{I}}(k')$. $\bar{\mathcal{I}}(k')$ is the sheaf restricted to the line ℓ, where ℓ is the projective line of $R_{1,0}$ denoted by $\mathbf{P}(R_{1,0})$ in Lemma 3.4.3. Let $\bar{I}_{k'}$ be the image of $I_{k,k'}$ in $R/\langle \ell \rangle$, and $\bar{\mathcal{I}}(k')$ the sheaf associated to $\bar{I}_{k'}$. Also note that $Tor_1^{\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}}(\mathcal{I}(k,k'), \mathcal{O}_{\mathbf{P}^1})$ is supported on $\mathbb{V}(\ell) = \mathbf{P}^1 \times \mathbf{P}^0$, and $\mathcal{I}(k,k')$ is locally free on $\mathbf{P}^1 \times \mathbf{P}^1 - \mathbb{V}(I)$, but $\mathbb{V}(\ell) \cap \mathbb{V}(I) = \emptyset$ by choice of ℓ. Hence $Tor_1^{\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}}(\mathcal{I}(k,k'), \mathcal{O}_{\mathbf{P}^1}) = 0$. Then we have the following exact sequence:

$$0 \to \mathcal{I}(k-1,k') \to \mathcal{I}(k,k') \to \bar{\mathcal{I}}(k') \to 0.$$

This gives the following diagram:

$$\begin{array}{ccccccc} & I_{k,k'} & \to & \bar{I}_{k'} & \to & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ \to & H^0(\mathcal{I}(k,k')) & \to & H^0(\bar{\mathcal{I}}(k')) & \to & H^1(\mathcal{I}(k-1,k')) & \to \\ & & H^1(\mathcal{I}(k,k')) & \to & H^1(\bar{\mathcal{I}}(k')) & \to & \end{array}$$

with exact rows. Suppose that $k \geq m$, $k' \geq 2n-1$. This says that $H^1(\bar{\mathcal{I}}(k')) = 0$. $I_{k,k'} \to \bar{I}_{k'}$ is onto, and $\bar{I}_{k'} \cong H^0(\bar{\mathcal{I}}(k'))$. Then the diagram gives an isomorphism

$$H^1(\mathcal{I}(k-1,k')) \cong H^1(\mathcal{I}(k,k')), \quad \forall k \geq m, \quad k' \geq 2n-1.$$

We can also tensor the following exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(0,-1) \to \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1} \to \mathcal{O}_{\mathbf{P}^1} \to 0$$

with $\mathcal{I}(k,k')$. By the same reasoning, it will give an exact sequence

$$0 \to \mathcal{I}(k,k'-1) \to \mathcal{I}(k,k') \to \bar{\mathcal{I}}(k) \to 0$$

whose long exact sequence in cohomology gives the following diagram:

$$\begin{array}{ccccccc} & I_{k,k'} & \to & \bar{I}_{k} & \to & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ \to & H^0(\mathcal{I}(k,k')) & \to & H^0(\bar{\mathcal{I}}(k)) & \to & H^1(\mathcal{I}(k,k'-1)) & \to \\ & & H^1(\mathcal{I}(k,k')) & \to & H^1(\bar{\mathcal{I}}(k)) & \to & \end{array}$$

with exact rows. Suppose that $k \geq 2m-1$ and $k' \geq n$. This says that $H^1(\bar{\mathcal{I}}(k)) = 0$. $I_{k,k'} \to \bar{I}_k$ is onto, and $\bar{I}_k \cong H^0(\bar{\mathcal{I}}(k))$. Then the diagram gives an isomorphism

$$H^1(\mathcal{I}(k,k'-1)) \cong H^1(\mathcal{I}(k,k')), \quad \forall k \geq 2m-1, \quad k' \geq n.$$

This implies that

$$H^1(\mathcal{I}(k-1,k'-1)) \cong H^1(\mathcal{I}(k,k')), \quad \forall k \geq 2m-1 \text{ and } k' \geq 2n-1.$$

This proves that $H^1(\mathcal{I}(k,k')) = 0 \ \ \forall k \geq 2m-2$ and $k' \geq 2n-2$. $\square$

Theorem 3.4.6. *Let $I \subset R = \mathbb{C}[s,u,t,v]$ is minimally generated by $r \geq 4$ bihomogeneous forms of bidegree (m,n) with $m,n \geq 1$, and assume $Z = \mathbb{V}(I) \subset \mathbf{P}^1 \times \mathbf{P}^1$ is finite. If $p \geq 2m-1$, and $p' \geq 2n-1$, then I is (p,p')-regular if and only if $\dim(R/I)_{p,p'} = \deg(Z)$ where $\deg(Z)$ denotes the degree of Z.*

Proof. When $p \geq 2m-1$ and $p' \geq 2n-1$, Theorem 3.4.5 implies $H^1(\mathcal{I}(p,p')) = 0$. The exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1} \to \mathcal{O}_Z \to 0$$

gives

$$0 \to H^0(\mathcal{I}(p,p')) \to H^0(\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(p,p')) \to H^0(\mathcal{O}_Z(p,p')) \to 0.$$

We always have the following exact sequence:

$$0 \to I_{p,p'} \to R_{p,p'} \to (R/I)_{p,p'} \to 0. \tag{3.6}$$

If I is (p,p')-regular, we have $I_{p,p'} = H^0(\mathcal{I}(p,p'))$, and $R_{p,p'} = H^0\mathcal{O}_Q(p,p'))$. By the 5-lemma, we have $(R/I)_{p,p'} = H^0(\mathcal{O}_Z(p,p'))$, thus $\dim(R/I)_{p,p'} = \dim H^0(\mathcal{O}_Z(p,p'))$. Since Z is finite,

$$\dim H^0(\mathcal{O}_Z) = \deg(Z) \quad \text{(see [39, page 140-142])}.$$

Since $\dim H^0(\mathcal{O}_Z) = \dim H^0(\mathcal{O}_Z(p,p'))$ when Z is finite. Therefore

$$\dim(R/I)_{p,p'} = \deg(Z).$$

On the other hand, suppose $\dim(R/I)_{p,p'} = \deg(Z)$. Since $H^2(\mathcal{I}(k,k')) = 0$ for all $k,k' \geq 0$, by the definition of (p,p')-regular, we only need to prove that

$$I_{p,p'} \cong H^0(\mathcal{I}(p,p')), \text{ and } H^1(\mathcal{I}(p-1,p'-1)) = 0.$$

If $p \geq 2m-1$ and $p' \geq 2n-1$, then $H^1(\mathcal{I}(p-1,p'-1)) = 0$ by Theorem 3.4.5. We know that $I_{p,p'} \to H^0(\mathcal{I}(p,p'))$ is injective, it is enough to show that

$\dim I_{p,p'} = \dim H^0(\mathcal{I}(p,p'))$. According to the exact sequence

$$0 \to H^0(\mathcal{I}(p,p')) \to R_{p,p'} \to H^0(\mathcal{O}_Z(p,p')) \to 0,$$

we know that

$$\dim H^0(\mathcal{I}(p,p')) = \dim R_{p,p'} - \dim H^0(\mathcal{O}_Z(p,p'))$$

$$= \dim R_{p,p'} - \deg(Z) = \dim R_{p,p'} - \dim(R/I)_{p,p'} = \dim I_{p,p'}.$$

The last equality is because of the exact sequence (3.6). Thus $I_{p,p'} \cong H^0(\mathcal{I}_{p,p'})$ and I is (p,p')-regular. □

Example 3.4.7. The following example shows the result of the computation by Theorem 3.4.6 is the same as the result of the computation by the free resolution of I.

$I = (u^2t^2v, u^2t^3 + suv^3, s^2tv^2, s^2v^3 + s^2t^3) \subset K[s,u,t,v]$. $\mathbb{V}(I) = (0:1;0:1) \subset \mathbf{P}^1 \times \mathbf{P}^1$. A computation with Singular shows $\dim(R/I)_{3,5} = \deg \mathbb{V}(I) = 2$. I is weakly $(3,5)$-regular. We have a free resolution for I as follows:

$$0 \longrightarrow R(-3,-6) \oplus R^2(-4,-5) \oplus R(-4,-6) \xrightarrow{\phi_2}$$

$$R(-2,-6) \oplus R^2(-3,-5) \oplus R^3(-4,-4) \oplus R(-3,-6) \xrightarrow{\phi_1} R^4(-2,-3)$$

$$\xrightarrow{\phi_0} I \longrightarrow 0,$$

where ϕ_0 is represented by the 1×4 matrix $[u^2t^2v, u^2t^3 + suv^3, s^2tv^2, s^2v^3 + s^2t^3]$, ϕ_1 is represented by 4×7 matrix

$$\begin{bmatrix} t^3+v^3 & ut^2 & uv^2 & u^2t & 0 & -suv+u^2v & 0 \\ 0 & -stv & st^2 & -s^2v & s^2t - suv & 0 & ut^3+sv^3 \\ 0 & sv^2 & -stv & 0 & suv & s^2t & 0 \\ -tv^2 & -uv^2 & 0 & 0 & -u^2v & -u^2t & 0 \end{bmatrix},$$

and ϕ_2 is represented by 7×4 matrix

$$\begin{bmatrix} u & 0 & 0 & 0 \\ -t & u & 0 & 0 \\ -v & 0 & s-u & ut \\ 0 & -t & 0 & -v^2 \\ 0 & -v & -t & 0 \\ 0 & 0 & v & 0 \\ 0 & 0 & 0 & -s \end{bmatrix}.$$

I is also a strongly $(3,5)$-regular.

3.5 Local Complete Intersection and Koszul Syzygies

Let X be a smooth surface over $\mathbb{C}$, and $\mathcal{I} \subset \mathcal{O}_X$ a coherent sheaf of ideals. Suppose that $Z = \mathbb{V}(\mathcal{I})$ is a finite set. We call the points of Z base points for $\mathcal{I}$. The reason for the terminology is that we consider that $\mathcal{I}$ comes from an ideal I in a polynomial ring, and we think of generators of I as providing a rational map $X \to \mathbf{P}^N$ and Z the base point locus of this map.

Definition 3.5.1. The base points are *local complete intersection* (LCI) if for every point $p \in Z$, $I\mathcal{O}_{X,p}$ is a complete intersection ideal, i.e., it is locally generated by two elements.

Recall that given elements $r_1, \cdots, r_n$ in any commutative ring R, a syzygy $(a_1, \cdots, a_n)$ is a relation $a_1 r_1 + \cdots + a_n r_n = 0$ for $a_i \in R$. A Koszul relation is one of the form $(r_j)r_i + (-r_i)r_j = 0$ for $i \neq j$. Let $\mathrm{Syz}(r_1, \cdots, r_n)$ be the submodule of R^n generated by the syzygies. Let $\mathrm{Kos}(r_1, \cdots, r_n) \subset \mathrm{Syz}(r_1, \cdots, r_n)$ be the submodule generated by the Koszul syzygies.

Consider the special case $R = \mathbb{C}[s,u,t]$ and I is generated by the homogeneous forms with $Z = \mathbb{V}(I) \subset \mathbf{P}^2$ a finite set of points.

Definition 3.5.2. A syzygy $(a_1, \cdots, a_n) \in \mathrm{Syz}(r_1, \cdots, r_n)$ *vanishes at all base points* if $a_i \in I^{\mathrm{sat}}$ where $I^{\mathrm{sat}} = \{r \in R : \langle s,t,v\rangle^k r \subset I \text{ for some } k\}$ for each i. If a_i is homogeneous of same degree d_i, this is equivalent to say a_i belongs to the ideal $I\mathcal{O}_{\mathbf{P}^2,p}(d_i)$ for all $p \in Z$, for all i.

The equivalence of the two conditions in the above definition follows from the facts: $I^{\mathrm{sat}}(d_i) = H^0(\mathbf{P}^2, I\mathcal{O}_{\mathbf{P}^2}(d_i))$ and $I^{\mathrm{sat}}(d_i) = \mathcal{O}_{\mathbf{P}^2,p}$ for all $p \notin Z$.

Cox and Schenck [11] have proved the following result:

Theorem 3.5.3. *If $I = \langle f_1, f_2, f_3\rangle \subset R = \mathbb{C}[s,t,u]$ where f_i is a homogeneous polynomial of degree d_i, the module of syzygies vanishing at Z is generated by the Koszul syzygies if and only if Z is a local complete intersection.*

We will extend this result to the bigraded case in this section.

Definition 3.5.4. Let $R = [s,u,t,v]$, where s,u have bidegree $(1,0)$ and t,v have bidegree $(0,1)$. Let $I = \langle a,b,c\rangle$, where a,b,c are bihomogeneous polynomials. A syzygy (A,B,C) on the generators of $I = (a,b,c)$ vanishes at the base point locus $Z = \mathbb{V}(I)$ if $A,B,C \in I^{\mathbf{m}\mathrm{sat}} = \{r \in R : \mathbf{m}^k r \subset I, \text{ for some } k\}$ where $\mathbf{m} = \langle st, sv, ut, uv\rangle$. If A,B,C are bihomogeneous forms of bidegree (d_i, d_i') for $i = 1,2,3$, this is equivalent to say that $A \in I\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1,p}(d_i,d_i')$, etc., for all $p \in Z$. The equivalence follows as before, noting that $I^{\mathbf{m}\mathrm{sat}}(d_i,d_i') = H^0(\mathbf{P}^1 \times \mathbf{P}^1, I\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(d_i,d_i'))$ and $I\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1,p}(d_i,d_i') = \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1,p}(d_i,d_i')$ for all $p \notin Z$.

Remark 3.5.5. Consider the exact sequence:

$$0 \longrightarrow \mathrm{Syz}(a,b,c) \longrightarrow \bigoplus_{i=1}^3 R(-d_i,-d_i') \xrightarrow{(a,b,c)} I \longrightarrow 0.$$

Since $\mathrm{Syz}(a,b,c)$ is a bigraded submodule of $\bigoplus_{i=1}^{3} R(-d_i,-d_i')$, we will say that $(A,B,C) \in \mathrm{Syz}(a,b,c)_{k,l}$ has bidegree (k,l). Note that as polynomials the bidegree of A is $(k-d_1, l-d_1')$, since $A \in R(-d_1,-d_1')_{k,l}$, etc. In other words, the polynomial expression $Aa+Bb+Cc$ has bidegree (k,l).

If all (d_i,d_i') are equal to a fixed pair (m,n), then a bihomogeneous syzygy (A,B,C) of bidegree (k,l) will have A,B,C all bihomogeneous of the same bidegree $(k-m,l-n)$. Some authors call this a syzygy of bidegree $(k-m,l-n)$. We will call (A,B,C) a syzygy of *pure degree* $(k-m,l-n)$.

For rest of the section, we will let $R = \mathbb{C}[s,u,t,v]$ be the bigraded coordinate ring of $\mathbf{P}^1 \times \mathbf{P}^1$, and consider the ideal $I = \langle f_1, f_2, f_3 \rangle \subset R$, where f_i is bihomogeneous of bidegree (d_i,d_i'). The f_i form a regular sequence in R if and only if the following Koszul complex is exact.

$$0 \to R\left(-\sum_{i=1}^{3} d_i, -\sum_{i=1}^{3} d_i'\right) \xrightarrow{\begin{bmatrix} f_3 \\ -f_2 \\ f_1 \end{bmatrix}} \bigoplus_{i<j} R(-d_i-d_j, -d_i'-d_j') \tag{3.7}$$

$$\xrightarrow{\begin{bmatrix} f_2 & f_3 & 0 \\ -f_1 & 0 & f_3 \\ 0 & -f_1 & -f_2 \end{bmatrix}} \bigoplus_{i=1}^{3} R(-d_i,-d_i') \xrightarrow{[f_1 \;\; f_2 \;\; f_3]} I \to 0 .$$

We will discuss the situation when $Z = \mathbb{V}(I) \subset \mathbf{P}^1 \times \mathbf{P}^1$ is a zero-dimensional subscheme, so that $I = \langle f_1, f_2, f_3 \rangle$ has codimension two in R. We call Z the base point locus of f_1, f_2, f_3. If I has codimension two, f_1, f_2, f_3 will no longer be a regular sequence, since R is Cohen-Macaulay.

Lemma 3.5.6. *If $I = \langle f_1, f_2, f_3 \rangle$ has codimension two in R, then* (3.7) *is exact except at $\bigoplus_{i=1}^{3} R(-d_i,-d_i')$. In particular, the Koszul complex of f_1, f_2, f_3 gives the* *exact sequences:*

$$0 \to R(-\sum_{i=1}^{3} d_i, -\sum_{i=1}^{3} d'_i) \to \bigoplus_{i<j} R(-d_i - d_j, -d'_i - d'_j) \to \bigoplus_{i=1}^{3} R(-d_i, -d'_i)$$

and

$$\bigoplus_{i=1}^{3} R(-d_i, -d'_i) \to I \to 0$$

Proof. The exactness of the first sequence will follow from the Buchsbaum-Eisenbud exactness criterion (page 500 [15]). To apply this criterion, we need to check the following rank conditions:

$$\text{rank} \bigoplus_{i<j} R(-d_i - d_j, -d'_i - d'_j) = \text{rank} \begin{bmatrix} f_2 & f_3 & 0 \\ -f_1 & 0 & f_3 \\ 0 & -f_1 & -f_2 \end{bmatrix} + \text{rank} \begin{bmatrix} f_3 \\ -f_2 \\ f_1 \end{bmatrix}, \quad (3.8)$$

$$\text{rank} R(-\sum_{i=1}^{3} d_i, -\sum_{i=1}^{3} d'_i) = \text{rank} \begin{bmatrix} f_3 \\ -f_2 \\ f_1 \end{bmatrix}, \quad (3.9)$$

(3.8) is true since both ranks are 3, and (3.9) since both ranks are 1. It is also necessary to check the following depth conditions: Let J be the ideal generated by the 2×2 minors of the matrix $\begin{bmatrix} f_2 & f_3 & 0 \\ -f_1 & 0 & f_3 \\ 0 & -f_1 & -f_2 \end{bmatrix}$, which is the ideal

$$J = \langle f_1 f_3, f_2 f_3, f_1 f_2, f_1^2, f_2^2, f_3^2 \rangle,$$

then

$$\text{depth} J = \text{codim} J = 2$$

since R is a Cohen-MaCaulay ring. The ideal generated by the 1×1 minor of the matrix $\begin{bmatrix} f_3 \\ -f_2 \\ f_1 \end{bmatrix}$ is just I, and

$$\text{depth} I = \text{codim} I = 2.$$

□

Definition 3.5.7. A *Koszul syzygy* on f_1, f_2, f_3 is an element of the submodule

$$K \subset \bigoplus_{i=1}^{3} R(-d_i, -d_i')$$

generated by the columns of the matrix

$$\begin{bmatrix} f_2 & f_3 & 0 \\ -f_1 & 0 & f_3 \\ 0 & -f_1 & -f_2 \end{bmatrix}$$

Corollary 3.5.8. *If $I = \langle f_1, f_2, f_3 \rangle$ is a codimension two ideal, then*

$$0 \to R(-\sum_{i=1}^{3} d_i, -\sum_{i=1}^{3} d_i') \to \bigoplus_{i<j} R(-d_i - d_j, -d_i' - d_j') \to K \to 0 \qquad (3.10)$$

Note: K is a proper submodule of the syzygy module S defined by the exact sequence

$$0 \to S \to \bigoplus_{i=1}^{3} R(-d_i, -d_i') \to I \to 0,$$

since $\text{codim} I = 2$, the Koszul complex is not exact.

According to [40, page 32], we have the following definition:

Definition 3.5.9. A submodule M of a finitely generated bigraded free $R-$module F is *saturated* if

$$M = \{x \in F | \mathbf{m} x \subset M\}$$

We define the *saturation* of M to be

$$M^{\mathbf{m}\text{sat}} = \{x \in F | \mathbf{m}^k x \subset M, \text{ for some } k\}.$$

where $\mathbf{m} = \langle st, sv, ut, uv \rangle$. We will use the notation M^{sat} for this, the ideal $\mathbf{m}$ being understood.

Remark 3.5.10. M is saturated if and only if $M = M^{\text{sat}}$.

Proof. Suppose M is saturated, we will prove that $M = M^{\text{sat}}$. Since $M \subset M^{\text{sat}}$, we only need to show $M^{\text{sat}} \subset M$ when M is saturated. Let $x \in M^{\text{sat}}$. There exists a k such that $\mathbf{m}^k x = \mathbf{m}\mathbf{m}^{k-1} x \subset M$. Since M is saturated, we have $\mathbf{m}^{k-1} x \subset M$. We can repeat the process until $\mathbf{m}x \subset M$. Therefore $x \in M$. Thus, $M^{\text{sat}} \subset M$, and we proved that $M = M^{\text{sat}}$.

On the other hand, suppose $M = M^{\text{sat}}$, if $x \in F$ such that $\mathbf{m}x \in M$, then $x \in M^{\text{sat}} = M$. Therefore, M is saturated. ☐

Proposition 3.5.11. *Let M be a bigraded submodule of a free $R = k[s, u, t, v]$-module of finite rank F. Let $\mathcal{M}$ be the corresponding coherent sheaf on $X = \mathbf{P}^1 \times \mathbf{P}^1$. Then*

$$M^{sat}_{k,l} = H^0(X, \mathcal{M}(k, l)).$$

Proof. For any finitely generated bigraded R-module we have an exact sequence (see [24])

$$0 \to H^0_{\mathbf{m}}(M) \to M \to \bigoplus_{(a,b)\in\mathbb{Z}^2} H^0(X, \mathcal{M}(a, b)) \to H^1_{\mathbf{m}}(M) \to 0.$$

We will show that $H^i_{\mathbf{m}}(M) = 0$ when $i = 0, 1$ if M is saturated. This is sufficient since both M and M^{sat} generate the same sheaf. Since $H^i_{\mathbf{m}}(R) = 0$, for $i = 0, 1$, and M is a submodule of a free R-module, it is clear that we have vanishing for

$i = 0$, and this does not depend on M being saturated. The long exact cohomology sequence for

$$0 \longrightarrow M \longrightarrow F \longrightarrow F/M \longrightarrow 0$$

has a piece

$$H^0_{\mathbf{m}}(F) \to H^0_{\mathbf{m}}(F/M) \to H^1_{\mathbf{m}}(M) \to H^1_{\mathbf{m}}(F).$$

Since the extreme terms are zero, we get an isomorphism

$$H^1_{\mathbf{m}}(M) \cong H^0_{\mathbf{m}}(F/M)$$

But the right-hand side is M^{sat}/M, proving our claim. □

We will use the following well-known results:

Proposition 3.5.12. *Let R be a Noetherian ring, $J \subset R$ an ideal, and M a finitely generated R-module. The following are equivalent:*

1. $\mathrm{Ass}(M) \subset \mathbb{V}(J)$.

2. $\mathrm{Supp}(M) \subset \mathbb{V}(J)$.

3. *There exists $n \geq 0$ such that $J^n M = 0$.*

When this is so, $H^0_J(M) = M$.

Proof. The equivalence of (1) and (2) follows from the fact that both $\mathrm{Ass}(M)$ and $\mathrm{Supp}(M)$ have the same minimal elements ([37, Theorem 1, p. 7]). Assume (3) holds, and let $\mathbf{p}$ be prime ideal in the support of M. Let $m/s \in M_{\mathbf{p}}$. If $\mathbf{p} \not\supseteq J$ there would exist $x \in J \setminus \mathbf{p}$, and clearly $x^n \in J^n \setminus \mathbf{p}$ for any $n \geq 1$. But then $x^n m = 0$, and this shows that $m/s = 0$, showing that $\mathbf{p}$ cannot be in the support of M. Conversely, assume (2) then ([37, Prop 3, p. 5])

$$\mathrm{Supp}(M) = \mathbb{V}(\mathrm{ann}(M)) \subset \mathbb{V}(J)$$

shows that $\sqrt{J} \subset \sqrt{\mathrm{ann}(M)}$, from which (3) follows easily.

To see the last statement, note that $H^0_J(M) = \{m \in M : J^k m = 0 \text{ for some } k\}$.

□

Remark 3.5.13. If M is a bigraded module over the ring $R = k[s, u, t, v]$, the elements of $\mathrm{Ass}(M)$ are bihomogeneous. Moreover, taking $J = \mathbf{m}$ the irrelevant ideal, the conditions of the previous proposition are easily seen to be equivalent to

4. There exists m, n such that $M_{k,l} = 0$ whenever $k \geq m$ and $l \geq n$, which we abbreviate by writing $(k, l) \gg (0, 0)$.

Definition 3.5.14. The *bigraded Hilbert polynomial* $P(M)$ of a finitely generated bigraded R-module M is the unique polynomial such that

$$P(M)(n, n') = \dim_{\mathbb{C}} M_{n,n'}$$

for all $n, n' >> 0$, where $M_{n,n'}$ is the bigraded piece of M in degree (n, n').

Note: if $\mathcal{M}$ is the sheaf of modules associated to M, for $n, n' \gg 0$, we have

$$P(M)(n, n') = \dim_{\mathbb{C}} M_{n,n'} = \dim H^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{M}(n, n')).$$

Remark 3.5.15.

$$P(M) = P(M^{\mathrm{sat}}).$$

Proof. Let $Q = M^{\mathrm{sat}}/M$. Since $\mathbf{m}^r Q = 0$ for some r, we must have $Q_{k,k'} = 0$ for $k, k' \gg 0$, as in the previous remark. Thus $P(M) = P(M^{\mathrm{sat}})$. □

Lemma 3.5.16. *Let $M \subset N \subset F$ be bigraded submodules where F is free of finite type. If M is saturated, then $M = N$ if and only if $P(M) = P(N)$.*

Proof. We will only show that $P(M) = P(N)$ implies that $M = N$. $P(M) = P(N)$ says that there exist n, n' such that $M_{k,k'} = N_{k,k'}$ for all $k \geq n, k' \geq n'$. Let $a_{p,p'} \in$

$N_{p,p'}$ where $p < n$, or $p' < n'$. We can find an α such that $\mathbf{m}^\alpha a_{p,p'} \subset N_{k,k'} = M_{k,k'}$ for some $k \geq n, k' \geq n'$. Since M is saturated, $M = M^{\text{sat}}$, thus $a_{p,p'} \in M_{p,p'}$. $\square$

Let S be the syzygy module, K the Koszul syzygy module and V the module of syzygies for f_1, f_2, f_3 vanishing at the base points Z of $I = \langle f_1, f_2, f_3 \rangle$.

Lemma 3.5.17. *K, V are submodules of $\bigoplus_{i=1}^3 R(-d_i, -d_i')$. V is a saturated submodule, and $K_{k,k'} = K^{\text{sat}}_{k,k'}$ when $(k - \sum_{i=1}^3 d_i + 1)(k' - \sum_{i=1}^3 d_i' + 1) \geq 0$.*

Proof. We know that $K \subset V \subset \bigoplus_{i=1}^3 R(-d_i, -d_i')$. We first consider V. By definition, we have

$$V = S \cap \bigoplus_{i=1}^{3} I^{\text{sat}}(-d_i, -d_i').$$

Note that $S = S^{\text{sat}}$, or equivalently, that S is saturated. To see this, let $(a, b, c) \in R^3$ such that $\mathbf{m}(a, b, c) \subset S$. This means that for all $h \in \mathbf{m}$ and $h(a, b, c) \in \text{Syz}(f_1, f_2, f_3)$. This says that $h(af_1 + bf_2 + cf_3) = 0$. But R has no zero divisors, thus $(a, b, c) \in S$, which shows that S is saturated. Since the intersection of saturated submodules is saturated, V is saturated.

We will show $K^{\text{sat}}_{k,k'} = K_{k,k'}$ for all $(k - \sum_{i=1}^3 d_i + 1)(k' - \sum_{i=1}^3 d_i' + 1) \geq 0$. Let $r = r_{k,k'} \in \bigoplus_{i=1}^3 R(-d_i, -d_i')$ with bidegree $(k - \sum_{i=1}^3 d_i + 1)(k' - \sum_{i=1}^3 d_i' + 1) \geq 0$ and satisfies $\mathbf{m}r \subset K$. We will show that $r \in K$. Let $L = K + Rr$. Consider the short exact sequence

$$0 \to K \to L \to L/K \to 0.$$

We get a long exact sequence in local cohomology

$$0 \to H^0_{\mathbf{m}}(K) \to H^0_{\mathbf{m}}(L) \to H^0_{\mathbf{m}}(L/K) \to H^1_{\mathbf{m}}(K) \to$$

Since $K \hookrightarrow \bigoplus_{i=1}^{3} R(-d_i, -d_i')$, $H^0_{\mathbf{m}}(K) = 0$. Also consider the exact local cohomology sequence of the exact sequence (3.10)

$$\bigoplus_{i<j} H^i_{\mathbf{m}}(R(-d_i - d_j, -d_i' - d_j'))_{k,k'} \to H^i_{\mathbf{m}}(K)_{k,k'} \to H^{i+1}_{\mathbf{m}}(R(-\sum_{i=1}^{3} d_i, -\sum_{i=1}^{3} d_i')).$$

$H^1_{\mathbf{m}}(K)_{k,k'} = 0$ if

$$\bigoplus_{i<j} H^1_{\mathbf{m}}(R(-d_i - d_j, -d_i' - d_j'))_{k,k'} = 0 \text{ and} \tag{3.11}$$

$$H^2_{\mathbf{m}}(R(-\sum_{i=1}^{3} d_i, -\sum_{i=1}^{3} d_i'))_{k,k'} = 0. \tag{3.12}$$

Since we know that R is strongly $(0,0)$-regular, Equation (3.11) holds for all k, k', and Equation (3.12) can be written as the following

$$H^2_{\mathbf{m}}(R)_{k-\sum_{i=1}^{3} d_i, k'-\sum_{i=1}^{3} d_i'} = H^1(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(k - \sum_{i=1}^{3} d_i, k' - \sum_{i=1}^{3} d_i')) = 0.$$

Now apply the Künneth formula, [35]:

$$\begin{aligned} &H^1(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(k - \sum_{i=1}^{3} d_i, k' - \sum_{i=1}^{3} d_i')) \\ &= \bigoplus_{i+j=1} H^i(\mathbf{P}^1, \mathcal{O}(k - \sum_{i=1}^{3} d_i)) \otimes H^j(\mathbf{P}^1, \mathcal{O}(k' - \sum_{i=1}^{3} d_i')) \end{aligned}$$

By Serre's computation of the cohomology of projective space and this last one clearly is 0, when $k = \sum_{i=1}^{3} d_i - 1$, $\forall l$, or $l = \sum_{i=1}^{3} d_i' - 1$, $\forall k$, or $k > \sum_{i=1}^{3} d_i - 1$, $l > \sum_{i=1}^{3} d_i' - 1$, or $k > \sum_{i=1}^{3} d_i - 1$, $l > \sum_{i=1}^{3} d_i' - 1$. i.e. $(k - \sum_{i=1}^{3} d_i + 1)(k' - \sum_{i=1}^{3} d_i' + 1) \geq 0$.

Therefore, we have

$$H^1_{\mathbf{m}}(K)_{k,k'} = 0, \quad \text{when } (k - \sum_{i=1}^{3} d_i + 1)(k' - \sum_{i=1}^{3} d_i' + 1) \geq 0.$$

Since $L \hookrightarrow \bigoplus_{i=1}^{3} R(-d_i, -d_i')$, $H^0_{\mathbf{m}}(L) = 0$. This implies that

$$H^0_{\mathbf{m}}(L/K)_{k,k'} = 0 \text{ when } (k - \sum_{i=1}^{3} d_i + 1)(k' - \sum_{i=1}^{3} d_i' + 1) \geq 0. \tag{3.13}$$

Clearly, $\mathbf{m}^\alpha L/K = 0$ for some α and by proposition (3.5.12) and the remark following, this implies that $\mathrm{Supp}(L/K) \subset \mathbb{V}(\mathbf{m})$. It is well-known that this last condition implies that $H^0_{\mathbf{m}}(L/K) = L/K$, and from equation (3.13) this gives $L_{k,k'} = K_{k,k'}$ when $(k - \sum_{i=1}^3 d_i + 1)(k' - \sum_{i=1}^3 d'_i + 1) \geq 0$. This means $r = r_{k,k'} \in K_{k,k'}$ and $K_{k,k'} = K^{\mathrm{sat}}_{k,k'}$ for those same indices, which was proved. □

Theorem 3.5.18. *Let $\mathcal{O}_p$ be the local ring of a point p in $\mathbf{P}^1 \times \mathbf{P}^1$, and let $\mathcal{I}_p \subset \mathcal{O}_p$ be a codimension two ideal. Then*

$$dim_{\mathbb{C}} \mathcal{I}_p/\mathcal{I}_p^2 \geq 2 dim_{\mathbb{C}} \mathcal{O}_p/\mathcal{I}_p.$$

Furthermore, equality holds if and only if $\mathcal{I}_p$ is a complete intersection in $\mathcal{O}_p$.

Proof. See [23, Folgerung 2.6 Page 154]. Note the proof of the theorem in [23] is for $\mathbf{P}^2$, but it only uses the local condition, the result is true for $\mathbf{P}^1 \times \mathbf{P}^1$. □

Theorem 3.5.19. *If $I = \langle f_1, f_2, f_3 \rangle \subset R$ has codimension two, then $K^{\mathrm{sat}} = V$ if and only if I is a local complete intersection.*

Proof. Since $K^{\mathrm{sat}} = V \Leftrightarrow P(K^{\mathrm{sat}}) = P(V)$, we will compute both $P(K^{\mathrm{sat}}), P(V)$.

$$P(K) = P(K^{\mathrm{sat}}) = \sum_{i<j} P(R(-d_i - d_j, -d'_i - d'_j)) - P(R(-\sum_{i=1}^3 d_i, -\sum_{i=1}^3 d'_i)).$$

$$\begin{aligned} & P(K^{\mathrm{sat}})(k,k') \\ = & \sum_{i<j} (k - d_i - d_j + 1)(k' - d'_i - d'_j + 1) - (k - \sum_{i=1}^3 d_i + 1)(k' - \sum_{i-1}^3 d'_i + 1) \\ = & \sum_{i=1}^3 (k - d_i + 1)(k' - d'_i + 1) - (k+1)(k'+1) \end{aligned}$$

Now consider $P(V)$. Since $V = S \cap \bigoplus_{i=1}^3 I^{\mathrm{sat}}(-d_i, -d'_i)$, and

$$0 \to S \to \bigoplus_{i=1}^3 R(-d_i, -d'_i) \to I \to 0,$$

we will have the exact sequence:

$$0 \to V \to \bigoplus_{i=1}^{3} I^{\mathrm{sat}}(-d_i, -d_i') \to II^{\mathrm{sat}} \to 0.$$

Then

$$P(V) = \sum_{i=1}^{3} P(I^{\mathrm{sat}}(-d_i, -d_i')) - P(II^{\mathrm{sat}}).$$

Since $0 \to I^{\mathrm{sat}} \to R \to R/I^{\mathrm{sat}} \to 0$ and $V(I^{\mathrm{sat}}) = V(I) = Z$ is zero-dimensional, we have

$$P(I^{\mathrm{sat}}) = P(R) - P(R/I^{\mathrm{sat}}) = P(R) - \deg(Z)$$

Therefore

$$P(V) = \sum_{i=1}^{3} P(R(-d_i, -d_i')) - 3\deg(Z) - P(II^{\mathrm{sat}}).$$

Note that I^2, II^{sat} have the same saturation. To see this, it is enough to show $II^{\mathrm{sat}} \subset (I^2)^{\mathrm{sat}}$, since $I^2 \subset II^{\mathrm{sat}}$. Let $f \in II^{\mathrm{sat}}$, so $f = \sum_{i=1}^{k} f_i g_i$ with $f_i \in I, g_i \in I^{\mathrm{sat}}$. But there exist an n such that $\langle st, sv, ut, uv\rangle^n g_i \subset I$ for all i, therefore $f \in (I^2)^{\mathrm{sat}}$. Thus from remark (3.5.15), we get $P(I^2) = P(II^{\mathrm{sat}})$. Now

$$P(V) = \sum_{i=1}^{3} P(R(-d_i, -d_i')) - 3\deg(Z) - P(I^2).$$

The exact sequences

$$0 \to I^2 \to R \to R/I^2 \to 0$$

and

$$0 \to I/I^2 \to R/I^2 \to R/I \to 0$$

give

$$P(I^2) = P(R) - P(R/I^2) = P(R) - P(I/I^2) - \deg(Z).$$

Therefore, we have

$$\begin{aligned}
& P(V)(k,k') \\
= & \sum_{i=1}^{3} P(R(-d_i,-d'_i))(k,k') - P(R)(k,k') - 2\deg(Z) + P(I/I^2)(k,k') \\
= & \sum_{i=1}^{3} (k-d_i+1)(k'-d'_i+1) - (k+1)(k'+1) - 2\deg(Z) + P(I/I^2)(k,k')
\end{aligned}$$

Comparing $P(K^{\text{sat}})$ and $P(V)$, we see

$$P(K^{\text{sat}}) = P(V) \iff P(I/I^2) = 2\deg(Z).$$

If $\mathcal{I}$ is the ideal sheaf of Z, then

$$\deg(Z) = \dim_{\mathbb{C}} H^0(Z, \mathcal{O}_Z) = \dim_{\mathbb{C}} H^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}/\mathcal{I}) = \sum_{p\in Z} \dim_{\mathbb{C}} \mathcal{O}_p/\mathcal{I}_p,$$

where $\mathcal{O}_p, \mathcal{I}_p$ is the localization at $p \in Z$. Since $\mathcal{I}/\mathcal{I}^2$ has zero dimensional support, we have

$$P(I/I^2) = \dim_{\mathbb{C}} H^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}/\mathcal{I}^2) = \sum_{p\in Z} \dim_{\mathbb{C}} \mathcal{I}_p/\mathcal{I}_p^2.$$

By Theorem 3.5.18, we know that

$$\dim_{\mathbb{C}} \mathcal{I}_p/\mathcal{I}_p^2 \geq 2\dim_{\mathbb{C}} \mathcal{O}_p/\mathcal{I}_p$$

for every $p \in Z$ with equality holds if and only if $\mathcal{I}_p$ is LCI. Therefore, we have

$$P(I/I^2) = 2\deg(Z) \Leftrightarrow \dim_{\mathbb{C}} \mathcal{I}_p/\mathcal{I}_p^2 = 2\dim_{\mathbb{C}} \mathcal{O}_p/\mathcal{I}_p, \quad \forall p \in Z,$$

and we conclude that

$$P(I/I^2) = 2\deg(Z) \Leftrightarrow I \text{ is LCI}.$$

□

Corollary 3.5.20. *If $I = \langle f_1, f_2, f_3 \rangle \subset R$ has codimension two, and the bidegree of f_i is (d_i, d'_i), then $K_{k,k'} = V_{k,k'} \subset \bigoplus_{i=1}^3 R(-d_i,-d'_i)_{k,k'}$ when $(k - \sum_{i=1}^3 d_i + 1)(k' - \sum_{i=1}^3 d'_i + 1) \geq 0$, if and only if I is a local complete intersection.*

Proof. (⇐) If I is LCI, then $K^{\mathrm{sat}} = V$ by the theorem. But $K_{k,k'} = K^{\mathrm{sat}}_{k,k'}$ when $(k - \sum_{i=1}^{3} d_i - 1)(k' - \sum_{i=1}^{3} d'_i + 1) \geq 0$, by Lemma 3.5.17.

(⇒) If $K_{k,k'} = V_{k,k'}$ when $(k - \sum_{i=1}^{3} d_i - 1)(k' - \sum_{i=1}^{3} d'_i - 1) \geq 0$, then $P(K) = P(K^{\mathrm{sat}}) = P(V)$. By Lemma 3.5.17, we have $K^{\mathrm{sat}} = V$. By Theorem 3.5.19, we have I is LCI. □

Example 3.5.21. The following example shows that not all syzygies vanishing at the base point are Koszul syzygies:

Let $I = \langle s^2v^2, u^2t^2, s^2t^2 \rangle$. The only base point of I is $p = (0 : 1; 0 : 1)$, and $I_p = \langle s^2, t^2 \rangle$. The base point is a local complete intersection. Consider the syzygies $(sut^4v, 0, -sut^2v^3)$ and $(0, s^4utv, -s^2u^3tv)$ of pure bidegree $(2, 5)$ and $(5, 2)$ respectively. By definition, they vanish on the base point, since $\mathbf{m}(sut^4v, 0, sut^2v^3) \subset I$, $\mathbf{m}(0, s^4utv, -s^2u^3tv) \subset I$. But neither one of them is a Koszul syzygy, since the Koszul syzygies are generated by

$$(u^2t^2, -s^2v^2, 0), (s^2t^2, 0, -s^2v^2), (0, s^2t^2, -u^2t^2).$$

4. Surface Implicitization

4.1 Introduction

This section will be devoted to a search for a generator of the ideal $I(\mathbb{V})$ where $\mathbb{V} \subset \mathbf{P}^3$ is a surface which is described parametrically by a parametrization $\phi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$. The map ϕ is given by

$$\phi(s:u;t:v) = (a(s:u;t:v) : b(s:u;t:v) : c(s:u;t:v) : d(s:u;t:v)),$$

where a, b, c, d are bihomogeneous polynomials of bidegree (m, n) in the ring $R = \mathbb{C}[s, u, t, v]$. Such parametrizations are sometimes referred to as tensor product parametrizations. Assume $\gcd(a, b, c, d) = 1$. If ϕ has no base points, that is $\mathbb{V}(a, b, c, d) = \emptyset$ in $\mathbf{P}^1 \times \mathbf{P}^1$, and is generically one-to-one, then the image of ϕ is a surface $S \subset \mathbf{P}^3$ of degree $2mn$, [9, Theorem 3.1], [10].

In the polynomial ring $\mathbb{C}[s, u, t, v, x_1, x_2, x_3, x_4] = R[x_1, x_2, x_3, x_4]$, consider the polynomial $Ax_1+Bx_2+Cx_3+Dx_4$ where $A, B, C, D \in R$ are bihomogeneous of the same bidegrees. If we fix a point $p = (s:u;t:v) \in \mathbf{P}^1\times\mathbf{P}^1$, then $A(p)x_1+B(p)x_2+C(p)x_3+D(p)x_4 = 0$ is an equation of a plane in $\mathbf{P}^3$. When the point p changes, we will have different equations of planes in $\mathbf{P}^3$. This suggests the following definition:

Definition 4.1.1. A *moving plane* on $\mathbf{P}^3$ is a polynomial of the form

$$Ax_1 + Bx_2 + Cx_3 + Dx_4$$

where x_1, x_2, x_3, x_4 are homogeneous coordinates on $\mathbf{P}^3$ and $A, B, C, D \in R$ are bihomogeneous of the same bidegree (k, l), which we will call the bidegree of the moving plane. We say the *moving plane follows the parametrization* ϕ if the graph of ϕ, $G(\phi) \subset \mathbb{V}(Ax_1 + Bx_2 + Cx_3 + Dx_4)$. Note that this means

$$A(p)a(p) + B(p)b(p) + C(p)c(p) + D(p)d(p) = 0, \quad \forall p \in \mathbf{P}^1 \times \mathbf{P}^1,$$

which is equivalent to

$$Aa + Bb + Cc + Dd = 0 \in \mathbb{C}[s, u, t, v]$$

where a, b, c, d are parameters of the surface. Thus the moving plane follows the parametrization ϕ if and only if

$$(A, B, C, D) \in \mathrm{Syz}(a, b, c, d)$$

where $\mathrm{Syz}(a, b, c, d)$ denotes the syzygy submodule of R^4 determined by a, b, c, d.

For the same reason, we give the following definitions:

Definition 4.1.2. A *moving quadric* is a polynomial of the form

$$Ax_1^2 + Bx_1x_2 + \cdots + Jx_4^2$$

where $A, \cdots, J \in R$ are bihomogeneous of the same bidegree (k, l), which we will call the bidegree of the moving quadric.

A *moving quadric follows the parametrization* ϕ, if

$$(A, B, \cdots, J) \in \mathrm{Syz}(a^2, ab, \cdots, d^2).$$

Consider moving planes and moving quadrics with bidegree $(m-1, n-1)$ which follow the parametrization ϕ. If $R_{k,l}$ denotes the bihomogeneous forms of bidegree (k, l), then the moving planes of bidegree $(m-1, n-1)$ make up the kernel of the map

$$MP : R^4_{m-1,n-1} \xrightarrow{(a,b,c,d)} R_{2m-1,2n-1}$$
$$(A, B, C, D) \rightarrow Aa + Bb + Cc + Dd,$$

where the map is represented by the $4mn \times 4mn$ matrix MP. The moving quadrics of this bidegree make up the kernel of the map

$$MQ : R^{10}_{m-1,n-1} \xrightarrow{(a^2,ab,\ldots,d^2)} R_{3m-1,3n-1}$$

$$(A, B, \cdots, J) \to Aa^2 + Bab + \cdots + Jd^2,$$

where the map is represented by the $9mn \times 10mn$ matrix MQ. If ϕ has no base points, then MP is an isomorphism [9, page 9], [13]. Thus there are no moving planes of bidegree $(m-1, n-1)$. Now consider MQ. Since $\dim R_{k,l} = (k+1)(l+1)$, so that $\dim R^{10}_{m-1,n-1} - \dim R_{3m-1,3n-1} = 10mn - 9mn = mn$, it follows that

$$\dim \mathrm{Syz}(a^2, \cdots, d^2)_{m-1,n-1} = mn \Longleftrightarrow MQ \text{ has maximal rank.}$$

If MQ has maximal rank, we will have mn linearly independent moving quadrics of bidegree $(m-1, n-1)$ which follow the parametrization ϕ. Each one of these mn moving quadrics Q_i $(1 \leq i \leq mn)$ can be written as

$$\begin{aligned} Q_i &= A_i x_1^2 + \cdots + J_i x_4^2 \\ &= (\sum_{j=0}^{m-1}\sum_{k=0}^{n-1} A_{i,jk} s^j t^k) x_1^2 + \cdots + (\sum_{j=0}^{m-1}\sum_{k=0}^{n-1} J_{i,jk} s^j t^k) x_4^2 \\ &= \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} (A_{i,jk} x_1^2 + \cdots + J_{i,jk} x_4^2) s^j t^k \\ &= \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} Q_{i,jk}(x_1, x_2, x_3, x_4) s^j t^k \end{aligned}$$

where $Q_{i,jk}$ is a quadric in x_1, x_2, x_3, x_4 with coefficients in $\mathbb{C}$. We arrange the $Q_{i,jk}$ into a square matrix M of size $mn \times mn$, where the columns of the matrix M are indexed by the monomial basis of $R_{m-1,n-1}$, namely $\{s^j t^k\}$, and the rows are indexed by the mn moving quadrics Q_i. Since each entry of M is a quadric in x_1, x_2, x_3, x_4, we may write

$$M = (Q_{i,jk}),$$

so that the determinant of M, $|M|$, is a polynomial in x_1, x_2, x_3, x_4 of degree $\leq 2mn$. The main result of [10] is

Theorem 4.1.3. *Suppose that $\phi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$ has no base points and is generically one-to-one. If MP has maximal rank, then so does MQ and furthermore, the image of ϕ is defined by the equation $|M| = 0$.*

Proof. See [9, p.7]. □

The goal of this chapter is to prove a similar result in the presence of base points. In the case that $\mathbf{P}^1 \times \mathbf{P}^1$ is replaced by $\mathbf{P}^2$, such as extension has already been done by Busé, Cox, and D'Andrea [6].

4.2 Base Point and Multiplicity

We are going to recall the multiplicity of a module. For details see, [15, p.280] and [4, Ch.4]. We first give some basic definitions.

Definition 4.2.1. Let $(R, \mathbf{m})$ be a local ring, M an R-module. An ideal $I \subset \mathbf{m}$ such that $\mathbf{m}^n M \subset IM$ for some n is called an *ideal of definition of* M.

The multiplicity of a module with respect to an ideal is defined via the Hilbert-Samuel polynomial:

Definition 4.2.2. Let $(R, \mathbf{m})$ be a Noetherian local ring, let $M \neq 0$ be a finitely generated R-module, and I an ideal of definition of M. Let $P(n) = P_{I,M}(n)$ be the Hilbert-Samuel polynomial. We may write $P(n)$ uniquely in the form

$$P(n) = \sum_{i=0}^{d} a_i F_i(n)$$

where $F_i(n) = \binom{n}{i}$ is the binomial coefficient regarded as a polynomial in n of degree i, the a_i are integers, and $a_d \neq 0$. The integer a_d is called the *multiplicity* of I on M, written $e(I, M)$. The degree of $P(n)$ is $d = \dim M - 1$, and the leading coefficient of $P(n)$ is $e(I, M)/d!$, where $e(I, M) > 0$.

Computing $e(I, M)$ is difficult. The following theorems will show that we can replace an arbitrary ideal of definition of M by an ideal J which is generated by a system of parameters of M such that $e(I, M) = e(J, M)$.

Definition 4.2.3. Let R be a Noetherian ring, I a proper ideal, and M a finite R-module. An ideal $J \subset I$ is called a *reduction ideal of* I with respect to M if $JI^nM = I^{n+1}M$ for some $n \gg 0$. J is called a *minimal reduction ideal* of I if J is a reduction ideal of I and J itself does not have any proper reductions. If J is a reduction ideal of I and is generated by a regular sequence, then J is a minimal reduction ideal of I.

Lemma 4.2.4. *Let $(R, \mathbf{m})$ be a Noetherian local ring, M a finite R-module, I an ideal of definition of M, and J a reduction ideal of I with respect to M. Then J is an ideal of definition of M, and $e(J, M) = e(I, M)$.*

Proof. See [4, p.182]. □

Theorem 4.2.5. *Let $(R, \boldsymbol{m})$ be a d-dimensional Noetherian local ring, suppose $R/\boldsymbol{m}$ is an infinite field, and let $\boldsymbol{q} = \langle u_1, \cdots, u_s \rangle$ be an $\boldsymbol{m}$-primary ideal. Then if $y_i = \sum a_{ij}u_j$ for $1 \leq i \leq d$ and $a_{ij} \in R$ are d 'sufficiently general' linear combinations of $u_1, \cdots, u_s$, the ideal $\boldsymbol{b} = \langle y_1, \cdots, y_d \rangle$ is a reduction of $\boldsymbol{q}$ and $\{y_1, \cdots, y_d\}$ is a system of parameters of R.*

Proof. See [29, p.112, Theorem 14.14]. □

Remark 4.2.6. If R is a K-algebra, and $R/\mathbf{m} = K$, then the linear combinations can be taken to be K-linear combinations.

Remark 4.2.7. If R is Cohen-Macaulay, then $\mathbf{x}$, a system of parameters in R, is equivalent to an R-regular sequence. In this case, to compute $e(I, M)$ is to compute

$e(\mathbf{x}, M)$, which is $\dim R/I$. For details, see [22, p.4, Propostion 1.5] and [32, p.312, Theorem 9].

Now we are going to talk about the intersection multiplicity. For details, see [21, p.361].

Definition 4.2.8. If C and D are curves in a projective space X with no common irreducible component, and if $p \in C \cap D$, then we define the *intersection multiplicity* $(C.D)_p$ of C and D at p to be the length of $\mathcal{O}_{p,X}/\langle f, g\rangle$, where f, g are local equations of C, D at p. The *length* is the same as the dimension of a $\mathbb{C}$-vector space, that is $\dim_{\mathbb{C}} \mathcal{O}_{p,X}/\langle f, g\rangle$. Moreover, $\{f, g\}$ is a system of parameters for $\mathcal{O}_{p,X}$, so $(C.D)_p = e(\langle f, g\rangle, \mathcal{O}_{p,X})$.

Remark 4.2.9. **Bezout's Theorem in $\mathbf{P}^1 \times \mathbf{P}^1$** Let C, D be curves in $\mathbf{P}^1 \times \mathbf{P}^1$ with no common components. If C has type (m_1, n_1), and D has type (m_2, n_2), where $C = \mathbb{V}(f)$ and $D = \mathbb{V}(g)$ with f, g bihomogeneous of bidegree (m_1, n_1) and (m_2, n_2), then

$$m_1 n_2 + m_2 n_1 = \sum_{p \in C \cap D} (C.D)_p$$

Proof. See [21, p.361, Example 1.4.3]. □

Now, we will give a formula for the multiplicity of the base points.

Definition 4.2.10. The *degree of a surface* in $\mathbf{P}^3$ is the cardinality of the intersection of the surface and a generic line in $\mathbf{P}^3$.

Definition 4.2.11. The *degree of a map* $\phi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$ is $|\phi^{-1}(y)|$ for a generic $y \in \mathrm{Im}(\phi) = S$. See [31, p.46]

Theorem 4.2.12. *Suppose $\phi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$ is defined by homogeneous polynomials a, b, c, d of bidegree (m, n) with no common factors. Let $Z = \{x \in \mathbf{P}^1 \times \mathbf{P}^1 : a(x) = b(x) = c(x) = d(x) = 0\}$ be a finite set of base points, $I = \langle a, b, c, d\rangle$, and* *let* $S = \overline{\phi(\mathbf{P}^1 \times \mathbf{P}^1 - Z)}$ *be the image. Assume* $\dim S = 2$*. Then*

$$2mn = \deg S \deg \phi + \sum_{p \in Z} e(I_p, \mathcal{O}_p) \tag{4.1}$$

where $\mathcal{O}_p = \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1, p}$*,* $I_p = \langle \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \rangle$*, where* $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ *are the elements in* $\mathcal{O}_p$ *determined by* a, b, c, d*, and* $e(I_p, \mathcal{O}_p)$ *is the multiplicity of* I_p *as a* $\mathcal{O}_p$*-module.*

Proof. Since base points are isolated, $\mathbb{V}(I_p) = \{p\}$ implies $\mathbf{m}_p^s \subset I_p$ for some s by Nullstellensatz where $\mathbf{m}_p \subset \mathcal{O}_p$ is the maximal ideal. Thus $e(I_p, \mathcal{O}_p)$ is defined, and we call it the multiplicity of the base point.

claim 1 : If $I_p = \langle \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \rangle$, then I_p has a reduction ideal J_p which is generated by a regular sequence of $\mathcal{O}_p$, and the generators of J_p can be chosen to be generic $\mathbb{C}$-linear combinations of $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$.

proof of claim 1: Applying Lemma 4.2.4, Theorem 4.2.5, and Remark 4.2.6, we can find a reduction ideal J_p of I_p, such that the generators of J_p are generic $\mathbb{C}$-linear combinations of $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ which form a system of parameters of $\mathcal{O}_p$. Since $\mathcal{O}_p$ is a regular local ring, it is a Cohen-Macaulay ring. By Remark 4.2.7, a system of parameters of $\mathcal{O}_p$ is a regular sequence. This says that J_p is a reduction ideal of I_p which is generated by a regular sequence, and we have

$$e(I_p, \mathcal{O}_p) = e(J_p, \mathcal{O}_p) = \dim(\mathcal{O}_p / J_p).$$

Thus claim 1 is proved.

Proof of Equation (4.1): $\phi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$ is defined by polynomials a, b, c, d of bidegree (m, n) and $S = \overline{\phi(\mathbf{P}^1 \times \mathbf{P}^1 - Z)}$ where $Z = \mathbb{V}(a, b, c, d)$. Consider $(\alpha, \beta) \subset \mathbf{P}^3 \times \mathbf{P}^3$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$ are in $\mathbf{P}^3$. Let

$$f_\alpha = \alpha_1 a + \alpha_2 b + \alpha_3 c + \alpha_4 d, \text{ and } g_\beta = \beta_1 a + \beta_2 b + \beta_3 c + \beta_4 d,$$

with

$$C_\alpha = \mathbb{V}(f_\alpha) \subset \mathbf{P}^1 \times \mathbf{P}^1, \text{ and } C_\beta = \mathbb{V}(g_\beta) \subset \mathbf{P}^1 \times \mathbf{P}^1.$$

Consider the inclusion map

$$i : \mathbf{P}^1 \times \mathbf{P}^1 - Z \hookrightarrow \mathbf{P}^3.$$

By Bertini's theorem the variety C_α is nonsingular outside the base point set Z for a "generic" α. That is, for α in an open dense subset U of $\mathbf{P}^3$, C_α is nonsingular outside Z. Now consider the inclusion map

$$C_\alpha \hookrightarrow \mathbf{P}^3.$$

If $\alpha \in U$, then C_α is a curve and the $\{g_\beta\}_{\beta \in \mathbf{P}^3}$ is a linear system on C_α. By Bertini's theorem, there is an open dense set $\tilde{U}$ such that if $\beta \in \tilde{U}$, then $C_\beta \cap C_\alpha$ is nonsingular outside of Z. Thus we can choose α, β such that C_α, C_β intersect transversally outside Z. By Bezout's theorem for $\mathbf{P}^1 \times \mathbf{P}^1$, we have

$$\begin{aligned} 2mn &= \sum_{p \in C_\alpha \cap C_\beta} (C_\alpha . C_\beta)_p = \sum_{p \in C_\alpha \cap C_\beta - Z} (C_\alpha . C_\beta)_p + \sum_{p \in Z} (C_\alpha . C_\beta)_p \\ &= \text{ the number of points in } (C_\alpha \cap C_\beta - Z) + \sum_{p \in Z} (C_\alpha . C_\beta)_p. \end{aligned}$$

For a generic line L in $\mathbf{P}^3$,

$$\deg S = \text{ the cardinality of } S \cap L.$$

Lines in $\mathbf{P}^3$ are parametrized by $\mathbf{P}^3$. Thus, there is an open dense subset $C_{\alpha,\beta,L}$ of $\mathbf{P}^3$, such that C_α, C_β intersect transversally, and $\deg S =$ the cardinality of $S \cap L$. That is,

$$\text{the number of points in } (C_\alpha \cap C_\beta - Z) = \text{ the cardinality of } \phi^{-1}(S \cap L)$$

and is the same as $\deg S \deg \phi$. Therefore, we will have

$$2mn = \deg S \deg \phi + \sum_{p \in Z} (C_\alpha . C_\beta)_p.$$

where $\deg\phi =$ the cardinality of a preimage of a generic point in S.

According to claim 1, there is an open dense subset of $\mathbf{P}^3$ such that for each $p \in Z$, J_p is generated by $\mathbb{C}$-linear combinations of $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ with $e(I_p, \mathcal{O}_p) = e(J_p, \mathcal{O}_p) = \dim(\mathcal{O}_p/J_p)$. Since we have finitely many base points, there is an open dense subset U of $\mathbf{P}^3$ such that if α, β, L are chosen from U, then $e(I_p, \mathcal{O}_p) = e(J_p, \mathcal{O}_p) = \dim(\mathcal{O}_p/J_p) = (C_\alpha.C_\beta)_p, \quad \forall p \in Z$. Therefore, we have

$$2mn = \deg S \deg\phi + \sum_{p\in Z}(C_\alpha.C_\beta)_p = \deg S \deg\phi + \sum_{p\in Z} e(I_p, \mathcal{O}_p).$$

□

4.3 Multiple Base Points of Total Multiplicity $k \le mn$

In this section, we will extend the method of moving quadrics to the case where multiple base points are present.

Throughout this section, ϕ will be a map $\phi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$ given by $\phi(s : u; t : v) = (a : b : c : d)$ where a, b, c, d are bihomogeneous polynomials of bidegree (m, n), and $I = \langle a, b, c, d\rangle$. Some conditions on ϕ related to the base points can be imposed. Among them are:

BPC1: $a(s, u, t, v), b(s, u, t, v), c(s, u, t, v), d(s, u, t, v)$ are bihomogeneous of bidegree m, n and linearly independent over $\mathbb{C}$.

BPC2: $\mathbb{V}(I)$ consists of a finite number of base points with total multiplicity $k \le mn$.

BPC3: the base points are LCI.

BPC4: $\dim_{\mathbb{C}}(R/I)_{2m-1,2n-1} = \deg(\mathbb{V}(I))$.

BPC5: $d \in \mathrm{sat}(a, b, c)$ and $\dim \mathrm{Syz}(a, b, c)_{m-1,n-1} = 0$.

We explain the base point conditions as follow:

1. Condition BPC1 says that $\mathrm{Im}(\phi)$ is not in any plane in $\mathbf{P}^3$.

2. The finiteness of $\mathbb{V}(I)$ in condition BPC2 is equivalent to the assumption that $\gcd(a, b, c, d) = 1$, and $k \le mn \Leftrightarrow \deg S \deg \phi \ge mn$ by Theorem 4.2.12.

3. The LCI condition will give the relationship between the syzygies vanishing on the base points and the Koszul syzygies. Since the base points are local complete intersection, the degree formula for the image of the parametrization given in the introduction involves the sum of the multiplicities of the base points. This equals $\deg(\mathbb{V}(I))$ only when $\mathbb{V}(I)$ is a local complete intersection.

4. Condition BPC4 is equivalent to the regularity conditions on I.

5. If we replace the input polynomials with generic linear combinations of them, condition BPC5 implies that the moving quadrics coming from the k linearly independent moving planes by multiplying by x_1, x_2, x_3, x_4, are linearly independent.

Lemma 4.3.1. *If* $\phi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$ *satisfies base points conditions BPC1, BPC2, BPC3, BPC4, then* $\dim \mathrm{Syz}(I)_{m-1,n-1} = k$.

Proof. Consider the following exact sequence:

$$0 \to \mathrm{Syz}(I)_{m-1,n-1} \to R^4_{m-1,n-1} \xrightarrow{(a,b,c,d)} R_{2m-1,2n-1} \to (R/I)_{2m-1,2n-1} \to 0.$$

We have

$$\begin{aligned} \dim \mathrm{Syz}(I)_{m-1,n-1} &= \dim(R/I)_{2m-1,2n-1} - \dim R_{2m-1,2n-1} + 4 \dim R_{m-1,n-1} \\ &= \dim(R/I)_{2m-1,2n-1}. \end{aligned}$$

Since at each base point, $\mathbb{V}(I)$is a local complete intersection, we have $\sum_{p \in \mathbb{V}(I)} e_p = \deg(\mathbb{V}(I)) = k$ where $e_p = e_p(I_p, \mathcal{O}_p)$. Thus, $\dim(R/I)_{2m-1,2n-1} = \deg(\mathbb{V}(I)) = k$. Therefore, we have $\dim \mathrm{Syz}(I)_{m-1,n-1} = k$. □

Remark 4.3.2. Under the hypothesis of Lemma 4.3.1, the condition

$$\dim \mathrm{Syz}(I)_{m-1,n-1} = k$$

means that there are exactly k linearly independent moving planes of bidegree $(m-1, n-1)$ which follow the parametrization ϕ.

Lemma 4.3.3. *If* $\phi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$ *satisfies the conditions BPC1, BPC2, BPC3, BPC4, then* $\dim \mathrm{Syz}(I^2)_{m-1,n-1} \geq mn + 3k$.

Proof. We will prove the following claims.

Claim 1: Under the hypothesis of the lemma, $\mathrm{sat}(I^2)$ is $(3m-1, 3n-1)$-regular, where $I = \langle a, b, c, d \rangle$ as usual.

Proof of the claim 1. Consider the following exact sequence:

$$0 \to \mathrm{Syz}(I^2)_{m-1,n-1} \to R^{10}_{m-1,n-1} \xrightarrow{(a^2,\cdots,d^2)} R_{3m-1,3n-1} \to (R/I^2)_{3m-1,3n-1} \to 0.$$

We have

$$\begin{aligned} \dim \mathrm{Syz}(I^2)_{m-1,n-1} &= \dim(R/I^2)_{3m-1,3n-1} - \dim R_{3m-1,3n-1} + 10 \dim R_{m-1,n-1} \\ &= \dim(R/I^2)_{3m-1,3n-1} + mn. \end{aligned}$$

Since $\dim(R/I)_{2m-1,2n-1} = \deg(\mathbb{V}(I)) = k$, we know that I is $(2m-1, 2n-1)$-regular by Theorem 3.4.6. Since $\mathbb{V}(I)$ is finite, the Krull dimension of $\dim R/I = 2$, see [24, Lemma 1.2]. By Proposition 3.3.2 and Proposition 3.3.3 we know that $\mathrm{sat}(I^2)$ is $((2-1)(2m-1)+m, (2-1)(2n-1)+n) = (3m-1, 3n-1)$-regular. Thus we showed that $\mathrm{sat}(I^2)$ is $(3m-1, 3n-1)$-regular.

Claim 2: For any ideal I, $P_{I^{sat}}(r, r') = P_I(r, r')$ where $P_I(r, r')$ is the bigraded Hilbert polynomial of I.

Proof of the Claim 2: Let $\mathbf{m}$ be the irrelevant of R. Consider the following exact sequence [24, Corollary 1.5]:

$$0 \to H^0_{\mathbf{m}}(I) \to I \to \oplus_{d,d'} H^0(\mathcal{I}(d, d')) \to H^1_{\mathbf{m}}(I) \to 0.$$

$H^0_{\mathbf{m}}(I) = (0 :_{\mathbf{m}} I) = 0$ since R is an integral domain. According to Remark 3.1.9, we have $H^1_{\mathbf{m}}(I) = I^{sat}/I$. This means that $I^{sat} = \oplus_{r,r'} H^0(\mathcal{I}(r, r'))$, and

$I^{sat}_{r,r'} = H^0(\mathcal{I}(r,r')) = I_{r,r'}$, for $r, r' \gg 0$. Since the bigraded Hilbert polynomial coincides with the bigraded Hilbert function which measures the length of $I_{r,r'}$ when $r, r' \gg 0$, we have $P_{I^{sat}}(r,r') = P_I(r,r')$. Thus we proved claim 2.

Claim 3: $\dim \mathrm{Syz}(I^2)_{m-1,n-1} \geq mn + 3k$.

Proof of the Claim 3: From the following exact sequence:

$$0 \to (I/I^2)_{r,r'} \to (R/I^2)_{r,r'} \to (R/I)_{r,r'} \to 0.$$

We have $\dim(R/I^2)_{r,r'} = \dim(R/I)_{r,r'} + \dim(I/I^2)_{r,r'}$. Since $\dim(R/I)_{2m-1,2n-1} = \deg(\mathbb{V}(I)) = k$, it follows that $\dim(R/I)_{r,r'} = k$ for $r \geq 2m-1, r' \geq 2n-1$. Hence $\dim(R/I)_{r,r'} = k + \dim(I/I^2)_{r,r'}$ for $r \geq 2m-1, r' \geq 2n-1$. Note, for $r, r' \gg 0$, $\dim(I/I^2)_{r,r'} = P_{I/I^2}(r,r')$ where $P_{I/I^2}(r,r')$ is the bigraded Hilbert polynomial of I/I^2. Note that support of (I/I^2) (which means the bigraded prime ideals such that $I_p/I_p^2 \neq 0$) is contained in $\mathbb{V}(I)$. This is because if $p \notin \mathbb{V}(I)$, then $I \nsubseteq p$, and $I_p \nsubseteq p\mathcal{O}_p$. Since $p\mathcal{O}_p$ is a maximal ideal, then elements in I_p are all invertible, which means that $I_p = \mathcal{O}_p$. Therefore $I_p/I_p^2 = 0$ for all $p \notin \mathbb{V}(I)$. Thus, the support of I/I^2 is contained in $\mathbb{V}(I)$. Since $\mathbb{V}(I)$ is finite, I/I^2 has a zero dimensional support, $\dim H^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}/\mathcal{I}^2) = \dim H^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}/\mathcal{I}^2(r,r'))$ where $\mathcal{I}$ is the sheaf associated to I, and $H^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}/\mathcal{I}^2(r,r')) = (I/I^2)_{r,r'}$ by [24, Theorem 1.6]. Therefore we have $P_{I/I^2}(r,r') = \dim_{\mathbb{C}} H^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}/\mathcal{I}^2) = \sum_{p\in\mathbb{V}(I)} \dim_{\mathbb{C}} \mathcal{I}_p/\mathcal{I}_p^2$. Therefore we have $\dim(I/I^2)_{r,r'} = \sum_{p\in\mathbb{V}(I)} \dim_{\mathbb{C}} \mathcal{I}_p/\mathcal{I}_p^2$. By Theorem 3.5.18, we know that

$$\sum_{p\in\mathbb{V}(I)} \dim_{\mathbb{C}} \mathcal{I}_p/\mathcal{I}_p^2 \geq 2 \sum_{p\in\mathbb{V}(I)} \dim_{\mathbb{C}} \mathcal{O}_p/\mathcal{I}_p \quad \forall p \in \mathbb{V}(I),$$

where $\mathcal{O}_p, \mathcal{I}_p$ are the localization at p, and the equality holds if only if I_p is a local complete intersection. Since $\mathbb{V}(I)$ is a local complete intersection, we have for

$r, r' >> 0$,

$$\dim(R/I^2)_{r,r'} = \dim(R/I)_{r,r'} + \dim(I/I^2)_{r,r'} = k + 2\sum_{p\in\mathbb{V}(I)} \dim_{\mathbb{C}} \mathcal{O}_p/\mathcal{I}_p = 3k. \quad (4.2)$$

By Claim 2, we know that $P_{I^{sat}}(r, r') = P_I(r, r')$. This says that $\dim(R/I^2)_{r,r'} = \dim(R/\mathrm{sat}(I^2))_{r,r'}$, for $r, r' \gg 0$. This combined with Equation (4.2) and $\mathrm{sat}(I^2)$ being $(3m-1, 3n-1)$-regular will give $\dim(R/\mathrm{sat}(I^2))_{3m-1,3n-1} = 3k$.

Since $I^2 \subset \mathrm{sat}(I^2)$, we have $\dim(R/I^2)_{3m-1,3n-1} \geq \dim(R/\mathrm{sat}(I^2))_{3m-1,3n-1} = 3k$. Therefore, Equation (4.2) becomes

$$\dim \mathrm{Syz}(I^2)_{m-1,n-1} = mn + \dim(R/I^2)_{3m-1,3n-1} \geq mn + 3k.$$

$\square$

Remark 4.3.4. Under the hypothesis of Lemma 4.3.3, the condition

$$\dim \mathrm{Syz}(I^2)_{m-1,n-1} \geq mn + 3k$$

means that there are at least $mn + 3k$ linearly independent moving quadrics of bidegree $(m-1, n-1)$ which follows the parametrization ϕ.

Lemma 4.3.5. *Let F be a map such that*

$$F : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^n$$

$F(x) = (F_0(x) : \cdots : F_n(x))$, for all $x \in \mathbf{P}^1 \times \mathbf{P}^1$, with F_i a bihomogeneous polynomial. Let $Z(F)$ denote the base point set of F, that is $Z(F) = \{x \in \mathbf{P}^1 \times \mathbf{P}^1 : F_0(x) = \cdots = F_n(x) = 0\}$. If $\dim F(\mathbf{P}^1 \times \mathbf{P}^1) = m$, then there exists a projection map

$$\pi : \mathbf{P}^n \to \mathbf{P}^m$$

such that $Z(\pi \circ F) = Z(F)$.

Proof. If $E \subset \mathbf{P}^n$ is a subspace of dimension d, then there exist $L_0, \cdots, L_{n-d-1}$ generic linear forms with $E = \mathbb{V}(L_0, \cdots, L_{n-d-1})$. Define $\pi_E : \mathbf{P}^n \to \mathbf{P}^{n-d-1}$ by $\pi_E(x) = (L_0(x) : \cdots : L_{n-d-1}(x))$ with $Z(\pi_E) = E$.

Claim : Suppose $E \cap F(\mathbf{P}^1 \times \mathbf{P}^1) = \emptyset$, then $Z(\pi_E \circ F) = Z(F)$.

Proof of the claim : We will first show that $Z(\pi_E \circ F) \subset Z(F)$. If $p \notin Z(F)$, then $(F_0(p) : \cdots : F_n(p)) \in \mathbf{P}^n - E$ since $F(\mathbf{P}^1 \times \mathbf{P}^1) \cap E = \emptyset$. This implies that some $L_i(F(p)) \neq 0$, so that $p \notin Z(\pi_E \circ F)$. Therefore $Z(\pi_E \circ F) \subset Z(F)$. On the other hand, if $p \in Z(F)$, then $F_i(p) = 0$ for all $i = 0, \cdots, n$, this says that all $L_j(F_0(p) : \cdots : F_n(p)) = 0$ for $j = 0, \cdots, n-d-1$. This says that $Z(F) \subset Z(\pi_E \circ F)$.

Let $V = F(\mathbf{P}^1 \times \mathbf{P}^1)$, assume $\dim V = m$, let E be the base point set of a generic linear system with $\dim E = n - m - 1$ and $E \cap V = \emptyset$, then $\pi_E : \mathbf{P}^n \to \mathbf{P}^{n-(n-m-1)-1} = \mathbf{P}^m$. □

Theorem 4.3.6. *Let $\phi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$, with $\phi(s : u; t : v) = (a : b : c : d)$, where $a, b, c, d \in \mathbb{C}[s, u, t, v]$ with bidegree (m, n), and the image is a surface. Let $Z(a, b, c, d)$ be the base point set of a, b, c, d, and let e_p the multiplicity of each base point $p \in Z(a, b, c, d)$. Then there are linear combinations a', b', c' of a, b, c, d with $Z(a', b', c') = Z(a, b, c, d)$, and the multiplicity of each base point is preserved.*

Proof. First, show that $Z(a', b', c') = Z(a, b, c, d)$. Consider the map $\phi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$, $\dim \phi(\mathbf{P}^1 \times \mathbf{P}^1) = 2$. In the proof of the Lemma 4.3.5, we can find $E \subset \mathbf{P}^3$ with $\dim E = n - m - 1 = 3 - 2 - 1 = 0$, that is $E = \{p\}$ with $p \notin \phi(\mathbf{P}^1 \times \mathbf{P}^1)$ where $E = \mathbb{V}(L_0(a, b, c, d), L_1(a, b, c, d), L_2(a, b, c, d))$ with L_i generic linear forms. Let $a' = L_0(a, b, c, d), b' = L_1(a, b, c, d), c' = L_2(a, b, c, d)$, then according to the Lemma 4.3.5 $Z(a, b, c, d) = Z(a', b', c')$.

Second, show the multiplicity of each base point is preserved. By the Theorem 4.2.5 and the Remark 4.2.6 we can find a'_d, b'_d generic $\mathbb{C}$-linear combinations of a_d, b_d, c_d, d_d such that the ideal $\langle a'_d, b'_d \rangle$ is a reduction ideal of $\langle a_d, b_d, c_d, d_d \rangle$, and $\{a'_d, b'_d\}$ is a system of parameters of $\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1, p}$, where a_d, b_d, c_d, d_d are in the local ring $\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1, p}$ with $p \in Z(a, b, c, d)$ determined by a, b, c, d. By Lemma 4.2.4, we know that $e(\langle a'_d, b'_d \rangle, \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1, p}) = e(\langle a_d, b_d, c_d, d_d \rangle, \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1, p})$. This says that the multiplicity of each base point is preserved. □

Corollary 4.3.7. *Suppose $a, b, c, d \in \mathbb{C}[s, u, t, v]$ are bihomogeneous of bidegree m, n with no common factor, and $\mathbb{V}(a, b, c, d)$ is a local complete intersection. If we replace a, b, c with generic linear combinations of a, b, c, d, then we have $\mathbb{V}(a, b, c) = \mathbb{V}(a, b, c, d)$ as subschemes and $d \in \mathrm{sat}(a, b, c)$.*

Proof. Let $p \in V(a, b, c, d)$ and let $I_p \subset \mathcal{O}_p$ be the ideal generated by a, b, c, d in the local ring $\mathcal{O}_p$ at p. By [4, Corollary 4.5.10], $\mathcal{O}_p$ has a system of parameters which generates a reduction ideal J_p for I_p. This system of parameters is a regular sequence if $\mathcal{O}_p$ is Cohen-Macaulay, [4, Theorem 2.12]. Furthermore, Theorem 4.2.5 shows that the system of parameters can be chosen to be generic linear combinations of generators of I_p.

Let $\tilde{I}_p$ be the ideal of $\mathcal{O}_p$ generated by the linear combinations of a, b, c, d. Then we have $J_p \subset \tilde{I}_p \subset I_p$, which gives the inequalities $e(J_p) \geq e(\tilde{I}_p) \geq e(I_p)$.

If I_p is a complete intersection, then it is the same as all of its reduction ideals. Thus $J_p = I_p$. This shows that $(\tilde{I})_p = I_p$. Let a', b', c' denote a generic linear combination of a, b, c, d, such that $\mathbb{V}(a', b', c') = \mathbb{V}(a, b, c, d)$ have the same scheme structure at p. When $\mathbb{V}(a, b, c, d)$ is a local complete intersection, this is true for all its points, and it says that $\mathbb{V}(a', b', c') = \mathbb{V}(a, b, c, d)$ as schemes. If we replace the input polynomials with generic linear combinations of them, we have $\mathbb{V}(a, b, c) =$

$\mathbb{V}(a,b,c,d)$ as schemes. This implies that $\mathrm{sat}(a,b,c) = \mathrm{sat}(a,b,c,d)$. Therefore, $d \in \mathrm{sat}(a,b,c)$. □

From now on we will replace a,b,c,d by a generic linear combination of a,b,c,d so that a,b,c has the properties indicated in Theorem 4.3.6 and Corollary 4.3.7. Recall the $4mn \times 4mn$ matrix MP which represents the map

$$MP : R^4_{m-1,n-1} \xrightarrow{(a,b,c,d)} R_{2m-1,2n-1}$$

$$(A,B,C,D) \to Aa + Bb + Cc + Dd.$$

If we replace the input polynomials by generic linear combinations of them, the rank of the coefficient matrix MP will not change.

Let

$$MC : R^3_{m-1,n-1} \xrightarrow{(a,b,c)} R_{2m-1,2n-1}$$

$$(A,B,C) \to Aa + Bb + Cc$$

MC is represented by a matrix of size $4mn \times 3mn$, and

$$\dim \mathrm{Syz}(a,b,c)_{m-1,n-1} = \dim \ker(MC).$$

Lemma 4.3.8. *If $\phi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$ satisfies the conditions BPC1, BPC2, BPC3, BPC5, then the matrix MC has maximal rank.*

Proof. Suppose MC does not have maximal rank, then $\dim \ker(MC) \neq 0$. This means $\dim \mathrm{Syz}(a,b,c)_{m-1,n-1} \neq 0$, which contradicts our assumption. Therefore, the matrix MC must have maximal rank. □

Let $V = V_1 \oplus V_2$ be the direct sum of two subspaces V_1 and V_2, and let $W \subset V$ be a subspace such that $V_1 \cap W = \{0\}$. Then the projection $\pi : V \to V_2$ along V_1 satisfies $\ker \pi = V_1$, and $\ker \pi|_W = W \cap V_1 = \{0\}$. In particular, $\pi|_W$ is injective, so that $\dim W = \dim \pi(W) := k$. Let $\mathcal{B} = \{v_1, \cdots, v_l\}$ be a given basis of V_2.

Lemma 4.3.9. *There is a subset* $\mathcal{B}_1 = \{v_{i_1}, \cdots, v_{i_k}\} \subset \mathcal{B}$ *and a basis* $\mathcal{C} = \{w_1, \cdots, w_k\}$ *of* W *such that*

$$\pi(w_e) = v_{i_e} + \bar{w}_e, \quad \textit{where } \bar{w}_e \in \operatorname{Span}(\mathcal{B} \setminus \mathcal{B}_1).$$

Proof. Let $\{\tilde{w}_1, \cdots, \tilde{w}_k\}$ be an arbitrary basis of W. Then

$$\pi(\tilde{w}_i) = \sum_{j=1}^{l} a_{ij} v_j.$$

Let $A = [a_{ij}]$. Then multiply A on the left by an invertible matrix P so that $PA = Q$, where Q is in reduced row echelon form. Since A is a $k \times l$ matrix which has rank$A = k$ (because $\dim \pi(W) = k$), there are k columns $i_1 < i_2 < \cdots < i_k$ which contain a leading 1 in rows 1 to k. Let $\mathcal{B}_1 = \{v_{i_1}, \cdots, v_{i_k}\}$. Let the basis $\mathcal{C} = \{w_1, \cdots, w_k\}$ be defined by

$$\begin{bmatrix} w_1 \\ \vdots \\ w_k \end{bmatrix} = P \begin{bmatrix} \tilde{w}_1 \\ \vdots \\ \tilde{w}_k \end{bmatrix},$$

i.e. $w_e = \sum_{j=1}^{k} p_{ej} \tilde{w}_j$. Then

$$\begin{aligned} \pi(w_e) &= \sum_{j=1}^{k} p_{ej} \pi(\tilde{w}_j) \\ &= \sum_{j=1}^{k} p_{ej} \sum_{r=1}^{l} a_{jr} v_r \\ &= \text{Row}_e Q \begin{bmatrix} v_1 \\ \vdots \\ v_l \end{bmatrix} \\ &= v_{i_e} + \bar{w}_e \end{aligned}$$

where $\bar{w}_e \in \operatorname{Span}(\mathcal{B} \setminus \mathcal{B}_1)$. □

If $P = A(s,u,t,v)x_1 + B(s,u,t,v)x_2 + C(s,u,t,v)x_3 + D(s,u,t,v)x_4$ in the ring $C[s,u,t,v,x_1,x_2,x_3,x_4]$ is any moving plane, and $L(x_1,x_2,x_3,x_4)$ is any homogeneous linear polynomial. Then $P \cdot L$ is a moving quadric. Moreover, if P follows ϕ, then $P \cdot L$ also follows ϕ. If $\mathcal{P}$ is a set of moving planes, then $\mathcal{P} \cdot L = \{P \cdot L : P \in \mathcal{P}\}$. Let $\mathcal{P}_{\phi,m-1,n-1}$ be the set of moving planes of bidegree $m-1, n-1$ which follow ϕ, i.e. $(A_{m-1,n-1}, B_{m-1,n-1}, C_{m-1,n-1}, D_{m-1,n-1}) \in \mathrm{Syz}(a,b,c,d)_{m-1,n-1}$.

Lemma 4.3.10. *Let* $\phi = (a : b : c : d)$ *as usual. Assume* $\mathrm{Syz}(a,b,c)_{m-1,n-1} = \{0\}$. *Let* $\mathcal{S} = \mathcal{P}_{\phi,m-1,n-1}$, *and let* $\dim \mathcal{S} = k$. *Then* $\mathcal{Q} = \sum_{i=1}^{4} \mathcal{S}x_i$ *is a vector space of moving quadrics which follow* ϕ, *and* $\dim \mathcal{Q} = 4k$.

Proof. Let $V = R^4_{m-1,n-1}$, $V_1 = R^3_{m-1,n-1}$, $V_2 = R_{m-1,n-1}$, $W = \mathcal{S}$, and $\mathcal{S} \cap V_1 = \mathrm{Syz}(a,b,c)_{m-1,n-1} = \{0\}$. Let $\mathcal{B} = \{s^\alpha t^\beta x_4 : 0 \le \alpha \le m-1, 0 \le \beta \le n-1\}$. According to Lemma 4.3.9, there exist $\{(\alpha_1,\beta_1), \cdots, (\alpha_k,\beta_k)\}$ and a basis $\{P_1, \cdots, P_k\}$ of $\mathcal{S}$ such that

$$P_i = s^{\alpha_i} t^{\beta_i} x_4 + \text{ other terms without } s^{\alpha_j} t^{\beta_j} x_4$$

where $i = 1,2,\cdots,k$, and $j = 1,2,\cdots,k$ but $j \ne i$. We claim that $\{P_i x_j\}_{i=1,j=1}^{i=k,j=4}$ are linear independent. We only need to show that if

$$\sum_{i=1}^{k} (\alpha_i x_1 + \beta_i x_2 + \gamma_i x_3 + \delta_i x_4) P_i = 0$$

where $\alpha_1, \cdots, \delta_k \in \mathbb{C}$, we must have $\alpha_1 = \cdots = \delta_k = 0$. Since P_k contains the term $s^{\alpha_i} t^{\beta_i} x_4$ term for each i, we must have

$$s^{\alpha_i} t^{\beta_i} (\alpha_i x_1 x_4 + \beta_i x_2 x_4 + \gamma_i x_3 x_4 + \delta_i x_4^2) = 0, \quad i = 1,2,\cdots,k.$$

Since we are in an integral domain, we must have $\alpha_i x_1 x_4 + \beta_i x_2 x_4 + \gamma_i x_3 x_4 + \delta_i x_4^2 = 0$ which says $\alpha_i = \beta_i = \gamma_i = \delta_i = 0$ for $i = 1,2,\cdots,k$. Therefore, the moving quadrics coming from the moving planes are linearly independent. Therefore $\dim \mathcal{Q} = 4k$.

□

Theorem 4.3.11. *Let $\phi = (a : b : c : d)$ as usual and assume the base point set satisfies BPC1 - BPC5, then* $\dim \mathrm{Syz}(I^2)_{m-1,n-1} = mn + 3k$.

Proof. By Lemma 4.3.3, $\dim \mathrm{Syz}(I^2)_{m-1,n-1} \geq mn + 3k$. Let $MQ : R^{10}_{m-1,n-1} \to R_{3m-1,3n-1}$ be the map $MQ(A, B, \cdots, J) = Aa^2 + Bab + \cdots + Jd^2$, so that

$$10mn - \mathrm{rank}MQ = \dim \mathrm{Syz}(I^2)_{m-1,n-1}$$

is the number of linearly independent moving quadrics.

Claim that $\mathrm{rank}MQ \geq 9mn - 3k$.

To see this, recall that in the proof of Lemma 4.3.10, we found an index set $\{(\alpha_1, \beta_1), \cdots, (\alpha_k, \beta_k)\}$ and a basis of moving planes $\{P_1, \cdots, P_k\}$ such that

$$P_i = s^{\alpha_i} t^{\beta_i} x_4 + \text{ other terms without } s^{\alpha_j} t^{\beta_j} x_4 \tag{4.3}$$

where $i = 1, 2, \cdots, k$ and $j = 1, 2, \cdots, k$ but $j \neq i$. The columns of MQ are indexed by

$$\Lambda = \{s^\alpha t^\beta x_i x_j : 0 \leq \alpha \leq m-1, 0 \leq \beta \leq n-1, 1 \leq i \leq j \leq 4\}.$$

Let

$$\Lambda_P = \{s^{\alpha_i} t^{\beta_i} x_j x_4, s^\alpha t^\beta x_4^2 : 0 \leq i \leq k, 0 \leq \alpha \leq m-1, 0 \leq \beta \leq n-1, 1 \leq j \leq 3\},$$

and let $\Lambda' = \Lambda \setminus \Lambda_P$, then $|\Lambda'| = 10mn - (mn + 3k) = 9mn - 3k$. Let MQ' be the matrix obtained from MQ by deleting the columns indexed by Λ_p. Thus the nonzero elements of $\ker MQ'$ corresponds to nontrivial syzygies:

$$Aa^2 + Bab + Cac + Dad + Eb^2 + Fbc + Gbd + Hc^2 + Icd = 0 \tag{4.4}$$

where $A, B, \cdots, I$ are bihomogeneous of bidegree $(m-1, n-1)$ and there are no terms $s^{\alpha_i} t^{\beta_i}$ in D, G, I. Since every term contains a, b, c, we obtain:

$$(Aa + Bb + Cc + Dd)a + (Eb + Fc + Gd)b + (Hc + Id)c = 0$$

The syzygy $((Aa+Bb+Cc+Dd),(Eb+Fc+Gd),(Hc+Id))$ of (a,b,c) has bidegree $(2m-1,2n-1)$ and it vanishes on the base point of (a,b,c) since $d \in \mathrm{sat}(a,b,c)$ (since the base point condition BPC5 is satisfied). Moreover, the base points of a,b,c are LCI of total multiplicity k. From Corollary 3.5.20 we have:

$$Aa + Bb + Cc + Dd = h_1c + h_2b$$
$$Eb + Fc + Gd = -h_2a + h_3c$$
$$Hc + Id = -h_1a - h_3b$$

for bihomogeneous polynomial h_1, h_2, h_3 of bidegree $(m-1, n-1)$. We can rewrite the above equations, and we get:

$$Aa + (B - h_2)b + Cc + Dd = 0, \tag{4.5}$$
$$h_2a + Eb + (F - h_3)c + Gd = 0, \tag{4.6}$$
$$h_1a + h_3b + Hc + Id = 0 \tag{4.7}$$

We know that $A, B, \cdots, I$ are bihomogeneous of bidegree $(m-1, n-1)$ and there are no $s^{\alpha_i}t^{\beta_i}$ terms in D, G, I. Thus each equation in (4.5) is a nontrivial syzygy on (a,b,c,d) which corresponds to a moving plane P with no $s^{\alpha_i}t^{\beta_i}x_4$ term with $1 \le i \le k$. But $\{P_1, \cdots, P_k\}$ is a basis of moving planes. Any nonzero moving plane $P = c_1P_1 + \cdots + c_kP_k$ must have some nonzero term $s^{\alpha_i}t^{\beta_i}x_4$, since if $c_i \neq 0$, then $s^{\alpha_i}t^{\beta_i}x_4$ appears.

Hence the nontrivial syzygies from (4.5) cannot exist. Thus $\ker MQ' = \{0\}$, so $\dim MQ \geq \mathrm{rank} MQ' = 9mn - 3k$. Therefore we proved our claim. Hence $\dim \mathrm{Syz}(I^2)_{m-1,n-1} = mn + 3k$. □

Remark 4.3.12. Under the hypothesis of Theorem 4.3.11, the condition

$$\mathrm{Syz}(I^2)_{m-1,n-1} = mn + 3k$$

means that there are exactly $mn + 3k$ linearly independent moving planes of bidegree $(m-1, n-1)$ which follow the parametrization ϕ.

Recall the construction of a $mn \times mn$ matrix M: the columns of the matrix M are indexed by the monomial basis of $R_{m-1,n-1}$; the rows of the matrix are indexed by certain linearly independent moving planes and moving quadrics. In particular, we will choose the basis of the moving planes as our k linearly independent moving planes, and we will choose $mn-k$ linearly independent moving quadrics which are complementary to the set of moving quadrics which are coming from the moving planes by multiplying by $\{x_i\}_{i=1}^4$.

Theorem 4.3.13. *Let $\phi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$ be the usual map and assume ϕ is generically one-to-one. If ϕ satisfies base point conditions BPC1-BPC5, then the image of ϕ is defined by the equation $|M| = 0$.*

Proof. By Lemma 4.3.1, $\dim \operatorname{Syz}(a,b,c,d)_{m-1,n-1} = k$, and there is a basis of the moving planes of the form

$$P_i = s^{\alpha_i} t^{\beta_i} x_4 + \text{ other terms without } s^{\alpha_j} t^{\beta_j} x_4$$

where $i = 1, 2, \cdots, k$, and $j = 1, 2, \cdots, k$ but $j \neq i$ by Lemma 4.3.10. By Theorem 4.3.11, rank$MQ = 9mn - 3k$, the moving quadrics of bidegree $(m-1, n-1)$ have the form:

$$\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} Q_{i,j}(x_1, x_2, x_3, x_4) s^i t^j = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \sum_{1 \leq p \leq q \leq 4} A_{ij,pq} x_p x_q \tag{4.8}$$

The moving quadrics follows the parametrization if and only if

$$\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} Q_{ij}(a,b,c,d) \equiv 0.$$

Let $V_{\Lambda'}$,V_{Λ_p} be subspaces of $V = R^{10}_{m-1,n-1}$ with basis Λ', Λ_P respectively, where

$$\Lambda = \{s^\alpha t^\beta x_i x_j : 0 \leq \alpha \leq m-1, 0 \leq \beta \leq n-1, 1 \leq i \leq j \leq 4\},$$

$$\Lambda_P = \{s^{\alpha_i}t^{\beta_i}x_jx_4, s^{\alpha}t^{\beta}x_4^2 : 0 \le i \le k, 0 \le \alpha \le m-1, 0 \le \beta \le n-1, 1 \le j \le 3\}.$$

and $\Lambda' = \Lambda \backslash \Lambda_P$, then $V = V_{\Lambda'} \oplus V_{\Lambda_p}$. $\mathrm{Syz}(I^2)_{m-1,n-1} \subset V$ satisfies $\mathrm{Syz}(I^2)_{m-1,n-1} \cap V_{\Lambda'} = \{0\}$. This means that $\pi : V \to V_{\Lambda_P}$ with $\pi(v_1 + v_2) = v_2$ is the projection onto V_{Λ_P} along $V_{\Lambda'}$. Then $\pi|_{\mathrm{Syz}(I^2)_{m-1,n-1}}$ is an isomorphism, since we have already proved that $\dim \mathrm{Syz}(I^2)_{m-1,n-1} = \dim V_{\Lambda_P} = mn + 3k$.

Let $Q_{x_jx_4,i} = \pi^{-1}(s^{\alpha_i}t^{\beta_i}x_jx_4)$, for $1 \le i \le k$, and $1 \le j \le 4$. Since $x_jP_i \in \mathrm{Syz}(I^2)_{m-1,n-1}$, and $\pi(x_jP_i) = s^{\alpha_i}t^{\beta_i}x_jx_4$, then x_jP_i must be $Q_{x_jx_4,i}$.

Let $Q_{\gamma,\delta} = \pi^{-1}(s^{\gamma}t^{\delta}x_4^2)$, where $(\gamma, \delta) \in \{(\alpha, \beta)\} \setminus \{(\alpha_i, \beta_i), 1 \le i \le k\}$. These $mn-k$ moving quadrics are not coming from the moving planes by multiplication by $\{x_i\}_{i=1}^4$. Thus we find $mn-k$ linearly independent moving quadrics which are not from multiplying a moving plane by $\{x_i\}_{i=1}^4$. These are listed below:

$$Q_{\gamma,\delta} = x_4^2 s^{\gamma}t^{\delta} + \text{ terms not involving } x_4^2,$$

$$\text{with } 0 \le \gamma \le m-1, 0 \le \delta \le n-1, \gamma \ne \alpha_1, \cdots, \alpha_k, \delta \ne \beta_1, \cdots, \beta_k.$$

We have k moving planes

$$P_i = x_4 s^{\alpha_i}t^{\beta_i} + \text{ other terms without } x_4 s^{\alpha_i}t^{\beta_i}, \quad i = 1, \cdots, k.$$

Now we construct a $mn \times mn$ matrix M as follow: the columns are indexed by monomial basis $R_{m-1,n-1}$; the rows are indexed by the basis of k moving planes and the basis of $mn-k$ moving quadrics; the entries of the matrix are the coefficients of the monomials $s^{\gamma}t^{\delta}$ where $0 \le \gamma \le m-1, 0 \le \delta \le n-1$ from the moving planes and the moving quadrics. Thus the matrix M is

$$\begin{bmatrix}
x_4^2+\cdots & & & & & & & & \\
& x_4^2+\cdots & & & & & & & \\
& & \ddots & & & & & & \\
c_{0,0}x_4+\cdots & c_{0,1}x_4+\cdots & c_{0,3}x_4+\cdots & x_4+\cdots & c_{\alpha_1,\beta_1+1}x_4+\cdots & & & & \\
& & & & \ddots & & & & \\
c_{k,0}x_4+\cdots & c_{k,1}x_4+\cdots & c_{k,3}x_4+\cdots & & x_4+\cdots & c_{\alpha_k,\beta_k+1}x_4+\cdots & & & \\
& & & & & & x_4^2+\cdots & & \\
& & & & & & & \ddots & \\
& & & & & & & & x_4^2+\cdots
\end{bmatrix}$$

Note, there are $mn - k$ rows of quadratic terms, and k row of linear term in x_1, x_2, x_3, x_4. In these $mn - k$ quadratic rows, the terms involving x_4^2 are on the diagonal entries only. The linear row has terms involving x_4 corresponding to the column indexed by monomial $s^{\alpha_i} t^{\beta_i}$, the entries corresponding to other columns may or may not involve x_4; we use $c_{i,j} \in \mathbb{C}$ to denote the coefficient of x_4 in those columns. In this case we can perform column operations to make the linear rows have entries with x_4 only in the column corresponding to $s^{\alpha_i} t^{\beta_i}$. After the operation, the matrix will look like the following:

$$\tilde{M} = \begin{bmatrix} x_4^2 + \cdots & & & & & & & & \\ & x_4^2 + \cdots & & & & & & & \\ & & \ddots & & & & & & \\ & & & x_4 + \cdots & & & & & \\ & & & & \ddots & & & & \\ & & & & & x_4 + \cdots & & & \\ & & & & & & x_4^2 + \cdots & & \\ & & & & & & & \ddots & \\ & & & & & & & & x_4^2 + \cdots \end{bmatrix}$$

$|\tilde{M}|$ contains the term x_4^{2mn-k}, which is from $2(mn - k) + k = 2mn - k$, since the diagonal entries give the highest degree in x_4. Thus $|\tilde{M}|$ is not identically zero. Since the off-diagonal entries have $mn - k$ rows of quadratic terms in x_1, x_2, x_3, x_4 and k rows of linear terms in x_1, x_2, x_3, x_4, the total degree of $|\tilde{M}|$ is $2mn - k$ in x_1, x_2, x_3, x_4. Moreover, by construction, the rows of the matrix are moving quadrics and moving planes that follow the surface. So for points on surface, the columns of M are linearly dependent, hence $|M|$ vanishes for points on the surface and $|M| = |\tilde{M}|$. On the other hand, the degree of the irreducible implicit equation of the surface is $2mn - k$. This comes from the formula:

$$\deg(\phi)\ \deg(S) = 2mn - \sum_{\text{base points}} \text{multiplicity of the base point.}$$

Since ϕ is generically one to one, and we have multiple base points with total multiplicity k, we will have $\deg S = 2mn - k$. Therefore $|M| = 0$ must be the implicit equation of the image of ϕ. □

Example 4.3.14. Consider the following parametrization:

$$a = u^2tv, b = u^2t^2 + suv^2, c = s^2tv, d = s^2v^2 + s^2t^2$$

Here $m = n = 2$, $(0 : 1; 0 : 1)$ is the base point of multiplicity one. We find one moving plane of bidegree $(1, 1)$ which is

$$stx_1 - sx_2 - tx_3 + x_4 = 0$$

and three linear independent moving quadric of bidegree $(1, 1)$ which are:

$$(-s + 1)x_1x_3 - tx_1x_4 + tx_2x_3 = 0$$

$$-tx_1x_3 - sx_1x_4 + (s + 1)x_2x_3 + tx_3^2 - x_3x_4 = 0$$

$$sx_1x_3 + stx_2x_3 - sx_2x_4 - x_3^2 - tx_2x_4 + x_4^2 = 0$$

We get our matrix M :

$$M = \begin{bmatrix} x_4 & -x_3 & x_2 & x_1 \\ x_1x_3 & x_2x_3 - x_1x_4 & -x_1x_3 & 0 \\ -x_3x_4 + x_2x_3 & x_3^2 - x_1x_3 & x_2x_3 - x_1x_4 & 0 \\ -x_3^2 + x_4^2 & -x_3x_4 & x_1x_3 - x_2x_4 & x_2x_3 \end{bmatrix}$$

$$|M| = -x_4^4x^3 + 2x_4^3x_1^2x_2x_3 + x_4^2x_1^2x_2^2x_3 + 4x_4^2x_1^3x_3^2 - 2x_4x_1^3x_2x_3^2 - x_4^2x_1x_2^2x_3^2$$

$$-2x_4x_1x_2^3x_3^2 + x_1^4x_3^3 - 6x_4x_1^2x_2x_3^3 + 2x_1^2x_2^2x_3^3 + x_2^4x_3^3 - 2x_1^3x_3^4 + 2x_1x_2^2x_3^4 + x_1^2x_3^5 = 0$$

which is the implicit equation.

Example 4.3.15. Consider the following parametrization:

$$(a,b,c,d) = (u^2t^2v, u^2t^3 + suv^3, s^2tv^2, s^2v^3 + s^2t^3).$$

Here, $m = 2, n = 3$ and $(0:1;0:1)$ is the only base point of multiplicity 2. From Singular we get 2 moving planes and 4 moving quadrics of bidegree $(1,2)$ as following:

$$-stx_1 + sx_2 + t^2x_3 - x_4 = 0$$

$$st^2x_1 - stx_2 + x_3 = 0$$

$$st^2x_1x_2 - (st-t)x_1x_3 - t^2x_1x_4 - stx_2^2 + tx_2x_4 = 0$$

$$stx_1x_2 - (s-1)x_1x_3 - tx_1x_4 - sx_2^2 + x_2x_4 = 0$$

$$-st^2x_1^2 + stx_1x_2 - (st+t)x_1x_4 + (st^2+t^2)x_2x_3 - tx_3^2 = 0$$

$$-stx_1^2 + sx_1x_2 - (s+1)x_1x_4 + (st+t)x_2x_3 - x_3^2 = 0$$

We form the matrix M:

$$M = \begin{bmatrix} -x_4 & 0 & x_3 & x_2 & -x_1 & 0 \\ x_3 & 0 & 0 & 0 & -x_2 & x_1 \\ 0 & x_2x_4 + x_1x_3 & 0 & -x_1x_3 - x_2^2 & x_1x_2 & 0 \\ x_1x_3 + x_2x_4 & -x_1x_4 & 0 & -x_1x_3 - x_2^2 & x_1x_2 & 0 \\ 0 & -x_3^2 - x_1x_4 & x_2x_3 & 0 & x_1x_2 - x_1x_4 & -x_1^2 + x_2x_3 \\ -x_3^2 - x_1x_4 & x_2x_3 & 0 & x_1x_2 - x_1x_4 & -x_1^2 + x_2x_3 & 0 \end{bmatrix}$$

After computing $|M|$, we get:

$$|M| = -x_4^5x_1^5 + 2x_4^4x_1^3x_2^2x_3 + 2x_4^3x_1^4x_2x_3^2 + x_4^2x_1^4x_2^2x_3^2 - x_4^3x_1x_2^4x_3^2 + 5x_4^2x_1^5x_3^3 - 2x_4x_1^5x_2x_3^3$$

$$-2x_4^2x_1^2x_2^3x_3^3 - 2x_4x_1^2x_2^4x_3^3 + x_1^6x_3^4 + 4x_4^2x_1^3x_2x_3^4 - 9x_4x_1^3x_2^2x_3^4 + 2x_1^3x_2^3x_3^4 +$$

$$x_2^6x_3^4 + 5x_4x_1^4x_3^5 - 6x_1^4x_2x_3^5 + 3x_1x_2^4x_3^5 + 3x_1^2x_2^2x_3^6 + x_1^3x_3^7$$

$|M| = 0$ gives the implicit equation. You can check by using Singular to do elimination.

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Vita

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