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Asymptotic Laplace transforms

Claudiu Mihai
Louisiana State University and Agricultural and Mechanical College, mihai2@math.lsu.edu

ASYMPTOTIC LAPLACE TRANSFORMS

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Abstract

In this work we discuss certain aspects of the classical Laplace theory that are relevant for an entirely analytic approach to justify Heaviside's operational calculus methods. The approach explored here suggests an interpretation of the Heaviside operator $\{\cdot\}$ based on the "Asymptotic Laplace Transform." The asymptotic approach presented here is based on recent work by G. Lumer and F. Neubrander on the subject. In particular, we investigate the two competing definitions of the asymptotic Laplace transform used in their works, and add a third one which we suggest is more natural and convenient than the earlier ones given. We compute the asymptotic Laplace transforms of the functions $t \mapsto e^{t^n}$ for $n \in \mathbb{N}$ and we show that elements in the same asymptotic class have the same asymptotic expansion at ∞. In particular, we present a version of Watson's Lemma for the asymptotic Laplace transforms.

Introduction

The Laplace transform theory has a rich history. It carries the name of a French mathematician, Pierre-Simon Marquis de Laplace, who used it in his monograph "Théory Analytique des Probabilités" from 1812. The Laplace transform of a function $f : [0, \infty) \to \mathbb{R}$ is the function $\hat{f} : \lambda \to \int_0^\infty e^{-\lambda t} f(t)\, dt$, where $\hat{f}$ is defined for all $\lambda \in \mathbb{C}$ with $Re\lambda > a$ and a depends on the growth of the function f or, more precisely, on the exponential growth of its antiderivative $F(t) := \int_0^t f(s)\, ds$. The most significant property of the Laplace transform is that, essentially, integration and differentiation become multiplication and division; i.e., if $\lambda \to \hat{f}(\lambda)$ is the Laplace transform of f, then it follows from $\int_0^\infty e^{-\lambda t} f'(t)\, dt = -f(0) + \lambda \int_0^\infty e^{-\lambda t} f(t)\, dt$ and $\int_0^\infty e^{-\lambda t} F(t)\, dt = \frac{1}{\lambda} \int_0^\infty e^{-\lambda t} f(t)\, dt$ that

$$\widehat{f'}(\lambda) = \lambda \hat{f}(\lambda) - f(0), \text{and}$$

$$\hat{F}(\lambda) = \frac{1}{\lambda} \hat{f}(\lambda).$$

With these properties, the Laplace transform changes linear integral and differential equations into algebraic equations which, at least theoretically, are easier to solve. In the 1900's, an English physicist, mathematician and electrical engineer, Oliver Heaviside (1850-1925), had set the ground for an "Operational Calculus" that allowed him to solve ordinary differential equations by transforming them into algebraic problems. In his monograph on "Electro-Magnetic Theory" ([14], Vol II, p 32) he says:

> "We have now to consider a number of problems which can be solved at once without going to the elaborate theory of Fourier series and integrals. In doing this, we shall have, preliminarily, to work by instinct,

> not by rigorous rules. We have to find out first how things go in the mathematics as well in the physics. When we have learnt the go of it we may be able to see our way to an understanding of the meaning of the processes, and bring them into alignment with other processes. And I must write here a caution. I may have to point out sometimes that my method leads to solutions much more simply than Fourier's method. I may, therefore, appear to be disparaging and endeavoring to supersede his work. But it is nothing of the sort. In a complete treatise on diffusion, Fourier's and other methods would come in side by side - not as antagonists, but as mutual friends helping one another. The limitations of the space forbid this, and I must necessarily keep Fourier series and integrals rather in the background. But this is not to be misunderstood in the sense suggested. No one admires Fourier more than I do. It is the only entertaining mathematical work I ever saw. Its lucidity has always been admired. But it was more than lucid. It was luminous."

Let us have a look how Heaviside's method leads to solutions of ordinary differential equations "much more simply than Fourier's method." Essentially, Heaviside postulated that there exists an one-to-one operator $\{\cdot\}$ from the world of functions defined on $[0, \infty)$ into a world of symbols containing the complex numbers, where neither world is made precise, with the following properties. The operator $\{\cdot\}$ is assumed (postulated) to be an algebra homomorphism; i.e., for all functions f, g and all $c \in \mathbb{C}$,

$$\{cf + g\} = c\{f\} + \{g\}, \tag{1}$$

$$\{f * g\} = \{f\}\{g\}, \tag{2}$$

where the multiplication $*$ is defined by the finite convolution product $(f*g)(t) = \int_0^t f(t-s)g(s)ds$. Heaviside assumes that in the world of symbols the usual properties of addition and multiplication will hold; i.e., the world of symbols is a field. Inspired by the Fourier transform which maps differentiation into multiplication, Heaviside postulates further that

$$\{f'\} = \{Df\} := \lambda\{f\} - f(0), \tag{3}$$

where λ is an unspecified, fixed symbol and where the number $f(0)$ is considered to be an element in the world of symbols. The postulates (1) and (3) imply that $\{0\} = \{1'\} = \lambda\{1\} - 1$, or

$$\{1\} = \frac{1}{\lambda}. \tag{4}$$

By (2), (4), and the fact that $(1*1)(t) = \int_0^t 1\,ds = t$, it follows that [1] $\{t\} = \{1*1\} = \{1\}\cdot\{1\} = \frac{1}{\lambda^2}$. Continuing in this fashion and using that $(1*t)(t) = \int_0^t s\,ds = \frac{1}{2}t^2$, we get $\{\frac{t^2}{2}\} = \{t*1\} = \{1*1*1\} = \frac{1}{\lambda^3}$. By induction, for all $n \in \mathbb{N}_0$,

$$\{\frac{t^n}{n!}\} = \{1*1*\ldots*1\} = \frac{1}{\lambda^{n+1}}. \tag{5}$$

In order to be able to assign a symbol to transcendental functions, Heaviside postulates further that in the world of symbols infinite addition poses no problem. As a consequence,

$$\{e^{-t}\} = \{1 - t + \frac{t^2}{2}\ldots\} = \{1\} - \{t\} + \{\frac{t^2}{2}\} + \ldots = \frac{1}{\lambda} - \frac{1}{\lambda^2} + \ldots = \frac{1}{\lambda+1},$$

or, more general,

$$\{e^{ct}\} = \frac{1}{\lambda - c}. \tag{6}$$

With these operational rules at hand, Heaviside has a simple, yet powerful framework to solve linear ordinary differential equation. As an example of the Heaviside

[1] For notational simplicity we write $\{t^n\}$ instead of $\{e_n\}$, where $e_n(t) := t^n$.

method consider the linear, first order, inhomogeneous initial value problem

$$f'(t) + f(t) = (2t+1)e^{t^2}, \quad f(0) = 1. \tag{7}$$

Using the postulated operator $\{\cdot\}$ defined above and its properties (1) $-$ (6), we see that

$$\begin{array}{rrcl}
 & f'(t) + f(t) & = & (2t+1)e^{t^2} \ \text{ and } \ f(0) = 1 \\
\Leftrightarrow & \{f'(t)\} + \{f(t)\} & = & \{(2t+1)e^{t^2}\} \text{ and } f(0) = 1 \\
\Leftrightarrow & \lambda\{f\} - 1 + \{f\} & = & \{(2t+1)e^{t^2}\} \\
\Leftrightarrow & (\lambda+1)\{f\} & = & 1 + \{(2t+1)e^{t^2}\} \\
\Leftrightarrow & \{f\} & = & \frac{1}{\lambda+1} + \frac{1}{\lambda+1}\{(2t+1)e^{t^2}\} \\
\Leftrightarrow & \{f\} & = & \{e^{-t}\} + \{e^{-t}\}\{(2t+1)e^{t^2}\} \\
\Leftrightarrow & \{f\} & = & \{e^{-t} + e^{-t} * (2t+1)e^{t^2}\} \\
 & & = & \{e^{-t} + \int_0^t e^{-(t-s)}(2s+1)e^{s^2}\, ds\} \\
 & & = & \{e^{-t} + e^{-t}\int_0^t (2s+1)e^{s^2+s}\, ds\} \\
 & & = & \{e^{-t} + e^{-t}[e^{t^2+t} - 1]\} \\
\Leftrightarrow & \{f\} & = & \{e^{t^2}\} \\
\Leftrightarrow & f(t) & = & e^{t^2}.
\end{array}$$

It is important to observe that by using Heaviside's "Operational Calculus", one can solve a differential equation with basic algebra and first semester calculus alone and, more important, that the procedure yields existence and uniqueness at the same time. It is worthwhile mentioning that Heaviside used this method for all differential equations appearing in his treatise on electro-magnetic theory and, even more remarkable, that his method always provides the right result. Since for mathematicians the result does not always justify the means, the mathematical community was not too impressed by what it saw. An anonymous Fellow of the

Royal Society in a letter to Sir Edmund T. Whittaker writes: (See, J.L.B. Cooper [7]).

> ”There was a sort of tradition that a Fellow of the Royal Society could print almost anything he liked in the Proceedings untroubled by referees: but when Heaviside had published two papers on his symbolic methods, we felt the line had to be drawn somewhere, so we put a stop to it”.

Looking once again at Heaviside postulates, natural questions that arises are: **What is the world of functions, what is the operator $\{\cdot\}$, and what is the world of symbols?**

As a first response to these questions, mathematicians like D.V. Widder and G. Doetsch developed the nowadays classical Laplace transform theory in the first half of the twentieth century (see [10], [26]). Laplace transform theory quickly replaced Heaviside's operational calculus as the engineers preferred method to solve linear differential equations. In Laplace transform theory, the world of functions is the set of all functions $f \in L^1_{loc}[0,\infty)$ with exponentially bounded antiderivative F, the operator $\{\cdot\}$ is the Laplace transform $f \mapsto \hat{f}$, and the world of symbols is the set of analytic functions r in a right half plane that are representable as Laplace transforms; i.e. $r = \hat{f}$ for some $f \in L^1_{loc}$. Although Laplace transform theory is a powerful tool with which one can treat most of the important linear engineering problems, it can only partially explain the success of Heaviside's operational methods. First of all, since Heaviside's world of functions contains rapidly growing functions like $t \mapsto e^{t^2}$, it is definitely larger than the set of all Laplace transformable functions. Second, Heaviside's world of symbols is a field, whereas the set of analytic functions that are representable as Laplace transform is not even a ring [2]. Third, since Laplace transform theory always requires artificial growth conditions, it cannot easily produce uniqueness results. In the 1950's, the Polish mathematician Jan Mikusiński developed a full mathematical foundation of Heaviside's operational calculus based on the fact that continuous functions defined on $[0,\infty)$ form an integral domain with respect to addition and convolution. In the Mikusiński algebraic foundation of Heaviside's operational calculus, functions and symbols are interpreted as elements of the quotient field constructed from the integral domain $(C[0,\infty),+,*)$ (see, for example, [19], [20]).

In this dissertation we discuss certain aspects of an alternative, entirely analytic approach to justify Heaviside's methods employing an interpretation of the Heaviside operator $\{\cdot\}$ based on the "Asymptotic Laplace Transform." This concept was developed around 1939 by the Argentinian mathematician J.C. Vignaux [27] and investigated among others by L. Berg [6], M. Cotlar [28], M. Deakin [8], W.A. Ditkin [9] and Yu. I. Lyubich [18]. The asymptotic approach presented here is based on more recent papers by G. Lumer and F. Neubrander on the subject ([16], [17]). In particular, we investigate the two competing definitions of the asymptotic Laplace transform used in these papers, and add a third one. Before studying the asymptotic Laplace transform in Chapter 2, we collect in Chapter 1 some of the background materials from classical Laplace transform theory, the convolution transform, and the recent convolution approach to generalized functions taken by B. Bäumer, G. Lumer, and F. Neubrander (see [3], [16], [17]). Our own results in Chapter 1 are concerned with the characterization of uniqueness sequences (see Proposition 1.9 and Theorem 1.10) and a more detailed investigation of the connection between the abscissa of convergence and the exponential growth bounds of

[2] We note that if $r(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\, dt$ and $u(\lambda) = \int_0^\infty e^{-\lambda t} g(t)\, dt$ for some $f, g \in L^1_{loc}([0,\infty), X)$ and some $\lambda \in \mathbb{C}$ such that $\int_0^\infty e^{-\lambda t} |f(t)|\, dt$ and $\int_0^\infty e^{-\lambda t} |g(t)|\, dt$ do not exist for λ, then $(f * g)(t) = \int_0^t f(t-s)g(s)\, ds$ might not be Laplace transformable at λ; i.e., $\int_0^\infty e^{-\lambda t}(f * g)(t)\, dt$ might not exists.(see [1]).

functions of bounded variation (see Proposition 1.2). Unfortunately, our attempt to generalize Bäumer's Phragmén-Mikusiński type inversion formula for the Laplace transform from real uniqueness sequences to complex uniqueness sequences was not successful, and we can only present a slightly more transparent proof of this crucial and far-reaching result. Since we still feel that a generalization to complex uniqueness sequences should be possible we collected our investigations of this topic in the Appendix.

In Chapter 2, our main contributions to the still developing theory of the asymptotic Laplace transform are as follows. We prove that all operational properties of the Laplace transform extend to the asymptotic settings whenever one uses the second definition given by G. Lumer and F. Neubrander in [17] or the new definition we add. With the new definition of the asymptotic Laplace transform given in this dissertation, all operational properties remain valid but the equivalence classes considered in our definition of the asymptotic Laplace transform are large enough to contain easily computable transforms; i.e., asymptotic Laplace transforms are more readily computable with our definition than with the definition given in [17]. We remark that with the first definition given by G. Lumer and F. Neubrander in [16] asymptotic Laplace transforms were defined by even larger equivalent classes (and therefore even more readily computable), but unfortunately, not all (only almost all) operational properties hold for the huge equivalent classes considered there. To demonstrate the usefulness of our definition, we compute the asymptotic Laplace transform of the functions $t \mapsto e^{t^n}$, $n \in \mathbb{N}$. As indicated above, the asymptotic Laplace transform $\{f\}$ of a function f consists of an equivalence class of analytic functions. We show that if one element in the asymptotic Laplace transform $\{f\}$ of $f \in L^1_{loc}([0,\infty), X)$ has an asymptotic expansion in terms of $\frac{1}{\lambda}$ as $\lambda \to \infty$, then any other element in the asymptotic class has the same asymptotic expansion in terms of $\frac{1}{\lambda}$ as $\lambda \to \infty$ (see Proposition 2.15). In this context, we also generalize Watson's Lemma for the asymptotic Laplace transform; i.e., we prove that if $f \in L^1_{loc}([0,\infty), X)$ has an asymptotic expansion in terms of $\{t^n\}_{n\in\mathbb{N}}$ as $t \to 0^+,\ f(t) \sim \sum_{n=0}^{\infty} c_n t^n$, then for any $u \in \{f\}$ we have that

$$u(\lambda) \sim \sum_{n=0}^{\infty} c_n \frac{n!}{\lambda^{n+1}} \quad \text{as} \quad \lambda \to \infty$$

(see Proposition 2.16). It is remarkable that Watson's Lemma holds no matter which definition of the asymptotic Laplace transform is chosen. We conclude the chapter with a brief description of the asymptotic Laplace transform of generalized functions and indicate a few applications.

1. The Classical Laplace Transform

In this chapter we will review, discuss, and expand results from the classical theory of the Laplace transform which are needed later. We are considering functions $f : \mathbb{R}_+ \to X$, where X is a complex Banach space and denote by $L^1_{loc}(\mathbb{R}_+, X)$ the space of all $f : \mathbb{R}_+ \to X$ which are Bochner integrable on $[0, T]$ for all $T > 0$. Clearly, the existence of the Laplace integral

$$\hat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)\, dt := \lim_{T\to\infty} \int_0^T e^{-\lambda t} f(t)\, dt \tag{1.1}$$

for $f \in L^1_{loc}(\mathbb{R}_+, X)$ and $\lambda \in \mathbb{C}$ depends on the growth of the function f. To discuss the existence of $\hat{f}$ and for the discussion of many other results in Laplace transform theory it is advantageous to use a generalization of of the Laplace integral; i.e, the Laplace-Stieltjes integral

$$\widehat{dF}(\lambda) := \int_0^\infty e^{-\lambda t}\, dF(t) := \lim_{T\to\infty} \int_0^T e^{-\lambda t} dF(t), \tag{1.2}$$

where $F : [0, \infty) \to X$ is a function of bounded semivariation (see below for a short summary of their basic properties and [1] for a complete summary of the properties of the Riemann-Stieltjes integral). If $F' = f$ almost everywhere for some $f \in L^1_{loc}([0, \infty), X)$, then (1.2) reduces to the Laplace transform (1.1). By the fundamental theorem of calculus, this holds if F is absolutely continuous and if either X is finite dimensional ($X = \mathbb{C}$ or $X = \mathbb{C}^n$), or X is reflexive, or X has the Radon-Nikodym property (see [1], Chapter 1, for a discussion of the Radon-Nicodym property). If[1] $F(t) = \sum_{i\geq 0} a_i \chi_{(t_i,\infty)}(t)$ with $a_i \in X$ and $0 \leq t_0 < t_1 < t_2 \ldots$, then (1.2) represents a Dirichlet series

$$\sum_{i\geq 0} a_i e^{-\lambda t_i}. \tag{1.3}$$

[1] clearly, χ_I denotes the characteristic function of $I \subset \mathbb{R}$.

In general, (1.2) is a true generalization of (1.1) and (1.3); i.e., there are functions $r : \mathbb{C} \supset \Omega \to X$ which have a representation (1.2) but which cannot be expressed either in form (1.1) or (1.3).

Recall that a function $F : [a, b] \to X$ is of bounded semivariation if there exists $M \geq 0$ such that $\|\sum_i (F(t_i) - F(s_i))\| \leq M$ for every choice of a finite number of non overlapping intervals (s_i, t_i) in $[a, b]$. A function F is of bounded variation on $[a, b]$ if there exists $M \geq 0$ such that $\sum_i \|(F(t_i) - F(t_{i-1})\| \leq M$ for every finite partition $a = t_0 < t_1 < \ldots < t_n = b$ of $[a, b]$. The set of functions $F : [a, b] \to X$ of bounded semivariation is denoted by $BSV([a, b], X)$. A function $F : \mathbb{R}_+ \to X$ is in $BSV_{loc}(\mathbb{R}_+, X)$ if it is of bounded semivariation on any compact subinterval of $\mathbb{R}_+$. Clearly, any function of bounded variation is of bounded semivariation, and in finite dimensions the two concepts are equivalent: i.e. every function $F : [a, b] \to \mathbb{C}$ which is of bounded semivariation is also of bounded variation (see [1], Section 1.9). However, in infinite dimensions there are functions of bounded semivariation which are not of bounded variation. Take, for instance, $F : [0, 1] \to L^\infty[0, 1]$ defined by $F(t) := \chi_{[0,t]}$. Then it is easy to check that F is not of bounded variation, but it is of bounded semivariation.

To discuss the domain of existence of $\hat{f}$ we introduce the abscissa of convergence of $\hat{f}$ by

$$\text{abs}(f) := \inf \left\{ Re\lambda : \hat{f}(\lambda) \text{ exists} \right\}. \tag{1.4}$$

It will be shown in Proposition 1.1 that the set of $\lambda \in \mathbb{C}$ for which the Laplace integral converges is either empty (i.e., $\text{abs}(f) = +\infty$), or a right half plane bounded to the left by $\text{abs}(f) < \infty$, or the entire complex plane (i.e., $\text{abs}(f) = -\infty$). To estimate $\text{abs}(f)$ in terms of $f : \mathbb{R}_+ \to X$ one considers the exponential growth bound

$$\omega(f) := \inf\left\{\omega \in \mathbb{R} : \sup_{t\geq 0}\left\|e^{-\omega t}f(t)\right\| < \infty\right\}. \tag{1.5}$$

Clearly, $\mathrm{abs}(f) \leq \omega(f)$. However, in general, $\mathrm{abs}(f) < \omega(f)$. In fact, take $f(t) = e^t e^{e^t} sin(e^{e^t})$. Then, clearly, $\omega(f) = +\infty$. However, since $F(t) = \int_0^t f(s)\,ds = -\cos(e^{e^t}) + \cos(e)$ is bounded, integration by parts shows that

$$\begin{aligned}\lim_{T\to\infty}\int_0^T e^{-\lambda t}f(t)\,dt &= \lim_{T\to\infty}\int_0^T e^{-\lambda t}F'(t)\,dt \\ &= \lim_{T\to\infty}[-e^{-\lambda T}\cos(e^{e^T}) - \cos e - \lambda\int_0^T e^{-\lambda t}F(t)\,dt]\end{aligned}$$

exists for $Re\lambda > 0$. This shows that, in general, $\mathrm{abs}(f) < \omega(f)$. Yet, the fundamental result about the existence of the Laplace integral states that $\mathrm{abs}(f)$ is always determined by the exponential growth of the normalized antiderivative

$$F_0(t) := F(t) - F(\infty) \ \ (t \geq 0),$$

where $F(t) := \int_0^t f(s)\,ds$, $F(\infty) := \lim_{t\to\infty} F(t) = \int_0^\infty dF(t)$ if the limit exists, and $F(\infty) := 0$ otherwise: i.e.,

$$F_0(t) = \begin{cases} -\int_t^\infty f(s)\,ds & \text{if the integral exists,} \\ \int_0^t f(s)\,ds & \text{otherwise.}\end{cases} \tag{1.6}$$

We will show in Proposition 1.2 below that $\mathrm{abs}(f) = \omega(F_0)$. To do so, we define

$$\mathrm{abs}(F) := \inf\left\{Re\lambda : \widehat{dF}(\lambda) \text{ exists}\right\}. \tag{1.7}$$

From

$$\int_0^T e^{-\lambda t}\,dF(t) = \int_0^T e^{-\lambda t}\,dF_0(t) = \int_0^T e^{-\lambda t}f(t)\,dt,$$

by letting $T \to \infty$, it follows that

$$\mathrm{abs}(f) = \mathrm{abs}(F_0) = \mathrm{abs}(F), \tag{1.8}$$

where F, F_0 are defined by (1.6).

Proposition 1.1. *Let* $F \in BSV_{loc}(\mathbb{R}_+, X)$*. Then* $\widehat{dF}(\lambda)$ *converges if* $Re\lambda >$ abs(F) *and diverges if* $Re\lambda <$ abs(F).

Proof. It follows from the definition of abs(F) that $\widehat{dF}(\lambda)$ does not exist if $Re\lambda <$ abs(F). For $\lambda_0 \in \mathbb{C}$ define $G_0(t) = \int_0^t e^{-\lambda_0 s} dF(s) \ \ (t \geq 0)$. Then

$$\int_0^t e^{-\lambda s} dF(s) = \int_0^t e^{-(\lambda-\lambda_0)s}\, dG_0(s) \quad (\lambda \in \mathbb{C},\ t \geq 0).$$

Integration by parts yields

$$\int_0^t e^{-\lambda s}\, dF(s) = e^{-(\lambda-\lambda_0)t} G_0(t) + (\lambda - \lambda_0) \int_0^t e^{-(\lambda-\lambda_0)s} G_0(s)\, ds. \tag{1.9}$$

If $\widehat{dF}(\lambda_0)$ exists, then G_0 is bounded. Therefore, $\widehat{dF}(\lambda)$ exists if $Re\lambda > Re\lambda_0$ and

$$\widehat{dF}(\lambda) = (\lambda - \lambda_0) \int_0^\infty e^{-(\lambda-\lambda_0)s} G_0(s) ds \quad (Re\lambda > Re\lambda_0). \tag{1.10}$$

This shows that $\widehat{dF}(\lambda)$ exists if $Re\lambda >$ abs(F). □

Also, we see from $\int_0^T e^{-\lambda t}\, dF(t) = F(T)e^{-\lambda T} - F(0) + \lambda \int_0^T e^{-\lambda t} F(t)\, dt$ that $\int_0^\infty e^{-\lambda t}\, dF(t)$ converges when $\|F(t)\| \leq Me^{\omega t}$ and $Re\lambda > \omega$ and then

$$\int_0^\infty e^{-\lambda t}\, dF(t) = \lambda \int_0^\infty e^{-\lambda t} F(t)\, dt - F(0) \qquad (Re\lambda > \omega). \tag{1.11}$$

Thus, for any function F of bounded semivariation

$$\text{abs}(F) \leq \omega(F).$$

Since abs(f) = abs(F), to prove abs$(f) = \omega(F)$ it remains to be shown that $\omega(F) \leq$ abs(F) for $F \in BSV_{loc}(\mathbb{R}_+, X)$. For $X = \mathbb{C}$, the following proposition is due to D.V. Widder [26].

Proposition 1.2. *Let* $F \in BSV_{loc}(\mathbb{R}_+, X)$.

(a) If $\int_0^\infty e^{-\lambda t}\, dF(t)$ *exists for* $\lambda = \lambda_0 = \gamma + i\delta$ *,* $\gamma > 0$*, then* $F(t) = o(e^{\gamma t})$ *as* $t \to \infty$.

(b) If $\int_0^\infty e^{-\lambda t}\, dF(t)$ *converges for* $\lambda = \lambda_0 = \gamma + i\delta$ *,* $\gamma < 0$*, then* $F(\infty)$ *exists and*

$$F_0(t) = F(t) - F(\infty) = o(e^{\gamma t}) \qquad as \;\; t \to \infty.$$

(c) $\omega(F_0) = \text{abs}(F)$.

(d) In particular, if $F(t) = \int_0^t f(s)\, ds$ *for some* $f \in L^1_{loc}([0,\infty), X)$*, then*

$$\text{abs}(f) < \infty \; if \; and \; only \; if \; \omega(F) < \infty.$$

Proof. (a) First define $G(t) := \int_0^t e^{-\lambda_0 s} dF(s)$. Then we have that

$$\int_0^t e^{\lambda_0 s} dG(s) = \int_0^t e^{-\lambda_0 s} e^{\lambda_0 s} dF(s) = \int_0^t dF(s) = F(t) - F(0).$$

On the other hand, integrating by parts we obtain

$$\int_0^t e^{\lambda_0 s}\, dG(s) = e^{\lambda_0 t} G(t) - \int_0^t G(s)\, d(e^{\lambda_0 s}) = e^{\lambda_0 t} G(t) - \lambda_0 \int_0^t G(s) e^{\lambda_0 s}\, ds.$$

So we have that $F(t) - F(0) = e^{\lambda_0 t} G(t) - \lambda_0 \int_0^t G(s) e^{\lambda_0 s}\, ds$. It follows that

$$[F(t) - F(0)] e^{-\lambda_0 t} = G(t) - \lambda_0 e^{-\lambda_0 t} \int_0^t G(s) e^{\lambda_0 s}\, ds.$$

Finally, taking the limit as $t \to \infty$, we get

$$\lim_{t\to\infty} [F(t) - F(0)] e^{-\lambda_0 t} = \lim_{t\to\infty} [G(t) - \lambda_0 e^{-\lambda_0 t} \int_0^t G(s) e^{\lambda_0 s} ds].$$

To finish the proof we have to show that

$$\lim_{t\to\infty} [G(t) - \lambda_0 e^{-\lambda_0 t} \int_0^t G(s) e^{\lambda_0 s} ds] = 0.$$

For this let $G(\infty) := \lim_{t\to\infty} \int_0^t e^{-\lambda_0 s} dF(s) = \int_0^\infty e^{-\lambda_0 s} dF(s)$ which exists from the hypothesis. Then we have

$$\begin{aligned} \lim_{t\to\infty} [G(t) - \lambda_0 e^{-\lambda_0 t} \int_0^t G(s) e^{\lambda_0 s}\, ds] &= G(\infty) - \lim_{t\to\infty} \lambda_0 e^{-\lambda_0 t} \int_0^t G(s) e^{\lambda_0 s}\, ds \\ &= \lim_{t\to\infty} \lambda_0 e^{-\lambda_0 t} \int_0^t (G(\infty) - G(s)) e^{\lambda_0 s} ds. \end{aligned}$$

At last, let $\epsilon > 0$. From the definition of $G(\infty)$ it follows that there exists a T_0 such that $\|G(\infty) - G(t)\| < \epsilon$ for $s > T_0$. Then

$$\begin{aligned}\lim_{t\to\infty} \lambda_0 e^{-\lambda_0 t} \int_0^t (G(\infty) - G(s))e^{\lambda_0 s}\, ds &= \lim_{t\to\infty} \lambda_0 e^{-\lambda_0 t}[\int_0^{T_0} (G(\infty) - G(s))e^{\lambda_0 s}\, ds \\ &+ \int_{T_0}^t (G(\infty) - G(s))e^{\lambda_0 s}\, ds].\end{aligned}$$

The first term in the limit will go to zero because it does not depend on t and the second term is bounded by ε because $\left\|\lambda_0 e^{-\lambda_0 t} \int_{T_0}^t (G(\infty) - G(s))e^{\lambda_0 s} ds\right\| \leq \lambda_0 e^{-\lambda_0 t} \int_{T_0}^t \epsilon\, e^{\lambda_0 s} ds = \epsilon\, e^{-\lambda_0 t}(e^{\lambda_0 t} - e^{\lambda_0 T_0})$. Since ϵ can be chosen arbitrarily small we have that

$$\lim_{t\to\infty} [F(t) - F(0)]e^{-\lambda_0 t} = 0$$

which implies that $F(t) = o(e^{\gamma t})$ $(as\ t \to \infty)$.

(b) Since $\gamma < 0$ it follows from Proposition 1.1 that

$$\lim_{T\to\infty} \int_0^T dF(t) = F(\infty) - F(0)$$

exists. So, the existence of $F(\infty)$ is assured. Next consider

$$F(\infty) - F(t) = \int_t^\infty e^{\lambda_0 s} dG(s),$$

where $G(t)$ is defined as in (a). Integration by parts gives

$$F(\infty) - F(t) = e^{-\lambda_0 t} G(t) - \lambda_0 \int_t^\infty e^{\lambda_0 s} G(s)\, ds.$$

Multiplying with $e^{-\lambda_0 t}$ and taking the limit we get

$$\lim_{t\to\infty} [F(\infty) - F(t)]e^{-\lambda_0 t} = -G(\infty) - \lim_{t\to\infty} \lambda_0 e^{-\lambda_0 t} \int_t^\infty e^{\lambda_0 s} G(s)\, ds$$

and the same argument as in (a) finishes the proof.

(c) From (a) it follows that, if $\text{abs}(F) > 0$ then $\omega(F_0) = \omega(F) \leq \text{abs}(F)$. From (b) it follows that, if $\text{abs}(F) < 0$ then $\omega(F_0) \leq \text{abs}(F)$. To finish the argument assume $\text{abs}(F) = 0$. There are two cases to consider. Case 1: If $\int_0^\infty dF(t)$ does not exists, then $F(\infty) = 0$ and $F_0(t) = F(t)$. Since $\text{abs}(F) = 0$, $\hat{F}$ exists for any $\lambda := \varepsilon + i\delta$ with $\varepsilon > 0$. Then by (a) it follows that $F(t) = o(e^{\varepsilon t})$ which implies $\omega(F_0) = \omega(F) < \varepsilon$. Taking the limit as $\varepsilon \to 0$ it follows that $\omega(F_0) = \omega(F) \le 0 = \text{abs}(F)$.

Case 2. If $\int_0^\infty dF(t)$ exists, then $F_0(t) = F(t) - F(\infty)$. As above

$$\|F_0(t)\| \le \|F(t)\| + \|F(\infty)\| = o(e^{\varepsilon t}) + \|F(\infty)\|.$$

for all $\varepsilon > 0$. Thus, $\omega(F_0) \le 0 = \text{abs}(F)$.

(d) Statement (d) follows now from the fact that $F'(t) = f(t)$ almost everywhere and therefore we have that $\widehat{dF}(\lambda) = \int_0^\infty e^{-\lambda t}\, dF(t) = \int_0^\infty e^{-\lambda t} f(t)\, dt = \hat{f}(\lambda)$. Hence, $\widehat{dF}(\lambda)$ exists if and only if $\hat{f}(\lambda)$ exists, and $\text{abs}(f) = \text{abs}(F) = \omega(F_0)$ (by (c)). Now, since $F_0(t) = F(t) - F(\infty)$ it follows that $\omega(F) < \infty$ if and only if $\omega(F_0) < \infty$. Hence $\omega(F) < \infty$ if and only if $\omega(F_0) = \text{abs}(F) = \text{abs}(f) < \infty$. □

The previous lemma will not hold for the case when $\lambda_0 \in i\mathbb{R}$ as it can be seen in the next example.

Example 1.3. Consider the Heaviside function $F(0) := 0$ and $F(t) := 1$ for $t > 0$. Then $\int_0^\infty e^{-\lambda t}\, dF(t)$ converges for any choice of $\lambda > 0$ but $F(t) \ne o(1)$ as $t \to \infty$. It is still to be noticed that $F(t) = o(e^{\epsilon t})$ for any choice of $\epsilon > 0$. Also, if $F(t) := 2$ for $t \in [0,1]$ and $F(t) := 2\sqrt{t}$ for $t > 1$ then, at $\lambda = ir$ $(r \ne 0)$, we have

$$\int_0^\infty e^{-\lambda t}\, dF(t) = \int_1^\infty \frac{e^{-irt}}{\sqrt{t}}\, dt.$$

Using integration by parts we have that

$$\int_1^\infty \frac{e^{-irt}}{\sqrt{t}}\, dt = \frac{1}{ir} - \frac{1}{2ir}\int_1^\infty \frac{e^{-irt}}{\sqrt{t^3}}\, dt.$$

So, $\hat{F}(ir)$ converges for all $r \ne 0$, yet $F(\infty)$ does not exists.

One of the main aims of this chapter is to describe and analyze uniqueness sequences for the Laplace transform. To do so we recall first the Riesz-Representation theorem for Lipschitz-continuous functions. We say that $F \in Lip_0([0,\infty),X)$ if $F(0) = 0$ and $\|F\|_{Lip} := \sup_{t,s\geq 0} \frac{\|F(t)-F(s)\|}{|t-s|} < \infty$. In order to see how the Laplace transform relates to the Laplace-Stieltjes transform, let $f \in L^\infty([0,\infty),X)$. Then $F(t) := \int_0^t f(s)\,ds$ is in $Lip_0([0,\infty),X)$ and the Laplace-Stieltjes transform reduces to the Laplace transform: i.e., $\int_0^\infty e^{-\lambda t}\,dF(t) = \int_0^\infty e^{-\lambda t} f(t)\,dt$. The key fact for the Laplace transform theory is the Riesz-Stieltjes representation of bounded linear operators from $L^1(0,\infty)$ into X. For a proof of this theorem see [1], Section 2.1.

Theorem 1.4. *(Riesz-Stieltjes Representation). There exists an isometric isomorphism* $\mathcal{R}_S : Lip_0([0,\infty),X) \to \mathcal{L}(L^1(0,\infty),X)$ *given by* $\mathcal{R}_S(F) := T$ *where* $Tg := \int_0^\infty g(t)\,dF(t)$ *for all continuous functions* $g \in L^1(0,\infty)$ *and* $T\chi_{[0,t]} = F(t)$ *for all* $t \geq 0$.

The Riesz-Stieltjes Representation Theorem is important because when considering the Laplace-Stieltjes transform $\mathcal{L}_S$ it allows us to see how properties of F and its transform $\widehat{dF}$ relate to each-other. Since

$$\widehat{dF}(\lambda) = T_F e^{-\lambda t} \quad \text{and} \quad F(t) = T_F \chi_{[0,t]}$$

for all $\lambda, t > 0$ it follows that the function F determines the operator T_F on the set of characteristic functions, which is total in $L^1(0,\infty)$. Therefore T_F, and in particular $T_F e^{-\lambda t} = \widehat{dF}(\lambda)$, is completely determined. On the other hand, any information on $\widehat{dF}(\lambda)$ for $\lambda > 0$ translates into information on T_F on the set of exponential functions which is also total[2] in $L^1(0,\infty)$ (see Proposition 1.8). Therefore, $\widehat{dF}$ determines the properties of T_F and, in particular, of $T_F\chi_{[0,t]} =$

[2]A set $D \subset X$ is called total in X if its linear span is dense in X.

$F(t)$, $(t \geq 0)$. The following results form complex analysis will be used, their proofs can be found in [22], pages 300-311.

Proposition 1.5. *Suppose that h is a bounded analytic function on the unit disc not identically zero, and $\alpha_1, \alpha_2, \ldots$ are the zeros of h. Then*

$$\sum_{n=1}^{\infty}(1 - |\alpha_n|) < \infty$$

Proposition 1.6. *Suppose that r_n is analytic on the region Ω for each $n \in \mathbb{N}$, that no r_n is identically zero there, and that*

$$\sum_{n=1}^{\infty} |1 - r_n(\lambda)| \tag{1.12}$$

converges uniformly on compact subsets of Ω. Then the product

$$r(\lambda) = \prod_{n=1}^{\infty} r_n(\lambda) \tag{1.13}$$

converges uniformly on compact subsets of Ω. Hence r is analytic on Ω. Moreover we have that

$$m(r, \lambda) = \sum_{n=1}^{\infty} m(r_n, \lambda), \qquad \lambda \in \Omega, \tag{1.14}$$

where $m(r, \lambda)$ is defined to be the multiplicity of the zero of r at λ. (If $r(\lambda) \neq 0$, then $m(r, \lambda) = 0$).

Motivated by Proposition 1.5 and the fact that the map $\lambda \mapsto \frac{\lambda-1}{\lambda+1}$ maps the open right half plane on the unit circle, we make the following definition.

Definition 1.7. *A sequence of distinct complex numbers $(\lambda_n)_{n\in\mathbb{N}}$ with no accumulation point and with the property that $Re\lambda_n \geq \gamma > 0$ for some $\gamma > 0$, is called a uniqueness sequence if*

$$\sum_{n=1}^{N} 1 - \frac{|\lambda_n - 1|}{|\lambda_n + 1|} \to \infty.$$

Proposition 1.8. *Let* $(\lambda_n)_{n\in\mathbb{N}}$ *be a uniqueness sequence. Then the exponential functions* $t \mapsto e^{-\lambda_n t}$ *are total in* $L^1(0,\infty)$.

Proof. a) Let $n_0 \in \mathbb{N}$ be such that $n_0 > \gamma$. The family of monomials

$$M = \{\frac{1}{n_0} t^{\frac{n}{n_0}} : n \in \mathbb{N}_0\}$$

contains the set $\{\frac{1}{n_0}, t, t^2 \ldots\}$. So, it is total in $C[0,1]$ and thus in $L^1(0,1)$. Define

$$\Phi : L^1(0,1) \to L^1(0,\infty) \text{ by } \Phi(f)(t) := n_0 e^{-n_0 t} f(e^{-n_0 t}).$$

Then Φ is an isometric isomorphism and

$$\Phi(\frac{1}{n_0} t^{\frac{n}{n_0}}) = n_0 e^{-n_0 t} \frac{1}{n_0} (e^{-n_0 t})^{\frac{n}{n_0}} = e^{-(n_0+n)t}.$$

Therefore, the family of monomials M is mapped onto the set of exponential functions

$$\mathcal{E} = \{e^{-n\cdot} : n \geq n_0\}.$$

Since M is total in $L^1(0,1)$ it follows that $\mathcal{E}$ is total in $L^1(0,\infty)$.

b) To finish the proof we show that $\mathcal{H}$, which is defined to be the closure of the linear span of the functions $\{e^{-\lambda_n \cdot}\}_{n\in\mathbb{N}}$, contains the set $\mathcal{E}$. If $\mathcal{H} \supseteq \mathcal{E}$, then the conclusion follows from a). Suppose that $\mathcal{H}$ does not contain the set $\mathcal{E}$. Then there exists $m \geq n_0$ such that $t \mapsto e^{-mt} \notin \mathcal{H}$. By the Hahn-Banach theorem there exists $\phi \in L^\infty(0,\infty) = L^1(0,\infty)^*$ such that $\phi(\mathcal{H}) \equiv 0$ and $\langle e^{-m\cdot}, \phi\rangle \neq 0$. Let $0 < \beta < \gamma$. The function

$$\lambda \to \Psi(\lambda) := \left\langle e^{-\lambda\cdot}, \phi \right\rangle = \int_0^\infty e^{-\lambda t} \phi(t)\, dt$$

is analytic and bounded for $Re\lambda \geq \beta$ and the function $\rho : z \to \frac{1+z}{1-z} + \beta$ is a conformal map between the unit disk and the half plane $\{Re\lambda > \beta\}$. Define $h(z) := \Psi(\rho(z))$ and $\mu_n := \rho^{-1}(\lambda_n) = \frac{\lambda_n - \beta - 1}{\lambda_n - \beta + 1}$. Then, h is analytic and bounded in the unit disk and

$h(\mu_n) = \Psi(\lambda_n) = \phi(e^{-\lambda_n \cdot}) = 0$ for all $n \in \mathbb{N}$. Let $a_n := 1 - |\mu_n| = \frac{|\lambda_n - \beta + 1| - |\lambda_n - \beta - 1|}{|\lambda_n - \beta + 1|}$ and $b_n := 1 - \frac{|\lambda_n - 1|}{|\lambda_n + 1|} = \frac{|\lambda_n + 1| - |\lambda_n - 1|}{|\lambda_n + 1|}$. Then, since $|\lambda + 1|^2 - |\lambda - 1|^2 = 4Re\lambda$, it follows that

$$\begin{aligned}
\frac{a_n}{b_n} &= \frac{|\lambda_n + 1| \left(|\lambda_n - \beta + 1| - |\lambda_n - \beta - 1|\right)}{\left(|\lambda_n + 1| - |\lambda_n - 1|\right) |\lambda_n - \beta + 1|} \\
&= \frac{|\lambda_n + 1|}{|\lambda_n - \beta + 1|} \cdot \frac{|\lambda_n - \beta + 1|^2 - |\lambda_n - \beta - 1|^2}{|\lambda_n + 1|^2 - |\lambda_n - 1|^2} \cdot \frac{|\lambda_n + 1| + |\lambda_n - 1|}{|\lambda_n - \beta + 1| + |\lambda_n - \beta - 1|} \\
&= \frac{|\lambda_n + 1|}{|\lambda_n - \beta + 1|} \cdot \frac{Re\lambda_n - \beta}{Re\lambda_n} \cdot \frac{|\lambda_n + 1| + |\lambda_n - 1|}{|\lambda_n - \beta + 1| + |\lambda_n - \beta - 1|}.
\end{aligned}$$

Since for $0 < \beta < \gamma$ and $Re\lambda_n \geq \gamma$ it follows that the distance of λ_n to -1 is larger than the distance of $\lambda_n - \beta$ to -1 and to 1; i.e., $|\lambda_n + 1| \geq |\lambda_n - \beta \pm 1|$. Moreover, it follows from $|\lambda_n + 1| \leq |\lambda_n - 1| + 2$ that $|\lambda_n - 1| \geq |\lambda_n + 1| - 2$. Thus,

$$\begin{aligned}
\frac{a_n}{b_n} &\geq \frac{Re\lambda_n - \beta}{Re\lambda_n} . \frac{|\lambda_n + 1| + |\lambda_n - 1|}{|\lambda_n - \beta + 1| + |\lambda_n - \beta - 1|} \\
&\geq \frac{Re\lambda_n - \beta}{Re\lambda_n} . \frac{2|\lambda_n + 1| - 2}{2|\lambda_n + 1|}. \\
&\geq (1 - \frac{\beta}{\gamma})(1 - \frac{1}{1+\gamma}) > 0.
\end{aligned}$$

Since $\sum_{n=1}^{N} b_n \to \infty$ as $N \to \infty$, we obtain that $\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} 1 - |\mu_n| \to \infty$ as $N \to \infty$. Using Proposition 1.5 it follows that $h(\mu) = 0$ for $|\mu| < 1$. But this implies that $\Psi(\lambda) = 0$ for $Re\lambda > \beta$, contradicting $\Psi(m) \neq 0$. Thus $\mathcal{H} = L^1(0, \infty)$.

□

Some examples of sequences satisfying the condition in the previous proposition are the equidistant sequences $\lambda_n = a + nb$ $(a, b > 0)$. Examples of sequences not satisfying the condition are given by $\lambda_n = n^\alpha$ for $\alpha > 1$ and $\lambda_n = 1 + in$. Next we will show that there are sequences satisfying the condition in the previous proposition, on any vertical line $x = \gamma > 0$, and on any line $y = \alpha x$ for any $\alpha \in \mathbb{R}$. Let $\lambda = (x, y) \in \mathbb{C}$ be such that $Re\lambda > 0$. Also let $A = (-1, 0)$ and

$B = (1, 0)$. Obviously, the distance from λ to A is larger than the distance from λ to B, therefore

$$0 \leq \frac{|\lambda - 1|}{|\lambda + 1|} = \epsilon < 1.$$

Now fix $0 < \epsilon < 1$. We are looking for the points $\lambda = (x, y)$ in the right half plane such that

$$0 \leq \frac{|\lambda - 1|}{|\lambda + 1|} = \epsilon$$

Therefore, we have $\frac{\sqrt{(x-1)^2+y^2}}{\sqrt{(x+1)^2+y^2}} = \epsilon$, so $\sqrt{(x-1)^2 + y^2} = \epsilon\sqrt{(x+1)^2 + y^2}$. Squaring both sides we obtain that

$$x^2(1 - \epsilon^2) - 2x(1 + \epsilon^2) + y^2(1 - \epsilon^2) + (1 - \epsilon^2) = 0.$$

Dividing by $(1 - \epsilon^2)$ and completing the squares we obtain that

$$\left(x - \frac{1 + \epsilon^2}{1 - \epsilon^2}\right)^2 + y^2 = \left(\frac{1 + \epsilon^2}{1 - \epsilon^2}\right)^2 - 1 = \frac{4\varepsilon^2}{(\varepsilon^2 - 1)^2}$$

which is a equation of a circle centered on the real axis at $(\frac{1+\epsilon^2}{1-\epsilon^2}, 0)$ with radius $\frac{2\varepsilon}{1-\epsilon^2}$. Therefore, the set of points $\lambda = (x, y)$ in the right half plane for which

$$\frac{|\lambda - 1|}{|\lambda + 1|} = \epsilon$$

is a circle centered at $(\frac{1+\epsilon^2}{1-\epsilon^2}, 0)$ with radius $\frac{2\epsilon}{1-\epsilon^2}$. We should note that as $\epsilon \to 1^-$ both the radius and the x-coordinate of the center are going to infinity. Define $\epsilon_n = \frac{n-1}{n}$. Then there is a circle C_n with center at $(\frac{2n^2-2n+1}{2n-1}, 0)$ and radius $\frac{2n^2-2n}{2n-1}$ such that for any choice of $\lambda_n \in C_n$ we have that

$$\sum_{n=1}^{N} 1 - \frac{|\lambda_n - 1|}{|\lambda_n + 1|} = \sum_{n=1}^{N} 1 - \epsilon_n = \sum_{n=1}^{N} \frac{1}{n} \to \infty$$

as $N \to \infty$. At this point we can show now that there are uniqueness sequences on any vertical line $x = \gamma$ with $\gamma > 0$ and on any line $y = \alpha x$ $(x > 0)$, where α is any real number. For this , let $\gamma > 0$. Then C_n intersects the line $x = \gamma$ at the points (γ, y_n) where

$$y_n = \pm\sqrt{(2n-1-\gamma)(\gamma - \frac{1}{2n-1})}.$$

Therefore, for any $\gamma > 0$ there exists an $n_\gamma \in \mathbb{N}$ such that the previous equation has two real roots for any $n \geq n_\gamma$. The same type of argument is used to show that C_n intersects any line $y = \alpha x$ for any $n \geq n_\alpha$ and, therefore there are uniqueness sequences on any line $y = \alpha x$.

We say that sequence of complex numbers $(\lambda_n)_{n\in\mathbb{N}}$ with $Re\lambda_n \geq \gamma > 0$ is called a Laplace uniqueness sequence if given a Laplace transformable function $f \in L^1_{loc}([0,\infty), X)$ with the property that $\hat{f}(\lambda_n) = 0$ for each n, implies $f \equiv 0$. In the following we will see how the concept of Laplace uniqueness sequence and uniqueness sequences as defined in Definition 1.7 are related. We need the following preliminary result for Lipschitz continuous functions, which is an extension of Theorem 1.7.3 in [1], which was first mentioned by Pastor in 1919 and is a special case of a result of Yu-Cheng Shen, see [23].

Proposition 1.9 (Uniqueness). *Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers with no accumulation point, $Re\lambda_n \geq \gamma > 0$, and $|arg(\lambda_n)| \leq \theta$ for some $\theta < \frac{\pi}{2}$ and all $n \in \mathbb{N}$. The following are equivalent.*

(i) $(\lambda_n)_{n\in\mathbb{N}}$ *is a uniqueness sequence.*

(ii) For all $F \in Lip_0([0,\infty), X)$ we have that $\widehat{dF}(\lambda_n) = 0$ for all $n \in \mathbb{N}$ if and only if $F = 0$.

Proof. To prove that (i) implies (ii) we have to show that if $\widehat{dF}(\lambda_n) = 0$ for any $n \in \mathbb{N}$ then $F \equiv 0$. To accomplish this task we combine Proposition 1.8 with the Riesz-Stieltjes Representation Theorem 1.4. By Proposition 1.8, if $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence such that $\sum_{n=1}^{N} 1 - \frac{|\lambda_n - 1|}{|\lambda_n+1|} \to \infty$ as $N \to \infty$, then the family of exponentials $\{e^{-\lambda_n \cdot}\}$ is dense in $L^1(0,\infty)$. Thus, the operator $T_F := T_F g := \int_0^\infty g(t)\, dF(t)$ has the property that

$$T_F(e^{-\lambda_n \cdot}) = \int_0^\infty e^{-\lambda_n t}\, dF(t) = \widehat{dF}(\lambda_n) = 0.$$

Since the induced operator is zero on a total set it follows that it is identically zero. Therefore the function F is identically zero which implies that $(\lambda_n)_{n\in\mathbb{N}}$ is a Laplace uniqueness sequence.

To prove that (ii) implies (i) we have to show that if $(\lambda_n)_{n\in\mathbb{N}}$ is a Laplace uniqueness sequence then $\sum_{n=1}^{N} 1 - \frac{|\lambda_n-1|}{|\lambda_n+1|} \to \infty$. We will prove this by contradiction. Suppose $\sum_{n=1}^{\infty} 1 - \frac{|\lambda_n-1|}{|\lambda_n+1|} < \infty$. For $\lambda_n = d_n e^{i\alpha_n}$ we have that

$$\frac{|\lambda_n+1| - |\lambda_n-1|}{\cos(\alpha_n)} = \frac{4}{\sqrt{1+\frac{2\cos(\alpha_n)}{d_n}+\frac{1}{d_n^2}}+\sqrt{1-\frac{2\cos(\alpha_n)}{d_n}+\frac{1}{d_n^2}}} \to 2 \text{ as } n \to \infty.$$

Thus, $\frac{|\lambda_n+1|-|\lambda_n-1|}{\cos(\alpha_n)} \geq 1$ for sufficiently large n. In particular, $|\lambda_n+1| - |\lambda_n - 1| \geq \cos(\alpha_n) \geq \cos(\theta)$ for sufficiently large n. Thus,

$$1 - \frac{|\lambda_n-1|}{|\lambda_n+1|} = \frac{|\lambda_n+1| - |\lambda_n-1|}{|\lambda_n+1|} > \frac{\cos(\theta)}{|\lambda_n+1|}.$$

Therefore, $\sum_{n=1}^{\infty} \frac{1}{|\lambda_n+1|} < \infty$ and for any $\varepsilon > 0$ we have that $\sum_{n=1}^{\infty} \frac{1}{|\lambda_n+\varepsilon+1|} < \sum_{n=1}^{\infty} \frac{1}{|\lambda_n+1|} < \infty$. Moreover,

$$\begin{aligned}\sum_{n=1}^{\infty} 1 - \frac{|\lambda_n+\varepsilon-1|}{|\lambda_n+\varepsilon+1|} &= \sum_{n=1}^{\infty} \frac{|\lambda_n+\varepsilon+1| - |\lambda_n+\varepsilon-1|}{|\lambda_n+\varepsilon+1|} \\ &\leq \sum_{n=1}^{\infty} \frac{2}{|\lambda_n+\varepsilon+1|}.\end{aligned}$$

Hence, $\sum_{n=1}^{\infty} 1 - \frac{|\lambda_n+\varepsilon-1|}{|\lambda_n+\varepsilon+1|} < \infty$.

For each $n \in \mathbb{N}$ define $\mu_n = \lambda_n + \varepsilon$ and $r_n(\lambda) := \frac{\mu_n - \lambda}{2+\mu_n+\lambda}$. Then r_n is analytic on the region $Re\lambda > 0$ and it has a zero at $\lambda = \mu_n$. Now we will check that the conditions in the Proposition 1.6 are satisfied. First we show that

$$\sum_{n=1}^{\infty} |1 - r_n(\lambda)| = \sum_{n=1}^{\infty} \left| \frac{2\lambda + 2}{2 + \mu_n + \lambda} \right| < \infty. \tag{1.15}$$

Let U be a compact subset of the region $Re\lambda > 0$ and let

$$a_n := \left| \frac{2\lambda + 2}{2 + \mu_n + \lambda} \right| \text{ and } b_n := \frac{1}{|\mu_n + 1|}. \tag{1.16}$$

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{|\mu_n + 1|\, |2\lambda + 2|}{|1 + \mu_n + 1 + \lambda|} = \lim_{n\to\infty} \frac{2\,|\lambda + 1|}{\left|1 + \frac{1+\lambda}{\mu_n+1}\right|}, \tag{1.17}$$

which exists for any $\lambda \in U$. Therefore $\sum_{n=1}^{\infty} |1 - r_n(\lambda)|$ converges uniformly on U. By Proposition 1.6 it follows that the product $r(\lambda) = \prod_{n=1}^{\infty} r_n(\lambda)$ converges uniformly on compact subsets of the open right half plane $Re\lambda > 0$. Hence $r(\lambda)$ is analytic on $Re\lambda > 0$. Moreover, since only a finite number of factors $r_n(\lambda)$ have $|r_n(\lambda)| > 1$ it follows that $0 \leq |r(\lambda)| < 1$. By Proposition 1.6, $r(\lambda) = 0$ if and only if $\lambda = \mu_n$ for some $n \in \mathbb{N}$. Now let $q(\lambda) := \frac{r(\lambda)}{\lambda}$. Then $|\lambda q(\lambda)| = |r(\lambda)| \leq 1$ for $Re\lambda > 0$. By [1], Theorem 2.5.1, it follows that there exists $f \in C_0(\mathbb{R}_+, X)$ such that

$$\sup_{t>0} \left\| \frac{f(t)}{t} \right\| < \infty \qquad \textit{and} \qquad q(\lambda) = \lambda \hat{f}(\lambda)$$

for $Re\lambda > 0$. Thus,

$$\hat{f}(\lambda) = \frac{q(\lambda)}{\lambda} = \frac{1}{\lambda} \prod_{n=1}^{\infty} \frac{\mu_n - \lambda}{2 + \mu_n + \lambda}.$$

Since $\sup_{t>0} \left\| \frac{f(t)}{t} \right\| < \infty$ it follows that $\|f(t)e^{-\varepsilon t}\| \leq M$ for any $\varepsilon > 0$. Therefore, $h(t) := f(t)e^{-\varepsilon t}$ has the Laplace transform $\hat{h}(\lambda) = \int_0^{\infty} e^{-(\lambda+\varepsilon)t} f(t)\, dt = \hat{f}(\lambda + \varepsilon)$. Since h is continuous and bounded it follows that $H(t) := \int_0^t h(s)\, ds$ belongs to $Lip_0([0, \infty), X)$ and we have that

$$\widehat{dH}(\lambda) = \hat{h}(\lambda) = \hat{f}(\lambda + \varepsilon).$$

Thus, $\widehat{dH}(\lambda_n) = \hat{f}(\lambda_n + \varepsilon) = \hat{f}(\mu_n) = 0$, for all $n \in \mathbb{N}$, which is a contradiction. This finishes the proof of the proposition. $\square$

Theorem 1.10. *Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers with $Re\lambda_n \geq \gamma > 0$ and $|arg(\lambda_n)| \leq \theta < \frac{\pi}{2}$ for all $n \in \mathbb{N}$. The following are equivalent.*

(i) $(\lambda_n)_{n\in\mathbb{N}}$ is a uniqueness sequence.

(ii) For all $f \in L^1_{loc}([0,\infty), X)$with $\text{abs}(f) < \gamma$ *we have that $\hat{f}(\lambda_n) = 0$ for all $n \in \mathbb{N}$ if and only if $f = 0$.*

Proof. The proof of this statement is a consequence of the previous proposition and the fact that there is a isometric isomorphism between $Lip_0([0,\infty), X)$ and $Lip_\omega([0,\infty), X)$ given by

$$I_\omega(F)(t) := \int_0^t e^{-\omega s} dF(s),$$

where $Lip_\omega([0,\infty), X) := \{F : \mathbb{R}_+ \to X : F(0) = 0, \|F\|_{Lip_\omega} < \infty\}$ and where $\|F\|_{Lip_\omega} := \sup_{t>s\geq 0} \frac{\|F(t)-F(s)\|}{\int_s^t e^{\omega r} dr}$. For a proof of this fact see [1], Section 2.4. Now, by Proposition 1.2, if $f \in L^1_{loc}(0,\infty)$ is Laplace transformable, then $F(t) := \int_0^t f(s)ds$ is absolutely continuous and exponentially bounded. Thus, $G(t) := \int_0^t F(s)\, ds \in Lip_\omega[0,\infty)$. Since

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\, dt = \lambda \int_0^\infty e^{-\lambda t} F(t)\, dt = \lambda^2 \int_0^\infty e^{-\lambda t} G(t)\, dt$$

it follows that $(\lambda_n)_{n\in\mathbb{N}}$ is a Laplace uniqueness sequence for f if and only if is a uniqueness sequence for G. Therefore $G(t) = 0$ implies $f(t) = 0$, which completes the proof. $\square$

Next we will present an inversion formula for the Laplace transform which is due to B. Bäumer (see [2] and [3]). Recall that a sequence $(\beta_n) \subset \mathbb{R}^+$ is a Müntz sequence provided that, for all $n \in \mathbb{N}$,

$$\beta_{n+1} - \beta_n \geq 1 \text{ and } \sum_{n=1}^{\infty} \frac{1}{\beta_n} = \infty. \tag{1.18}$$

We should note that Müntz sequences are uniqueness sequences since

$$\sum_{n=1}^{\infty} 1 - \frac{|\beta_n - 1|}{|\beta_n + 1|} = \sum_{n=1}^{\infty} 1 - \frac{\beta_n - 1}{\beta_n + 1} = \sum_{n=1}^{\infty} \frac{2}{\beta_n + 1} = \infty$$

because $\sum_{n=1}^{\infty} \frac{1}{\beta_n}$ diverges. With this inversion formula one can prove many important properties of the Laplace and convolution transform (see below). We have tried to generalize this theorem for real Müntz sequences $(\beta_n)_{n\in\mathbb{N}}$ to any complex uniqueness sequence, unfortunately without much success. See the Appendix for some partial results. For technical reasons, we state the inversion formula first for the finite Laplace transform $q(\lambda) = \int_0^T e^{-\lambda t} f(t)\, dt$. Later on, in Corollary 1.15, we will state it for the Laplace Transform $\hat{f}(\lambda) = \int_0^{\infty} e^{-\lambda t} f(t)\, dt$.

Theorem 1.11 (Phragmén-Mikusiński Inversion). *Let $f \in C_0([0,T], X)$ for some $T > 0$ and $q(\lambda) := \int_0^T e^{-\lambda t} f(t)\, dt$. Let $(\beta_n)_{n\in\mathbb{N}}$ be a Müntz sequence. Then*

$$f(t) = \lim_{n\to\infty} \sum_{i=1}^{N_n} \alpha_{n,i} q(\beta_{ni}) e^{\beta_{ni} t}, \tag{1.19}$$

where N_n is chosen such that $\sum_{j=1}^{N_n} \frac{1}{\beta_{nj}} \geq T$ and $\alpha_{n,i}$ are defined by

$$\alpha_{n,i} := \beta_{ni} e^{-\beta_{ni} \sum_{j=1}^{N_n} \frac{1}{\beta_{nj}}} \prod_{j=1, j\neq i}^{N_n} \frac{\beta_{nj}}{\beta_{nj} - \beta_{ni}}. \tag{1.20}$$

Moreover, the limit in (1.19) is uniform on $[0,s]$ for all $0 < s < T$.

Proof. First we show that for any $T > 0$ and any $n \in \mathbb{N}$ there exists an N_n such that $\sum_{j=1}^{N_n} \frac{1}{\beta_{nj}} \geq T$. Since $\frac{1}{\beta_{nj}} > \frac{1}{\beta_{nj+i}}$ for any $1 \leq i \leq n$ we have that

$$\begin{aligned} n\sum_{j=1}^{\infty} \frac{1}{\beta_{nj}} &= \sum_{j=1}^{\infty} \frac{1}{\beta_{nj}} + \ldots + \sum_{j=1}^{\infty} \frac{1}{\beta_{nj}} \quad \text{(n times)} \\ &\geq \sum_{j=1}^{\infty} \frac{1}{\beta_{nj+1}} + \ldots + \sum_{j=1}^{\infty} \frac{1}{\beta_{nj+n}} \\ &= \sum_{k=n+1}^{\infty} \frac{1}{\beta_k} = \infty. \end{aligned}$$

Since n is fixed we obtain that $\sum_{j=1}^{\infty} \frac{1}{\beta_{nj}} = \infty$ which assures the existence of N_n. Let $c_n := \sum_{j=1}^{N_n} \frac{1}{\beta_{nj}}$. The main idea of this proof is to construct a sequence of functions ϕ_n which will "converge" to the δ-function. This will be done by showing that the antiderivatives $\Phi_n := 1 * \phi_n$ converge to the Heaviside function $H : t \to \chi_{(0,\infty)}(t)$, where the convergence will be pointwise for any $t \neq 0$ and uniform for $|t| > \varepsilon$. We define ϕ_n by

$$\phi_n(t) := \begin{cases} \beta_n e^{-\beta_n(\cdot)} * \beta_{n2} e^{-\beta_{2n}(\cdot)} * \ldots * \beta_{nN_n} e^{-\beta_n N_n(\cdot)}(t + c_n) & \text{for } t \geq -c_n, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the shifted antiderivative of ϕ_n given by

$$\psi_n := 1 * \beta_n e^{-\beta_n(\cdot)} * \beta_{n2} e^{-\beta_{2n}(\cdot)} * \ldots * \beta_{nN_n} e^{-\beta_n N_n(\cdot)}.$$

It's Laplace transform is

$$\begin{aligned} \int_0^{\infty} e^{-\lambda t} \psi_n(t)\, dt &= \frac{1}{\lambda} \frac{\beta_n}{\lambda + \beta_n} \cdots \frac{\beta_{nN_n}}{\lambda + \beta_{nN_n}} \\ &= \gamma_{n,0} \frac{1}{\lambda} + \gamma_{n,1} \frac{\beta_n}{\lambda + \beta_n} + \ldots + \gamma_{nN_n} \frac{\beta_{nN_n}}{\lambda + \beta_{nN_n}}, \end{aligned}$$

where the coefficients $\gamma_{n,i}$ are found using the partial fraction decomposition and are given by

$$\gamma_{n,i} = -\prod_{j=1, j\neq i}^{N_n} \frac{\beta_{nj}}{\beta_{nj} - \beta_{ni}}.$$

Here we have used the operational property of the Laplace transform of mapping convolution into multiplication. Finally taking the inverse Laplace transform, and knowing that the inverse Laplace transform of $\frac{1}{\lambda+\beta_{ni}}$ is $e^{-\beta_{ni}t}$, we have that

$$\psi_n(t) = 1 + \sum_{i=1}^{N_n} \gamma_{n,i} e^{-\beta_{ni}t} \tag{1.21}$$

for $t \geq 0$. Since $c_n = \sum_{j=1}^{N_n} \frac{1}{\beta_{nj}}$, the antiderivative of ϕ_n, denoted by Φ_n, is given by

$$\begin{aligned}
\Phi_n(t) = \psi_n(t + c_n) &= 1 + \sum_{i=1}^{N_n} \gamma_{n,i} e^{-\beta_{ni}(t+c_n)} \\
&= 1 - \sum_{i=1}^{N_n} \prod_{j=1, j\neq i}^{N_n} \frac{\beta_{nj}}{\beta_{nj} - \beta_{ni}} e^{-\beta_{ni}c_n} e^{-\beta_{ni}t} \\
&= 1 - \sum_{i=1}^{N_n} \prod_{j=1, j\neq i}^{N_n} \frac{\beta_{nj}}{\beta_{nj} - \beta_{ni}} e^{-\beta_{ni}\sum_{j=1}^{N_n} \frac{1}{\beta_{nj}}} e^{-\beta_{ni}t}.
\end{aligned}$$

By definition, $\alpha_{n,i} = \beta_{ni} e^{-\beta_{ni}\sum_{j=1}^{N_n} \frac{1}{\beta_{nj}}} \prod_{j=1, j\neq i}^{N_n} \frac{\beta_{nj}}{\beta_{nj}-\beta_{ni}}$. Thus,

$$\Phi_n(t) = 1 - \sum_{i=1}^{N_n} \frac{\alpha_{n,i}}{\beta_{ni}} e^{-\beta_{ni}t}$$

for all $t \geq -c_n$. To show that Φ_n converges to the Heaviside function we need some estimates for $|\alpha_{n,i}|$. For this consider

$$\begin{aligned}
\ln\left|\frac{\alpha_{n,i}}{\beta_{ni}}\right| &= \ln\left| e^{-\beta_{ni}\sum_{j=1}^{N_n} \frac{1}{\beta_{nj}}} \prod_{j=1, j\neq i}^{N_n} \frac{\beta_{nj}}{\beta_{nj} - \beta_{ni}} \right| \\
&= -\beta_{ni} \sum_{j=1}^{N_n} \frac{1}{\beta_{nj}} + \sum_{j=1, j\neq i}^{N_n} \ln\left|\frac{\beta_{nj}}{\beta_{nj} - \beta_{ni}}\right| \\
&= -\beta_{ni} \sum_{j=1}^{N_n} \frac{1}{\beta_{nj}} + \sum_{j=1}^{i-1} \ln \frac{\beta_{nj}}{\beta_{ni} - \beta_{nj}} + \sum_{j=i+1}^{N_n} \ln \frac{\beta_{nj}}{\beta_{nj} - \beta_{ni}} \\
&= S_1 + S_2 + S_3.
\end{aligned}$$

We estimate first S_1+S_3. For this consider the function $g(t) := -\frac{\beta_{ni}}{t} + \ln \frac{t}{t-\beta_{ni}}$. Since $\lim_{t\to\infty} g(t) = 0$ and $g'(t) = \frac{-\beta_{ni}^2}{t^2(t-\beta_{ni})}$ it follows that g is a positive and decreasing function on the interval (β_{ni}, ∞). Using this and the fact that $\beta_{ni} + n(j-i) \leq \beta_{nj}$

implies $g(\beta_{ni}+n(j-i)) > g(\beta_{nj})$; i.e., $-\frac{\beta_{ni}}{\beta_{nj}}+\ln\frac{\beta_{nj}}{\beta_{nj}-\beta_{ni}} < -\frac{\beta_{ni}}{\beta_{ni}+n(j-i)}+\ln\frac{\beta_{ni}+n(j-i)}{n(j-i)}$ for any $j > i$ we obtain that

$$\begin{aligned}
S_1+S_3 &= -\beta_{ni}\sum_{j=1}^{N_n}\frac{1}{\beta_{nj}}+\sum_{j=i+1}^{N_n}\ln\frac{\beta_{nj}}{\beta_{nj}-\beta_{ni}} \\
&\le \sum_{j=i+1}^{N_n}-\frac{\beta_{ni}}{\beta_{nj}}+\ln\frac{\beta_{nj}}{\beta_{nj}-\beta_{ni}} \\
&\le \sum_{j=i+1}^{N_n}\left(-\frac{\beta_{ni}}{\beta_{nj}+n(j-i)}+\ln\frac{\beta_{ni}+n(j-i)}{n(j-i)}\right) \\
&\le \sum_{j=i+1}^{\infty}\left(-\frac{\beta_{ni}}{\beta_{ni}+n(j-i)}+\ln\frac{\beta_{ni}+n(j-i)}{n(j-i)}\right) \\
&= \sum_{j=1}^{\infty}-\frac{\beta_{ni}}{\beta_{ni}+nj}+\ln\frac{\beta_{ni}+nj}{nj} \\
&< \int_0^{\infty}\left(-\frac{\beta_{ni}}{\beta_{ni}+nt}+\ln\frac{\beta_{ni}+nt}{nt}\right)dt.
\end{aligned}$$

Finally, making the substitution $t=\frac{\beta_{ni}u}{n}$ we obtain that

$$S_1+S_3 \le \frac{\beta_{ni}}{n}\int_0^{\infty}\left(-\frac{1}{1+u}+\ln\frac{1+u}{u}\right)du = \frac{\beta_{ni}}{n}\left(u\ln\frac{1+u}{u}\right)\Big|_0^{\infty}.$$

Since $\left(u\ln\frac{1+u}{u}\right)\Big|_0^{\infty}=1$ it follows that $S_1+S_3 \le \frac{\beta_{ni}}{n}$. It remains to estimate S_2. For this we observe that $\beta_{nj} \le \beta_{ni}-n(i-j)$ for $j < i$ and that the function $h(t) := \frac{t}{\beta_{ni}-t}$ is increasing on $(0,\beta_{ni})$. Therefore,

$$\frac{\beta_{nj}}{\beta_{ni}-\beta_{nj}} \le \frac{\beta_{ni}-n(i-j)}{\beta_{ni}-(\beta_{ni}-n(i-j))} = \frac{\beta_{ni}-n(i-j)}{n(i-j)}.$$

Now it follows that

$$\begin{aligned}
S_2 &\le \sum_{j=1}^{i-1}\ln\frac{\beta_{ni}-n(i-j)}{n(i-j)} = \sum_{j=1}^{i-1}\ln\frac{\beta_{ni}-nj}{nj} \\
&\le \int_0^{i-1}\ln\frac{\beta_{ni}-nt}{nt}\,dt = \frac{\beta_{ni}}{n}\int_0^{n(i-1)/\beta_{ni}}\ln(\frac{1}{t}-1)\,dt.
\end{aligned}$$

Since $\ln(\frac{1}{t}-1)<0$ if $t\in(\frac{1}{2},1)$ it follows that

$$S_2 \le \frac{\beta_{ni}}{n}\int_0^{1/2}\ln(\frac{1}{t}-1)\,dt = \frac{\beta_{ni}}{n}\int_0^{1/2}\frac{1}{1-t}\,dt = \frac{\beta_{ni}\ln 2}{n}.$$

Finally, we have that

$$S_1 + S_2 + S_3 \le \frac{\beta_{ni}\ln 2}{n} + \frac{\beta_{ni}}{n} = \frac{\beta_{ni}(1+\ln 2)}{n} \le \frac{2\beta_{ni}}{n}.$$

Thus,

$$|\alpha_{ni}| \le \beta_{ni} e^{\frac{2}{n}\beta_{ni}}. \tag{1.22}$$

Now we are able to show that $\Phi_n(t) \to 1$ for all $t > 0$. Let $t > 0$. Then

$$\begin{aligned} |\Phi_n(t) - 1| &\le \sum_{i=1}^{N_n} \left|\frac{\alpha_{n,i}}{\beta_{ni}}\right| e^{-\beta_{ni}t} \le \sum_{i=1}^{\infty} e^{\frac{2\beta_{ni}}{n}} e^{-\beta_{ni}t} \\ &\le \sum_{i=1}^{\infty} e^{-\frac{\beta_{ni}t}{2}} \le \sum_{i=1}^{\infty} e^{\frac{-nit}{2}} = \frac{e^{-nt/2}}{1-e^{-nt/2}}. \end{aligned}$$

where the inequalities hold for $n > \frac{4}{t}$. Therefore, $\Phi_n(t) \to 1$ as $n \to \infty$ uniformly for $t > \epsilon > 0$. Next we have to show that $\Phi_n(t) \to 0$ for $t < 0$. We will show this in two steps. First we show that

$$\int_0^\infty e^{-t}\Phi_n(t)dt \to 1.$$

This follows from the fact that the convolution of two positive functions is positive. So ϕ_n is positive and $\Phi_n = 1 * \phi_n$ is positive and monotonically increasing. The second step is to show that

$$\int_{-\infty}^\infty e^{-t}\Phi_n(t)dt \to 1$$

which will imply again from the positivity and monotonicity that $\Phi_n(t) \to 0$ for all $t < 0$. Using the well known inequalities $1 + t < e^t$ $(t > 0)$ and $e^t < \frac{1}{1-t}$ $(0 < t < 1)$ we obtain that

$$\frac{\beta_{ni}+1}{\beta_{ni}} = 1 + \frac{1}{\beta_{ni}} < e^{\frac{1}{\beta_{ni}}} < \frac{1}{1 - 1/\beta_{ni}} = \frac{\beta_{ni}}{\beta_{ni}-1}.$$

Multiplying by $\frac{\beta_{ni}}{\beta_{ni}+1}$ we have that

$$1 < \frac{\beta_{ni}}{\beta_{ni}+1} e^{\frac{1}{\beta_{ni}}} < \frac{\beta_{ni}^2}{\beta_{ni}^2 - 1} < 1 + \frac{1}{(ni)^2 - 1} < e^{\frac{1}{(ni)^2-1}}.$$

Since $\Phi_n = 0$ for $t < -c_n$ it follows that

$$\int_{-\infty}^{\infty} e^{-t}\Phi_n(t)dt = \int_{-c_n}^{\infty} e^{-t}\Phi_n(t)dt = e^{c_n}\int_{0}^{\infty} e^{-t}\Phi_n(t-c_n)dt$$
$$= e^{\sum_{i=1}^{N_n} 1/\beta_{ni}} \prod_{i=1}^{N_n} \frac{\beta_{ni}}{\beta_{ni}+1} = \prod_{i=1}^{N_n} \frac{\beta_{ni}}{\beta_{ni}+1} e^{1/\beta_{ni}}.$$

Therefore,

$$1 \le \int_{-\infty}^{\infty} e^{-t}\Phi_n(t)dt = \prod_{i=1}^{N_n} \frac{\beta_{ni}}{\beta_{ni}+1} \le \prod_{i=1}^{\infty} e^{1/((ni)^2-1)} = e^{\sum_{i=1}^{\infty} \frac{1}{(ni)^2-1}}.$$

Since $\sum_{i=1}^{\infty} \frac{1}{(ni)^2-1} = \frac{1}{n^2}\sum_{i=1}^{\infty} \frac{1}{(i)^2-1/n^2}$ it follows that $\int_{-\infty}^{\infty} e^{-t}\Phi_n(t)dt \to 1$ which completes the argument. The only thing left to complete the proof is the converges of the inversion formula: i.e.

$$\left\| f(t) - \sum \alpha_{n,i} q(\beta_{ni}) e^{\beta_{ni}t} \right\| \to 0 \text{ as } n \to \infty.$$

Using that $q(\beta_{ni}) = \int_0^T e^{-\beta_{ni}s} f(s)ds$ we obtain that

$$\left\| f(t) - \sum \alpha_{n,i} q(\beta_{ni}) e^{\beta_{ni}t} \right\| = \left\| f(t) - \int_0^T \sum \alpha_{n,i} e^{\beta_{ni}t} e^{-\beta_{ni}s} f(s)ds \right\|$$
$$= \left\| f(t) - \int_0^T \phi_n(s-t) f(s)ds \right\|.$$

Let $\epsilon > 0$. Then there exists a $\delta > 0$ such that $\|f(t) - f(s)\| < \epsilon$ for $|t-s| < 2\delta$. Also choose a n_0 such that $\Phi_n(-\delta) + 1 - \Phi_n(\delta) < \epsilon$ for all $n > n_0$. Then for $t \in [\delta, T-\delta]$ we have that

$$\left\| f(t) - \int_0^T \phi_n(s-t) f(s)ds \right\| \le \int_0^{t-\delta} \phi_n(s-t)\|f(s)\|ds + \int_{t+\delta}^{T} \phi_n(s-t)\|f(s)\|ds$$
$$+ \left\| f(t) - \int_{t-\delta}^{t+\delta} \phi_n(s-t) f(t)ds \right\| + \int_{t-\delta}^{t+\delta} \phi_n(s-t)\|f(t) - f(s)\|ds$$
$$\le \|f\| \int_0^{t-\delta} \phi_n(s-t)ds + \|f\| \int_{t+\delta}^{T} \phi_n(s-t)ds$$
$$+ \|f\| \left| 1 - \int_{t-\delta}^{t+\delta} \phi_n(s-t)ds \right| + \epsilon \int_{t-\delta}^{t+\delta} \phi_n(s-t)ds$$
$$\le \|f\| (\Phi_n(-\delta) - \Phi_n(-t)) + \|f\| (\Phi_n(T-t) - \Phi_n(\delta))$$
$$+ \|f\| |1 - \Phi_n(\delta) + \Phi_n(-\delta)| + \epsilon(\Phi_n(t+\delta) - \Phi_n(t-\delta)) \le \epsilon(3\|f\| + 1).$$

Finally, using that $f(0) = 0$ we obtain for $t \in [0, \delta]$ that

$$\begin{aligned}
&\left\| f(t) - \int_0^T \phi_n(s-t) f(s) ds \right\| \\
&\qquad \leq \|f(t)\| + \int_0^{t+\delta} \phi_n(s-t) \|f(s)\| \, ds + \int_{t+\delta}^T \phi_n(s-t) \|f(s)\| \, ds \\
&\qquad\qquad \leq \epsilon + \epsilon(\Phi_n(\delta) - \Phi_n(t)) + \|f\| \left(\Phi_n(T-t) - \phi_n(\delta) \right) \leq \epsilon(2 + \|f\|).
\end{aligned}$$

Therefore, $\sum_{i=1}^{N_n} \alpha_{n,i} q(\beta_{ni}) e^{\beta_{ni} t}$ converges uniformly to $f(t)$ on $[0, s]$. □

The Phragmén-Mikusiński inversion theorem has some remarkable properties and corollaries. To state them we have to introduce the notion of functions of exponential decay $T > 0$.

Definition 1.12. *Let $T > 0$. We say that $r : (\omega, \infty) \to X$ is of exponential decay T, if $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|r(\lambda)\| \leq -T$. To simplify notation we will write $r \approx_T 0$ whenever r is of exponential decay T.*

Observe that a function is of exponential decay T if for all $\varepsilon > 0$ there exists $\lambda_0 > \omega$ such that $\|r(\lambda)\| \leq M e^{\lambda(-T+\varepsilon)}$ for any $\lambda > \lambda_0$. Examples of functions which are of exponential decay T for all $T > 0$ are $\lambda \mapsto e^{-\lambda \ln \lambda}$ and $\lambda \mapsto e^{-\lambda^\alpha}$ for $\alpha > 1$. We show next that if $f \in L^1_{loc}([0, \infty), X)$ is Laplace transformable then $\int_T^\infty e^{-\lambda t} f(t) \, dt$ is of exponential decay T. In particular, for any Laplace transformable $f \in L^1_{loc}([0, \infty), X)$ with $supp(f) \subset [T, \infty)$ it follows that $\hat{f}(\lambda)$ is of exponential decay T.

Lemma 1.13. *Let $f \in L^1_{loc}[0, \infty)$ such that $\hat{f}(\lambda)$ exists for $\lambda > \omega > 0$ and let $T > 0$. Then $a(\lambda) := \int_T^\infty e^{-\lambda t} f(t) dt$ is of exponential decay T; i.e., $a(\lambda) \approx_T 0$.*

Proof. Let $F(t) := \int_0^t f(s) ds$. From Proposition 1.2 it follows that $\hat{f}$ exists for $Re\lambda > \omega > 0$ if and only if there exist $M > 0$ such that $\|F(t)\| \leq M e^{\omega t}$. Let

$T \geq 0$, using integration by parts we obtain

$$\int_T^\infty e^{-\lambda t} f(t)dt = -e^{-\lambda T}F(T) + \lambda \int_T^\infty e^{-\lambda t}F(t)\, dt. \tag{1.23}$$

Therefore, there exists a constant $C_T > 0$ such that

$$\begin{aligned} \| \int_T^\infty e^{-\lambda t} f(t)\, dt\| &= \| -e^{-\lambda T}F(T) + \lambda \int_T^\infty e^{-\lambda t}F(t)\, dt\| \\ &\leq e^{-\lambda T}\|F(T)\| + M\frac{\lambda}{\lambda-\omega}e^{-(\lambda-\omega)T} \\ &\leq C_T e^{-\lambda T} \end{aligned}$$

for all $\lambda > 2\omega$. This shows that $a(\lambda) = \int_T^\infty e^{-\lambda t} f(t)\, dt \approx_T 0$ for all $T > 0$. □

The next corollary states that the Phragmén-Mikusiński inversion formula is invariant to perturbation of exponential decay; i.e., if for $t \in [0, T)$,

$$f(t) = \lim_{n\to\infty} \sum_{i=1}^{N_n} \alpha_{n,i} q(\beta_{ni}) e^{\beta_{ni}t}$$

for some function q then

$$f(t) = \lim_{n\to\infty} \sum_{i=1}^{N_n} \alpha_{n,i} \tilde{q}(\beta_{ni}) e^{\beta_{ni}t}$$

for all perturbed functions $\tilde{q} = q + r$, where r is a perturbation of exponential decay T.

Corollary 1.14 (Perturbation). *Let $r : (\omega, \infty) \to X$ and let $(\beta_n)_{n\in\mathbb{N}}$ be a Müntz sequence. If r is of exponential decay $T > 0$, then*

$$\lim_{n\to\infty} \sum_{i=1}^{N_n} \alpha_{n,i} e^{\beta_{ni}t} r(\beta_{ni}) = 0$$

for all $0 \leq t < T$ and the limit is uniform for all $t \in [0, s]$ and $0 < s < T$.

Proof. Let $t \in [0, T)$. Then $T > \frac{2T+t}{3}$. Since r is of exponential decay T it follows that there exists a n_0 such that $e^{-\frac{T-t}{3}\beta_n} < \frac{1}{\beta_n}$ and $\|r(\beta_n)\| \leq \frac{M}{\beta_n} e^{-\frac{2T+t}{3}\beta_n}$ and such that $\frac{2}{n_0} < \frac{T-t}{3}$. From 1.22 we know that $|\alpha_{n,i}| < \beta_{ni} e^{\frac{2}{n}\beta_{ni}} < \beta_{ni} e^{\frac{T-t}{3}\beta_{ni}}$. Therefore

$$\begin{aligned} \left\| \sum_{i=1}^{N_n} \alpha_{n,i} e^{\beta_{ni} t} r(\beta_{ni}) \right\| &\leq \sum_{i=1}^{N_n} \beta_{ni} e^{\frac{T-t}{3}\beta ni} e^{\beta_{ni} t} \frac{M}{\beta_{ni}} e^{-\frac{2T+t}{3}\beta_{ni}} \\ &\leq M \sum_{i=1}^{\infty} e^{\beta_{ni}(\frac{T-t}{3}+t-\frac{2T+t}{3})} \leq M \sum_{i=1}^{\infty} e^{-\frac{T-t}{3}\beta_{ni}} \\ &\leq M \sum_{i=1}^{\infty} e^{-\frac{T-t}{3}ni} = M \frac{e^{-\frac{T-t}{3}n}}{1-e^{-\frac{T-t}{3}n}} \to 0 \end{aligned}$$

as $n \to \infty$ uniformly for all $t \in [0, s]$ and for all $0 < s < T$. □

Corollary 1.15 (Phragmén-Mikusiński Inversion, Part II).

Let $f \in C_0([0,\infty), X)$ be Laplace transformable and let $\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\, dt$. Let $(\beta_n)_{n\in\mathbb{N}}$ be a Müntz sequence and let $\alpha_{n,i}$ and N_n be as in Theorem 1.11. Then, for all $t \geq 0$,

$$f = \lim_{n\to\infty} \sum_{i=1}^{N_n} \alpha_{n,i} \hat{f}(\beta_{ni}) e^{\beta_{ni}(\cdot)},$$

where the limit is uniform on compact sets.

Proof. The proof is an immediate consequence of the Theorem 1.11, Lemma 1.13 and Corollary 1.14. □

The following theorem is another consequence and it will characterize the maximal interval $[0, T]$ on which a function can vanish in terms of the growth of the Laplace transform evaluated at Müntz points.

Theorem 1.16 (Support Theorem). *Let $0 \leq T$ and let $f \in L^1_{loc}([0,\infty), X)$ be Laplace transformable. Then the following are equivalent:*

(i) Every Müntz sequence $(\beta_n)_{n\in\mathbb{N}}$ satisfies $\limsup_{n\to\infty} \frac{1}{\beta_n} \ln \left\| \hat{f}(\beta_n) \right\| = -T$.

(ii) For every Müntz sequence $(\beta_n)_{n\in\mathbb{N}}$ there exists a Müntz subsequence $(\beta_{n_k})_{k\in\mathbb{N}}$ satisfying

$$\lim_{k\to\infty} \frac{1}{\beta_{n_k}} \ln \left\| \hat{f}(\beta_{n_k}) \right\| = -T.$$

(iii) There exists a Müntz sequence $(\beta_n)_{n\in\mathbb{N}}$ *satisfying*

$$\limsup_{n\to\infty}\frac{1}{\beta_n}\ln\left\|\hat{f}(\beta_n)\right\| = -T.$$

(iv) $f(t)=0$ *almost everywhere on* $[0,T]$ *and* $T\in supp(f)$.

(v) $\limsup_{\lambda\to\infty}\frac{1}{\lambda}\ln\left\|\hat{f}(\lambda)\right\| = -T$.

Proof. We show first that $(i)\Rightarrow(ii)\Rightarrow(iii)\Rightarrow(iv)\Rightarrow(i)$, and then that (iv) is equivalent with (v).

Suppose (i) holds, and let $(\beta_n)_{n\in\mathbb{N}}$ be a Müntz sequence. For each $\varepsilon=\frac{1}{k},\ k\in\mathbb{N}$ we denote by $(\beta_n^\varepsilon)_{n\in\mathbb{N}}$ a subsequence of $(\beta_n)_{n\in\mathbb{N}}$ obtained by dropping all the elements of $(\beta_n)_{n\in\mathbb{N}}$ for which $\left\|\hat{f}(\beta_n)\right\|\le e^{-(T+\varepsilon)\beta_n}$. The dropped sequence denoted by α_n^ε has the property that $\limsup_{n\to\infty}\frac{1}{\alpha_n^\varepsilon}\ln\left\|\hat{f}(\alpha_n^\varepsilon)\right\|\le-(T+\varepsilon)$, and by (i) it cannot be a Müntz sequence, so it converges. Hence the remaining sequence will still satisfy the Müntz condition. Next, for $\varepsilon=1$ we pick the first k_1 elements of the $(\beta_n^1)_{n\in\mathbb{N}}$ such that $\sum_{i=1}^{k_1}\frac{1}{\beta_i^1}\ge1$. For $\varepsilon=\frac{1}{2}$ we pick k_2 elements such that $\sum_{i=1}^{k_1}\frac{1}{\beta_i^1}+\sum_{i=1}^{k_2}\frac{1}{\beta_i^{1/2}}\ge2$. Continuing this process in the same fashion we obtain a Müntz sequence which has the property that $\lim_{k\to\infty}\frac{1}{\beta_{n_k}}\ln\left\|\hat{f}(\beta_{n_k})\right\|=-T$. Clearly, $(ii)\to(iii)$.

Suppose that (iii) holds. Since $f\in L^1_{loc}([0,\infty),X)$ is Laplace transformable it follows that $F(t):=\int_0^t f(s)\,ds$ is in $C_0([0,T],X)$ and by 1.23 for $T=0$, $\hat{F}(\lambda)=\frac{1}{\lambda}\hat{f}(\lambda)$. Let $t\in[0,T)$, let $\hat{F_T}(\lambda):=\int_0^T e^{-\lambda t}F(t)\,dt$ and $a(\lambda):=\int_T^\infty e^{-\lambda t}F(t)\,dt.$. Then, by the Phragmén-Mikusiński inversion theorem

$$\begin{aligned}F(t) &= \lim_{n\to\infty}\sum_{i=1}^{N_n}\alpha_{n,i}\hat{F_T}(\beta_{ni})e^{\beta_{ni}t} = \lim_{n\to\infty}\sum_{i=1}^{N_n}\alpha_{n,i}(\hat{F}(\beta_{ni})-r(\beta_{ni}))e^{\beta_{ni}t}\\ &= \lim_{n\to\infty}\sum_{i=1}^{N_n}\alpha_{n,i}\hat{F}(\beta_{ni})e^{\beta_{ni}t}-\lim_{n\to\infty}\sum_{i=1}^{N_n}\alpha_{n,i}r(\beta_{ni})e^{\beta_{ni}t},\end{aligned}$$

where the second term goes to zero by Lemma 1.13 and Corollary 1.14. It remains to be shown that the fist term goes to zero. This follows from

$$\begin{aligned}\left\|\sum_{i=1}^{N_n}\alpha_{n,i}\hat{F}(\beta_{ni})e^{\beta_{ni}t}\right\| &\leq \sum_{i=1}^{N_n}\left\|\alpha_{n,i}\hat{F}(\beta_{ni})e^{\beta_{ni}t}\right\| \\ &\leq \sum_{i=1}^{N_n}\beta_{ni}e^{\frac{T-t}{3}\beta ni}e^{\beta_{ni}t}\frac{1}{\beta_{ni}}\left\|\hat{f}(\beta_{ni})\right\| \\ &\leq \sum_{i=1}^{\infty}e^{\frac{T-t}{3}\beta ni}e^{\beta_{ni}t}e^{-\frac{2T+t}{3}\beta_{ni}} \\ &\leq \sum_{i=1}^{\infty}e^{-\frac{T-t}{3}ni}=\frac{e^{-\frac{T-t}{3}n}}{1-e^{-\frac{T-t}{3}n}}\to 0.\end{aligned}$$

Hence $F(t)=0$ for all $t\in[0,T)$. Thus, $f=0$ on $[0,T)$. To show that $T\in supp(f)$, suppose that $f=0$ almost everywhere on $[0,T+\varepsilon)$. Then it follows that $\hat{f}(\lambda)=\int_{T+\varepsilon}^{\infty}e^{-\lambda t}f(t)\,dt$ which is of exponential decay $T+\varepsilon$ contradicting (iii). Thus, (iv) holds.

Finally, suppose (iv) holds. Then, from Lemma 1.13 we have that $\limsup_{\lambda\to\infty}\frac{1}{\lambda}\ln\left\|\hat{f}(\lambda)\right\|\leq -T$. Let $(\beta_n)_{n\in\mathbb{N}}$ be a Müntz sequence such that

$$\limsup_{\lambda\to\infty}\frac{1}{\beta_n}\ln\left\|\hat{f}(\beta_n)\right\|\leq -T-\varepsilon.$$

Then we will have again that $F(t)=\int_0^t f(s)\,ds=0$ for $t\in[0,T+\varepsilon)$, contradicting $T\in supp(f)$, and therefore (i) holds. Moreover, suppose that (v) does not hold; i.e.,

$$\limsup_{\lambda\to\infty}\frac{1}{\lambda}\ln\left\|\hat{f}(\lambda)\right\|<-T.$$

This contradicts (i) so, (v) must be true. Finally, assuming that (v) holds, it follows that (iii) holds. Thus, (iv) holds and the proof is finished. □

A crucial corollary of the Support Theorem 1.16 is the following result due to E. C. Titchmarsh (see [25]).

Corollary 1.17. *Let* $0 < T$*, and let* $k \in L^1_{loc}[0,\infty)$ *and* $f \in L^1_{loc}([0,\infty),X)$ *be Laplace transformable. Then* $k * f = 0$ *on* $[0,T]$ *implies that there exists* $x_1, x_2 > 0$ *with* $x_1 + x_2 \geq T$ *such that* $k = 0$ *almost everywhere on* $[0, x_1]$ *and* $f = 0$ *almost everywhere on* $[0, x_2]$*.*

Proof. Let $x_1 := -\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \left|\hat{k}(\lambda)\right|$ and $x_2 := -\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \left\|\hat{f}(\lambda)\right\|$. Then, by the Support Theorem 1.16, k and f are zero on $[0, x_1]$ and $[0, x_2]$ respectively. Furthermore, by the same theorem there exists a Müntz subsequence such that $x_1 := -\lim_{n\to\infty} \frac{1}{\beta_n} \ln \left|\hat{k}(\beta_n)\right|$ and $x_2 := -\lim_{n\to\infty} \frac{1}{\beta_n} \ln \left\|\hat{f}(\beta_n)\right\|$. Since

$$\lim_{n\to\infty} \frac{1}{\beta_n} \ln \left\|\hat{k}(\beta_n)\hat{f}(\beta_n)\right\| = \lim_{n\to\infty} \frac{1}{\beta_n} \ln \left|\hat{k}(\beta_n)\right| + \lim_{n\to\infty} \frac{1}{\beta_n} \ln \left\|\hat{f}(\beta_n)\right\| = -x_1 - x_2$$

we obtain that $x_1 + x_2 \geq T$. □

Another remarkable corollary is the celebrated Titchmarsh-Foias Theorem. For numerical functions (i.e., $X = \mathbb{C}$), the equivalence of (i) and (ii) was proved by E.C. Titchmarsh in 1926 (see [25]) and the equivalence of (i) and (iii) by C. Foias in 1961 (see [12]). For Banach space valued functions, the following result is due to B. Bäumer, G. Lumer and F. Neubrander (see [2], [3] and [5]).

Corollary 1.18 (Titchmarsh-Foias). *Let* $k \in L^1[0,T]$ *and consider the convolution operator* $T_k : f \mapsto k * f$ *from* $C([0,T],X)$ *into* $C_0([0,T],X)$*. Then the following are equivalent:*

(i) $0 \in supp(k)$;

(ii) T_k *is one-to-one;*

(iii) *The range of* T_k *is dense.*

Proof. The fact that (i) is equivalent with (ii) follows essentially from the Corollary 1.17. Suppose that $0 \in supp(k)$; then k is not zero on any interval $[0, \varepsilon]$, for $\varepsilon > 0$.

Then, by Corollary 1.17 it follows that $f = 0$ on the interval $[0, x_1]$ with $x_1 \geq T$ whenever $k * f = 0$ on $[0, T]$. This shows that T_k is one-to-one. For the converse suppose that T_k is one-to-one, but $0 \notin supp(k)$; i.e., $k(t) = 0$ for $t \in [0, \varepsilon]$. Then, $\int_0^t k(t-s)f(s)\,ds = \int_0^{t-\varepsilon} k(t-s)f(s)\,ds$. Now choose f with $supp(f) \subset [1-\varepsilon, 1]$. Then $k * f = 0$ but $f \neq 0$ contradicting that T_k is one-to-one. It remains to be shown that (i) is equivalent with (iii). We will show first that (i) implies (iii). Let β_n be a Müntz sequence which satisfies the condition $\lim_{n\to\infty} \frac{1}{\beta_n} \ln \left\| \hat{k}(\beta_n) \right\| = 0$ and let $\epsilon_n := \max\{\frac{2}{\sqrt{n}}, -\frac{2}{\beta_n} \ln \left\| \frac{\hat{k}(\beta_n)}{\beta_n} \right\|\}$ and define

$$g_n(t) := \sum_{i=1}^{N_n} \alpha_{n,i} \frac{\hat{f}(\beta_{ni})}{\hat{k}(\beta_{ni})} e^{\beta_{ni}(t-\epsilon_n)}$$

where N_n and $\alpha_{n,i}$ are defined in Theorem 1.11. Clearly $g_n \in C([0,T], X)$ and we will show that $k * g_n \to f$ uniformly on compacts intervals.

$$\begin{aligned}
(k * g_n)(t) &= \sum_{i=1}^{N_n} \alpha_{n,i} \frac{\hat{f}(\beta_{ni})}{\hat{k}(\beta_{ni})} \int_0^t e^{\beta_{ni}(t-\epsilon_n-s)} k(s)ds \\
&= \sum_{i=1}^{N_n} \alpha_{n,i} \frac{\hat{f}(\beta_{ni})}{\hat{k}(\beta_{ni})} e^{\beta_{ni}(t-\epsilon_n)} \int_0^t e^{-\beta_{ni}s} k(s)ds \\
&= \sum_{i=1}^{N_n} \alpha_{n,i} \frac{\hat{f}(\beta_{ni})}{\hat{k}(\beta_{ni})} e^{\beta_{ni}(t-\epsilon_n)} \left[\hat{k}(\beta_{ni}) - \int_t^T e^{-\beta_{ni}s} k(s)ds \right] \\
&= \sum_{i=1}^{N_n} \alpha_{n,i} \frac{\hat{f}(\beta_{ni})}{\hat{k}(\beta_{ni})} e^{\beta_{ni}(t-\epsilon_n)} \hat{k}(\beta_{ni}) \\
&- \sum_{i=1}^{N_n} \alpha_{n,i} \frac{\hat{f}(\beta_{ni})}{\hat{k}(\beta_{ni})} e^{\beta_{ni}(t-\epsilon_n)} \int_t^T e^{-\beta_{ni}s} k(s)ds
\end{aligned}$$

By Theorem 1.11 the first sum converges to f, therefore to finish the argument we have to show that the second sum goes to zero. Using the estimates for $\alpha_{n,i}$ from the Theorem 1.11 and the definition of ϵ_n we obtain that

$$\left\| \sum_{i=1}^{N_n} \alpha_{n,i} \frac{\hat{f}(\beta_{ni})}{\hat{k}(\beta_{ni})} e^{\beta_{ni}(t-\epsilon_n)} \int_t^T e^{-\beta_{ni}s} k(s)ds \right\| \leq$$

$$\begin{aligned}
&\leq \sum_{i=1}^{\infty} \beta_{ni} e^{2\beta_{ni}/n} \frac{C}{\beta_{ni} e^{-\beta_{ni}\epsilon_n/2}} e^{\beta_{ni}(t-\epsilon_n)} e^{-\beta_{ni}t} \\
&= C \sum_{i=1}^{\infty} e^{\beta_{ni}(2/n-\epsilon_n/2)} \\
&\leq C \frac{e^{2-\sqrt{n}}}{1-e^{2-\sqrt{n}}} \to 0
\end{aligned}$$

To complete the proof we finally have to show that (iii) implies (ii). Let $f \in C([0,T],X)$. Suppose $k * f = 0$. We will show that $f = 0$ which will complete the proof. Since the range of T_k is dense it follows that there exists a sequence g_n such that $k*g_n \to Id$. Then it follows that $0 = k*f = \lim k*f*g_n = \lim k*g_n*f = Id*f$. Therefore $\int_0^t (t-s)f(s)\,ds = 0$ for all $t \geq 0$ and thus $f = 0$. □

Before continuing, we would like to destile an important fact from the proof above.

Corollary 1.19 (Inversion of the Convolution Transform). *Let $k \in L^1[0,T]$ with $0 \in supp(k)$, and let $(\beta_n)_{n\in\mathbb{N}}$ be a Müntz sequence such that*

$$\lim_{n\to\infty} \frac{1}{\beta_n} \ln \left|\hat{k}(\beta_n)\right| = 0.$$

Then, for all $f \in C([0,T],X)$ we have that

$$k * g_n \to f$$

uniformly on $[0,T]$, where

$$g_n(t) := \sum_{i=1}^{N_n} \alpha_{n,i} \frac{\hat{f}(\beta_{ni})}{\hat{k}(\beta_{ni})} e^{\beta_{ni}(t-\varepsilon_n)},$$

and where the constants $N_n, \alpha_{n,i}$ are defined as in Phragmén-Mikusiński Inversion Theorem 1.11 and $\varepsilon_n := \max\{\frac{2}{\sqrt{n}}, -\frac{2}{\beta_n} \ln \left\|\frac{\hat{k}(\beta_n)}{\beta_n}\right\|\}$.

Let $f \in C_0([0,T],X)$ and $k \in L^1[0,T]$ with $0 \in supp(k)$. To find a solution g of the equation

$$k * g = f$$

based on the corollary above, we have to find a topology on $C([0,T],X)$ for which the functions g_n converge towards a function g. This will be done in the reminder of this chapter where we define the field of generalized functions via linear extensions of the spaces $C([0,a],X)$.

A linear extension of a Banach space X is, by definition, a completion of X with respect to a new norm. In our set-up, new norms are generated through one-to-one operators $T \in \mathcal{L}(X,Y)$, where $\mathcal{L}(X,Y)$ denotes the Banach space of all bounded linear operators from X into a Banach space Y. The following lemma is essential for our construction.

Proposition 1.20. *Let X, Y be Banach spaces and let $T \in \mathcal{L}(X,Y)$ be one-to-one. On X define a new norm by $\|x\|_T := \|Tx\|$. Then $\| \, \|_T$ is a new norm and if X^T denotes the completion of X with respect to $\| \, \|_T$ we have that*

$$\begin{array}{ccc} X^T & \xrightarrow{\widetilde{T}} & \overline{Im(T)} \\ \uparrow & & \uparrow \\ X & \xrightarrow{T} & Im(T) \end{array}$$

i.e. X is continuously embedded in X^T, and the operator T extend to an isometric isomorphism $\widetilde{T}$ between X^T and $\overline{Im(T)}$. In particular $(X, \| \, \|_T)$ is already a Banach space if and only if $Im(T)$ is closed in Y.

Proof. Clearly, $\| \, \|_T$ is a new norm. The identity map $X \hookrightarrow X^T$ is continuous since $\|x\|_T = \|Tx\| \leq \|T\| \|x\|$. Next we need to show that $\widetilde{T}$ is an isometric isomorphism. Let $x \in X^T$. Then there exists a sequence $x_n \in X$ such that $x_n \to x$ in X^T and therefore x_n is Cauchy in X^T and Tx_n is a Cauchy sequence in Y. Hence $\lim_{n\to\infty} Tx_n$ exists and equals y. Now, suppose that $v_n \to x$ as well. Then, the same argument shows that $\lim_{n\to\infty} Tv_n = w$. Finally,

$$\|y - w\|_T = \lim_{n\to\infty} \|Tx_n - Tv_n\| = \lim_{n\to\infty} \|x_n - v_n\|_T = \|z - z\|_T = 0.$$

Therefore $\widetilde{T}(x) := \lim_{n\to\infty} Tx_n$ is a well defined extension of T mapping X^T into $\overline{Im(T)}$ which is linear and

$$\left\|\widetilde{T}x\right\| = \lim_{n\to\infty} \|Tx_n\| = \lim_{n\to\infty} \|x_n\|_T = \|x\|_T$$

for all $x \in X^K$. That shows that $\widetilde{T}$ is an isometry and is one-to-one. It remains to be shown that $\widetilde{T}$ is onto. Consider $y \in \overline{Im(T)}$ and the sequence $y_n \in Im(T)$ with $y_n \to y$ in Y. Let $x_n \in X$ be the corresponding sequence with $Tx_n = y_n$. Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X^T. Since X^T is complete let $x = \lim_{n\to\infty} x_n$. Then

$$\widetilde{T}(x) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} y_n = y.$$

Therefore we have shown that $\widetilde{T}$ is an isometric isomorphism between X^T and $\overline{Im(T)}$. To finish the proof we need to show that X is complete under the norm $\| \, \|_T$ if and only if $Im(T)$ is closed in Y. Suppose $(X, \| \, \|_T)$ is complete. Let $y \in \overline{Im(T)}$. Then there exists a convergent sequence $(y_n)_{n\in\mathbb{N}}$ such that $y_n \in Y$ and $y_n \to y$. Therefore there exists a sequence $x_n \in X$ for any $n \in \mathbb{N}$ such that $Tx_n = y_n$. so, we have

$$\lim_{n\to\infty} \|y_n - y\| = \lim_{n\to\infty} \|Tx_n - y\| = 0.$$

Since Tx_n converges it follows that $(x_n)_{n\in\mathbb{N}}$ is Cauchy in $(X, \| \, \|_T)$. By completeness, it follows that there exists an x such that $x_n \to x$ and $Tx = y$. So, $y \in Im(T)$. Conversely, suppose that $Im(T)$ is closed. We need to show that any Cauchy sequence with respect to $\| \, \|_T$ converges. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Then Tx_n is a Cauchy sequence in Y and because of the closeness of $Im(T)$ it follows that $Tx_n \to y$ for some $y \in Im(T)$. Hence, there exists an $x \in X$ such that $Tx = y$ and $x_n \to x$. Therefore $(X, \| \, \|_T)$ is a Banach space and the proof is complete. □

Now, if $k \in L^1[0,a]$ and $0 \in supp(k)$ then it follows from Corollary 1.18 that the convolution operator $T_k : f \mapsto k * f$ is a one-to-one operator defined on $C([0,a], X)$

with a dense range in $C_0([0,a],X)$. Therefore, by Proposition 1.20, we obtain the following diagram

$$\begin{array}{ccc} C([0,a],X)^{T_k} & \xrightarrow{\widetilde{T_k}} & C_0([0,a],X) = \overline{Im(T_k)} \\ \uparrow & & \uparrow \\ C([0,a],X) & \xrightarrow{T_k} & C_0([0,a],X), \end{array}$$

where $\widetilde{T_k}$ is an isometric isomorphism between the completion of $C([0,a],X)$ with the norm $\|f\|_{T_k} := \|T_k f\|$ and $C_0([0,a],X)$. For example, if $k(t) = 1$, then the antiderivative operator $T_k : f \mapsto 1 * f$ defines an isometric isomorphism $\widetilde{T_k}$ between $C([0,a],X)^{T_k}$ and $C_0([0,a],X)$. In particular, for any $f \in C_0([0,a],X)$ there exists a unique generalized derivative $f' := \widetilde{T_k}^{-1} f \in C([0,a],X)^{T_k}$. More general, one can consider the convolution operators induced by $k_\alpha : t \mapsto \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $\alpha > 0$. Since $k_\alpha \in L^1[0,1]$ we denote by T_α the operator $f \mapsto k_\alpha * f$. Then $T_\alpha(T_\beta(f)) = k_\alpha * (k_\beta * f) = (k_\alpha * k_\beta) * f = T_{\alpha+\beta}$. For a detailed exposition of these facts see [2], Sections II.2 and II.4. As an immediate consequence of this construction we obtain the following consequence from Corollary 1.19.

Corollary 1.21. *Let $k \in L^1[0,T]$ with $0 \in supp(k)$. Then, for all $f \in C_0([0,a],X)$ there exists a unique $g \in C([0,T],X)^{T_k}$ such that*

$$k * g := \tilde{T}_k g = f.$$

Moreover $g = T_k - \lim g_n$, where the sequence $g_n \in C([0,a],X)$ is defined as in Corollary 1.19.

When one looks at arbitrary generalized function spaces; i.e., $C[0,a]^{T_{k_i}}$ with $i = 1,2$ induced by $k_1, k_2 \in L^1[0,1]$ the following shed some light on the structure of these spaces. We identify elements $f \in C[0,a]^{T_{k_1}}$ and $\tilde{f} \in C[0,a]^{T_{k_2}}$ if they have the same embedding in $C[0,a]^{T_{k_1 * k_2}}$; i.e., $f = \tilde{f}$ if

$$k_2 * (k_1 * f) = k_1 * (k_2 * \tilde{f}). \tag{1.24}$$

By doing so, one is able to define the convolution between the two generalized functions $f \in C[0,a]^{T_{k_1}}$ and $g \in C[0,a]^{T_{k_2}}$ by

$$f * g := \widetilde{T}^{-1}_{k_1 * k_2}((k_1 * f) * (k_2 * g)) \in C[0,1]^{T_{k_1 * k_2}}.$$

Clearly, the definition also works if one of the functions is $X-$ valued. Since $(f * k_1) * (g * k_2) = (g * k_2) * (f * k_1)$ it follows that $f * g = g * f$. Suppose that f can be identified with $\tilde{f} \in C[0,a]^{T_{k_3}}$; i.e., $k_3 * (k_1 * f) = k_1 * (k_3 * \tilde{f})$. Then $f * g$ can be identified with $\tilde{f} * g$. To see this, observe that $f * g \in \widetilde{T}^{-1}_{k_1 * k_2}((k_1 * f) * (k_2 * g))$ can be identified with $\tilde{f} * g \in \widetilde{T}^{-1}_{k_3 * k_2}((k_3 * \tilde{f}) * (k_2 * g))$ if and only if

$$\begin{aligned}
& (k_3 * k_2) * (k_1 * k_2) * (f * g) = (k_1 * k_2) * (k_3 * k_2) * (\tilde{f} * g) \\
\Leftrightarrow \quad & (k_3 * k_2) * (k_1 * f) * (k_2 * g) = (k_1 * k_2) * (k_3 * \tilde{f}) * (k_2 * g) \\
\Leftrightarrow \quad & k_2 * [k_3 * (k_1 * f)] * (k_2 * g) = k_2 * [k_1 * (k_3 * \tilde{f})] * (k_2 * g).
\end{aligned}$$

Moreover, for all $f \in C[0,a]^{T_{k_2}}$ we have that $k_1 * f \in C[0,a]^{T_{k_2}}$ and, for $h := T^{-1}_{k_1 * k_1 * 1}(1 * k_1)$ we have that

$$\begin{aligned}
h * (k_1 * f) &= \widetilde{T}^{-1}_{k_1 * k_1 * 1 * k_2}(k_1 * k_1 * 1 * h * k_2 * (k_1 * f)) \\
&= \widetilde{T}^{-1}_{k_1 * k_1 * 1 * k_2}(1 * k_1 * k_2 * (k_1 * f)) \\
&= f.
\end{aligned}$$

This shows that that the convolution $g = k_1 * f$ has a convolution inverse h (i.e., $h * g = f$) if $0 \in supp(k_1)$. In order to obtain a convolution inverse for functions k_1 with $0 \notin supp(k_1)$, let us consider the convolution with a function $k \in L^1_{loc}[0,\infty)$

with $k = 0$ on some interval $[0, a]$ and $a \in supp(k)$. Then we have that

$$\begin{aligned}(k * f)(t) &= \int_0^t f(t-s)k(s)\,ds = \int_a^t f(t-s)k(s)\,ds \\ &= \int_0^{t-a} f(t-a-s)k(s+a)\,ds = (k_{-a} * f)(t-a),\end{aligned}$$

where $k_{-a}(t) := k(t+a)$ is the shift of k to the left by a. This means that convoluting with a function which is zero on $[0, a]$ is the same as convoluting with the left-shifted function k_{-a} and then right-shifting the result. Since the shift operations are not invertible in intervals or half lines we will adjust our functions by identifying them with their zero continuations onto $(-\infty, \infty)$. In considering functions on $(-\infty, \infty)$ the shift operations become invertible. Next, we adapt the the definition of the convolution operator for functions on half lines. For $k \in L^1_{loc}[a, \infty)$ with $a \in supp(k)$ and $f \in C[b, \infty)$ with $b \in supp(f)$ we define

$$k * f : t \mapsto \int_{-\infty}^{\infty} k(t-s)f(s)\,ds,$$

where for notational simplicity both k and f are identified with their zero continuations on $(-\infty, \infty)$. We remark here that, for $a = b = 0$ this convolution is the convolution we have discussed so far. Now for k, f as above we have that

$$\begin{aligned}(k * f)(t) &= \int_{-\infty}^{\infty} k(t-s)f(s)\,ds = \int_b^{t-a} k(t-s)f(s)\,ds \\ &= \int_0^{t-a-b} k(t-b-s)f(s+b)\,ds \\ &= \int_0^{t-a-b} k(t-a-b-s+a)f(s+b)\,ds \\ &= \int_0^{t-a-b} k_{-a}(t-a-b-s)f_{-b}(s)\,ds \\ &= (k_{-a} * f_{-b})(t-a-b) = (k_{-a} * f_{-b})_{a+b}(t).\end{aligned}$$

First observe that $k * f \equiv 0$ on $(-\infty, a+b)$. Second, since k_{-a} and f_{-b} are both zero on $(-\infty, 0)$ and the convolution transform is one-to-one on $C[0, \infty)$ we obtain that the convolution on the space of continuous functions with $supp(f) \subset [a,\infty)$ for some $a \in \mathbb{R}$ has no zero divisors. In particular, if $k_1 \in L^1_{loc}[a,\infty)$ and $f \in C[b,\infty)$ with $a \in supp(k_1)$ and $b \in supp(f)$, then the convolution operator is given by

$$T_k f = S_{a+b} \circ T_{k_{-a}} \circ S_{-b} f,$$

where S_{a+b} denotes the right shift operator $f \mapsto f(\cdot - a - b)$, S_{-b} is the left shift operator $f \mapsto f(\cdot + b)$ and where f, k are identified with their zero continuation to $(-\infty,\infty)$. At this point, we also extend spaces like $C[a,\infty)$ to spaces $C[a,\infty)^{T_k}$ via the operators $T_k : C[a,\infty) \to C[a,\infty)$ with

$$T_k f : f \mapsto \int_a^t k(t-s) f(s)\, ds$$

for $k \in C[0,\infty)$ by taking the completion of the space $C[a,\infty)$ with respect to the seminorms $\|f\|_{\alpha,T_k} := \sup_{t\in[a,\alpha]} \|T_k f(t)\|$. For any generalized function $f \in C_0[b,\infty)^{T_k}$ there is a natural embedding in $C_0[a,\infty)^{T_k}$ for any $a \le b$ just by identifying $k * f$ with its zero extension to the left and then taking T_k^{-1}. In a similar manner with the finite interval case we identify $f \in C_0[a,\infty)^{T_{k_1}}$ with $g \in C_0[b,\infty)^{T_{k_2}}$ if $k_2 * k_1 * f = k_1 * k_2 * g$, where, again the functions $k_2 * k_1 * f$ and $k_1 * k_2 * g$ are considered as functions in $C(-\infty,\infty)$, extending them with zero to the left.

Theorem 1.22 (The field of generalized scalar functions). *Let*

$$\mathcal{F} := \{f \in C_0[a,\infty)^{T_k} : a \in \mathbb{R}, k \in C[0,\infty) \ \textit{with} \ 0 \in supp(k)\},$$

and define

$$f * g := T^{-1}_{k_1 * k_2}(k_1 * f * k_2 * g)$$

for $f \in C_0[a,\infty)^{T_{k_1}}$ and $g \in C_0[a,\infty)^{T_{k_2}}$. Then $\mathcal{F}$ is a field with respect to addition and convolution.

Proof. Clearly, $\mathcal{F}$ is an additive group. The convolution is defined for all $f, g \in \mathcal{F}$ by the embedding of the functions with different domains in $C(-\infty, \infty)$ extending them with zero to the left. If $f*g = 0$ then we have that $k_1 * f * k_2 * g = 0$. Therefore, either $k_1 * f = 0$ or $k_2 * g = 0$ which will imply that $f = 0$ or $g = 0$. Next we show that there exists an identity element with respect to the convolution. For this, let $f \in C_0[a, \infty)^{T_k}, f \neq 0$ for some $k \in C[0, \infty)$ with $0 \in supp(k * f)$. Define $h := (f * k)_{-a}$. Then $h \in C[0, \infty)$ with $0 \in supp(h)$. Define $f^{-1} := T_h^{-1}(k_{-a})$. Then for any $g \in \mathcal{F},\ g \in C_0[a, \infty)^{T_{k_1}}$ we have that

$$f * f^{-1} * g = T^{-1}_{k*h*k_1}(k * f * h * T_h^{-1}(k_{-a}) * k_1 * g) = T^{-1}_{k*h*k_1}(h_{-a} * k_{-a} * k_1 * g) = g.$$

Therefore $f * f^{-1}$ is the identity with respect to the convolution. Since, $f*(g+h) = f * g + f * h$ it finally follows that $\mathcal{F}$ is a field. □

In this construction we have only considered scalar valued functions. The vector valued generalized functions cannot form a field, since the convolution of two $X-$valued functions is not defined if the space X has no algebra structure. However, we can consider spaces $C_0([a, \infty), X)^{T_k}$ as above and we can define the convolution of a scalar valued generalized function with a vector valued generalized function. In this way, the vector valued generalized functions form a vector space over the field of scalar valued generalized functions.

Corollary 1.23 (The vector space of generalized functions). *Let X be a Banach space and let $\mathcal{F}$ be the field of generalized scalar functions. Then*

$$\mathcal{V} := \{f \in C([a, \infty), X)^{T_k} \ : a \in \mathbb{R}, k \in C[0, \infty) \ \ with \ \ 0 \in supp(k)\}$$

is a vector space over $\mathcal{F}$ where scalar multiplication of a vector $f \in \mathcal{V}$ with some $h \in \mathcal{F}$ is defined by

$$h * f := \widetilde{T}^{-1}_{k_1 * k_2}(k_1 * h * k_2 * f),\ where\, h \in C[a, \infty)^{T_{k_1}},\ f \in C[b, \infty)^{T_{k_2}}.$$

2. Asymptotic Laplace Transforms

The purpose of this chapter is to present asymptotic versions of the Laplace transform which have the benefit that all $f \in L^1_{loc}([0,\infty);X)$ and all generalized functions $f \in C([0,\infty),X)^{T_k}$ become transformable while, depending on the definition, all or almost all operational properties of the classical Laplace transform remain valid. The concept of asymptotic Laplace transforms has its roots in the theory of asymptotic power series of analytic functions; see, e.g., R. Remmert [21]. Asymptotic Laplace transforms were considered first by J.C. Vignaux [27] and further investigated by J.C Vignaux and M. Cotlar [28], W.A. Ditkin [9], L. Berg [6], Yu. I. Lyubich [18] and M. Deakin [8]. Since slightly less transparent definitions were used, the method and the scope of its applicability remained largely unnoticed. In 1999, G. Lumer and F. Neubrander [16] revisited the asymptotic Laplace transform and applied the concept to verify general theorems for distributions and hyperfunctions semigroups. Although the Lumer-Neubrander definition of the asymptotic Laplace transform was superior to earlier versions, it still had the disadvantage that the operational property $\hat{f}'(\lambda) = (\widehat{-tf(t)})$ did not extend to the asymptotic setting. To remedy this defect, G. Lumer and F. Neubrander gave yet another definition of the Asymptotic Laplace transform in [17]. In this chapter we will prove the operational properties of the asymptotic Laplace Transform as defined by G. Lumer and F. Neubrander in [17], a task that was omitted there. We will also propose a third definition of the asymptotic Laplace transform. Our definition is located somewhere "in-between" the two definitions proposed by G. Lumer and F. Neubrander, and enjoys the good properties of both: i.e., all operational Laplace transform properties are valid and Laplace transforms are "easily computed". As we have seen in Proposition 1.1, a function $f \in L^1_{loc}([0,\infty);X)$ is Laplace transformable if and only if its antiderivative $F : t \mapsto \int_0^t f(s)ds$ is exponentially bounded. In this case the Laplace transform $\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt = \lambda \int_0^\infty e^{-\lambda t} F(t)dt$ is analytic and bounded in a right half plane.

To state the first definition of the asymptotic Laplace transform given by G. Lumer and F. Neubrander in [16] we need the following notations. We denote by Σ a *post-sectorial region* in $\mathbb{C}$; i.e., an open subset of the right half plane such that for all angles $0 < \phi_0 < \frac{\pi}{2}$ there exists $r_0 > 0$ such that $\lambda = re^{i\phi} \in \Sigma$ for all $r > r_0$ and $|\phi| < \phi_0$.

We also denote by $\mathcal{A}(\Sigma, X)$ the set of all analytic, $X-$valued functions defined on some postsector Σ. Recall from Definition 1.12 that $a \approx_T 0$ if a is of exponential decay T; i.e.,

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|a(\lambda\| \leq -T.$$

Definition 2.1. *Let $0 < T < \infty$. The T-asymptotic Laplace transform (of type one) of $f \in L^1_{loc}([0,\infty), X)$ is given by*

$$\begin{aligned} \{f\}_1^T &:= \{r \in \mathcal{A}(\Sigma, X) : \; r(\lambda) - \int_0^T e^{-\lambda t} f(t)\, dt \approx_T 0\} \\ &= \int_0^T e^{-\lambda t} f(t)\, dt + \{0\}_1^T, \end{aligned}$$

where $\{0\}_1^T = \{a \in \mathcal{A}(\Sigma, X) : \;$ such that $a \approx_T 0\}$. The asymptotic Laplace transform (of type one) is defined as

$$\begin{aligned} \{f\}_1 &:= \bigcap_{T>0} \{f\}_1^T \\ &= \{r \in \mathcal{A}(\Sigma, X) : \; r(\lambda) - \int_0^T e^{-\lambda t} f(t)\, dt \approx_T 0 \; for\; all\; T > 0\}. \end{aligned}$$

Observe that, if $r \in \{f\}_1$ then, for all $T > 0$, there exists a_T such that

$$r(\lambda) - \int_0^T e^{-\lambda t} f(t)\, dt =: a_T(\lambda) \approx_T 0.$$

Clearly, since $\lambda \mapsto \int_0^T e^{-\lambda t} f(t)\,dt$ is entire and $r \in \mathcal{A}(\Sigma, X)$, it follows that $a_T \in \mathcal{A}(\Sigma, X)$. Since this definition does not include any restrictions on the growth of the elements in the asymptotic class outside of the real line, the equivalence classes are huge as soon as they are nonempty (which they always are, see the Existence and Uniqueness Theorem 2.11 below). The lack of growth restriction on $r \in \mathcal{A}(\Sigma, X)$ allows us to easily compute the asymptotic Laplace transform (of type one) of the functions $t \mapsto e^{t^2}$ and $t \mapsto e^{t^n}$ $(n \in \mathbb{N})$.

Example 2.2. Let Γ be the directed rectangle consisting of the paths $\Gamma_1 := \{t \in [0,T]\}, \Gamma_2(N) := \{t = T + ir; r \in [0,N]\}, \Gamma_3(N) := \{t = r + iN; r \in [0,T]\}$, and $\Gamma_4(N) := \{t = ir; r \in [0,N]\}$. Since $\int_{\Gamma_3(N)} e^{-\lambda t} e^{t^2}\,dt = \int_0^T e^{-\lambda(r+iN)} e^{r^2 - N^2 + 2irN}\,dr$ $\to 0$ as $N \to \infty$, it follows that

$$\int_0^T e^{-\lambda t} e^{t^2}\,dt = \lim_{N\to\infty} \left(-\int_{\Gamma_2(N)} + \int_{\Gamma_4(N)} \right) e^{-\lambda t} e^{t^2}\,dt.$$

Define

$$r(\lambda) := \lim_{N\to\infty} \int_{\Gamma_4(N)} e^{-\lambda t} e^{t^2}\,dt = i \int_0^\infty e^{-i\lambda r} e^{-r^2}\,dr.$$

Then $r(\lambda)$ is entire and $r(\lambda) - \int_0^T e^{-\lambda t} e^{t^2}\,dt = a(\lambda)$, where the remainder $\lambda \to a(\lambda)$ is given by $a(\lambda) := \lim_{N\to\infty} \int_{\Gamma_2(N)} e^{-\lambda t} e^{t^2}\,dt = i \int_0^\infty e^{-\lambda(T+ir)} e^{T^2 - r^2 + 2iTr}\,dr$. For $\lambda > 0$, the estimate $\|a(\lambda)\| \le e^{-\lambda T} e^{T^2} \int_0^\infty e^{-r^2}\,dr$ implies that $r_2 \in \{e^{t^2}\}_1$; that is

$$\{e^{t^2}\}_1 = i \int_0^\infty e^{-i\lambda r} e^{-r^2}\,dr + \{0\}_1, \tag{2.1}$$

where

$$\{0\}_1 = \{a \in \mathcal{A}(\Sigma, X) : \text{ such that } a \approx_T 0 \; for\ all\ T > 0\}. \tag{2.2}$$

Following almost the same argument, we can also compute the asymptotic Laplace transform of the function $t \mapsto e^{t^n}$ as follows. Let Γ be the directed path consisting of the paths $\Gamma_1 := \{t \in [0,T]\},\ \Gamma_2 := \{t = T + ir;\ r \in [0, T\sin(\pi/n)]\},\ \Gamma_3 := \{t =$

$re^{i\cos(\pi/n)}$; $r \in [0, \frac{T}{\cos(\pi/n)}]\}$. Then, by Cauchy's theorem, we have that

$$\int_{\Gamma_1} e^{-\lambda t + t^n}\, dt + \int_{\Gamma_2} e^{-\lambda t + t^n}\, dt + \int_{\Gamma_3} e^{-\lambda t + t^n}\, dt = 0.$$

Define

$$\begin{aligned} r(\lambda) := \lim_{T\to\infty} - \int_{\Gamma_3} e^{-\lambda t + t^n}\, dt &= -e^{i\frac{\pi}{n}} \int_0^\infty e^{-\lambda r e^{i\frac{\pi}{n}}} e^{r^n e^{i\pi}}\, dr \\ &= -e^{i\frac{\pi}{n}} \int_0^\infty e^{-\lambda r(\cos(\frac{\pi}{n}) + i\sin(\frac{\pi}{n}))} e^{-r^n}\, dr. \end{aligned}$$

Then r is entire and

$$r(\lambda) - \int_0^T e^{-\lambda t} e^{t^n}\, dt = a(\lambda),$$

where

$$a(\lambda) := \int_{\Gamma_2} e^{-\lambda t + t^n}\, dt + e^{i\frac{\pi}{n}} \int_{T/\cos(\pi/n)}^\infty e^{-\lambda r(\cos(\pi/n) + i\sin(\pi/n)) - r^n}\, dr.$$

To show that $r \in \{e^{t^2}\}_1$ we need to show that $a \approx_T 0$. Note that

$$\begin{aligned} |a(\lambda)| &= \left| \int_{\Gamma_2} e^{-\lambda t + t^n}\, dt + e^{i\frac{\pi}{n}} \int_{T/\cos(\pi/n)}^\infty e^{-\lambda r(\cos(\pi/n) + i\sin(\pi/n)) - r^n}\, dr \right| \\ &\le \left| i \int_0^{T\sin(\frac{\pi}{n})} e^{-\lambda(T+ir)} e^{(T+ir)^n}\, dr \right| + \left| e^{i\frac{\pi}{n}} \int_{\frac{T}{\cos(\frac{\pi}{n})}}^\infty e^{-\lambda r(\cos(\frac{\pi}{n}) + i\sin(\frac{\pi}{n})) - r^n}\, dr \right| \\ &= I_1 + I_2 \end{aligned}$$

For I_1 we have that

$$I_1 \le e^{-\lambda T} \int_0^{T\sin(\pi/n)} e^{(T+ir)^n}\, dr = M_T e^{-\lambda T}.$$

For I_2 we also have that

$$I_2 = \int_{\frac{T}{\cos(\frac{\pi}{n})}}^\infty e^{-\lambda r\cos(\frac{\pi}{n}) - r^n}\, dr \le \int_{\frac{T}{\cos(\frac{\pi}{n})}}^\infty e^{-\lambda r\cos(\frac{\pi}{n})}\, dr = \frac{e^{-\lambda T}}{\lambda\cos(\frac{\pi}{n})}.$$

Thus, $|a(\lambda)| \le M_T e^{-\lambda T} + \frac{e^{-\lambda T}}{\lambda\cos(\frac{\pi}{n})} \le e^{-\lambda T}(M_T + \frac{1}{\lambda\cos(\frac{\pi}{n})})$. Therefore $a \approx_T 0$ and $r \in \{e^{t^n}\}_1$. In particular

$$\{e^{t^n}\} = -e^{i\frac{\pi}{n}} \int_0^\infty e^{-\lambda r(\cos(\frac{\pi}{n}) + i\sin(\frac{\pi}{n}))} e^{-r^n}\, dr + \{0\}_1, \tag{2.3}$$

where $\{0\}_1$ is defined by (2.2).

As we will see later in this chapter, with this definition all but one of the operational properties of the Laplace transform extend to the asymptotic Laplace transform. The one operational property of the Laplace transform that does not extend is $\hat{f}'(\lambda) \neq (\widehat{-tf(t)})$.

Example 2.3. Let $X = \mathbb{C},\ 1 < \alpha < 2$, and $r(\lambda) := e^{-\lambda^\alpha} \sin(e^{\lambda^2})$. Clearly, we have that $r \approx_T 0$ for any $T > 0$. Thus, $r \in \{0\}_1$. But

$$\begin{aligned}\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln |r'(\lambda)| &= \limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \left| -e^{-\lambda^\alpha} \lambda^{\alpha-1} \sin(e^{\lambda^2}) + 2\lambda e^{\lambda^2-\lambda^\alpha} \cos(e^{\lambda^2}) \right| \\ &= +\infty.\end{aligned}$$

Therefore, $r' \notin \{0\}_1 = \{-t0\}_1$. Hence, in general $\{f\}_1' \neq \{-tf\}_1$.

To repair this defect, in the next definition of the asymptotic Laplace transform given by G. Lumer and F. Neubrander, a smaller class of analytic functions appears; namely, functions of minimal exponential type on some postsector Σ.

We say that an analytic function $u : \mathbb{C} \supset \Sigma \to X$ is of *minimal exponential type on a postsector* Σ if for every sector

$$\Sigma_\phi = \left\{ \lambda : |arg(\lambda)| \leq \phi < \frac{\pi}{2}, \lambda \neq 0 \right\}$$

and all $\omega > 0$ there exist $M > 0$ such that $\|u(\lambda)\| \leq Me^{\omega|\lambda|}$ in $\Sigma \cap \Sigma_\phi$. We denote by $O_\Sigma(X)$ the set of $X-$*valued* analytic functions of minimal exponential type on a given postsector Σ. Also, we denote by $O(\Sigma, X) := \bigcup_\Sigma O_\Sigma(X)$ the set of analytic functions that are of minimal exponential type on some postsector Σ. Clearly, $O(\Sigma, X)$ and $O_\Sigma(X)$ are closed under addition, multiplication and scalar multiplication; i.e., if $u_1 \in O_{\Sigma_1}(X)$ and $u_2 \in O_{\Sigma_2}(X)$ then $u_3 = u_1 + u_2 \in O_{\Sigma_3}(X)$ and $u_3 = u_1 \cdot u_2 \in O_{\Sigma_3}(X)$ where $\Sigma_3 \subset \Sigma_1 \cap \Sigma_2$. With these notations at hand we can state the second definition of the asymptotic Laplace transform given in [17].

Definition 2.4. *For $0 < T < \infty$, the T-asymptotic Laplace transform (of type two) of $f \in L^1_{loc}([0,\infty), X)$ is given by the set of analytic functions r defined on some post-sectorial region Σ with values in X which are of the minimal exponential type and are asymptotically equal to the finite Laplace transform of f; i.e.,*

$$\begin{aligned} \{f\}_2^T &:= \{r \in O(\Sigma, X) : r(\lambda) - \int_0^T e^{-\lambda t} f(t)\,dt \approx_T 0\} \\ &= \int_0^T e^{-\lambda t} f(t)\,dt + \{0\}_2^T, \end{aligned}$$

where $\{0\}_2^T = \{a \in O(\Sigma, X)$ such that $a \approx_T 0\}$. The asymptotic Laplace transform (of type two) is defined as the intersection of the sets $\{f\}_2^T$; i.e.,

$$\begin{aligned} \{f\}_2 &:= \cap_{T>0}\{f\}_2^T \\ &= \{r \in O(\Sigma, X) : r(\lambda) - \int_0^T e^{-\lambda t} f(t)\,dt \approx_T 0 \text{ for all } T > 0\}. \end{aligned}$$

Observe again that if $r \in \{f\}_2$, then $r \in O(S, X)$. Let $a_T(\lambda) := r(\lambda) - \int_0^T e^{-\lambda t} f(t)\,dt \approx_T 0$. Since $\lambda \mapsto \int_0^T e^{-\lambda t} f(t)\,dt$ is entire and bounded for $Re\lambda > 0$, it follows that $a_T \in O(\Sigma, X)$. That is, as in Definition 2.1, the remainder a_T inherits the regularity of r.

As we will see in Proposition 2.13 below, the fact that the elements in the asymptotic equivalence class of $\int_0^T e^{-\lambda t} f(t)\,dt$ are assumed to be of minimal exponential type is strong enough to ensure the validity of the operational property $\{f\}' = \{-tf\}$. As a first indication why this might be true consider the function $a(\lambda) = e^{-\lambda^\alpha}\sin(e^{\lambda^2})$ discussed in the (counter) example above. Then a is not of minimal exponential type on any cut sector. For $\lambda = re^{i\theta}$ we have that

$$\begin{aligned} a(\lambda) &= e^{-r^\alpha e^{i\theta}} \frac{e^{ir^2 e^{i2\theta}} - e^{-ir^2 e^{i2\theta}}}{2} \\ &= e^{-r^\alpha(\cos(\alpha\theta) + i\sin(\alpha\theta))} \frac{e^{ir^2(\cos(2\theta)+i\sin(2\theta))} - e^{-ir^2(\cos(2\theta)+i\sin(2\theta))}}{2} \\ &= e^{-r^\alpha(\cos(\alpha\theta) + i\sin(\alpha\theta))} \frac{e^{ir^2\cos(2\theta) - r^2\sin(2\theta))} - e^{-ir^2\cos(2\theta) + r^2\sin(2\theta))}}{2} \end{aligned}$$

Using the equality $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2Re(z_1 \cdot \overline{z_2})$ we obtain that

$$|a(\lambda)| = e^{-r^\alpha \cos(\alpha\theta)} \sqrt{(e^{-r^2 \sin 2\theta})^2 + (e^{r^2 \sin 2\theta})^2 - 2\cos^2(r^2 \cos 2\theta))}$$

Therefore, we have

$$\begin{aligned} \frac{|a(\lambda)|}{e^{\omega r}} &= e^{-r^\alpha \cos(\alpha\theta) - \omega r} \sqrt{(e^{-r^2 \sin 2\theta})^2 + (e^{r^2 \sin 2\theta})^2 - 2\cos^2(r^2 \cos 2\theta))} \\ &= e^{-r^\alpha \cos(\alpha\theta) - \omega r + r^2 \sin 2\theta} \sqrt{\frac{(e^{-r^2 \sin 2\theta})^2}{(e^{r^2 \sin 2\theta})^2} + 1 - \frac{2\cos^2(r^2 \cos 2\theta))}{(e^{r^2 \sin 2\theta})^2}} \end{aligned}$$

Assuming $\frac{\pi}{2} > \theta > 0$ we obtain that

$$\lim_{r \to \infty} \frac{|a(\lambda)|}{e^{\omega r}} = \infty.$$

Thus, a is not of minimal exponential type on any cut sector and, therefore, Example 2.3 is not an example disproving $\{f\}_2' = \{-tf\}_2$. In fact, as we will see below, this operation property holds for $\{\cdot\}_2$ and for $\{\cdot\}_3$.

Before we start proving the operational properties of the asymptotic Laplace transform $\{f\}_1$ and $\{f\}_2$, let us add yet another definition of the asymptotic Laplace transform. This definition is based on our observation that the use of postsectors by G. Lumer and F. Neubrander is not necessary in order for these operational properties to be valid. To state the third definition we need the following notation. Denote by S_ϕ an open cut sector on the right half plane; i.e.,

$$S_\phi := \{\lambda : \lambda = re^{i\theta}, \ \ r > r_0, \ |\theta| < \phi < \frac{\pi}{2}\}.$$

We say that an analytic function $u : S_\phi \to X$ is of *minimal exponential type on* S_ϕ if for all $\omega > 0$ there exists $M > 0$ such that $\|u(\lambda)\| \le Me^{\omega|\lambda|}$ in S_ϕ. We denote by $O_{S_\phi}(X)$ the set of $X-$valued analytic functions of minimal exponential type on a given open cut sector S_ϕ. Also, we denote by $O(S, X) := \bigcup_{S_\phi} O_{S_\phi}(X)$ the set of analytic functions that are of minimal exponential type on some open cut sector S_ϕ. Again, $O(S,X)$ and $O_{S_\phi}(X)$ are closed under addition and scalar multiplication; i.e., if $u_1 \in O_{S_1}(X)$ and $u_2 \in O_{S_2}(X)$ then $u_3 = u_1 + u_2 \in O_{S_3}(X)$ and $u_3 = u_1 \cdot u_2 \in O_{S_3}(X)$ where $S_3 = S_1 \cap S_2$.

Definition 2.5. *For $0 < T < \infty$, the T-asymptotic Laplace transform (of type three) of $f \in L^1_{loc}([0,\infty), X)$ is given by the set of analytic functions r defined on some sectorial region S with values in X which are of the minimal exponential type and are asymptotically equal to the finite Laplace transform of f; i.e.,*

$$\begin{aligned} \{f\}_3^T &:= \{r \in O(S,X) : r(\lambda) - \int_0^T e^{-\lambda t} f(t)\,dt \approx_T 0\} \\ &= \int_0^T e^{-\lambda t} f(t)\,dt + \{0\}_3^T, \end{aligned}$$

where $\{0\}_3^T = \{a \in O(S,X)$ such that $a \approx_T 0\}$. The asymptotic Laplace transform (of type three) is defined as the intersection of the sets $\{f\}_3^T$; i.e.,

$$\begin{aligned} \{f\}_3 &:= \cap_{T>0}\{f\}_3^T \\ &= \{r \in O(S,X) : r(\lambda) - \int_0^T e^{-\lambda t} f(t)\,dt \approx_T 0 \text{ for all } T > 0\}. \end{aligned}$$

We remark here that with these definitions we have the following inclusions

$$\{f\}_2 \subset \{f\}_3 \text{ and } \{f\}_2 \subset \{f\}_1. \tag{2.4}$$

Since the functions of minimal exponential type play an important role in what follows, we give some examples of such functions.

Example 2.6. (a) The functions

$$u(\lambda) = \lambda^n e^{c\lambda^\gamma}$$

are of minimal exponential type in any postsector contained in the region $\{\lambda : |\lambda| > 1\} \cap \{\lambda : Re\lambda > 0\}$ for any $n \in \mathbb{R}$, $\gamma \in [-1, 1)$ and $c \in \mathbb{R}$. Notice that u is also of minimal exponential type for $\gamma = 1$ provided that $c \leq 0$.

To see this, it is enough to show that the two factors are of minimal exponential type. Clearly, for $n \leq 0$, λ^n is of minimal exponential type since it is bounded for $|\lambda| \geq \varepsilon > 0$. For $n > 0$, consider the positive real function $g(x) = \frac{x^n}{e^{\omega x}}(x \geq 0)$ for some $\omega > 0$. Then $g(0) = 0 = \lim_{x\to\infty} g(x) = 0$ and it follows from $g'(x) = [nx^{n-1} - \omega x^n]e^{-\omega x}$ that g attains its maximum at $x = \frac{n}{\omega}$. Since $\frac{n}{e^n} \leq e^{-1}$ for all n it follows that $x^n \leq \frac{n}{\omega e^n}e^{\omega x} \leq \frac{1}{\omega}e^{-1}e^{\omega x}$. Therefore,

$$|\lambda|^n \leq \frac{1}{\omega}e^{-1}e^{\omega|\lambda|} \text{ for } |\lambda| \geq \epsilon > 0.$$

Next, the function $u(\lambda) = e^{c\lambda^\gamma}$ is of minimal exponential type for $\gamma \in [-1, 1)$. We have that $|u(\lambda)| = e^{c|\lambda|^\gamma \cos(\gamma \arg(\lambda))} \leq e^{c|\lambda|^\gamma}$. Since $\frac{e^{c|\lambda|^\gamma}}{e^{\omega|\lambda|}} = e^{|\lambda|[c|\lambda|^{\gamma-1}-\omega]}$ and $\lim_{|\lambda|\to\infty} |\lambda|\,[c\,|\lambda|^{\gamma-1} - \omega] = -\infty$ it follows that for $|\lambda| > 1$ there exists a $M > 0$ such that $e^{|\lambda|[c|\lambda|^{\gamma-1}-\omega]} \leq e^M$. Thus $|u(\lambda)| \leq e^M e^{\omega|\lambda|}$ for $|\lambda| > 1$.

(b) The function

$$u(\lambda) := \ln(\lambda) := \ln|\lambda| + iarg(\lambda)$$

with $-\pi < arg(\lambda) < \pi$ (the principal branch of the logarithm) is of minimal exponential type in any postsector contained in the region $\{\lambda : |\lambda| > 1\} \cap \{\lambda : Re\lambda > 0\}$. To see this observe that $|\ln(\lambda)| \leq \ln(|\lambda|) + |arg(\lambda)| \leq |\lambda| + \pi$ for $|\lambda| \geq 1$. Thus, for all $\omega > 0$ there exits $M > 0$ such that $|\ln\lambda| \leq Me^{\omega|\lambda|}$ for all $|\lambda| \geq 1$.

(c) The function

$$u(\lambda) := e^{-\lambda\ln\lambda}$$

is of minimal exponential type in any postsector contained in the region $\{\lambda : |\lambda| > 1\} \cap \{\lambda : Re\lambda > 0\}$. To see this,let $\omega > 0$ and $\lambda = re^{i\alpha}$. Then

$$Re(-\lambda\ln\lambda) = Re[(-r\cos\alpha - ir\sin\alpha)(\ln r + i\alpha)] = -r\cos\alpha\ln r + r\alpha\sin\alpha$$

and therefore

$$\frac{|u(\lambda)|}{e^{\omega|\lambda|}} = e^{-r\cos\alpha\ln r + r\alpha\sin\alpha - \omega r}.$$

The real function $g(\alpha) := -r\cos\alpha\ln r + r\alpha\sin\alpha - \omega r$ is an even function so, it will be enough to study g for $\alpha \in [0, \theta]$ where $0 < \theta < \frac{\pi}{2}$. Since $g'(\alpha) = r\sin\alpha\ln r + r\sin\alpha + r\alpha\cos\alpha > 0$, it follows that $g(\alpha)$ is increasing for $\alpha \in [0, \theta]$. Because g is even it follows that $\max_{|\alpha|\le\theta} g(\alpha) = g(\theta)$. Finally, since $\lim_{r\to\infty}(-r\cos\theta\ln r + r\theta\sin\theta - \omega r) = -\infty$, it follows that there exists a $M > 0$ such that

$$e^{-r\cos\alpha\ln r + r\alpha\sin\alpha - \omega r} \le e^{g(\theta)} \le e^{M}$$

for all $\alpha \in [-\theta, \theta]$.

(d) The function

$$u(\lambda) := e^{-\lambda^2}$$

is of minimal exponential type on the sector $S_{\frac{\pi}{4}} := \{\lambda : \lambda = re^{i\theta}, \ \ |\theta| < \frac{\pi}{4}\}$, but is not of minimal exponential type on any postsector Σ. To see this observe that $|u(\lambda)| = e^{-r^2\cos 2\alpha}$. It follows that u is of minimal exponential type for $|\alpha| < \frac{\pi}{4}$. If $\alpha > \frac{\pi}{4}$, then $u(\lambda)$ is not of minimal exponential type.

We should note here that, by definition, all examples of analytic functions of minimal exponential type on some postsector are of minimal exponential type on all open cut sectors with a sufficiently large cut. Next we will prove a useful lemma which describes the behavior of functions of minimal exponential type which are also of exponential decay T.

Lemma 2.7. *Let $a \in O(S_\phi, X)$. The following are equivalent*

(a) $a \approx_T 0$; *i.e.,* $\limsup_{\lambda\to\infty} \frac{1}{r}\ln\|a(\lambda)\| \le -T$,

(b) $\limsup_{r\to\infty} \frac{1}{r}\ln\|a(re^{i\theta})\| \le -T\cos\theta$ *for all* $-\phi < \theta < \phi$.

Proof. Clearly, (b) implies (a). To show that (a) implies (b) we define the Phragmén-Lindelöf function h by

$$h(\phi) := \limsup_{r\to\infty} \frac{1}{r} \ln \|a(re^{i\phi})\| < \infty. \tag{2.5}$$

To complete this proof we need the following lemma taken form E.C. Titchmarch [24], Chapter 5, Section 5.7.

Lemma 2.8. *Let $\alpha < \phi_1 < \phi_2 < \beta$, and $\phi_2 - \phi_1 < \pi$, and let $h(\phi_1) \le h_1, \quad h(\phi_2) \le h_2$. Let $H(\phi)$ be the function of the form $a\cos\phi + b\sin\phi$ which takes the values h_1, h_2 at ϕ_1, ϕ_2. i.e.,*

$$H(\phi) = \frac{h_1 \sin\phi_2 - h_2 \sin\phi_1}{\sin(\phi_2 - \phi_1))} \cos\phi + \frac{h_1 \cos\phi_2 - h_2 \cos\phi_1}{\sin(\phi_1 - \phi_2))} \sin\phi,$$

which takes the values h_1, h_2 at ϕ_1, ϕ_2. Then

$$h(\phi) \le H(\phi) \ \ (for\ all\ \phi\ such\ that\ \phi_1 \le \phi \le \phi_2).$$

Using this lemma with $\phi_1 = 0, \ \phi_2 = \frac{\pi}{2} - \mu$, where $0 < \mu < \frac{\pi}{2}$, we have that $h(\phi_1) = h(0) \le -T$; i.e., $h_1 = -T$ and $h(\phi_2) = \limsup_{r\to\infty} \frac{1}{r} \ln \|a(re^{i\phi_2})\| \le \omega$, for any $\omega > 0$; i.e., $h_2 = 0$. Therefore, $a = H(0) = -T$ and $0 = H(\frac{\pi}{2} - \mu)$ which gives $b = \frac{T\cos(\frac{\pi}{2}-\mu)}{\sin(\frac{\pi}{2}-\mu)}$. Hence,

$$h(\phi) \le H(\phi) = -T\cos\phi + \frac{T\cos(\frac{\pi}{2} - \mu)}{\sin(\frac{\pi}{2} - \mu)} \sin\phi,$$

for all $0 < \phi < \pi/2 - \mu$. Thus, taking the limit as $\mu \to 0$ we have that

$$h(\phi) \le -T cos\phi \quad \text{for } |\phi| < \pi/2.$$

This finishes the proof of the Lemma 2.7. ☐

Lemma 2.9. *Let $f \in L^1_{loc}([0,\infty), X)$ such that $\hat{f}(\lambda)$ exists for $\lambda > \omega > 0$. Then $\hat{f} \in \{f\}_i$ for $i = 1, 2, 3$. Moreover* (*a*) $\hat{f}$ *is of minimal exponential type on the half plane* $Re\lambda > \omega+\varepsilon$*, for any* $\varepsilon > 0$ *and*

(*b*) $a(\lambda) := \int_T^\infty e^{-\lambda t} f(t)dt$ *is of minimal exponential type on the half plane* $Re\lambda > \omega + \varepsilon$ *and* $a \approx_T 0$ *for any* $T > 0$.

Proof. (a) Let $F(t) := \int_0^t f(s)ds$. Since $\hat{f}(\lambda)$ exists for $Re\lambda > \omega > 0$ if and only if there exist $M > 0$ such that $\|F(t)\| \leq Me^{\omega t}$, integration by parts yields

$$\int_0^\infty e^{-\lambda t} f(t)\, dt = +\lambda \int_0^\infty e^{-\lambda t} F(t)\, dt. \tag{2.6}$$

Therefore, $\left\|\hat{f}(\lambda)\right\| \leq |\lambda| \int_0^\infty e^{-Re\lambda t} Me^{\omega t} dt \leq M\frac{|\lambda|}{Re\lambda - \omega}$. This shows that $\hat{f}$ is of minimal exponential type on the half plane $\{\lambda : Re\lambda > \omega + \varepsilon\}$. Therefore, $\hat{f} \in \{f\}_2 \subset \{f\}_3$ and $\hat{f} \in \{f\}_1$.

To prove (*b*) observe that $a(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\chi_{(T,\infty)}(t)\, dt$. Therefore, a is of minimal exponential type on the half plane $Re\lambda > \omega + \varepsilon$. By Lemma 1.13 it follows that $a \approx_T 0$. □

Remark 2.10. *Let* $f \in L^1_{loc}([0,\infty), X)$*. Then the finite Laplace transform*

$$\hat{f}_T(\lambda) := \int_0^\infty e^{-\lambda t} f(t)\chi_{[0,T]}(t)dt = \int_0^T e^{-\lambda t} f(t)\, dt$$

is bounded on the half plane $\{\lambda : Re\lambda > 0\}$*. In particular* $\hat{f}_T(\lambda) \in \{f\}_i^T$ *for* $i = 1, 2, 3$.

In the following we will prove that all operational properties of the Laplace transform extend to the asymptotic Laplace transform. Clearly, $f \to \{f\}_i$ is linear: i.e.,$\{(f+g)\}_i = \{f\}_i + \{g\}_i$ and $\{cf\}_i = c\{f\}_i$ for $f, g \in L^1_{loc}([0,\infty), X)$, $c \in \mathbb{C}$ and $i = 1, 2, 3$. Also notice that $\approx_T$ defines an equivalence relation on $O(\Sigma, X)$. Furthermore, if $r \in O(\Sigma, X)$ and $q \in O(\Sigma, C)$ with $r \approx_T 0$ and $q \approx_S 0$, then $qr \approx_{T+S} 0$. The same holds for $O(S, X)$ and $\mathcal{A}(\Sigma, X)$.

Theorem 2.11 (Existence and Uniqueness). [1] *Let* $f, g \in L^1_{loc}([0,\infty); X)$ *and* $0 < T \leq \infty$. *Then for* $i = 1, 2, 3$

(a) $\emptyset \neq \{f\}_i \subset \{f\}_i^T$;

(b) *if* $\hat{f}$ *exists, then* $\hat{f} \in \{f\}_i$;

(c) $\{f\}_i^T \cap \{g\}_i^T \neq \emptyset$ *if and only if* $f = g$ *a.e. on* $[0, T]$.

Proof. (a) By definition $\{f\}_i \subset \{f\}_i^T$. We remark here that in order to prove statements (a) and (b) it is sufficient to prove them for the smaller equivalence class; i.e., if (a) and (b) hold for $\{f\}_2$ then they hold for $\{f\}_3$ and $\{f\}_1$. For $f \in L^1_{loc}([0,\infty), X)$ and $0 < T < \infty$ define $f_T := f \cdot \chi_{[0,T]}$. Then the finite Laplace transform $\hat{f}_T(\lambda) = \int_0^T e^{-\lambda t} f(t)\,dt = \int_0^\infty f_T(t)\,dt$ is an entire function of minimal exponential type (even bounded on $Re\lambda > 0$) and

$$\{f\}_2^T = \hat{f}_T + \{0\}_2^T \neq \emptyset, \tag{2.7}$$

where $\{0\}_2^T := \{a \in O(\Sigma; X) : a \approx_T 0\}$. Let $0 < T' < T < \infty$, $F(t) := \int_0^t f(s)\,ds$ $(t \geq 0)$ and $h(\lambda) := \int_{T'}^T e^{-\lambda t} f(t)\,dt = e^{-\lambda T} F(T) - e^{-\lambda T'} F(T') + \lambda \int_{T'}^T e^{-\lambda t} F(t)\,dt$. Since F is bounded on $[T', T]$, it follows that $h \approx_{T'} 0$. Thus

$$\{f\}_2^T = \hat{f}_{T'} + h + \{0\}_2^T \subset \hat{f}_{T'} + h + \{0\}_2^{T'} = \hat{f}_{T'} + \{0\}_2^{T'} = \{f\}_2^{T'}.$$

This shows that

$$\{f\}_2 = \cap_{T>0} \{f\}_2^T \subset \{f\}_2^T \subset \{f\}_2^{T'} \text{ if } 0 < T' < T < \infty.$$

[1]This theorem is essentially due to Vignaux [27] who extended Ritt's Theorem about asymptotic series to asymptotic integrals; see, e.g., R. Remmert [21], 9.6.4 and also J.C. Vignaux and M. Cotlar [28], L. Berg [6]. In the present form, the result was proved for $\{\,\}_1$ by G. Lumer and F. Neubrander in [16].

To see that the asymptotic Laplace transform $\{f\}_2$ of f is well defined (i.e., $\{f\} \neq \emptyset$ for all $f \in L^1_{loc}((0,\infty),X)$), define

$$r(\lambda) := \lambda \int_0^\infty e^{-\lambda t}(1 - e^{-\frac{d(\lambda)}{G(t)}})\, F(t)\, dt, \tag{2.8}$$

where $G(t) := \max\{\|F(t)\|, 1\}$, and $\lambda \to d(\lambda)$ is an analytic function to be chosen below such that $r \in \{f\}_T$ for all $T > 0$. Since $|1 - e^{-z}| = |z \int_0^1 e^{-zt}\, dt| \le |z|$ for all $z \in C$ with $Re(z) \ge 0$, it follows that $\|e^{-\lambda t}(1 - e^{-\frac{d(\lambda)}{G(t)}})\, F(t)\| \le e^{-Re(\lambda)t}|d(\lambda)|$ for all $t \ge 0$ and $\lambda \in \Omega := \{\lambda \in C : Re(\lambda) > 0 \text{ and } Re(d(\lambda)) \ge 0\}$. If we assume that Ω contains a post-sectorial region and that $d(\lambda)$ is of minimal exponential type it follows that $r \in O(\Sigma, X)$ from the growth estimate

$$\|r(\lambda)\| \le \frac{1}{Re(\lambda)}|\lambda d(\lambda)|.$$

Consider $a(\lambda) = r(\lambda) - \int_0^T e^{-\lambda t} f(t)\, dt = r(\lambda) + \lambda e^{-\lambda T} F(T) - \lambda \int_0^T e^{-\lambda t} F(t) dt$ $= \lambda \int_T^\infty e^{-\lambda t}(1 - e^{-\frac{d(\lambda)}{G(t)}}) F(t)\, dt - \lambda \int_0^T e^{-\lambda t} e^{-\frac{d(\lambda)}{G(t)}} F(t)\, dt + \lambda e^{-\lambda T} F(T) = a_1(\lambda) - a_2(\lambda) + a_3(\lambda)$. Then $r \in \{f\}$ if and only if $a \approx_T 0$ for all $T > 0$. Clearly, $a_3(\lambda) = \lambda e^{-\lambda T} F(T)$ satisfies $a_3 \approx_T 0$ for all $T > 0$. Consider $a_2(\lambda) = \lambda \int_0^T e^{-\lambda t} e^{-\frac{d(\lambda)}{G(t)}} F(t)\, dt$. If $\lambda \in \Omega \cap R_+$, then $\|a_2(\lambda)\| \le MT\lambda e^{-Re(d(\lambda))/M}$, where $M = \sup_{0 \le t \le T} \|G(t)\|$, and therefore $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|a_2(\lambda)\| \le -\liminf_{\lambda\to\infty} \frac{1}{\lambda} Re(d(\lambda))$. Thus, to ensure that $a_2 \approx_T 0$ for all $T > 0$ we assume that $\frac{1}{\lambda} Re(d(\lambda)) \to \infty$ as $\lambda \to \infty$. Finally, consider $a_3(\lambda) = \lambda \int_T^\infty e^{-\lambda t}(1 - e^{-\frac{d(\lambda)}{G(t)}}) F(t)\, dt$. Then $\|a_3(\lambda)\| \le |d(\lambda)| e^{-\lambda T}$ for all $\lambda \in \Omega \cap R_+$. This shows that $a_3 \approx_T 0$ for all $T > 0$ if $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln(|d(\lambda)|) \le 0$. The considerations above show that r as defined in 2.8 is an element of $\{f\}$ if the damping function d is analytic on a post-sectorial subregion Σ of the open right half-plane, $Re(d(\lambda)) \ge 0$ for all $\lambda \in \Sigma$, $\frac{1}{\lambda} Re(d(\lambda)) \to \infty$ as $\lambda \to \infty$, and $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln(|d(\lambda)|) \le 0$.

An example of a damping function with these properties is given by

$$d(\lambda) = \lambda \ln(\lambda).$$

From the Example 2.6 it follows that $d(\lambda)$ is of minimal exponential type. Now for $\lambda = re^{i\theta}$ we have that $Re(d(\lambda)) = r \ln r \cos\theta - r\theta \sin\theta$ which implies that $Re(d(\lambda)) \geq 0$ for all $\lambda \in \Sigma$, $\frac{1}{\lambda} Re(d(\lambda)) \to \infty$ as $\lambda \to \infty$ and that $\Sigma = \{\lambda \in C : Re(\lambda) > 1\} \cap \{\lambda = re^{i\alpha} : r > e^{\theta \tan(\theta)}, -\pi/2 < \theta < \pi/2\}$. Since $|d(\lambda)| = \sqrt{r^2(\ln r)^2 + r^2\theta^2}$ it follows that $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln(|d(\lambda)|) = 0$, so $d(\lambda) = \lambda \ln \lambda$ satisfies all the required conditions.

(b) Follows from Lemma 2.9.

(c) Clearly, the approximate Laplace transform $\{f\}_i^T = \int_0^T e^{-\lambda t} f(t)\, dt + \{0\}_i^T$ of any $f \in L^1_{loc}([0,\infty), X)$ comprises a "large" set of analytic functions. Nevertheless, the sets are disjoint for different f. If (c) holds for $\{f\}_1$ then it holds for $\{f\}_2$. To see this let $0 < T \leq \infty$, and $r \in \{f\}_1^T \cap \{g\}_1^T$. Then there exist $a, b \approx_T 0$ such that

$$r(\lambda) = \int_0^T e^{-\lambda t} f(t)\, dt + a(\lambda) = \int_0^T e^{-\lambda t} g(t)\, dt + b(\lambda)$$

for all $\lambda > \omega$. Thus, $\hat{h}_T(\lambda) = \int_0^T e^{-\lambda t} h(t)\, dt = c(\lambda)$, where $h := f - g$, $c := b - a$, and $c \approx_T 0$. Now statement (c) follows from the Proposition 1.16; see also B. Bäumer and F. Neubrander [4]. The proof for $\{f\}_3$ is identical with the one above. $\square$

The usefulness of the Laplace transform in applications to differential and integral equations is due to the fact that it maps differentiation, integration, and more generally, convolution onto multiplication. Next it will be shown that these crucial operational properties extend to the asymptotic Laplace transform as well. Recall that if $f \in L^1_{loc}([0,\infty), X)$ and , $g \in L^1_{loc}([0,\infty), \mathbb{R})$ then

$$f * g : t \to \int_0^t f(t-s) g(s)\, ds \quad (t \geq 0)$$

denotes the *convolution* of f and g. For all scalar, locally Lebesgue integrable functions functions g, h and locally Bochner integrable functions f the convolution

$f*g$ is in $L^1_{loc}[0,\infty)$, $f*g = g*f$, and $(g*h)*f = g*(h*f)$.[2] To simplify notation we define $\{f\}_i^\infty := \{f\}$.

Proposition 2.12. *Let $0 < T \le \infty$, $i \in \{1,2,3\}$, and $f \in L^1_{loc}([0,\infty);X)$. If f is absolutely continuous and f' exists a.e., then*

$$\{f'\}_i^T = \lambda\{f\}_i^T - f(0).$$

If $g \in L^1_{loc}([0,\infty);\mathbb{C})$, then

$$\{f*g\}_i^T = \{f\}_i^T \cdot \{g\}_i^T.$$

Proof. Let $0 < T < \infty$. If $q \in \{f'\}_i^T$, then there exist $\omega > 0$ and $a \approx_T 0$ such that $q(\lambda) = \int_0^T e^{-\lambda t} f'(t)\,dt + a(\lambda) = \lambda \left[\int_0^T e^{-\lambda t} f(t)\,dt + \frac{1}{\lambda}e^{-\lambda T} f(T) + \frac{1}{\lambda}a(\lambda)\right] - f(0)$ for all $\lambda > \omega$. Thus $q \in \lambda\{f\}_i^T - f(0)$. Conversely, if $r \in \lambda\{f\}_i^T - f(0)$ then there exist $\omega > 0$ and $a \approx_T 0$ such that $r(\lambda) = \lambda\left[\int_0^T e^{-\lambda t} f(t)\,dt + a(\lambda)\right] - f(0) = \int_0^T e^{-\lambda t} f'(t)\,dt - e^{-\lambda T} f(T) + \lambda a(\lambda)$ for all $\lambda > \omega$. Thus, $r \in \{f'\}_i^T$. This shows that $\{f'\}_i^T = \lambda\{f\}_i^T - f(0)$ for all $0 < T < \infty$ and thus also for $T = \infty$.

To prove the other statement, let $0 < T < \infty$, $r(\lambda) = \int_0^T e^{-\lambda t} f(t)\,dt + a(\lambda)$, and $q(\lambda) = \int_0^T e^{-\lambda s} g(s)\,ds + b(\lambda)$, where $a, b \approx_T 0$. Define $w(\lambda) := a(\lambda)\int_0^T e^{-\lambda s} g(s)\,ds + b(\lambda)\int_0^T e^{-\lambda t} f(t)\,dt + a(\lambda)b(\lambda)$. Then $w \approx_T 0$ and

$$\begin{aligned}
r(\lambda)q(\lambda) &= \int_0^T \int_0^T e^{-\lambda(t+s)} f(t)g(s)\,dtds + w(\lambda) \\
&= \int_0^T \int_s^{T+s} e^{-\lambda t} f(t-s)g(s)\,dtds + w(\lambda) \\
&= \int_0^T \int_0^t e^{-\lambda t} f(t-s)g(s)\,dsdt + \int_T^{2T} \int_{t-T}^T e^{-\lambda t} f(t-s)g(s)\,dsdt + w(\lambda) \\
&= \int_0^T e^{-\lambda t}(f*g)(t)\,dt + e^{-\lambda T}\int_0^T e^{-\lambda t}\int_t^T f(t+T-s)g(s)\,dsdt + w(\lambda).
\end{aligned}$$

Thus, $\{f\}_i^T \cdot \{g\}_i^T \subset \{f*g\}_i^T$. If $r \in \{f*g\}_i^T$, then $r(\lambda) = \int_0^T e^{-\lambda t}(f*g)(t)\,dt + a(\lambda)$ for some $a \approx_T 0$. Since $\int_0^T e^{-\lambda t}(f*g)(t)\,dt = \left(\int_0^T e^{-\lambda t} f(t)\,dt\right)\left(\int_0^T e^{-\lambda s} g(s)\,ds\right) -$

[2] For a proof of the scalar case, see N. Dunford and J. Schwartz [11], VIII.1.24, 633-636.

$e^{-\lambda T}\int_0^T e^{-\lambda t}\int_t^T f(t+T-s)g(s)\,dsdt$ it follows that $r\in\{f\}_i^T\{g\}_i^T+\{0\}_i^T$. This shows that $\{f*g\}_i^T\subset\{f\}_i^T\{g\}_i^T+\{0\}_i^T\subset\{f*g\}_i^T+\{0\}_i^T=\{f*g\}_i^T$. Thus, $\{f*g\}_i^T=\{f\}_i^T\cdot\{g\}_i^T+\{0\}_i^T$, or $\{f\}_i^T\cdot\{g\}_i^T=\{f*g\}_i^T+\{0\}_i^T=\{f*g\}_i^T$. $\square$

Proposition 2.13. *Let $f\in L^1_{loc}([0,\infty),X)$ and $i\in\{2,3\}$. Then*

$$\{-tf\}_i=\{f\}_i'.$$

Proof. Let $u\in\{f\}_i$ be analytic and of minimal exponential type on a postsector Σ or an open cut sector S. Let $T>0$. Consider $a:=u-\hat{f}_T$. Then a is of minimal exponential type since it is a difference of two functions of minimal exponential type and $a'(\lambda)=u'(\lambda)-\hat{f}_T'(\lambda)=u'(\lambda)-(\widehat{-tf})_T(\lambda)$. To show that $u'\in\{-tf\}_2$ it will be enough to show that a' is of minimal exponential type on some adjusted postsector Σ' and that $a'\approx_T 0$. Take a small $r>0$ and define $\Sigma':=\{\lambda\in\Sigma$ such that the disc $\Omega_r(\lambda):=\{\xi:|\xi-\lambda|\le r\}\subset\Sigma\}$. Then Σ' is a postsector and for $\lambda\in\Sigma'$ and $\xi\in\delta\Omega_r$ we have that $\xi=re^{iarg(\xi)}+\lambda$ and $\|a(\xi)\|\le M_\omega e^{\omega|\xi|}$, for sufficiently large ξ. By Cauchy's Theorem,

$$\|a'(\lambda)\|=\left\|\frac{1}{2\pi i}\int_{\delta\Omega_r}\frac{a(\xi)}{(\xi-\lambda)^2}d\xi\right\|\le\frac{1}{r}\max_{\xi\in\delta\Omega_r}\|a(\xi)\|\le\frac{M_\omega e^{\omega|\xi|}}{r}=\frac{M_\omega e^{\omega r}}{r}e^{\omega|\lambda|}.$$

This shows that a' is of minimal exponential type. It remains to be shown that $a'\approx_T 0$. Using that $a\approx_T 0$, $\xi=|\xi|\,e^{i\theta_\lambda}$, we obtain that for any $\varepsilon>0$

$$\begin{aligned}\|a'(\lambda)\| &\le \frac{1}{r}\max_{\xi\in\delta\Omega_r}\|a(\xi)\|\\ &\le \frac{1}{r}e^{(-T+\varepsilon)|\xi|\cos\theta_\xi}\\ &\le \frac{1}{r}e^{(-T+\varepsilon)(\lambda+r)\cos\theta_\xi}.\end{aligned}$$

Thus,

$$\frac{1}{\lambda}\ln\|a'(\lambda)\|\le\frac{1}{\lambda}\ln(\frac{1}{r}e^{(-T+\varepsilon)\cos\theta_\xi(\lambda+r)})=(-T+\varepsilon)\cos\theta_\xi(1+\frac{r}{\lambda})-\frac{1}{\lambda}\ln r.$$

Now taking the limit as $\lambda \to \infty$ it follows that $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|a'(\lambda)\| \leq -T + \varepsilon$ since $\theta_\xi \to 0$ as $\lambda \to \infty$. Since ε was arbitrary, we conclude that $a' \approx_T 0$ and thus $u \in \{-tf\}_2$. One should note that the previous argument shows that if a function u is of minimal exponential type on some postsector Σ then the same holds for u' on a smaller postsector Σ' and that $a \approx_T 0$ implies $a' \approx_T 0$.

For the inclusion $\{-tf\}_2 \subset \{f\}'_2$, let $T > 0$ and $u \in \{-tf\}$. Then $u(\lambda) = \int_0^T -tf(t)e^{-\lambda t}dt + a(\lambda)$, where $a \approx_T 0$. It follows that $u(\lambda) = (\int_0^T f(t)e^{-\lambda t}dt)' + a(\lambda)$. It remains to be shown that there exits a $A(\lambda)$ of minimal exponential type and $A \approx_T 0$ such that $A'(\lambda) = a(\lambda)$. Consider $A(\lambda) := \int_\lambda^\infty a(\mu)d\mu = \lim_{\lambda_1\to\infty} \int_\lambda^{\lambda_1} a(\mu)d\mu$ where we integrate along the ray $\Gamma := \{\mu = se^{i\alpha} : \ \alpha = \arg \lambda\}$. Then $A(\lambda)$ is analytic and, by Lemma 2.7, for all $\varepsilon > 0$ there exists $r_0 > 0$ such that $\|a(\mu)\| \leq e^{(-T+\varepsilon)s}$ for all $s > r_0$. Thus,

$$\begin{aligned} \|A(\lambda)\| &\leq \lim_{\lambda_1\to\infty} \int_\lambda^{\lambda_1} \|a(\mu)\| \, d\mu \\ &\leq \lim_{r_1\to\infty} \int_r^{r_1} e^{-Ts\cos\alpha+\varepsilon s} \, ds \\ &\leq \lim_{r_1\to\infty} \left. \frac{e^{-Ts\cos\alpha+\varepsilon s}}{-T\cos\alpha+\varepsilon} \right|_r^{r_1} \\ &= \frac{e^{-Tr\cos\alpha+\varepsilon r}}{T\cos\alpha-\varepsilon} \end{aligned}$$

Therefore $A(\lambda)$ is of minimal exponential type. Moreover, on rays $\lambda = re^{i\alpha}$ we have that

$$\limsup_{r\to\infty} \frac{1}{r} \ln \left\|A(re^{i\alpha})\right\| = \limsup_{r\to\infty} \frac{1}{r}(-Tr\cos\alpha - \varepsilon r - \ln(T\cos\alpha - \varepsilon)) \leq -T\cos\alpha.$$

Now using Lemma 2.7 it follows that $A \approx_T 0$. To complete the proof we have to show that $A'(\lambda) = a(\lambda)$. Consider $\Gamma := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ where, $\Gamma_1 := \{\xi : \xi = re^{i\alpha}, |\lambda| \leq r \leq r_1\}$, $\Gamma_2 := \{\xi : \xi = (1-t)(Re\lambda_1 + iRe\lambda_1 \tan\alpha) + t(Re\lambda_1 + iRe\lambda_1 \tan\beta), \ \beta = arg(\lambda + h), t \in [0,1]\}$, $\Gamma_3 := \{\xi : \xi = re^{i\beta}, Re\lambda_1 \cos\beta \geq r \geq$

$|\lambda+h|\}, \Gamma_4 := \{\xi := r\lambda+(1-r)(\lambda+h) : r \in [0,1]\}$. Then by Cauchy's Theorem we have that $\int_\Gamma a(\mu)d\mu = 0$. Since, $\left|\int_{\Gamma_2} a(\mu)d\mu\right| \leq ML \leq e^{-TRe\lambda_1} Re\lambda_1(\tan\beta - \tan\alpha)$ where $M := \max_{\mu\in\Gamma_2} |a(\mu)|$ and L is the length of the curve Γ_2 it follows that $\int_{\Gamma_2} a(\mu)d\mu \to 0 \quad as\ \lambda_1 \to \infty$. Finally,

$$\begin{aligned}
\lim_{h\to 0}\left\|\frac{A(\lambda+h)-A(\lambda)}{h} - a(\lambda)\right\| &= \lim_{h\to 0}\left\|\frac{1}{h}[\int_{\lambda+h}^\infty a(\mu)\,d\mu - \int_\lambda^\infty a(\mu)\,d\mu] - a(\lambda)\right\| \\
&\leq \lim_{h\to 0}\left\|\frac{1}{h}\int_\lambda^{\lambda+h} a(\mu)d\mu - a(\lambda)\right\| \\
&\leq \lim_{h\to 0}\left\|\frac{1}{h}\int_\lambda^{\lambda+h} a(\mu) - a(\lambda)d\mu\right\| \\
&\leq \frac{1}{h}\left\|a(\mu)-a(\lambda)\right\| h \\
&\leq \epsilon,
\end{aligned}$$

by the continuity of a. Therefore $A'(\lambda) = a(\lambda)$. Replacing Σ by S and $\{\cdot\}_2$ by $\{\cdot\}_3$ all arguments remain valid. This finishes the proof of the proposition. □

In Example 2.2 we have seen that

$$\{e^{t^2}\}_1 = i\int_0^\infty e^{-i\lambda r}e^{-r^2}\,dr + \{0\}_1.$$

Unfortunately, it is difficult to see whether $r_0 : \lambda \mapsto i\int_0^\infty e^{-i\lambda r}e^{-r^2}\,dr$ is of minimal exponential type on a certain open cut sector S_θ. We suspect not, but in absence of a strong argument for or against the statement $r_0 \in \{e^{t^2}\}_2$ or $r_0 \in \{e^{t^2}\}_3$ we go a third road and discuss other functions in $\{e^{t^2}\}$. As shown in the proof of Theorem 2.11, the function

$$r_1(\lambda) := \lambda\int_0^\infty e^{-\lambda t}(1 - e^{-\frac{\lambda\ln(\lambda)}{G(t)}})\,F(t)\,dt$$

is an asymptotic Laplace transform of f (i.e., $r_1 \in \{f\}$), where $F(t) = \int_0^t e^{s^2}\,ds$ and $G(t) := \max\{\|F(t)\|, 1\}$. Since r_1 is not a classical function of analysis one might want to find more familiar functions in $\{f\}$. We present one further method, motivated by a method used by L. Berg [6], of constructing elements of $\{e^{t^2}\}$ which depend more on the special nature of the function $f(t) = e^{t^2}$. Consider the function

$$r_2(\lambda) := \int_0^{\lambda/2} e^{-\lambda t + t^2}\, dt.$$

Since the function $\lambda \mapsto e^{-\lambda t + t^2}$ is analytic in $\mathbb{C}$ it follows that $r_2(\lambda)$ is analytic in $\mathbb{C}$ and that the integral is independent of the path chosen [15]. Moreover, $r(\lambda)$ is of minimal exponential type on the open cut sector $S_{\frac{\pi}{4}}$, since for the parametrization $t = s\frac{\lambda}{2}$ we have that $r(\lambda) = \frac{\lambda}{2}\int_0^1 e^{-\frac{\lambda^2}{2}s + \frac{\lambda^2}{4}s^2} ds$ and therefore

$$\|r_2(\lambda)\| \leq \left|\frac{\lambda}{2}\right| \int_0^1 \left|e^{-\frac{\lambda^2}{2}s + \frac{\lambda^2}{4}s^2}\right| ds.$$

Now for $\lambda = a + ib$ we have that $\left|e^{-\frac{\lambda^2}{2}s + \frac{\lambda^2}{4}s^2}\right| = e^{-\frac{a^2-b^2}{2}s + \frac{a^2-b^2}{4}s^2}$ which is a positive decreasing function on $[0, 1]$ with minimum at $s = 1$, provided that $a^2 - b^2$ is positive. So for λ in the sector $S_{\frac{\pi}{4}} := \{\lambda = re^{i\theta}, \frac{-\pi}{4} \leq \theta \leq \frac{\pi}{4}\}$ we have that $\|r_2(\lambda)\| \leq \left|\frac{\lambda}{2}\right|$ which implies the minimal exponential type. Now for λ a real positive number the convex function $t \to e^{-\lambda t + t^2}$ attains its minimum at $t = \frac{\lambda}{2}$. It follows that $r_2(\lambda) - \int_0^T e^{-\lambda t} e^{t^2}\, dt = \int_T^{\lambda/2} e^{-\lambda t + t^2}\, dt \leq (\frac{\lambda}{2} - T)e^{-\lambda T + T^2}$ for all sufficiently large $\lambda > 0$. Therefore, $r_2 \in \{f\}_3^T$ for all $T > 0$ and thus $r_2 \in \{e^{t^2}\}_3$; i.e.,

$$\{e^{t^2}\}_3 = \int_0^{\lambda/2} e^{-\lambda t + t^2}\, dt + \{0\}_3. \tag{2.9}$$

This example can be generalized for any function of the type $t \mapsto e^{t^n}$ for $n > 1$ as follows. Define

$$r(\lambda) := \int_0^{(\frac{\lambda}{n})^{\frac{1}{n-1}}} e^{-\lambda t + t^n}\, dt.$$

Since $\lambda \mapsto e^{-\lambda t + t^n}$ is analytic in $\mathbb{C}$ it follows that r is analytic in $\mathbb{C}$ and the integral is independent of the path chosen. For the parametrization $t = s(\frac{\lambda}{n})^{\frac{1}{n-1}}$ we obtain that

$$r(\lambda) = (\frac{\lambda}{n})^{\frac{1}{n-1}} \int_0^1 e^{-\lambda(\frac{\lambda}{n})^{\frac{1}{n-1}} s + (\frac{\lambda}{n})^{\frac{n}{n-1}} s^n}\, ds.$$

Therefore, $\|r(\lambda)\| \leq \left|(\frac{\lambda}{n})^{\frac{1}{n-1}}\right| \int_0^1 \left|e^{-\lambda(\frac{\lambda}{n})^{\frac{1}{n-1}}s+(\frac{\lambda}{n})^{\frac{n}{n-1}}s^n}\right| ds$. Since

$$\left|e^{-\lambda(\frac{\lambda}{n})^{\frac{1}{n-1}}s+(\frac{\lambda}{n})^{\frac{n}{n-1}}s^n}\right| = \left|e^{-\frac{\lambda^{\frac{n}{n-1}}}{n^{\frac{1}{n-1}}}(s-\frac{1}{n}s^n)}\right| = e^{-\frac{Re\lambda^{\frac{n}{n-1}}}{n^{\frac{1}{n-1}}}(s-\frac{1}{n}s^n)}$$

and the fact that the function $f(s) = e^{-c(s-\frac{1}{n}s^n)}$ is a positive decreasing function on $[0,1]$ for $c > 0$ it follows that $\|r(\lambda\| \leq \left|(\frac{\lambda}{n})^{\frac{1}{n-1}}\right|$. Hence r is of minimal exponential type. It remains to be shown that $r \in \{e^{t^n}\}_3$. Again, for λ a real positive number, the convex function $t \mapsto e^{-\lambda t+t^n}$ attains its minimum at $t = (\frac{\lambda}{n})^{\frac{1}{n-1}}$, hence

$$r(\lambda) - \int_0^T e^{-\lambda t}e^{t^n}\,dt = \int_T^{(\frac{\lambda}{n})^{\frac{1}{n-1}}} e^{-\lambda t+t^n}\,dt \leq ((\frac{\lambda}{n})^{\frac{1}{n-1}} - T)e^{-\lambda T+T^n}$$

for all sufficiently large $\lambda > 0$. Therefore, $r \in \{e^{t^n}\}_3^T$ for any $T > 0$ and thus $r(\lambda) \in \{e^{t^n}\}_3$. We should notice here that the examples 2.2 and 2.9 and are valid only when we consider the asymptotic Laplace transform as being the class of analytic functions of minimal exponential type on a sector. It appears that working with analytic functions on sectors will simplify the procedure of finding examples, and it is one of the main ideas of this dissertation to adopt this definition instead of the ones given earlier by G. Lumer and F. Neubrander. We turn now to the question about the characteristic properties of an equivalence class $\{f\}_i$. We will prove below that if one of the members of the asymptotic class $\{f\}$ has an asymptotic expansion in terms of $\frac{1}{\lambda}$ as $\lambda \to \infty$ then all members of the asymptotic Laplace transform have the same asymptotic expansion. The investigation of this topic is what follows next.

Recall that $f(x) = O(g(x))$ as $x \to \infty$ if there exists $a > 0$ such that $\|f(x)\| \leq M\,\|g(x)\|$ for $x \in [a,\infty)$. Also, $f(x) = o(g(x))$ as $x \to \infty$ if for each $\varepsilon > 0$ there exists a_ε such that $\|f(x)\| \leq \varepsilon\,\|g(x)\|$ for all $x \in [a_\varepsilon,\infty)$. The following properties of the order symbols hold.

1 If c_1, c_2 are constants, then $c_1 O(g) + c_2 O(g) = O(g)$.

2 If c_1, c_2 are constants, then $c_1 o(g) + c_2 o(g) = o(g)$.

3 $O(O(g)) = O(g)$.

4 $O(o(g)) = o(O(g)) = o(o(g)) = o(g)$.

5 $O(f)O(g) = O(fg)$.

6 $O(f)o(g) = o(fg)$.

A finite or infinite sequence of functions $\{\phi_n(x)\}_{n\in\mathbb{N}}$ defined on an interval (a, ∞) is called an asymptotic sequence as $x \to \infty$ if the following two conditions are satisfied:

(1) $\phi_n(x) \neq 0$, for $n = 1, 2 \ldots$

(2) $\phi_{n+1}(x) = o(\phi_n(x))$, as $x \to \infty$.

Obvious examples of asymptotic sequences are given by $\phi_n(x) = \frac{1}{x}$ or $\phi_n(x) = e^{-nx}$. We say that a function$f(x)$ has an asymptotic development to N terms with respect to the asymptotic sequence $\{\phi_n(x)\}_{n\in\mathbb{N}}$ if there exists constants $c_1, c_2, \ldots, c_N$ such that

$$f(x) = c_1\phi_1(x) + \ldots + c_N\phi_N(x) + o(\phi_N(x)) \text{ as } x \to \infty.$$

In the case that $f(x)$ has an asymptotic development to N terms for any $N \in \mathbb{N}$, we say that $f(x)$ has an asymptotic expansion in terms of the sequence $\{\phi_n(x)\}_{n\in\mathbb{N}}$ and write

$$f(x) \sim \sum_{n=1}^{\infty} c_n\phi_n(x) \text{ as } x \to \infty.$$

The following theorem is essentially due to G. Lumer (unpublished).

Theorem 2.14. *Let $f : [0, \infty) \to X$ extend to an entire analytic function $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots$. Then each $r \in \{f\}_i$ admits the uniquely determined asymptotic expansion at ∞ in $\frac{1}{\lambda^n}$*

$$\{f\}(\lambda) \sim \frac{a_0}{\lambda} + \frac{a_1}{\lambda^2} + \frac{2!a_2}{\lambda^3} + \ldots \tag{2.10}$$

Proof. We must show that there exists a sequence of vectors $(a_n)_{n\in\mathbb{N}},\ a_n \in X$, such that for any $u \in \{f\}_i$ and any $n \geq 0$,

$$u(\lambda) = \frac{a_0}{\lambda} + \ldots + \frac{n!a_n}{\lambda^{n+1}} + o(\frac{1}{\lambda^{n+1}}) \tag{2.11}$$

as $\lambda \to \infty$. We do this in several steps.

a) Let $T > 0$, and $n \in \mathbb{N}$ be fixed. Then $\hat{f}_T(\lambda) \approx_T u(\lambda)$, where

$$\hat{f}_T(\lambda) = \int_0^T e^{-\lambda t} f(t)dt = I_1 + I_2 \tag{2.12}$$

with

$$\begin{aligned} I_1 &:= \int_0^T e^{-\lambda t}(a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n)dt \\ I_2 &:= \int_0^T e^{-\lambda t}(a_{n+1}t^{n+1} + \ldots)dt \end{aligned}$$

Since $f(z)$ is entire we have that $\sqrt[j]{\|a_j\|} \to 0$; i.e for any $\epsilon > 0$ there exists $n_\epsilon > n$ such that

$$\|a_j\| \leq (\epsilon)^j \tag{2.13}$$

for $j \geq n_\epsilon$. Let $0 < \alpha < 1$ and choose $\epsilon > 0$ such that $T\epsilon \leq \alpha$. Write $I_2 = I_2^{**} + I_2^*$, where

$$I_2^* := \int_0^T e^{-\lambda t} \sum_{m \geq n_\epsilon} a_m t^m dt$$

and

$$I_2^{**} := \int_0^T e^{-\lambda t} \sum_{m=n+1}^{n_\epsilon - 1} a_m t^m dt.$$

Let $\sup_{t\in[0,T]} \left\|\sum_{m=n+1}^{n_\varepsilon-1} a_m t^{m-n-1}\right\| := M(n_\varepsilon, T)$. Then

$$\|I_2^{**}\| \leq M(n_\epsilon, T)\int_0^T e^{-\lambda t} t^{n+1} dt \leq \frac{M(n_\varepsilon, T)(n+1)!}{\lambda^{n+2}}. \tag{2.14}$$

In particular, $\|I_2^{**}\| = o(\frac{1}{\lambda^{n+2}})$ as $\lambda \to \infty$. For I_2^* we have

$$\begin{aligned} \left\|\sum_{m\geq n_\epsilon} a_m t^m\right\| &\leq t^{n_\epsilon}(\epsilon^{n_\epsilon} + \epsilon^{n_\epsilon+1}t + \epsilon^{n_\epsilon+2}t^2 + \ldots) \\ &\leq \frac{\alpha^{n_\epsilon}}{T^{n_\epsilon}} t^{n_\epsilon}(1 + \alpha + \alpha^2 + \ldots) \\ &= \frac{\alpha^{n_\epsilon}}{T^{n_\epsilon}}\frac{1}{1-\alpha} t^{n_\epsilon}. \end{aligned}$$

Thus,

$$\begin{aligned} \|I_2^*\| &\leq \int_0^T e^{-\lambda t}\left\|\sum_{m\geq n_\epsilon} a_m t^m\right\| dt \\ &\leq \frac{\alpha^{n_\epsilon}}{T^{n_\epsilon}}\frac{1}{1-\alpha}\frac{(n_\epsilon)!}{\lambda^{n_\epsilon+1}} \\ &\leq \frac{M^*}{\lambda^{n+2}} \end{aligned}$$

if $\lambda > 1$ and $M := \frac{\alpha^{n_\epsilon}}{T^{n_\epsilon}}\frac{(n_\epsilon)!}{1-\alpha}$. In particular,

$$\|I_2\| = O(\frac{1}{\lambda^{n+2}}) = o(\frac{1}{\lambda^{n+1}}) \tag{2.15}$$

as $\lambda \to \infty$.

b) To estimate $I_1 = \int_0^T e^{-\lambda t}(a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n)dt$ define

$$\begin{aligned} E_1(\lambda) : &= \int_0^\infty e^{-\lambda t}(a_0 + \ldots + a_n t^n)dt - I_1 \\ &= \int_T^\infty e^{-\lambda t}(a_0 + a_1 t + \ldots + a_n t^n)dt. \end{aligned}$$

Let $M_1 := \sup_{t\in[T,\infty)} \left\|\frac{a_0}{t^n} + \ldots + a_n\right\|$. Then

$$\begin{aligned} \|E_1(\lambda)\| &\leq \int_T^\infty e^{-\lambda t} t^n \left\|\frac{a_0}{t^n} + \frac{a_1}{t^{n-1}} + \ldots + a_n\right\| dt \\ &\leq M_1 \int_T^\infty e^{-\lambda t} t^n dt \\ &\leq M_1 e^{-\lambda T}(\frac{T^n}{\lambda} + \frac{nT^{n-1}}{\lambda^2} + \ldots + \frac{n!}{\lambda^{n+1}}) \\ &\leq M_2(n, T)e^{-\lambda T} \end{aligned}$$

where $M_2(n,T) = \sup_{\lambda \geq 1} \left| \frac{T^n}{\lambda} + \frac{nT^{n-1}}{\lambda^2} + \ldots + \frac{n!}{\lambda^{n+1}} \right| = T^n + nT^{n-1} + \ldots + n!$

To finish the proof, let $a(\lambda) := u(\lambda) - \hat{f}_T(\lambda) \approx_t 0$. In particular, for all $\mu > 0$ there exists $M_\mu > 0$ such that $\|a(\lambda)\| \leq M_\mu e^{-\lambda(T+\mu)}$. This implies that $\|a(\lambda)\| = O(e^{-\lambda(T+\mu)})$ for all $\mu > 0$. Therefore,

$$\left\| u(\lambda) - (\frac{a_0}{\lambda} + \frac{a_1}{\lambda^2} + \ldots + \frac{n! a_n}{\lambda^{n+1}}) \right\|$$

$$\begin{aligned}
&\leq \left\| u(\lambda) - \hat{f}_T(\lambda) \right\| + \left\| I_1(\lambda) + I_2(\lambda) - (\frac{a_0}{\lambda} + \frac{a_1}{\lambda^2} + \ldots + \frac{n! a_n}{\lambda^{n+1}}) \right\| \\
&\leq \left\| u(\lambda) - \hat{f}_T(\lambda) \right\| + \|I_2(\lambda)\| + \|E_1(\lambda)\| \\
&\leq O(e^{-\lambda(T+\mu)}) + o(\frac{1}{\lambda^{n+1}}) + O(e^{-\lambda T}) \\
&= o(\frac{1}{\lambda^{n+1}})
\end{aligned}$$

for any $\mu > 0$. Hence 2.11 holds so the theorem is proved. □

This theorem has inspired us to prove the following result, which in essence states that if one of the functions in the elements of the equivalence class of $\{f\}$ has an asymptotic expansion in terms of $\frac{1}{\lambda}$ as $\lambda \to \infty$, then any other element of the equivalence class has the same asymptotic expansion.

Proposition 2.15. *Let $r_1, r_2 \in \{f\}_i$ for some $f \in L^1_{loc}([0,\infty), X)$ be such that r_1 has an asymptotic expansion in terms of $\frac{1}{\lambda}$ as $\lambda \to \infty$. Then r_2 has the same asymptotic expansion in terms of $\frac{1}{\lambda}$ as $\lambda \to \infty$.*

Proof. Suppose that $r_1, r_2 \in \{f\}_i$ and

$$r_1(\lambda) = a_0 + \frac{a_1}{\lambda} + \ldots + \frac{a_n}{\lambda^n} + o(\frac{1}{\lambda^{n+1}}).$$

from the definition of the asymptotic Laplace transform we have that for any $T > 0$

$$r_i(\lambda) - \int_0^T e^{-\lambda t} f(t) dt \approx_T 0 \tag{2.16}$$

for $i = 1, 2$. We need to show that

$$r_2(\lambda) - a_0 - \frac{a_1}{\lambda} - \ldots - \frac{a_n}{\lambda^n} = o(\frac{1}{\lambda^{n+1}})$$

as $\lambda \to \infty$. Since

$$\left\| r_2(\lambda) - a_0 - \frac{a_1}{\lambda} - \ldots - \frac{a_n}{\lambda^n} \right\| =$$

$$\begin{aligned}
&= \left\| r_2(\lambda) - \int_0^T e^{-\lambda t} f(t)dt + \int_0^T e^{-\lambda t} f(t)dt - r_1(\lambda) + r_1(\lambda) - a_0 - \ldots - \frac{a_n}{\lambda^n} \right\| \\
&\leq \left\| r_2(\lambda) - \int_0^T e^{-\lambda t} f(t)dt \right\| + \left\| \int_0^T e^{-\lambda t} f(t)dt - r_1(\lambda) \right\| \\
&\quad + \left\| r_1(\lambda) - a_0 - \ldots - \frac{a_n}{\lambda^n} \right\| \\
&\leq e^{-\lambda T} + e^{-\lambda T} + o(\frac{1}{\lambda^{n+1}}) \\
&= o(\frac{1}{\lambda^{n+1}})
\end{aligned}$$

we have that

$$r_2(\lambda) = a_0 + \frac{a_1}{\lambda} + \ldots + \frac{a_n}{\lambda^n} + o(\frac{1}{\lambda^{n+1}})$$

and the proof is complete. □

Recall that $f(x) = O(g(x))$ as $x \to 0^+$ if there exists $a > 0$ such that $\|f(x)\| \leq M \|g(x)\|$ for $x \in (0, a)$. Also, $f(x) = o(g(x))$ as $x \to 0$ if for each $\varepsilon > 0$ there exists a_ε such that $\|f(x)\| \leq \varepsilon \|g(x)\|$ for all $x \in (0, a_\varepsilon)$ Next we prove a generalization of Watson's Lemma for the asymptotic Laplace transform.

Theorem 2.16 (Watson's Lemma). *Suppose that $f \in L^1_{loc}([0, \infty), X)$ has an asymptotic expansion in terms of $\{t^n\}_{n\in\mathbb{N}}$ as $t \to 0^+$; i.e., suppose that*

$$f(t) \sim \sum_{n=0}^{\infty} c_n t^n.$$

Then, for any $u \in \{f\}_i$ we have that

$$u(\lambda) \sim \sum_{n=0}^{\infty} c_n \frac{n!}{\lambda^{n+1}} \quad as \ \lambda \to \infty \tag{2.17}$$

Proof. We need to show that

$$\left\| u(\lambda) - \sum_{n=0}^{N} c_n \frac{n!}{\lambda^{n+1}} \right\| = o(\frac{1}{\lambda^{N+1}}) \text{ as } \lambda \to \infty$$

for any $N > 0$. Let $T > 0$. Since $\sum_{n=0}^{N} c_n \frac{n!}{\lambda^{n+1}} = \int_0^\infty \sum_{n=0}^{N} e^{-\lambda t} c_n t^n \, dt$ we have that

$$\begin{aligned}
\left\| u(\lambda) - \sum_{n=0}^{N} c_n \frac{n!}{\lambda^{n+1}} \right\| &= \left\| u(\lambda) - \sum_{n=0}^{N} \int_0^\infty e^{-\lambda t} c_n t^n \, dt \right\| \\
&= \left\| u(\lambda) - \int_0^T \sum_{n=0}^{N} c_n t^n e^{-\lambda t} dt - \int_T^\infty \sum_{n=0}^{N} c_n t^n e^{-\lambda t} dt \right\| \\
&\leq \left\| u(\lambda) - \int_0^T \sum_{n=0}^{N} c_n t^n e^{-\lambda t} dt \right\| + \left\| \int_T^\infty \sum_{n=0}^{N} c_n t^n e^{-\lambda t} dt \right\| \\
&= I_1 + I_2
\end{aligned}$$

Now

$$\begin{aligned}
I_1 &= \left\| u(\lambda) - \int_0^T e^{-\lambda t} f(t)\, dt + \int_0^T e^{-\lambda t} f(t)\, dt - \int_0^T \sum_{n=0}^{N} c_n t^n e^{-\lambda t} \, dt \right\| \\
&\leq \left\| u(\lambda) - \int_0^T e^{-\lambda t} f(t)\, dt \right\| + \left\| \int_0^T e^{-\lambda t} (f(t) - \sum_{n=0}^{N} c_n t^n) \, dt \right\| \\
&\leq \|a(\lambda)\| + \left\| \int_0^T e^{-\lambda t} (f(t) - \sum_{n=0}^{N} c_n t^n) \, dt \right\|
\end{aligned}$$

From the definition of the asymptotic Laplace transform it follows that for any $\varepsilon > 0$ $\|a(\lambda)\| \leq e^{-\lambda(T-\varepsilon)}$ for sufficiently large la and from the fact that $f(t) \sim \sum_{n=0}^{\infty} c_n t^n$ and by using Proposition 1.16 for the function $(f(t) - \sum_{n=0}^{N} c_n t^n)\chi_{[r,T]}$ it follows that

$$\left\| \int_0^T e^{-\lambda t} (f(t) - \sum_{n=0}^{N} c_n t^n) \, dt \right\|$$

$$
\begin{aligned}
&\leq \int_0^r e^{-\lambda t} \left\| f(t) - \sum_{n=0}^{N} c_n t^n \right\| dt + \left\| \int_r^T e^{-\lambda t} (f(t) - \sum_{n=0}^{N} c_n t^n)\, dt \right\| \\
&\leq \int_0^r e^{-\lambda t} c_N t^N \, dt + \left\| \int_r^T e^{-\lambda t} (f(t) - \sum_{n=0}^{N} c_n t^n) \chi_{[r,T]}\, dt \right\| \\
&\leq \int_0^\infty e^{-\lambda t} c_N t^N \, dt + M e^{-\lambda r}\, dt \\
&= c_N \frac{(N)!}{\lambda^{N+1}} + M e^{-\lambda r} \\
&= o(\frac{1}{\lambda^{N+2}}).
\end{aligned}
$$

It remains to be shown that $I_2 = o(\frac{1}{\lambda^{n+1}})$.

$$
\begin{aligned}
I_2 &= \left\| \int_T^\infty \sum_{n=0}^{N} c_n t^n e^{-\lambda t} dt \right\| \\
&\leq M_1 \int_T^\infty e^{-\lambda t} t^n dt \\
&\leq M_1 e^{-\lambda T} (\frac{T^n}{\lambda} + \frac{nT^{n-1}}{\lambda^2} + \ldots + \frac{n!}{\lambda^{n+1}}) \\
&\leq M_2(n,T) e^{-\lambda T},
\end{aligned}
$$

where $M_2(n,T) = \sup_{\lambda \geq 1} \left| \frac{T^n}{\lambda} + \frac{nT^{n-1}}{\lambda^2} + \ldots + \frac{n!}{\lambda^{n+1}} \right| = T^n + nT^{n-1} + \ldots + n!$ In particular $I_2 = O(e^{-\lambda T})$ and therefore $I_2 = o(\frac{1}{\lambda^{n+1}})$. Thus

$$
\begin{aligned}
\left\| u(\lambda) - \sum_{n=0}^{N} c_n \frac{n!}{\lambda^{n+1}} \right\| &\leq o(\frac{1}{\lambda^{N+1}}) + O(e^{-\lambda T}) \\
&= o(\frac{1}{\lambda^{N+2}})
\end{aligned}
$$

for any $N > 0$ and the proof is complete. □

In the next part of the dissertation we will briefly describe how to take asymptotic Laplace transforms of generalized functions $f \in C([a'\infty), X)^{T_k}$. To simplify the notation, in this section we will drop the index used for different definitions of the asymptotic Laplace transform and remark that, if used consistently any of these definitions can be used.

Definition 2.17. *Let $k \in C[0,\infty)$ with $0 \in supp(k)$ and $f \in C_0([0,\infty),X)^{T_k}$. Then $k * f \in C_0[0,\infty),X)$ and we define the asymptotic Laplace transform of f to be*

$$\{f\} := \frac{\{k * f\}}{\{k\}}.$$

The functions $r = \frac{p}{q}$ with $p \in \{k * f\}$ and $q \in \{k\}$ in the asymptotic class are going to be meromorphic functions in a postsectorial (sectorial region) with possible poles at the zeros of $q \in \{k\}$. Since $0 \in supp(k)$ it follows that $\{k\} \neq \{0\}$. Thus, the zeros of $q \in \{k\}$ cannot form a uniqueness sequence.

We need to show that this extension of the asymptotic Laplace transform is well defined and that the operational properties of the asymptotic Laplace transform extend to the generalized function case. Recall from Chapter 1 that we identify two generalized functions $f \in C_0([0,\infty),X)^{T_{k_1}}$ and $g \in C_0([0,\infty),X)^{T_{k_2}}$ if

$$k_2 * k_1 * f = k_1 * k_2 * g,$$

considered as functions on $C(-\infty,\infty)$ by continuing them with zero to the left. Suppose now that f, g as above can be identified. Then

$$\begin{aligned}
\{f\} &= \frac{\{k_1 * f\}}{\{k_1\}} = \frac{\{(k_1 * f) * k_2\}}{\{k_1 * k_2\}} \\
&= \frac{\{(k_1 * f * k_2)\}}{\{k_1 * k_2\}} = \frac{\{k_1 * (g * k_2)\}}{\{k_1 * k_2\}} \\
&= \frac{\{(g * k_2)\}}{\{k_2\}} = \{g\}.
\end{aligned}$$

where we have used the operational properties of the asymptotic Laplace transform which are valid for any functions in $C_0([0,\infty),X)$.

We will show next that the asymptotic Laplace transform is linear. Suppose that $f \in C_0([0,\infty),X)^{T_{k_1}}$ and $g \in C_0([0,\infty),X)^{T_{k_2}}$ and $k_1, k_2 \in C[0,\infty)$ with $0 \in$

$supp(k_1) \cap supp(k_2)$. Then $f + g = \tilde{T}^{-1}_{k_1 * k_2}(k_2 * k_1 * f + k_1 * k_2 * g)$ and thus

$$\begin{aligned}
\{f + g\} &= \frac{\{(k_2 * k_1 * f + k_1 * k_2 * g)\}}{\{k_1 * k_2\}} \\
&= \frac{\{(k_2 * k_1 * f)\} + \{(k_1 * k_2 * g)\}}{\{k_1 * k_2\}} \\
&= \frac{\{k_2\}\{(k_1 * f)\} + \{k_1\}(k_2 * g)\}}{\{k_1\}\{k_2\}} \\
&= \frac{\{(k_1 * f)\}}{\{k_1\}} + \frac{\{(k_2 * g)\}}{\{k_2\}} \\
&= \{f\} + \{g\}
\end{aligned}$$

Recall that one cannot define the convolution of two vector valued generalized function. However, in the case that one of them is scalar valued the following operational property holds. Let $f \in C_0([0, \infty), X)^{T_{k_2}}$ and $h \in C_0[0, \infty)^{T_{k_1}}$. Then the convolution $h * f := \tilde{T}^{-1}_{k_1 * k_2}(k_1 * h * k_2 * f)$. Therefore,

$$\{h * f\} = \frac{\{k_1 * h * k_2 * f\}}{\{k_1 * k_2\}} = \frac{\{k_1 * h\}}{\{k_1\}} \frac{\{k_2 * f\}}{\{k_2\}} = \{h\}\{f\}.$$

In Chapter 1 we have shown that the scalar valued generalized functions

$$\mathcal{F} := \{f \in C_0[a, \infty)^{T_k} : a \in \mathbb{R}, k \in C[0, \infty) \text{ with } 0 \in supp(k)\},$$

form a field and that the vector valued generalized functions form a vector space over that field.

$$\mathcal{V} := \{f \in C([a, \infty), X)^{T_k} : a \in \mathbb{R}, k \in C[0, \infty) \text{ with } 0 \in supp(k)\}$$

is a vector space over $\mathcal{F}$ where scalar multiplication of a vector $f \in \mathcal{V}$ with some $h \in \mathcal{F}$ is defined by

$$h * f := \widetilde{T}^{-1}_{k_1 * k_2}(k_1 * h * k_2 * f),$$

In order to extend the definition of the asymptotic Laplace transform for a generalized function $f \in C_0([a, \infty), X)^{T_{k_1}}$ we first observe that for $f \in L^1_{loc}([0, \infty), X)$

we have that the asymptotic Laplace transform of the shifted function

$$f_a : t \mapsto \begin{cases} 0 & \text{if } 0 \le t \le a \\ f(t-a) & \text{otherwise,} \end{cases}$$

satisfies $\{f_a\} = e^{-\lambda a}\{f\}$ since

$$\begin{aligned} \{f_a\}^T &= \int_0^T e^{-\lambda t} f_a(t)\, dt + \{0\}^T \\ &= \int_a^T e^{-\lambda t} f(t-a)\, dt + \{0\}^T \\ &= e^{-\lambda a} \int_0^{T-a} e^{-\lambda s} f(s)\, ds + \{0\}^T \\ &= e^{-\lambda a} [\int_0^{T-a} e^{-\lambda s} f(s)\, ds + e^{\lambda a}\{0\}^T] \\ &= e^{-\lambda a}\{f\}^{T-a}. \end{aligned}$$

We should note that $e^{\lambda a}\{0\}^T = \{0\}^{T-a}$ since for $a \approx_T 0$ we have that

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \left\| e^{\lambda a} a(\lambda) \right\| \le a + \limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \| a(\lambda \| = a - T = -(T-a).$$

With this property at hand we can define the asymptotic Laplace transform of a generalized function $f \in C_0([a,\infty), X)^{T_k}$ for some $k \in C[0,\infty)$ with $0 \in supp(k)$, as

$$\{f\} := e^{-\lambda a} \frac{\{k * f_{-a}\}}{\{k\}}.$$

The asymptotic class is now formed by meromorphic functions defined on a post-sector (or on a open cut sector), except at the poles. We denote the space of meromorphic functions with $\mathcal{M}(\Sigma, X)$ and respectively $\mathcal{M}(S, X)$. We also remark that $\mathcal{M}(\Sigma, X)$ is a vector space over $\mathbb{C}$ as well as over $\mathcal{M}(\Sigma, \mathbb{C})$. First, we need to show that the asymptotic Laplace transform is well defined for generalized functions. Recall that we identify $f \in C([a,\infty), X)^{T_{k_1}}$ with $g \in C([b,\infty), X)^{T_{k_2}}$ if

$$k_2 * k_1 * f = k_1 * k_2 * g,$$

considered as functions on $C(-\infty,\infty))$ by continuing them with zero to the left. Then we have that

$$\begin{aligned}\{f\} &= e^{-\lambda a}\frac{\{(k_1 * f)_{-a}\}}{\{k_1\}} = e^{-\lambda a}\frac{\{(k_1 * f)_{-a} * k_2\}}{\{k_1 * k_2\}} \\ &= e^{-\lambda(a+b)}\frac{\{(k_1 * f * k_2)_{-(a+b)}\}}{\{k_1 * k_2\}} = e^{-\lambda b}\frac{\{k_1 * (g * k_2)_{-b}\}}{\{k_1 * k_2\}} \\ &= e^{-\lambda b}\frac{\{(g * k_2)_{-b}\}}{\{k_2\}} = \{g\}.\end{aligned}$$

Proposition 2.18. *The map* $\{\cdot\} : (\mathcal{V}, +, *) \to \mathcal{M}(\Sigma, X)$ *has the following properties*

(*a*) $\{f + g\} = \{f\} + \{g\}$.

(*b*) $\{h * g\} = \{h\}\{g\}$,

for any $f \in C_0([0,\infty), X)^{T_{k_1}}$, $g \in C_0([b,\infty), X)^{T_{k_2}}$, $h \in C_0[a,\infty)^{T_{k_1}}$ *and* $k_1, k_2 \in C[0,\infty)$ *with* $0 \in supp(k_1) \cap supp(k_2)$.

Proof. Without loss of generality we can assume that $a \leq b$. By definition, $f + g = \tilde{T}^{-1}_{k_1 * k_2}(k_2 * k_1 * f + k_1 * k_2 * g)$. Therefore, using the properties of the asymptotic Laplace transform for continuous functions we obtain statement (a) as follows

$$\begin{aligned}\{f + g\} &= e^{-\lambda a}\frac{\{(k_2 * k_1 * f + k_1 * k_2 * g)\}}{\{k_1 * k_2\}} \\ &= e^{-\lambda a}\frac{\{k_2\}\{(k_1 * f)_a\} + \{k_1\}\{(k_2 * g)_a\}}{\{k_1 * k_2\}} \\ &= e^{-\lambda a}\frac{\{k_2\}\{(k_1 * f)_a\}}{\{k_1\}\{k_2\}} + e^{-\lambda a}\frac{\{k_1\}\{(k_2 * g)_a\}}{\{k_1\}\{k_2\}} \\ &= e^{-\lambda a}\frac{\{(k_1 * f)_a\}}{\{k_1\}} + e^{-\lambda a}e^{-\lambda(b-a)}\frac{\{(k_2 * g)_b\}}{\{k_2\}} \\ &= \{f\} + \{g\}.\end{aligned}$$

To prove statement (b) recall that for $h \in C_0[a,\infty)^{T_{k_1}}$ and $g \in C_0([b,\infty), X)^{T_{k_2}}$ the scalar multiplication in the vector space $\mathcal{V}$ is defined as

$$h * g : \tilde{T}^{-1}_{k_1 * k_2}(k_1 * h * k_2 * g).$$

Thus,

$$\begin{aligned}\{h*g\} &= e^{-\lambda(a+b)}\frac{\{(k_2*k_1*h*k_1*k_2*g)_{a+b}\}}{\{k_2*k_1\}}\\ &= e^{-\lambda a}\frac{\{k_2\}\{(k_1*h)_a\}}{\{k_2\}\{k_1\}}e^{-\lambda b}\frac{\{k_1\}\{(k_2*g)_b\}}{\{k_2\}\{k_1\}}\\ &= \{h\}\{g\}.\end{aligned}$$

□

Since integrating is the same as convoluting with the Heaviside function $1 = \chi_{[0,\infty)}$ it follows that we can define the antiderivative F of a generalized function $f \in C_0([a,\infty),X)^{T_k}$ as $F := 1 * f$. It follows from Proposition 2.18 that

$$\{F\} := \{1 * f\} = \{1\}\{f\} = \frac{1}{\lambda}\{f\}.$$

Thus, $\{f\} = \lambda\{F\}$ and we have that $\{f'\} = \lambda\{f\}$. In other words in the space of generalized functions multiplying with λ corresponds to differentiation while dividing by λ corresponds to integrating the generalized function.

We conclude this dissertation with some applications of the asymptotic Laplace transform.

Example 2.19. Consider once again, the linear, first order, inhomogeneous initial value problem

$$f'(t) + f(t) = (2t+1)e^{t^2}, \quad f(0) = 1. \tag{2.18}$$

Taking now the asymptotic Laplace transform in both sides of the equation and using its properties we obtain

$$
\begin{aligned}
f'(t) + f(t) &= (2t+1)e^{t^2} \;\text{ and }\; f(0) = 1 \\
\Leftrightarrow \quad \{f'(t)\} + \{f(t)\} &= \{(2t+1)e^{t^2}\} \text{ and } f(0) = 1 \\
\Leftrightarrow \quad \lambda\{f\} - 1 + \{f\} &= \{(2t+1)e^{t^2}\} \\
\Leftrightarrow \quad (\lambda+1)\{f\} &= 1 + \{(2t+1)e^{t^2}\} \\
\Leftrightarrow \quad \{f\} &= \tfrac{1}{\lambda+1} + \tfrac{1}{\lambda+1}\{(2t+1)e^{t^2}\} \\
\Leftrightarrow \quad \{f\} &= \{e^{-t}\} + \{e^{-t}\}\{(2t+1)e^{t^2}\} \\
\Leftrightarrow \quad \{f\} &= \{e^{-t} + e^{-t} * (2t+1)e^{t^2}\} \\
&= \{e^{-t} + \textstyle\int_0^t e^{-(t-s)}(2s+1)e^{s^2}\, ds\} \\
&= \{e^{-t} + e^{-t} \textstyle\int_0^t (2s+1)e^{s^2+s}\, ds\} \\
&= \{e^{-t} + e^{-t}[e^{t^2+t} - 1]\} \\
\Leftrightarrow \quad \{f\} &= \{e^{t^2}\} \\
\Leftrightarrow \quad f(t) &= e^{t^2}.
\end{aligned}
$$

Hence, taking the asymptotic Laplace transform yields existence and uniqueness of the solution of the equation. We remark here that using any of the definition of the asymptotic Laplace transform will yield the same solution since the property $\{f\}' = \{-tf\}$ is not required to solve this equation. The next example deals with a more general class of ordinary differential equations.

Example 2.20. Consider the Laplace equation

$$(a_2 t + b_2)y''(t) + (a_1 t + b_1)y'(t) + (a_0 t + b_0)y(t) = 0, \tag{2.19}$$

where a_i, b_i are given complex or real numbers with $a_2 \neq 0,\ t \geq 0$. We may assume that $b_2 = 0$ by taking $(t - \frac{b_2}{a_2})$ as our new variable. Therefore we discuss the equation

$$(a_2 t)y''(t) + (a_1 t + b_1)y'(t) + (a_0 t + b_0)y(t) = 0, \tag{2.20}$$

with $a_2 \neq 0$ and initial conditions $y'(0) = y(0) = 0$.

First observe that

$$ty''(t) = [ty(t)]'' - 2y'(t) \text{ and } ty'(t) = [ty(t)]' - y(t).$$

It follows from $\{f'\} = \lambda\{f\} - f(0)$ and from $\{f\}' = \{-tf\}$ that

$$\{ty''(t)\} = \{[ty(t)]''\} - 2\{y'(t)\} = \lambda\{[ty(t)]'\} - 2\lambda\{y(t)\} = -\lambda^2\{y(t)\}' - 2\lambda\{y(t)\}$$

$$\{ty'(t)\} = \{[ty(t)]' - y(t)\} = \lambda\{ty(t)\} - \{y(t)\} = -\lambda\{y(t)\}' - \{y(t)\}$$

$$\{y''(t)\} = \lambda^2\{y(t)\}$$

$$\{y'(t)\} = \lambda\{y(t)\}$$

Thus, taking the asymptotic Laplace transform on both sides of the equation we have

$$\{(a_2t)y''(t) + (a_1t + b_1)y'(t) + (a_0t + b_0)y(t)\} = \{0\}.$$

Using the formulas derived above we obtain

$$a_2[-\lambda^2\{y\}' - 2\lambda\{y\}] + a_1[-\lambda\{y\}' - \{y\}] + b_1\lambda\{y\} - a_0\{y\}' + b_0\{y\} = \{0\}.$$

Thus,

$$\{y\}'(-a_2\lambda^2 - a_1\lambda - a_0) + \{y\}((b_1 - 2a_2)\lambda + b_0 - a_1) = \{0\}. \text{ Therefore,}$$

$$\begin{aligned} \frac{\{y\}'}{\{y\}} &= \frac{(b_1 - 2a_2)\lambda + b_0 - a_1}{a_2\lambda^2 + a_1\lambda + a_0} \\ &= \frac{\gamma_1}{\lambda - \lambda_1} + \frac{\gamma_2}{\lambda - \lambda_2} \end{aligned}$$

where, λ_1, λ_2 are the distinct roots (in the case they exists) of the equation $a_2\lambda^2 + a_1\lambda + a_0 = 0$, and γ_1, γ_2 are obtain by using the partial fraction decomposition

$$\gamma_1 = \frac{\lambda_1(b_1 - 2a_2) + b_0 - a_1}{\lambda_1 - \lambda_2} \text{ and}$$

$$\gamma_2 = \frac{\lambda_2(b_1 - 2a_2) + b_0 - a_1}{\lambda_2 - \lambda_1}.$$

Hence,

$$\begin{aligned}\ln\{(y(t))\} &= \int \frac{\gamma_1}{\lambda-\lambda_1} + \frac{\gamma_2}{\lambda-\lambda_2}\, d\lambda \\ &= \gamma_1 \ln(\lambda-\lambda_1) + \gamma_2 \ln(\lambda-\lambda_2) + c.\end{aligned}$$

It follows that

$$\{y(t)\} \ni e^{\gamma_1 \ln(\lambda-\lambda_1)+\gamma_2 \ln(\lambda-\lambda_2)+c} = \tilde{c}(\lambda-\lambda_1)^{\gamma_1}(\lambda-\lambda_2)^{\gamma_2}.$$

It follows from Example 2.6 that the function $r : \lambda \mapsto (\lambda-\lambda_1)^{\gamma_1}(\lambda-\lambda_2)^{\gamma_2}$ is a function of minimal exponential type on a certain open cut sector or postsector. Now, let $q_1(\lambda) := (\lambda-\lambda_1)^{\gamma_1}$ and $q_2(\lambda) := (\lambda-\lambda_2)^{\gamma_2}$. If $\gamma_i < 0$ then $q_i(\lambda) = \hat{h}_i(\lambda)$ where, $h_i(t) = \frac{1}{\Gamma(\gamma_i)} t^{-\gamma_i-1} e^{\lambda_i t}$.

If $\gamma_i \geq 0$, then

$$\frac{1}{(\lambda-\lambda_i)^{\gamma_i+2}} q_i(\lambda) = \frac{1}{(\lambda-\lambda_i)^2} \in \{te^{\lambda_i t}\}.$$

Let $k_i(t) := \frac{t^{\gamma_i+1}}{\Gamma(\gamma_i+2)} e^{\lambda_i t}$. Then

$$\hat{k}_i(\lambda) = \frac{1}{(\lambda-\lambda_i)^{\gamma_i+2}}$$

and $\hat{k}_i(\lambda) q_i(\lambda) \in \{te^{\lambda_i t}\}$. Let $h_i := T_{k_i}^{-1}(te^{\lambda_i t}) \in C[0,\infty)^{T_{k_i}}$. Then $k_i * h_i = te^{\lambda_i t}$. Thus,

$$\hat{k}_i(\lambda) q_i(\lambda) \in \{k_i * h_i\}$$

and therefore $q_i(\lambda) \in \frac{\{k_i * h_i\}}{\hat{k}_i} = \{h_i\} \subset \frac{\{k_i * h_i\}}{\{k_i\}} = \{h_i\}$. Thus,

$$\{y(t)\} = \tilde{c}\{h_1\}\{h_2\} \text{ and therefore, } y = \tilde{c}(h_1 * h_2).$$

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Appendix. Remarks on Complex Version of Phragmén-Mikusiński Inversion

In this Appendix we have collected some partial results we obtained trying to extend the Phragmén-Mikusiński inversion for sequences in the complex plane. Although we were not successful, we strongly believe that an inversion formula based on uniqueness sequences in the complex plane should be valid. Let $f \in C_0([0,T],X)$, and let $q(\lambda) := \int_0^\infty e^{-\lambda t} f(t)dt$. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a uniqueness sequence such that $Re\lambda_n$ is a Müntz sequence. Let N_n be such that $c_n := \sum_{k=1}^{N_n} \frac{1}{Re\lambda_{nk}} \geq T$. Since $Re\lambda_n$ is a Müntz sequence the same argument will hold to show that such an N_n exists. We proceed in the same manner defining the functions ϕ_n by:

$$\phi_n(t) = \begin{cases} \lambda_n e^{-\lambda_n(\cdot)} * \lambda_{n2} e^{-\lambda_{2n}(\cdot)} * \ldots * \lambda_{nN_n} e^{-\lambda_n N_n(\cdot)}(t+c_n) & \text{for } t \geq -c_n, \\ 0 & \text{otherwise.} \end{cases}$$

Following the same idea we want to show that ϕ_n converges to the $\delta-$ function: i.e. The antiderivative $\Phi_n := 1 * \phi_n$ converges to the Heaviside function $H(t) = \chi_{(0,\infty)}(t)$. Let $\psi_n := 1 * \lambda_n e^{-\lambda_n(\cdot)} * \lambda_{n2} e^{-\lambda_{2n}(\cdot)} * \ldots * \lambda_{nN_n} e^{-\lambda_n N_n(\cdot)}$. Taking the Laplace transform of it we get

$$\begin{aligned} \psi_n(\lambda) &= \int_0^\infty e^{-\lambda t}\psi_n(t)dt = \frac{1}{\lambda}\frac{\lambda_n}{\lambda+\lambda_n}\cdots\frac{\lambda_{nN_n}}{\lambda+\lambda_{nN_n}} \\ &= \frac{1}{\lambda} + \sum_{k=1}^{N_n} \gamma_{n,k}\frac{1}{\lambda+\lambda_{nk}} \end{aligned}$$

where the $\gamma_{n,k}$ are obtained using the partial fractions method and are given by:

$$\gamma_{n,k} = -\prod_{j=1, j\neq k}^{N_n} \frac{\lambda_{nj}}{\lambda_{nj}-\lambda_{nk}}$$

Since the inverse Laplace transform of $\frac{1}{\lambda+\lambda_{nk}}$ is equal to $e^{-\lambda_{nk}t}$ we obtain that

$$\psi_n(t) = 1 + \sum_{k=1}^{N_n} \gamma_{n,k} e^{-\lambda_{nk}t} \text{ for all } t \geq 0.$$

Therefore

$$\begin{aligned}\Phi_n(t) &= \psi_n(t+c_n) = 1 + \sum_{k=1}^{N_n} \gamma_{n,k} e^{-\lambda_{nk}(t+c_n)} \\ &= 1 - \sum_{k=1}^{N_n} \frac{\alpha_{n,k}}{\lambda_{nk}} e^{-\lambda_{nk} t}\end{aligned}$$

Where the $\alpha_{n,k}$ are defined to be

$$\begin{aligned}\alpha_{n,k} &:= \lambda_{nk} e^{-\lambda_{nk} \sum_{j=1}^{N_n} \frac{1}{Re\lambda_{nj}}} \prod_{j=1, j\neq k}^{N_n} \frac{\lambda_{nj}}{\lambda_{nj}-\lambda_{nk}} \\ &= -\lambda_{nk}\gamma_{n,k} e^{-\lambda_{nk} \sum_{j=1}^{N_n} \frac{1}{Re\lambda_{nj}}}\end{aligned}$$

Next we will show that

$$\left|\frac{\alpha_{n,k}}{\lambda_{nk}}\right| \leq e^{\frac{1+\ln 2}{n} Re\lambda_{nk}}$$

For the argument to work we will need an extra assumption on the sequence $(\lambda_n)_{n\in\mathbb{N}}$: i.e. all λ_n should lie on a line passing through the origin.

$$\begin{aligned}\ln\left|\frac{\alpha_{n,k}}{\lambda_{nk}}\right| &= \ln\left|\gamma_{n,k} e^{-\lambda_{nk} \sum_{j=1}^{N_n} \frac{1}{Re\lambda_{nj}}}\right| \\ &= -Re\lambda_{nk} \sum_{j=1}^{N_n} \frac{1}{Re\lambda_{nj}} + \sum_{j=1}^{k-1} \ln\left|\frac{\lambda_{nj}}{\lambda_{nj}-\lambda_{nk}}\right| + \sum_{j=i+1}^{N_n} \ln\left|\frac{\lambda_{nj}}{\lambda_{nj}-\lambda_{nk}}\right| \\ &\leq -Re\lambda_{nk} \sum_{j=1}^{N_n} \frac{1}{Re\lambda_{nj}} + \sum_{j=1}^{k-1} \ln\frac{|\lambda_{nj}|}{|\lambda_{nk}|-|\lambda_{nj}|} + \sum_{j=i+1}^{N_n} \ln\frac{|\lambda_{nj}|}{|\lambda_{nj}|-|\lambda_{nk}|} \\ &= -Re\lambda_{nk} \sum_{j=1}^{N_n} \frac{1}{Re\lambda_{nj}} + \sum_{j=1}^{k-1} \ln\frac{1}{\frac{Re\lambda_{nk}}{Re\lambda_{nj}}-1} + \sum_{j=k+1}^{N_n} \ln\frac{1}{1-\frac{Re\lambda_{nk}}{Re\lambda_{nj}}} \\ &= -Re\lambda_{nk} \sum_{j=1}^{N_n} \frac{1}{Re\lambda_{nj}} + \sum_{j=1}^{k-1} \ln\frac{Re\lambda_{nj}}{Re\lambda_{nk}-Re\lambda_{nj}} \\ &\quad + \sum_{j=k+1}^{N_n} \ln\frac{Re\lambda_{nj}}{Re\lambda_{nj}-Re\lambda_{nk}}\end{aligned}$$

Since $Re\lambda_n$ is a Müntz sequence it follows that

$$\ln\left|\frac{\alpha_{n,k}}{\lambda_{nk}}\right| \leq \frac{Re\lambda_{nk}(1+\ln 2)}{n}$$

Next we show that $\Phi_n \to 1$ for all $t > 0$.

$$\begin{aligned}
|\Phi_n(t) - 1| &= \left| \sum_{j=1}^{N_n} \frac{\alpha_{n,j}}{\lambda_{nj}} e^{-\lambda_{nj} t} \right| \\
&\leq \sum_{j=1}^{N_n} \left| \frac{\alpha_{n,j}}{\lambda_{nj}} \right| e^{-Re\lambda_{nj} t} \\
&\leq \sum_{j=1}^{\infty} e^{\frac{Re\lambda_{nj} 2}{n}} e^{-Re\lambda_{nj} t} \\
&\leq \sum_{j=1}^{\infty} e^{\frac{-Re\lambda_{nj} t}{2}} \\
&\leq \sum_{j=1}^{\infty} e^{-\frac{njt}{2}} = \frac{e^{-nt/2}}{1 - e^{-nt/2}}
\end{aligned}$$

Therefore $\Phi_n(t) \to 1$ as $n \to \infty$.

Vita

Claudiu Mihai was born on October 25, 1967, in Gheraesti Village, Neamt, Romania. He finished his undergraduate studies at Bucharest University, June 1993. In August 1999 he came to Louisiana State University to pursue graduate studies in mathematics. He earned a master of science degree in mathematics from Louisiana State University in May 2001. He is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in December 2004.