2005

Multiscale strain analysis

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MULTISCALE STRAIN ANALYSIS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by

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B.A., Eastern Illinois University, 2001
M.S., Louisiana State University, 2002
August 2005
Dedication

To my parents, Robert A. Breitzman and Alice M. Breitzman, who pushed me through the good times, carried me through the bad times, and helped me become every bit the person I am today. THANK YOU.
Acknowledgments

So many people have contributed to my ability to produce this work. First and foremost, I thank my advisor, Professor Robert Lipton. His love of doing “good science” has been an inspiration to my young career. It has truly been a privilege to have learned from such a mathematician. He has also supported me financially through grants from the Air Force Office of Scientific Research and the National Science Foundation.

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Abstract

The mathematical homogenization and corrector theory relevant to prestressed heterogeneous materials in the linear-elastic regime is discussed. A suitable corrector theory is derived to reconstruct the local strain field inside the composite. Based on this theory, we develop an inexpensive numerical method for multi scale strain analysis within a prestressed heterogeneous material. The theory also provides a characterization of the macroscopic strength domain. The strength domain places constraints on the homogenized strain field which guarantee that the actual strain in the heterogeneous material lies inside the strength domain of each material participating in the structure.
Chapter 1

Introduction

Composite materials are increasingly finding use in structural applications. One example is the light-weight high-strength fiber reinforced composites used to make aircraft wings. Composite technology, however, is plagued by the inability to accurately characterize their structures and predict their strengths. The reason for this problem is simple. Composites are not homogeneously composed. For example, fiber composites consist of a relatively soft resin material surrounding stiff reinforcement fibers. Thus the elastic and inelastic response of a composite is directly affected by its microstructure as well as by the external loading. To make matters more complicated, composites often incur prestress, which is caused by the mismatch of thermal expansion coefficients between the phases that comprise the composite microstructure. The thesis work presented here develops a methodology for the quantitative assessment of the local strain states and failure initiation inside prestressed composite materials.
The full computation of the stress and strain fields inside a heterogeneous prestressed elastic body is a daunting task. One way to attack this problem is through the use of homogenization theory (see [1] and [2]). Homogenization theory provides a way to compute the average stress and strain fields inside a linearly elastic composite material. Here the complicated heterogeneous microstructure is replaced with an effective elastic solid that possesses the same locally averaged stresses and strains as the original. To fix ideas, we consider a periodic microstructure. In this context, the homogenization process replaces the complicated periodic phase-dependent local elasticity tensors with a constant effective elasticity tensor. The homogenized boundary value problem is expressed in terms of the effective elastic tensor and provides the means for computing the macroscopic or averaged stress and strain fields in the composite. The homogenized boundary value problem for multi-phase prestressed elastic composites is presented in Chapter 3. The explicit methodology for computing the homogenized tensors and deriving the homogenized boundary value problem is provided in Chapter 4.

It is clear that there is a loss of information incurred by the homogenization process. It is simply impossible to capture the complicated strain field inside a heterogeneous composite using a homogeneous representation. In order to more accurately resolve the local details, one seeks to reconstruct the local strain field from the homogenized strain field. To this end, one can appeal to the corrector theory of homogenization (see [1], [3], [4], and [5]). The corrector theory provides the means to locally reconstruct the actual strain up to an error that vanishes in $L^p$, $1 \leq p \leq 2$ as $\epsilon$ tends to zero. Here the value of
p depends upon the regularity of the microstructure (see [3]). A formal notion of quantities similar to correctors has recently been applied to analyze field fluctuations near singularities (see [6], [7], [8], [9], and [10]). In these formal treatments, the corrector problem is posed, however, the precise sense of how the actual fields are approximated by the corrector problem is not considered. For random microstructures, Luciano and Willis analyze the boundary layer behavior of the stress and strain field using approximate trial fields of Hashin-Shtrikman type (see [11]). In the absence of prestress, a corrector theory based on the differentiability of $G$-limits is introduced by Lipton ([12], [13], and [14]) and is used to rigorously bound the extent of highly stressed zones near singularities due to reentrant corners and other stress risers.

In this thesis, a new corrector theory is developed for the the quantitative assessment of local strain states inside prestressed heterogeneous materials. The corrector theory forms the basis for a new multi scale assessment method for prediction of failure initiation inside prestressed composite materials. We begin by introducing the notion of failure initiation inside a homogeneous material. Failure initiation is described in terms of a strength domain (see [15] and [16]) that is given by a prescribed bounded open set in strain space. The displacements inside the material remain elastic as long as the strain lies within the strength domain. The boundary of the strength domain is called the failure envelope. Failure initiation occurs when the strain lies on this surface. The strength domain of a particular material is characterized by an inequality of the form

$$f(\xi) < t.$$
Here $\xi$ represents the strain and the function $f$ is referred to as the failure criteria. Failure initiation occurs when $f(\xi) = t$. The choice of failure criteria depends upon the material under consideration. The primary failure criteria used in this work is a generalization of the maximum distortion energy theory [17] used for ductile metals (see [18]). This failure criteria addresses both volume (dilatational) change and shape (deviatoric) change. Precise formulas for the failure criteria are given in terms of the dilatational and deviatoric strain invariants, see equations (3.11) and (3.12) in Section 3.3. An accounting for the distortion and dilation also enables the incorporation of effects due to the elastic anisotropy of the materials.

The strength domains of the individual materials making up the composite will be referred to as the microscopic strength domains. The failure criteria that describe the strength domains for each of the materials will be referred to as the microscopic failure criteria. The macroscopic strength domain is defined in terms of suitable constraints placed on the macroscopic (homogenized) strain field that guarantee the actual local strain in the composite lies within the microscopic strength domains of each of the materials participating in the composite structure. The macroscopic strength domain is obtained by bounding the actual local strain fields in terms of the $L^\infty$ norm of the corrector fields. The macroscopic strength domains are presented in Theorem 3.3.1 and in Theorem 3.3.2. These theorems are proved in Chapter 4.

The macroscopic strength domains developed in this thesis pave the way for a multi scale strain analysis method. This method is described in Chapter 5. It is a numerically inexpensive method for strain assessment that allows one
to predict prestressed composite failure based on the homogenized strain field. The homogenization step relies on the analysis of a suitable volume element containing a representative sample of the microstructure. In the literature, this volume element is referred to as a *representative volume element*, or RVE. For periodic microstructures, the RVE is given by the unit period cell for the microstructure. The unit cell is used to compute the effective properties for the composite. These effective properties are then used on the macroscopic level to predict the homogenized strain field. The corrector theory enables a scale-linkage between the macroscopic and microscopic scales. As developed in Section 3.2 and in Section 3.3, this linkage is used to bound the amplification of the homogenized strain by the microstructure. Knowing the maximum possible strain states occurring in the composite (on the microstructural level), one can predict the strength domain of the composite based on given microscopic failure criteria. The strength domain is given in Theorem 3.3.1 and Theorem 3.3.2.

Consistent with the macroscopic strength domain, we present bounds on the microscopic strain state given in terms of the homogenized strain (see Corollary 3.3.3 and Corollary 3.3.4). We refer to these bounds as the multi scale bounds. Individually, we refer to them as the first invariant bound and second invariant bound, respectively, see Section 3.3.

Chapters 6 - 8 apply the theory to fiber reinforced multi-ply layups. Chapter 6 provides a comparison between the multi scale strain analysis method and a direct numerical simulation of a free edge problem. In this chapter, we study the accuracy of the multi scale strain analysis method in areas of
uniform periodicity, as well as in areas of abrupt physical changes (a free edge
and a ply interface). Chapter 7 applies the theory to an 8-ply symmetric lam-
inate with an open hole. The surface of the hole is a free edge, and strain
concentrations arise near this surface. The problem illustrates the difference
between the invariants of the homogenized strain field and the multi scale
bounds. More importantly, the problem highlights the effects of the prestress
on the composite. We finish by comparing the multi scale strain analysis
method for two different fiber reinforced microstructures. Chapter 8 examines
the failure prediction of the multi scale strain analysis method. It compares
the predictions of the multi scale bounds with the macroscopic stress failure
criteria given by Hashin and max stress. Prestress is taken into account and
predictions of highly stressed zones are given for two applied strain states.
Chapter 2

Stress-Strain Law and the PDE of Elastic Equilibrium

In this chapter, we introduce the equations of linear elasticity for heterogeneous prestressed materials. We begin with the notions of displacement, strain, and stress. We note that the space $\text{Sym}(\mathbb{R}^3)$ is the space of symmetric $3 \times 3$ matrices with real coefficients.

Composites are treated mathematically as three dimensional heterogeneous continua. Material points inside the composite displace in response to applied forces and thermally induced stresses. The displacement is said to be elastic if the material points return to their initial position upon the removal of the imposed force. In this treatment, all displacements will be taken to be elastic. The displacement is represented mathematically as a vector field. The relative displacement of material points inside a body is described by the strain tensor, which is given in terms of the gradient of the displacement. The calculation of forces at any point inside the composite is obtained through the use of the stress tensor field, $\sigma(x)$. The stress tensor at each point is a linear transform, and the force per unit area acting on the plane passing through $x$
with normal \( n \) is given by \( \sigma(x)n \). For linear elastic materials, the stress is a linear function of the strain. It is often the case that additional internal forces are generated inside heterogeneous materials due to the differences in thermal expansion between the component materials. Such internally generated forces are referred to as prestresses. We will incorporate the effect of thermally induced prestresses into our analysis.

2.1 Stress-Strain Relations

When a body is subjected to a mechanical load, the points inside the body will experience an elastic displacement. We call this elastic displacement function \( u \). For the elastic transmission problems studied within, we have the elastic displacement \( u : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \), and \( u \in H^1(\Omega)^3 \), where \( \Omega \) is a bounded, connected, open set in \( \mathbb{R}^3 \) with a Lipschitz boundary.

From the elastic displacement function, we define the elastic strain by

\[
(e(u))_{ij} = \frac{\nabla u + (\nabla u)^T}{2} = \frac{u_{i,j} + u_{j,i}}{2}, \quad \text{where} \quad u_{i,j} = \frac{\partial u_{x_i}}{\partial x_j}.
\]

For future reference, we provide a basis for \( Sym(\mathbb{R}^3) \). The basis elements used are referred to as

\[
e^{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e^{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e^{33} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]
The stress is a linear function of the strain, and is defined by the second-order tensor

\[
\begin{align*}
    e^{12} &= \begin{bmatrix}
        0 & \frac{1}{2} & 0 \\
        \frac{1}{2} & 0 & 0 \\
        0 & 0 & 0
    \end{bmatrix}, \\
    e^{33} &= \begin{bmatrix}
        0 & 0 & \frac{1}{2} \\
        0 & 0 & 0 \\
        \frac{1}{2} & 0 & 0
    \end{bmatrix}, \\
    e^{23} &= \begin{bmatrix}
        0 & 0 & 0 \\
        0 & 0 & \frac{1}{2} \\
        0 & \frac{1}{2} & 0
    \end{bmatrix}.
\end{align*}
\]

The stress is a linear function of the strain, and is defined by the second-order tensor

\[
\sigma_{ij} = C_{ijkl} ((e (u))_{kl}),
\]

where the stiffness matrix, \( C \), is a fourth-order tensor satisfying the eigen-value constraint \( 0 < \lambda < C < \Lambda \).

### 2.2 PDE for Elastic Displacement with Prestress

One main contribution of this work is the treatment of prestress for composite materials. Prestress is a form of nonmechanical load felt by a body. The most common cause of prestress in composites is the relatively high cure temperatures required for processing. The materials are heated up, catalyzing chemical reactions and bonding between the various phases of the composite. Once bonding is complete, the newly-formed composite must be cooled. However, the materials used in the composite typically have different coefficients of thermal expansion (CTE). These coefficients relate the expansion of the material to the change in temperature. Once bonded, the individual materials are not able to expand and contract freely with the temperature change. This
restriction creates internal, nonmechanical stress inside the composite, which can influence the strength and ultimately the failure envelope of the composite.

We now formulate the prestress boundary-value problem. The problem can be formulated as an $N$-phase elastic transmission problem with prestress caused by the mismatch of CTE between the materials. However, to fix ideas, we illustrate the problem in the context of two-phase fiber reinforced materials. The set $\Omega$ is a bounded, connected, open set in $\mathbb{R}^3$ with a Lipschitz-continuous boundary $\Gamma$. It is filled with two material phases, denoted $F$ and $M$. For consistent terminology, material $F$ will be referred to as the fiber material and material $M$ as the matrix material. When necessary, material properties from the fiber phase and matrix phase will be identified by the superscripts $f$ and $m$, respectively. Further, the set $\Omega$ is divided into $K$ subdomains $\omega_i$ (see Figure 2.1(a)) such that

1. Each set $\omega_i$ is open for $1 \leq i \leq K$

2. $\overline{\Omega} = \bigcup_{i=1}^{M} \overline{\omega_i}$

3. The microstructure of each set $\omega_i$ is $\epsilon$-periodic (see Figure 2.1(b))

The elastic stiffness $C$ is piecewise constant and takes the values $C^f$ in the fiber phase and $C^m$ in the matrix phase. The inelastic strain is caused by the mismatch in CTE between the matrix and fiber phases. The inelastic strain
is given by

\[
\begin{bmatrix}
\alpha_{11} \Delta T & 0 & 0 \\
0 & \alpha_{22} \Delta T & 0 \\
0 & 0 & \alpha_{33} \Delta T
\end{bmatrix},
\]

(2.4)

where \( \alpha_{ij} = \begin{cases} 
\alpha_{ij}^f, & \text{in the fiber phase} \\
\alpha_{ij}^m, & \text{in the matrix phase.}
\end{cases} \)

The inelastic strain \( \varepsilon \) is piecewise constant, and takes the values \( \varepsilon^f \) and \( \varepsilon^m \) in the fiber and matrix phases.

In order to conveniently describe the local elastic stiffness and inelastic strain, we introduce the functions \( C(x, y) \) and \( e(x, y) \) defined on \( \Omega \times Q \). Here \( Q \) denotes the unit cube in \( \mathbb{R}^3 \). The functions are \( Q \)-periodic in the second
variable and are defined by

\begin{align}
C(x, y) &= C^{(i)}(y) \quad \text{for } x \in \omega_i \\
e(x, y) &= e^{(i)}(y) \quad \text{for } x \in \omega_i,
\end{align}

where \( C^{(i)}(y) = \begin{cases} C^f & \text{in the fiber} \\ C^m & \text{in the matrix} \end{cases} \) and \( e^{(i)}(y) = \begin{cases} e^f & \text{in the fiber} \\ e^m & \text{in the matrix} \end{cases} \).

The local elastic stiffness and inelastic strain inside the composite are given by \( C^e(x) = C(x, x/\varepsilon) \) and \( e^e(x) = e(x, x/\varepsilon) \). The stress and strain inside the composite are related by

\begin{align}
\sigma^e_{ij} = C^e_{ijkl} \left( (e^e)_{kl} - e^e_{kl} \right).
\end{align}

The boundary of the domain \( \Omega \) is split into two subsets, \( \Gamma_0 \) and \( \Gamma_1 \). On \( \Gamma_0 \), the displacement is set to zero. A traction load is prescribed on \( \Gamma_1 \). The boundary value problem is given by

\begin{align}
\begin{cases}
\text{div} (\sigma^e) = f & \text{in } \Omega \\
u^e = 0 & \text{on } \Gamma_0 \\
\sigma^e \mathbf{n} = g & \text{on } \Gamma_1.
\end{cases}
\end{align}

In order to give the weak form of (2.7), we suppose that the functions \( f \) and \( g \) are elements of \( L^2(\Omega)^3 \) and \( L^2(\Gamma_1)^3 \), respectively. We define the space \( \mathbf{V}(\Omega) = \{ v \in H^1(\Omega)^3 \mid v = 0 \text{ on } \Gamma_0 \} \), equipped with the \( H^1(\Omega)^3 \) norm. The
solution \( u^\varepsilon \) in \( \mathbf{V}(\Omega) \) of the weak formulation satisfies

\[
(2.8) \quad \int_{\Omega} C^\varepsilon \left( e\left( u^\varepsilon \right) - e^\varepsilon \right) : e(v) \, dx = \int_{\Omega} f \cdot v \, dx + \int_{\Gamma_1} g \cdot v \, dS,
\]

for all \( v \) in \( \mathbf{V}(\Omega) \). The existence of a solution \( u^\varepsilon \) to (2.8) is given by the Lax-Milgram Theorem.
Chapter 3

The Homogenization Theory

The focus of this chapter is the presentation of the periodic homogenization and corrector theory that lies at the core of the multi scale strain analysis method. The method is based upon homogenization theory and the development of the corrector theory for multiphase elastic composites with prestress due to a mismatch in coefficients of thermal expansion. Our approach proceeds in two steps: the first provides the homogenized equations of elastic equilibrium given in terms of an effective elastic tensor relating macroscopic stresses to macroscopic strains. The second step provides a macroscopic strength domain such that if the homogenized strain lies in this domain, then the actual strain in the composite lies within the strength domain of each material participating in the composite structure.
### 3.1 Homogenization of Heterogeneous Elastic Media with Prestress

In order to capture the effects of an applied load, the homogenized properties of the composite must be computed. Indexing by the superscript $\epsilon$, the behavior of the elastic displacement $u^\epsilon$ must be considered in the limit of vanishing $\epsilon$.

The elastic displacement solves the equilibrium equation

$$-\text{div} \left( C_{ijkl}^\epsilon \left( (e \left( u^\epsilon \right))_{kl} - e_{kl}^i \right) \right) = f \text{ in } \Omega, \ \text{with } u^\epsilon = 0 \text{ on } \Gamma_0 \text{ and } \sigma^\epsilon \cdot n = g \text{ on } \Gamma_1. $$

The non-mechanical prestress has been denoted by $e^\epsilon$. Here $C_{ijkl}$ is the piecewise constant elastic tensor taking constant values in each material.

Taking into account the prestress, the homogenized properties are given by the following theorem.

**Theorem 3.1.1 (Homogenization Theorem for Periodic Microstructure).** Suppose $\Omega$ is a bounded, connected, open set in $\mathbb{R}^3$ with a Lipschitz-continuous boundary.

Then, for all $f$ in $L^2(\Omega)^3$ and $g$ in $L^2(\Gamma_1)^3$, $u^\epsilon \rightharpoonup u^0$ weakly in $V(\Omega)$ as $\epsilon \to 0$ and as $\epsilon \to 0$, we have

\[
\begin{align*}
    &u^\epsilon \to u^0 \quad \text{strong in } L^2(\Omega)^3 \\
    &e \left( u^\epsilon \right) \rightharpoonup e \left( u^0 \right) \quad \text{weakly in } L^2(\Omega)^{3\times 3} \\
    &C^\epsilon \left( e \left( u^\epsilon \right) - e^\epsilon \right) \rightharpoonup C^{E}_{ijkl} \left( (e \left( u^0 \right))_{kl} \right) - H^e_{ij} \quad \text{weakly in } L^2(\Omega)^{3\times 3},
\end{align*}
\]

where

\[
C^{E}_{ijkl}(x) = \frac{1}{|Q|} \int_Q \left( C'_{ijmn}(y) \left( \left( e \left( w^{kl} \right) \right)_{mn} + \tau^{ij}_{mn} \right) \right) dy, \text{ for } x \in \omega^e,
\]
\[ H_{ij}^e(x) = \frac{1}{|Q|} \int_Q C_{mnop}^{ij}(y) \left( (e \left( w^{ij} \right))_{op} + \varepsilon_{ij}^{op} \right) : e_{mn}(y) dy, \text{ for } x \text{ in } \omega_\ell, \]

where \( w^{ij}(y) \) is the \( Q \)-periodic solution of

\[ \text{div} \left( C^{ij}(y) \left( e \left( w^{ij} \right) + \varepsilon^{ij} \right) \right) = 0, \]

and \( \varepsilon^{ij} \) is a basis element of \( \text{Sym}(\mathbb{R}^3) \). Moreover, \( u^0 \) solves the limit differential equation

\[
\begin{aligned}
-\text{div} \left( C^E \left( e (u^0) \right) - H^e \right) &= f \quad \text{in } \Omega \\
u^0 &= 0 \quad \text{on } \Gamma_0 \\
(C^E (e (u^0)) - H^e) \mathbf{n} &= g \quad \text{on } \Gamma_1
\end{aligned}
\]

Theorem 3.1.1 is proved in Section 4.2.

### 3.2 Prestress and Corrector Theory

Because of the inhomogeneity of composites, homogenization is not enough to accurately characterize the strain states within. Note that the homogenization theory provides only weak convergence of the strains \( (e(u)) \) in \( L^2 \). In order to provide macroscopic failure criteria, one requires strong convergence of the strains to a suitable limit. The strong \( L^1 \) convergence is provided by constructing a suitable corrector theory for each subdomain \( \omega_i \) in \( \Omega \).

The corrector matrix \( P(y)\varepsilon \) is given by the equation

\[
(3.1) \quad P(y)\varepsilon = e \left( w^\varepsilon \right)(y) + \varepsilon,
\]
where $w^\varphi$ is the $Q$-periodic solution of \( \text{div} \left( C^{(i)}(y) \left( e \left( w^\varphi \right) + \varepsilon \right) \right) = 0 \). Note next that any strain $\varepsilon$ can be written as a linear combination of basis strains. Thus for any strain $\varepsilon$, we have

\[
(3.2) \quad \varepsilon = \sum_{i,j=1}^{3} \varepsilon_{ij} e^{ij},
\]

where subscripts indicate coordinate scalars and superscripts indicate basis strains. Using this process, the strain $e \left( w^\varphi \right)(y)$ can be written as the linear combination

\[
(3.3) \quad e \left( w^\varphi \right)(y) = \varepsilon_{ij} e \left( w^{ij} \right)(y).
\]

Combining this work, we have

\[
(3.4) \quad P(y)\varepsilon = \sum_{i,j=1}^{3} \varepsilon_{ij} P(y)e^{ij}
\]

for any strain $\varepsilon$. Considering the limit displacement solution $u^0$, we have

\[
(3.5) \quad P(y) e \left( u^0 \right) = \sum_{i,j=1}^{3} \left( e \left( u^0 \right) \right)_{ij} P(y)e^{ij}.
\]

We next introduce the matrix $\varphi$ by declaring

\[
(3.6) \quad \varphi = \sum_{i,j=1}^{3} \varphi_{ij}(x)e^{ij},
\]

where $\varphi_{ij}(x) \in C^\infty(\omega_i)$ and $\varphi_{ij} e^{ij} \in \text{Sym}(\mathbb{R}^3)$. Moreover, the $Q$-periodic
function $\eta$ gives the non-mechanical strain and solves the equation

$$\text{(3.7)} \quad \text{div} \left( C^{(i)}(y) (e(\eta) - e(y)) \right) = 0.$$ 

We introduce the Principal Corrector Result.

**Theorem 3.2.1 (Principal Corrector Result).** For any $C^\infty_c(\omega)$ function $\phi$, we have

$$\lim_{\epsilon \to 0} \int_{\omega_i} \phi C^\epsilon \left( e\left( u^\epsilon \right) - \left( P^\epsilon \varphi + e(\eta^\epsilon) \right) \right) : \left( e\left( u^\epsilon \right) - \left( P^\epsilon \varphi + e(\eta^\epsilon) \right) \right) \, dx$$

$$= \int_{\omega_i} \phi C^E \left( e\left( u^0 \right) - \varphi \right) : \left( e\left( u^0 \right) - \varphi \right) \, dx.$$

To motivate Theorem 3.2.1, we give some insight into its application. Suppose that $u^0 \in C^\infty(\omega)^3$. Then $e\left( u^0 \right)$ has the form of $\varphi$. Taking $\varphi = e(\varphi)$ in the theorem, we find

$$\text{(3.8)} \quad \lim_{\epsilon \to 0} \int_{\omega_i} \left\{ \phi C^\epsilon \left( e\left( u^\epsilon \right) - \left( P^\epsilon \left( e\left( u^0 \right) \right) + e(\eta^\epsilon) \right) \right) : \left( e\left( u^\epsilon \right) - \left( P^\epsilon \left( e\left( u^0 \right) \right) + e(\eta^\epsilon) \right) \right) \right\} \, dx = 0,$$

and thus, noting the eigen-value constraint on $C^\epsilon$,

$$\text{(3.9)} \quad \lim_{\epsilon \to 0} \lambda \int_{\omega_i} \left| e\left( u^\epsilon \right) - \left( P^\epsilon \left( e\left( u^0 \right) \right) + e(\eta^\epsilon) \right) \right|^2 \, dx = 0,$$

which gives that

$$\text{(3.10)} \quad \lim_{\epsilon \to 0} \left( e\left( u^\epsilon \right) - \left( P^\epsilon \left( e\left( u^0 \right) \right) + e(\eta^\epsilon) \right) \right) = 0 \quad \text{in} \quad L^2_{\text{loc}}(\omega)^{3 \times 3},$$
where $L^{2}_{\text{loc}}(\omega_i)^{3\times3} = \{ u : \omega_i \to \mathbb{R} \mid u \in L^{2}(\omega)^{3\times3} \text{ for each } \omega \subset \subset \omega_i \}$. Note that the symbol $\subset \subset$ means compactly contained. Thus the actual strain in the composite $e(u')$ is forced arbitrarily close to the corrected macroscopic strain $P_e(e(u^0)) + e(\eta')$ by refining the microstructure.

### 3.3 Macroscopic Strength Domain

Here we present a macroscopic strength domain such that if the homogenized, or macroscopic strain lies in this domain, then the actual strain in the composite lies within the strength domain of each material participating in the composite structure. The strength domain of each material in the composite is assumed to be the same. It is described by an open, bounded domain in the space of $3 \times 3$ strain tensors. For the problem considered here, the strength domain is described to be the set given by the intersection of a linear and a quadratic constraint. The linear constraint determines a half-space bounded by a hyperplane, while the quadratic constraint determines an ellipsoid (see Figure 3.1). For any $3 \times 3$ strain $\eta$, the linear and quadratic constraints are denoted by $\mathcal{L}(\eta) \leq t$ and $\Pi(\eta) \leq t$, respectively.

We introduce the dilatational strain invariant

$\mathcal{L} : Sym(\mathbb{R}^{3}) \rightarrow \mathbb{R}$ by the formula

$$L(e(u')) = \text{tr} (e(u')) ,$$

where $\text{tr}(X)$ is the trace of the matrix $X$. We also introduce the deviatoric
strain invariant $\Pi : \text{Sym} (\mathbb{R}^3) \to \mathbb{R}$ by the formula

$$
(3.12) \quad \Pi (e (u^e)) = \frac{3}{2} |e (u^e)|^2 - \frac{1}{2} (L (e (u^e)))^2,
$$

where $|e (u^e)|^2 = \sum_{i,j=1}^{3} (e (u^e))_{ij}^2$. We note that the dilatational and deviatoric strain invariants are often referred to as the first and second strain invariants, respectively.

The set of strains which satisfies both $L (\eta) < t$ and $\Pi (\eta) < t$ is called the strength domain. The boundary of the strength domain is referred to as the failure envelope. In the sequel, we refer to this strength domain as the microscopic strength domain. We shall also refer to (3.11) and (3.12) as the microscopic failure criteria.

Our goal is to establish suitable constraints on the macroscopic strain $e (u^0)$ that ensure that the actual strain $e (u^e)$ lies inside the microscopic strength domain. This idea is rigorously stated in Theorem 3.3.1 and Theorem 3.3.2 below.

![Figure 3.1: Common Strength Domain of each Material](image-url)
The macroscopic failure criteria $L^M$ and $\Pi^M$ are defined to be

\[(3.13) \quad L^M \left( e \left( u^0 \right) \right) = \sup_{y \in Q} L \left( P(y) \left( e \left( u^0 \right) \right) + e \left( \eta \right) \right)\]

and

\[(3.14) \quad \Pi^M \left( e \left( u^0 \right) \right) = \sup_{y \in Q} \Pi \left( P(y) \left( e \left( u^0 \right) \right) + e \left( \eta \right) \right).\]

We now present the two main results for the characterization of the macroscopic strength domain. In the following results, $| \cdot |$ denotes three-dimensional volume and $\bar{\omega}$ denotes a subset of $\omega_i$. For $A \subset \bar{\omega}$, the set $\bar{\omega} \setminus A$ denotes the set of all the points in $\bar{\omega}$ that are not in $A$.

**Theorem 3.3.1 (Dilatational Constraint for the Macroscopic Strength Domain).** If $L^M \left( e \left( u^0 \right) \right) < t$ on $\bar{\omega}$, then for every positive number $\delta$ there is a set $A \subset \bar{\omega}$ with $|\bar{\omega} \setminus A| < \delta$ and a positive number $\epsilon_0$ such that whenever $\epsilon < \epsilon_0$, we have $L \left( e \left( u' \right) \right) < t$ on $A$.

This theorem is proved in Section 4.4.

**Theorem 3.3.2 (Deviatoric Constraint for the Macroscopic Strength Domain).** If $\Pi^M \left( e \left( u^0 \right) \right) < t$ on $\bar{\omega}$, then for every positive number $\delta$ there is a set $A \subset \bar{\omega}$ with $|\bar{\omega} \setminus A| < \delta$ and a positive number $\epsilon_0$ such that whenever $\epsilon < \epsilon_0$, we have $\Pi \left( e \left( u' \right) \right) < t$ on $A$.

This theorem is proved in Section 4.4.

Recall that the first and second strain invariants refer to the dilatational and deviatoric constraints, respectively. With the previous macroscopic con-
straints, we have the First Strain Invariant Bound Theorem and the Second Strain Invariant Bound Theorem presented below as corollaries.

**Corollary 3.3.3 (First Strain Invariant Bound Theorem).** Given $\delta, \gamma > 0$, for each set $\tilde{\omega}$ in the composite there exists a positive length scale $\epsilon_0$ such that for all microstructural scales $\epsilon < \epsilon_0$, the volume of the set in $\tilde{\omega}$ with $\mathcal{L}(\epsilon u'(x)) \geq \mathcal{L}^M(\epsilon u^0(x)) + \gamma$ is less than $\delta$.

This corollary is proved in Section 4.4.

**Corollary 3.3.4 (Second Strain Invariant Bound Theorem).** Given $\delta, \gamma > 0$, for each set $\tilde{\omega}$ in the composite there exists a positive length scale $\epsilon_0$ such that for all microstructural scales $\epsilon < \epsilon_0$, the volume of the set in $\tilde{\omega}$ with $\Pi(\epsilon u'(x)) \geq \Pi^M(\epsilon u^0(x)) + \gamma$ is less than $\delta$.

This corollary is proved in Section 4.4.
Chapter 4

Proofs

In this chapter, we justify the main results of Chapter 3. We begin with the analogue of the Div-Curl theorem for linear elasticity.

4.1 Div-Curl Theorem

We begin by presenting a fundamental result which will be used several times in the derivation of the homogenization theory. For notational purposes, the symbols “→” and “−” denote strong convergence and weak convergence, respectively. Moreover, $M_3(\mathbb{R})$ denotes the space of $3 \times 3$ matrices with real coefficients.

Theorem 4.1.1 (Div-Curl Theorem for Linear Elasticity). Let $\Omega$ be an open subset of $\mathbb{R}^3$. Suppose that $e : H^1(\Omega)^3 \to \text{Sym}(\mathbb{R}^3)$ is given by

$$(e(v))_{ij} = \frac{v_{i,j} + v_{j,i}}{2}, \text{ where } v_{i,j} = \frac{\partial v_{xi}}{\partial x_j}.$$
Suppose also that

1. There is a sequence \( \{v^\epsilon\}_{\epsilon > 0} \subset H^1(\Omega)^3 \) and a \( H^1(\Omega)^3 \) function \( v \) such that
\( v^\epsilon \rightharpoonup v \) in \( H^1(\Omega)^3 \),

and

2. There is a sequence \( \{\eta^\epsilon\}_{\epsilon > 0} \subset L^2(\Omega)^{3 \times 3} \) and a \( L^2(\Omega)^{3 \times 3} \) function \( \eta \) such that
   
   (a) \( \eta^\epsilon = (\eta^\epsilon)^T \), for every \( \epsilon \)
   
   (b) \( \eta^\epsilon \rightharpoonup \eta \) in \( L^2(\Omega)^{3 \times 3} \)
   
   (c) \( -\text{div} \eta^\epsilon = f \), with \( f \in W^{-1,2}(\Omega)^3 \).

Then \( e(v^\epsilon) : \eta^\epsilon \rightharpoonup e(v) : \eta \) in the sense of distributions. That is, for every \( C_c^\infty(\Omega) \) function \( \phi \),

\[
\lim_{\epsilon \to 0} \int_{\Omega} (e(v^\epsilon) : \eta^\epsilon) \phi \, dx = \int_{\Omega} (e(v) : \eta) \phi \, dx.
\]

Proof. We begin by noting two facts. First, the convergence in (1) gives that \( e(v^\epsilon) \rightharpoonup e(v) \) in \( L^2(\Omega)^{3 \times 3} \). Second, from (2)(b) we get that \( -\text{div} \eta = f \). To simplify appearances, we introduce the following notation. Let

\( \otimes : \mathbb{R}^3 \times \mathbb{R}^3 \to M_3(\mathbb{R}) \) be given by the formula \( (a \otimes b)_{ij} = a_i b_j \). Then, for any \( C_c^\infty(\Omega) \) function \( \phi \) and for any \( H^1(\Omega)^3 \) function \( v \), the product rule gives

\[
(4.1) \quad e(\phi v) = \frac{\nabla \phi \otimes v + v \otimes \nabla \phi}{2} + \phi e(v).
\]

We will denote the term \( \frac{\nabla \phi \otimes v + v \otimes \nabla \phi}{2} \) by \( \nabla \phi \otimes v \).
To begin the proof, let $\phi$ be a given $C^\infty_c(\Omega)$ function. Form the product

\[(4.2) \quad \int_\Omega \phi \eta^\varepsilon : e(v^\varepsilon) \, dx.\]

Using the product rule (4.1) for $\phi e(v^\varepsilon)$ and noting that $\eta^\varepsilon : (\nabla \phi \otimes v^\varepsilon) = \eta^\varepsilon \nabla \phi \cdot v^\varepsilon$, we rewrite (4.2) as

\[(4.3) \quad \int_\Omega \eta^\varepsilon : e(\phi v^\varepsilon) \, dx - \int_\Omega \eta^\varepsilon \nabla \phi \cdot v^\varepsilon \, dx.\]

Consider a cluster point of the sequence $\{\int_\Omega (e(v^\varepsilon) : \eta^\varepsilon) \, dx\}_{\varepsilon > 0}$. Then, so long as both limits exist,

\[(4.4) \quad \lim_{\varepsilon \to 0} \int_\Omega \phi \eta^\varepsilon : e(v^\varepsilon) \, dx = \lim_{\varepsilon \to 0} \int_\Omega \eta^\varepsilon : e(\phi v^\varepsilon) \, dx - \lim_{\varepsilon \to 0} \int_\Omega \eta^\varepsilon \nabla \phi \cdot v^\varepsilon \, dx.\]

We now show that both limits exist and complete the proof of the theorem. Note that for each $\varepsilon$, the product $\phi v^\varepsilon$ is a $H^1(\Omega)^3$ function. Moreover, $\phi v^\varepsilon \to \phi v$ in $H^1(\Omega)^3$. Thus

\[(4.5) \quad \lim_{\varepsilon \to 0} \int_\Omega \eta^\varepsilon : e(\phi v^\varepsilon) \, dx = \lim_{\varepsilon \to 0} \langle f, \phi v^\varepsilon \rangle = \langle f, \phi v \rangle,\]

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $W^{-1,2}(\Omega)^3$ and $H^1(\Omega)^3$. On the other hand, $\nabla \phi \otimes v^\varepsilon \to \nabla \phi \otimes v$ in $L^2(\Omega)^{3 \times 3}$ and $\eta^\varepsilon \to \eta$ in $L^2(\Omega)^{3 \times 3}$. Thus

\[(4.6) \quad \lim_{\varepsilon \to 0} \int_\Omega \eta^\varepsilon \nabla \phi \cdot v^\varepsilon \, dx = \int_\Omega \eta \nabla \phi \cdot v \, dx.\]
To complete the proof, recall that

\begin{equation}
\langle f, \phi v \rangle = \int_{\Omega} \eta : e(\phi v) \, dx.
\end{equation}

Then, combining the results of (4.5) and (4.6) and using the product rule (4.1), we have for every $C_c^\infty(\Omega)$ function $\phi$, that

\begin{equation}
\lim_{\epsilon \to 0} \int_{\Omega} (e(v^\epsilon) : \eta^\epsilon) \phi \, dx = \int_{\Omega} (e(v) : \eta) \phi \, dx,
\end{equation}

completing the proof.

4.2 Homogenization Theorem for Periodic Microstructure

Here we justify the homogenization theorem (Theorem 3.1.1), establishing the method by which we obtain our homogenized coefficients and the inelastic strains.

\textit{Proof.} The first convergence proved is $u^\epsilon \rightharpoonup u^0$ in $V(\Omega)$. Note that if $u^\epsilon \rightharpoonup u^0$ in $V(\Omega)$, then $u^\epsilon \rightarrow u^0$ in $L^2(\Omega)^3$.

Consider the sequence of solutions $\{u^\epsilon\}_{\epsilon > 0} \subset V(\Omega)$. We show there exists a weakly converging subsequence. The outline is to show that $\{u^\epsilon\}_{\epsilon > 0}$ is a norm-bounded sequence in $V(\Omega)$. Then, since $V(\Omega)$ is a reflexive Banach space, we appeal to the compactness theorem of Banach-Alaoglu to find a $V(\Omega)$ limit function $u^0$, completing the convergence proofs. Finally, we show that $u^0$ solves the limit differential equation, completing the proof of the theorem.
To see that \( \{ u^\varepsilon \}_{\varepsilon > 0} \) is bounded in the norm, we begin with the weak form

\[
\int_{\Omega} (C^\varepsilon (e (u^\varepsilon)) - C^\varepsilon e^\varepsilon) : e(v) = \langle f, v \rangle + \int_{\Gamma_1} g \cdot v \, dS.
\]

Making the substitution \( v = u^\varepsilon \) in (4.9), and applying the Cauchy-Schwarz inequality, we have

\[
\lambda \| e (u^\varepsilon) \|^2_{L^2} \leq \sqrt{\int_{\Omega} |C^\varepsilon (e (u^\varepsilon))|^2} \cdot \sqrt{\int_{\Omega} |e^\varepsilon|^2} + \| f \|_{W^{-1,2}(\Omega)^3} \cdot \| u^\varepsilon \|_{H^1(\Omega)^3} + \| g \|_{H^{-1/2}(\Gamma_1)^3} \cdot \| u^\varepsilon \|_{H^{1/2}(\Gamma_1)^3}.
\]

Noting that \( \| u^\varepsilon \|_{H^{1/2}(\Gamma_1)^3} \leq c \| e^\varepsilon \|_{H^1(\Omega)^3} \) and \( \| u^\varepsilon \|_{H^1(\Omega)^3} \leq k \| e (u^\varepsilon) \|_{L^2(\Omega)^3} \), where the constant \( k \) takes into account the constant from Korn’s inequality, we have

\[
\| u^\varepsilon \|_{H^1(\Omega)^3} \leq \frac{k}{\lambda} \left( \Lambda \| e^\varepsilon \|_{L^2(\Omega)^3} + k \| f \|_{W^{-1,2}(\Omega)^3} + c \| g \|_{H^{-1/2}(\Gamma_1)^3} \right).
\]

But \( e^\varepsilon \) is piecewise constant, and thus has a finite 2-norm on \( \Omega \). Hence the sequence \( \{ u^\varepsilon \}_{\varepsilon > 0} \) is bounded in the norm, and is thus a compact set by Banach-Alaoglu. Since the set is compact, there is a subsequence, also denoted \( \{ u^\varepsilon \}_{\varepsilon > 0} \), and a \( \mathbf{V}(\Omega) \) function \( u^0 \) with \( u^\varepsilon \to u^0 \) in \( \mathbf{V}(\Omega) \). Passing to yet another subsequence if necessary, the Rellich-Kondrachov compactness theorem gives that \( u^\varepsilon \to u^0 \) in \( L^2(\Omega)^3 \), and \( e (u^\varepsilon) \to e (u^0) \) in \( L^2(\Omega)^{3 \times 3} \).

With these convergence properties, the remaining piece of the theorem is to show that \( u^0 \) solves the limit equation. To accomplish this task, we introduce the unit period cell problem. Given the six basis strains \( e^{ij} \), we have the

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$Q$-periodic solution $w^{ij}$ of

\begin{equation}
\text{div}_y \left( C^{(i)}(y) \left( e_y \left( w^{ij}(y) + e^{ij} \right) \right) \right) = 0,
\end{equation}

where a $y$-subscript indicates differentiation with respect the unit cell coordinate $y$, instead of the macroscopic coordinate $x$. Our goals are to show that $u^0$ solves

\begin{equation}
\int_\Omega \left( C^E \left( e \left( u^0 \right) \right) - H^e \right) : e(v) = \langle f, v \rangle + \int_{\Gamma_1} g \cdot v \, dS,
\end{equation}

and $(C^\epsilon (e(\epsilon u))) - C^\epsilon e^\epsilon \rightarrow (C^E (e(u^0)) - H^e)$, in $L^2(\Omega)^{3 \times 3}$.

The first step is to see that $\{C^\epsilon (e(\epsilon u))\}_{\epsilon > 0}$ is bounded in the $L^2(\Omega)^{3 \times 3}$ norm. From the previous result, there is a positive real number $B$ such that, for every $\epsilon$, we have $\|e(\epsilon u)\|_{L^2} \leq B$. Using the eigen-value constraint on the stiffness matrix, we have the uniform upper bound $\Lambda^2 B$ on the 2-norms of the sequence. Applying the result of Banach-Alaoglu, there is a subsequence weakly converging to a limit $M(x)$ in $L^2(\Omega)^{3 \times 3}$, i.e. $C^\epsilon (e(\epsilon u)) \rightharpoonup M$ in $L^2(\Omega)^{3 \times 3}$.

In addition to the previous sequence, the sequence $\{C^\epsilon e^\epsilon\}_{\epsilon > 0}$ is weakly converging to its average over $Q$, denoted $\langle C e \rangle$, in the weak-\* topology of $L^\infty(\Omega)^{3 \times 3}$. To see this fact, note that $e^\epsilon$ is $Q$-periodic. Thus, passing to a subsequence if necessary, Banach-Alaoglu gives a $L^2$ weak limit. But local periodicity ensures that

\begin{equation}
C^\epsilon e^\epsilon \rightharpoonup \langle C e \rangle(x) = \int_Q C(x, y)e(x, y)dy, \text{ in } L^\infty \text{ weak-\*}.
\end{equation}
Thus passing to a subsequence and taking the $\epsilon$-limit in the weak form (4.9), we have

\begin{equation}
(4.15) \quad \int_{\Omega} (M(x) - \langle Ce \rangle(x)) : e(v) = \langle f, v \rangle + \int_{\Gamma_1} g \cdot v \, dS,
\end{equation}

for every $v \in \mathbf{V}(\Omega)$. The goal now is to identify the quantity $M(x) - \langle Ce \rangle(x)$ in each subdomain $\omega_i$.

Let $\overline{\varepsilon}$ be a given constant strain. Let $w^\overline{\varepsilon}$ be the $Q$-periodic solution of $\text{div} \left( C^{\overline{\varepsilon}}(y) \left( e \left( w^\overline{\varepsilon} \right) + \overline{\varepsilon} \right) \right) = 0$. Also, let $\phi \in C_c^\infty(\omega_i)$ be given. Taking $v = \phi \left( \epsilon w^\overline{\varepsilon} \left( \frac{x}{\epsilon} \right) + \overline{\varepsilon} x \right)$ in (4.9), we have

\begin{equation}
(4.16) \quad \int_{\omega_i} C^\varepsilon \left( e \left( u^\varepsilon \right) - e^\varepsilon \right) : e \left( \phi \left( \epsilon w^\overline{\varepsilon} \left( \frac{x}{\epsilon} \right) + \overline{\varepsilon} x \right) \right) \, dx = \left\langle f, \phi \left( \epsilon w^\overline{\varepsilon} \left( \frac{x}{\epsilon} \right) + \overline{\varepsilon} x \right) \right\rangle.
\end{equation}

Taking $\tilde{v}(x) = \epsilon w^\overline{\varepsilon} \left( \frac{x}{\epsilon} \right) + \overline{\varepsilon} x$ and using the product rule (4.1) on $e(\phi \tilde{v})$, we have

\begin{equation}
(4.17) \quad \int_{\omega_i} C^\varepsilon \left( e \left( u^\varepsilon \right) - e^\varepsilon \right) : \left( \nabla \phi \otimes \tilde{v} \right) \, dx + \int_{\omega_i} C^\varepsilon \left( e \left( u^\varepsilon \right) - e^\varepsilon \right) : \phi e \left( \tilde{v} \right) \, dx = \langle f, \phi \tilde{v} \rangle.
\end{equation}

Noting the symmetry of $C^\varepsilon \left( e \left( u^\varepsilon \right) - e^\varepsilon \right)$, we have

\begin{equation}
(4.18) \quad \int_{\omega_i} C^\varepsilon \left( e \left( u^\varepsilon \right) - e^\varepsilon \right) \nabla \phi \left( \epsilon w^\overline{\varepsilon} \left( \frac{x}{\epsilon} \right) + \overline{\varepsilon} x \right) \, dx \\
+ \int_{\omega_i} C^\varepsilon \left( e \left( u^\varepsilon \right) - e^\varepsilon \right) : \left( e_y \left( w^\overline{\varepsilon} \left( \frac{x}{\epsilon} \right) + \overline{\varepsilon} \right) \phi \right) \, dx = \left\langle f, \phi \left( \epsilon w^\overline{\varepsilon} \left( \frac{x}{\epsilon} \right) + \overline{\varepsilon} x \right) \right\rangle.
\end{equation}

At this point, it is necessary to depart from (4.18) and to consider the quantity $e_y \left( w^\overline{\varepsilon} \left( \frac{x}{\epsilon} \right) + \overline{\varepsilon} \right)$. Thus for each $\mathbf{V}(\Omega)$ function $v$ with support in $\omega_i$,
form

\begin{equation}
\int_{\omega_i} C\left(\frac{x}{\epsilon}\right) \left(e_y (w^\epsilon) \left(\frac{x}{\epsilon}\right) + \bar{e}\right) : e(v) \, dx.
\end{equation}

Expanding with the divergence theorem and rescaling the unit cell problem, we easily deduce that

\begin{equation}
\int_{\omega_i} C^\epsilon \left(e_y (w^\epsilon) \left(\frac{x}{\epsilon}\right) + \bar{e}\right) : e(v) \, dx = 0.
\end{equation}

Now let $\phi \in C^\infty_c(\omega_i)$. Taking $v = \phi u^\epsilon$ and using the product rule (4.1) on $e(\phi u^\epsilon)$, we obtain the identity

\begin{equation}
\int_{\omega_i} C^\epsilon \left(e_y (w^\epsilon) \left(\frac{x}{\epsilon}\right) + \bar{e}\right) \nabla \phi \cdot u^\epsilon \, dx + \int_{\omega_i} C^\epsilon \left(e_y (w^\epsilon) \left(\frac{x}{\epsilon}\right) + \bar{e}\right) : e(u^\epsilon \phi) \, dx = 0.
\end{equation}

Briefly recapping, we have the following convergence properties.

1. $u^\epsilon \to u^0$ in $L^2(\Omega)^3$

2. $e(u^\epsilon) \to e(u^0)$ in $L^2(\Omega)^{3\times3}$

3. $C^\epsilon (e_y (w^\epsilon) \left(\frac{x}{\epsilon}\right) + \bar{e}) \rightharpoonup C^E(x) \bar{e}$ in $L^\infty(\Omega)$ weak-

4. $C^\epsilon (e(u^\epsilon)) - e^\epsilon \to M(x) - (Ce)(x)$ in $L^2(\Omega)^{3\times3}$

5. $\epsilon w^\epsilon \to 0$ in $L^2(\Omega)^{3\times3}$

6. $C^\epsilon e^\epsilon : (e_y (w^\epsilon) \left(\frac{x}{\epsilon}\right) + \bar{e}) \rightharpoonup H^\epsilon(x) : \bar{e}$ in $L^\infty(\Omega)$ weak-

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Looking back to (4.18), we expand the term

\[(4.22) \quad \int_{\omega_i} C^\varepsilon (e (u^\varepsilon) - e^\varepsilon) : \left( e_y (w^\varepsilon) \left( \frac{x}{\varepsilon} \right) + \bar{\varepsilon} \right) \phi \, dx \]

and obtain

\[(4.23) \quad \int_{\omega_i} C^\varepsilon (e (u^\varepsilon)) : \left( e_y (w^\varepsilon) \left( \frac{x}{\varepsilon} \right) + \bar{\varepsilon} \right) \phi \, dx - \int_{\omega_i} C^\varepsilon e^\varepsilon : \left( e_y (w^\varepsilon) \left( \frac{x}{\varepsilon} \right) + \bar{\varepsilon} \right) \phi \, dx.\]

Substituting this expansion, taking limits in (4.18), and using the Div-Curl theorem, we have

\[(4.24) \quad \int_{\omega_i} (M(x) - \langle Ce \rangle(x)) \nabla \phi \cdot \bar{\varepsilon} x \, dx + \int_{\omega_i} e (u^0) : \left( \int_Q C(y) \left( e_y (w^\varepsilon) (y + \bar{\varepsilon}) dy \right) \phi \, dx - \int_{\omega_i} H^\varepsilon : \bar{\varepsilon} \, dx = \langle f, \phi \bar{\varepsilon} x \rangle.\]

Using the product rule (4.1) on \( e (\phi \bar{\varepsilon} x) \) and noting that \( \phi \in C_c^\infty (\omega_i) \), we can rewrite (4.24) as

\[(4.25) \quad \int_{\omega_i} \left[ - (M - \langle Ce \rangle) + C^E \left( e (u^0) \right) + H^\varepsilon \right] : \bar{\varepsilon} \phi \, dx = 0,\]

for every \( C_c^\infty (\omega_i) \) function \( \phi \). Thus for every \( 3 \times 3 \) strain \( \bar{\varepsilon} \), we have

\( - (M - \langle Ce \rangle) + C^E \left( e (u^0) \right) + H^\varepsilon \) : \( \bar{\varepsilon} = 0 \), for almost every \( x \) in \( \omega_i \). Hence

\( M - \langle Ce \rangle = C^E \left( e (u^0) \right) - H^\varepsilon \), completing the proof. \( \square \)
4.3 Principal Corrector Result

In this section, we prove Theorem 3.2.1, the Principal Corrector Result. Before giving the proof, we introduce the $Q$-periodic function $\eta$ that gives the elastic response to the fluctuating prestress. Without loss of generality, we restrict our attention to a subdomain $\Omega_i$. The $Q$-periodic function $\eta$ is the solution of

\[(4.26) \quad \text{div} \left( C^{(i)}(y) \left( e(\eta) - e^{(i)}(y) \right) \right) = 0.\]

We now prove Theorem 3.2.1.

Proof. Consider the quantity

\[(4.27) \quad \int_{\Omega_i} \phi C^e(x) \left( e(u^e) - (P^e \overline{\varphi} + e(\eta^e)) \right) : \left( e(u^e) - (P^e \overline{\varphi} + e(\eta^e)) \right) \, dx.\]

We now identify good div-curl quantities are rearrange terms in (4.27). The theorem is then proved by passing to the limit. The quantities with good divergence are $C^e(P^e \overline{\varphi})$, $C^E(e(u^e))$, $C^e(e(\eta^e) - e^e)$, and $C^e(e(u^e) - e^e)$. The quantities with good curl are $e(w^{ij}) \left( \frac{x_j}{\varepsilon} \right) + e^{ij}$, $e(\eta^e)$, $e(u^e)$, and $e(w^{(x)ij}) \left( \frac{x_j}{\varepsilon} \right)$.

Note also that $P^e \overline{\varphi} = \sum_{i,j=1}^{3} \overline{\varphi}_{ij}(x) P^e e^{ij}$.

Adding and subtracting the term $e^e$ in (4.27) gives

\[(4.28) \quad \int_{\Omega_i} \phi C^e(x) \left( e(u^e) - e^e - (P^e \overline{\varphi} + e(\eta^e) - e^e) \right) : \left( e(u^e) - (P^e \overline{\varphi} + e(\eta^e)) \right) \, dx.\]
Expanding (4.28) yields the nine terms

\[(4.29) \quad \int_{\omega_i} \phi C^e (e (u^\tau) - e^\tau) : e (u^\tau) \, dx - \sum_{i,j=1}^{3} \int_{\omega_i} \phi \varphi_{ij} C^e (e (u^\tau) - e^\tau) : P^e e^{ij} \, dx \]

\[- \int_{\omega_i} \phi C^e (e (u^\tau) - e^\tau) : e (\eta^\tau) \, dx - \sum_{i,j=1}^{3} \int_{\omega_i} \phi \varphi_{ij} C^e (P^e e^{ij}) : e (u^\tau) \, dx \]

\[+ \sum_{i,j,k,l=1}^{3} \int_{\omega_i} \phi \varphi_{ij} \varphi_{kl} C^e (P^e e^{ij}) : P^{e kl} \, dx + \sum_{i,j=1}^{3} \int_{\omega_i} \phi \varphi_{ij} C^e (P^e e^{ij}) : e (\eta^\tau) \, dx \]

\[- \int_{\omega_i} \phi C^e (e (\eta^\tau) - e^\tau) : e (u^\tau) \, dx + \sum_{i,j=1}^{3} \int_{\omega_i} \phi \varphi_{ij} C^e (e (\eta^\tau) - e^\tau) : P^e e^{ij} \, dx \]

\[+ \int_{\omega_i} \phi C^e (e (\eta^\tau) - e^\tau) : e (\eta^\tau) \, dx. \]

Applying the Div-Curl Theorem to each term and passing to the limit in (4.29) gives

\[(4.30) \quad \int_{\omega_i} \phi C^E (e (u^0) - \varphi) : (e (u^0) - \varphi) \, dx - \int_{\omega_i} \phi H^E : e (u^0) \, dx \]

\[+ \int_{\omega_i} \phi H^E : \varphi \, dx - \int_{\omega_i} \phi \left( \int_{Q} C(y) (e(\eta)(y) - e(y)) \, dy \right) : e (u^0) \, dx \]

\[+ \int_{\omega_i} \phi \left( \int_{Q} C(y) (e(\eta)(y) - e(y)) \, dy \right) : \varphi \, dx. \]

Since the first term is the desired result, we show the other terms cancel.

From equation (4.26), we note for every Q-periodic function w, that

\[(4.31) \quad \int_{Q} C^{(i)}(y) (e(\eta)(y) - e^{(i)}(y)) : e(w)(y) \, dy = 0. \]
Now consider the term

\[(4.32) \quad \int_Q C^{(i)}(y) \left( e(\eta) - e^{(i)} \right) : \left( e \left( w^{ij} \right) + e^{ij} \right) dy. \]

Using the previous observation, we rewrite (4.32) as

\[(4.33) \quad \int_Q C^{(i)}(y) \left( e(\eta) - e^{(i)} \right) : e^{ij} dy. \]

On the other hand, using result (4.12) and the symmetry of $C^{(i)}(y)$, the quantity (4.32) can also be be rewritten

\[(4.34) \quad - \int_Q C^{(i)}(y) \left( e \left( w^{ij} \right) + e^{ij} \right) : e^{(i)}(y) dy = -H^\varepsilon. \]

The result follows on substitution of (4.34) into (4.30). \[\square\]

As was seen in Section 3.2, if $u^0$ is a $C^\infty(\omega_i)^3$ function, then we get immediately that the actual strain can be made arbitrarily close to the macroscopic strain in the composite. In general, however, we have that $u^0$ is only a $H^1(\omega_i)$ function. Thus we approximate the $H^1(\omega_i)$ function $u^0$ by a $C^\infty(\omega_i)^3$ function to achieve the general result.

To begin, let $\delta > 0$ be given. Let $z^\varepsilon$ be given by

\[(4.35) \quad z^\varepsilon = e \left( u^\varepsilon \right) - \left( P^\varepsilon \left( e \left( u^0 \right) \right) + e \left( \eta^\varepsilon \right) \right). \]

Then our goal is to show that for every $\bar{\omega}$ which is compactly contained in $\omega_i$, 

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that

\[
\lim_{\varepsilon \to 0} \| z^\varepsilon \|_{L^1(\omega)} \leq \delta, \tag{4.36}
\]

which would give that

\[
\lim_{\varepsilon \to 0} \| z^\varepsilon \|_{L^1(\omega_1)} = 0. \tag{4.37}
\]

Note that convergence in the 1-norm gives convergence in measure. Thus the result of this argument gives results about the volume of the highly-stressed region of the composite.

Since \( H^1(\omega_i) \) is the completion of \( C^\infty(\omega_i)^3 \) in the \( W^{1,2}(\omega_i)^3 \) norm, there is a \( C^\infty(\omega_i)^{3\times3} \) function \( \varphi \) such that

\[
\| e \left( u^0 \right) - e \left( \varphi \right) \|_{L^2(\omega_i)}^2 < \gamma, \tag{4.38}
\]

with \( 0 < \gamma \leq \delta^2 \left( \sqrt{\frac{\Lambda|\hat{\omega}|}{\lambda}} + \sqrt{k|\omega_i|} \cdot \| e_y \left( w^{ij} \right)(y) + e^{ij}_y \|_{L^2(Q)} \right)^{-2} \),

and \( k \) is a constant. Next, we write \( z^\varepsilon = z_1^\varepsilon - z_2^\varepsilon \), where

\[
z_1^\varepsilon = e \left( u^\varepsilon \right) - (P^\varepsilon \varphi + e \left( \eta^\varepsilon \right)) \quad \text{and} \quad z_2^\varepsilon = P^\varepsilon \left( e \left( u^0 \right) \right) - P^\varepsilon \varphi. \]

Then given \( \hat{\omega} \) as above, we have \( \| z^\varepsilon \|_{L^1(\omega)} \leq \| z_1^\varepsilon \|_{L^1(\omega)} + \| z_2^\varepsilon \|_{L^1(\omega)} \).

We now get upper bounds on \( \lim_{\varepsilon \to 0} \| z_1^\varepsilon \|_{L^1(\omega)} \) and \( \lim_{\varepsilon \to 0} \| z_2^\varepsilon \|_{L^1(\omega)} \). To begin, choose \( \phi \in C^\infty_0(\omega_i) \) such that \( \phi = 1 \) on \( \hat{\omega} \) and \( 0 \leq \phi \leq 1 \) on \( \omega_i \setminus \hat{\omega} \).

Using the Cauchy inequality, we have that

\[
\| z_1^\varepsilon \|_{L^1(\omega)} \leq \sqrt{\| \hat{\omega} \|} \cdot \| z_1^\varepsilon \|_{L^2(\omega)}. \tag{4.39}
\]

35
Using the eigen-value constraint on $C^{e}$ and the previous result to estimate $\|z_{1}^{\epsilon}\|_{L^{2}(\bar{\omega})}$, we see that

\[
\lim_{\epsilon \to 0} \|z_{1}^{\epsilon}\|_{L^{1}(\bar{\omega})} \leq \sqrt{\frac{\gamma|\Lambda|}{\lambda}}.
\]

(4.40)

Now since $\bar{\omega} \subset \omega_{i}$, we use linearity, the triangle inequality, and Cauchy inequality to see that

\[
\|P^{e}(e(u^{0})) - P^{e}v\|_{L^{1}(\bar{\omega})} \leq \sum_{i,j=1}^{3} (\|P^{e}e^{ij}\|_{L^{2}(\omega_{i})} \cdot \|(e(u^{0}))_{ij} - v_{ij}\|_{L^{2}(\omega_{i})})
\]

By definition of $\gamma$, we have $\|(e(u^{0}))_{ij} - v_{ij}\|_{L^{2}(\omega_{i})} < \sqrt{\gamma}$. Thus we estimate $\|P^{e}e^{ij}\|_{L^{2}(\omega_{i})}^{2}$.

To estimate this 2-norm, we must first do some work. Denote by $\{Q_{l}^{f}\}_{l \geq 1}$ a countable cover of $\mathbb{R}^{3}$ by $\epsilon^{3}$-volume unit cells formed by placing the center of $Q_{1}^{f}$ at the origin, and then stacking unit cells around it. Let $Q^{e} = \{Q_{l}^{f}, l \geq 1 \mid Q_{l}^{f} \cap \Omega \neq \emptyset\}$. Since $\omega_{i}$ is bounded, there exists a ball $B^{e}$ such that $\bigcup_{Q^{f} \in Q^{e}} Q_{l}^{f} \subset B^{e}$. Now let $B^{Q^{e}}$ be a ball such that $B^{e} \subset B^{Q^{e}}$ and $|B^{Q^{e}}| = k|\omega_{i}|$, for some positive integer $k$. Then the set $Q^{e}$ has at most $\frac{k|\omega_{i}|}{\epsilon^{3}}$ elements. Recalling the definition of the corrector $P(y)$, and noting that for $Q_{l}^{f} \in Q^{e}$, we have $\|P^{e}e^{ij}\|_{L^{2}(Q^{f})} = \|e_{y}(w^{ij})(y) + e^{ij}\|_{L^{2}(Q^{f})}$, then

\[
\|P^{e}e^{ij}\|_{L^{2}(\omega_{i})}^{2} \leq k|\omega_{i}| \cdot \|e_{y}(w^{ij})(y) + e^{ij}\|_{L^{2}(Q^{f})}^{2}
\]

(4.42)
which is a bound independent of \( \epsilon \). Thus we have

\[
\lim_{\epsilon \to 0} \| z' \|_{L^1(\bar{\omega})} \leq \sqrt{k \gamma |\omega|} \cdot \| e_y (w^{ij}) (y) + e^{ij} \|_{L^2(Q)}.
\]

Combining (4.40) and (4.43), we see that

\[
\lim_{\epsilon \to 0} \| z' \|_{L^1(\bar{\omega})} \leq \sqrt{\gamma \left( \frac{\Lambda |\bar{\omega}|}{\lambda} + \sqrt{k |\omega|} \cdot \| e_y (w^{ij}) (y) + e^{ij} \|_{L^2(Q)} \right)} < \delta,
\]

completing the argument.

### 4.4 Macroscopic Failure Criteria

In this section, we establish Theorem 3.3.1 and Theorem 3.3.2. We begin with the **Dilatational Constraint for the Macroscopic Strength Domain**.

**Proof.** Note that if \( \mathcal{L} (z^\epsilon) \leq 0 \), then \( \mathcal{L} (e(u^\epsilon)) < t \). Thus assume \( \mathcal{L} (z^\epsilon) > 0 \). Since \( z^\epsilon \) is independent of the variable \( y \), we have

\[
\mathcal{L} (e(u^\epsilon)) = \mathcal{L} \left( P^\epsilon \left( e(u^0) \right) + e(\eta^\epsilon) + z^\epsilon \right) \leq \mathcal{L}^M \left( e(u^0) \right) + \mathcal{L} (z^\epsilon).
\]

By assumption, there is a positive real number \( \gamma \) such that

\[
\mathcal{L}^M \left( e(u^0) \right) + \gamma < t.
\]

Applying Egoroff’s Theorem, for every positive number \( \delta \) there is a set \( A \subset \bar{\omega} \) with \( |\bar{\omega} \setminus A| < \delta \) such that \( e(u^\epsilon) \) converges to \( P^\epsilon \left( e(u^0) \right) + e(\eta^\epsilon) \)
uniformly on $A$. Hence there is a positive number $\epsilon_0$ such that whenever $\epsilon < \epsilon_0$, we have $|\mathcal{L}(z^\epsilon)| < \gamma$ on $A$. Choosing $\epsilon_0$ in this way, the result follows.

We now prove the *Deviatoric Constraint for the Macroscopic Strength Domain*.

**Proof.** Note first that $\Pi(\cdot) \geq 0$. Hence $\Pi(z^\epsilon) \geq 0$. Since $z^\epsilon$ is independent of the variable $y$, we have

$$
(4.47) \quad \Pi(e(u^\epsilon)) = \Pi(P^\epsilon(e(u^0)) + e(\eta^\epsilon) + z^\epsilon) \leq \Pi^M(e(u^0)) + \Pi(z^\epsilon).
$$

By assumption, there is a positive real number $\gamma$ such that

$$
(4.48) \quad \Pi^M(e(u^0)) + \gamma < t.
$$

Applying Egoroff’s Theorem, for every positive number $\delta$ there is a set $A \subset \bar{\omega}$ with $|\bar{\omega} \setminus A| < \delta$ such that $e(u^\epsilon)$ converges to $P^\epsilon(e(u^0)) + e(\eta^\epsilon)$ uniformly on $A$. Hence there is a positive number $\epsilon_0$ such that whenever $\epsilon < \epsilon_0$, we have $|\Pi(z^\epsilon)| < \gamma$ on $A$. Choosing $\epsilon_0$ in this way, the result follows.

The proofs of Corollaries 3.3.3 and 3.3.4 are identical to the proofs of Theorems 3.3.1 and 3.3.2, respectively.
Chapter 5

The Multi Scale Strain Analysis Method

The multi scale strain analysis method is a numerically inexpensive method for strain assessment that allows one to predict the failure of prestressed heterogeneous materials based on the homogenized strain field. The equilibrium equation of the prestressed heterogeneous system is given by

\[
-\text{div}\{C_{ijkl}^\varepsilon [\varepsilon(u^\varepsilon)_{kl} - \varepsilon_{kl}^\varepsilon]\} = f \quad \text{in } \Omega
\]

\[
\begin{align*}
\varepsilon^\varepsilon &= 0 & \text{on } \Gamma_0 \\
\sigma^\varepsilon \mathbf{n} &= g & \text{on } \Gamma_1
\end{align*}
\]

Here \(f\) represents the body force and the function \(g\) specifies a normal traction on a portion of the boundary. The multi scale strain analysis method consists of the following three steps.
5.1 Determine the Effective Properties

The first step is to determine the effective elasticity tensor and the effective prestress for the heterogeneous material in each subdomain. For periodic microstructures, this task is accomplished by analyzing the unit period cell \((Q)\) of each subdomain. Determination of the effective elasticity tensor requires solving six problems on each unit cell, corresponding to unit loads for each of the six basis strains. For a given subdomain \(\omega_i\), the six equations are given by

\[
\text{div}\{C^{(i)}(y)\left(e\left(w^{ij}\right) + \overrightarrow{e}^{ij}\right)\} = 0,
\]

where the solutions \(w^{ij}\) are \(Q\)-periodic and \(\overrightarrow{e}^{ij}\) is the unit load for one of the six basis strains. The effective elasticity tensor \((C^E)\) in \(\omega_i\) is computed from these solutions by averaging over the unit cell according the the formula

\[
C^E_{ijkl} = \frac{1}{|Q|} \int_Q \{C^{(i)}_{ijmn}(y) \left(\left(e\left(w^{kl}\right)\right)_{mn} + \overrightarrow{e}^{ij}_{mn}\right)\} dy.
\]

The effective prestress is also computed from the basis solutions \(w^{ij}\). The effective prestress \((H^e)\) in \(\omega_i\) is computed from these solutions by averaging over the unit cell according to the formula

\[
H^e_{ij} = \frac{1}{|Q|} \int_Q \{C^{(i)}_{mnop}(y) \left(\left(e\left(w^{ij}\right)\right)_{op} + \overrightarrow{e}^{ij}_{op}\right) : e^{(i)}_{mn}(y)\} dy,
\]

where \(e^{(i)}_{mn}(y)\) is the inelastic strain (see equation (2.4)).
A seventh problem on each unit cell is required for Section 5.3. The inelastic strain problem is given by

\[
\text{div} \left\{ C^{(i)}(y) \left[ e(\eta) - e^{(i)}(y) \right] \right\} = 0,
\]

where \( e^{(i)}(y) \) is the inelastic strain (see equation (2.4)) and \( \eta \) is \( Q \)-periodic.

### 5.2 Macroscopic Problem

The next step is to solve the macroscopic problem using the effective properties computed in Section 5.1. Here the heterogeneous microstructure is replaced by a homogeneous continua having the homogenized properties of the heterogeneous material. The displacement solution to the macroscopic problem is the homogenized displacement vector field, denoted \( u^0 \). The macroscopic problem is given by the equation

\[
\begin{align*}
-\text{div} \left( C^E (e (u^0)) - H^c \right) &= f \quad \text{in } \Omega \\
u^0 &= 0 \quad \text{on } \Gamma_0 \\
(C^E (e (u^0)) - H^c) \cdot n &= g \quad \text{on } \Gamma_1
\end{align*}
\]

### 5.3 Compute the Macroscopic Failure Criteria

The final step is to bound the microscopic failure criteria in terms of the macroscopic strain field \( e (u^0) \) computed in Section 5.2. To recover the information about the microstructure that was lost in the homogenization process,
we correct the macroscopic strain field using the basis solutions \( w^{ij} \) and basis strains \( \varepsilon^{ij} \) of Section 5.1. The corrected macroscopic strain is given by

\[
P(y) \left( e\left( u^0 \right) \right) = \sum_{i,j=1}^{3} \left( e\left( u^0 \right) \right)_{ij} P(y)\varepsilon^{ij},
\]

where

\[
P(y)\varepsilon^{ij} = e\left( w^{ij} \right) (y) + \varepsilon^{ij}.
\]

The microscopic failure criteria is given by two invariants of the strain tensor. The dilatational and deviatoric microscopic failure criteria are given by the functions

\[
\mathcal{L} \left( e\left( u^\varepsilon \right) \right) = \text{tr} \left( e\left( u^\varepsilon \right) \right)
\]

and

\[
\Pi \left( e\left( u^\varepsilon \right) \right) = \frac{3}{2} |e\left( u^\varepsilon \right)|^2 - \frac{1}{2} \left( \mathcal{L} \left( e\left( u^\varepsilon \right) \right) \right)^2,
\]

respectively. Here \( e\left( u^\varepsilon \right) \) is the actual strain in the heterogeneous material, and

\[|e\left( u^\varepsilon \right)|^2 = \sum_{i,j=1}^{3} (e\left( u^\varepsilon \right))_{ij}^2.\]

The macroscopic failure criteria are given by the formulas

\[
\mathcal{L}^M \left( e\left( u^0 \right) \right) = \sup_{y \in Q} \mathcal{L} \left( P(y) \left( e\left( u^0 \right) \right) + e\left( \eta \right) \right)
\]
and

\[ \Pi^M (e(u^0)) = \sup_{y \in Q} \Pi \left( P(y) (e(u^0)) + e(\eta) \right). \]

Theorems 3.3.1 and 3.3.2 ensure these macroscopic failure criteria are accurate for small enough microstructures (see Section 3.3). Thus for small enough \( \epsilon \), if \( \mathcal{L}^M (e(u^0)) < t \) on \( A \), then \( \mathcal{L} (e(u^0)) < t \) on \( A \), except for a controllably small set (similarly for \( \Pi^M (e(u^0)) \)). This method provides a way to determine the macroscopic strength domain of the heterogeneous material.
Chapter 6

Multi Scale Analysis of Free Edge Composite Laminates

In this chapter, we compare the macroscopic failure criteria with a direct numerical simulation of fiber reinforced composite laminates. A laminate is a composite having multiple layers. Each layer consists of fibers having possibly different orientations. A lamina refers to a single layer of the layered composite. We consider fiber orientations lying strictly in the \( x-y \) plane. The fiber orientation angle is given with respect to the \( x \)-axis. The specific sequence of fiber orientation angles associated with each layer in a laminate is listed starting with the top most layer and proceeding downward. For example, \([0/90/0]\) means a three-layer composite having top layer with fiber orientation angle \( 0^\circ \), followed by a layer with fiber orientation angle \( 90^\circ \), followed by a bottom layer with fiber orientation angle \( 0^\circ \). For future reference, a \( s \) subscript implies a symmetry condition. Thus \([0/ +45/-45/90]_s\) denotes an eight-layer composite with symmetry between the two \( 90^\circ \) layers.

We show that the macroscopic failure criteria are accurate for subdomains where the fiber microstructure exhibits uniform periodicity. We consider free
edge problems of both the [0/matrix/0] (unidirectional/matrix) and [0/90/0] (crossply) construction.

The purpose of this problem is to verify the accuracy of the multi scale strain analysis method for a model of an actual fiber reinforced composite. Since our homogenization scheme is based on periodic homogenization, we created full 3D models that have periodic fiber geometries on the lamina level. The multi scale strain analysis method is found to be quite accurate in the central areas of a ply, where the microstructure exhibits uniform periodicity. This study, however, is also directed to document and assess the accuracy of the method in the vicinity of critical regions of the composite where the microstructure does not exhibit periodicity. These regions are of great importance to structural designers because of the strain concentrations created by abrupt changes in mechanical properties (e.g. ply interfaces) or external boundary conditions (e.g. free edges). The calculations within are carried out by the FORTRAN-based code B-Spline Analysis method (BSAM), which has been developed by the University of Dayton Research Institute, under contract to the Air Force Research Lab, Materials Directorate, Non-Metallic Materials Division.

Because strain fields can be large near a free edge, free surfaces are generally avoided whenever possible in component design. They have, however, remained a standard problem on which to test strain field predictions. As a result, much experimentation and numerical simulation has been performed on free edge composite laminates. In 1974, Pagano and Rybicki [19] introduced a set of free edge boundary value problems. In light of technological advances,
Pagano and Yuan [20] revisited the problems. Although they studied large-radius fibers, their work provides the motivation for this study.

6.1 Unit Cell and Constitutive Material Properties

In this thesis, representative volume element (RVE) models for fiber reinforced composite laminates are developed. The unit cell model is a unidirectional composite of IM7/5250-4 materials (see Figure 6.1). The unit cell for this model is cubic. The unit cell contains a single fiber occupying 60% of the cell by volume. Taking the fiber direction along the $x$-axis, the thermo-mechanical properties of the IM7 fiber (graphite) and 5250-4 matrix (bismaleimide), are summarized in Table 6.1. The constitutive material properties can be found in [21]. The values obtained from our homogenization process are listed as Composite properties in Table 6.1.

![RVE Model of IM7/5250-4 Composite](image)

Figure 6.1: RVE Model of IM7/5250-4 Composite
Table 6.1: Thermomechanical Properties of IM7 Fiber, 5250-4 Matrix, and IM7/5250-4 Composite

<table>
<thead>
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<th>IM7 Fiber</th>
<th>5250-4 Matrix</th>
<th>Composite</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{xx}$</td>
<td>276 GPa</td>
<td>3.45 GPa</td>
<td>167 GPa</td>
</tr>
<tr>
<td>$E_{yy}, E_{zz}$</td>
<td>27.6 GPa</td>
<td>3.45 GPa</td>
<td>11.0 GPa</td>
</tr>
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<td>$\nu_{xy}, \nu_{xz}$</td>
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<td>0.35</td>
<td>0.32</td>
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<tr>
<td>$\nu_{yz}$</td>
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<td>0.35</td>
<td>0.51</td>
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<tr>
<td>$G_{xy}, G_{xz}$</td>
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<td>1.28 GPa</td>
<td>5.33 GPa</td>
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<tr>
<td>$G_{yz}$</td>
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<td>1.28 GPa</td>
<td>2.72 GPa</td>
</tr>
<tr>
<td>$\alpha_{xx}$</td>
<td>$-3.6 \times 10^{-6}/^\circ$C</td>
<td>$46.8 \times 10^{-6}/^\circ$C</td>
<td>$3.72 \times 10^{-6}/^\circ$C</td>
</tr>
<tr>
<td>$\alpha_{yy}, \alpha_{zz}$</td>
<td>$5.04 \times 10^{-6}/^\circ$C</td>
<td>$46.8 \times 10^{-6}/^\circ$C</td>
<td>$24.3 \times 10^{-6}/^\circ$C</td>
</tr>
</tbody>
</table>

6.2 Comparison of Multi Scale Analysis with Full Numerical Simulation

To provide comparison to the multi scale strain analysis method, fully three-dimensional models of laminate sections with explicit modeling of fiber and matrix were constructed. Figure 6.2 illustrates these 3D models. The size of the microstructure, $\epsilon$, is given by the edge length of the cubic RVE. For this problem, $\epsilon = 11.4 \, \mu$m. The dimensions are one unit cell width ($\epsilon$) deep in the $x$-direction, 10$\epsilon$ in the $y$-direction, and 9$\epsilon$ in the $z$-direction. The 0$^\circ$ ply on top has $z$-thickness of 6$\epsilon$. In Figure 6.2(a) and 6.2(b), the fibers are represented by red. Note that the surface $z = 0$ corresponds to the ply interface. The boundary conditions were applied as follows. On the $x = 0$ and $x = \epsilon$ surfaces, periodicity conditions were applied along with 1% tensile strain loading in the $x$-direction. Displacement conditions $u_z = 0$ on the surface $z = -3\epsilon$ (bottom of the model) and $u_y = 0$ on the surface $y = 10\epsilon$ were applied so that the 3D
model corresponds to a quarter model of an infinitely long (in the $x$-direction) three-ply symmetric laminate. The remaining two surfaces were free edges.

To compute the macroscopic failure criteria, we follow the steps outlined in Chapter 5, Sections 5.1 - 5.3. We first compute the effective stiffness $C^E(x)$ and effective prestress $H^e(x)$ for each lamina. We then solve the macroscopic problem to obtain the homogenized strain field $e(u^0)$. Last, we compute the macroscopic failure criteria $L^M (e(u^0))$ and $\Pi^M (e(u^0))$.

The fiber geometry used for the multi scale analysis macroscopic problem was similarly composed as a quarter model of an infinitely long three-ply symmetric laminate. The model dimensions were identical to those for the direct simulation except for the depth in the $x$-direction. Instead of a single unit cell length, a length of $10^4\epsilon$ was used to approximate an infinite sheet. Boundary conditions were applied identically except again in the $x$-direction, where the 1% tensile strain was applied via displacement on the surfaces $x = 0$ and $x = 10^4\epsilon$. A schematic of the free edge problem is provided in Figure 6.3.
6.3 Direct Numerical Simulation and Macroscopic Failure Criteria

Here we compare the direct numerical simulations with the multi scale strain analysis method. The sample points for the macroscopic failure criteria $\mathcal{L}^M (e(u^0))$ and $\Pi^M (e(u^0))$ are given by horizontal lines in the $y-z$ plane. The points are chosen in the middle of the laminate (with respect to $x$). Thus $x = 5c \times 10^3$. Sample points for the direct simulation are shown in green in Figure 6.4. In general, they follow the same path as the points for the multi scale bounds. When they reach a fiber/matrix interface, however, they follow the circular interface to remain in the matrix phase of the material. The $x$-coordinate of each sample point for the direct numerical simulation is $0.5c$, corresponding to the middle (with respect to $x$) of the laminate.
We plot the invariants of the strain tensor as a function of the distance from the free edge surface \( y = 0 \). These plots are shown at five different horizontal surfaces in the 0° ply (\( z = 0, 0.5\varepsilon, 1.5\varepsilon, 2.5\varepsilon, \) and \( 3\varepsilon \)) in Figures 6.5 - 6.14. We compare the invariants of the actual strain in the composite (\( \mathcal{L}(e(u^e)) \) and \( \Pi(e(u^e)) \)) with the macroscopic failure criteria \( \mathcal{L}^M(e(u^0)) \) and \( \Pi^M(e(u^0)) \).

Referring to Figures 6.5(a), 6.7(a), 6.9(a), 6.11(a), and 6.13(a), the macroscopic failure criteria \( \mathcal{L}^M(e(u^0)) \) remains close to \( \mathcal{L}(e(u^e)) \) everywhere in the unidirectional/matrix laminate. In these figures, \( \mathcal{L}^M(e(u^0)) \) is seen to over estimate the microscopic failure criteria \( \mathcal{L}(e(u^e)) \) in the unidirectional/matrix laminate, accounting for the strain concentrations on the fiber perimeter. In Figures 6.5(b), 6.7(b), 6.9(b), 6.11(b), and 6.13(b), the macroscopic failure criteria \( \mathcal{L}^M(e(u^0)) \) is seen to lie above \( \mathcal{L}(e(u^e)) \) in the crossply laminate, except for the one fiber nearest the free edge and ply interface.

Referring to Figures 6.6(a), 6.8(a), 6.10(a), 6.12(a), and 6.14(a), \( \Pi^M(e(u^0)) \) lies above \( \Pi(e(u^e)) \) everywhere in the unidirectional/matrix laminate. In Figures 6.6(b), 6.8(b), 6.10(b), 6.12(b), and 6.14(b), the second invariant (\( \Pi(e(u^e)) \)) is under predicted by \( \Pi^M(e(u^0)) \). The characteristic under pre-
diction of the second invariant in the crossply model requires further investigation. In Figures 6.6(b), 6.8(b), 6.10(b), 6.12(b), and 6.14(b), the largest invariant values are seen to occur on the fiber/matrix interface. It is these locations where the second invariant is under predicted. It is plausible that the anisotropy effects of the 90° ply simply create a larger transition (boundary) layer between plies (as in [22] and [20]). Finite calculation resources were a limiting factor in this study. Since ply thickness was only six unit cells thick (whereas a normal amount would be roughly twenty), increasing ply thickness could answer this question about the boundary layer and the second invariant. Except on the ply interface itself, shrinking the size of the microstructure would also make the bound accurate (as shown by Corollaries 3.3.3 and 3.3.4). Regardless, more investigation needs to be done concerning the second invariant near a ply interface.

6.4 Summary

The multi scale strain analysis method gives accurate predictions in central areas of the ply, where the microstructure exhibits uniform periodicity. In addition, the multi scale bounds hold in the proximity a free edge. The multi scale bounds are accurate within two fibers of the free edge. From the size of the microstructure, this implies the multi scale bounds are accurate to within 22.8 μm (0.0008976 in.) of the free edge surface.
Figure 6.5: 1\textsuperscript{st} Strain Invariant at $z = 3\epsilon$
Figure 6.6: 2\textsuperscript{nd} Strain Invariant at $z = 3\epsilon$
Figure 6.7: 1\textsuperscript{st} Strain Invariant at \( z = 2.5\varepsilon \)
Figure 6.8: 2\textsuperscript{nd} Strain Invariant at $z = 2.5\epsilon$
Figure 6.9: 1st Strain Invariant at $z = 1.5\epsilon$
Figure 6.10: 2\textsuperscript{nd} Strain Invariant at $z = 1.5\epsilon$
Figure 6.11: 1st Strain Invariant at $z = 0.5\epsilon$
Figure 6.12: 2\textsuperscript{nd} Strain Invariant at $z = 0.5\epsilon$
Figure 6.13: 1\textsuperscript{st} Strain Invariant at $z = 0$

(a) Unidirectional/Matrix

(b) Crossply
Figure 6.14: 2\textsuperscript{nd} Strain Invariant at $z = 0$

(a) Unidirectional/Matrix

(b) Crossply
Chapter 7

Multi Scale Analysis of a Symmetric Laminate with an Open Hole

The results of Chapter 6 verify the accuracy of the multi scale bounds in volumes of uniformly periodic microstructure, as well as in the vicinity of a free edge. This chapter illustrates the difference between the strain invariants of the homogenized strain field and the multi scale bounds. More importantly, it highlights the effects of the prestress on the composite. We finish by comparing the multi scale strain analysis method for two different fiber reinforced microstructures. We examine a symmetric 8-ply open hole composite laminate system in uniaxial tension.

A bolted joint is one standard way to join separate components in a structure. Use of these devices, however, complicates the strain field in the vicinity of the bolt or hole. This phenomenon is evidenced by the fact that fastened joints are frequently the source of structural failure and load carrying capacity loss in aerospace vehicles [23]. The goal is to study the strain field in the plies and in the vicinity of the hole for a simple rectangular plate with an open hole.
The surface of the hole is a free edge. Thus it is important to get accurate predictions of the strain field in this locale. Chapter 6 shows that strain fields can be large at ply interfaces. The combination of a free edge and ply interface generates out-of-plane strains, leading to delamination failure. Because of their use in structural applications, strain and stress fields for filled holes, pinned holes, and open holes have been widely studied. In [24], Iarve examined layered composites under uniaxial tension. His method of polynomial spline approximation accurately predicted interlaminar stresses. In [25], Iarve and Pagano once again examine composite laminates under uniaxial loading. Using polynomial B-spline approximations for the displacement, the method allows determination of the coefficient of the singular term in the vicinity of the hole edge.

7.1 Specimen Study

For this problem, we consider a 4-in. x 1-in. rectangular plate (see Figure 7.1). Each ply in the laminate has thickness 0.005 in., and thus the 8-ply specimen is 0.04 in. thick in the z-direction. The plate is punched through with a 0.25 in. diameter hole in the center of the plate. The 8-ply laminate is an IM7/977-3 \([0/ + 45/ - 45/90]_s\) composite. The symmetry plane corresponds to the surface \(z = 0\). Taking the fiber direction along the \(x\)-axis, the thermo-mechanical properties of the IM7 fiber and 977-3 matrix are summarized in Table 7.1. The values obtained from our homogenization process are listed as Composite properties in Table 7.1. We note that the unit Msi is Megapounds.
The problem is modeled as a 4-ply laminate (see Figure 7.2). A symmetry condition ensures correspondence to the 8-ply laminate. The specimen is subjected to a 1% tension strain in the $x$-direction, applied as a displacement condition ($u_x = 0.04$) on the surface $x = 2$. In addition, the $y$ displacement was restricted on the surface $x = 2$. Similarly, the $x$ displacement and $y$ displacement were restricted on the surface $x = -2$. Midplane symmetry is provided for the laminate by the boundary condition $u_z = 0$ on the surface $z = 0$. 

Figure 7.1: $x - y$ Schematic of Open Hole Plate

per square inch (1 Msi = 10$^6$ psi). Moreover, the stress-free temperature of the composite is 350$^\circ$ F. Thus to take into account the prestress, $\Delta T = -270^\circ$ F.
### Table 7.1: Thermomechanical Properties of IM7 Fiber, 977-3 Matrix, and IM7/977-3 Composite

<table>
<thead>
<tr>
<th></th>
<th>IM7 Fiber</th>
<th>977-3 Matrix</th>
<th>Composite</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{xx}$</td>
<td>39.3 Msi</td>
<td>0.55 Msi</td>
<td>23.8 Msi</td>
</tr>
<tr>
<td>$E_{yy}, E_{zz}$</td>
<td>2.50 Msi</td>
<td>0.55 Msi</td>
<td>1.43 Msi</td>
</tr>
<tr>
<td>$\nu_{xy}, \nu_{xz}$</td>
<td>0.32</td>
<td>0.36</td>
<td>0.33</td>
</tr>
<tr>
<td>$\nu_{yz}$</td>
<td>0.20</td>
<td>0.36</td>
<td>0.34</td>
</tr>
<tr>
<td>$G_{xy}, G_{xz}$</td>
<td>4.00 Msi</td>
<td>0.202 Msi</td>
<td>0.718 Msi</td>
</tr>
<tr>
<td>$G_{yz}$</td>
<td>1.20 Msi</td>
<td>0.202 Msi</td>
<td>0.427 Msi</td>
</tr>
<tr>
<td>$\alpha_{xx}$</td>
<td>$-0.60 \times 10^{-6} / ^\circ F$</td>
<td>$32.0 \times 10^{-6} / ^\circ F$</td>
<td>$-2.87 \times 10^{-7} / ^\circ F$</td>
</tr>
<tr>
<td>$\alpha_{yy}, \alpha_{zz}$</td>
<td>$4.60 \times 10^{-6} / ^\circ F$</td>
<td>$32.0 \times 10^{-6} / ^\circ F$</td>
<td>$18.8 \times 10^{-6} / ^\circ F$</td>
</tr>
</tbody>
</table>

---

**Figure 7.2: 4-Ply Layup Model**

---

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7.2 Open Hole Simulations

Lamina level effective properties were determined from the cubic unit cell model as in Chapter 6. These effective properties were used to solve the macroscopic open-hole tension problem. Macroscopic failure criteria $\mathcal{L}^M (e(u^0))$ and $\Pi^M (e(u^0))$ were computed at four $z$-heights in the model, corresponding to the middle (with respect to $z$) of each ply. Computations were performed with and without prestress, and were carried out once again using BSAM.

7.2.1 Comparison of Macroscopic Failure Criteria with Invariants of the Homogenized Strain

We begin by showing the differences that can exist between the macroscopic failure criteria $\mathcal{L}^M (e(u^0))$ and $\Pi^M (e(u^0))$ and the invariants of the homogenized strain field given by $\mathcal{L} (e(u^0))$ and $\Pi (e(u^0))$. Figures 7.3 and 7.4 portray the contours for $\mathcal{L} (e(u^0))$, $\mathcal{L}^M (e(u^0))$, $\Pi (e(u^0))$, and $\Pi^M (e(u^0))$ for a $0^\circ$ uni-directional 2-ply symmetric laminate. The sample points lie on the symmetry plane. The values given by $\mathcal{L} (e(u^0))$ and $\Pi (e(u^0))$ are far smaller than those given by $\mathcal{L}^M (e(u^0))$ and $\Pi^M (e(u^0))$. It is clear from Figures 7.3 and 7.4 that the invariants of the homogenized strain severely under predict the actual invariants of the strain field inside the composite.

7.2.2 Prestress Effects

Here we examine the effects of the prestress on the laminate. Figures 7.5 - 7.8 portray the macroscopic failure criteria $\mathcal{L}^M (e(u^0))$ in each ply with and without prestress. Note first the slight asymmetry in the plot of $\mathcal{L}^M (e(u^0))$ in
(a) Invariant of the Homogenized Strain Field $\mathcal{L}(e(u^0))$

(b) Macroscopic Failure Criteria $\mathcal{L}^M(e(u^0))$

Figure 7.3: Dilatational Comparison for a Unidirectional $0^\circ$ Laminate
Figure 7.4: Deviatoric Comparison for a Unidirectional $0^\circ$ Laminate
the $0^\circ$ ply (Figure 7.5). This phenomenon is not seen in the unidirectional case (Figures 7.3 and 7.4). It can be explained by the presence of the plies of varying fiber orientation. Directly below the $0^\circ$ ply is a ply with $45^\circ$ fiber orientation (Figure 7.6), and thus the strain state of the $0^\circ$ ply feels this presence.

The prestress appears to increase the first strain invariant values almost everywhere on the $0^\circ$ and $\pm 45^\circ$ plies by one order of magnitude (Figures 7.5 - 7.7). This general phenomenon of increased strain is evident in the $90^\circ$ ply (Figure 7.8), however, the prestress only increases the first invariant by a few tenths.

The macroscopic failure criteria $\Pi^M(e(u^0))$ is shown in Figures 7.9 - 7.12. It is again clear from the figures that the prestress increases $\Pi^M(e(u^0))$ on nearly the entire plate. The prestress does not, however, appear to significantly increase the maximum of $\Pi^M(e(u^0))$ inside the ply. It is seen that the prestress increases the volume of the sets feeling the higher strain states. This volume can be crucial in failure predictions, and thus its increase is significant. It is important to note that the $\Pi^M(e(u^0))$ values in the $\pm 45^\circ$ plies are about twice as large as the values in the $0^\circ$ and $90^\circ$ plies. This trend is because the off-axis fiber orientation angles clearly experience larger shear strains.

We make one final remark about Figures 7.5 - 7.12. The largest strains in each ply are concentrated very near the hole edge. Hence contours approaching these extreme values are not visible in these figures. The invariant contours plotted show the general trends of the invariants in each ply, but do not represent the extreme invariant values that occur. For the first invariant, the largest level-line plotted in each figure is roughly 50% of the maximum
invariant in the ply. For the second invariant, the largest level-line shown in each set of figures is about 20% of the maximum invariant in the ply.

7.3 Comparison of Cubic and Hexagonal Representative Volume Elements

The theory developed in Chapters 3 and 4 can be used for any microstructure that is periodic. Two common periodic microstructures are the cubic RVE presented in Chapter 6 and the hexagonal RVE presented below. In this section, we compare the previous results obtained for the 8-ply open hole problem using the cubic RVE (see Figure 7.13(a)) with new results obtained using the hexagonal RVE (see Figure 7.13(b)). To maintain consistent macroscopic properties, both representative volume elements have a 5\(\mu\)m-radius fiber and a 60% fiber volume fraction. In addition, the volume of each RVE is equal.

The macroscopic problem remains the same as in the previous section. We have an 8-ply IM7/977-3 \([0/ + 45/ − 45/90]_s\) laminate and apply a 1% tension via a displacement boundary condition on the surface \(x = 2\). Macoscopic failure criteria are computed on the surfaces \(z = 0.0025, 0.0075, 0.0125, 0.0175\), corresponding to the middle (with respect to \(z\)) of each ply in the model. We do calculations with and without prestress. We note that the level lines plotted in the hexagonal figures are the same invariant values as those plotted in the corresponding cubic unit cell figure. Thus the color of the figures can be used to directly compare the results.

The macroscopic failure criteria \(\mathcal{L}^M (e(u^0))\), with and without prestress, are shown for the hexagonal RVE in Figures 7.14 - 7.17. As was seen with
(a) Macroscopic Failure Criteria $\mathcal{L}^M (e (u^0))$ without Prestress

(b) Macroscopic Failure Criteria $\mathcal{L}^M (e (u^0))$ with Prestress

Figure 7.5: Macroscopic Failure Criteria $\mathcal{L}^M (e (u^0))$ in the $0^\circ$ Ply

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(a) Macroscopic Failure Criteria $\mathcal{L}^M (e (u^0))$ without Prestress

(b) Macroscopic Failure Criteria $\mathcal{L}^M (e (u^0))$ with Prestress

Figure 7.6: Macroscopic Failure Criteria $\mathcal{L}^M (e (u^0))$ in the $45^\circ$ Ply
Figure 7.7: Macroscopic Failure Criteria $\mathcal{L}^M (e(u^0))$ in the $-45^\circ$ Ply
(a) Macroscopic Failure Criteria $\mathcal{L}^M (e (u^0))$ without Prestress

(b) Macroscopic Failure Criteria $\mathcal{L}^M (e (u^0))$ with Prestress

Figure 7.8: Macroscopic Failure Criteria $\mathcal{L}^M (e (u^0))$ in the 90° Ply
Figure 7.9: Macroscopic Failure Criteria $\Pi^M(e(u^0))$ in the $0^\circ$ Ply
(a) Macroscopic Failure Criteria $\Pi^M(e(u^0))$ without Prestress

(b) Macroscopic Failure Criteria $\Pi^M(e(u^0))$ with Prestress

Figure 7.10: Macroscopic Failure Criteria $\Pi^M(e(u^0))$ in the 45° Ply
(a) Macroscopic Failure Criteria $\Pi^M(e(u^0))$ without Prestress

(b) Macroscopic Failure Criteria $\Pi^M(e(u^0))$ with Prestress

Figure 7.11: Macroscopic Failure Criteria $\Pi^M(e(u^0))$ in the $-45^\circ$ Ply
(a) Macroscopic Failure Criteria $\Pi^M(e(u^0))$ without Prestress

(b) Macroscopic Failure Criteria $\Pi^M(e(u^0))$ with Prestress

Figure 7.12: Macroscopic Failure Criteria $\Pi^M(e(u^0))$ in the 90° Ply
the cubic RVE, the prestress appears to raise $\mathcal{L}^M (e(u^0))$ by one order of magnitude in all plies but the 90° ply. Generalizing from the results of the cubic RVE and hexagonal RVE, the bounds suggest that the prestress can increase $\mathcal{L}^M (e(u^0))$ by one order of magnitude. Comparing Figures 7.5 - 7.8 with Figures 7.14 - 7.17, the hexagonal RVE $\mathcal{L}^M (e(u^0))$ values are obviously smaller than for the cubic RVE over most of each ply. Maximum predicted values, however, were slightly larger in the ±45° plies in the hexagonal RVE.

The macroscopic failure criteria $\Pi^M (e(u^0))$, with and without prestress, are shown for the hexagonal RVE in Figures 7.18 - 7.21. Once again, the prestress raises $\Pi^M (e(u^0))$ on nearly the entire ply for the hexagonal RVE. As was the case for the cubic RVE, the prestress in the hexagonal RVE simply increases the volume of the sets feeling the larger values of $\Pi^M (e(u^0))$. Comparing Figures 7.9 - 7.12 with Figures 7.18 - 7.21, we see that the volumes of
high $\Pi^M (e(u^0))$ values are smaller in the hexagonal RVE than in the cubic RVE.

### 7.4 Summary

We have illustrated the difference between the macroscopic failure criteria versus the invariants of the homogenized strain. Moreover, we have reconfirmed that prestress effects can be significant and thus must be considered for multiphase composites. We note that the cure temperature of the IM7/977-3 is relatively low. Thus the effects of the prestress seen here should be relatively small when compared to composites with higher cure temperatures. Future communications will investigate composites of different cure temperatures. We have demonstrated the macroscopic failure criteria using two representative volume elements. The hexagonal RVE has been shown to characteristically produce macroscopic failure criteria in the ply smaller than those produced by the cubic RVE.
(a) Macroscopic Failure Criteria $L^M(e(u^0))$ without Prestress

(b) Macroscopic Failure Criteria $L^M(e(u^0))$ with Prestress

Figure 7.14: Macroscopic Failure Criteria $L^M(e(u^0))$ in the 0° Ply
(a) Macroscopic Failure Criteria $\mathcal{L}^M(e(u^0))$ without Prestress

(b) Macroscopic Failure Criteria $\mathcal{L}^M(e(u^0))$ with Prestress

Figure 7.15: Macroscopic Failure Criteria $\mathcal{L}^M(e(u^0))$ in the 45° Ply
(a) Macroscopic Failure Criteria $\mathcal{L}^M (e (u^0))$ without Prestress

(b) Macroscopic Failure Criteria $\mathcal{L}^M (e (u^0))$ with Prestress

Figure 7.16: Macroscopic Failure Criteria $\mathcal{L}^M (e (u^0))$ in the $-45^\circ$ Ply

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Figure 7.17: Macroscopic Failure Criteria $\mathcal{L}^M(e(u^0))$ in the $90^\circ$ Ply
Figure 7.18: Macroscopic Failure Criteria $\Pi^M(e(u^0))$ in the 0° Ply
(a) Macroscopic Failure Criteria $\Pi^M (e (u^0))$ without Prestress

(b) Macroscopic Failure Criteria $\Pi^M (e (u^0))$ with Prestress

Figure 7.19: Macroscopic Failure Criteria $\Pi^M (e (u^0))$ in the 45° Ply

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(a) Macroscopic Failure Criteria $\Pi^M(e(u^0))$ without Prestress

(b) Macroscopic Failure Criteria $\Pi^M(e(u^0))$ with Prestress

Figure 7.20: Macroscopic Failure Criteria $\Pi^M(e(u^0))$ in the $-45^\circ$ Ply
Figure 7.21: Macroscopic Failure Criteria $\Pi^M (e (u^0))$ in the 90° Ply
Chapter 8

Failure Analysis of a Symmetric Laminate with an Open Hole

In this chapter, we use the macroscopic failure criteria to predict the over-strained zone in the composite. Here, the over-strained (or over-stressed) zone is any material point in the composite for which the failure criteria meets or exceeds the failure value. We compare the macroscopic failure criteria predictions with the failure predictions of the Hashin and max stress failure criteria. These (and other) failure criteria can be found in [26], [27], [28], [29], and [30].

Many failure criteria that are currently in use are based on the homogenized stress (or strain) at the lamina level. The Hashin and max stress failure criteria are two commonly used examples of this type of failure criteria. They do not attempt to recover information about the local stress fields. The corrector theory developed in this thesis provides extra information about the local strain field in the composite. With these ideas in mind, we compare the failure predictions on an 8-ply symmetric laminate with an open hole. We
illustrate the overstressed zone predictions of the three failure criteria under two uniaxial tension loads.

8.1 Composite Specimen and Failure Values

The specimens are IM7/977-3 $[0/ + 45/-45/90]_s$ composite laminates having a cubic periodic microstructure (see Figure 7.13(a)) with a 60% fiber volume fraction. The individual constituent and composite (homogenized) mechanical properties are given in Table 7.1. We chose the cubic unit cell over the hexagonal unit cell because the macroscopic failure criteria were generally larger with the cubic cell (see Section 7.3).

8.1.1 Specimen Design and Boundary Conditions

The composite model is identical to the model used in Chapter 7. We have a 4 in. x 1 in. x 0.04 in. rectangular composite laminate, with a 0.25 in. diameter hole (see Figure 7.1) in the center. The laminate is subjected to 0.3% and 0.6% uniaxial tension strains in the $x$-direction. We restrict the $x$ and $y$ displacements on the surface $x = -2$ by specifying $u_x = u_y = 0$, and we restrict the $y$ displacement on the surface $x = 2$. The tensions are applied by the displacement conditions $u_x = 0.012$ and $u_x = 0.024$ on the surface $x = 2$, respectively. We model the 8-ply laminate as a 4-ply laminate, with the symmetry plane corresponding to the surface $z = 0$. Hence we specify the displacement boundary condition $u_z = 0$ on the surface $z = 0$ (see Figure 7.2). Once again the hole surface is a free edge.
8.1.2 Failure Values for IM7/977-3 Composite

Unidirectional specimens of the IM7/977-3 composite material were prepared according to manufacturer guidelines. These specimens were then tested to failure to determine the macroscopic failure stresses for the composite. The measured failure stresses are listed in Table 8.1. The unit ksi is kilopounds per square inch. Thus 1 ksi = 10^3 psi.

<table>
<thead>
<tr>
<th>Failure Mode</th>
<th>Stress (ksi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Longitudinal Tension Failure Stress</td>
<td>$X_t = 410$</td>
</tr>
<tr>
<td>Longitudinal Compression Failure Stress</td>
<td>$X_c = 330$</td>
</tr>
<tr>
<td>Transverse Tension Failure Stress</td>
<td>$Y_t = 9.6$</td>
</tr>
<tr>
<td>Transverse Compression Failure Stress</td>
<td>$Y_c = 40$</td>
</tr>
<tr>
<td>In-plane Shear Failure Stress</td>
<td>$S = 16$</td>
</tr>
<tr>
<td>Out-of-plane Shear Failure Stress</td>
<td>$ST = 19$</td>
</tr>
</tbody>
</table>

Both the Hashin and max stress failure criteria are based on stress at the lamina level. Thus the homogenized stress field is used to compute the value of the failure criteria. In formulas (8.2) and (8.1), the notation $|\cdot|$ means absolute value. Using the failure values in Table 8.1, a material point in the
composite lamina is over-stressed by the Hashin criteria if

\[
\begin{align*}
\max \quad & 
\begin{cases}
H_1 = \left( \frac{\sigma_{11}}{X_t} \right), & H_2 = \sqrt{\left( \frac{\sigma_{11}}{X_c} \right)^2 + \left( \frac{\sigma_{12}}{S} \right)^2}, \\
H_3 = \sqrt{\left( \frac{\sigma_{22}}{Y_t} \right)^2 + \left( \frac{\sigma_{12}}{S} \right)^2}, \\
H_4 = \sqrt{\left( \frac{\sigma_{22}}{Y_c} \right)^2 + \left( \frac{\sigma_{12}}{S} \right)^2}, \\
H_5 = \sqrt{\left( \frac{\sigma_{33}}{Y_t} \right)^2 + \left( \frac{\sigma_{13}}{ST} \right)^2 + \left( \frac{\sigma_{23}}{ST} \right)^2}, \\
H_6 = \sqrt{\left( \frac{\sigma_{33}}{Y_c} \right)^2 + \left( \frac{\sigma_{13}}{ST} \right)^2 + \left( \frac{\sigma_{23}}{ST} \right)^2}
\end{cases}
\end{align*}
\]

and is over-stressed by the max stress criteria if

\[
\begin{align*}
\max \quad & 
\begin{cases}
M_1 = \left( \frac{\sigma_{11}}{X_t} \right), & M_2 = \left( \frac{-\sigma_{11}}{X_c} \right), \\
M_3 = \left( \frac{\sigma_{22}}{Y_t} \right), & M_4 = \left( \frac{-\sigma_{22}}{Y_c} \right), \\
M_5 = \left( \frac{\sigma_{33}}{Y_t} \right), & M_6 = \left( \frac{-\sigma_{33}}{Y_c} \right), \\
M_7 = \left| \frac{\sigma_{13}}{ST} \right|, \\
M_8 = \left| \frac{\sigma_{23}}{ST} \right|, & M_9 = \left| \frac{\sigma_{12}}{S} \right|
\end{cases}
\end{align*}
\]

We note that failure criteria concerned with tension failure \((X_t \text{ and } Y_t)\) are only evaluated when the principal stress component in the criterion \((\sigma_{ii})\) is greater than zero. Similarly, failure criteria concerned with compression failure \((X_c \text{ and } Y_c)\) are only evaluated when the principal stress component is less than zero. We plot \(H_3, H_4, H_5, \text{ and } H_6\) for the Hashin criteria and \(M_3, M_4, M_5, M_6, \text{ and } M_9\) for the max stress criteria.

We introduce the critical values for the strain invariant failure criteria, \(t_L\) and \(t_{II}\). We take \(t_L\) to be the dilatational failure value of the matrix phase of
the composite and $t_\Pi$ to be the deviatoric failure value of the matrix phase of
the composite. The critical values for the strain invariant failure criteria are
given in Table 8.2.

Table 8.2: Critical Values for the Strain Invariant Failure Criteria for the
IM7/977-3 Composite

<table>
<thead>
<tr>
<th>Dilatational Failure Value</th>
<th>$t_L = 0.0237$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deviatoric Failure Value</td>
<td>$t_\Pi = 0.125$</td>
</tr>
</tbody>
</table>

Using these failure values, a material point in the IM7/977-3 lamina is over-
strained by the macroscopic failure criteria if

$$(8.3) \quad \max \left\{ I_1 = \left( \frac{L^M (e (u^0))}{t_L} \right), \ I_2 = \left( \frac{\Pi^M (e (u^0))}{t_\Pi} \right) \right\} \geq 1.$$

In what follows, we plot both $I_1$ and $I_2$.

8.2 Failure Analysis Results

Failure criteria were computed on the four surfaces $z = 0.0025, 0.0075, 0.0125,$
and 0.0175, corresponding to the middle (with respect to $z$) of each ply. For
a given failure criteria $f$, the location determined by $f \geq 1$ represents the
over-stressed zone. Prestress is taken into account on all results.

8.2.1 Results for 0.3% Tension Strain

The IM7/977-3 composite laminate fails between 0.6% and 0.7% tension strain.
To capture the order of ply failure, we illustrate the over-stressed zones first
at 0.3% tension. Figures 8.1 - 8.6 portray the over-stressed zones at 0.3% tension given by the macroscopic failure criteria, Hashin failure criteria, and max stress failure criteria. In each figure, the over-stressed zone is indicated by yellow.

In Figures 8.1 - 8.6, we see that the three failure theories agree that the 90° ply is the first ply to incur large over-stressed volumes. At only 0.3% strain, nearly the entire 90° ply is over-stressed (see Figures 8.2(b), 8.4(b), and 8.6(b)). The ±45° plies show far smaller over-stressed zones (Figures 8.1(b), 8.2(a), 8.3(b), 8.4(a), 8.5(b), and 8.6(a)). The plots of the 90° and ±45° over-stressed zones suggest that the failure is via transverse tension ($I_1$, $H_3$, and $M_9$).

The 0° ply has extremely small over-stressed zones at the hole edge (Figures 8.1(a), 8.3(a), and 8.5(a)). The dashed-line boundary around the macroscopic failure criteria over-strained zone (Figure 8.1(a)) indicates failure by the $I_2$ criterion (shear failure). The solid-line boundary around the Hashin over-stressed zone (Figure 8.3(a)) indicates failure via the $H_3$ criterion. This criterion is a quadratic combination of transverse tension and in-plane shear forces. The boundary around the max stress over-stressed zone (Figure 8.5(a)) indicates failure by the $M_9$ criterion, which is the in-plane shear force. Thus the results suggest that the three theories agree that the shear forces will initiate failure at the hole edge in the 0° ply.
Figure 8.1: Macroscopic Failure Criteria Over-strained Zone at 0.3% Strain
Figure 8.2: Macroscopic Failure Criteria Over-strained Zone at 0.3% Strain
Figure 8.3: Hashin Over-stressed Zone at 0.3% Strain
Figure 8.4: Hashin Over-stressed Zone at 0.3% Strain
(a) Over-stressed Zone in the $0^\circ$ Ply

(b) Over-stressed Zone in the $45^\circ$ Ply

Figure 8.5: Max Stress Over-stressed Zone at 0.3% Strain

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Figure 8.6: Max Stress Over-stressed Zone at 0.3% Strain
8.2.2 Results for 0.6% Tension Strain

The trends initiated at 0.3% continue as the tension reaches 0.6% strain. Figures 8.7 - 8.12 portray the over-stressed zones predicted by the macroscopic failure criteria, Hashin failure criteria, and max stress failure criteria. The 90° and ±45° plies all have large over-stressed zones (Figures 8.7(b), 8.8(a), 8.8(b), 8.9(b), 8.10(a), 8.10(b), 8.11(b), 8.12(a), and 8.12(b)). In addition, the 0° ply over-stressed zones are more pronounced (Figures 8.7(a), 8.9(a), and 8.11(a)).

The small over-stressed zones very near the hole edge in the 0° ply are the shear over-stressed zones seen at 0.3% strain. These shear stresses near the hole edge produce the characteristic cracks running parallel to the fibers in the 0° ply. The slightly larger over-stressed zones in the 0° ply appear to be transverse tension failure. These over-stressed zones are believed to be an effect of the ply stacking sequence. Future communications will investigate the effects of the ply stacking sequence on these predictions.

8.3 Summary

The predictions of the multi scale strain analysis method have been shown to be consistent with the Hashin and max stress failure criteria. In the case of the 0° ply, the multi scale strain analysis method is more conservative (in terms of strength) than the Hashin and max stress methods. The failure theories agree on the failure mode of each over-stressed region.
Figure 8.7: Macroscopic Failure Criteria Over-strained Zone at 0.6% Strain
Figure 8.8: Macroscopic Failure Criteria Over-strained Zone at 0.6% Strain
Figure 8.9: Hashin Over-stressed Zone at 0.6% Strain
(a) Over-stressed Zone in the $-45^\circ$ Ply

(b) Over-stressed Zone in the $90^\circ$ Ply

Figure 8.10: Hashin Over-stressed Zone at 0.6% Strain
(a) Over-stressed Zone in the 0° Ply

(b) Over-stressed Zone in the 45° Ply

Figure 8.11: Max Stress Over-stressed Zone at 0.6% Strain
Figure 8.12: Max Stress Over-stressed Zone at 0.6% Strain
Chapter 9

Conclusions

We have considered a multi-phase linear elastic composite material with periodic microstructure and prestress. In terms of the thermomechanical properties of the individual materials, we have provided a method by which one can accurately characterize the microscopic strain field in the composite via the macroscopic strain field. Moreover, we have developed macroscopic failure criteria for the homogenized strain that guarantees that the actual strain in the composite lies inside the strength domain of the individual materials. The multi scale strain analysis method has been shown to be reliable for actual composite systems with finite $\epsilon$. The macroscopic failure criteria are seen to bound the actual strain invariants in areas of uniform periodicity, even in the presence of a free edge in multi-ply composite laminates.

The practical piece missing from this theory is the prediction of the scale of the microstructure ($\epsilon_0$) when Theorem 3.3.1 and Theorem 3.3.2 hold. Future work will focus on establishing this length scale.
We also seek to extend this method beyond the realm of linear elastic materials. Current work focuses on the homogenization and strain bounding of linear viscoelastic media. Viscoelastic materials are important because many composites are composed of a viscoelastic matrix. The viscous property associated with these materials adds the time dimension to the homogenization problem.
Bibliography


Appendix A

Function Spaces

$C^\infty_c(\Omega)$  
Infinitely differentiable functions with compact support in $\Omega$

$W^{k,p}(\Omega)$  
Sobolev space in which $k^{th}$ derivatives are $p^{th}$ power summable (note that $W^{1,2}(\Omega) = H^1(\Omega)$)

$W^{-1,2}(\Omega)$  
Dual space of $W^{1,2}(\Omega)$

$H^1(\Omega)^3$  
$\{v = (v_1, v_2, v_3) \mid v_i \in H^1(\Omega), \text{ for } 1 \leq i \leq 3\}$

$V(\Omega)$  
$\{v \in H^1(\Omega)^3 \mid v = 0 \text{ on } \Gamma_0\}$, where $\Gamma_0 \subset \Gamma$ and $\Gamma = \partial \Omega$

$L^p(\Omega)$  
Space of $p^{th}$ power summable functions

$L^p_{\text{loc}}(\Omega)$  
Space of locally $p^{th}$ power summable functions

$L^\infty(\Omega)$  
Space of essentially bounded functions

$L^p(\Omega)^{3 \times 3}$  
$3 \times 3$ matrices for which each entry is a $L^p(\Omega)$ function

$L^\infty(\Omega)$ weak-$*$  
Space $L^\infty(\Omega)$ equipped with the weak-$*$ topology

$M_3(\mathbb{R})$  
Space of real valued $3 \times 3$ matrices

$\text{Sym}(\mathbb{R}^3)$  
Space of real valued symmetric $3 \times 3$ matrices
Appendix B
List of Symbols

\( \varepsilon(\mathbf{x}) \) \quad \text{Strain tensor field at a point } \mathbf{x} \in \mathbb{R}^3

\( \sigma(\mathbf{x}) \) \quad \text{Stress tensor field at a point } \mathbf{x} \in \mathbb{R}^3

\| \cdot \|_A \quad \text{Norm on function space } A

|\omega| \quad \text{Lebesgue norm on } \mathbb{R}^3 \text{ of the set } \omega

|\varepsilon(u^*)|^2 \quad \text{Matrix norm given by } \sum_{i,j=1}^{3} (\varepsilon(u^*))_{ij}^2

\langle \cdot, \cdot \rangle \quad \text{Duality pairing between the spaces } W^{-1,2}(\Omega)^3 \text{ and } H^1(\Omega)^3

A : B \quad \text{Dot product for matrices. That is, } A : B = (A_i \cdot B_j)(\alpha_i \cdot \beta_j), \text{ for } A = A_i \alpha_i \text{ and } B = B_j \beta_j

\omega \setminus A \quad \text{Set subtraction. The set of all points in } \omega \text{ that are not in } A

a \otimes b \quad \text{Given by } (a \otimes b)_{ij} = a_i b_j, \text{ for any } a, b \in \mathbb{R}^3

\nabla \phi \otimes v \quad \text{Given by } \frac{\nabla \phi \otimes v + v \otimes \nabla \phi}{2}, \text{ for } \phi \in C_c^\infty(\Omega) \text{ and } v \in H^1(\Omega)^3

Q \quad \text{Denotes the unit period cell for a periodic microstructure}

[A/B/C/D]_s \quad \text{Denotes a fiber reinforced composite having fiber orientation angles } A, B, C, D \text{ from top to bottom, with symmetry surface at the bottom of layer } D
\( \Delta T \)  
Temperature change

\( X_t, X_c, Y_t, Y_c, S, ST \)  
Critical stress values. If any of these values are reached, the composite breaks

\( t_{\Sigma}, t_{II} \)  
Critical strain invariant values. If either of these values are reached, the composite breaks
Vita

Timothy Donald Breitzman was born on September 18, 1978, in Bloomington, Illinois, United States of America. He grew up in Washington, Illinois, and graduated from Washington Community High School in 1997. He completed his undergraduate studies in mathematics with secondary-level teacher certification and chemistry (also with secondary-level teacher certification) at Eastern Illinois University in 2001. He then came to Louisiana State University, where he completed a Master of Science degree in mathematics in 2002. Under the direction of Professor Robert Lipton, this dissertation is the culmination of his graduate study at Louisiana State University for the degree of Doctor of Philosophy.