2012

Modeling of dynamics of gyroscopic systems

Siddharth Sharma
Louisiana State University and Agricultural and Mechanical College, sshar13@lsu.edu

Follow this and additional works at: https://digitalcommons.lsu.edu/gradschool_theses

Part of the Mechanical Engineering Commons

Recommended Citation
https://digitalcommons.lsu.edu/gradschool_theses/3923

This Thesis is brought to you for free and open access by the Graduate School at LSU Digital Commons. It has been accepted for inclusion in LSU Master's Theses by an authorized graduate school editor of LSU Digital Commons. For more information, please contact gradetd@lsu.edu.
MODELING OF DYNAMICS OF GYROSCOPIC SYSTEMS

A Thesis

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Master of Science in Mechanical Engineering

in

The Department of Mechanical and Industrial Engineering

by

Siddharth Sharma
B. Tech., Rajasthan Technical University, 2010
December 2012
I wish to dedicate this thesis to my sister Shruti Sharma who is eyeing to embark upon her PhD program in Fall 2013.

Shruti is currently pursuing her Bachelor of Technology degree in Metallurgical Engineering and Materials Science at Indian Institute of Technology (IIT), Bombay, India. I hope this dedication will serve as an inspiration for Shruti to continue to follow her dream of a doctorate degree.
Acknowledgements

I convey my sincere thanks to Dr. Yitshak Ram for irrigating my research aptitude by asking questions, for showing absolute faith in me and my abilities, and for supporting me in all my endeavors in the past couple of years. The insights of the discussions with Dr. Ram will have a profound impact on my life. I wish to thank him for showering his personal attention on me and for sharing his passion for research. I will always rejoice his teachings on the physics involved in rotor dynamics and the analytical and mathematical treatment of the subject. Thank you for everything, Dr. Ram.

I express my deepest regards for Dr. Marcio de Queiroz for his consistent support. Dr. de Queiroz is credited to firmly steer my boat to the shore and I am highly indebted to him for making my Masters degree a reality. I thank him for his comments and revisions of my thesis document. I earnestly put across my heartiest greetings to Siemens Industry, Inc. facility located in Cincinnati, Ohio, USA for funding my Master’s thesis research work. I am glad to be mentored by Mr. Sumit Singhal, Research and Development Engineer, Siemens Dynamowerk, Berlin, Germany. Sumit’s comments and his vision about the implications of my research work to the practical world shaped the industrial outlook for my project. I also wish to thank Dr. Guoqiang Li and Dr. Su-Seng Pang for being my Graduate Committee members and for evaluating my research work.

A special note of appreciation goes to Ms. Diane Morgan for patiently helping me with all the departmental paperwork so far. The LSU Transit facility receives warm thanks for driving me free and safe to and back from my lab during night hours. The questions from my undergraduate students of the Dynamics course and Design Lab were instrumental in making me understand the subjects deeply. Therefore, I thank them all to allow me to teach them as a student of the subjects!

I feel out of words and full of emotions when I think of thanking my family. I can only say that I am blessed to be born to the good souls of Mrs. Mamta Sharma and Mr. Pradeep Kumar Sharma. I feel special to be the brother of Shruti Sharma. The overwhelming love of my grandparents, Mrs. Usha Rani Sharma and Mr. Shiv Raj Singh Sharma is beyond any limits. I also cherish the consistent support of all my relatives. I take great pride in belonging to the Indian culture which respects education so high in rank that families which have traditionally tried to keep their members close by, now they don’t ever think to restrain their children from going to the other side of the planet for world class higher education.

I am thankful to Mr. Vishva Raj Bangad who has played the role of my elder brother in the US and has helped me every bit in establishing a hold in this illustrious nation. His wife, Ms. Haritma Sodhi and their son Daksh Raj Bangad have endorsed upon me the feeling of a family member and I am happily grateful to them. Mr. Prithu Sharma, a doctoral candidate at MIT, walks away with particular thanks for all his help and professional guidance throughout the last decade of my life. The mere presence of Mr. Pankaj P Kulkarni in my life defines the meaning of friendship for me.

LSU has a beautiful campus and my stay would have been incomplete without finding a soul-friend in Sai Sashankh Rao. Pratap Rao has proved to be a good companion and a perfect host along with Sai. I also thank Pranaya Pokharel, Arpitha Prakash and Narendra...
Na for being so nice to me. The reassuring smiles shared by Manoj and Ramanathan during my initial period at LSU will stay in my memories forever.

Last but not the least, I wish to acknowledge my undergraduate program teachers who introduced me to the world of mechanical engineering. I will always remember Dr. K. V. S. Rao, Mr. Praveen Bhandari, Mr. Anant Ballal, Dr. Sandeep Parashar, Dr. S. K. Rathore, Dr. Sanjeev Mishra, Dr. Arvind Dwivedi and Mr. Rajiv Rajora for nurturing my inquisitiveness through their lectures, showcasing the art of teaching and proving as points of excellence till my graduation in 2010.
**Table of Contents**

Acknowledgements ........................................................................................................... iii

List of Tables ..................................................................................................................... vii

List of Figures ................................................................................................................... viii

Abstract ............................................................................................................................ ix

Chapter 1  Introduction .................................................................................................... 1
  1.1 Literature Review .................................................................................................... 1
  1.2 Motivation .............................................................................................................. 2
  1.3 The Concept of Stability and Root Locus Technique ........................................ 2

Chapter 2  Equations of Relative Position, Velocity and Acceleration ....................... 6
  2.1 Transformation of Coordinates ............................................................................ 8

Chapter 3  Modeling of a Rotating Shaft with No Displacement at Bearing End ........ 10

Chapter 4  Modeling of a Mass-less Rotating Shaft with Finite Displacement at Bearing
  End ................................................................................................................................. 13
  4.1 Symmetrical Shaft and Symmetrical Bearings (SSSB) ........................................ 13
    4.1.1 Modeling of SSSB System ............................................................................. 13
    4.1.2 Equations of Motion for SSSB System ....................................................... 14
    4.1.3 Mathematical and Analytical Treatment ..................................................... 15
    4.1.4 Root Locus ................................................................................................. 17
    4.1.5 Stability ................................................................................................. 19
    4.1.6 Parametric Investigation of Stability ......................................................... 20

Chapter 5  Modeling of a Shaft Having Mass and Rotating with Finite Displacement at
  Bearing End .................................................................................................................... 24
  5.1 Root Locus ......................................................................................................... 26
  5.2 Stability ............................................................................................................. 31
  5.3 Parametric Investigation of Stability ................................................................. 31

Chapter 6  Conclusions ................................................................................................. 35
  6.1 Future Work ......................................................................................................... 36

References ....................................................................................................................... 37

Appendix A Codes for Chapter 4 (mass-less shaft) ......................................................... 38

Appendix B Codes for Chapter 5 (shaft with mass) ......................................................... 44
List of Tables

Table 4.1: Poles of the mass-less shaft system at the stability boundary………………..19

Table 4.2: Stability boundary of Smith’s system………………………………………….20

Table 4.3: Parametric investigation of stability at lower critical speed based on damping coefficients in a mass-less shaft system………………………………………………….20

Table 4.4: Parametric investigation of stability at higher critical speed based on damping coefficients in a mass-less shaft system………………………………………………….21

Table 4.5: Parametric investigation of stability at lower critical speed based on stiffness coefficients in a mass-less shaft system………………………………………………….21

Table 4.6: Parametric investigation of stability at higher critical speed based on stiffness coefficients in a mass-less shaft system………………………………………………….22

Table 4.7: Parametric investigation of stability at lower critical speed based on rotor mass in a mass-less shaft system………………………………………………………………22

Table 4.8: Parametric investigation of stability at higher critical speed based on rotor mass in a mass-less shaft system………………………………………………………...23

Table 5.1: Poles of the shaft with mass system at the stability boundary………………..31

Table 5.2: Parametric investigation of stability at critical speed based on damping in a shaft with mass system…………………………………………………………………...32

Table 5.3: Parametric investigation of stability at critical speed based on stiffness coefficients in a shaft with mass system……………………………………………………….33

Table 5.4: Parametric investigation of stability at critical speed based on rotor mass in a shaft with mass system………………………………………………………………….33

Table 5.5: Parametric investigation of stability at critical speed based on shaft mass in a shaft with mass system………………………………………………………………….34
List of Figures

Figure 1.1: Stable, unstable and neutral equilibriums………………………………………3

Figure 1.2: (a) Single degree of freedom spring mass system………………………….………….3
  (b) Free body diagram of mass \( m \) in Figure 1.2 (a)

Figure 1.3: (a) Single degree of freedom spring mass damper system…………………………….4
  (b) Free body diagram of mass \( m \) in Figure 1.3 (a)

Figure 2.1: Stationary and rotating coordinate system……………………………………….6

Figure 2.2: Transformation of coordinates…………………………………………………..8

Figure 3.1: (a) System of shaft rotating in clamped roller bearing………………………….10
  (b) Free body diagram of rotor \( B \)

Figure 4.1: (a) System of mass-less shaft rotating in bearing with finite displacement at
  the bearing end…………………………………………………………………….13
  (b) Planar coordinate system of shaft motion
  (c) Free body diagram of mass-less shaft \( AB \) (moments not shown)
  (d) Free body diagram of rotor \( B \) of mass \( M \)

Figure 4.2: Comparison of root locus of Smith’s and our poles at lower angular speeds.17

Figure 4.3: Comparison of root locus of Smith’s and our poles at higher angular speeds18

Figure 5.1: (a) System of shaft of mass \( m \) rotating in bearing with finite displacement at
  the bearing end…………………………………………………………………….24
  (b) Planar coordinate system of shaft motion
  (c) Free body diagram of shaft \( AB \) of mass \( m \) (moments not shown)
  (d) Free body diagram of rotor \( B \) of mass \( M \)

Figure 5.2: (a) Root locus of shaft having mass and rotating at very slow speed………..26
  (b) Enlarged view of roots in Figure 5.2(a)

Figure 5.3: (a) Root locus of shaft having mass and rotating at slow speed……………..28
  (b) Enlarged view of roots in Figure 5.3(a)

Figure 5.4: (a) Root locus of shaft having mass and rotating at faster speed……………29
  (b) Enlarged view of roots in Figure 5.4(a)

Figure 5.5: (a) Root locus of shaft having mass and rotating at high speed……………30
  (b) Enlarged view of roots in Figure 5.5(a)
Abstract

The amplitude of vibrations of a rotating system operating near or at its critical speed grows without bound. This poses a serious problem for the rotating machinery industry from the perspective of design, maintenance, vibration troubleshooting and stability of the rotating equipments.

In this thesis, we wish to develop an exact method of analysis for the system of a massless symmetrical shaft rotating in symmetrical bearings (SSSB) with finite displacement at bearing end. The method so developed is expected to be extended to the SSSB model of a shaft having mass. The stability analysis is performed for both the systems. The parametric study of the effect of mass, damping and stiffness characteristics of the shaft-bearing system on the stability of the system is also carried out. This thesis closely examines the works carried out by D. M. Smith (1933) for the SSSB case and the destabilizing, counter-intuitive phenomenon of internal damping of rotating shaft mentioned by Crandall.

As a starting point, we analyze the undamped SSSB system with no displacement at bearing end, i.e., the system rotates in clamped roller bearings. This system serves as an introductory model for the mass-less SSSB system and for shaft with mass SSSB system. The basic theory to derive the equations of motion for this system is based on the development of equations of relative position, velocity and acceleration. The powerful concept of the transformation of coordinates from rotational coordinate system to stationary coordinate system (or vice versa) is also showcased.

The thesis starts with an introduction to the concept of stability for systems. The concept of stability culminates in the determination of the stability boundary of the system which is based on the nature of poles of the system. The conclusions and future work are reported in the end.
Chapter 1

Introduction

Rotordynamics is the study of forces acting on a rotating member and its behavior under their influence. The theory of Dynamics acts as a tool in understanding the physics of a rotating system and the theory of Vibration serves as a powerful instrument to mathematically quantify the behavior of a rotor. In the simplest terms, a rotor is a shaft rotating in bearings with a mass mounted on it. One of the most common functions of such a system is to generate power. Electric motors, turbines, compressors, pumps and computer disk storage are some examples of rotating systems.

A good understanding of the behavior of the rotor is critical to a rotordynamicist who is responsible for the proper functioning, diagnosis and preventive actions taken for the rotating equipments in the industry. The study of gyroscopic systems presents an opportunity of application of the principles of mechanical engineering to rotating systems. A rotating system is fundamentally different from a non-rotating system because of the presence of the Coriolis Effect. As laid out further in the text, the Coriolis component plays a key role in the determination of stability boundary of a rotating system. The counter-intuitive phenomenon of destabilizing nature of internal damping of the shaft also invigorates curiosity.

The concept of critical speed of a rotating system bears equivalence to resonance in a non-rotating system. Critical speed occurs when the rotational speed of the shaft coincides with its natural frequency. The shapes of the vibrating rotor obtained at critical speeds are referred as its mode shapes.

1.1 Literature Review

Rankine (1869) was the first one to analyze a shaft rotating in bearings. However, he did not take Coriolis acceleration into account. Dunkerley is credited with the coinage of the term ‘Critical Speed’ in 1895. Föppl (1895), a German civil engineer, successfully analyzed the undamped model of a circular shaft with a centrally located single disk. He showed that stable operation was possible above the rotating speed mentioned by Rankine. Unfortunately, he published his research in the journal Der Civilingenieur which was read very little by the rotordynamics community of his time.

Kerr (1916) showed experimental evidence of the existence of a second critical speed. Jeffcott was appointed by the Royal Society of London in 1919 to bring forward the fallacy between Rankine’s theory and Kerr’s work. He analyzed a model similar to Föppl’s. He also included damping in his model. His analysis was published in a widely read English journal. His results of the existence of supercritical operation verified the works of Föppl, Kerr and the Swedish engineer, Laval who operated a single stage impulse turbine super critically at 42,000 rpm in 1883. A single disk rotor is now widely called as a Jeffcott rotor in order of appreciation of the work performed by him.

D. M. Smith performed a ground breaking work in his paper titled as, ‘The motion of a rotor carried by a flexible shaft in flexible bearings’ which was published in the Royal Society of London, Series A in 1933. He called the shaft and bearings as symmetric when it had uniform elastic and damping characteristics in all transverse directions. Amongst his other conclusions, he approximated that a symmetrical mass-less shaft with a rotor
disk mounted on it and rotating in symmetrical bearings in presence of rotary damping, results in a determinant equation of degree four, which is obtained from the equation of motion of the shaft. He further added that when both rotary and stationary damping are present, the system never regains stability after crossing a certain transition speed which is higher than the critical speed of the system.

It is well known that an increase in damping in a non-rotating system will produce stabilizing effects. However, this does not always hold true for a rotating system. Crandall (1980) mentioned the destabilizing, counter-intuitive phenomenon of internal damping of shaft in his work, ‘Physical explanations of the destabilizing effect of damping in rotating parts’. Internal damping of a shaft can be experienced by performing bending or fatigue loading on a metal wire which heats it up. Before Crandall, it was A.L. Kimball, Jr. (1925) who related damping forces in a bent shaft to its whirling motion for supercritical operation.

The author recommends referring to the works of Nelson (2007) and Swanson et al. (2005) to gain basic understanding of rotordynamics. Malcolm Leader’s work mentioned in the ‘References’ gives a practical insight into rotordynamics. A timeline on rotordynamics is available in the short term course document of Tiwari (2008).

1.2 Motivation

The motivation of this thesis is to develop a model similar to Smith’s model of a symmetrical shaft rotating in symmetrical bearings and perform an exact analytical treatment of the model. Smith made certain approximations in his work and reached a fourth order determinant equation for a mass-less shaft. The author is interested in verifying the accuracy of Smith’s approximations by performing an exact analysis. The model is expected to be extended to include the mass of the shaft and the stability analysis will be performed. The author finds the foundation of the thesis by deriving the equations of motion for a shaft rotating in clamped bearings and compares it to the equations presented in the book, ‘Principles of Structural Stability’ by Hans Ziegler (1968).

The objective of this thesis is to study the rotodynamic aspects of rotors and develop an understanding of critical speed, stability boundary and the effects of mass, damping and stiffness of the shaft and the bearings on the critical speed values of rotor by performing parametric investigation.

1.3 The Concept of Stability and Root Locus Technique

The stability analysis of a system is critically important as such to avoid large vibrations in the system at the stability boundary. Stability boundary occurs when at least a pair of poles of the system becomes purely imaginary.

The introductory example helpful in understanding the concept of stability is shown in Figure 1.1. The ball rolls down the curve under the action of gravity and subsequently settles down to a point of equilibrium in Figure 1.1(a). This point is called the point of stable equilibrium because if the ball is disturbed again, it will always roll down to the same point. The explanation is a consequence of our observation of the physical world around us.
Contrastingly in Figure 1.1(b), the ball is resting at a point on the crest of the curve. Even a slight disturbance initiates motion and the ball rolls down the curve, moreover it never returns to the same point again. This is called a condition of unstable equilibrium.

However, a ball lying on a flat surface in Figure 1.1(c) is said to be in neutral equilibrium. The ball always comes to rest at a new position whenever it is disturbed. There exist infinite points of equilibrium on the flat surface.

Building on this visual and intuitive example, we examine a single degree of freedom spring-mass system which is shown in Figure 1.2(a).

The equation of motion derived from the free body diagram in Figure 1.2(b) is,

\[ m\ddot{x}(t) + kx(t) = 0 \]  

(1.1)

where \( m \) is the mass of the system, \( k \) is the stiffness of the linear spring, \( x \) is the position coordinate referenced at the equilibrium position of the block and \( t \) represents time.
The periodic motion of the system shown in Figure 1.2(a) can be trapped as a solution of the form,

\[ x(t) = A \sin(\omega_n t + \phi) \]

(1.2)

where the constants \( A \) and \( \phi \) are called as amplitude and phase shift respectively. \( \omega_n \) is the natural frequency of motion such that

\[ \omega_n = \sqrt{\frac{k}{m}}. \]

(1.3)

The characteristic equation, which corresponds to the non-trivial solutions of the system, is obtained by substituting \( x(t) = Ae^{\lambda t} \) in (1.1). This leads to

\[ \lambda^2 + \frac{k}{m} = 0. \]

(1.4)

It is evident from the above equation that the root \( \lambda \) of the characteristic equation is purely imaginary as long as \( k \) and \( m \) are positive and therefore \( \lambda \) lies across the quadrature axis of the complex plane. It can be seen from (1.2) that the solution is bounded by the constant \( A \). Thus, the system is said to be marginally stable. More specifically, the system is at the stability boundary when the roots of its characteristic equation become purely imaginary.

However, if \( k \) is negative, the solution becomes unbounded as \( t \) increases and the system becomes unstable which is governed by

\[ x(t) = A \sinh(\omega_n t) + B \cosh(\omega_n t). \]

(1.5)

A more realistic vibration model can be studied by the addition of damping forces.

Figure 1.3: (a) Single degree of freedom spring mass damper system

(b) Free body diagram of mass \( m \) in Figure 1.3 (a)

The equation of motion of a single degree of freedom spring-mass-damper system shown in Figure 1.3 (b) can be written as
\[ m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0 \quad (1.6) \]

where \( c \) is the damping coefficient of the system.

The characteristic equation is obtained by substituting \( x(t) = Ae^{\lambda t} \) in (1.6) which leads to

\[ \lambda^2 + \frac{c}{m} \lambda + \frac{k}{m} = 0. \quad (1.7) \]

A system governed by (1.6) is said to be stable if the roots of the characteristic equation (1.7) are purely imaginary or in other words, the roots lie along the imaginary axis of the complex plane. The system will be unstable if the real part of the roots is positive, or to say that the roots lie in the right half side of the complex plane. When the roots have their real parts as negative, it makes the system asymptotically stable. The path of the roots traced on a complex plane is called ‘Root Locus’. Root locus serves as an important tool in determination of the stability boundary of the system when the roots of the characteristic equation change the sign of their real part.
Chapter 2

Equations of Relative Position, Velocity, and Acceleration

In this section, the basic theory for chapter 3 is developed which will be used to derive the equations of motion of a shaft rotating in clamped roller bearings.

Let \( xyz \) represent a stationary coordinate system in Figure 2.1. The rotating coordinate system \( \chi \nu \zeta \) rotates with an angular velocity \( \omega \) with respect to the \( xyz \) stationary coordinate system. The origins of \( xyz \) and \( \chi \nu \zeta \) coordinate systems are at points \( O \) and point \( A \), respectively.

![Figure 2.1: Stationary and rotating coordinate system](image)

The vectors \( \mathbf{r}_A \) and \( \mathbf{r}_B \) are the position vectors of points \( A \) and point \( B \) originating from origin \( O \), respectively. The position of point \( B \) with respect to origin \( A \) is \( \mathbf{r}_{B/A} \).

By the triangle law of vector addition, we have

\[
\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}. \tag{2.1}
\]

The operator rule of differentiation is defined as

\[
\left( \frac{d\mathbf{p}}{dt} \right)_{xyz} = \left( \frac{d\mathbf{p}}{dt} \right)_{\chi\nu\zeta} + \omega \times \mathbf{p} \tag{2.2}
\]

where \( \mathbf{p} \) represents any vector and \( t \) represents time.

Differentiating (2.1) with respect to time \( t \) in the \( xyz \) coordinate system gives

\[
\mathbf{v}_B = \mathbf{v}_A + \left( \frac{d\mathbf{r}_{B/A}}{dt} \right)_{xyz} \tag{2.3}
\]

where \( \mathbf{v}_B \) and \( \mathbf{v}_A \) are the velocities of \( B \) and \( A \) in the \( xyz \) coordinate system respectively.

Applying (2.2) to (2.3), we have

\[
\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} + \omega \times \mathbf{r}_{B/A} \tag{2.4}
\]
where \( \mathbf{v}_{B/A} \) is the velocity of \( B \) with respect to \( A \) in the \( \chi\nu\zeta \) coordinate system.

Similarly, differentiation of (2.4) with respect to time \( t \) in the \( xyz \) coordinate system leads to

\[
\begin{align*}
\mathbf{a}_B &= \mathbf{a}_A + \mathbf{a}_{B/A} + \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}_{B/A}) + 2 \mathbf{\omega} \times \mathbf{v}_{B/A} + \dot{\mathbf{\omega}} \times \mathbf{r}_{B/A} \tag{2.5}
\end{align*}
\]

where \( \mathbf{a}_B \) and \( \mathbf{a}_A \) are the accelerations of \( B \) and \( A \) in the \( xyz \) coordinate system, respectively. The acceleration of \( B \) with respect to \( A \) in the \( \chi\nu\zeta \) coordinate system is \( \dot{\mathbf{a}}_{B/A} \). The rate of change of angular velocity \( \mathbf{\omega} \) of the \( \chi\nu\zeta \) coordinate system with respect to time is \( \dot{\mathbf{\omega}} \).

By definition, the vectors \( \mathbf{r}_A, \mathbf{r}_B, \mathbf{r}_{B/A} \) and \( \mathbf{\omega} \) are written in matrix form as

\[
\begin{align*}
\mathbf{r}_A &= \begin{pmatrix} x_A \\ y_A \\ z_A \end{pmatrix}, \quad \mathbf{r}_B &= \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{r}_{B/A} &= \begin{pmatrix} \chi \\ \nu \\ \zeta \end{pmatrix}, \quad \mathbf{\omega} &= \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}. \tag{2.6}
\end{align*}
\]

It follows that

\[
\begin{align*}
\mathbf{a}_A &= \begin{pmatrix} \dot{x}_A \\ \dot{y}_A \\ \dot{z}_A \end{pmatrix}, \quad \mathbf{a}_B &= \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}, \quad \mathbf{v}_{B/A} &= \begin{pmatrix} \dot{\chi} \\ \dot{\nu} \\ \dot{\zeta} \end{pmatrix}, \quad \mathbf{a}_{B/A} &= \begin{pmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{pmatrix}. \tag{2.7}
\end{align*}
\]

Let \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) be the unit vectors of the \( xyz \) coordinate system. Thus, we have

\[
\begin{align*}
\mathbf{\omega} \times \mathbf{r}_{B/A} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ \chi & \nu & \zeta \end{vmatrix} = \begin{pmatrix} \omega_z \nu - \omega_y \chi \\ -\omega_x \zeta + \omega_y \chi \\ \omega_x \nu - \omega_z \chi \end{pmatrix}. \tag{2.8}
\end{align*}
\]

Rearranging the right hand side of above equation, we obtain

\[
\begin{align*}
\mathbf{\omega} \times \mathbf{r}_{B/A} &= \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \begin{pmatrix} \chi \\ \nu \\ \zeta \end{pmatrix}. \tag{2.9}
\end{align*}
\]

It follows that

\[
\begin{align*}
\mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}_{B/A}) &= \begin{bmatrix} -\omega_y^2 - \omega_z^2 & \omega_y \omega_x & \omega_z \omega_x \\ \omega_x \omega_y & -\omega_x^2 - \omega_z^2 & \omega_x \omega_y \\ \omega_x \omega_z & \omega_y \omega_z & -\omega_x^2 - \omega_y^2 \end{bmatrix} \begin{pmatrix} \chi \\ \nu \\ \zeta \end{pmatrix}, \tag{2.10}
\end{align*}
\]
2.1 Transformation of Coordinates

Let $Ox$ and $A\chi$ in Figure 2.1 be coincident such that origin $O$ of the $xyz$ stationary coordinate system coincides with origin $A$ of the $\chi\nu\zeta$ rotating coordinate system. Thus, the plane perpendicular to $Ox$ and $A\chi$ consists of $Oy, Oz, A\nu$ and $A\zeta$ with origin at $A$ as shown in Figure 2.2.

\[2\omega \times v_{r/A} = 2 \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \tag{2.11}\]

and

\[\omega \times r_{r/A} = \begin{bmatrix} 0 & -\dot{\omega}_z & \dot{\omega}_y \\ \dot{\omega}_z & 0 & -\dot{\omega}_x \\ -\dot{\omega}_y & \dot{\omega}_x & 0 \end{bmatrix} \begin{bmatrix} \chi \\ \nu \\ \zeta \end{bmatrix}. \tag{2.12}\]

The stationary coordinate system transforms to rotating coordinate system by an angle $\theta = \omega t$. This transformation of coordinates is governed by the following equations:

\[y = \nu \cos \theta - \zeta \sin \theta \tag{2.13}\]

and

\[z = \nu \sin \theta + \zeta \cos \theta. \tag{2.14}\]

The set of equations (2.13) and (2.14) can be arranged as

\[
\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{bmatrix} \begin{bmatrix} \nu \\ \zeta \end{bmatrix}. \tag{2.15}\]

The inverse of the matrix with trigonometric elements in (2.15) leads to

\[
\begin{bmatrix} \nu \\ \zeta \end{bmatrix} = \begin{bmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}. \tag{2.16}\]
Differentiating equation (2.16) with respect to time gives

\[
\begin{pmatrix}
\dot{\nu} \\
\dot{\zeta}
\end{pmatrix} =
\begin{pmatrix}
\cos \omega t & \sin \omega t \\
-\sin \omega t & \cos \omega t
\end{pmatrix}
\begin{pmatrix}
\dot{y} \\
\dot{z}
\end{pmatrix}
+ \omega
\begin{pmatrix}
-\sin \omega t & \cos \omega t \\
-\cos \omega t & -\sin \omega t
\end{pmatrix}
\begin{pmatrix}
y \\
z
\end{pmatrix}.
\]

(2.17)

The set of equations (2.16) and (2.17) will be used in chapter 4 to transform the equation of motion of a symmetrical shaft which rotates in symmetrical bearings with a finite displacement at bearing end, from rotating coordinates to stationary coordinates.
Chapter 3

Modeling of a Rotating Shaft with No Displacement at Bearing End

The system shown in Figure 3.1(a) consists of a rotor mounted on a shaft that is rotating with constant angular velocity $\omega$ in clamped roller bearings. The stationary coordinate system $xyz$ and the rotating coordinate system $\chi \nu \zeta$ originate at the common origin $A$. The mass of the system is taken to be the mass of the rotor. The stiffness of the shaft represents the stiffness of the system. The shaft has a stiffness coefficient $k_\nu$ in the $\nu$ direction and a stiffness coefficient $k_\zeta$ in the $\zeta$ direction. The bearings are clamped roller bearings with zero displacement of the shaft end in it.

![Diagram of system](image)

Figure 3.1: (a) System of shaft rotating in clamped roller bearing  
(b) Free body diagram of rotor $B$

The origin $A$ of the $xyz$ and $\chi \nu \zeta$ coordinate systems represents the centre of the cross section of the shaft in the clamped roller bearings as shown in Figure 3.1(a). The effect of gravity on the system is neglected in this model. The point $B$ locates the centre of mass of the rotor, which is deflected from its equilibrium position when the shaft is rotating at constant angular velocity $\omega$. The equilibrium position of the rotor coincides with point $A$ in Figure 3.1(b). The rotating coordinate system $\chi \nu \zeta$ sweeps an angle $\theta = \omega t$ from the stationary coordinate system $xyz$.

The coordinate systems defined in chapter 2 are employed in this model. The points $O$ and $A$ in Figure 2.1 converge to a single point in Figure 2.2, reducing equation (2.6) to

$$
\begin{align*}
\mathbf{r}_A &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\
\mathbf{r}_B &= \begin{pmatrix} 0 \\ y \\ z \end{pmatrix}, \\
\mathbf{r}_{B/A} &= \begin{pmatrix} 0 \\ \nu \\ \zeta \end{pmatrix}, \\
\mathbf{\omega} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\end{align*}
$$

(3.1)

From equations (2.7), (2.10), (2.11), (2.12) and (3.1), it follows that
\[
a_A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad a_B = \begin{pmatrix} 0 \\ \ddot{y} \\ \ddot{z} \end{pmatrix}, \quad v_{B/A} = \begin{pmatrix} 0 \\ \dot{v} \\ \dot{\zeta} \end{pmatrix}, \quad a_{B/A} = \begin{pmatrix} 0 \\ \ddot{v} \\ \ddot{\zeta} \end{pmatrix}, \quad \dot{\omega} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]
(3.2)

\[
\omega \times (\omega \times r_{B/A}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\omega^2 & 0 \\ 0 & 0 & -\omega^2 \end{bmatrix} \begin{pmatrix} 0 \\ \nu \\ \zeta \end{pmatrix} = \begin{pmatrix} 0 \\ -\omega^2 \nu \\ -\omega^2 \zeta \end{pmatrix},
\]
(3.3)

\[
2\omega \times v_{B/A} = 2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\omega & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ \dot{v} \\ \dot{\zeta} \end{pmatrix} = 2 \begin{pmatrix} 0 \\ -\omega \dot{\zeta} \\ \omega \dot{\nu} \end{pmatrix}
\]
(3.4)

and

\[
\omega \times r_{B/A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \chi \\ \nu \\ \zeta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]
(3.5)

The results of (3.2), (3.3), (3.4) and (3.5) when assembled into (2.5) give

\[
a_B = \begin{pmatrix} 0 \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \dot{v} & \dot{\zeta} \\ 0 & \dot{\nu} & \dot{\zeta} \end{bmatrix} + \begin{pmatrix} -\omega^2 \nu \\ -\omega \dot{\zeta} \\ \omega \dot{\nu} \end{pmatrix} = \begin{pmatrix} 0 \\ \ddot{v} - 2\omega \dot{\zeta} - \omega^2 \nu \\ \ddot{\zeta} + 2\omega \dot{\nu} - \omega^2 \zeta \end{pmatrix}.
\]
(3.6)

Newton’s second law of motion for mass B

\[
\sum \vec{F} = M \vec{a}_B
\]
(3.7)
can be written in the matrix form using Figure 3.2(b) and (3.6)

\[
\begin{bmatrix} 0 \\ -k_y \nu \\ -k_z \zeta \end{bmatrix} = M \begin{pmatrix} 0 \\ \ddot{v} - 2\omega \dot{\zeta} - \omega^2 \nu \\ \ddot{\zeta} + 2\omega \dot{\nu} - \omega^2 \zeta \end{pmatrix}
\]
(3.8)

where \( M \) is the mass of the rotor \( B \).

The matrix form of (3.8) yields

\[
\begin{align*}
M \ddot{v} - 2M \omega \dot{\zeta} + (k_y - M \omega^2) \nu &= 0 \\
M \ddot{\zeta} + 2M \omega \dot{\nu} + (k_z - M \omega^2) \zeta &= 0
\end{align*}
\]
(3.9)

which appears as (1.56) on page 17 of the book, ‘Principles of Structural Stability’ published in 1968 with Hans Ziegler as its author. This validates our method of derivation of equations of motion as the modeling of this model is similar to the one used by Zeigler but the method of derivation of equations of motion is different,
The equation (3.9) is a very important equation as it involves the Coriolis term \(-2M\omega \dot{\zeta}, 2M\omega \dot{v}\). It is well known that if the coefficients of acceleration, velocity and position of a damped system are all positive then the system is asymptotically stable. It is the presence of the Coriolis term which makes it a requirement for a rotating system to be analyzed for stability considerations. The Coriolis term in (3.9) is responsible for the origin of gyroscopic effects in the system.

The model presented in this chapter is the simplest and it serves as a window to understand the more complex models mentioned in chapter 4 and chapter 5.
Chapter 4

Modeling of a Mass-less Rotating Shaft with Finite Displacement at Bearing End

4.1 Symmetrical Shaft and Symmetrical Bearings (SSSB)

A shaft or a bearing is called symmetrical when it has the same damping and stiffness coefficients in all perpendicular directions. For example, a shaft with a key-way or an oval shaft is not considered as symmetrical.

4.1.1 Modeling of SSSB System

The mass-less shaft in Figure 4.1(a) has a finite displacement at the bearing end resulting in $A(AA_{A})$ to be different from the origin $O$ as shown in Figure 4.1(b). Point $B$ is the center of mass of the rotor deflected from its equilibrium position. The shaft rotates with an angular velocity $\omega$ such that $\dot{\theta} = \omega t$.

The stiffness and damping coefficients of the symmetric bearings are defined as $k'$ and $\sigma'$ respectively. The symbols $k''$ and $\sigma''$ are used to denote the stiffness and damping coefficients of the symmetric shaft respectively as shown in Figure 4.1(b).

Figure 4.1: (a) System of mass-less shaft rotating in bearing with finite displacement at the bearing end (b) Planar coordinate system of shaft motion (c) Free body diagram of mass-less shaft $AB$ (moments not shown) (d) Free body diagram of rotor $B$ of mass $M$. 
4.1.2 Equations of Motion for SSSB System

Let a force $\mathbf{f}_M$ act on rotor $B$ in the $y-z$ coordinate system. The free body diagram of the rotor in Figure 4.1(d) results in

$$
\mathbf{f}_M = k\begin{bmatrix}
-\cos \omega t & \sin \omega t \\
-\sin \omega t & -\cos \omega t
\end{bmatrix}(\dot{y} - \dot{y}_A) + \sigma \begin{bmatrix}
-\cos \omega t & \sin \omega t \\
-\sin \omega t & -\cos \omega t
\end{bmatrix}(\dot{z} - \dot{z}_A).
$$

(4.1)

Using (2.16) and (2.17) into (4.1), the transformation of coordinates yields

$$
\mathbf{f}_M = -\begin{bmatrix}
\sigma'' & 0 \\
0 & \sigma''
\end{bmatrix}(\dot{y} - \dot{y}_A) - \begin{bmatrix}
k' & 0 \\
0 & k'
\end{bmatrix}(\dot{\zeta} - \dot{\zeta}_A) + \omega \begin{bmatrix}
0 & \sigma'' \\
-\sigma'' & 0
\end{bmatrix}(y - y_A) + z - z_A)
$$

(4.2)

where $(y, z)$ are the coordinates of $B$ in the stationary coordinate system.

It thus follows from Newton’s second law for the rotor of mass $M$ that

$$
M \begin{bmatrix}
\ddot{y} \\
\ddot{z}
\end{bmatrix} + \begin{bmatrix}
\sigma'' & 0 \\
0 & \sigma''
\end{bmatrix}\begin{bmatrix}
\dot{y} \\
\dot{z}
\end{bmatrix} + \begin{bmatrix}
k' & 0 \\
0 & k'
\end{bmatrix}\begin{bmatrix}
\dot{\zeta} \\
\dot{\zeta}
\end{bmatrix} + \omega \begin{bmatrix}
0 & \sigma'' \\
-\sigma'' & 0
\end{bmatrix}\begin{bmatrix}
y \\
z - z_A
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
$$

(4.3)

The above equation can be rearranged as

$$
\begin{bmatrix}
M & 0 \\
0 & M
\end{bmatrix}\begin{bmatrix}
\ddot{y} \\
\ddot{z}
\end{bmatrix} + \begin{bmatrix}
\sigma'' & 0 \\
0 & \sigma''
\end{bmatrix}\begin{bmatrix}
\dot{y} \\
\dot{z}
\end{bmatrix} - \begin{bmatrix}
k' & 0 \\
0 & k'
\end{bmatrix}\begin{bmatrix}
\dot{\zeta} \\
\dot{\zeta}
\end{bmatrix} + \omega \begin{bmatrix}
0 & \sigma'' \\
-\sigma'' & 0
\end{bmatrix}\begin{bmatrix}
y \\
z - z_A
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
$$

(4.4)

The free body diagram of the mass-less shaft in Figure 4.1(c) gives

$$
-k\begin{bmatrix}
\ddot{y}_A \\
\ddot{z}_A
\end{bmatrix} + \begin{bmatrix}
k'' & 0 \\
0 & k''
\end{bmatrix}\begin{bmatrix}
\dot{y} - \dot{y}_A \\
\dot{z} - \dot{z}_A
\end{bmatrix} + \omega \begin{bmatrix}
0 & \sigma'' \\
-\sigma'' & 0
\end{bmatrix}\begin{bmatrix}
y_A \\
z - z_A
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
$$

(4.5)

by virtue of (4.2). The above equation can be arranged as

$$
\begin{bmatrix}
\sigma'' & 0 \\
0 & \sigma''
\end{bmatrix}\begin{bmatrix}
\ddot{y} \\
\ddot{z}
\end{bmatrix} + \begin{bmatrix}
\sigma'' & 0 \\
0 & \sigma''
\end{bmatrix}\begin{bmatrix}
\dot{y} \\
\dot{z}
\end{bmatrix} + \omega \begin{bmatrix}
0 & \sigma'' \\
-\sigma'' & 0
\end{bmatrix}\begin{bmatrix}
y \\
z - z_A
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
$$

(4.6)
It is important to note that (4.1) is expressed in rotating coordinates and (4.6) presents itself in stationary coordinates. To be able to write a meaningful combined equation of motion for the shaft-rotor system, it is imperative to write (4.1) in stationary coordinates. This is achieved by making use of (2.16) and (2.17) which governs the transformation of coordinates.

The equation of motion for the shaft-rotor system is obtained by combining equations (4.4) and (4.6) as

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & M & 0 \\
0 & 0 & M & 0 \\
\end{bmatrix}
\begin{bmatrix}
y_A' \\
y_A' \\
y' \\
\dot{z} \\
\end{bmatrix}
+ \begin{bmatrix}
\kappa + \kappa'' & 0 & -\kappa'' & 0 \\
0 & \kappa + \kappa'' & 0 & -\kappa'' \\
-\kappa'' & 0 & \kappa'' & 0 \\
0 & -\kappa'' & 0 & \kappa'' \\
\end{bmatrix}
\begin{bmatrix}
y_A'' \\
y'' \\
y'' \\
\dot{z} \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

(4.7)

The above equation takes the form

\[
M\ddot{x} + C\dot{x} + (K + \omega G)x = 0
\]

(4.8)

where \(M\), \(C\), \(K\) and \(G\) represents mass, damping, stiffness and gyroscopic matrices, respectively and \(x\) is the position vector.

4.1.3 Mathematical and Analytical Treatment

The characteristic equation obtained by substituting a solution of the form \(x = Xe^{ns}\) in (4.8) leads to

\[
[k^2M + sC + (K + \omega G)]X = 0
\]

(4.9)

where \(s\) are called the poles of the shaft-rotor system and \(X\) represents a non-zero column vector. This approach is similar to the explanation for obtaining (1.4) and (1.7) in section 1.3 of chapter 1. (4.9) is basically representing the determinant of a quadratic eigen-value problem. It is evident from (4.9) and (4.7) that the proposed model has a characteristic equation of order six because all elements in the first two rows of the mass matrix are zero.

The characteristic equation (4.9) can be denoted as

\[
\phi(\omega,s) = \sum_{i=0}^{6} a_i s^k.
\]

(4.10)
where

\begin{align}
\alpha_0 &= \kappa' \frac{\kappa''^2}{(\kappa' + \kappa'')^2} + \omega^2 \sigma^2 \frac{\kappa''^2}{(\kappa' + \kappa'')^4}, \\
\alpha_1 &= 2\kappa' \left( \sigma' \left( \kappa''^2 + \omega^2 \sigma^2 \right) + \sigma'' (\kappa' \kappa'') \right), \\
\alpha_2 &= 2M \kappa' \left( \left( \kappa''^2 + \omega^2 \sigma^2 \right) + \kappa' \kappa'' + \kappa' \sigma'' (\kappa' \sigma'' + 4 \kappa'' \sigma') + \sigma''^2 (\kappa'^2 + \omega^2 \sigma^2) \right), \\
\alpha_3 &= 2M \left( \kappa' (\kappa' \sigma'' + 2 \kappa'' (\sigma' + \sigma'')) + \sigma' (\kappa''^2 + \omega^2 \sigma^2) \right) + 2 \sigma' \sigma'' (\kappa' \sigma'' + \kappa'' \sigma'), \\
\alpha_4 &= M^2 \left( \left( \kappa''^2 + \omega^2 \sigma^2 \right) + \kappa' (\kappa'' + 2 \kappa''^3) \right) \\
&+ 2M \left( \kappa' \sigma'' (\sigma' + \sigma'') + (\kappa' + \kappa'') \sigma'' \sigma' \right) + \sigma' \sigma''^2, \\
\alpha_5 &= 2M \left( \sigma' + \sigma'' \right) \left( M (\kappa' + \kappa'') + \sigma'' \sigma' \right), \text{ and} \\
\alpha_6 &= M^2 \left( \sigma' + \sigma'' \right)^2,
\end{align}

are obtained as shown in Appendix C.

The characteristic polynomial corresponding to equations (5) and (6) in D. M. Smith’s work; titled as ‘The motion of a rotor carried by a flexible shaft in flexible bearings’, which was published in the Royal Society of London, Series A in 1933; is

\[ \hat{\phi}(\omega, s) = \sum_{i=0}^{4} \hat{\alpha}_4 s^4 \]

where

\begin{align}
\hat{\alpha}_0 &= \frac{\kappa' \sigma''^2}{(\kappa' + \kappa'')^2} + \omega^2 \sigma^2 \frac{\kappa''^2}{(\kappa' + \kappa'')^4}, \\
\hat{\alpha}_1 &= \frac{2\kappa' \kappa'' (\sigma''^2 + \kappa''^2 \sigma')}{(\kappa' + \kappa'')^3}, \\
\hat{\alpha}_2 &= \frac{2M \kappa' \sigma''}{(\kappa' + \kappa'')} + \frac{\left( \sigma' \kappa''^2 + \sigma''^2 \kappa'' \sigma' \right)^2}{(\kappa' + \kappa'')^4}, \\
\hat{\alpha}_3 &= \frac{2M (\kappa' \sigma'' + \kappa'' \sigma')}{(\kappa' + \kappa'')^2}, \text{ and}
\end{align}
\[ \hat{\alpha}_4 = M^2 \]  
(4.23)

are obtained as shown in Appendix C.

### 4.1.4 Root Locus

A comparison between the poles of (4.10) and that of Smith’s from (4.18) is performed using the root locus technique. The coefficients mentioned above are used to generate the Matlab code which obtains the root locus. The code is presented in APPENDIX A.

Smith’s poles in Figure 4.2 show good agreement with the poles of (4.10) when \( \sigma \) is small as compared to \( \kappa \) and \( \omega \) is not too large. However in Figure 4.3, when \( \omega \) is increased and \( \sigma \) is still kept small in comparison to \( \kappa \), Smith’s poles start to loose agreement with the poles of (4.10) as shown below.

---

**Figure 4.2**: Comparison of root locus of Smith’s and our poles at lower angular speeds

It is worth mentioning that the system represented in Figure 4.2 reaches the stability boundary for \( 0 < \omega < 10 \) when the poles of the system become purely imaginary for the first time. A system is stable when all of its poles are in the left half of the complex plane, poles lying on the imaginary axis represent the stability boundary and poles in the right
half of the complex plane make the system unstable. The nature of poles determines the stability of a system because the poles appear in the exponent of the exponential function term of the suggested solutions; see (1.4), (1.7) and (4.9). The poles are related to the boundedness of solution.

The deviation in the root loci can be attributed to the number of poles that appear in (4.10) and (4.18). The model developed in this thesis is an exact model which results in a characteristic polynomial of degree six. However, Smith used an approximation for stationary damping and rotary damping coefficients. These two approximations can be directly located just below (5) and (6) in his work that we have repeatedly mentioned. Using these approximations, Smith ended up with a determinantal equation of degree four.

The bending of the root locus of (4.10) in Figure 4.3 raises the natural question, whether the system ever regains stability? If it does, the observations, for the system to never regain stability, made by Smith based on his approximations will not hold true. The answer to this question lies in the determination of stability boundary of the system which is investigated in the next section.

Figure 4.3: Comparison of root locus of Smith’s and our poles at higher angular speeds
4.1.5 Stability

In the stability problem we want to find the values of $\omega$ where the system changes its state of equilibrium from stable to unstable and vice versa. This happens when two conjugate poles become purely imaginary.

Let

$$s = \alpha + i\beta$$

(4.24)

where $\alpha$ and $\beta$ are real numbers. If $s$ is purely imaginary then $\alpha = 0$. Since $s$ is a root of equation (4.10), substituting $s = \beta i$ both the real and imaginary parts of equation (4.10) reduces to zero and it lead to

$$
\begin{align*}
-\alpha_0\beta^5 + \alpha_4\beta^4 - \alpha_2\beta^2 + \alpha_0 &= 0 \\
\alpha_4\beta^5 - \alpha_3\beta^3 + \alpha_0\beta &= 0
\end{align*}
$$

(4.25)

The two equations in (4.25) in conjunction with the coefficient $a_\omega(\omega)$ defined above allow us to solve for the two unknowns $\omega$ and $s = \pm \beta i$ that define the stability boundary.

For the system defined by (4.7), (4.10) has eight real solutions

$$
\begin{align*}
\{\omega = 6.695510048 \quad s = \pm 1.825839254i\} \\
\{\omega = 18632.40800 \quad s = \pm 2.235978512i\}
\end{align*}
$$

(4.26)

The root locus along with the values of all poles at the stated $\omega$ indicates that the system is unstable for $6.695510048 < \omega < 18632.40800$ and stable for other positive values of $\omega$. Therefore, $\omega = 6.695510048$ and $\omega = 18632.40800$ are the two critical speeds for the proposed system.

Table 4.1: Poles of the mass-less shaft system at the stability boundary

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$\omega = 6.695510048$</th>
<th>$\omega = 18632.40800$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_{1,2}$</td>
<td>$\pm 1.8258 i$</td>
<td>$\pm 2.2360 i$</td>
</tr>
<tr>
<td>$s_{3,4}$</td>
<td>$-0.0122 \pm 1.8258 i$</td>
<td>$-0.0120 \pm 2.2356 i$</td>
</tr>
<tr>
<td>$s_{5,6}$</td>
<td>$-299.99 \pm 4.0172 i$</td>
<td>$-299.99 \pm 11179.4 i$</td>
</tr>
</tbody>
</table>

Now, with Smith’s formulation the stability boundary is determined by solving

$$
\begin{align*}
\hat{\alpha}_4\beta^4 - \hat{\alpha}_2\beta^2 + \hat{\alpha}_0 &= 0 \\
-\hat{\alpha}_3\beta^3 + \hat{\alpha}_1\beta &= 0
\end{align*}
$$

(4.27)
in conjunction with the definition of $\hat{\alpha}_2(\omega)$.

The root locus along with the values of all poles at the stated $\omega$ indicates that the system is stable for $0 < \omega < 6.694378239$ and unstable for the other positive values of $\omega$.

Table 4.2: Stability boundary of Smith’s system

<table>
<thead>
<tr>
<th>$s$</th>
<th>$\omega = 6.694378239$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_{1,2}$</td>
<td>$\pm 1.8257i$</td>
</tr>
<tr>
<td>$s_{3,4}$</td>
<td>$-0.0122 \pm 1.8257i$</td>
</tr>
</tbody>
</table>

4.1.6 Parametric Investigation of Stability

Now that we have an exact model, we wish to study the effect of change in stiffness and damping characteristics of shaft and bearings on the stability boundary of a mass-less shaft system. The impact of alteration of the mass of the rotor disk on the stability of the system is also studied.

Example 1:

Fixed parameters: $M = 10$, $\kappa' = 50$, $\kappa'' = 100$, $\omega = 6.695510048$

Table 4.3: Parametric investigation of stability at lower critical speed based on damping coefficients in a mass-less shaft system

<table>
<thead>
<tr>
<th>$\sigma'$</th>
<th>$\sigma''$</th>
<th>$s_{1,2}$</th>
<th>$s_{3,4}$</th>
<th>$s_{5,6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.3</td>
<td>$\pm 1.8258i$</td>
<td>$-0.012222 \pm 1.8258i$</td>
<td>$-299.99 \pm 4.0172i$</td>
</tr>
<tr>
<td>0.15</td>
<td>0.3</td>
<td>$0.001111 \pm 1.8258i$</td>
<td>$-0.011110 \pm 1.8258i$</td>
<td>$-333.33 \pm 4.4637i$</td>
</tr>
<tr>
<td>0.25</td>
<td>0.3</td>
<td>$-0.001111 \pm 1.8259i$</td>
<td>$-0.013333 \pm 1.8258i$</td>
<td>$-272.73 \pm 3.6520i$</td>
</tr>
<tr>
<td>0.2</td>
<td>0.25</td>
<td>$-0.000741 \pm 1.8258i$</td>
<td>$-0.010926 \pm 1.8258i$</td>
<td>$-333.33 \pm 3.7197i$</td>
</tr>
<tr>
<td>0.2</td>
<td>0.35</td>
<td>$0.000743 \pm 1.8259i$</td>
<td>$-0.013517 \pm 1.8258i$</td>
<td>$-272.73 \pm 4.2608i$</td>
</tr>
</tbody>
</table>

This example demonstrates that when $\omega$ reached its first critical speed, where the system transferred from stable state to unstable state, a decrease in the damping coefficient of the symmetrical bearing makes the system unstable while an increase in the damping coefficient of the symmetrical bearing makes the system stable. This is the normal behavior for non-rotating systems. However, contrary to the common behavior of damping, a decrease in the damping coefficient of the symmetrical shaft makes the system stable and an increase in the damping coefficient of the symmetrical shaft makes the system unstable.

Example 2:

Fixed parameters: $M = 10$, $\kappa' = 50$, $\kappa'' = 100$, $\omega = 18632.40800$
Table 4.4: Parametric investigation of stability at higher critical speed based on damping coefficients in a mass-less shaft system

<table>
<thead>
<tr>
<th>$\sigma'$</th>
<th>$\sigma''$</th>
<th>$s_{1,2}$</th>
<th>$s_{3,4}$</th>
<th>$s_{5,6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.3</td>
<td>$\pm 2.2360i$</td>
<td>$-0.019986 \pm 2.2356i$</td>
<td>$-299.99 \pm 11179.4i$</td>
</tr>
<tr>
<td>0.15</td>
<td>0.3</td>
<td>0.002499 $\pm 2.2359i$</td>
<td>$-0.017488 \pm 2.2357i$</td>
<td>$-333.33 \pm 12421.6i$</td>
</tr>
<tr>
<td>0.25</td>
<td>0.3</td>
<td>$-0.002499 \pm 2.2360i$</td>
<td>$-0.022484 \pm 2.2356i$</td>
<td>$-272.71 \pm 10163.1i$</td>
</tr>
<tr>
<td>0.2</td>
<td>0.25</td>
<td>0.001998 $\pm 2.2359i$</td>
<td>$-0.021979 \pm 2.2355i$</td>
<td>$-333.32 \pm 10351.3i$</td>
</tr>
<tr>
<td>0.2</td>
<td>0.35</td>
<td>$-0.001428 \pm 2.2360i$</td>
<td>$-0.018562 \pm 2.2357i$</td>
<td>$-272.72 \pm 11857.0i$</td>
</tr>
</tbody>
</table>

This example demonstrates that when $\omega$ reached its second critical speed, where the system transferred from unstable state to stable state, the behavior of damping, for both the shaft and the bearing, resembles the normal behavior for non rotating systems, i.e., increase in damping stabilizes the system and a decrease in damping makes it unstable.

Example 3:

Fixed parameters: $M = 10$, $\sigma' = 0.2$, $\sigma'' = 0.3$, $\omega = 6.695510048$

Table 4.5: Parametric investigation of stability at lower critical speed based on stiffness coefficients in a mass-less shaft system

<table>
<thead>
<tr>
<th>$\kappa'$</th>
<th>$\kappa''$</th>
<th>$s_{1,2}$</th>
<th>$s_{3,4}$</th>
<th>$s_{5,6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>100</td>
<td>$\pm 1.8258i$</td>
<td>$-0.012222 \pm 1.8258i$</td>
<td>$-299.99 \pm 4.0172i$</td>
</tr>
<tr>
<td>45</td>
<td>100</td>
<td>$-0.000711 \pm 1.7618i$</td>
<td>$-0.011690 \pm 1.7617i$</td>
<td>$-289.99 \pm 4.0173i$</td>
</tr>
<tr>
<td>55</td>
<td>100</td>
<td>0.000661 $\pm 1.8838i$</td>
<td>$-0.012762 \pm 1.8838i$</td>
<td>$-309.99 \pm 4.0173i$</td>
</tr>
<tr>
<td>50</td>
<td>95</td>
<td>0.000521 $\pm 1.8101i$</td>
<td>$-0.012672 \pm 1.8100i$</td>
<td>$-289.99 \pm 4.0173i$</td>
</tr>
<tr>
<td>50</td>
<td>105</td>
<td>$-0.000472 \pm 1.8405i$</td>
<td>$-0.011827 \pm 1.8405i$</td>
<td>$-309.99 \pm 4.0173i$</td>
</tr>
</tbody>
</table>

This example demonstrates that when $\omega$ reached its first critical speed, where the system transferred from stable state to unstable state, a decrease in the stiffness coefficient of the symmetrical bearing makes the system stable while an increase in the stiffness coefficient of the symmetrical bearing makes the system unstable. Again, this is contrary to the normal behavior for non-rotating systems. However, a decrease in the stiffness coefficient of the symmetrical shaft makes the system unstable and an increase in the stiffness coefficient of the symmetrical shaft makes the system stable. This observation of the behavior of stiffness coefficient of the symmetrical shaft at lower critical speed resembles the normal behavior of non-rotating systems.

Example 4:

Fixed parameters: $M = 10$, $\sigma' = 0.2$, $\sigma'' = 0.3$, $\omega = 18632.40800$
Table 4.6: Parametric investigation of stability at higher critical speed based on stiffness coefficients in a mass-less shaft system

<table>
<thead>
<tr>
<th>$\kappa'$</th>
<th>$\kappa''$</th>
<th>$s_{1,2}$</th>
<th>$s_{3,4}$</th>
<th>$s_{5,6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>100</td>
<td>$\pm 2.2360i$</td>
<td>$-0.019986 \pm 2.2356i$</td>
<td>$-299.99 \pm 11179.4i$</td>
</tr>
<tr>
<td>45</td>
<td>100</td>
<td>$-0.001461 \pm 2.1213i$</td>
<td>$-0.018527 \pm 2.1209i$</td>
<td>$-289.99 \pm 11179.4i$</td>
</tr>
<tr>
<td>55</td>
<td>100</td>
<td>$0.001537 \pm 2.3451i$</td>
<td>$-0.021521 \pm 2.3447i$</td>
<td>$-309.99 \pm 11179.4i$</td>
</tr>
<tr>
<td>50</td>
<td>95</td>
<td>$0.0000003 \pm 2.2360i$</td>
<td>$-0.019987 \pm 2.2356i$</td>
<td>$-289.99 \pm 11179.4i$</td>
</tr>
<tr>
<td>50</td>
<td>105</td>
<td>$-0.0000003 \pm 2.2360i$</td>
<td>$-0.019986 \pm 2.2356i$</td>
<td>$-309.99 \pm 11179.4i$</td>
</tr>
</tbody>
</table>

This example demonstrates that when $\omega$ reached its second critical speed, where the system transferred from unstable state to stable state, a decrease in the stiffness coefficient of the symmetrical bearing makes the system stable while an increase in the stiffness coefficient of the symmetrical bearing makes the system unstable. This is contrary to the normal behavior for non-rotating systems. However, a decrease in the stiffness coefficient of the symmetrical shaft makes the system unstable and an increase in the stiffness coefficient of the symmetrical shaft makes the system stable. This observation of the behavior of stiffness coefficient of the symmetrical shaft at higher critical speed resembles the normal behavior of non-rotating systems.

Example 5:

Fixed parameters: $\kappa' = 50$, $\kappa'' = 100$, $\sigma' = 0.2$, $\sigma'' = 0.3$, $\omega = 6.695510048$

Table 4.7: Parametric investigation of stability at lower critical speed based on rotor mass in a mass-less shaft system

<table>
<thead>
<tr>
<th>$M$</th>
<th>$s_{1,2}$</th>
<th>$s_{3,4}$</th>
<th>$s_{5,6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$\pm 1.8258i$</td>
<td>$-0.012222 \pm 1.8258i$</td>
<td>$-299.99 \pm 4.0172i$</td>
</tr>
<tr>
<td>9.5</td>
<td>$-0.000163 \pm 1.8732i$</td>
<td>$-0.012702 \pm 1.8732i$</td>
<td>$-299.99 \pm 4.0173i$</td>
</tr>
<tr>
<td>10.5</td>
<td>$0.000144 \pm 1.7818i$</td>
<td>$-0.011783 \pm 1.7818i$</td>
<td>$-299.99 \pm 4.0173i$</td>
</tr>
</tbody>
</table>

This example demonstrates that when $\omega$ reached its first critical speed, where the system transferred from stable state to unstable state, a decrease in the mass of the rotor makes the system stable while an increase in the mass of the rotor makes the system unstable. This resembles the normal behavior for non-rotating systems.

Example 6:

Fixed parameters: $\kappa' = 50$, $\kappa'' = 100$, $\sigma' = 0.2$, $\sigma'' = 0.3$, $\omega = 18632.40800$
Table 4.8: Parametric investigation of stability at higher critical speed based on rotor mass in a mass-less shaft system

<table>
<thead>
<tr>
<th>$M$</th>
<th>$s_{1,2}$</th>
<th>$s_{3,4}$</th>
<th>$s_{5,6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$\pm 2.2360i$</td>
<td>$- 0.019986 \pm 2.2356i$</td>
<td>$- 299.99 \pm 11179.4i$</td>
</tr>
<tr>
<td>9.5</td>
<td>$- 0.000266 \pm 2.2941i$</td>
<td>$- 0.020772 \pm 2.2937i$</td>
<td>$- 299.99 \pm 11179.4i$</td>
</tr>
<tr>
<td>10.5</td>
<td>$0.000235 \pm 2.1820i$</td>
<td>$- 0.019270 \pm 2.1817i$</td>
<td>$- 299.99 \pm 11179.4i$</td>
</tr>
</tbody>
</table>

This example demonstrates that when $\omega$ reached its second critical speed, where the system transferred from unstable state to stable state, the behavior of the mass of rotor resembles to the normal behavior of mass for non-rotating systems, i.e., an increase in mass of rotor destabilizes the system and a decrease in mass of rotor makes the system stable.

The parametric study performed in this section will be used to compare the results of a similar section of parametric study in the next chapter which deals with a shaft of mass $m$ system. The conclusions from this study are presented in chapter 6.
Chapter 5
Modeling of a Shaft Having Mass and Rotating with Finite Displacement at Bearing End

We extend the model developed in the previous chapter by adding mass \( m \) of the shaft in the SSSB model.

Figure 5.1: (a) System of shaft of mass \( m \) rotating in bearing with finite displacement at the bearing end (b) Planar coordinate system of shaft motion, (c) Free body diagram of shaft \( AB \) of mass \( m \) (moments not shown) (d) Free body diagram of rotor \( B \) of mass \( M \).

Let a force \( \mathbf{f}_m \) act on the shaft in the \( y - z \) coordinate system. The free body diagram of shaft in Figure 5.1(c) results in

\[
\mathbf{f}_m = -\sigma_y \begin{pmatrix} \dot{y}_A \\ \dot{z}_A \end{pmatrix} - k_y \begin{pmatrix} y_A \\ z_A \end{pmatrix} + \begin{pmatrix} \sigma'' & 0 \\ 0 & \sigma'' \end{pmatrix} \begin{pmatrix} \dot{y} - \dot{y}_A \\ \dot{z} - \dot{z}_A \end{pmatrix} + \begin{pmatrix} \kappa'' & 0 \\ 0 & \kappa'' \end{pmatrix} + \omega \begin{pmatrix} 0 & 0 \\ 0 & -\sigma'' \end{pmatrix} \begin{pmatrix} y - y_A \\ z - z_A \end{pmatrix}
\]

by virtue of (4.2). It thus follows from Newton’s second law for the shaft of mass \( m \) that
where
\[ \alpha_0 = \kappa'^2 (\kappa''^2 + \omega^2 \sigma''^2) \],
\[ \alpha_1 = 2k' (\sigma' (\kappa''^2 + \omega^2 \sigma''^2) + \sigma'' (\kappa' \kappa'')) \],
\[ \alpha_2 = (\kappa''^2 + \omega^2 \sigma''^2) (2k' (m + M) + \sigma'^2) + k' (\kappa' (2k'' M + \sigma''^2) + 4k'' \sigma' \sigma'') \]
\[ \alpha_3 = 2(m + M)\left(\kappa''^2 + \omega^2 \sigma^* \sigma' + 2\kappa' \kappa'' \sigma^* \right) + 2(M \kappa' + \sigma' \sigma^*) \left(\kappa'' \sigma'' + \kappa'' \sigma' \right) + 2\kappa' \kappa'' \sigma' \sigma'' M' \]  
(5.9)

\[ \alpha_4 = (\kappa''^2 + \omega^2 \sigma^* \sigma') \left(m + M \right)^2 + 2\sigma'' \left(m + M \right) \left(\kappa' \sigma'' + 2\sigma' \kappa'' \right) + (\kappa'M + \sigma' \sigma'')^2 + 2M \sigma'' \left(\kappa' \sigma'' + \kappa'' \sigma' \right) + \kappa' \kappa'' \sigma' \sigma'' \sigma'' M \left(3m + 2M \right) \]  
(5.10)

\[ \alpha_5 = 2M^2 \left(\kappa' + \kappa'' \right) \left(\sigma' + \sigma'' \right) + M \left(3m \left(\kappa' \sigma'' + \kappa'' \sigma' \right) + 2\sigma' \sigma'' \left(\sigma' + \sigma'' \right) \right) + 2m \sigma'' \left(\kappa'' \left(m + 2M \right) + \sigma' \sigma'' \right) \]  
(5.11)

\[ \alpha_6 = M^2 \left(\sigma' + \sigma'' \right)^2 + m \left(M^2 \left(\kappa' + \kappa'' \right) + M \left(m \kappa'' + 3 \sigma' \sigma'' \right) + \sigma'' \left(m + 2M \right) \right), \]  
(5.12)

\[ \alpha_7 = mM \left(\sigma' M + \sigma'' \left(m + M \right) \right), \]  
(5.13)

\[ \alpha_8 = 0.25m^2 M^2. \]  
(5.14)

which are obtained as shown in Appendix C.

5.1 Root Locus

The whole motivation to add mass to the shaft model of previous chapter is to find out the region of stability of the new system. A small mass \( m \) is introduced as the mass of the shaft and root locus of the new poles of the system was plotted.

Figure 5.2: (a) Root locus of shaft having mass and rotating at very slow speed
Figure 5.2(a) shows two pairs of conjugate roots in the left half of the complex plane, which indicates a stable equilibrium for the shaft rotating at very slow speeds. However, two more pairs of conjugate roots appear near the imaginary axis and needs to be examined carefully. The region of roots near the imaginary axis is enlarged and shown in Figure 5.2(b).

It can be clearly seen in Figure 5.2(b) that the root locus crosses the imaginary axis of the complex plane when the shaft rotates between $0 < \omega < 10$. In particular, the root locus starts at the point where $\omega = 0$ and spreads on both sides. Therefore, it is concluded that the system was initially stable but lost its stability at some particular $\omega$.

As soon as the region of instability is encountered, the natural question arises: When does the system regains stability? The range of $\omega$ is increased and the new root locus is plotted this time in Figure 5.3(a).

It is observed that the root locus of the roots present in the far left in Figure 5.2(a) crosses the imaginary axis as $\omega$ is increased. The enlarged view of the roots present near the imaginary axis is shown in Figure 5.3(b). These roots continue to spread more or less as a line with some roots present in the right hand side of the complex plane. Therefore, the system still remains unstable for $10 < \omega < 100$. 

![Enlarged Root Locus for 0 ≤ ω ≤ 10](image)
Figure 5.3: (a) Root locus of shaft having mass and rotating at slow speed

Figure 5.3: (b) Enlarged view of roots in Figure 5.3(a)
The search for the stability boundary of the system is extended by increasing the range of \( \omega \) of the system. The system largely remains unstable as per Figure 5.4(a).

Figure 5.4: (a) Root locus of shaft having mass and rotating at faster speed

(Enlarged) Root Locus for \( 0 \leq \omega \leq 1000 \)
Figure 5.4: (b) Enlarged view of roots in Figure 5.4(a)
The bending of the root locus in Figure 5.4(a) and the progress of completion of the ring by roots in Figure 5.4 (b), which is an enlarged view of roots near the imaginary axis in Figure 5.4(a), gives the necessary motivation for the existence of stability boundary which can be obtained for higher values of $\omega$.

The root locus for higher values of $\omega$ is plotted in Figure 5.5 (a). The bending of the root locus in Figure 5.4(a) takes the shape of a curve which never crosses the imaginary axis of the complex plane again.

![Root Locus for $0 \leq \omega \leq 20000$](image_url)

Figure 5.5: (a) Root locus of shaft having mass and rotating at high speed

![Enlarged Root Locus for $0 \leq \omega \leq 20000$](image_url)

Figure 5.5: (b) Enlarged view of roots in Figure 5.5(a)
The ring to be formed by root locus mentioned in Figure 5.4(b) approaches its closure in Figure 5.5(b). However, its crossing of the imaginary axis lost its importance because of the presence of roots in the right half side of the complex plane in Figure 5.5(a).

In nutshell, the shaft of mass\( m \) system never regains stability after losing it for the first time.

### 5.2 Stability

As it is clear from the discussion above that the proposed system in this chapter arrives at the stability boundary only once. We substitute (4.24) in (5.5) and follow on the same lines of philosophy which leads us to (4.25). Therefore, we end up with

\[
\begin{align*}
\alpha_4 \beta^8 & - \alpha_6 \beta^6 + \alpha_4 \beta^4 - \alpha_2 \beta^2 + \alpha_0 = 0 \\
- \beta^7 & + \alpha_3 \beta^5 - \alpha_3 \beta^3 + \alpha_1 \beta = 0
\end{align*}
\]

The set of two equations in (5.15) in conjunction with the coefficient \( \alpha_i(\omega) \) defined above allow us to solve for the two unknowns \( \omega \) and \( s = \pm \beta i \). Thus, the twelve real solutions that exists for (5.5) are

\[
\begin{align*}
\{\omega = 6.733791818 \quad s = \pm 1.820783650 i\} & \quad \{\omega = -6.733791818 \quad s = \pm 1.820783650 i\} \\
\{\omega = 91.11607897 \quad s = \pm 54.92363736 i\} & \quad \{\omega = -91.11607897 \quad s = \pm 54.92363736 i\} \\
\{\omega = 18448.07536 \quad s = \pm 2.224880446 i\} & \quad \{\omega = -18448.07536 \quad s = \pm 2.224880446 i\}
\end{align*}
\]

The root locus along with the values of all poles at the stated \( \omega \) indicates that the system is stable for \( 0 \leq \omega \leq 6.733791818 \) and unstable for all other values of \( \omega \).

Table 5.1: Poles of the shaft with mass system at the stability boundary

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( s )</th>
<th>( \omega = 6.733791818 )</th>
<th>( \omega = 91.11607897 )</th>
<th>( \omega = 18448.07536 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_{1,2} )</td>
<td>( s_{3,4} )</td>
<td>( s_{5,6} )</td>
<td>( s_{7,8} )</td>
<td></td>
</tr>
<tr>
<td>( \pm 1.8208i )</td>
<td>( -0.0122 \pm 1.8208i )</td>
<td>( -4.6511 \pm 54.6953i )</td>
<td>( -5.3968 \pm 54.6954i )</td>
<td></td>
</tr>
<tr>
<td>( \pm 54.9236i )</td>
<td>( -10.0479 \pm 54.9241i )</td>
<td>( -0.0850 \pm 1.8367i )</td>
<td>( 0.0729 \pm 1.8371i )</td>
<td></td>
</tr>
<tr>
<td>( \pm 2.2248i )</td>
<td>( -0.0197 \pm 2.2245i )</td>
<td>( -238.3069 \pm 239.6092i )</td>
<td>( 228.2667 \pm 239.6089i )</td>
<td></td>
</tr>
</tbody>
</table>

### 5.3 Parametric Investigation of Stability

Similar to the analysis carried out in section 4.1.6, the parametric investigation for the shaft with mass model is performed. It should be pointed out that the system under consideration has only one value of critical speed \( \omega = 6.733791818 \), while the system of
mass-less shaft had two critical speeds. The stiffness, damping and mass characteristics will be studied in this section with the help of examples.

Example 1:

Fixed parameters: \( m = 0.1 \), \( M = 10 \), \( \kappa' = 50 \), \( \kappa'' = 100 \), \( \omega = 6.733791818 \)

Table 5.2: Parametric investigation of stability at critical speed based on damping in a shaft with mass system

<table>
<thead>
<tr>
<th>( \sigma' )</th>
<th>( \sigma'' )</th>
<th>( s_{1,2} )</th>
<th>( s_{3,4} )</th>
<th>( s_{5,6} )</th>
<th>( s_{7,8} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.3</td>
<td>( \pm 1.8208i )</td>
<td>-0.1222 ( \pm 1.8208i )</td>
<td>-4.6511 ( \pm 54.6953i )</td>
<td>-5.3968 ( \pm 54.6954i )</td>
</tr>
<tr>
<td>0.15</td>
<td>0.3</td>
<td>0.0011 ( \pm 1.8208i )</td>
<td>-0.0111 ( \pm 1.8208i )</td>
<td>-4.1525 ( \pm 54.7389i )</td>
<td>-4.8976 ( \pm 54.7389i )</td>
</tr>
<tr>
<td>0.25</td>
<td>0.3</td>
<td>-0.0011 ( \pm 1.8208i )</td>
<td>-0.0133 ( \pm 1.8207i )</td>
<td>-5.1496 ( \pm 54.6471i )</td>
<td>-5.8960 ( \pm 54.6472i )</td>
</tr>
<tr>
<td>0.2</td>
<td>0.25</td>
<td>-0.0007 ( \pm 1.8208i )</td>
<td>-0.0109 ( \pm 1.8207i )</td>
<td>-4.2087 ( \pm 54.7389i )</td>
<td>-4.8296 ( \pm 54.7390i )</td>
</tr>
<tr>
<td>0.2</td>
<td>0.35</td>
<td>0.0007 ( \pm 1.8208i )</td>
<td>-0.0134 ( \pm 1.8208i )</td>
<td>-5.0933 ( \pm 54.6471i )</td>
<td>-5.9640 ( \pm 54.6471i )</td>
</tr>
</tbody>
</table>

This example demonstrates agreement with the comments mentioned below Example 1 of the previous chapter which hold true for the explanation of behavior of the poles of the system considered in Table 1.2. The counter-intuitive phenomenon of internal damping reported in Example 1 of previous chapter is again characterized in this example.

It is to be noted that the critical speed of shaft with mass system is near to the lower critical speed of the mass-less shaft system. Therefore, it becomes obvious to compare Example 2 and Example 3 of this chapter to Examples 3 and Example 5 of the previous chapter, respectively.

Further, the nature of poles of the shaft with mass system in Example 2 and Example 3 of this chapter bears resemblance to the comments mentioned in Example 3 and Example 5 of the previous chapter, respectively.

Example 2:

Fixed parameters: \( m = 0.1 \), \( M = 10 \), \( \sigma' = 0.2 \), \( \sigma'' = 0.3 \), \( \omega = 6.733791818 \)
Table 5.3: Parametric investigation of stability at critical speed based on stiffness coefficients in a shaft with mass system

<table>
<thead>
<tr>
<th>$\kappa'$</th>
<th>$\kappa''$</th>
<th>$s_{1,2}$</th>
<th>$s_{3,4}$</th>
<th>$s_{5,6}$</th>
<th>$s_{7,8}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>100</td>
<td>$\pm 1.8208i$</td>
<td>-0.0122$\pm 1.8208i$</td>
<td>-4.6511$\pm 54.6953i$</td>
<td>-5.3968$\pm 54.6954i$</td>
</tr>
<tr>
<td>45</td>
<td>100</td>
<td>$-0.0007 \pm 1.7566i$</td>
<td>-0.0116$\pm 1.7566i$</td>
<td>-4.6446$\pm 53.7756i$</td>
<td>-5.4031$\pm 53.7756i$</td>
</tr>
<tr>
<td>55</td>
<td>100</td>
<td>$0.0007 \pm 1.8788i$</td>
<td>-0.0127$\pm 1.8788i$</td>
<td>-4.6572$\pm 55.6000i$</td>
<td>-5.3907$\pm 55.6000i$</td>
</tr>
<tr>
<td>50</td>
<td>95</td>
<td>$0.0005 \pm 1.8051i$</td>
<td>-0.0126$\pm 1.8051i$</td>
<td>-4.6447$\pm 53.7647i$</td>
<td>-5.4032$\pm 53.7647i$</td>
</tr>
<tr>
<td>50</td>
<td>105</td>
<td>$-0.0005 \pm 1.8353i$</td>
<td>-0.0118$\pm 1.8353i$</td>
<td>-4.6572$\pm 55.6104i$</td>
<td>-5.3906$\pm 55.6104i$</td>
</tr>
</tbody>
</table>

Example 3:

Fixed parameters: $m = 0.1, \kappa' = 50, \kappa'' = 100, \sigma' = 0.2, \sigma'' = 0.3, \omega = 6.733791818$

Table 5.4: Parametric investigation of stability at critical speed based on rotor mass in a shaft with mass system

<table>
<thead>
<tr>
<th>$M$</th>
<th>$s_{1,2}$</th>
<th>$s_{3,4}$</th>
<th>$s_{5,6}$</th>
<th>$s_{7,8}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$\pm 1.8208i$</td>
<td>-0.0122$\pm 1.8208i$</td>
<td>-4.6511$\pm 54.6953i$</td>
<td>-5.3968$\pm 54.6954i$</td>
</tr>
<tr>
<td>9.5</td>
<td>$-0.0002 \pm 1.8678i$</td>
<td>-0.0126$\pm 1.8678i$</td>
<td>-4.6522$\pm 54.7033i$</td>
<td>-5.3981$\pm 54.7033i$</td>
</tr>
<tr>
<td>10.5</td>
<td>$0.0001 \pm 1.7771i$</td>
<td>-0.0117$\pm 1.7771i$</td>
<td>-4.6501$\pm 54.6882i$</td>
<td>-5.3955$\pm 54.6882i$</td>
</tr>
</tbody>
</table>

Example 4:

A new variation that can be studied in this model of shaft with mass $m$ is the parametric investigation of the effect of the mass of the shaft on the stability boundary of the system.

Fixed parameters: $M = 10; \kappa' = 50; \kappa'' = 100; \sigma' = 0.2; \sigma'' = 0.3; \omega = 6.733791818$
Table 5.5: Parametric investigation of stability at critical speed based on shaft mass in a shaft with mass system

<table>
<thead>
<tr>
<th>$m$</th>
<th>$s_{1,2}$</th>
<th>$s_{5,6}$</th>
<th>$s_{7,8}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\pm 1.8208i$</td>
<td>$- 4.6511 \pm 54.6953i$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$- 0.0122 \pm 1.8208i$</td>
<td>$- 5.3968 \pm 54.6954i$</td>
<td></td>
</tr>
<tr>
<td>0.09</td>
<td>$0.0000003 \pm 1.8213i$</td>
<td>$- 5.1865 \pm 57.6111i$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$- 0.0122 \pm 1.8213i$</td>
<td>$- 5.9724 \pm 57.6111i$</td>
<td></td>
</tr>
<tr>
<td>0.11</td>
<td>$- 0.0000003 \pm 1.8203i$</td>
<td>$- 4.2138 \pm 52.1843i$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$- 0.0122 \pm 1.8202i$</td>
<td>$- 4.9250 \pm 52.1843i$</td>
<td></td>
</tr>
</tbody>
</table>

This example demonstrates that when $\omega$ reached its critical speed, where the system transferred from stable state to unstable state, the behavior of the mass of shaft is contrary to the normal behavior of mass for non-rotating systems. In this example, a decrease in the mass of the shaft destabilizes the system and an increase in mass of shaft makes the system stable.

The conclusions from this study are presented in the next chapter.
Chapter 6
Conclusions

The following are the concluding remarks for Chapter 4:

We proposed an exact model based on (4.7) which lead us to the characteristic equation of order six for the system. Smith’s model, based on approximations, ended up with a characteristic equation of degree four. His claim that a symmetrical mass-less shaft rotating in symmetrical bearings with deflection at bearing supports and a disk mounted on the shaft, will never regain stability after loosing it for the first time, is found to be incorrect as the poles of the proposed exact model crosses the imaginary axis as per (4.26).

A symmetrical mass-less shaft rotating in symmetrical bearings with finite displacement at bearing end will have two critical speeds and there exists a region of instability between the two critical speeds.

A mass-less SSSB system with finite displacement at bearing end will behave in the following manner when it reaches the first critical speed or to say that it reaches the stability boundary for the first time.

- The system can be stabilized by
  - Increasing the damping of the symmetrical bearings
  - Decreasing the damping of the symmetrical shaft
  - Decreasing the stiffness of the symmetrical bearings
  - Increasing the stiffness of the symmetrical shaft
  - Decreasing the mass of the rotor disk
  provided all other parameters are kept unchanged.

A mass-less SSSB system with finite displacement at bearing end will behave in the following manner when it reaches the second critical speed or to say that it reaches the stability boundary for the second time.

- The system can be stabilized by
  - Increasing the damping of the symmetrical bearings
  - Increasing the damping of the symmetrical shaft
  - Decreasing the stiffness of the symmetrical bearings
  - Increasing the stiffness of the symmetrical shaft
  - Decreasing the mass of the rotor disk
  provided all other parameters are kept unchanged.

Following are the conclusions from Chapter 5:

A system of a symmetrical shaft having mass and rotating in symmetrical bearings with a rotor mounted on it, results in a characteristic equation of order eight. The system
transfers from stable equilibrium to the unending region of instability when it encounters the stability boundary for the first time. The other way of acknowledging the stated conclusion is that the system remains unstable after crossing its critical speed value.

However, the following steps help to stabilize the system when it reaches stability boundary:

- Increasing the damping of the symmetrical bearings
- Decreasing the damping of the symmetrical shaft
- Decreasing the stiffness of the symmetrical bearings
- Increasing the stiffness of the symmetrical shaft
- Decreasing the mass of the rotor disk
- Increasing the mass of the shaft

provided all other parameters are kept unchanged.

The counter-intuitive phenomenon of internal damping which destabilizes a rotating system and has been historically reported by A. L. Kimball and S. H. Crandall in their works is also encountered in the proposed model in the form of the damping of shaft.

The values presented in Table 4.5 and Table 5.3 indicates the existence of a similar counter-intuitive phenomenon for bearing stiffness. An increase in the bearing stiffness is reported to cause destabilization of the system whenever the system encounters stability boundary, i.e. at both critical speeds for mass-less shaft SSSB system and at the single value of critical speed for the SSSB system of shaft having mass.

6.1 Future Work

The proposed model can be improved by taking moments into consideration for the free body diagrams shown in Figure 4.1 and Figure 5.1. The reason that the author has not included the moments in the free body diagrams is because he anticipated that the problem might become mathematically too involved.

An exact analysis of the case of Unsymmetrical Shaft and Unsymmetrical Bearings (USUB) can also be performed and the results can be compared to that of Smith’s model. However, the time dependent coefficients which appear in the equations of motion for the USUB system presents a mathematical exercise.
References


Appendix A

Codes for Chapter 4 (mass-less shaft)

A.1 Maple Code for Characteristic Polynomial (4.10)

```maple
restart;
with(LinearAlgebra):
MM:=<<0 | 0 | 0 | 0> , <0 | 0 | 0 | 0> , <0 | 0 | M | 0> ,
          <0 | 0 | 0 | M>>:
C:=<<c1+c2 | 0 | -c2 | 0> , <0 | c1+c2 | 0 | -c2> , <-c2 |
       0 | c2 | 0> , <0 | -c2 | 0 | c2>>:
K:=<<k1+k2 | 0 | -k2 | 0> , <0 | k1+k2 | 0 | -k2> , <-k2 |
       0 | k2 | 0> , <0 | -k2 | 0 | k2>>:
G:=<<0 | c2 | 0 | -c2> , <-c2 | 0 | c2 | 0> , <0 | -c2 | 0 |
       c2>, <c2 | 0 | -c2 | 0>>:
P:=MM*s^2+C*s+K+w*G:
p:=Determinant(P):
collect(p,s):
```

A.2 Maple Code for Smith’s Characteristic Polynomial (4.18)

```maple
restart;
with(LinearAlgebra):
c1_eq:=c1*(k2/(k1+k2))^2:
c2_eq:=c2*(k1/(k1+k2))^2:
k_eq:=k1*k2/(k1+k2):
MM:=<<M | 0> , <0 | M>>:
C:=<<c1_eq+c2_eq | 0> , <0 | c1_eq+c2_eq>>:
K:=<<k_eq | 0> , <0 | k_eq>>:
G:=<<0 | c2_eq> , <-c2_eq | 0>>:
P:=s^2*MM*s^2+C*s+K+w*G:
p:=Determinant(P):
collect(p,s):
```

A.3 Matlab Code for Root Locus shown in Figure 4.2

```matlab
clear all
k1=50;k2=100;c1=0.2;c2=0.3;M=10;
k=0;
for w=0:1:10,
k=k+1;
a6=M^2*(c1+c2)^2;
a5=2*c1^2*M^2+2*c1*k1*M^2+2*c2*k2*M^2+2*c1*k2*M^2+2*c1*c2^2*M+...
   2*c2*k1*M^2;
a4=2*c2^2*k1*M+4*c1*k1*M^2+2*c2^2*k1^2*M+4*c1*c2*M*k2+...
```

38
\[2*c1^2*M*k2+k2^2*M^2+2*k1*k2*M^2+w^2*c2^2*M^2;\]
\[a3=4*c1*k1*M*k2+2*c1*k1*c2^2+2*k1^2*M*c2+2*c1*w^2*c2^2*M^2+\]
\[2*c1^2*c2*k2+2*c2^2*M^2+2*k1^2*M*k2*k2+2*c1*M*k2^2+\]
\[4*c1*k1*c2*k2+2*c1^2*M*k2^2+w^2*c2^2*M;\]
\[a2=2*c1*k1*w^2*c2^2+2*c1*k1*k2^2+2*c1*w^2*c2^2*M^2+\]
\[2*c1^2*k2+2*c2^2*M^2+2*c1*k2^2+2*c1*w^2*c2^2;\]
\[a1=2*c1*k1*w^2+c1*k1*k2^2+2*c1*w^2*c2^2*M^2+\]
\[2*c1^2*k2+2*c2^2*M^2+2*c1*k2^2+2*c1*w^2*c2^2;\]
\[a0=k1^2*w^2+c2^2*M^2+2*k1^2*M*k2^2+2*c1*w^2*c2^2;\]
\[p=[a6 a5 a4 a3 a2 a1 a0];\]
\[s=roots(p);\]
\[S(k,:)=s. '';\]
\[end\]
\[plot(S(:,3:6),'*b')\]
\[grid on\]
\[hold on\]
\[%Root locus of Smith’s poles;\]
\[clear all\]
\[k1=50;k2=100;c1=0.2;c2=0.3;M=10;\]
\[k=0;\]
\[for w=0:1:10,\]
\[k=k+1;\]
\[a4=4*M^2*k1*k2^3+M^2*k1^4+4*M^2*k1^3*k2+6*M^2*k1^2*k2^2+M^2*k2^4;\]
\[a3=2*M*k1^2*c1*k2^2+2*M*k2^4*c1+4*M*k1^3*k2*c2+2*M*k2^2*c2*k1^2+\]
\[2*M*k1^4*c2+4*M*k1*k2^3+c1;\]
\[a2=6*M*k1^3*k2^2+2*M*k1^4*k2+c2^2*k1^4+6*M*k1^2*k2^3+\]
\[2*M*k1^2*k2^2+2*M*k2^4*k1+c1^2*k1^2*k2^4;\]
\[a1=2*c2*k1^4*k2^2+2*c1*k2^3*k1^2+2*c1*k2*k1^3+k2^2;\]
\[a0=k1^2*k2^4+w^2*c2^2*k1^4+k1^4*k2^2+2*k1^3*k2^3;\]
\[p=[a4 a3 a2 a1 a0];\]
\[s=roots(p);\]
\[S(k,:)=s. '';\]
\[end\]
\[plot(S,'or')\]
\[grid on\]

A.4 Maple Code for Evaluating Stability Boundary in (4.26)

```maple
restart;
with(LinearAlgebra):
MM:=<<0 | 0 | 0 | 0>, <0 | 0 | 0 | 0>, <0 | 0 | M | 0>, <0 | 0 | 0 | M>>;
C:=<<c1+c2 | 0 | -c2 | 0>, <0 | c1+c2 | 0 | -c2>, <-c2 | 0 | c2 | 0>, <0 | -c2 | 0 | c2>>;
K:=<<k1+k2 | 0 | -k2 | 0>, <0 | k1+k2 | 0 | -k2>, <-k2 | 0 | k2 | 0>, <0 | -k2 | 0 | k2>>;
G:=<<0 | c2 | 0 | -c2>, <-c2 | 0 | c2 | 0>, <0 | -c2 | 0 | c2>, <c2 | 0 | -c2 | 0>>;
P:=MM*s^2+C*s+K+w*G;
p:=Determinant(P);
collect(p,s):
```

39
> a6:=M^2*(c1+c2)^2:
> a5:=2*c1^2*M*c2+2*c1*k1*M^2+2*c2*k2*M^2+2*c1*k2*M^2+2*c1*c2
> +2*M^2*c1*k1*M^2:
> a4:=2*c2^2*k1*M+4*c1*k1*M*c2+c1^2*c2+2*k1^2*M^2+4*c1*c2*M*k
> 2+2*c1^2*M^2+k2+2*M^2+k1^2*M^2+w^2*c2^2*M^2:
> a3:=4*c1*k1*M^2+2*c1*k1*c2^2+2*c1*k1^2*M^2+2*c1*w^2*c2^2*M+2
> k1^2*c2^2+4*c2*k1*M^2+2*c1*M^2+2*c1*k1*M^2:
> a2:=2*c1^2*M^2+4*c1*k1*M^2+4*c1*c2*M^2+2*k1^2*M^2+2*k1*M^2:
> a1:=2*c1*k1*w^2*c2^2+2*c1*k1^2*M^2+2*c1^2*c2^2+2*c1*k1^2*M^2:
> a0:=k1^2*w^2+c2^2+2*k1^2*w^2+c2^2:
> Eq1:=-a6*b^6+a4*b^4-a2*b^2+a0:
> Eq2:=a5*b^5-a3*b^3+a1*b:
> k1:=50:k2:=100:c1:=0.2:c2:=0.3:M:=10:
> solve({Eq1,Eq2},{b,w}):  

A.5 Maple Code for Evaluating Stability Boundary of Smith’s system in Table 4.2  

> restart;
> with(LinearAlgebra):
> c1_eq:=c1*(k2/(k1+k2))^2:
> c2_eq:=c2*(k1/(k1+k2))^2:
> k_eq:=k1*k2/(k1+k2):
> MM:=<<M | 0> , <0 | M>>:
> C:=<<c1_eq+c2_eq | 0> , <0 | c1_eq+c2_eq>>:
> K:=<<k_eq | 0> , <0 | k_eq>>:
> G:=<<0 | c2_eq> , <-c2_eq | 0>>:
> P:=s^2*MM+s*C+K+w*G:
> p:=Determinant(P):
> collect(p,s):
> a4:=(4*M^2*k1*k2^3+M^2*k1^4+4*M^2*k1^3*k2+6*M^2*k1^2*k2^2+M
> ^2*k2^4)/(k1+k2)^4:
> a3:=(2*M*k1^2*c1*k2^2+2*M*k2^4*c1+4*M*k1^3*k2+c2+2*M*k2^2*c
> k1^2+2*M*k1^4*c2+4*M*k1*k2^3+c1)/(k1+k2)^4:
> a2:=(6*M*k1^3*k2^2+2*M*k1^4*k2+c2*2*k1^4+6*M*k1^2*k2^3+c1
> k2^2*c2*k1^2+2*M*k2^4*k1+4*c1*k2^2)/(k1+k2)^4:
\[ a_1 := \frac{(2c2k1^4k2+2c1k2^3k1^2+2c1k2^4k1+2c2k1^3k2^2)}{(k1+k2)^4} \]
\[ a_0 := \frac{(k1^2k2^4+w^2c2^2k1^4+k1^4k2^2+2k1^3k2^3)}{(k1+k2)^4} \]
\[ Eq1 := a4b^4-a2b^2+a0 \]
\[ Eq2 := -a3b^3+a1b \]
\[ k1 := 50; k2 := 100; c1 := 0.2; c2 := 0.3; M := 10; \]
\[ \text{solve}\{\text{Eq1, Eq2}\},\{b, w\} \]

A.6 Matlab Code for the roots presented in Table 4.1

```matlab
clear all
k1=50; k2=100; M=10; c1=0.2; c2=0.3;
w=6.695510048;
%w=18632.40800;
a6=M^2*(c1+c2)^2;
a5=2*c1^2*M*c2+2*c1*k1*M^2+2*c2*k2*M^2+2*c1*k2*M^2+2*c1*c2^2*M+2*c2*k1*M^2;
a4=2*c2^2*k1*M^4+c1*k1*M*c2+c1^2*c2^2+2k1^2*M^2+4*c1*c2*M^2+2...c1^2*M^2+c2^2*M^2+w^2*c2^2*M^2;
a3=4*c1*k1*M*k2^2+2*c1*k1^2+c2+2*c1^2*M*c2+2*c1*w^2+c2^2*M+...c1^2*M^2+c2^2+c1^2*M^2+c2^2;
a2=2*k1*w^2+c2^2*M^2+2*k1^2*M^2+2*c1^2*M^2+c2^2+4*c1*k1*c2+k2^2+...c1^2*M^2+c2^2+2*c1^2*M^2+c2^2;
a1=2*c1*k1+w^2+2*c1*k1^2+2*c1*k1^2+2+c1^2*k2^2+2*k1^2+2*c2*k2;
a0=k1^2*w^2+2*c2^2+2*k1^2+2*k2^2;
p=[a6 a5 a4 a3 a2 a1 a0];
s=roots(p)
```

A.7 Matlab Code for the Smith’s roots presented in Table 4.2

```matlab
clear all
k1=50; k2=100; M=10; c1=0.2; c2=0.3; w=6.694378239;
a4=4*M^2*k1*k2^3+M^2*k1^4+4*M^2*k1^3*k2+6*M^2*k1^2*k2^2+M^2*k2^4;
a3=2*M^2*k1^2+c1*k2^2+2*M^2*k2^2+4*c1*M^2*k1^2+4*M^2*k1^2+2*M^2*k2^2+2...c1^2*M^2+c2^2+4*M^2*k1^2+2*M^2+c1^2*c2^2+c1^2*M^2+c2^2;
a2=6*M^2*k1^3*k2^2+2*M^2*k1^4*k2+c2^2+k1^4+6*M*k1^2+k2^3+...2*c1*k2^2+2*M^2*k2^2+4*k1+c1^2*M^2+4;
a1=2*c2*k1^4*k2+2*c1*k2^3+k1^2+2+c1*k2^4+k1+2+c2*k1^2+2;
a0=k1^2+k2^4+w^2+2*c2^2+2*k1^4+k1^4+k2^2+2*k1^3+k2^3;
p=[a4 a3 a2 a1 a0];
s=roots(p)
```

A.8 Matlab Code for parametric analysis of stability presented in Table 4.3 and Table 4.4

```matlab
clear all
```
A.9 Matlab Code for parametric analysis of stability presented in Table 4.5 and Table 4.6

```matlab
clear all
c1=0.2; c2=0.3; M=10;
w=6.695510048;
%M=18632.40800;
k1=50;
%c1=0.15;
%c1=0.25;
k2=100;
%c2=0.25;
%c2=0.35;
M=10;
w=6.695510048;
%w=18632.40800;
c1=0.2;
%c1=0.15;
%c1=0.25;
c2=0.3;
%c2=0.25;
%c2=0.35;
a6=M^2*(c1+c2)^2;
a5=2*c1^2*M*c2+2*c1*k1*M^2+2*c2*k2*M^2+2*c1*k2*M^2+2*c1*c2^2*M+2*c2*k1*M^2;
a4=2*c2^2*k1*M+4*c1*k1*M^2+2*c1*c2^2*M+4*c1*c2*M*k2+...
2*c1^2*M*k2^2+2*M^2+2*k1*k2*M^2+w^2*c2^2*M^2;
a3=4*c1*k1*M^2+2*c1*k1*c2^2+2*c1^2*M^2+4*c1^2*k1*M^2+4*c1*c2*M^2+...
2*c1^2*M^2+2*k1*k2*M^2+w^2*c2^2*M^2;
a2=2*c1^2*M^2+2*c1*k1*M^2+4*c1*c2*M^2+2*k1^2*M^2+4*c1*k1*c2*M^2+...
2*c1^2*M^2+2*k1*k2*M^2+w^2*c2^2*M^2;
a1=2*c1*k1*M^2+2*c1*k1^2*M^2+2*c1^2*M^2+4*c1^2*k1*M^2+4*c1*c2*M^2+...
2*c1^2*M^2+2*k1*k2*M^2+w^2*c2^2*M^2;
a0=k1^2*M^2+2*c1^2*M^2+4*c1*k1*M^2+2*k1^2*M^2+4*c1^2*k1*M^2+4*c1*c2*M^2+...
2*c1^2*M^2+2*k1*k2*M^2+w^2*c2^2*M^2;
p=[a6 a5 a4 a3 a2 a1 a0];
s=roots(p)
```

A.10 Matlab Code for parametric analysis of stability presented in Table 4.7 and Table 4.8

```matlab
clear all
c1=0.2; c2=0.3; M=10;
w=6.695510048;
%M=18632.40800;
k1=50;
%c1=0.15;
%c1=0.25;
k2=100;
%c2=0.25;
%c2=0.35;
M=10;
w=6.695510048;
%w=18632.40800;
c1=0.2;
%c1=0.15;
%c1=0.25;
c2=0.3;
%c2=0.25;
%c2=0.35;
a6=M^2*(c1+c2)^2;
a5=2*c1^2*M*c2+2*c1*k1*M^2+2*c2*k2*M^2+2*c1*k2*M^2+2*c1*c2^2*M+2*c2*k1*M^2;
a4=2*c2^2*k1*M+4*c1*k1*M^2+2*c1*c2^2*M+4*c1*c2*M*k2+...
2*c1^2*M*k2^2+2*M^2+2*k1*k2*M^2+w^2*c2^2*M^2;
a3=4*c1*k1*M^2+2*c1*k1*c2^2+2*c1^2*M^2+4*c1^2*k1*M^2+4*c1*c2*M^2+...
2*c1^2*M^2+2*k1*k2*M^2+w^2*c2^2*M^2;
a2=2*c1^2*M^2+2*c1*k1*M^2+4*c1*c2*M^2+2*k1^2*M^2+4*c1*k1*c2*M^2+...
2*c1^2*M^2+2*k1*k2*M^2+w^2*c2^2*M^2;
a1=2*c1*k1*M^2+2*c1*k1^2*M^2+2*c1^2*M^2+4*c1^2*k1*M^2+4*c1*c2*M^2+...
2*c1^2*M^2+2*k1*k2*M^2+w^2*c2^2*M^2;
a0=k1^2*M^2+2*c1^2*M^2+4*c1*k1*M^2+2*k1^2*M^2+4*c1^2*k1*M^2+4*c1*c2*M^2+...
2*c1^2*M^2+2*k1*k2*M^2+w^2*c2^2*M^2;
p=[a6 a5 a4 a3 a2 a1 a0];
s=roots(p)
```
clear all
k1=50; k2=100; c1=0.2; c2=0.3;
w=6.695510048;
%M=18632.40800
M=10;
%M=9.5;
%M=10.5;
a6=M^2*(c1+c2)^2;
a5=2*c1^2*M*c2+2*c1*k1*M^2+2*c2*k2*M^2+2*c1*k2*M^2+2*c1*c2^2*M+2*c2*k1*M^2;
a4=2*c2^2*k1*M+4*c1*k1*M*c2+2*c2*k1*M^2+4*c1*c2*M^2+2*c1*k1*M^2+2*c2^2*M^2;
a3=4*c1*k1*M^2+2*k1^2*M^2+2*c1*%w^2+2*c2^2*M^2;
a2=2*k1^2*M^2+4*c1*k1*M^2+2*c1*k1*c2^2+2*c1*M*c2+2*c2^2*M^2+2*c1*M*k2^2;
a1=2*c1*k1*%w^2+2*c2^2*M^2+2*k1^2*M^2+2*c1*k1*c2^2+2*c1*M*k2^2+2*c2^2*M^2+2*c1*k1*c2^2;
a0=k1^2*%w^2+2*c1^2*M^2+2*c1*k1*%w^2+2*c1^2*M^2+2*c1*K^2+2*c2^2*M^2+2*c1*k1*c2^2;
p=[a6 a5 a4 a3 a2 a1 a0];
s=roots(p)
Appendix B

Codes for Chapter 5 (shaft with mass)

B.1 Maple Code for Characteristic Polynomial (5.5)

```maple
restart;
with(LinearAlgebra):
MM := <<0.5*m | 0 | 0.5*m | 0>, <0 | 0.5*M | 0 | 0.5*m>, <0 | 0 | M | 0>, <0 | 0 | 0 | M>>;
C := <<c1+c2 | 0 | -c2 | 0>, <0 | c1+c2 | 0 | -c2>, <-c2 | 0 | c2 | 0>, <0 | -c2 | 0 | c2>>;
K := <<k1+k2 | 0 | -k2 | 0>, <0 | k1+k2 | 0 | -k2>, <-k2 | 0 | k2 | 0>, <0 | -k2 | 0 | k2>>;
G := <<0 | c2 | 0 | -c2>, <-c2 | 0 | c2 | 0>, <0 | -c2 | 0 | c2>>;
P := s^2*MM+s*C+(K+w*G):
p := Determinant(P):
collect(p, s):
```

B.2 Matlab Code for Root Locus shown in Figure 5.2

```matlab
clear all
k1=50;k2=100;c1=0.20;c2=0.30;M=10;m=0.1;
k=0;
for w=0:1:10,
    k=k+1;
    a8=0.25*m^2*M^2;
a7=m*c2*M^2+m*c1*M^2+m^2*M*c2;
a6=2*m*c2^2*M+m^2*c2^2+c1^2*M^2+c2^2*k2*M^2+2*m*k1*M^2;
a5=2*m*c1*M^2+2*m^2*m*c1*c2+2*m^2*2*M^2;
a4=3*m*k1*M*k2^2+2*m^2*M^2+4*m*c2^2+8*m^2*M^2+4*m*c2^2+8*m^2*M^2;
a3=4*m*k1*M*k2^2+4*m^2*M^2+4*m^2*M^2+4*m^2*M^2;
a2=2*m^2*M^2+4*m^2*M^2+4*m^2*M^2+4*m^2*M^2;
a1=4*m^2*M^2+4*m^2*M^2+4*m^2*M^2+4*m^2*M^2;
a0=m^2*M^2+4*m^2*M^2+4*m^2*M^2+4*m^2*M^2;
p=[a8 a7 a6 a5 a4 a3 a2 a1 a0];
s=roots(p);
S(k,:)=s.';
end
plot(S(:,1:4),'.');hold on
```

44
B.3 Maple Code leading to (5.16)

```maple
> restart;
> with(LinearAlgebra):
> MM:=<<0.5*m | 0 | 0.5*m | 0> , <0 | 0.5*m | 0 | 0.5*m> ,
  <0 | 0 | M | 0> , <0 | 0 | 0 | M>>:
> C:=<<c1+c2 | 0 | -c2 | 0> , <0 | c1+c2 | 0 | -c2> , <-c2 | 0 | c2 | 0> , <0 | -c2 | 0 | c2>>:
> K:=<<k1+k2 | 0 | -k2 | 0> , <0 | k1+k2 | 0 | -k2> , <-k2 | 0 | k2 | 0> , <0 | -k2 | 0 | k2>>:
> G:=<<0 | c2 | 0 | -c2> , <-c2 | 0 | c2 | 0> , <0 | -c2 | 0 | c2> , <c2 | 0 | -c2 | 0>>:
> P:=s^2-MM+s*C+(K+w*G):
> p:=Determinant(P):
> collect(p,s):
> a8:=0.25*m^2*M^2:
> a7:=m*c2*M^2+m*c1*M^2+m^2*M*c2:
> a6:=2*m*c2^2*M+m^2*c2^2+m^2*M+k2*M^2+m*k2*M^2+m*k1*M^2
  +2*m*c1^2*M+c2^2*M^2+3*m*c1*M*c2+c2^2*M^2:
> a5:=2*m*c1+c2^2+2*c2*k2*M^2+2*m^2*c2^2*k2+2*c2*k1*M^2+2*c1^2*<
  M^2+3*m*k1*M*c2+2*c1^2*M^2+4*m*c2*M^2+k2+2*c1*c2^2*M^2+3*m*c1
  *M^2+2*c1*c2^2+c2^2>M:
> a4:=3*m*k1*M^2+2*c1^2*M^2+2*m^2*w^2+2*m^2*M+k1*M^2+m^2*M+k2^2
  +2*m^2*M^2+4*c1^2+k1*M^2+2*m^2*M^2+k2^2*M^2+4*c1^2*M^2+2*k1^2*M^2
  +2*k1*M^2+4*c1*M^2+2*c1^2*M^2+2*c1^2*M^2+2*c1^2*M^2:
> a3:=4*m+k1*M^2+k2^2+2*m^2+c1^2*M^2+2*m^2+c1^2*M^2+2*m^2+c1^2*<M^2+2*
  c1^2>M^2:
> a2:=2*m*k1*w^2+c2^2+4*k1*k2*c1^2+c2^2+M+k1^2+c2^2
  +c1^2*w^2+c2^2+c1^2*k2^2+c1^2*k2^2+2*k1^2*M^2+2*m^2+k1^2
  +2*m^2+k1^2*M^2:
> a1:=2*c1+k1*w^2+c2^2+c2^2+2*k1^2+c2^2+2*c1+k1^2:
> a0:=k1^2+2*k1^2*w^2+c2^2:
> Eq1:=a8*b^8-a6*b^6+a4*b^4-a2*b^2+a0:
> Eq2:=-a7*b^7+a5*b^5-a3*b^3+a1*b:
> k1:=50:k2:=100:c1:=0.2:c2:=0.3:M:=10:m:=0.1:
```
B.4 Matlab Code leading to Table 5.1

clear all
k1=50; k2=100; c1=0.20; c2=0.30; M=10; m=0.1;
w=6.733791818;
%w=91.11607897;
%w=18448.07536;
a8=0.25*m^2*M^2;
a7=m*c2*M^2+m*c1*M^2+m^2*M*c2;
a6=2*m*c2^2*M+m^2*M^2+c1*M^2+m*k2*M^2+c2*M^2+...
  2*c1*c2*M^2+3*m*c1*M*c2+c2*M^2;
a5=2*m*c1+c^2+2*c2*k2*M^2+2*m^2*c2*k2+2*c2*k1*M^2+...
  2*c1*k2*M^2+3*m*k1*M*c2+2*c1^2*M*c2+4*m*c2*M*k2+...
  2*c1+c^2+M^3+m*c1*M*k2+2*c1*k1*M^2;
a4=3*m*k1*M^2+2*c1*M^2+2*m^2*w^2*c2^2*M^2+2*m*k1*c2^2+...
  m^2*w^2+c2^2+w^2*c1*M^2+4*c1*k1*M*c2+2*m*M^2+2*m^2*M^3;
a3=4*m^2+c1*M^2+2*c1*M^2+2*m*1+*w^2*c2^2+...
  4*c1*k1*M^2+4*c1*M^2+M^2+2*c2^2+c1^2*w^2+c2^2+...
  2*c1^2*M^2+2+2+c1*k1*M^2;
a2=2+m*k1*M^2+2*c2^2+c1^2*M^2+2+2+k1*M^2+2*m^2*c2^2+...
  c1^2*w^2+c2^2+2+c1^2*w^2+2+c1*k1*M^2+2;
a1=2*c1*k1*M^2+2+c1^2+2*c2^2; a0=1*k1*M^2+2+2+2+c1*k1^2+2;
p=[a8 a7 a6 a5 a4 a3 a2 a1 a0];
s=roots(p)

B.5 Matlab Code leading to Table 5.2

clear all
k1=50; k2=100; M=10; m=0.1; w=6.733791818;
c1=0.20;
c2=0.30;
%c1=0.15;
%c2=0.25;
%c2=0.35;
a8=0.25*m^2*M^2;
a7=m*c2*M^2+m*c1*M^2+m^2*M*c2;
a6=2*m*c2^2*M+m^2*M^2+c1*M^2+m*k2*M^2+c2*M^2+...
  2*c1*c2*M^2+3*m*c1*M*c2+c2*M^2;
a5=2*m*c1+c^2+2*c2*k2*M^2+2*m^2*c2*k2+2*c2*k1*M^2+...
  2*c1*k2*M^2+3*m*k1*M*c2+2*c1^2*M*c2+4*m*c2*M*k2+...
  2*c1+c^2+M^3+m*c1*M*k2+2*c1*k1*M^2;
a4=3*m*k1*M^2+2*c1*M^2+2*m^2*w^2*c2^2*M^2+2*m*k1*c2^2+...
  m^2*w^2+c2^2+w^2*c1*M^2+4*c1*k1*M*c2+2*m*M^2+2*m^2*M^3;
a3=4*m^2+c1*M^2+2*c1*M^2+2*m*1+*w^2*c2^2+...
  4*c1*k1*M^2+4*c1*M^2+M^2+2*c2^2+c1^2*w^2+c2^2+...
  2*c1^2*M^2+2+2+c1*k1*M^2;
a2=2+m*k1*M^2+2*c2^2+c1^2*M^2+2+2+k1*M^2+2*m^2*c2^2+...
  c1^2*w^2+c2^2+2+c1^2*w^2+2+c1*k1*M^2+2;
a1=2*c1*k1*M^2+2+c1^2+2*c2^2; a0=1*k1*M^2+2+2+2+c1*k1^2+2;
p=[a8 a7 a6 a5 a4 a3 a2 a1 a0];
s=roots(p)
\[ a_2 = 2m k_1 w^2 c_2^2 + 4k_1 k_2 c_1 c_2 + 2k_1 w^2 c_2^2 M + k_1^2 c_2^2 + \ldots \\
\quad c_1^2 w^2 c_2^2 c_1^2 + k_1^2 c_2^2 + 2k_1 w^2 c_2^2 M k_2 + 2 c_1 k_1 k_2^2 + 2 k_1^2 M k_2^2; \\
a_1 = 2c_1 k_1 w^2 c_2^2 + 2k_1^2 c_2^2 + 2k_1 k_2^2 c_1 + k_1 k_2^2; \\
a_0 = k_1^2 k_2^2 c_1^2 + 2k_1 w^2 c_2^2; \\
p = [a_8 \ a_7 \ a_6 \ a_5 \ a_4 \ a_3 \ a_2 \ a_1 \ a_0]; \\
s = \text{roots}(p) \\

B.6 Matlab Code leading to Table 5.3

clear all \\
c1 = 0.20; c2 = 0.30; M = 10; m = 0.1; w = 6.733791818; \\
k1 = 50; \\
k2 = 100; \\
\% k1 = 45; \\
\% k1 = 55; \\
\% k2 = 95; \\
\% k2 = 105; \\
a8 = 0.25 m^2 M^2; \\
a7 = 2m c_2^2 + m c_1 M^2 + 2 m^2 M c_2; \\
a6 = 2m c_2^2 + m c_1 M^2 + c_1 M^2 + 2m k_2 M^2 + m k_1 M^2 + \ldots \\
\quad 2c_1 k_2 M^2 + 3m c_1 M c_2 + 2c_2^2 M^2; \\
a5 = 2m c_1^2 + 2c_2^2 + 2c_1 c_2 + 2c_1 M^2 + 4c_1^2 M c_2 + 4m c_2 M c_2 + \ldots \\
\quad 2c_1 c_2 M^2 + 3m c_1 M c_2 + 2c_2^2 M c_2; \\
a4 = 3m k_1 M k_2 + 2 c_1^2 M k_2 + 2 m w^2 c_2^2 M^2 + 2 m k_1 c_2^2 + \ldots \\
\quad m^2 c_1^2 + 2w^2 c_2^2 + 4 c_1 k_1 M^2 + 2 c_2 M^2 + 2c_1^2 M c_2 + 2 c_1 M c_2 + \ldots \\
\quad 2c_1 c_2 c_1 M^2 + 3m c_1 c_2 + 2 c_1 k_2 M^2 + 2 c_1 k_1 M^2 + \ldots \\
\quad 2c_1^2 M k_2 + 2 c_2 M c_2 + 2 c_1 M c_2 + 2 c_1 M^2; \\
a3 = 4c_1 k_1 M^2 + 4 c_1 M k_2 + 4 c_1 k_1 M k_2 + 2 c_1 c_2 + 2 m c_1 w^2 c_2^2 + \ldots \\
\quad 2 c_1^2 c_2 M c_2 + 2 c_1 c_2^2; \\
a2 = 2m c_1 k_1 w^2 c_2^2 + 4k_1 k_2 c_1 c_2 + 2k_1 w^2 c_2^2 M + k_1^2 c_2^2 + \ldots \\
\quad 2c_1 w^2 c_2^2 + c_1^2 k_2^2 + c_1^2 M^2 + 2 c_1 k_2^2 M + 2 c_1 k_1 M^2 + \ldots \\
\quad 2c_1^2 M k_2 + 2 c_2 M c_2 + 2 c_1 M c_2 + 2 c_1 M^2; \\
a1 = 2c_1 k_1 w^2 c_2^2 + 2k_1 k_2 c_1 c_2 + 2k_1^2 c_1 + k_1 k_2^2; \\
a0 = k_1^2 k_2^2 + 2 c_1^2 M k_2 + 2 c_1 M c_2 + 2 c_1 M^2 + \ldots \\
\quad c_1^2 c_2^2 + 4m c_1 c_2 + 2 c_1 k_2 c_2 + 2 c_1 M^2; \\
p = [a_8 \ a_7 \ a_6 \ a_5 \ a_4 \ a_3 \ a_2 \ a_1 \ a_0]; \\
s = \text{roots}(p) \\

B.7 Matlab Code leading to Table 5.4

clear all \\
c1 = 0.20; c2 = 0.30; m = 0.1; k1 = 50; k2 = 100; w = 6.733791818; \\
M = 10; \\
\% M = 9.5; \\
\% M = 10.5; \\
a8 = 0.25 m^2 M^2; \\
a7 = 2m c_2^2 + m c_1 M^2 + 2m^2 M c_2; \\
a6 = 2m c_2^2 + m c_1 M^2 + c_1 M^2 + 2m k_2 M^2 + m k_1 M^2 + \ldots \\
\quad 2c_1 k_2 M^2 + 3m c_1 M c_2 + 2c_2^2 M^2; \\
a5 = 2m c_1^2 + 2c_2^2 + 2c_1 c_2 + 2c_1 M^2 + 4c_1^2 M c_2 + 4m c_2 M c_2 + \ldots \\
\quad 2c_1 c_2 c_1 M^2 + 3m c_1 c_2 + 2 c_1 k_2 M^2 + 2 c_1 k_1 M^2 + \ldots \\
\quad 2c_1^2 M k_2 + 2 c_2 M c_2 + 2 c_1 M c_2 + 2 c_1 M^2; \\
a4 = 3m k_1 M k_2 + 2 c_1^2 M k_2 + 2 m w^2 c_2^2 M^2 + 2 m k_1 c_2^2 + \ldots \\
\quad m^2 c_1^2 + 2w^2 c_2^2 + 4 c_1 k_1 M^2 + 2 c_2 M^2 + 2c_1^2 M c_2 + 2 c_1 M c_2 + \ldots \\
\quad 2c_1 c_2 c_1 M^2 + 3m c_1 c_2 + 2 c_1 k_2 M^2 + 2 c_1 k_1 M^2 + \ldots \\
\quad 2c_1^2 M k_2 + 2 c_2 M c_2 + 2 c_1 M c_2 + 2 c_1 M^2; \\
a3 = 4c_1 k_1 M^2 + 4 c_1 M k_2 + 4 c_1 k_1 M k_2 + 2 c_1 c_2 + 2 m c_1 w^2 c_2^2 + \ldots \\
\quad 2 c_1^2 c_2 M c_2 + 2 c_1 c_2^2; \\
a2 = 2m k_1 w^2 c_2^2 + 4k_1 k_2 c_1 c_2 + 2k_1 w^2 c_2^2 M + k_1^2 c_2^2 + \ldots \\
\quad 2c_1 w^2 c_2^2 + c_1^2 k_2^2 + c_1^2 M^2 + 2 c_1 k_2^2 M + 2 c_1 k_1 M^2 + \ldots \\
\quad 2c_1^2 M k_2 + 2 c_2 M c_2 + 2 c_1 M c_2 + 2 c_1 M^2; \\
a1 = 2c_1 k_1 w^2 c_2^2 + 2k_1 k_2 c_1 c_2 + 2k_1^2 c_1 + k_1 k_2^2; \\
a0 = k_1^2 k_2^2 + 2 c_1^2 M k_2 + 2 c_1 M c_2 + 2 c_1 M^2 + \ldots \\
\quad c_1^2 c_2^2 + 4m c_1 c_2 + 2 c_1 k_2 c_2 + 2 c_1 M^2;
\[ a_3 = 4m k_1 c_2 k_2 + 2c_1 k_1 c_2 + 2c_1 w^2 c_2 + 2c_1 k_2 w^2 c_2 + \ldots \]
\[ = 4c_1 k_1 M k_2 + 4c_2 k_1 M k_2 + 2m c_1 k_1 k_2 + 2m c_1 w^2 c_2 + 2M c_1 k_2 w^2 + \ldots \]
\[ + 2k_1^2 M c_1 + 2c_1^2 c_2 k_2; \]
\[ a_2 = 2m k_1 w^2 c_2 + 2c_1 k_2 c_2 + 2c_1 k_1 w^2 c_2 + 2k_1^2 M k_2 + 2c_1 k_1 k_2 w^2 + \ldots \]
\[ + c_1^2 w^2 + 2c_1^2 c_2 k_2 + 2c_1^2 c_2 k_2 + 2c_1 k_2 w^2 + 2c_1 k_1 k_2 w^2; \]
\[ a_1 = 2c_1 k_2 w^2 c_2 + 2k_1 k_2 c_2 + 2c_1 k_1 k_2 w^2; \]
\[ a_0 = k_1^2 k_2^2 + k_1^2 w^2 c_2; \]
\[ p = [a_8 \ a_7 \ a_6 \ a_5 \ a_4 \ a_3 \ a_2 \ a_1 \ a_0]; \]
\[ s = \text{roots}(p) \]

**B.8 Matlab Code leading to Table 5.5**

```matlab
clear all

c1=0.20; c2=0.30; M=10; k1=50; k2=100; w=6.733791818;
m=0.10;
%m=0.09;
%m=0.11;

a8=0.25*m^2*M^2;
a7=m*c2*M^2+m*c1*M^2+m^2*M*c2;
a6=2*m*c2^2*M+m^2*c2^2+c1^2*M^2+m*k2*M^2+m^2*M*k2+m1*M^2+\ldots
  2*c1^2*M^2+3*m*c1^2*M^2+c2^2+2*M^2;
a5=2*m*c1^2*M^2+2*c2^2+k2^2*c1^2+2*m^2*c2^2+2*c2^2+k1*M^2+\ldots
  2*c1*c2^2+3*m+c1^2*M^2+c2+4*m^2*c2^2*M^2+\ldots
  c1^2*M^2+3*m+c1^2*M^2+c2^2+2*M^2;
a4=3*m*k1*M^2+2+c1^2*M^2+k2^2*m+c2+2*M^2+c1^2+2*M^2+\ldots
  m^2+c2^2+2*m^2+c2^2+2*M^2+4*c1*k1*M^2+c2+2*M^2+c2^2+\ldots
  k2^2+M^2+4*c1^2*M^2+c2^2+2*M^2+c2^2+2*M^2+2*k1*k2*M^2+\ldots
  c1^2+c2+4*m^2+c1^2+c2^2+2*M^2+\ldots
  4*m^2+c1^2+c2^2+2*M^2+c1^2+2*M^2+\ldots
  2*k1^2+M^2+c2^2+2*M^2+\ldots
a3=4*m^2+k1^2+c2^2+2*c1+k1^2+c2+c1^2+c2^2+2*k1^2+M+1^2+2*c1^2+c2^2+\ldots
  c1^2+c2^2+2+c1+k1^2+c2+c2^2+2*k1^2+M+1^2+2*c1^2+c2^2+\ldots
a2=2*m^2+k1^2+c2^2+2+c1+k1^2+c2+c1^2+c2^2+2*k1^2+M+1^2+2*c1^2+c2^2+\ldots
  c1^2+c2^2+2+c1+k1^2+c2+c2^2+2*k1^2+M+1^2+2*c1^2+c2^2+\ldots
a1=2*m^2+k2^2+2+c1^2+c2^2+2+c1+k1^2+c2+c2^2+2*k1^2+M+1^2+2*c1^2+c2^2+\ldots
a0=k1^2+M+1^2+2+c1^2+c2^2+2+c1+k1^2+c2+c2^2+2*k1^2+M+1^2+2*c1^2+c2^2+\ldots
p=[a8 a7 a6 a5 a4 a3 a2 a1 a0];
s=\text{roots}(p)
```
Appendix C
Simplification of Coefficients of Characteristic Polynomials

C.1 Simplification of coefficients of characteristic polynomial (4.10), which results from A.1 and appears in A.3

\[ \alpha_0 = \kappa^2 \omega^2 \sigma^2 + \kappa^2 \omega^2 \sigma^2 \]
\[ = \kappa^2 (\kappa^2 \omega^2 + \omega^2 \sigma^2) \quad (4.11) \]

\[ \alpha_1 = 2 \sigma \kappa \omega^2 \sigma^2 + 2 \sigma \kappa \kappa^2 \omega^2 + 2 \kappa^2 \sigma^2 \kappa^* \]
\[ = 2 \kappa^2 (\kappa^2 \sigma^2 + \omega^2 \sigma^2) + \sigma^* (\kappa^2 \kappa^*) \quad (4.12) \]

\[ \alpha_2 = 2 \kappa^2 \omega^2 \sigma^2 M + 2 \kappa^2 M \kappa^* + 2 \kappa M \kappa^2 + 2 \kappa^2 \sigma^2 M + 2 \kappa \sigma \sigma^2 \kappa^2 + \omega^2 \sigma^2 \kappa^2 + 4 \sigma \kappa \kappa^2 \sigma^2 + \sigma \omega^2 \sigma^2 \kappa^2 \]
\[ = 2 \kappa M \kappa^2 + 2 \kappa \omega^2 \sigma^2 M + 2 \kappa^2 M \kappa^* + \left( \kappa^2 \sigma^2 + 4 \sigma \kappa \sigma^2 \kappa^* \right) + \left( \sigma \omega^2 \kappa^2 + \sigma \omega^2 \omega^2 \sigma^2 \kappa^2 \right) \]
\[ = 2 \kappa^2 \left( \kappa^2 + \omega^2 \sigma^2 \right) + \kappa^2 \kappa^* + \kappa^2 \sigma^2 (\kappa^2 \sigma^2 + 4 \sigma \kappa \sigma^2 \kappa^* + \sigma \omega^2 \sigma^2 \kappa^2) + \sigma \omega^2 \sigma^2 \kappa^2 \]
\[ = 2 \kappa^2 \left( \kappa^2 + \omega^2 \sigma^2 \right) + \kappa^2 \kappa^* + \kappa^2 \sigma^2 \kappa^2 \quad (4.13) \]

\[ \alpha_3 = 4 \sigma \kappa M \kappa^* + 2 \sigma \kappa \sigma^2 M + 2 \kappa^2 M \sigma^2 + 2 \sigma \omega^2 \sigma^2 M \]
\[ + 2 \sigma \omega^2 \sigma^2 \kappa^2 + 4 \sigma \kappa M \kappa^* + 2 \sigma \omega^2 \kappa^2 \]
\[ = 2 \sigma \kappa M \kappa^* + 2 \sigma \kappa \sigma^2 M + 2 \kappa^2 M \sigma^2 + 2 \sigma \omega^2 \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M \]
\[ + 4 \sigma \kappa M \kappa^* + 4 \sigma \kappa M \kappa^* \]
\[ = 2 \sigma \kappa M \kappa^* + 2 \sigma \kappa \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M \]
\[ + 4 \kappa^2 \kappa^* \left( \sigma^2 + \sigma^2 \right) \]
\[ = 2 \sigma \kappa M \kappa^* + 2 \sigma \kappa \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M + 4 \kappa^2 \kappa^* \left( \sigma^2 + \sigma^2 \right) \]
\[ = 2 \sigma \kappa \sigma^2 M + 2 \sigma \kappa \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M \]
\[ + 4 \kappa^2 \kappa^* \left( \sigma^2 + \sigma^2 \right) \]
\[ = 2 \sigma \kappa \sigma^2 M + 2 \sigma \kappa \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M \]
\[ + 2 \sigma \omega^2 \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M \]
\[ + 4 \kappa^2 \kappa^* \left( \sigma^2 + \sigma^2 \right) \]
\[ = 2 \sigma \kappa \sigma^2 M + 2 \sigma \kappa \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M \]
\[ + 2 \sigma \omega^2 \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M \]
\[ + 2 \sigma \omega^2 \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M \]
\[ + 4 \kappa^2 \kappa^* \left( \sigma^2 + \sigma^2 \right) \]
\[ = 2 \sigma \kappa \sigma^2 M + 2 \sigma \kappa \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M \]
\[ + 2 \sigma \omega^2 \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M \]
\[ + 2 \sigma \omega^2 \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M \]
\[ + 4 \kappa^2 \kappa^* \left( \sigma^2 + \sigma^2 \right) \]
\[ = 2 \sigma \kappa \sigma^2 M + 2 \sigma \kappa \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M \]
\[ + 2 \sigma \omega^2 \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M \]
\[ + 2 \sigma \omega^2 \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M \]
\[ + 4 \kappa^2 \kappa^* \left( \sigma^2 + \sigma^2 \right) \]
\[ = 2 \sigma \kappa \sigma^2 M + 2 \sigma \kappa \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M \]
\[ + 2 \sigma \omega^2 \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M \]
\[ + 2 \sigma \omega^2 \sigma^2 M + 2 \sigma \omega^2 \sigma^2 M \]
\[ + 4 \kappa^2 \kappa^* \left( \sigma^2 + \sigma^2 \right) \]
\[ (4.14) \]
\[ \alpha_4 = \sigma'^2 \sigma'' + 2 \sigma''^2 \kappa'M + 4 \sigma' \sigma'M \kappa'' + 2 \sigma'^2 M \kappa'' + 4 \sigma' \kappa'M \sigma'' \\
+ \left( \kappa'^2 + \omega^2 \sigma''^2 \right) M^2 + \kappa'^4 (\kappa'' + 2 \kappa')M^2 \\
= \sigma'^2 \sigma''^2 + \left( \sigma''^2 \kappa' + 2 \sigma' \sigma'' \kappa'' + \sigma'^2 \kappa'' + 2 \sigma' \kappa' \sigma'' \right) 2M \\
+ \left( \kappa'^2 + \omega^2 \sigma''^2 \right) M^2 + \kappa'^4 (\kappa'' + 2 \kappa')M^2 \\
= \sigma'^2 \sigma''^2 + \left( \sigma''^2 \kappa' + 2 \sigma' \kappa' \sigma'' + \sigma'^2 \kappa'' + 2 \sigma' \sigma'' \kappa'' \right) 2M \\
+ \left( \kappa'^2 + \omega^2 \sigma''^2 \right) M^2 + \kappa'^4 (\kappa'' + 2 \kappa')M^2 \\
= \sigma'^2 \sigma'^2 + \left( \kappa' \sigma'' + \sigma' + \sigma'' \right) + \kappa'' \sigma' \left( \sigma''' + \sigma'' \right) + \kappa'' \sigma'' \sigma'' \right) 2M \\
+ \left( \kappa'^2 + \omega^2 \sigma''^2 \right) M^2 + \kappa'^4 (\kappa'' + 2 \kappa')M^2 \\
= \sigma'^2 \sigma''^2 + \left( \kappa' \sigma'' + \sigma' + \sigma'' \right) + \kappa'' \sigma' \left( \sigma''' + \sigma'' \right) + \kappa'' \sigma'' \sigma'' \right) 2M \\
+ \left( \kappa'^2 + \omega^2 \sigma''^2 \right) M^2 + \kappa'^4 (\kappa'' + 2 \kappa')M^2 \\
= M^2 \left( \kappa' \sigma'' + \sigma' + \sigma'' \right) + \kappa' \kappa'' \sigma'' \sigma'' \right) 2M \\
+ 2M \left( \kappa' \sigma'' + \sigma' + \sigma'' \right) \left( \kappa' \sigma'' + \sigma' + \sigma'' \right) + \sigma'^2 \sigma''^2 \\
\]

(4.15)

\[ \alpha_5 = 2 \sigma'^2 M \sigma'' + 2 \sigma' \kappa'M^2 + 2 \sigma'' \kappa'M^2 + 2 \sigma' \kappa'M^2 + 2 \sigma'' \kappa'M^2 + 2 \sigma' \kappa'M^2 + 2 \sigma'' \kappa'M^2 + 2 \sigma'' \kappa'M^2 \\
= 2 \sigma' \kappa'M^2 + 2 \sigma'' \kappa'M^2 + 2 \sigma' \kappa'M^2 + 2 \sigma'' \kappa'M^2 + 2 \sigma'^2 \kappa'M^2 + 2 \sigma'' \kappa'M^2 \\
= 2M^2 \left( \sigma' \kappa' + \sigma'' \kappa'' + \sigma' \kappa' + \sigma'' \kappa'' \right) + 2M \sigma' \sigma'' \left( \sigma'' + \sigma' \right) \\
= 2M^2 \left( \sigma' \kappa' + \sigma'' \kappa'' + \sigma' \kappa' + \sigma'' \kappa'' \right) + 2M \sigma' \sigma'' \left( \sigma'' + \sigma' \right) \\
= 2M^2 \left( \sigma' \left( \kappa' + \kappa'' \right) + \sigma'' \left( \kappa'' + \kappa' \right) \right) + 2M \sigma' \sigma'' \left( \sigma'' + \sigma' \right) \\
= 2M \left( \sigma' + \sigma'' \right) \left( \kappa' \kappa'' + \sigma' \sigma'' \right) \\
\]

(4.16)

\[ \alpha_6 = M^2 \left( \sigma' + \sigma'' \right) \]

(4.17)

C.2 Simplification of coefficients of characteristic polynomial (4.18), which results from A.2 and appears in A.3

\[ \hat{\alpha}_0 = \frac{\kappa'^4 \kappa^4 + \omega^2 \sigma''^2 \kappa^4 + \kappa'^4 \kappa^2 + 2 \kappa^3 \kappa^3}{\left( \kappa'' + \kappa'' \right)^4} \\
= \frac{\kappa'^4 \kappa^4 + \kappa'^4 \kappa^2 + 2 \kappa^3 \kappa^3 + \omega^2 \sigma''^2 \kappa'^2}{\left( \kappa'' + \kappa'' \right)^4} \]

50
\[ \hat{\alpha}_0 = \frac{\kappa'^2 (\kappa''^2 + \kappa'^2 + 2 \kappa' \kappa'') + \omega^2 \sigma''^2 \kappa'^2}{(\kappa' + \kappa'')^4} \]

\[ = \frac{\kappa'^2 \kappa''^2}{(\kappa' + \kappa'')^2} + \frac{\omega^2 \sigma''^2 \kappa'^4}{(\kappa' + \kappa'')^4} \] (4.19)

\[ \hat{\alpha}_1 = \frac{2 \sigma'' \kappa'^4 \kappa'' + 2 \sigma' \kappa''^3 \kappa'^2 + 2 \sigma' \kappa''^2 \kappa' + 2 \sigma'' \kappa''^3 \kappa'^2}{(\kappa' + \kappa'')^4} \]

\[ = \frac{2(\sigma'' \kappa'^4 \kappa'' + \sigma'' \kappa''^3 \kappa'^2 + \sigma' \kappa''^2 \kappa' + \sigma'' \kappa''^2 \kappa'^2)}{\left(\kappa' + \kappa''\right)^4} \]

\[ = \frac{2(\kappa' \kappa'' \sigma' (\kappa' + \kappa'' + \kappa'' \sigma' (\kappa' + \kappa''))}{\left(\kappa' + \kappa''\right)^4} \]

\[ = \frac{2(\kappa' + \kappa'') \kappa' \left(\kappa'' \sigma' + \kappa'' \sigma'\right)}{\left(\kappa' + \kappa''\right)^4} \]

\[ = \frac{2 \kappa' \kappa' \left(\kappa'' \sigma' + \kappa'' \sigma'\right)}{(\kappa' + \kappa'')^3} \] (4.20)

\[ \hat{\alpha}_2 = \frac{\left(6 M \kappa'^2 \kappa''^2 + 2 M \kappa'^2 \kappa'' + \sigma''^2 \kappa'^4 + 6 M \kappa'^2 \kappa''^3 + 2 \sigma' \kappa''^2 \sigma'' \kappa'^2 + 2 M \kappa''^4 \kappa' + \sigma''^2 \kappa''^4\right)}{(\kappa' + \kappa'')^4} \]

\[ = \frac{\left(6 M \kappa'^2 \kappa''^2 + 2 M \kappa'^2 \kappa''^3 + 2 M \kappa'^4 \kappa'' + 2 M \kappa''^4 \kappa' + \sigma''^2 \kappa'^2 + \sigma''^2 \kappa''^4 + 2 \sigma' \kappa''^2 \sigma'' \kappa'^2\right)}{(\kappa' + \kappa'')^4} \]

\[ = \frac{\left(3 \kappa'^2 \kappa''^2 + 3 \kappa'^2 \kappa''^3 + \kappa'^4 \kappa'' + \kappa''^4 \kappa'\right) 2 M + \left(\sigma' \kappa''^2 + \sigma'' \kappa'^4\right)}{(\kappa' + \kappa'')^4} \]

\[ = \frac{\left(2 M \left(3 \kappa'^2 \kappa''^2 (\kappa' + \kappa'') + \kappa' \k'' (\k''^3 + \k''^3)\right) + \left(\sigma' \k''^2 + \sigma'' \k'^4\right)^2\right)}{(\kappa' + \k''^4)} \]

\[ = \frac{\left(2 M \left(3 \k''^3 (\k' + \k'') + \k' \k'' \left(\k''^3 + \k''^3\right)\right) + \left(\sigma' \k''^2 + \sigma'' \k'^4\right)^2\right)}{(\k' + \k'')^4} \]

\[ = \frac{\left(2 M \left(\k' + \k''\right) \k' \k'' \left(3 \k' + \k'' + \left(\k''^2 + \k''^2 - \k' \k''\right)\right) + \left(\sigma' \k''^2 + \sigma'' \k'^4\right)^2\right)}{(\k' + \k'')^4} \]
\[ \hat{\alpha}_2 = \frac{2M(k' + k'')(k' + k'')(k' + k'')(k' + k'')^2 + (\sigma'k'^2 + \sigma'k'^2)^2}{(k' + k')^4} \]
\[ = \frac{2Mk'k''}{(k' + k')^4} + \frac{(\sigma'k'^2 + \sigma'k'^2)^2}{(k' + k')^4} \]  
\[ (4.21) \]

\[ \hat{\alpha}_3 = \frac{2Mk'^2\sigma'k'^2 + 2Mk'^4\sigma' + 4Mk'^3k''\sigma'' + 2Mk'^2k''k'^2 + 2Mk'^4\sigma'' + 4Mk'k'^3\sigma'\sigma'^4}{(k' + k'')^4} \]
\[ = \frac{2M(k'^2\sigma'k'^2 + k'^4\sigma' + 2k'^3k''\sigma'' + k'^2k''\sigma'^2 + k'^4\sigma'' + 2k'k'^3\sigma')}{(k' + k')^4} \]
\[ = \frac{2M((k'^2\sigma'k'^2 + k'^4\sigma' + 2k'k'^3\sigma') + (2k'^3k''\sigma'' + k'^2\sigma'k'^2 + k'^4\sigma'))}{(k' + k')^4} \]
\[ = \frac{2M(k'^2\sigma'k'^2 + 2k'k'^3)}{(k' + k'')^4} \]
\[ = \frac{2M(k'^2\sigma''(k' + k'')^2 + k'^4\sigma''(k' + k''))}{(k' + k'')^4} \]
\[ = \frac{2M(k'^2\sigma''(k' + k'')^2 + k'^4\sigma''(k' + k'')^2)}{(k' + k'')^4} \]  
\[ (4.22) \]

\[ \hat{\alpha}_4 = \frac{(4M^2k'k'^3 + M^2k'^4 + 4M^2k'^3k'' + 6M^2k'^2k''^2 + M^2k'^4)}{(k' + k'')^4} \]
\[ = \frac{M^2(4k'k'^3 + 4k'^3k'' + 6k'^2k''^2 + k'^4 + k'^4)}{(k' + k'')^4} \]
\[ = \frac{M^2(3k'k'^3 + 3k'^3k'' + 6k'^2k''^2 + k'k'^3 + k'^3k'' + k'^4 + k'^4)}{(k' + k'')^4} \]
\[ = \frac{M^2((3k'k'^3 + 3k'^3k'' + 6k'^2k''^2) + (k'k'^3 + k'^4) + (k'^3k'' + k'^4))}{(k' + k'')^4} \]
\[ = \frac{M^2(3k'k'^3 + k'^2 + 2k'k'' + k'(k'^3 + k'^3) + k''(k'^3 + k'^3))}{(k' + k'')^4} \]
\[ = \frac{M^2(3k'k'^3(k' + k'')^3 + (k'^3 + k'^3)(k' + k'')}{(k' + k'')^4} \]
\[ = \frac{M^2(k' + k')(3k'k''(k' + k'') + (k'^3 + k'^3))}{(k' + k'')^4} \]
\[ = \frac{M^2(k' + k')(k' + k'')^3}{(k' + k'')^4} = M^2 \]  
\[ (4.23) \]
C.3 Simplification of coefficients of characteristic polynomial (5.5), which results from B.1 and appears in B.2

\[ \alpha_0 = \kappa^2 \kappa^2 + \kappa^2 \omega^2 \sigma^2 \]
\[ = \kappa^2 \left( \kappa^2 + \omega^2 \sigma^2 \right) \]
(5.6)

\[ \alpha_1 = 2\sigma' k' \omega^2 \sigma^2 + 2\kappa^2 \sigma'' \kappa'' + 2\sigma' k' \kappa^2 \]
\[ = 2\sigma' k' \kappa^2 + 2\sigma' k' \omega^2 \sigma^2 + 2\kappa^2 \sigma'' \kappa'' \]
\[ = 2\sigma' k' \left( \kappa^2 + \omega^2 \sigma^2 \right) + 2\kappa^2 \sigma'' \kappa'' \]
\[ = 2k' \left( \sigma' \left( \kappa^2 + \omega^2 \sigma^2 \right) + \sigma'' \left( \kappa' \kappa'' \right) \right) \]
(5.7)

\[ \alpha_2 = 2m k' \omega^2 \sigma^2 + 2m \kappa^2 \kappa'' + 2m k' \kappa'' + 2\kappa^2 \omega^2 \sigma^2 M + \kappa^2 \sigma'' \kappa'' + \sigma'' \omega^2 \sigma^2 + \sigma''^2 \kappa^2 \]
\[ + 2\kappa^2 \kappa^2 M \kappa'' + 2m k' \kappa'' + 2\kappa^2 M \kappa'' \]
\[ = 2m k' \kappa^2 + 2m k' \omega^2 \sigma^2 + 2\kappa^2 M \kappa'' + 2\kappa^2 \omega^2 \sigma^2 M + \sigma^2 \kappa^2 + \sigma'' \omega^2 \sigma^2 \]
\[ + \kappa^2 \sigma'' \kappa'' + 2\kappa^2 \kappa^2 M \kappa'' + 4\kappa^2 \kappa'' \sigma'' \sigma'' \]
\[ = 2m k' \left( \kappa^2 + \omega^2 \sigma^2 \right) + 2 \kappa^2 \left( \kappa^2 + \omega^2 \sigma^2 \right) + \sigma^2 \left( \kappa^2 + \omega^2 \sigma^2 \right) \]
\[ + \kappa^2 \sigma'' \kappa'' + 2\kappa^2 \kappa^2 M \kappa'' + 4\kappa^2 \kappa'' \sigma'' \sigma'' \]
(5.8)

\[ \alpha_3 = 4m k' \sigma'' \kappa'' + 2\sigma' M \kappa'' + 2\sigma' k' \sigma'' + 2m \sigma' \omega^2 \sigma^2 + 4\sigma' k' M \kappa'' + 4\sigma'' k' M \kappa'' + 2m \sigma' k'' \]
\[ + 2\sigma' \omega^2 \sigma^2 M + 2\kappa^2 M \omega^2 \sigma^2 + 2\sigma''^2 \kappa'' \]
\[ = 2\sigma' M \kappa^2 + 2\sigma' \omega^2 \sigma^2 M + 2m \sigma' \omega^2 \sigma^2 \]
\[ + 4m k' \sigma'' \kappa'' + 4\sigma'' k' M \kappa'' + 2\sigma' k' \sigma'' + 2\sigma''^2 \kappa'' + 4\sigma' k' M \kappa'' + 2\kappa^2 M \sigma^2 \]
\[ = 2\sigma' M \left( \kappa^2 + \omega^2 \sigma^2 \right) + 2m \sigma' \left( \kappa^2 + \omega^2 \sigma^2 \right) \]
\[ + 4m k' \sigma'' \kappa'' + 4\sigma'' k' M \kappa'' + 2\sigma' k' \sigma'' + 2\sigma''^2 \kappa'' + 4\sigma' k' M \kappa'' + 2\kappa^2 M \sigma^2 \]
\[ = \left( \kappa^2 + \omega^2 \sigma^2 \right) \sigma' \left( m + M \right) \]
\[ + (4\kappa' \sigma'' \kappa' \left( m + M \right) + 2 \sigma'' \sigma'' \left( \sigma' \kappa' + \sigma' \kappa'' \right) + M k' \left( \sigma'' \kappa' + \sigma'' \kappa'' \right) + \sigma' k' \kappa'' \left( m + M \right) \]
\[ = \left( \kappa^2 + \omega^2 \sigma^2 \right) \sigma' + \left( 4\kappa' \sigma'' \kappa' \right) \left( m + M \right) + 2 \left( \sigma'' \sigma'' + M k' \right) \left( \sigma' \kappa' + \sigma' \kappa'' \right) + 2 \sigma' k' \kappa'' \left( m + M \right) \]
\[ = 2 \left( m + M \right) \left( \kappa^2 + \omega^2 \sigma^2 \right) \sigma' + \left( 2 \kappa' \sigma'' \kappa' \right) + 2 \left( M k' + \sigma' \sigma'' \right) \left( \sigma' \kappa' + \sigma'' \kappa'' \right) + 2 \k' \k'' \sigma' M \]
(5.9)
\[ \alpha_4 = 3m\kappa'M\kappa'' + 2\sigma'^2M\kappa'' + 2m\omega^2\sigma'^2M + 2m\kappa'\sigma'^2 + m^2\omega\sigma''M^2 + \omega\sigma''M^2 + 4\sigma'\kappa'M\sigma'' + 2m\kappa''M^2 + \kappa'^2M^2 + 4\sigma'\sigma''\kappa'' + m^2\kappa''M^2 + \kappa'^2M^2 + 2\kappa'k'M^2 + \sigma'^2\sigma'' + 4m\sigma'\sigma''\kappa'' + 2\sigma''\kappa'M \]

\[ = m^2\kappa'' + m^2\omega\sigma'' + \kappa'^2M^2 + \omega^2\sigma''M^2 + 2m\kappa''M^2 + 2m\omega^2\sigma''M \]

\[ + 2m\kappa'\sigma'' + 2\sigma'^2\kappa'M + 4m\sigma'\sigma''\kappa'' + 4\sigma'\sigma''\kappa'' \]

\[ + \kappa'^2M^2 + \sigma'^2\sigma'' + 2\sigma'\k'\k'M\sigma'' \]

\[ + 2\sigma'^2\k'M'' + 2\sigma'k'M\sigma'' \]

\[ + 2\k'\k''M^2 + 2m\k'\k''M + m\k'\k'' \]

\[ = m\left(\kappa'' + \omega^2\sigma''\right) + M^2\left(\kappa'' + \omega^2\sigma''\right) + 2mM\left(\kappa'' + \omega^2\sigma''\right) \]

\[ + 2\k'\sigma''(m + M) + 4\sigma'\sigma''\kappa'(m + M) \]

\[ + \left(\k'\k'' + \sigma''\right)^2 \]

\[ + 2m\sigma'\left(\sigma'\k' + \k''\right) \]

\[ + 2\k'\k''M(m + M) + m\k'\k'' \]

\[ = \left(\kappa'' + \omega^2\sigma''\right)(m + M)^2 + 2\sigma''(m + M)\left(\kappa'\sigma'' + 2\sigma'\k''\right) + \left(k'\k'' + \sigma'\sigma''\right)^2 \]

\[ + 2M\sigma'\left(\sigma'\k'' + \k''\right) + k'k''M(3m + 2M) \]

\[ = \left(\kappa'' + \omega^2\sigma''\right)(m + M)^2 + 2\sigma''(m + M)\left(\kappa'\sigma'' + 2\sigma'\k''\right) + \left(k'\k'' + \sigma'\sigma''\right)^2 \]

\[ + 2M\sigma'\left(\k'\sigma'' + \k''\sigma''\right) + k'k''M(3m + 2M) \]

\[ (5.10) \]

\[ \alpha_5 = 2m\sigma'\sigma'' + 2\sigma''k'M^2 + 2m^2\sigma'k'' + 2\sigma''k'M^2 + 2\sigma'k''M^2 + 3m\k'\k''M\sigma'' + 2\sigma'^2M\sigma'' \]

\[ + 4m\sigma''M\k'' + 2\sigma'\sigma''^2M + 3m\sigma'M\k'' + 2\sigma'k'M^2 \]

\[ = 2\sigma''k'M^2 + 2\sigma''k'M^2 + 2\sigma'k''M^2 + 2\sigma'k'M^2 \]

\[ + 3m\k'M\sigma'' + 3m\sigma'M\k'' \]

\[ + 2\sigma'^2M\sigma'' + 2\sigma'\sigma''^2M \]

\[ + 4m\sigma''M\k'' + 2m^2\sigma''k'' + 2m\sigma'\sigma''^2 \]
\[
\alpha_5 = 2M^2(\sigma'' \kappa'' + \sigma'' \kappa' + \sigma \kappa'' + \sigma \kappa') \\
+ 3mM(\kappa' \sigma'' + \sigma' \kappa'') \\
+ 2M \sigma' \sigma''(\sigma' + \sigma'') \\
+ 2m \sigma''(2M \kappa'' + m \kappa'' + \sigma' \sigma'') \\
= 2M^2(\sigma''(\kappa'' + \kappa') + \sigma'(\kappa'' + \kappa')) \\
+ 3mM(\kappa' \sigma'' + \sigma' \kappa'') \\
+ 2M \sigma' \sigma''(\sigma' + \sigma'') \\
+ 2m \sigma''(\kappa''(2M + m) + \sigma' \sigma'') \\
= 2M^2(\kappa' + \kappa'')(\sigma' + \sigma'') \\
+ 3mM(\kappa' \sigma'' + \sigma' \kappa'') \\
+ 2M \sigma' \sigma''(\sigma' + \sigma'') \\
+ 2m \sigma''(\kappa''(m + 2M) + \sigma' \sigma'') \\
= 2M^2(\kappa' + \kappa'')(\sigma' + \sigma'') + M(3m(\kappa' \sigma'' + \kappa'' \sigma') + 2\sigma' \sigma''(\sigma' + \sigma'')) \\
+ 2m \sigma''(\kappa''(m + 2M) + \sigma' \sigma'') \\
(5.11)
\]

\[
\alpha_6 = 2m \sigma'' \sigma' M + m^2 \sigma''^2 + \sigma''^2 M^2 + m \kappa'' M^2 + m^2 M \kappa'' + m \kappa'' M^2 + 2 \sigma' \sigma'' M^2 \\
+ 3m \sigma M \sigma'' + \sigma''^2 M^2 \\
= \sigma''^2 M^2 + 2 \sigma' \sigma'' M^2 + \sigma''^2 M^2 \\
+ m^2 \sigma''^2 + 2m \sigma'' M \\
+ m \kappa'' M^2 + m \kappa'' M^2 \\
+ 3m \sigma M \sigma'' + m^2 M \kappa'' \\
= M^2(\sigma' + \sigma'')^2 \\
+ m \sigma''^2 (m + 2M) \\
+ m M^2 (\kappa'' + \kappa') \\
+ m M (3 \sigma' \sigma'' + m \kappa'') \\
= M^2(\sigma' + \sigma')^2 + m(M^2(\kappa' + \kappa'') + M(m \kappa'' + 3 \sigma' \sigma'') + \sigma''^2 (m + 2M)) \\
(5.12)
\]
\[ \alpha_7 = m\sigma^2 M^2 + m\sigma M^2 + m^2 M\sigma^2 \]
\[ = mM(\sigma^2 M + \sigma^2 (M + M)) \]  
\[ (5.13) \]

\[ \alpha_8 = 0.25m^2 M^2 \]  
\[ (5.14) \]
Vita

Siddharth Sharma was born in 1986 in Ghaziabad, Uttar Pradesh, India. He received his school education from Saint John’s Senior Secondary School, Kota, Rajasthan, India. He completed his Bachelor of Technology degree in Mechanical Engineering from University College of Engineering, Rajasthan Technical University in 2010. He entered the Master of Science in Mechanical Engineering program at Louisiana State University, Baton Rouge, United States of America in Fall 2010. Currently, he is pursuing an year long internship at the large AC ANEMA motor manufacturing facility of Siemens Industry, Incorporated located in Cincinnati, Ohio, United States of America. He successfully defended his thesis on October 29, 2012 and will graduate with a Master’s degree on the Commencement Day of December 14, 2012. Siddharth wishes to make a career in research, development and/ or engineering departments of rotating equipment industry.