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Mathematical Models for Interest Rate Dynamics

Xiaoxue Shan

Louisiana State University and Agricultural and Mechanical College, sxxmao@hotmail.com

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MATHEMATICAL MODELS FOR INTEREST RATE DYNAMICS

A Thesis

Submitted to the Graduate Faculty of the
Louisiana State University and
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by

Xiaoxue Shan

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Abstract

We present a study of mathematical models of interest rate products. After an introduction to the mathematical framework, we study several basic one-factor models, and then explore multifactor models. We also discuss the Heath-Jarrow-Morton model and the LIBOR Market model. We conclude with a discussion of some modified models that involve stochastic volatility.

Chapter 1

Introduction

In this work we present a study of mathematical models of interest rate products. We focus entirely on mathematical aspects of the models and do not discuss discretization and numerical implementation of any model.

We begin our work in Chapter 2 with an introduction to the mathematical framework and financial concepts for studying models of financial instruments, such as zero-coupon bond, volatility, martingale property, pricing measures, numeraire and stochastics.

In chapter 3, we introduce the basic one-factor models, such as Ho-Lee model, Vasicek model, Cox-Ingersoll-Ross (CIR) model, Hull-White model and the Black-Derman-Toy model. They are the simplest models for bond prices that are based on stochastic differential equations for the evolution of the short rate.

In the following chapter, we describe multifactor models, since one-factor models cannot adequately explain the full complexity of the yield curve dynamics. We study the two-factor Hull and White model, Green's functions and a multifactor Gaussian model.

Then, we turn to present the Heath-Jarrow-Morton model and the LIBOR Market model. The HJM model is a method for pricing of interest rate sensitive contingent claims under a stochastic term structure of interest rates, and it is based on the equivalent martingale measure technique, and gives an initial forward rate curve and potential stochastic processes for its following movements. The LIBOR Market model is specified in terms of the dynamics of prices of market instruments,

and it is in contrast to short rate models and the HJM model that is specified in terms of instantaneous forward rates.

In Chapter 6, we discuss some works modifying the HJM model and the LIBOR Market model to include stochastic volatility.

Chapter 2

Mathematical Framework

In this chapter we describe the mathematical framework for studying models of financial instruments and we also describe some financial concepts. We use the framework from Sengupta [37], which we summarize.

2.1 Pricing measures

We now outline some of the central concepts and definitions from finance and related mathematics that we need. A *zero-coupon bond* is a financial instrument in which the buyer makes a payment at a certain time to the seller and at a later time, called the *maturity* date, the seller makes a payment to the buyer; the seller makes no other payment to the buyer (thus, there are no intermediate ‘coupons’). We work with *risk free* zero-coupon bonds; for these bonds there is no possibility of default, and on the maturity date the seller definitely makes a payment to the buyer as agreed on at the time the bond is initiated. If the payment at maturity date T is one unit of currency then the price of this bond at time $t < T$ is denoted

$$P(t, T)$$

in terms of the same currency at time t . For simplicity we will generally use the term ‘bond’ to mean a zero-coupon risk-free bond in some specific currency.

Following [37], we model the set of all market *scenarios* by a set Ω . An element $\omega \in \Omega$ describes a market scenario, by which we mean a specific trajectory of the entire market through all time. Market *events* are described by subsets of Ω , forming a σ -algebra \mathcal{F} of subsets of Ω . When we speak of a financial instrument we always mean an instrument that has a particular time-stamp on it: for example, a

specific bond at a specific given time. A *numeraire* is a financial instrument that may be used to measure the price of other instruments; for example, one dollar at a given specific time is a numeraire. Working with a fixed choice of numeraire, we denote by

$$\mathbb{Q}(A)$$

the market price of a hypothetical market instrument that pays one unit of numeraire if an event A occurs and nothing if it does not occur. The probability measure \mathbb{Q} , which is well defined under market equilibrium, is called the *risk neutral* or *market pricing* measure. If a financial instrument has value $X(\omega)$ in market scenario ω , relative to a given numeraire, then its price, with no information known about the market, is the expectation

$$\mathbb{E}_{\mathbb{Q}}[X],$$

where \mathbb{Q} is the risk neutral measure for the particular numeraire. Recall that the instrument, as well as the numeraire, has a specific time stamp and so there is no need here to use notation such as X_t with t being time; however, there will be situations where we will need to stress that time t and then we may use the notation X_t .

The price at time t of a zero-coupon risk free bond, with unit maturity payoff at maturity time T , in market scenario ω may be expressed as

$$p(t, T; \omega) = e^{-\int_t^T r_s(\omega) ds},$$

where $r_s(\omega)$ is the *short rate* or *spot rate* at time s in scenario ω . Thus the price at time 0 of a T -maturity bond is

$$P(0, T) = \mathbb{E}_{\mathbb{Q}_0} [p(0, T)] \tag{2.1}$$

where \mathbb{Q}_0 is the risk neutral measure when the numeraire is cash at time 0.

Often we have to make distinctions between expectations based on information available at different times. We denote by

$$\mathcal{F}_t$$

be the σ -algebra of events generated by the functions X_s , the time- s prices of all market instruments that are defined at time s in terms of all numeraires of interest, for all time instants $s \leq t$. Thus \mathcal{F}_t describes all information available up to time t . Then the σ -algebras \mathcal{F}_t form a *filtration* in the sense that

$$\mathcal{F}_s \subset \mathcal{F}_t \quad \text{if } s \leq t.$$

Thus, the price of this bond at time t is

$$P(t, T) = \mathbb{E}_{\mathbb{Q}_t} \left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right] \quad (2.2)$$

where \mathbb{Q}_t is risk neutral measure with cash at time t as numeraire. It will be important to keep in mind that when we speak of *price as seen at time t* , or something equivalent to this, we mean the risk neutral expectation of the value conditional on information available at time t :

$$X_t = \mathbb{E}_{\mathbb{Q}}[X \mid \mathcal{F}_t] \quad (2.3)$$

where X is the random variable whose value $X(\omega)$ in market scenario ω is the price relative to the chosen numeraire and \mathbb{Q} is the corresponding market pricing measure (we assume $\mathbb{E}_{\mathbb{Q}}|X| < \infty$). The process $t \mapsto X_t$ is a *martingale* in the sense that

$$\mathbb{E}_{\mathbb{Q}}[X_t \mid \mathcal{F}_s] = X_s \quad \text{if } s \leq t \quad (2.4)$$

Note that all prices here are measured with respect to the same numeraire and so no ‘discounting’ is needed.

The *forward measure* \mathbb{Q}_T is the risk neutral measure when the numeraire is cash at time T . This is related to \mathbb{Q}_0 as follows. The price in time-0 cash of an instrument whose value is $X(\omega)$ in time- T money in scenario ω is

$$\mathbb{E}_{\mathbb{Q}_T}[X] = \frac{\mathbb{E}_{\mathbb{Q}_0}[p(0, T)X]}{\mathbb{E}_{\mathbb{Q}_0}[p(0, T)]} \quad (2.5)$$

This is an example of a *change of numeraire* formula.

The change of numeraire formula (2.5) works also for conditional expectations:

$$\mathbb{E}_{\mathbb{Q}_t}[X | \mathcal{F}_s] = \frac{\mathbb{E}_{\mathbb{Q}_0}[p(0, t)X | \mathcal{F}_s]}{\mathbb{E}_{\mathbb{Q}_0}[p(0, t) | \mathcal{F}_s]} \quad (2.6)$$

for any $t, s \geq 0$ and non-negative or integrable random variable X ; this result, drawn from [37, Theorem 24.5.2], is proved below in Proposition 2.1. Note that if $t \leq s$ then $p(0, t)$ is \mathcal{F}_s -measurable and so the right side reduces to $\mathbb{E}_{\mathbb{Q}_0}[X | \mathcal{F}_s]$.

The *forward price* at time 0 of the T_2 -maturity bond at time T_1 is the expected value

$$FP(T_1, T_2) = \mathbb{E}_{\mathbb{Q}_{T_1}}[P(T_1, T_2)], \quad (2.7)$$

and this can be shown ([37, eqn (9.12)]), using the change of numeraire formula (2.5), to be equal to

$$FP(T_1, T_2) = \frac{P(0, T_2)}{P(0, T_1)} \quad (2.8)$$

This is the expected time- T_1 price of a T_2 -maturity bond, as estimated at time 0. More generally, working at an initial time $t \leq T_1 \leq T_2$ we have, on using the change of numeraire formula (2.6), the forward price as seen at time t for a T_2 -maturity bond to be purchased at time T_1 :

$$FP(t; T_1, T_2) = \mathbb{E}_{\mathbb{Q}_{T_1}}[P(T_1, T_2) | \mathcal{F}_t] = \frac{P(t, T_2)}{P(t, T_1)} \quad (2.9)$$

where $t \leq T_1 \leq T_2$.

The rate of interest over the time period $[T_1, T_2]$, as seen at time 0, is

$$f(T_1, T_2) = \frac{[P(0, T_1) - P(0, T_2)]/P(0, T_2)}{T_2 - T_1} \quad (2.10)$$

and is called the *forward rate* for the period $[T_1, T_2]$. Letting $T_2 \rightarrow T_1$ gives the instantaneous *forward rate*:

$$f(0, T) = -\frac{\partial \ln P(0, T)}{\partial T} \quad (2.11)$$

The *yield* of a T -maturity bond, at time t , is the value of r for which

$$P(t, T) = e^{-r(T-t)},$$

and is thus given by

$$R(t, T) = \frac{1}{T-t} \ln P(t, T) \quad (2.12)$$

The *term structure* of interest rates, at time t , is the curve

$$T \mapsto P(t, T)$$

for all maturity dates $T \geq t$. When this information is displayed as a graph of yields, instead of bond prices, against time to maturity, it is called the *yield curve*.

2.2 Change of numeraire

The discussion and all results and their proofs are taken from [37].

Let \mathbb{Q}_X and \mathbb{Q}_Y be the probability measures corresponding to numeraires X and Y , respectively. We will determine the relationship between \mathbb{Q}_X and \mathbb{Q}_Y .

Let v_X^Y be the value of X in terms of Y :

$$X \text{ is worth } v_X^Y(\omega) \text{ units of } Y \text{ in market scenario } \omega \in \Omega. \quad (2.13)$$

In general, v_X^Y is a *random* quantity, as it *depends on the market scenario*. In what follows we will assume v_X^Y is always positive.

Consider now an event B , and the instrument I_B which pays off one unit of the numeraire X if B occurs and pays 0 otherwise. It is worth

$$\mathbb{Q}_X(B)$$

units of the numeraire X .

Now the price, in units of Y , for a unit of X is

$$\mathbb{E}_{\mathbb{Q}_Y}[v_X^Y]$$

So

$$I_B \text{ is worth } \mathbb{Q}_X(B)\mathbb{E}_{\mathbb{Q}_Y}[v_X^Y] \text{ units of } Y \quad (2.14)$$

Now look at I_B from the point of view of Y . The payoff of I_B , measured in units of Y , is described by the random variable

$$v_X^Y 1_B$$

So its value is

$$\mathbb{E}_{\mathbb{Q}_Y}[v_X^Y 1_B] \quad (2.15)$$

units of Y .

Equating the two answers (2.14) and (2.15), we see that

$$\mathbb{Q}_X(B) = \frac{\mathbb{E}_{\mathbb{Q}_Y}[v_X^Y 1_B]}{\mathbb{E}_{\mathbb{Q}_Y}[v_X^Y]} = \frac{\int v_X^Y 1_B d\mathbb{Q}_Y}{\int v_X^Y d\mathbb{Q}_Y} \quad (2.16)$$

This expresses \mathbb{Q}_X -probabilities in terms of the measure \mathbb{Q}_Y .

For an instrument whose value in any scenario ω is $f(\omega)$ then:

$$\int f d\mathbb{Q}_X = \frac{\int f v_X^Y d\mathbb{Q}_Y}{\int v_X^Y d\mathbb{Q}_Y} \quad (2.17)$$

This can be expressed symbolically as

$$d\mathbb{Q}_X = \frac{1}{\mathbb{E}_{\mathbb{Q}_Y}[v_X^Y]} v_X^Y d\mathbb{Q}_Y \quad (2.18)$$

Proposition 2.1. *Suppose \mathcal{F} is a σ -algebra of subsets of a non-empty set Ω , and suppose \mathbb{Q}_X and \mathbb{Q}_Y are probability measures on \mathcal{F} related by*

$$\mathbb{Q}_X(B) = \frac{\mathbb{E}_{\mathbb{Q}_Y}[v_X^Y 1_B]}{\mathbb{E}_{\mathbb{Q}_Y}[v_X^Y]} \quad (2.19)$$

for all $B \in \mathcal{F}$, where v_X^Y is an \mathcal{F} -measurable positive function on Ω . Let \mathcal{A} be any sub- σ -algebra of \mathcal{F} . Then

$$\mathbb{E}_{\mathbb{Q}_X}[f|\mathcal{A}] = \frac{\mathbb{E}_{\mathbb{Q}_Y}[fv_X^Y|\mathcal{A}]}{\mathbb{E}_{\mathbb{Q}_Y}[v_X^Y|\mathcal{A}]} \quad (2.20)$$

for all \mathcal{F} -measurable functions f for which either side exists.

Before working out the proof let us apply this to forward prices of bonds. Let $0 \leq t \leq T_1 \leq T_2$, and let Y be unit cash at time T_2 (or, the T_2 -maturity bond) and X be unit cash at time 0. Then

$$v_X^Y = p(0, T_2)$$

Hence, by (2.20) we have

$$\mathbb{E}_{\mathbb{Q}_{T_2}} \left[\frac{1}{p(T_1, T_2)} \middle| \mathcal{F}_t \right] = \frac{\mathbb{E}_{\mathbb{Q}_0} [p(0, T_2) \frac{1}{p(T_1, T_2)} | \mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}_0} [p(0, T_2) | \mathcal{F}_t]} = \frac{\mathbb{E}_{\mathbb{Q}_0} [p(0, T_1) | \mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}_0} [p(0, T_2) | \mathcal{F}_t]}$$

which means

$$\mathbb{E}_{\mathbb{Q}_{T_2}} \left[\frac{1}{p(T_1, T_2)} \middle| \mathcal{F}_t \right] = \frac{P(t, T_1)}{P(t, T_2)} \quad (2.21)$$

We can understand this relation by a ‘no-arbitrage’ argument. Suppose that at time t we buy a T_1 -maturity bond, paying amount $P(t, T_1)$, and then at time T_1 invest the resulting 1 unit of cash into a money market account which generates amount

$$\frac{1}{p(T_1, T_2; \omega)}$$

in any scenario ω , so that its expected value at time T_2 is

$$\mathbb{E}_{\mathbb{Q}_{T_2}} \left[\frac{1}{p(T_1, T_2)} \middle| \mathcal{F}_t \right]$$

On the other hand if we invested the initial amount $P(t, T_1)$ into a T_2 -maturity bond, the resulting payoff at time T_2 would be $\frac{1}{P(t, T_2)}P(t, T_1)$. This justifies the identity (2.21) directly.

Proof. Let

$$g = \frac{v_X^Y}{\mathbb{E}_{\mathbb{Q}_Y}(v_X^Y)}$$

Then the conversion formula (2.19) reads:

$$\mathbb{E}_{\mathbb{Q}_X}(1_B) = \mathbb{E}_{\mathbb{Q}_Y}(1_B g) \tag{2.22}$$

Taking linear combinations, monotone limits, and the usual procedure, we have then

$$\mathbb{E}_{\mathbb{Q}_X}(f) = \mathbb{E}_{\mathbb{Q}_Y}(fg) \tag{2.23}$$

valid for all measurable functions f for which either side exists.

We will show that

$$\mathbb{E}_{\mathbb{Q}_X}[f|\mathcal{A}] = \frac{\mathbb{E}_{\mathbb{Q}_Y}[fg|\mathcal{A}]}{\mathbb{E}_{\mathbb{Q}_Y}[g|\mathcal{A}]}, \tag{2.24}$$

from which (2.20) follows by substituting in $g = \frac{v_X^Y}{\mathbb{E}_{\mathbb{Q}_Y}(v_X^Y)}$.

We may and will assume that f is non-negative, since the general case can be broken down to this case.

Now consider any \mathcal{A} -measurable non-negative function h . Then, using the properties of conditional expectations repeatedly, we have:

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_X} \left[h \frac{\mathbb{E}_{\mathbb{Q}_Y} [fg|\mathcal{A}]}{\mathbb{E}_{\mathbb{Q}_Y} [g|\mathcal{A}]} \right] &= \mathbb{E}_{\mathbb{Q}_Y} \left[gh \frac{\mathbb{E}_{\mathbb{Q}_Y} [fg|\mathcal{A}]}{\mathbb{E}_{\mathbb{Q}_Y} [g|\mathcal{A}]} \right] \quad \text{by (2.23)} \\
&= \mathbb{E}_{\mathbb{Q}_Y} \left[\mathbb{E}_{\mathbb{Q}_Y} \left[gh \frac{\mathbb{E}_{\mathbb{Q}_Y} [fg|\mathcal{A}]}{\mathbb{E}_{\mathbb{Q}_Y} [g|\mathcal{A}]} \middle| \mathcal{A} \right] \right] \\
&= \mathbb{E}_{\mathbb{Q}_Y} \left[h \frac{\mathbb{E}_{\mathbb{Q}_Y} [fg|\mathcal{A}]}{\mathbb{E}_{\mathbb{Q}_Y} [g|\mathcal{A}]} \mathbb{E}_{\mathbb{Q}_Y} [g|\mathcal{A}] \right] \\
&= \mathbb{E}_{\mathbb{Q}_Y} [\mathbb{E}_{\mathbb{Q}_Y} [hfg|\mathcal{A}]] \\
&= \mathbb{E}_{\mathbb{Q}_Y} [ghf] \\
&= \mathbb{E}_{\mathbb{Q}_X} (hf) \quad \text{again by (2.23)}
\end{aligned}$$

Thus, we have

$$\mathbb{E}_{\mathbb{Q}_X} (hf) = \mathbb{E}_{\mathbb{Q}_X} \left[h \frac{\mathbb{E}_{\mathbb{Q}_Y} [fg|\mathcal{A}]}{\mathbb{E}_{\mathbb{Q}_Y} [g|\mathcal{A}]} \right]$$

valid for all non-negative \mathcal{A} -measurable functions h . This proves that the conditional expectation $\mathbb{E}_{\mathbb{Q}_X} (f|\mathcal{A})$ is given by (2.24). QED

We specialize to the case where the random variable v_X^Y is log-normal.

Consider an asset X whose price $S = v_X^Y$, at some given time, relative to a numeraire Y has a log-normal distribution with respect to the pricing measure \mathbb{Q}_Y , i.e. $\log S$ is a Gaussian variable relative to the measure \mathbb{Q}_Y . We prove that S is also log-normally distributed with respect to the measure \mathbb{Q}_X . This is a very important observation.

Theorem 2.2. *Let S be the price of an asset X relative to a numeraire Y , and suppose that $\log S$ has Gaussian distribution with mean m and variance σ^2 , relative to the measure \mathbb{Q}_Y . Then $\log S$ is Gaussian with mean $m + \sigma^2$ and variance σ^2 , relative to \mathbb{Q}_X .*

Observe that the mean changes when the measure is changed.

The theorem above follows by using the conversion-of-numeraires relation (2.16) and taking $L = \log S$ and $P = \mathbb{Q}_B$ in the following result:

Lemma 2.3. *Let L be a random variable which is Gaussian with mean m and variance σ^2 with respect to a probability measure \mathbb{P} . Let \mathbb{P}' be the measure specified by*

$$\mathbb{P}'(B) = \frac{\mathbb{E}_{\mathbb{P}}(e^L 1_B)}{\mathbb{E}_{\mathbb{P}}(e^L)} \quad (2.25)$$

for all events B . Then the distribution of L with respect to \mathbb{P}' is Gaussian with mean $m + \sigma^2$ and variance σ^2 .

The proof is straightforward and is given in [37].

2.3 Stochastics

The models we will describe are given by means of stochastic differential equations. Some of these describe the time evolution of the short-rate r_t . For this purpose we will use standard *Brownian motion* or *Wiener process*

$$[0, \infty) \times \Omega \rightarrow \mathbb{R}^N : t \mapsto W_t(\omega)$$

This is a measurable function on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, such that almost-surely the paths

$$t \mapsto W_t(\omega)$$

are continuous, each W_t is an \mathbb{R}^N -valued, the components of which are independent Gaussians of mean 0 and variance t , and, for any $s, t \in [0, \infty)$ with $s < t$, the increment

$$W_t - W_s$$

is independent of

$$\sigma(W_u : u \leq s).$$

When we have a given market filtration $(\mathcal{F}_t)_{t \geq 0}$, we require that W_t be \mathcal{F}_t -measurable for all $t \geq 0$.

A *stochastic process* $\{X_t\}_{t \in I}$, with ‘time’ t running over some interval $I \subset \mathbb{R}$, on a measurable space (Ω, \mathcal{F}) , with values in \mathbb{R}^d , is a mapping

$$\Omega \times I \rightarrow \mathbb{R}^d : (\omega, t) \mapsto X_t(\omega)$$

such that each X_t is \mathcal{F} -measurable. If $\{\mathcal{F}_t\}_{t \in I}$ is a filtration on Ω , then the process $\{X_t\}_{t \in I}$ is said to be *adapted* to $\{\mathcal{F}_t\}_{t \in I}$ if X_t is \mathcal{F}_t -measurable for every $t \in I$. Most processes of interest also satisfy the condition that X , as mapping defined on $\Omega \times I \rightarrow \mathbb{R}^d$ is measurable with respect to the product σ -algebra $\mathcal{F} \otimes \text{Borel}(\mathbb{R}^d)$. A stochastic process $\{X_t\}_{t \in I}$ is a *martingale* with respect to a filtration $\{\mathcal{F}_t\}_{t \in I}$, and a probability measure \mathbb{Q} on a σ -algebra \mathcal{F} containing \mathcal{F}_t for all $t \in I$, if $\mathbb{E}[|X_t|] < \infty$ for all $t \in I$ and

$$\mathbb{E}[X_t | \mathcal{F}_s] = X_s \quad \text{for all } s, t \in I \text{ with } s \leq t, \quad (2.26)$$

where on the left is the conditional expectation of X_t on the σ -algebra \mathcal{F}_s . All stochastic processes of interest to us in this work are continuous in the sense that the path

$$t \mapsto X_t(\omega)$$

is continuous for almost every ω .

We will use Itô stochastic integrals. Briefly, for stochastic processes $\{X_t\}_{t \in [a,b]}$ and $\{Y_t\}_{t \in [a,b]}$, the *stochastic integral*

$$\int_a^b X_t dY_t \quad (2.27)$$

is the limit in probability of ‘Riemann sums’

$$\sum_{j=1}^N X_{t_{j-1}} (Y_{t_j} - Y_{t_{j-1}})$$

as the mesh $\max_{1 \leq j \leq N} |t_j - t_{j-1}|$ of the partition $a = t_0 < \dots < t_N = b$ goes to 0. A *stochastic differential* associates to each subinterval $[a, b]$ of some interval I a random variable, all defined in some given probability space. Thus, the stochastic differential dY associates to each interval $[a, b]$ the random variable

$$\int_a^b dY = \int_a^b dY_t = Y(b) - Y(a),$$

and the product

$$XdY$$

associates to the interval $[a, b]$ the random variable

$$\int_a^b X dY = \int_a^b X_t dY_t.$$

The *product* of two differentials ϕ and ψ is the differential that associates to $[a, b]$ the limit in probability of the sums

$$\sum_{j=1}^N \phi([t_{j-1}, t_j])\psi([t_{j-1}, t_j])$$

as the mesh of the partition $a = t_0 < \dots < t_N = b$ goes to 0. Then there is the famous Itô differential calculus for Brownian motion specified by

$$(dW_t)^2 = dt, \quad (dW_t)dt = 0 = (dt)dW_t, \quad (dt)^2 = 0. \quad (2.28)$$

We will often work with *stochastic differential equations* of the form

$$dX_t = m(t)dt + \sigma(t)dW_t,$$

in which the process $t \mapsto m(t)$ is called the *drift* and $\sigma(t)$, or sometimes its square, is called the *volatility*. For $X(t)$ to be a martingale the drift term must be 0.

Recalling the martingale property of prices given in (2.4), we see that if X_t is the price at time t of an instrument, relative to a fixed numeraire, then the drift term in dX_t is 0.

Chapter 3

Basic One-factor Models

In this chapter we describe some of the simplest models for bond prices that are based on stochastic differential equations for the evolution of the short rate. Recall that the short rate at time t is a random variable $r(t)$, and the price of a T -maturity bond is

$$P(0, T) = \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_0^T r(t) dt} \right],$$

where $\mathbb{E}_{\mathbb{Q}}$ is the expectation value with respect to the market pricing measure \mathbb{Q} . Using this the parameters in any given model for the stochastic process $t \mapsto r(t)$, can be calibrated to known values of $P(0, T)$, bond prices available in the market.

3.1 Ho-Lee model

The Ho-Lee model [17] was introduced by Thomas S.Y. Ho and Sang Bin Lee in 1986. In its original form it was a discrete-time model for the time evolution of forward rates. We discuss the continuum-time version of this model as a short-rate model. The stochastic process of the short rate r in this model is:

$$dr(t) = \theta(t)dt + \sigma dW(t) \tag{3.1}$$

In this equation, σ is the instantaneous standard deviation of the short-term interest rate, and it is a constant; $\theta(t)$ is a deterministic function of time. Thus

$$r(t) = r(0) + \Theta(t) + \sigma W(t) \tag{3.2}$$

where

$$\Theta(t) = \int_0^t \theta(s) ds$$

The price of a T -maturity bond is

$$P(0, T) = \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_0^T r(t) dt} \right] = \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_0^T \Theta(t) dt - \sigma \int_0^T W(t) dt} \right] \quad (3.3)$$

From the formula [37, eqn (5.4)]

$$\mathbb{E}[e^X] = e^{\mathbb{E}[X] + \frac{1}{2} \text{var}(X)}, \quad (3.4)$$

where X is any Gaussian random variable, we then have

$$P(0, T) = e^{-\int_0^T \Theta(t) dt} e^{\frac{1}{2} \sigma^2 \text{var}[\int_0^T W(t) dt]}$$

The variance is the integral:

$$\mathbb{E}_{\mathbb{Q}} \left[\int_0^T \int_0^T W(s)W(t) ds dt \right],$$

and on using the Brownian motion expectation formula

$$\mathbb{E}[W(s)W(t)] = \min\{s, t\}$$

we have

$$\begin{aligned} \text{var} \left[\int_0^T W(t) dt \right] &= 2 \int_0^T \left(\int_s^T s dt \right) ds \\ &= 2 \int_0^T s(T-s) ds \\ &= 2 \left[T \frac{T^2}{2} - \frac{T^3}{3} \right] \\ &= \frac{T^3}{3} \end{aligned} \quad (3.5)$$

Then from (3.3) we have

$$P(0, T) = e^{-\int_0^T \Theta(t) dt + \frac{1}{6} \sigma^2 T^3} \quad (3.6)$$

Then the instantaneous forward rate is

$$f(0, T) = \frac{\partial -\ln P(0, T)}{\partial T} = \Theta(T) - \frac{1}{2} \sigma^2 T^2 \quad (3.7)$$

Taking the derivative gives

$$\partial_T f(0, T) = \theta(T) - \sigma^2 T$$

and so

$$\theta(T) = \partial_T f(0, T) + \sigma^2 T \tag{3.8}$$

This shows how to construct the drift function $\theta(\cdot)$ from market data: using the known term structure of interest rates as seen in a yield curve we can calculate the instantaneous forward rates $f(0, T)$ and from that we obtain the drift term $\theta(T)$ by using (3.8). Once this is known, and the volatility σ is also calibrated to market data, the model gives complete information about the evolution of the short rate $r(t)$, from which prices of options can be computed.

Although the Ho-Lee model is simple to use, it has the shortcoming that the volatility σ has a fixed value, and is thus not flexible enough to fit the term structure of volatility (time-varying volatility).

3.2 Vasicek Model

This model [40] was introduced in 1977 by Oldrich Vasicek. It is a mathematical model describing the evolution of interest rates, assuming that there is just one source of market risk that governs interest rate movements. Vasicek [40] described a general method of using stochastic calculus to relate a general short rate model described by a stochastic differential equation to bond prices, and presented this model as an example of the general theory.

The stochastic process of the interest rate $r(t)$ under the risk neutral measure \mathbb{Q} is assumed to be:

$$dr(t) = a(b - r(t))dt + \sigma dW(t) \tag{3.9}$$

where $t \mapsto W(t)$ is a standard one-dimensional Brownian motion process, and a , b , and σ are positive parameters, characterized as follows:

- b is the ‘long-term mean level’, the long run equilibrium value towards which the interest rate reverts.
- a is the ‘speed of reversion’, the rate at which such trajectories will regroup around b in time; a is non-negative and needs to be positive to ensure stability around the long term value.
- σ is the volatility of the interest rate, with higher σ indicating more randomness.

The parameters a and σ tend to oppose each other, increasing σ increases the amount of randomness entering the system, while increasing a increases the speed at which the system will stabilize statistically around the long term mean b with the variance also determined by a . The drift factor $a(b - r(t))$ represents the expected instantaneous change of the interest rate at time t . For instance, when $r(t)$ is below b and a is positive, the drift term $a(b - r(t))$ becomes positive, creating a tendency for the interest rate to move upwards and toward equilibrium.

Lemma 3.1. *The stochastic differential equation*

$$dr(t) = a(b - r(t))dt + \sigma dW(t) \quad (3.10)$$

is solved by

$$r(t) = r(0)e^{-at} + \int_0^t abe^{-a(t-s)}ds + \sigma \int_0^t e^{-a(t-s)}dW(s) \quad (3.11)$$

Proof. Consider the process:

$$y(t) = r(t)e^{at}$$

then we get the differential:

$$\begin{aligned}
dy(t) &= d(r(t)e^{at}) \\
&= ae^{at}r(t)dt + e^{at}dr(t) \\
&= e^{at}(ar(t)dt + a(b - r(t))dt + \sigma dW(t)) \\
&= abe^{at}dt + e^{at}\sigma dW(t)
\end{aligned}$$

Integrating both sides from 0 to t , and get:

$$y(t) - y(0) = \int_0^t abe^{as}ds + \sigma \int_0^t e^{as}dW(s)$$

Thus,

$$r(t)e^{at} = r(0) + \int_0^t abe^{as}ds + \sigma \int_0^t e^{as}dW(s)$$

then we get:

$$\begin{aligned}
r(t) &= r(0)e^{-at} + \int_0^t abe^{-a(t-s)}ds + \sigma \int_0^t e^{-a(t-s)}dW(s) \\
&= r(0)e^{-at} + b(1 - e^{-at}) + \sigma \int_0^t e^{-a(t-s)}dW(s)
\end{aligned} \tag{3.12}$$

□

A disadvantage of the Vasicek model is that it is theoretically possible for the interest rate to become negative.

3.3 Cox-Ingersoll-Ross (CIR) Model

The Cox-Ingersoll-Ross model (or CIR model) [8] was introduced in 1985 by John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross. In contrast to the Vasicek model, the CIR model keeps the interest rate positive. The stochastic process for the interest rate $r(t)$ under the risk neutral measure \mathbb{Q} is ([8, eqn (17)]):

$$dr(t) = a(b - r(t))dt + \sigma\sqrt{r(t)}dW_t \tag{3.13}$$

In this equation, W_t is also a Wiener process modelling the random market risk factor. The parameters a and b are positive. The condition

$$2ab \geq \sigma^2 \quad (3.14)$$

is imposed to ensure that the interest rate $r(t)$ remains positive if the initial value is positive.

The drift factor of the Cox Ingersoll Ross Model is $a(b - r(t))$, and it is the same as the drift factor of the Vasicek Model. It ensures that the average reversion of rate of interest is in the direction of the long run value b .

The probability density of the interest rate at time s , conditional on its value at the current time t , is of the form:

$$f(r(s), s; r(t), t) = ce^{-u-v} \left(\frac{v}{u}\right)^{q/2} I_q(2\sqrt{uv}) \quad (3.15)$$

where

$$\begin{aligned} c &= \frac{2a}{\sigma^2(1 - e^{-a(s-t)})} \\ u &= cr(t)e^{-a(s-t)} \\ v &= cr(s) \\ q &= \frac{2ab}{\sigma^2} - 1 \end{aligned}$$

and $I_q(\cdot)$ is the modified Bessel function of the first kind of order q .

The expected value and the variance of $r(s)$ is of the form:

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[r(s)|r(t)] &= r(t)e^{-a(s-t)} + b(1 - e^{-a(s-t)}) \\ \text{var}[r(s)|r(t)] &= r(t) \left(\frac{\sigma^2}{a}\right) (e^{-a(s-t)} - e^{-2a(s-t)}) + b \left(\frac{\sigma^2}{2a}\right) (1 - e^{-a(s-t)})^2 \end{aligned} \quad (3.16)$$

As a approaches infinity, the conditional mean is close to b , and the variance is close to 0; while as a approaches 0, the mean is close to the current interest rate and the variance to $\sigma^2 r(t)(s - t)$.

The bond price

$$P(t, T) = \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_t^T r(s) ds} \mid \mathcal{F}_t \right]$$

works out to

$$P(t, T) = A(t, T)e^{-B(t, T)r(t)} \quad (3.17)$$

where

$$\begin{aligned} A(t, T) &= \left(\frac{2\gamma \cdot e^{[(a+\gamma)(T-t)]/2}}{(\gamma + a)(e^{\gamma(T-t)} - 1) + 2\gamma} \right)^{2ab/\sigma^2} \\ B(t, T) &= \left(\frac{e^{\gamma(T-t)} - 1}{(\gamma + a)(e^{\gamma(T-t)} - 1) + 2\gamma} \right) \\ \gamma &= \sqrt{a^2 + 2\sigma^2} \end{aligned} \quad (3.18)$$

(This is from [8, eqn (23)] with the risk premium λ set to 0.)

The bond price is a decreasing convex function of the interest rate, an increasing function of time and a decreasing function of maturity. For the mean interest rate level b , the bond price is a decreasing convex function of b . If the interest rate is greater (less) than b , the bond price is an increasing concave (decreasing convex) function of the speed of reversion a . The bond price is an increasing concave function of the variance σ^2 .

The dynamics of the bond price is of the form:

$$dP = rPdt - B(t, T)\sigma\sqrt{r}PdW_t \quad (3.19)$$

3.4 Hull-White Model

The Hull-White model [19] is a one-factor interest rate model introduced by John C. Hull and Alan White in 1990. This model is extended from the Vasicek and CIR models and consistent with both the current term structure of spot or forward interest rates and the current term structure of interest rate volatilities.

In this model, it is assumed that the short rate is normally distributed and subject to mean reversion. The essence of the mean reversion is the assumption that both a stock's high and low prices are temporary and the price will tend to move to the average price over time. This mean reversion parameter is consistent with the empirical observation that the long rates are less volatile than short rates. The normal distribution results in greatly fast computation times relative to competing the no-arbitrage yield curve models.

The stochastic process of the extended Vasicek model is ([19, eqn (2)] with $\beta = 0$):

$$dr(t) = (\theta(t) - a \cdot r(t))dt + \sigma(t)dW(t) \quad (3.20)$$

where

- $\theta(t)$ is a function of time that determines the average direction in which $r(t)$ moves;
- a is the mean reversion rate that governs the relationship between the short and long rate volatilities;
- σ is the volatility measure of the short rate;
- $W(t)$ is a standard Wiener process with respect to the risk-neutral measure \mathbb{Q} .

We will discuss Gaussian models, of which the Hull-White model (3.20) is a special case after suitable change of variables, extensively in Chapter 4.

The stochastic process of the extended CIR model is ([19, eqn (2)] with $\beta = 0.5$):

$$dr(t) = (\theta(t) - a \cdot r(t))dt + \sigma(t)\sqrt{r}dW(t) \quad (3.21)$$

If the mean reversion parameter a is equal to zero, then the Hull-White model reduces to Ho-Lee model.

3.5 The Black-Derman-Toy model

The Black-Derman-Toy model [2], which was introduced by Fischer Black, Emanuel Derman and Bill Toy, is a one-factor model that uses the short rate, the annualized one period interest rate, to price bond options, swaptions and other interest rate derivatives.

The standard Black-Derman-Toy model is created to take as inputs both the existing term structure of zero-coupon yields and the term structure of yield volatilities for the same bonds using a binomial lattice framework. As the model varies the implied interest rate distribution matches an observed interest rate volatility curve at each time step.

The stochastic differential equation of the model for the short rate is:

$$d\ln(r) = \left[\theta(t) + \frac{\sigma'(t)}{\sigma(t)} \ln(r) \right] dt + \sigma(t)dW_t, \quad (3.22)$$

where

- r is the instantaneous short rate at time t ;
- $\sigma(t)$ is the instant time-dependent short rate volatility;
- $\theta(t)$ is the value of the underlying asset when the option expires;
- W_t is a standard Brownian motion under a risk-neutral probability measure;
- $\frac{\sigma'(t)}{\sigma(t)}$ is the mean reversion term.

If the short rate volatility σ is time-independent, σ becomes a constant, and the model is reduced to:

$$d\ln(r) = \theta(t)dt + \sigma dW_t \quad (3.23)$$

One of the main advantages of the Black-Derman-Toy model is that it is a lognormal model, so that it can capture a realistic term structure of interest rate volatilities. To realize this feature, the short-term rate volatility is allowed to vary over time, and the drift in the interest rate movements is dependent on the level of rates. Though the interest rate mean reversion is not explicit, this property is introduced by the term structure of volatilities.

Chapter 4

Multifactor Gaussian Models

One-factor models cannot adequately explain the full complexity of the yield curve dynamics, and so it is natural to develop multifactor models. Rebonato [35] describes the utility and importance of models involving several factors. In this chapter, we will review a multifactor Gaussian model for the short rate from Sengupta [37, pages 23-30, 117-122, 141-144], which generalizes the two-factor Hull and White Model discussed before.

4.1 The two-factor Hull-White model

For the purpose of overcoming the limitations of the one-factor model, there is a two-factor framework [21] for Hull and White's extended Vasicek model. The model used two Brownian motions W_1 and W_2 , which satisfy $dW_1 \cdot dW_2 = \rho dt$. The stochastic process is of the form:

$$dr(t) = [\theta(t) + u(t) - a \cdot r(t)]dt + \sigma_1 dW_1(t) \quad (4.1)$$

where $u(t)$ is a stochastic function and an additional disturbance term whose mean reverts to zero. It follows the process:

$$du = -budt + \sigma_2 dW_2(t) \quad (4.2)$$

with $u(0) = 0$.

Now we rewrite (4.1) and (4.2) as:

$$dr(t) = [\theta(t) + a(u'(t) - r(t))]dt + \sigma_1 dW_1(t)$$

$$du'(t) = b(0 - u'(t))dt + \sigma_2' dW_2$$

These two equations imply that the short rate reverts to the time-varying level $u'(t)$ with the constant reversion speed a , and the reversion level reverts to 0 with constant reversion speed of b in turn.

The price of a discount bond is:

$$P(t, T) = A(t, T)e^{-B(t, T)r - C(t, T)u} \quad (4.3)$$

where

$$B(t, T) = \frac{1 - e^{-a(T-t)}}{a}$$

$$C(t, T) = \frac{be^{-a(T-t)} - ae^{-b(T-t)} + a - b}{ab(a - b)}$$

The duration $B(t, T)$ is dependent only on the reversion speed of the short interest rate, and the term $C(t, T)$ is dependent on both a and b .

4.2 Green's functions

The Green's function is an entity of central importance in models for the short-rate that depend on an underlying Markov process. Consider a market model in which the state of the market at any given time is described by a point y in some *state space* \mathcal{S} ; usually, the state space is \mathbb{R}^N , for some N . We assume that the state space is equipped with a fixed background measure, such as Lebesgue measure or counting measure (if the set is discrete).

Suppose our market model specifies the stochastic.

The Green's function $G_{\mathbb{Q}}(s, x; t, y)$, as a function of y , is the probability density of the market state y_t , given that $y_s = x$, under the pricing measure \mathbb{Q} . The distribution of y_t is assumed to depend on the market events at all times prior to an instant $s < t$ only through the market state y_s . In this case, we have a transition probability function:

$$G_{\mathbb{Q}}(s, y; t, B), 0 \leq s \leq t$$

where B is any measurable subset of the state space \mathcal{S} , and the points y in \mathcal{S} describe the various states the market might be in at any time:

$$\mathbb{Q}[y_t \in B | \sigma\{y_u : u \leq s\}] = G_{\mathbb{Q}}(s, y_s; t, B)$$

The transition probability $G_{\mathbb{Q}}(s, y_s; t, B)$ is the price of an instrument which yields a unit of numeraire at time t if the event B occurs and nothing otherwise, and given that the market state is y_s at time s .

In all models, the configuration space \mathcal{S} is equipped with a σ -algebra \mathcal{F} and a reference measure m . For example, \mathcal{S} might be a finite set, with m is the counting measure on the σ -algebra \mathcal{F} of all subsets of \mathcal{S} . In other examples, $\mathcal{S} = \mathbb{R}^N$, with m being the Lebesgue measure on the σ -algebra \mathcal{F} of Lebesgue-measurable sets. Then, the transition probability $G_{\mathbb{Q}}(s, y_s; t, B)$ is assumed to be:

$$G_{\mathbb{Q}}(s, y; t, B) = \int_B G_{\mathbb{Q}}(s, y_s; t, z) dm(z) \quad (4.4)$$

where $G_{\mathbb{Q}}(s, y; t, z)$ is a transition density function.

For the derivative pricing measure \mathbb{Q} , there are three types of numeraires:

- time- t cash for an asset delivered at the future date T ;
- time- T cash for an asset delivered at the future date T ;
- an underlying asset at time- t .

The notations of the pricing measures and Green's functions for these three numeraires are:

The Green's functions $G_{\mathbb{Q}_T}^{\text{for}}$ and $G_{\mathbb{Q}_t}^{\text{nd}}$ are used to determine the forward and non-discounted future prices and $G_{\mathbb{Q}_t^*}^{\text{shifted}}$ is different from the forward measure just in that the parameters of the distribution are shifted.

TABLE 4.1. Green's Function and Numeraires

Green's Function	Numeraire for the Pricing Measure \mathbb{Q}	Specification
$G_{\mathbb{Q}_T}^{\text{for}}(t, x; T, y)$	Time- T cash	$E_{\mathbb{Q}_T}(f(y_T) \mathcal{F}_t) = \int_{R^N} G_{\mathbb{Q}_T}^{\text{for}}(t, y_t; T, y) f(y) dy$
$G_{\mathbb{Q}_T}^{\text{nd}}(t, x; T, y)$	Time- t cash	$E_{\mathbb{Q}_t}(f(y_T) \mathcal{F}_t) = \int_{R^N} G_{\mathbb{Q}_T}^{\text{nd}}(t, y_t; T, y) f(y) dy$
$G_{\mathbb{Q}_t^*}^{\text{shifted}}(s, x; t, y)$	Asset delivered at time t	$E_{\mathbb{Q}_t^*}(f(y_t) \mathcal{F}_s) = \int_{R^N} G_{\mathbb{Q}_t^*}^{\text{shifted}}(s, y_s; t, y) F(y) dy$

For any $\lambda = (\lambda_1, \dots, \lambda_N) \in R^N$, consider the function:

$$f_\lambda(t, y) = f_\lambda(t, y; T) = \mathbb{E}_{\mathbb{Q}_s} \left[e^{-\int_t^T r(u) du - \sum_{j=1}^N \lambda_j \cdot y_j(T)} | \mathcal{F}_t \right]$$

where $t \leq T$ and $y = y(t)$. We assume the process $t \rightarrow y(t)$ has the property of Markov, which means that the conditional expectation is a function of $y = y(t)$.

Proposition 4.1.

$$f_\lambda(t, y(t)) = \mathbb{E}_{\mathbb{Q}_s} \left[e^{-\int_t^T r(u) du} e^{-\lambda \cdot y(T)} | \mathcal{F}_t \right] \quad (4.5)$$

for all $0 \leq s \leq t \leq T$, and

$$\mathbb{E}_{\mathbb{Q}_s} \left[e^{-\lambda y(T)} | \mathcal{F}_t \right] = \frac{f_\lambda(t, y(t))}{f_0(t, y(t))} \quad (4.6)$$

and the process:

$$t \rightarrow e^{-\int_0^t r(u) du} f_\lambda(t, y(t)) \quad (4.7)$$

is a martingale concerning the measure \mathbb{Q}_0 and the family of σ -algebra \mathcal{F}_t .

Proof. we have known:

$$d\mathbb{Q}_t = \frac{1}{\mathbb{E}_{\mathbb{Q}_s}(p(s, t))} p(s, t) d\mathbb{Q}_s$$

and then we have:

$$\begin{aligned}
f_\lambda(t, y(t)) &= \mathbb{E}_{\mathbb{Q}_t} \left[e^{-\int_t^T r(u)du} e^{-\lambda \cdot y(T)} | \mathcal{F}_t \right] \\
&= \frac{\mathbb{E}_{\mathbb{Q}_s} [p(s, t)p(t, T)e^{-\lambda \cdot y(T)} | \mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}_s} [p(s, t) | \mathcal{F}_t]} \\
&= \frac{p(s, t)\mathbb{E}_{\mathbb{Q}_s} [p(t, T)e^{-\lambda \cdot y(T)} | \mathcal{F}_t]}{p(s, t)} \\
&= \mathbb{E}_{\mathbb{Q}_s} [p(t, T)e^{-\lambda \cdot y(T)} | \mathcal{F}_t]
\end{aligned}$$

therefore,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_T} [e^{-\lambda \cdot y(T)} | \mathcal{F}_t] &= \frac{E_{\mathbb{Q}_0} [p(0, T)e^{-\lambda \cdot y(T)} | \mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}_0} [p(0, T) | \mathcal{F}_t]} \\
&= \frac{p(0, t)\mathbb{E}_{\mathbb{Q}_0} [e^{-\int_t^T r(s)ds} e^{-\lambda \cdot y(T)} | \mathcal{F}_t]}{p(0, t)\mathbb{E}_{\mathbb{Q}_0} [p(t, T) | \mathcal{F}_t]} \\
&= \frac{f_\lambda(t, y(t))}{f_0(t, y(t))}
\end{aligned}$$

□

The preceding result provides the Laplace transform of the Green's function for the forward measure and from this the forward measure can be determined.

4.3 Multifactor Gaussian Model

The market state space for this model is \mathbb{R}^N . It means that the market state at any time is coordinated by N factors y_1, y_2, \dots, y_N . The evolution of the market is described by a process:

$$t \rightarrow (y_1(t), \dots, y_N(t)) \quad (4.8)$$

The zero-coupon bond is the instrument that determines the interest rate. The price of T -maturity zero-coupon bond at time t is:

$$P(t, T) = \mathbb{E}_{\mathbb{Q}_t} [p(t, T) | \mathcal{F}_t], t \leq T. \quad (4.9)$$

where \mathbb{Q}_t is the pricing measure of time- t cash numeraire, \mathcal{F}_t is the σ -algebra of market events up till time t , $p(t, T)$ is the random variable illustrating the scenario-dependent discount factor. For any market scenario $\omega \in \Omega$, the discount factor $p(t, T; \omega)$ is of the form:

$$p(t, T; \omega) = e^{-\int_t^T r(u; \omega) du}$$

where $r(u; \omega)$ is the spot rate at time u when the market scenario ω is realized.

In this multifactor model, we assume that the spot interest rate $r(t)$ is the sum of a deterministic factor $\alpha_0(t)$ plus the certain fluctuating factors y_1, \dots, y_N , and of the form:

$$r(t) = \alpha_0(t) + \sum_{j=1}^N y_j(t)$$

where $\alpha_0(\cdot)$ is a deterministic term and independent of the market state, and y_j satisfy the following stochastic differential equations:

$$dy_j(t) = -a_j y_j(t) dt + \sigma_j dW_j(t) \quad (4.10)$$

where a_j, σ_j are bigger than 0, and W_j are Brownian motions concerning the pricing measure \mathbb{Q}_0 and of the form:

$$dW_j(t) dW_k(t) = \rho_{jk} dt, \quad (4.11)$$

Assuming $\rho_{jj}=1$, and the equation for y_j has the the feature of mean-reverting.

Let

$$f_{\lambda, \mu}(t, y) = \mathbb{E}_{\mathbb{Q}_0} \left[e^{-\sum_{j=1}^N \mu_j \int_t^T y_j(u) du - \sum_{j=1}^N \lambda_j y_j(T)} | y(t) = y \right]$$

For any $s \leq t$, \mathbb{Q}_0 can be replaced with \mathbb{Q}_s , and this will not affect the value of $f_{\lambda, \mu}(t, y)$.

Then, let

$$g(t) = e^{-\int_0^t \mu \cdot y(s) ds} f(t, y(t)) = \mathbb{E}_{\mathbb{Q}_0} \left[e^{-\sum_{j=1}^N \mu_j \int_0^T y_j(s) ds - \sum_{j=1}^N \lambda_j y_j(T)} | \mathcal{F}_t \right]$$

The process $t \rightarrow g(t)$ is a martingale under the measure \mathbb{Q}_0 . By Ito,

$$dg(t) = [**]e^{-\int_0^t \mu \cdot y(s) ds} dt + *dW(t)$$

where (writing f for $f_{\lambda, \mu}$)

$$[**] = -\mu \cdot y(t)f(t, y(t)) + \frac{\partial f(t, y(t))}{\partial t} - \sum_{j=1}^N a_j y_j \frac{\partial f(t, y(t))}{\partial y_j} + \frac{1}{2} \sum_{j,k=1}^N \sigma_j \sigma_k \rho_{jk} \frac{\partial^2 f}{\partial y_j \partial y_k}$$

Since $t \rightarrow g(t)$ is a martingale, the drift term $[**]=0$, and of the form:

$$\begin{aligned} & -\mu \cdot y(t)f(t, y(t)) + \frac{\partial f(t, y(t))}{\partial t} - \sum_{j=1}^N a_j y_j \frac{\partial f(t, y(t))}{\partial y_j} \\ & + \frac{1}{2} \sum_{j,k=1}^N \sigma_j \sigma_k \rho_{jk} \frac{\partial^2 f}{\partial y_j \partial y_k} = 0 \end{aligned} \quad (4.12)$$

and the boundary condition is:

$$f(T, y) = e^{-\lambda \cdot y}$$

We try a solution:

$$f(t, y) = e^{\tilde{A}_{\lambda, \mu}(t, T) - \sum_{j=1}^N B_j^{\lambda, \mu}(t, T) y_j} \quad (4.13)$$

We substitute (4.13) into the partial differential equation (4.12) and get:

$$\begin{aligned} & \frac{1}{2} \sum_{j,k=1}^N \sigma_j \sigma_k \rho_{jk} B_j^{\lambda, \mu} B_k^{\lambda, \mu} + \sum_{j=1}^N a_j y_j B_j^{\lambda, \mu} \\ & - \sum_{j=1}^N \mu_j y_j + \left(\frac{\partial \tilde{A}_{\lambda, \mu}}{\partial t} - \frac{\partial B^{\lambda, \mu}}{\partial t} \cdot y \right) = 0 \end{aligned}$$

Let the coefficient of y_j and the constant term be zero, we get:

$$\frac{\partial B_j^{\lambda, \mu}}{\partial t} = a_j B_j^{\lambda, \mu} - \mu_j \quad (4.14)$$

and

$$\frac{\partial \tilde{A}_{\lambda, \mu}}{\partial t} = -\frac{1}{2} \sum_{j,k=1}^N \sigma_j \sigma_k \rho_{jk} B_j^{\lambda, \mu} B_k^{\lambda, \mu} \quad (4.15)$$

with the boundary conditions:

$$\tilde{A}_{\lambda,\mu}(T, T) = 0$$

$$B_j^{\lambda,\mu}(T, T) = \lambda_j$$

The differential equation of $B_j^{\lambda,\mu}$ implies:

$$\frac{\partial}{\partial t} \left[e^{a_j(T-t)} B_j^{\lambda,\mu}(t, T) \right] = -\mu_j e^{a_j(T-t)} \quad (4.16)$$

and then,

$$\begin{aligned} & B_j^{\lambda,\mu}(T, T) - e^{a_j\tau} B_j^{\lambda,\mu}(t, T) \\ &= -\mu_j \int_t^T e^{a_j(T-s)} ds \\ &= -\mu_j \int_0^{T-t} e^{a_j u} du \end{aligned}$$

Then it gives:

$$B_j^{\lambda,\mu}(t, T) = \left(\lambda_j - \frac{\mu_j}{a_j} \right) e^{-a_j\tau} + \frac{\mu_j}{a_j}$$

where $\tau = T - t$.

Now we substitute the value of $B_j^{\lambda,\mu}$ into the equation (4.15), and get:

$$\begin{aligned} \frac{\partial \tilde{A}_{\lambda,\mu}}{\partial t} &= -\frac{1}{2} \sum_{j,k=1}^N \sigma_j \sigma_k \rho_{jk} \left[\left(\lambda_j - \frac{\mu_j}{a_j} \right) \left(\lambda_k - \frac{\mu_k}{a_k} \right) e^{-(a_j+a_k)\tau} \right. \\ &\quad \left. + 2 \left(\lambda_j - \frac{\mu_j}{a_j} \right) \frac{\mu_k}{a_k} e^{-a_j\tau} + \frac{\mu_j \mu_k}{a_j a_k} \right] \end{aligned}$$

Then, we use the integration and $\tilde{A}_{\lambda,\mu}(T, T) = 0$, and obtain:

$$\tilde{A}_{\lambda,\mu} = \frac{1}{2} \sum_{j,k=1}^N \sigma_j \sigma_k \rho_{jk} [*]$$

where

$$\begin{aligned} [*] &= \left(\lambda_j - \frac{\mu_j}{a_j} \right) \left(\lambda_k - \frac{\mu_k}{a_k} \right) \frac{1 - e^{-(a_j+a_k)\tau}}{a_j + a_k} \\ &\quad + 2 \left(\lambda_j - \frac{\mu_j}{a_j} \right) \frac{\mu_k}{a_k} \frac{1 - e^{-a_j\tau}}{a_j} + \frac{\mu_j \mu_k}{a_j a_k} \tau \end{aligned}$$

Let $\lambda = 0$ and $\mu_j = 1$, we obtain the price of T-maturity bond at time t is of the form:

$$P(t, T) = e^{A(t, T) - B(t, T) \cdot y(t)} \quad (4.17)$$

where

$$\begin{aligned} A(t, T) = & - \int_t^T \alpha_0(s) ds + \frac{1}{2} \sum_{j, k=1}^N \sigma_j \sigma_k \rho_{jk} \\ & \times \left[\frac{1 - e^{-(a_j + a_k)\tau}}{a_j a_k (a_j + a_k)} - \frac{2}{a_j a_k} \frac{1 - e^{-a_j \tau}}{a_j} + \frac{1}{a_j a_k} \tau \right] \end{aligned}$$

and

$$B_j(t, T) = \frac{1 - e^{-a_j \tau}}{a_j} \quad (4.18)$$

with $\tau = T - t$.

Review the Green's function, we have:

$$\frac{\mathbb{E}_{\mathbb{Q}_0} \left(e^{-\int_t^T \mu y(s) ds - \lambda y(T)} | \mathcal{F}_t \right)}{\mathbb{E}_{\mathbb{Q}_0} \left(e^{-\int_t^T \mu y(s) ds} | \mathcal{F}_t \right)} = e^{\tilde{A}_{\lambda, \mu} - \tilde{A}_{0, \mu} - (\tilde{B}^{\lambda, \mu} - \tilde{B}^{0, \mu}) \cdot y(t)}$$

Then we obtain:

$$\begin{aligned} & \tilde{A}_{\lambda, \mu} - \tilde{A}_{0, \mu} - (\tilde{B}^{\lambda, \mu} - \tilde{B}^{0, \mu}) y(t) \\ & = \frac{1}{2} \sum_{j, k=1}^N \sigma_j \sigma_k \rho_{jk} [**] - \sum_{j=1}^N \lambda_j e^{-a_j \tau} y_j(t) \end{aligned}$$

where

$$[**] = \left(\lambda_j \lambda_k - 2 \lambda_j \frac{\mu_k}{a_k} \right) \frac{1 - e^{-(a_j + a_k)\tau}}{a_j + a_k} + 2 \lambda_j \cdot \frac{\mu_k}{a_k} \cdot \left(\frac{1 - e^{-a_j \tau}}{a_j} \right)$$

If a N -dimensional variable (X_1, \dots, X_N) is Gaussian, then by the Gaussian integration formula:

$$\mathbb{E}(e^X) = e^{\mathbb{E}(X) + \frac{1}{2} \text{var}(X)}$$

we obtain:

$$\mathbb{E}(e^{-\sum_{j=1}^N \lambda_j X_j}) = e^{-\sum_{j=1}^N \mathbb{E}(X_j) \lambda_j + \frac{1}{2} \sum_{j=1}^N \text{cov}(X_j, X_k) \lambda_j \lambda_k}$$

where

$$\text{cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))]$$

After summarizing these observations, we can conclude that the distribution of $y(t)$ under the measure \mathbb{Q}_t , which is conditional on all information up to time u , has the N-dimensional Gaussian density function:

$$G_{\mathbb{Q}_t}^{\text{for}}(u, y; t, x) = (2\pi \det C)^{-N/2} e^{-\frac{(x-m)C^{-1}(x-m)}{2}} \quad (4.19)$$

where $[C_{jk}]$ is the variance-covariance matrix, and the mean of $y_j(T)$ is $m = (m_1, \dots, m_N)$ with:

$$m_j = \frac{1}{2} \sum_{k=1}^N \sigma_j \sigma_k \rho_{jk} \frac{1}{a_k} \times \left[\frac{1 - e^{-(a_j + a_k)(t-u)}}{a_j + a_k} - \frac{1 - e^{-a_j(t-u)}}{a_j} \right] + e^{-a_j(t-u)} y_j(u)$$

and C_{jk} is a function of $t - u$ and of the form:

$$C_{jk} = \text{cov}(y_j(T), y_k(T)) = \rho_{jk} \sigma_j \sigma_k \frac{1 - e^{-(a_j + a_k)(t-u)}}{a_j + a_k}$$

The distribution of $y(t)$ under the measure \mathbb{Q}_u , which is conditional on all information up to time u , has the N-dimensional Gaussian density function, and with the mean $m^{\text{nd}} = (m_1^{\text{nd}}, \dots, m_N^{\text{nd}})$, where

$$m_j^{\text{nd}} = e^{-a_j(t-u)} y_j(u) \quad (4.20)$$

Since that $\log P(t, T)$ is Gaussian with respect to the measure \mathbb{Q}_t , now we apply the Black-Scholes formula to obtain the price of a call option.

We review the Black-Scholes Model. This model was described by Fischer Black and Myron Scholes, and used for pricing the stock options. For the underlying asset, such as a stock, this model has the following assumptions: there is no arbitrage

opportunity; the interest rate is deterministic; the stock price follows a geometric Brownian motion with constant drift and volatility; the underlying asset does not pay dividends, and there are no splits during the considered time interval. This model was extended by Robert Merton to the case that the assets pay dividends. The Black-Scholes formula calculates the price of European put and call options. In terms of the Black-Scholes formula, the call price is:

$$C = S_0 N(d_+) - e^{-rt} K N(d_-) \quad (4.21)$$

where N is the standard Gaussian distribution function, S_0 is the spot price of the underlying asset at time 0, K is the strike price, r is the risk-free interest rate on a t -maturity bond, and d_+ , d_- are of the form:

$$d_+ = \frac{1}{\sigma\sqrt{t}} \log[e^{rt} S_0 / K] + \frac{1}{2} \sigma \sqrt{t}$$

$$d_- = \frac{1}{\sigma\sqrt{t}} \log[e^{rt} S_0 / K] - \frac{1}{2} \sigma \sqrt{t}$$

where σ is the volatility of returns of the underlying asset.

Now we use this Black-Scholes formula to get the call price. The variance of $\log P(t, T)$ is:

$$\begin{aligned} \text{var}(\log P(t, T)) &= \sum_{j,k=1}^N C_{jk}(t) B_j(t, T) B_k(t, T) \\ &= \sum_{j,k=1}^N \rho_{jk} \sigma_j \sigma_k (1 - e^{-a_j \tau}) (1 - e^{-a_k \tau}) \frac{1 - e^{-(a_j + a_k)t}}{a_j a_k (a_j + a_k)} \end{aligned}$$

Then the price of a call option on a T -maturity bond is:

$$\text{Price} = P(0, t) [FP(t, T) N(d_+) - K N(d_-)] \quad (4.22)$$

where K is the strike, t is the expiration, and $N(x)$ is the Gaussian cumulative probability:

$$N(x) = \int_{-\infty}^x \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy, \quad (4.23)$$

and

$$d_+ = \frac{\log \frac{FP(t,T)}{K} + \frac{s^2}{2}}{s}$$
$$d_- = \frac{\log \frac{FP(t,T)}{K} - \frac{s^2}{2}}{s}$$

with $s^2 = \text{var}(\log P(t, T))$

Chapter 5

The HJM Model and Market Model

5.1 Introduction

The Heath-Jarrow-Morton (HJM) model [16] originates from the work of David Heath, Robert A. Jarrow and Andrew Morton in the late 1980s, and it is a method for pricing of interest rate sensitive contingent claims under a stochastic term structure of interest rates. The methodology is based on the equivalent martingale measure technique, and gives an initial forward rate curve and potential stochastic processes for its following movements. This method imposes its stochastic structure directly on the evolution of the forward rate curve, and does not require an ‘inversion of the term structure’ to eliminate the market prices of risk, and it has a stochastic spot rate process with multiple stochastic factors influencing the term structure. The HJM framework describes the evolution of the entire forward rate curve, while the short-rate models only focus on the short rate. The HJM model is a step towards a more realistic and complex model in comparison to the Ho-Lee model, which introduced the focus on forward rate evolution.

The general HJM process is essentially non-Markovian, that is, path dependent: an up move for the yield curve followed by a down move does not lead to the same result as a down move followed by an up move. A non-trivial HJM process cannot be mapped onto a recombining tree; therefore bushy trees, or Monte Carlo paths, are the tools available to the practitioner for pricing and hedging options. Path-dependent options almost impossibly dealt with using in higher dimensions, are tackled very easily using the MC methodology. Compound, or American options, on the other hand, efficiently dealt with by backwards induction using finite-

differences grids or recombining lattices, become very arduous to evaluate using Monte Carlo techniques.

5.2 The HJM model

Let $P(t, T)$ be the price at time t of a zero-coupon bond maturing at time T . Recall the instantaneous forward rate

$$f(t, T) = -\frac{\partial \ln P(t, T)}{\partial T} \quad (5.1)$$

The bond price $P(t, T)$ is related to the forward rates by

$$P(t, T) = e^{-\int_t^T f(t, u) du}$$

The HJM model was originally formulated [16, section 3] as a direct specification of the evolution of the forward rate:

$$f(t, T) - f(0, T) = \int_0^t \alpha(v, T) dv + \sum_{j=1}^n \int_0^t \sigma_j(v, T) dZ_j(v) \quad (5.2)$$

where the initial term structure of forward rates $f(0, T)$, for $T \geq 0$, is assumed known, α is a suitably well-behaved drift function, the coefficients σ_i are also well-behaved volatility functions, and

$$t \mapsto (Z_1(t), \dots, Z_n(t))$$

is an \mathbb{R}^n -valued standard Wiener process on some probability space (not necessarily the risk neutral measure). The dynamics of the short rates is then given by

$$r(t) = f(0, t) + \int_0^t \alpha(v, t) dv + \int_0^t \sigma(v, t) \cdot dZ(t) \quad (5.3)$$

where we have used the dot product of the vector ‘functions’

$$\sigma = (\sigma_1, \dots, \sigma_n)$$

and $dZ(\cdot)$. The exact conditions on the drift and volatility terms are stated in [16] and these are essentially the ones needed to prove existence of the integrals in (5.2).

We now turn to a formulation in terms of the bond prices, following the exposition in Rebonato [35, section 17.2]. In this form, the HJM model is stated as:

$$dP(t, T) = r(t)P(t, T)dt + v(t, T, P(t, T))dW(t) \quad (5.4)$$

where $v(t, T)$ reflects volatility, $r(t)$ is the short rate at time t , and $t \mapsto W(t)$ is standard Wiener process with respect to the risk neutral measure \mathbb{Q}_0 , where the numeraire is time-0 cash. For convenience of exposition we focus on the case where the process W is one-dimensional. Before proceeding further let us see informally why the drift term in (5.4) must be $r(t)P(t, T)$. Let

$$b(t, T) = e^{-\int_0^t r(s) ds} P(t, T),$$

which is the time- t price of a T -maturity bond with the price discounted down to time 0. Then

$$\begin{aligned} b(t, T) &= e^{-\int_0^t r(s) ds} \mathbb{E}_{\mathbb{Q}_t} \left[e^{-\int_t^T r(s) ds} \mid \mathcal{F}_t \right] \\ &= e^{-\int_0^t r(s) ds} \mathbb{E}_{\mathbb{Q}_0} \left[e^{-\int_t^T r(s) ds} \mid \mathcal{F}_t \right] \end{aligned} \quad (5.5)$$

(using change of numeraire)

$$= \mathbb{E}_{\mathbb{Q}_0} \left[e^{-\int_0^T r(s) ds} \mid \mathcal{F}_t \right]$$

This implies that

$$t \mapsto b(t, T)$$

is a martingale with respect to \mathbb{Q}_0 . Hence the drift term in

$$d_t b(t, T)$$

should be zero. Now

$$d_t b(t, T) = e^{-\int_0^t r(s) ds} [-r(t)dt] P(t, T) + e^{-\int_0^t r(s) ds} d_t P(t, T)$$

Hence the drift term on $d_t P(t, T)$ must exactly cancel $[-r(t)dt]P(t, T)$:

$$d_t P(t, T) = r(t)P(t, T)dt + \text{martingale differential} \quad (5.6)$$

This explains the formula (5.4).

Let

$$y(t, T) = \ln(P(t, T))$$

Using Itô's lemma we have:

$$d_t y(t, T) = d(\ln P(t, T)) = \left[r(t) - \frac{v(t, T, P)^2}{2P(t, T)^2} \right] dt + \frac{v(t, T)}{P(t, T)} dW(t) \quad (5.7)$$

Then we have

$$\begin{aligned} d_t [\ln P(t, T_2) - \ln P(t, T_1)] &= \frac{1}{2} \left[\frac{v(t, T_1, P)^2}{P(t, T_1)^2} - \frac{v(t, T_2, P)^2}{P(t, T_2)^2} \right] dt \\ &\quad + \left[\frac{v(t, T_2, P)}{P(t, T_2)} - \frac{v(t, T_1, P)}{P(t, T_1)} \right] dW(t) \end{aligned} \quad (5.8)$$

Recall that

$$f(t, T_1, T_2) = \frac{\ln P(t, T_1) - \ln P(t, T_2)}{T_2 - T_1} \quad (5.9)$$

Combining with (5.8), we obtain:

$$d[f(t, T_1, T_2)] = \frac{1}{2} \left[\frac{\frac{v(t, T_2, P)^2}{P(t, T_2)^2} - \frac{v(t, T_1, P)^2}{P(t, T_1)^2}}{T_2 - T_1} \right] dt + \left[\frac{\frac{v(t, T_1, P)}{P(t, T_1)} - \frac{v(t, T_2, P)}{P(t, T_2)}}{T_2 - T_1} \right] dW(t) \quad (5.10)$$

Introducing the notation

$$\xi(t, T, P) = \frac{v(t, T, P)}{P} \quad (5.11)$$

we have

$$\begin{aligned}
d_t[f(t, T_1, T_2)] &= \left[\frac{\xi(t, T_2, P(t, T_2))^2 - \xi(t, T_1, P(t, T_1))^2}{2(T_2 - T_1)} \right] dt \\
&+ \left[\frac{\xi(t, T_1, P(t, T_1)) - \xi(t, T_2, P(t, T_2))}{T_2 - T_1} \right] dW(t)
\end{aligned} \tag{5.12}$$

Letting $T_2 \rightarrow T_1$, and assuming conditions are met for proceeding to the limit, we have

$$d_t f(t, T) = \xi \left[\left(\frac{\partial \xi}{\partial T} \right)_P + \left(\frac{\partial \xi}{\partial P} \right)_T \frac{\partial P}{\partial T} \right] dt + \left[\left(\frac{\partial \xi}{\partial T} \right)_P + \left(\frac{\partial \xi}{\partial P} \right)_T \frac{\partial P}{\partial T} \right] dW(t) \tag{5.13}$$

Here the right side is evaluated at $P = P(t, T)$.

Equation (5.13) establishes the relationship between the volatility of discount bonds and drifts of forward rates with respect to the risk-neutral measure.

Writing

$$S(t, T) = \frac{v(t, T, P(t, T))}{P(t, T)} \tag{5.14}$$

we have

$$v(t, T, P(t, T)) = S(t, T)P(t, T)$$

and equation (5.13) becomes:

$$d_t f(t, T) = S \frac{\partial S}{\partial T} dt + \frac{\partial S}{\partial T} dW(t) \tag{5.15}$$

Thus, remarkably, the risk neutral *drift in the forward rate is completely determined by the volatility* (as a function of the maturity and initiation times) in this model.

5.3 The LIBOR Market Model

5.3.1 Introduction

The LIBOR Market model is specified in terms of the dynamics of prices of market instruments. This is in contrast to short rate models and the HJM model that is

specified in terms of instantaneous forward rates. Consider maturity dates

$$T_0 = 0 < T_1 < \dots < T_{N+1}$$

In a forward rate agreement for the time space $[T_j, T_{j+1}]$, struck at time $t \leq T_j$, the buyer will pay an agreed-upon price $FP(t; T_j, T_{j+1})$ in time- T_j cash at time T_j to purchase a T_{j+1} -maturity bond that will yield unit amount in time- T_{j+1} cash at time T_{j+1} . The *forward rate*, as seen at time t , is then

$$L_j(t) = \frac{1}{T_{j+1} - T_j} \left[\frac{1 - FP(t; T_j, T_{j+1})}{FP(t; T_j, T_{j+1})} \right]$$

which is the same as

$$L_j(t) = \frac{1}{T_{j+1} - T_j} \left[\frac{P(t, T_j)}{P(t, T_{j+1})} - 1 \right]$$

We have seen this earlier in (2.10). In view of the forward price relation (2.21) we see then that

$$L_j(t) = \mathbb{E}_{\mathbb{Q}_{T_{j+1}}} [l_j | \mathcal{F}_t] \tag{5.16}$$

where l_j is the random variable expressing the forward rate in each scenario:

$$l_j = \frac{1}{T_{j+1} - T_j} \left[\frac{1}{p(T_j, T_{j+1})} - 1 \right] \tag{5.17}$$

In the LIBOR Market model the forward rate $L_j(t)$ is a martingale relative to $\mathbb{Q}_{T_{j+1}}$, and so

$$dL_j(t) = \sigma_j(t)L_j(t)dW_{j+1}(t) \tag{5.18}$$

where $t \mapsto W_{j+1}(t)$ is a standard Wiener process with respect to the pricing measure $\mathbb{Q}_{T_{j+1}}$.

5.3.2 No-arbitrage assumption

From this section, we change to use the historical measure, which resulting in the appearance of MPR (market price of risk), and it is from the work of R. Pietersz [34].

The No-arbitrage assumption for the LIBOR market model is:

Assume that there exists a locally bounded process φ^{MPR} , such that

$$\mu_i(t) = \beta_i(t) \cdot \varphi^{MPR}(t), 0 \leq t \leq T_i, i = 1, \dots, N. \quad (5.19)$$

where MPR indicates 'market price of risk', and the market price of risk is the quotient of expected rate of return over the amount of uncertainty. The process φ^{MPR} can be used to construct an equivalent martingale measure for the LIBOR market model. Moreover, if this process is almost uniquely defined by (5.19) at all times, then the LIBOR market will be complete.

5.3.3 Measures and numeraire

1. Spot LIBOR measure

The spot LIBOR portfolio of the bonds is specified as follows:

- At time T_0 , start with 1 euro, buy $\frac{1}{B_1(T_0)}$ T_1 -maturity bonds.
- at time T_1 , receive $\frac{1}{B_1(T_0)}$ euros, buy $\frac{1}{B_1(T_0)B_2(T_1)}$ T_2 -maturity bonds;
- at time T_2 , receive $\frac{1}{B_1(T_0)B_2(T_1)}$ euros, buy $\frac{1}{B_1(T_0)B_2(T_1)B_3(T_2)}$ T_3 -bonds;
- generally, at time T_i , for $i \in \{1, \dots, N\}$, receive $\frac{1}{B_1(T_0)\dots B_i(T_{i-1})}$ euros, buy $\frac{1}{B_1(T_0)\dots B_{i+1}(T_i)}$ of T_{i+1} -maturity bonds.

The value $B(t)$ of the spot LIBOR portfolio is:

$$B(t) = \frac{B_{i+1}(t)}{\prod_{j=1}^{i+1} B_j(T_{j-1})}, \quad \text{for } t \in [T_i, T_{i+1}]. \quad (5.20)$$

The stochastic differential of the spot LIBOR price process is:

$$\frac{dB(t)}{B(t)} = \mu_{i(t)}(t)dt + \beta_{i(t)}(t) \cdot dW(t), \quad 0 \leq t \leq T. \quad (5.21)$$

for some drift and volatility.

Under the spot LIBOR measure, quotients of asset price processes over the spot LIBOR portfolio price process become martingales. Therefore, by Lemma 5.1 below, we get the stochastic differential of a bond price over the numeraire price:

$$\frac{d(B_i(t)/B(t))}{B_i(t)/B(t)} = ((\mu_i(t) - \mu_{i(t)}(t)) - (\beta_i(t) - \beta_{i(t)})\beta_{i(t)}(t)) dt \quad (5.22)$$

$$+ (\beta_i(t) - \beta_{i(t)})dW(t), 0 \leq t \leq T_i. \quad (5.23)$$

Lemma 5.1. *Let X and Y be continuous positive semi-martingales relative to some given filtration, satisfying:*

$$\frac{dX(t)}{X(t)} = \mu_X(t)dt + \beta_X(t) \cdot dW(t), \quad (5.24)$$

$$\frac{dY(t)}{Y(t)} = \mu_Y(t)dt + \beta_Y(t) \cdot dW(t). \quad (5.25)$$

where $t \mapsto W(t)$ is \mathbb{R}^N -valued standard Wiener process, adapted to the filtration, and μ and β are continuous adapted processes, with the β being \mathbb{R}^N -valued. Then,

$$\begin{aligned} \frac{d(X/Y)(t)}{(X/Y)(t)} &= (\mu_X(t) - \mu_Y(t) - (\beta_X(t) - \beta_Y(t)) \cdot \beta_Y(t)) dt \\ &+ (\beta_X(t) - \beta_Y(t)) \cdot dW(t) \end{aligned} \quad (5.26)$$

The proof is by a straight-forward application of Itô's lemma.

Define the process φ^{spot} by:

$$\varphi^{spot}(t) = \varphi^{MPR}(t) - \beta_{i(t)}(t), 0 \leq t \leq T \quad (5.27)$$

Define the local martingale M by:

$$M(t) = \int_0^t \varphi^{spot}(s)dW(s) \quad (5.28)$$

Define the process $W^{\mathbb{Q}_{spot}}$ by:

$$W^{\mathbb{Q}_{spot}}(t) = W(t) + \int_0^t \varphi^{spot}(s) ds \quad (5.29)$$

where $W^{\mathbb{Q}_{spot}}$ is a local martingale under \mathbb{Q}_{spot} .

The stochastic differential equation of bond price processes over the spot LIBOR price process is:

$$\begin{aligned} \frac{d(B_i(t)/B(t))}{B_i(t)/B(t)} &= ((\mu_i(t) - \mu_{i(t)}(t)) - (\beta_i(t) - \beta_{i(t)})\beta_{i(t)}(t)) dt \\ &+ (\beta_i(t) - \beta_{i(t)})(dW^{\mathbb{Q}_{spot}}(t) - \varphi^{spot}(t)dt) \\ &= (\beta_i(t) - \beta_{i(t)})dW^{\mathbb{Q}_{spot}}(t) \end{aligned}$$

This quotients are martingales under $\mathbb{Q}_{spot}(t)$, thus $\mathbb{Q}_{spot}(t)$ is the spot LIBOR measure.

2. Terminal LIBOR measure

In this condition, the numeraire is one of the bonds, called B_{n+1} , and $n \in 1, \dots, N$. Quotients of asset price processes over the bond price process need to become martingales under the terminal measure. In particular, $\frac{B_n}{B_{n+1}}$ will be a martingale. Therefore, the affine transformation of $\frac{B_n}{B_{n+1}}$, the n th LIBOR forward rate, will be a martingale under the terminal measure.

We get the stochastic differential equation of bond price over the numeraire price by Lemma 5.1:

$$\frac{d(B_i(t)/B(t))}{B_i(t)/B(t)} = [(\mu_i(t) - \mu_{n+1}(t)) - (\beta_i(t) - \beta_{n+1})\beta_{n+1}(t)] dt \quad (5.30)$$

$$+ (\beta_i(t) - \beta_{n+1})dW(t), 0 \leq t \leq T_i. \quad (5.31)$$

The processes $\varphi^{T_{n+1}}$, the measure $\mathbb{Q}_{T_{n+1}}$, and the Brownian motion $W^{\mathbb{Q}_{T_{n+1}}}$ are defined by:

$$\varphi^{T_{n+1}}(t) = \varphi^{MPR}(t) - \beta_{n+1}(t), \quad (5.32)$$

$$W^{\mathbb{Q}_{T_{n+1}}} = W(t) + \int_0^t \varphi^{T_{n+1}}(s) ds, \quad (5.33)$$

$$\frac{d\mathbb{Q}_{T_{n+1}}}{dP}(t) = e^{\int_0^t \varphi^{T_{n+1}}(s) dW(s) - \frac{1}{2} \int_0^t \|\varphi^{T_{n+1}}(s)\|^2 ds} \quad (5.34)$$

and $\varphi^{T_{n+1}}$ satisfies:

$$\mu_{V_1}(t) - \mu_{V_2}(t) - (\beta_{V_1}(t) - \beta_{V_2}(t)) \beta_{n+1}(t) \quad (5.35)$$

$$= (\beta_{V_1}(t) - \beta_{V_2}(t)) \varphi^{T_{n+1}}(t) \quad (5.36)$$

where V_1 and V_2 are portfolio price processes.

The stochastic differential equations for the bond price processes over the $(n + 1)$ th bond price process are satisfying:

$$\begin{aligned} \frac{d(B_i(t)/B_{n+1}(t))}{B_i(t)/B_{n+1}(t)} &= \left((\mu_i(t) - \mu_{n+1}(t)) - (\beta_i(t) - \beta_{n+1}(t)) \beta_{n+1}(t) \right) dt \\ &\quad + (\beta_i(t) - \beta_{n+1}(t)) \cdot (dW^{\mathbb{Q}_{T_{n+1}}}(t) - \varphi^{T_{n+1}}(t) dt) \\ &= (\beta_i(t) - \beta_{n+1}(t)) dW^{\mathbb{Q}_{T_{n+1}}}(t) \end{aligned}$$

It shows that the above quotients are martingales under $\mathbb{Q}_{T_{n+1}}$, therefore $\mathbb{Q}_{T_{n+1}}$ is the n th terminal measure.

Chapter 6

The HJM Model and Market Model with Stochastic Volatility

6.1 Introduction

In this concluding chapter we take a brief look at some models that have been proposed as modifications of the HJM and Libor Market models.

6.2 The HJM Model with Stochastic Volatility

A difficulty with the Heath-Jarrow-Morton model is that it involves non-Markovian processes, and thus the methods from the theory of partial differential equation are not directly usable. Monte Carlo simulation is the practical method of solution. In 1999, to overcome these problems, Chiarella and Kwon [6] considered a class of HJM term structure models with stochastic volatility and discussed ways of transforming these models to Markovian systems. These models provide one way of incorporating stochastic volatility into the HJM framework, and accordingly provide themselves to fixed solutions for bonds and bond option prices. In those transformed systems, the wanted properties of the earlier Markovian models and the HJM model exist conjointly, and provide useful settings to study interest rate derivatives.

The volatility processes in the standard HJM framework are path dependent and stochastic. However, the stochastic volatility processes are driven by Wiener processes independent of the Wiener processes which drive the underlying asset price process, or the forward rate process. In the case of the one-dimensional HJM model, the forward rate process is:

$$df(t, T) = \sigma^*(t, T)dt + \sigma(t, T)dW^f(t) \quad (6.1)$$

where $W^f(t) = (W_1^f(t), \dots, W_n^f(t))$ is a standard Wiener process, and $0 \leq t \leq T$, and both σ^* and σ are stochastic.

One direction is to assume a volatility processes of the form:

$$\sigma(t, T) = \sigma(t, t)e^{-\int_t^T \lambda(u)du} \quad (6.2)$$

where the function λ is deterministic and have been used to generate a useful set of Markovian interest models, in which a formula of closed form for the bond price is available. It should be noted that an extra assumption $\sigma(t, t) = \sigma(t, r(t))$ must be made in order to get a Markovian model.

Chiarella and Kwon assumed a volatility process to introduce the stochastic volatility into the standard HJM model, and the form of the volatility process is:

$$\sigma(t, T) = v^\gamma(t)\mathfrak{S}(t, f)e^{-\int_0^t \lambda(u)du} \quad (6.3)$$

where $f = (f(t, t + \tau_1), f(t, t + \tau_2), \dots, f(t, t + \tau_m))$ is a vector of the finite set of fixed tenor forward rates with $0 \leq \tau_1 \leq \tau_2 \leq \dots \leq \tau_m$, the functions $\mathfrak{S}(t, f)$ and $\lambda(u)$ are both deterministic; the appropriate parameter γ allows the volatility or variance to be modeled as a stochastic process. The stochastic process $v(t)$ is of the form:

$$dv(t) = \kappa(t)[\bar{v}(t) - v(t, w)]dt + \pi(t)v^\epsilon(t)dW^v(t) \quad (6.4)$$

where the functions $\kappa(t)$, $\bar{v}(t)$ and $\pi(t)$ are deterministic; ϵ is a constant; $W^v(t)$ is a standard Wiener process.

Chiarella and Kwon used one-dimensional stochastic volatility models to illustrate the volatility processes. For example, a volatility specification is:

$$\sigma(t, T) = \sigma_0 \sqrt{v(t)r(t)} e^{-\lambda(T-t)} \quad (6.5)$$

where σ_0 , λ are both constants, and $v(t)$ is satisfying:

$$dv(t) = \kappa[\bar{v} - v(t)]dt + \pi\sqrt{v(t)}dW^v(t) \quad (6.6)$$

where π , \bar{v} and κ are constants.

The main contribution of Chiarella and Kwon are how the stochastic dynamics can be reduced to a Markovian form, and the specification of stochastic volatility processes driven by additional Wiener processes which are independent of those driving the forward rate dynamics in the normal HJM framework.

In 2000, Chiarella and Kwon extended their theory to present how a stochastic dynamics can be turned into Markov form and obtained a counterpart of the complete stochastic volatility model of Hobson and Rogers (1998) [18] in the HJM framework.

6.3 The Market Model with Stochastic Volatility

6.3.1 Joshi and Rebonato Model (2001)

In 2001, Mark Joshi and Riccardo Rebonato [27] presented an extension of the LIBOR market model, which allowed for forward rates with stochastic instantaneous volatilities. They showed that almost all the useful approximations, which are used in the deterministic situation, can be extended to the stochastic volatility model successfully.

In the standard LIBOR market model the deterministic volatility of the forward rates is

$$\sigma_{\text{inst}}(T, t)$$

where t is the current time and T is the maturity of a given forward rate.

Joshi and Rebonato refer to a particular specification:

$$\sigma_{\text{inst}}(t, T) = k(T) \cdot g(T - t) \quad (6.7)$$

where $\sigma_{\text{inst}}(t, T)$ is the total instantaneous volatility of the T -maturity forward rate at time t , $g(T - t)$ is a time-homogeneous component, and $k(T)$ is a forward-rate specific part. To make sure of correct pricing of caplets for any choice of $g(\cdot)$, they impose the condition:

$$k(T)^2 = \frac{\sigma_{\text{black}}(T)^2 \cdot T}{\int_0^T g(u, T)^2 du} \quad (6.8)$$

and the time-homogeneous function $g(T - t)$ is satisfying:

$$g(T - t) = (a + b(T - t))e^{-c(T-t)} + d \quad (6.9)$$

Joshi and Rebonato's stochastic extension of the LIBOR market model is:

$$\frac{d(f_i(t) + \alpha)}{f_i(t) + \alpha} = \mu_i^\alpha(f, t)dt + \sigma_i^\alpha(t, T_i)dW_i(t) \quad (6.10)$$

$$\sigma_i^\alpha(t, T_i) = [a_t + b_t(T_i - t)]e^{-c_t(T_i-t)} + d_t \quad (6.11)$$

$$da_t = RS_a(a - RL_a)dt + \sigma_a dz_a \quad (6.12)$$

$$db_t = RS_b(b - RL_b)dt + \sigma_b dz_b \quad (6.13)$$

$$d[\ln(c_t)] = RS_c(\ln(c) - RL_c)dt + \sigma_c dz_c \quad (6.14)$$

$$d[\ln(d_t)] = RS_d(\ln(d) - RL_d)dt + \sigma_d dz_d \quad (6.15)$$

$$E[dz_i dz_a] = 0$$

$$E[dz_i dz_b] = 0$$

$$E[dz_i dz_c] = 0$$

$$E[dz_i dz_d] = 0$$

$$E[dz_b dz_a] = 0; E[dz_c dz_a] = 0; E[dz_d dz_a] = 0$$

$$E[dz_b dz_c] = 0; E[dz_b dz_d] = 0$$

$$E[dz_d dz_c] = 0$$

where $RS_a, RS_b, RS_c, RS_d, RL_a, RL_b, RL_c, RL_d$ respectively denote the reversion speeds and reversion levels of the relative coefficients a, b and their logarithms $\ln(c), \ln(d)$; and $\sigma_a, \sigma_b, \sigma_c, \sigma_d$ are their volatilities.

6.3.2 Wu and Zhang Model (2002)

One advantage of the market model is that it validates the use of the Black formula for caplet and swaption prices, and this relationship facilitated fast calibration of the standard model. But the disadvantage of the standard market model is that it only generates flat implied volatility curves, while the implied volatility curves of the LIBOR markets often have the shape of a smile or skew.

In 2002, Wu and Zhang [41] considered using the stochastic volatilities to extend the standard market model by adopting a stochastic factor for the forward-rate volatilities, and making this factor follow a square-root process under the risk-neutral measure:

$$df_j(t) = f_j(t)\sqrt{V(t)}\gamma_j(t) \left[dZ_t - \sqrt{V(t)}\sigma_{j+1}(t)dt \right] \quad (6.16)$$

$$dV(t) = \kappa(\theta - V(t))dt + \epsilon\sqrt{V(t)}dW_t \quad (6.17)$$

where $f_j(t)$ is the forward rate $f(t; T_j, T_{j+1})$ (with notation as in the LIBOR Market model), the variables κ, θ and ϵ are state-independent, W_t is a Brownian motion,

Z_t is a vector of independent Wiener processes under the risk-neutral measure, $\gamma_j(t)$ is a function of zero-coupon bond volatilities and of the form:

$$\gamma_j(t) = \frac{1 + \Delta T_j f_j(t)}{\Delta T_j f_j(t)} [\sigma(t, T_j) - \sigma(t, T_{j+1})] \quad (6.18)$$

and it is regarded as the volatility vector for $f_j(t)$.

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Vita

Xiaoxue Shan was born on January 2 1987, in Zibo City, Peoples Republic of China. She finished her undergraduate studies at Beijing University of Chemical Technology in June 2009 where she eared a Bachelor of Science in Mathematics. In August 2009 she came to Louisiana State University to pursue graduate studies in mathematics. She is currently a candidate for the degree of Master of Science in mathematics with concentration in finance, which will be awarded in May 2012.