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Performance based switching control for single input linear time invariant systems

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PERFORMANCE BASED SWITCHING CONTROL FOR SINGLE INPUT LINEAR TIME INVARIANT SYSTEMS

A Thesis

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Abstract

In the present research, we propose a new control strategy to improve the performance of a linear time invariant single input system. We consider a performance index that is dependent only on the controlled variable and try to improve it over that obtained when using a stable state feedback control. Since the control law depends on the performance of the controlled variable, we call it Performance Based Switching Control(PBSC). Switching between two state feedbacks is used to achieve this improvement in performance. The switching strategy combines the best features of both strategies.

The switching surface is actually composed of two subspace which intersect on the null space of the switching matrix. Analysis of each of the these surfaces shows that sliding might be present on one of them. However, if both state feedbacks are designed using optimal quadratic regulators with different weights in the control effort, sliding motion has not been detected in any of the experiments performed.

The new approach can also yield a fuel efficient control that uses a system's unstable behavior to its advantage. Overall PBSC can be used to improve output performance of a system.

Chapter 1
Introduction

Switched systems are composed of a group of sub-systems guided by a switching law that governs the change among these subsystems. Use of switching in control was proved to give better performance when compared to the performance of a system without switching control. It is generally acknowledged that the work done on variable structures systems by Emelyanov and his co-researchers [10] in 1950's marked the beginning of the idea of using switching in control. The last two decades have seen much development in switching systems. Nowadays, researchers are interested in hybrid systems, which employ the principle of logic based switching. The present research concentrates on developing a switching criterion, which would improve the output performance of the system over the performance obtained by using standard state feedback. Most conventional control strategies attempt to provide a tradeoff between the control cost and the performance of the controlled variables. For example, the Linear Quadratic Regulator (LQR) attains this tradeoff by minimizing a quadratic cost given by Eq. 1.1.

$$J = \int_0^\infty (\|u\|^2 + \|y\|^2) dt \tag{1.1}$$

However, it does not necessarily give a good performance. It gives a performance that is acceptable to some extent. In order to obtain optimal performance of the controlled variable, the index given by Eq. 1.2 should be minimized.

$$J = \int_0^\infty \|y\|^2 dt \tag{1.2}$$

The problem of minimizing the above cost Eq. 1.2 does not have an optimal solution. Cheap control [15], [33] provided a solution to this problem to some extent.

Here, we propose a new switching control strategy that provides better performance than state feedback control. This type of switching control might be unsuitable for Boeing 747 aircrafts, but it could prove useful in high performance aircrafts like fighter planes. The present research is a part of the project, Aircraft safety: Control Upset Management, sponsored by NASA and Louisiana Board of Regeants under the ESPCOR-2000 program.

1.1 Basic Case Study-The Double Integrator

Before formulating our approach, we analyze the phase portrait of a basic double integrator system using unstable and stable feedback. The unstable feedback places the poles at $p_1 = -10$, $p_2 = 0.2$ while stable feedback places them at $p_{1,2} = -1 \pm j$.

Fig 1.1 shows the phase portrait of the system with the above mentioned pole placements. It can be seen that the operating point of the system with unstable feedback tends to move towards the origin in 2nd and 3rd quadrants. This result suggests that "there is a region where the unstable feedback is better than stable feedback." The above combination of stable and unstable feedbacks was simulated for an initial condition $(x_1, x_2) = (10, -1)$ in the region of "advantageous unstable control". The plant was allowed to start with unstable feedback. The controller was switched to stable feedback at the instant when the plant starts diverging from the origin. The phase plane trajectory of this switched system when compared to the unswitched case is given in Fig. 1.2 and the time response in Fig. 1.3.

From the above illustration, it is clear that there could exist regions where unstable feedback is advantageous. Here it should be noted that the idea of making the plant unstable to improve the response of the system might sound alarming in view of the system safety. If implemented, it may require special reliability consideration. A safer approach might consider switching between two stable strategies. In any event, the experiments raise the following questions.

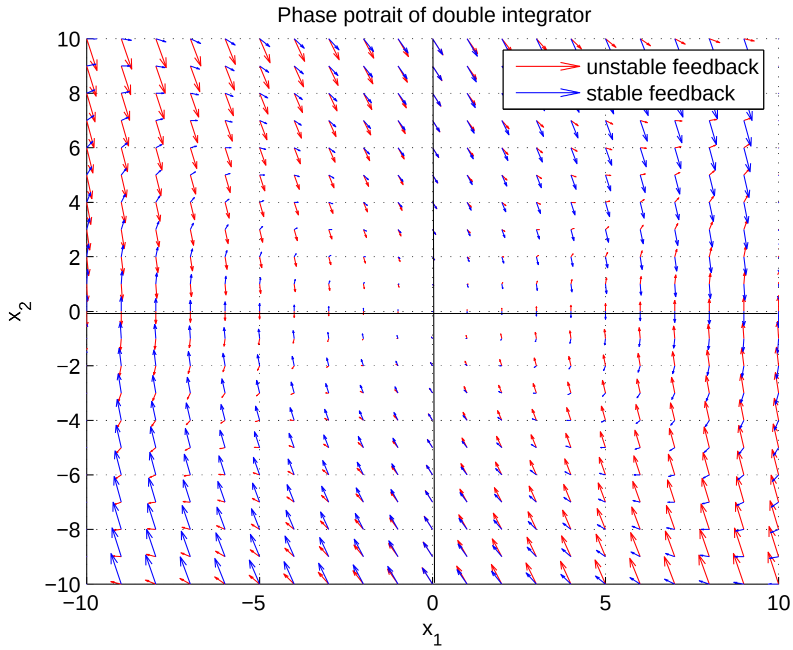

FIGURE 1.1. Phase portrait of the system with unstable poles at $p_1, p_2 = -10, 0.2$ respectively

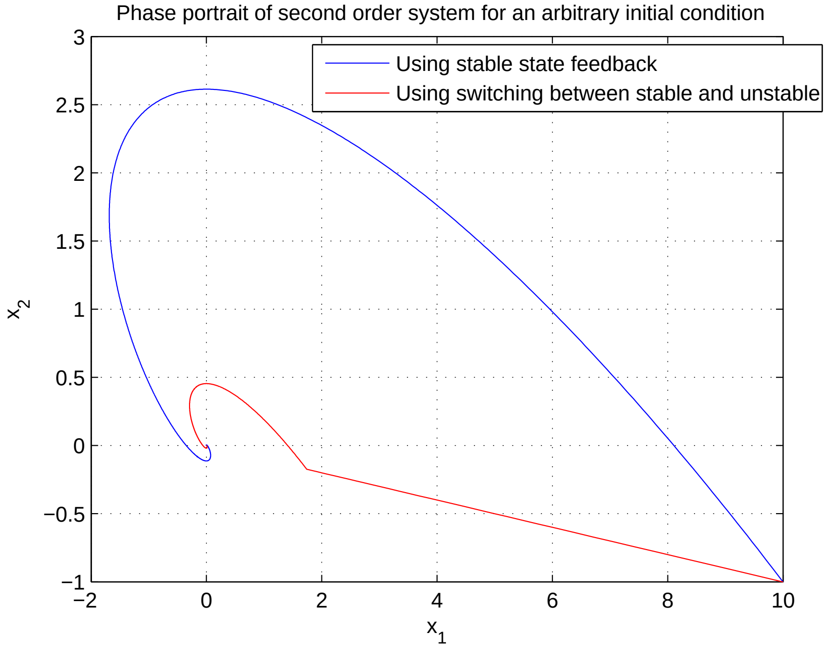

FIGURE 1.2. Phase portrait for an initial condition in the region where unstable feedback was useful

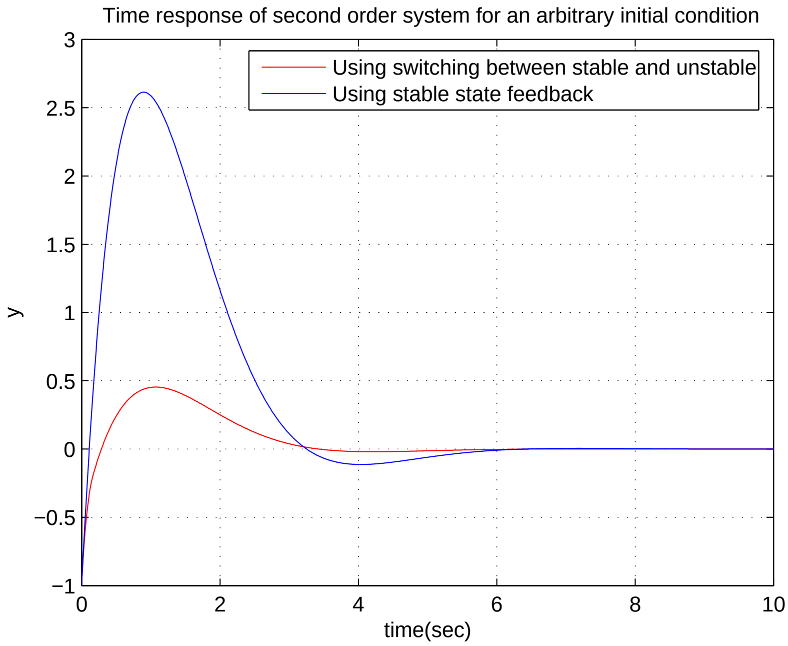

FIGURE 1.3. Time response of the controlled variable for an initial condition in the region where unstable feedback was useful

- How to quantify the effect or improvement of switching control strategy?

- How to determine in what region of the state space, it is more advantageous to use alternate feedback?

- How to avoid high frequency oscillations of the manipulated variable?

- How to guarantee the stability of the overall configuration?

In the course of the work we provide a logical answer to all the above questions.

1.2 Thesis Outline

The thesis is outlined as follows. Chapter 2 summarizes the contributions of different researchers in variable structure control, switched systems and hybrid systems. We list some of the drawbacks that have motivated us in the present research.

Chapter 3 explains the mathematical formulation of the proposed performance based switching control. The derivation and analysis of the switching strategy is explained here. We also derive some results concerning sliding motion on the switching surface.

In chapter 4, we illustrate the proposed method with the help of simulations performed on systems of order two, three and four. The conclusion and the future work are presented in the subsequent chapter.

Chapter 2
Literature Review

Recent decades have seen considerable development in nonlinear control. Many nonlinear techniques have been formulated and implemented for linear as well as nonlinear systems. Among these nonlinear techniques, control techniques using switching are found to be applicable in most of the industries. The ability of switched systems to model most of the uncertainties could be an obvious reason for this popularity [24]. Of all the switching strategies so far developed, variable structure control is most popular.

2.1 Variable Structure Control

According to literature so far published [19],[38], Variable Structure Control (VSC) was first proposed by the Russian researcher Emelyanov and others in the 1950's [10]. Variable structure control is a class of switching feedback control in which the gain in each feedback path switches among a set of values. In other words, the control is allowed to switch at any instant from one member of a set of possible continuous functions of state to another. V. Utkin [38] cited many works which utilized the idea of changing structures even before VSC became widely known [7], [16], [13], [12], [23], [25], [28], [29], [36], [34], [40]. A survey paper [21] cited the outstanding works of Androve el. al. [4] and Flugge-Lotz [14] in phase plane method, which laid the foundation for the emergence of VSC. The theory of VSC is explained in terms of phase portraits in most of the papers [38], [19]. Much of the research done in VSC deals with VSC with sliding mode. In [19], Utkin explains that VSC without sliding planes are difficult to analyze and their properties are established only for few special cases.

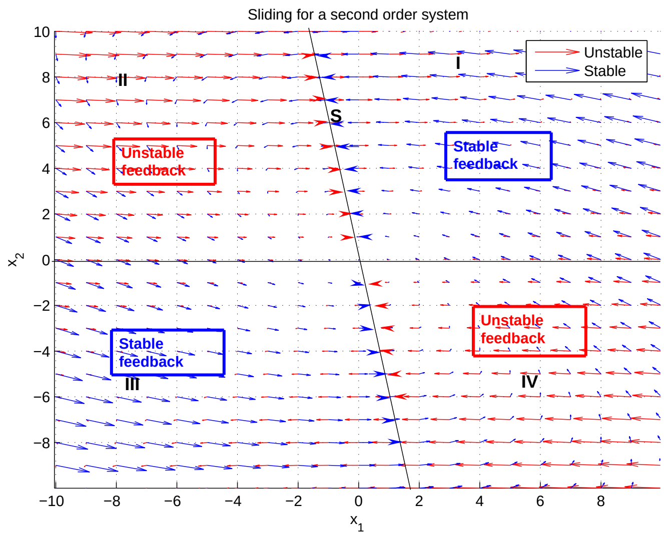

FIGURE 2.1. Sliding in second order system

2.1.1 Sliding Mode Control

The principle of sliding mode control can be explained by taking the example of a linear time invariant (LTI), single input single output (SISO) second order system.The phase portraits of such a system with stable as well as unstable feedbacks are given in Fig. 2.1. Assume that unstable feedback is used in quadrants II and IV while stable feedback is used in quadrants I and III. Assume that the system is starting in quadrant I. As the states hit the switching surface S, the feedback switches to unstable. But, the unstable feedback also forces the states towards the switching surface. As the result, the system oscillates about the switching surface at infinitely high frequency as shown in Fig. 2.2[1].

This high frequency switching is termed “sliding action” [19]. Filipov [11] developed the conditions for the existence of sliding mode based on the theory of differential equation with discontinuities. These conditions showed that the motion in sliding plane might be stable even when some of the structures employed are unstable. The invariance of sliding mode with respect to the plants parameters makes it suitable for time varying plants. H. K. Khalil proved the stability of slid-

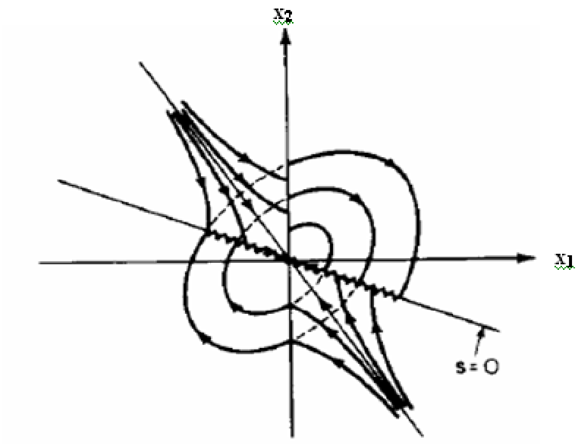

FIGURE 2.2. Visualization of sliding action along sliding surface
[V. I. Utkin, Variable structure control system with Sliding Modes, *IEEE Trans. on. Aut. Control*, AC-22(2), 1977]

ing mode systems using Lyapunov stability in his work [22]. Ideal sliding motion is characterized by high frequency oscillation about the switching line [19], [38]. Theoretically, the frequency of oscillation is infinite. Practically, the presence of time delays causes the systems to oscillate at unusually high frequency, producing the unwanted effect called chattering (Fig. 2.3 [38]).

Chattering is the major problem encountered in variable structure control systems with sliding modes as it reduces control accuracy and increases wear and tear in physical systems [20]. Much work has been done and many methods like continuous approximation of switching function, dynamic adjustment of switching function, chatter reduction using state observer etc., have been proposed for chatter reduction [20], [18].

In [31], the authors discussed chattering reduction in an induction motor with VSC. A low pass filter was used to smoothen the output of the controller before applying it to the plant. This however gave a sluggish transient response. A variable

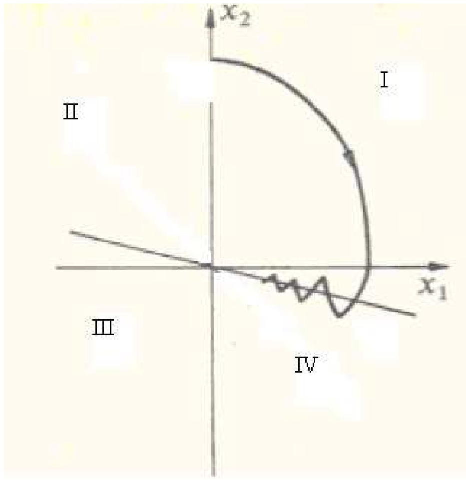

FIGURE 2.3. Chattering in second order system
[V. I. Utkin, Variable structure control system with Sliding Modes, *IEEE Trans. on. Aut. Control*, AC-22(2), 1977]

bandwidth filter improved the response to some extent. In [32] chattering was eliminated by dropping the discontinuous term. But, this decreased the robustness of the system. The use of disturbance estimator improved the robustness to some extend. The work of J. Slotine [35] cited in [32] reduced the chattering by using a boundary layer around the sliding surface. When the states are with in this boundary layer, the VSC behaves like a high gain controller.

In [18] chattering was reduced by making the width of the boundary layer variable. A large boundary layer width eliminates chattering and smaller width improves the robustness. The width of the boundary layer is allowed to change to provide a reasonable compromise between chatter reduction and robustness. It has been proved in [18] that larger boundary layer width is required when ever the states are far away from the origin, regardless of how close the states are to the sliding surface. The author also provides a comparative study of fixed boundary layer sliding and the above variable boundary layer sliding. Difficulty in online calculation and adjustment of the width of the boundary layer are significant drawbacks of the method.

An adaptive chatter reduction method discussed in [17] uses a fuzzy controller to adjust the magnitude of the switching term in the presence of perturbation. In the absence of perturbation, chattering does not exist. In all the chatter reduction methods discussed above, chattering was assumed to occur when the switching frequency is finite. In [39], the author D. Z. Sun, addressed the problem of chatter reduction when the switching frequency was infinite. A state observer was used to estimate the states in the above case. These states are used in sliding mode controller. This method may not be the same as the true sliding mode. But, the error is less if we assume that the unmodelled dynamics are fast. Yet another method proposed in [18] augments the plant so that the chattering control is present in the low power section of the plant. This is similar to the method proposed in [17]. All the methods described above reduce chattering. It is impossible to eliminate chattering without considerable loss of robustness.

2.2 Switched Systems and Hybrid Systems

Sliding mode control system is a special class of switched system, where, the switching occurs at very high rate. This makes sliding mode control unsuitable for many of flight control applications. For example it cannot be used in control of flights like Boeing 747 actuated by hydraulic actuators.

This situation has motivated us to investigate other kinds of switched systems so as to answer the question "Is there a switching strategy that improves the system's output performance by switching between two strategies, but is free from chattering?"

Different authors treated the theory of switched systems in different ways. For example Daniel Liberzon [9], [8] used differential geometric approach to find sufficient conditions for asymptotic stability of linear time invariant system. These conditions were derived based on the proposition that the switched system would be stable for an arbitrary switching signal if there exists a common quadratic Lyapunov function for the family of the linear systems. Similarly linear matrix inequalities were used to analyze the stability of continuous time system with state delays [2]. The problem of stabilization of the switched systems was effectively investigated in [41]. If the switching signal were not a design variable, then the problem would be to design a feedback that stabilizes the system under all possible switching signals. As cited in [42] the partial solution to this design problem was given by [43]. However, this partial solution could not explain the stability of the switched system.

The survey paper [42] summarizes the recent development in the analysis and synthesis of switched linear control systems. The authors delineate the basic concepts such as controllability, observability, system structural decomposition, feedback controller design for stabilization and optimal control with reference to switched systems. In the words of the authors, "Controllability deals with whether or not the system is controllable through the input and switching signals. Similarly, observability deals with whether or not the initial state can be observed through the inputs, outputs and switching signal." It has been shown in [41] that switched systems follow the principle of duality.

In yet another paper by Zhendong [37] it was proved that switched linear systems can be structurally decomposed in terms of controllability and observability, in the same manner as the linear time invariant systems.

In [42], the feedback strategies were classified into two categories. If the switching signal is also a design variable, it is required to design both switching criterion as well as feedback law otherwise it is enough to design the feedback law in order to make the closed loop system stable. This is similar to the scheduling and routing tasks of hybrid systems mentioned in [27].

G. Bartolini et al. [30] called the logic driven sliding mode control as hybrid control. A hybrid system is a dynamic system that contains continuous as well as discrete variables. Much literature has been published in hybrid systems during recent decades. A. Ferrate, L. Magnani and R. Scattlolini propose a hybrid variable structure control strategy in their work [5]. But, the strategy applies for variable structure control with sliding modes. We are mostly interested in switched systems with variable structure and switching ruled described by logical expressions.

In [27], A. S. Morse discusses another kind of discontinuous control called logic based control. It could be considered as a combination of discontinuous control and digital control. A logic based switching control includes logical components along with the conventional control components like integrators, summers etc. The author describes various kinds of logic based switching, namely prerouted switching, hysteresis switching, dwell time switching and cyclic switching. Each of these methods decides the controller to be placed in the feedback to achieve the desired performance. The multi-controller architecture shown in Fig. 2.4[27] was used. The controlled variable, or the plant output, drives a bank of controllers, each of which generates a feedback signal. The control signal is a piecewise continuous signal, the generation of which was governed by a supervisor system. At any instant of time, the control signal from only one of the controllers is applied to the plant. We used a similar type of controller architecture in our research. The architecture could be further simplified by using a single controller with adjustable parameters, instead of a bank of controllers. This is called state sharing.

The task of the switching controller in case of a multi-controller is twofold: deciding when to switch or scheduling and deciding which controller to use. In

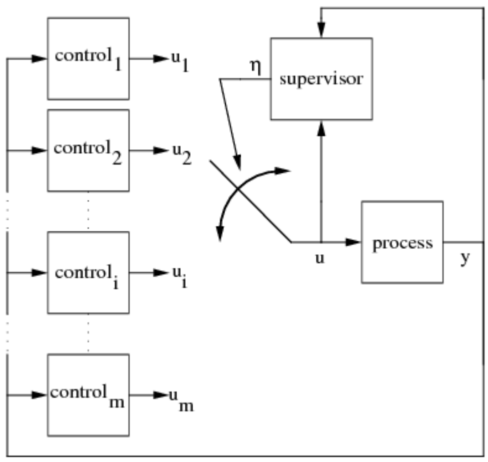

FIGURE 2.4. Multi-controller architecture
[A.S. Morse, Logic based switching]

case of prerouting switching, the order in which the controllers are to be switched is decided in advance. The switching logic only needs to decide when to switch. In case of hysteresis, dwell time and cyclic switching, the control logic performs both scheduling and routing. All the methods of switching proposed above require that each of the controllers of the controller bank should stabilize the systems. However, this does not imply the stability of overall switched system. Lyapunov analysis can be used to solve this problem of stability to certain extent.

M. Barnicky in his paper [6] presents the stability analysis of switched and hybrid systems. The switched systems can be "variable structured" or "multi-modal". The stability analysis holds for the switched systems, which have finite switches in finite time. The number of switches are chosen by a controller, computer, a human operator or may be a function of time or state or both. If the number of switches is a function of time or state then the system is said to be autonomous. If it chosen by other processes like controller, computer or a human operator the system is said to be controlled. Thus an autonomous hybrid system is one in which the states change discontinuously when they hit a certain boundary. Similarly, the states change discontinuously in response to a control command, which might have an associated cost. In the present work, the number of switches depends on the states and hence can be classified as autonomous hybrid systems or autonomous switched system.

Inspite of its advantages, switching control is faced with many unsolved challenges. The problem encountered in switching control where discussed in [24]. Finding the conditions of existence, identification of the switching signal and construction of such signals are the three main problems encountered in the design of switched systems. The above problems are discussed with respect to the multi-controller architecture described in [26]. A higher switching time would ensure system stability. However, with the development of computer controlled systems, it has become necessary to analyze the stability of the switched system for fast switching signals. Apart from this, none of the researchers talked about the presence of sliding in case of switching system. We attempt to analyze the presence of sliding along with the stability for a system switching between two feedbacks.

Chapter 3
Problem Statement

In this section we present, in mathematical terms, the problem being discussed. The aim of this research is to formulate a switching control strategy, which would improve the performance of the controlled variable over the conventional state feedback control. Usually, the state feedback control aims at obtaining a compromise between control cost and performance. However, there are certain applications where performance is the primary concern. For example, systems like fighter planes may require faster response at any control cost. The present control method could be very useful when the output performance rather than the control cost is our primary concern. Since our control law depends on the performance of the controlled variable, we call it Performance Based Switching Control(PBSC).

3.1 Mathematical Formulation

In the present research we consider linear time invariant systems. All the control laws are assumed to be state feedbacks. The system can be written as

$$\dot{x} = Ax+Bu \quad x \in \Re^n, \quad u \in \Re^m \tag{3.1}$$

$$u = -Kx \tag{3.2}$$

$$y = Cx \quad y \in \Re^p \tag{3.3}$$

Most conventional control strategies attempt to provide a tradeoff between the control cost and the performance of the controlled variables. For example the Linear Quadratic Regulator (LQR) attains this tradeoff by minimizing a quadratic cost J_o given by Eq. 3.4.

$$J_o = \int_0^\infty (\|u\|^2 + r\|y\|^2)dt \tag{3.4}$$

However it does not necessarily give good performance on the controlled variable. It gives a performance that is acceptable to some extent. In order to obtain optimal performance of the controlled variable, the quadratic cost J given by Eq. 3.5 should be minimized.

$$J = \int_0^\infty \|y\|^2 dt \tag{3.5}$$

Since the problem of minimization of the above cost, Eq. 3.5, does not have an optimal solution in L_2, we do not make any such attempt here. We try to find a control strategy that would give lesser value of the performance index (J) measured using Eq. 3.5, compared to the equivalent cost obtained when using state feedback control strategy.

3.2 Control Law Derivation

In this section we formulate our approach to find the control strategy that gives performance better than state feedback control. Let us consider an observable LTI plant

$$\dot{x} = Ax + Bu \quad x \in \Re^n, \quad u \in \Re^m \tag{3.6}$$

$$y = Cx \quad y \in \Re^p \tag{3.7}$$

The state feedback control law is given by

$$u = -Kx \tag{3.8}$$

We consider two controller gains namely, K_s and K_u. The controller K_s is the primary controller and K_u is the alternate controller. K_s is selected such that it stabilizes the system asymptotically. There are no restriction imposed on the controller K_u. Our goal is to find the switching strategy such that the performance index J given in Eq. 3.5 is better than when only primary state feedback is used.

Let

$$A_s = A - BK_s \tag{3.9}$$

$$A_u = A - BK_u \tag{3.10}$$

For a stable matrix A we have the following well known result.

Theorem 3.1. *The matrix A is stable if and only if for any positive definite matrix Q there exists a unique positive definite system matrix P such that*

$$A^T P + PA = -Q. \tag{3.11}$$

From the above theorem we can state the following corollary

Corollary 3.1.1. *A Lyapunov function for $\dot{x} = Ax$ is given by Eq. 3.12*

$$V = x^T P x \tag{3.12}$$

Remark : For an observable system, taking $Q = C^T C$ will still give a positive definite solution to the Lyapunov equation, Eq. 3.11

In the present case for the stable matrix A_s we can write Eq. 3.11 as

$$\Gamma_0 A_s + A_s^T \Gamma_0 = -C^T C \tag{3.13}$$

where Γ_0 is positive definite. We can show that

Lemma 1. *For a stable system*

$$\dot{x} = \text{Ax+Bu} \quad x \in \Re^n, \quad u \in \Re^m \tag{3.14}$$

$$y = Cx \quad y \in \Re^p \quad \textit{and} \quad x(0) = x_0 \tag{3.15}$$

the performance of the controlled variable can be can given as

$$J = \int_0^\infty \|y\|^2 dt = \langle x_0, \Gamma_0 x_0 \rangle \tag{3.16}$$

Proof. For any system we can have

$$J = \int_0^\infty y^T y dt \tag{3.17}$$

$$= \int_0^\infty x^T C^T C x dt \quad (\because y = Cx) \tag{3.18}$$

For a systems starting from an initial condition x_0 and using primary feedback we have

$$x = e^{A_s t} x_0 \tag{3.19}$$

If the system is observable, the cost in Eq. 3.17 can be written as

$$\begin{aligned} J &= \int_0^\infty x_0^T e^{A_s t} C^T C e^{A_s t} x_0 dt \\ &= \int_0^\infty x_0^T e^{A_s t} Q e^{A_s t} x_0 dt \\ &= -\int_0^\infty x_0^T e^{A_s t} (\Gamma_0 A_s + A_s^T \Gamma_0) e^{A_s t} x_0 \\ &= \int_0^\infty \frac{d\langle x_0 e^{A_s t}, \Gamma_0 x_0 e^{A_s t}\rangle}{dt} \\ &= \langle x_0, \Gamma_0 x_0 \rangle \end{aligned} \tag{3.20}$$

Thus, for an initial state x_0 the cost of using stable feedback is

$$J(x_0) = \langle x_0, \Gamma_0 x_0 \rangle \tag{3.21}$$

□

We now proceed to derive the switching control law that is of interest to us

Theorem 3.2. *Given two alternative state variable feedbacks, K_s, K_u, where $A - BK_s$ is known to be asymptotically stable. For an initial state $x = \xi$, the use of alterative strategy K_u for some time $\tau > 0$ will be beneficent if*

$$\sigma(\xi, \tau) = \langle \xi, e^{A_u^T \tau} (\Gamma(\tau) - \Gamma_0) e^{A_u \tau} \xi \rangle > 0 \tag{3.22}$$

where $\Gamma(\tau)$ *is the solution, at time* τ*, of the differential equation*

$$\dot{\Gamma} + A_u^T \Gamma + \Gamma A_u \;=\; -C^T C; \quad t \geq 0, \quad \Gamma(0) = \Gamma_0 \tag{3.23}$$

and Γ_0 *is the solution of the Lyapunov equation*

$$\Gamma_0 A_s + A_s^T \Gamma_0 \;=\; -C^T C \tag{3.24}$$

Proof. Consider the differential equation.

$$\dot{\Gamma} + A_u^T \Gamma + \Gamma A_u \;=\; -C^T C; \quad t \geq 0, \quad \Gamma(0) = \Gamma_0 \tag{3.25}$$

Pre multiplying and post multiplying the equation Eq. 3.25 by $e^{A_u^T t}$ and $e^{A_u t}$ respectively,

$$e^{A_u^T t}(\dot{\Gamma} + A_u^T \Gamma + \Gamma A_u) e^{A_u t} \;=\; -e^{A_u^T t} C^T C e^{A_u t}$$

From the above it is clear that

$$\frac{d(e^{A_u^T t} \Gamma e^{A_u t})}{dt} \;=\; -e^{A_u^T t} C^T C e^{A_u t} \tag{3.26}$$

For any give $\tau > 0$, quadratic performance of the controlled variable when using the alternate gain K_u in the interval $[0, \tau]$ and starting from the initial state ξ_0 is

$$J_u \;=\; \langle \xi_0, (\int_0^\tau e^{A_u^T t} C^T C e^{A_u t}) \xi_0 \rangle \tag{3.27}$$

$$\;=\; \langle \xi_0, \Gamma(0) \xi_0 \rangle - \langle x_s, \Gamma(\tau) x_s \rangle \tag{3.28}$$

where $x_s = e^{A_u \tau} \xi_0$ is the state at τ using the strategy K_u

The performance of the system that switches to primary gain at time τ is given by $\langle x_s, \Gamma_0 x_s \rangle$. Assume that the system starts from a initial state ξ_0 and uses the alternate gain initially. The cost of the system starting with alternate gain and switching to primary gain at some time τ is

$$J_s = \langle \xi_0, \Gamma(0)\xi_0 \rangle - \langle x_s, \Gamma(\tau) x_s \rangle + \langle x_s, \Gamma_0 x_s \rangle \tag{3.29}$$

Given the choice of initial condition for $\Gamma(t)$, the first term is the cost of using the primary feedback K_s, from the initial time $t = 0$ and never switching to the alternative strategy. Hence, the use of alternative feedback would be beneficial only if there exists a time $\tau > 0$ such that

$$\langle x_s, \Gamma(\tau) x_s \rangle - \langle x_s, \Gamma_0 x_s \rangle = \langle x_s, (\Gamma(\tau) - \Gamma_0) x_s \rangle > 0 \tag{3.30}$$

For an initial condition $\xi \in \Re^n$, we can write Eq. 3.30 as

$$\sigma(\xi, \tau) = \langle \xi, e^{A_u^T \tau} (\Gamma(\tau) - \Gamma_0) e^{A_u \tau} \xi \rangle > 0 \tag{3.31}$$

□

The above theorem shows that if the alternate feedback is applied for τ seconds, the performance of the system is improved. From the above theorem it was difficult to establish a time of switching. One alternative is to maximize the function σ, which might be difficult to implement. Here we can note that since $\Gamma(0) = \Gamma_0$, the function $\sigma(\xi, \tau)$ is zero for $\tau = 0$. Hence if the derivative of σ at $\tau = 0$ is positive, then we can say that there will be a time interval of length $\tau > 0$ where the function σ will be positive. Thus we can state the following corollary

Corollary 3.2.1. *If for an initial state, ξ, the following conditions holds*

$$\frac{d\sigma(\xi, \tau)}{d\tau}|_{\tau=0} = -\langle \xi, (A_u^T \Gamma_0 + \Gamma_0 A_u + C^T C)\xi \rangle > 0 \tag{3.32}$$

then for small enough values of τ the function σ will be positive and the alternative control is beneficial.

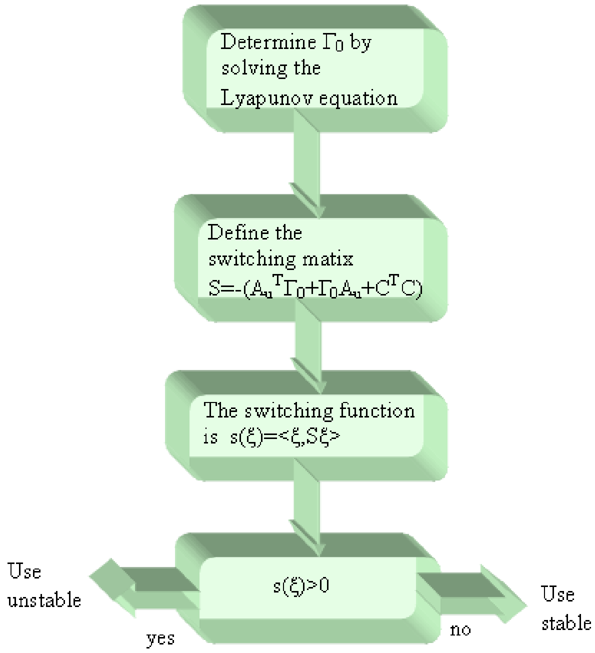

FIGURE 3.1. Diagrammatic representation of the control law

The control strategy can be explained in the following steps (Fig. 3.1):

- Define $K_s, \quad K_u$ with $A_s = A - BK_s$ asymptotically stable.

- Determine Γ_0 by solving the algebraic Lyapunov equation

  $A_s^T\Gamma_0 + \Gamma_0 A_s = -C^T C$

- Using $A_u = A - BK_u$, define the switching matrix.

$$S = -(A_u^T\Gamma_0 + \Gamma_0 A_u + C^T C) \qquad (3.33)$$

- Using alternate control K_u if $s(x) = \langle x, Sx\rangle > 0$.

The switching region is the set of all states which satisfy $s = 0$.

3.2.1 Stability of the Switching Strategy

In this section we prove the asymptotic stability of performance based switching control.

Theorem 3.3. *If the system is observable and sliding does not exist, the feedback control system using the switching strategy defined above is asymptotically stable.*

Proof. Firstly, we assume that the switching occurs ideally; i.e., when $s(x) \leq 0$ the primary feedback, is in place and when $s(x) > 0$ the alternative feedback K_u is used without any delays. Recall that

$$s(x) = -\left\langle x, (A_u^T\Gamma_0 + \Gamma_0 A_u + C^T C)x\right\rangle \tag{3.34}$$

Let

$$A_c = \gamma A_u + (1-\gamma)A_s \tag{3.35}$$

for $\gamma = \{0, 1\}$. When $s(x) > 0$ we use alternate feedback, hence $\gamma = 1$. When $s(x) < 0$, we use primary feedback and hence $\gamma = 0$. In the absence of sliding, $\dot{x} = A_c x$ at any instant of time. Consider the positive definite function

$$V(x) = \langle x, \Gamma_0 x\rangle \tag{3.36}$$

For Lyapunov stability consider

$$\begin{aligned}
\dot{V} &= \frac{d\langle x, \Gamma_0 x\rangle}{dt} \\
&= \langle x, (A_c^T\Gamma_0 + \Gamma_0 A_c)x\rangle \\
&= \langle x, A_c^T\Gamma_0 x\rangle + \langle x, \Gamma_0 A_c x\rangle \\
&= \langle x, (\gamma A_u + (1-\gamma)A_s)\Gamma_0 x\rangle + \langle x, \Gamma_0(\gamma A_u + (1-\gamma)A_s)x\rangle \\
&= \langle x, \gamma A_u^T\Gamma_0 x\rangle + \langle x, (1-\gamma)A_s^T\Gamma_0 x\rangle + \langle x, \gamma\Gamma_0 A_u x\rangle + \langle x, \gamma\Gamma_0 A_s x\rangle \\
&= \gamma\langle x, (A_u^T\Gamma_0 + \Gamma_0 A_u)x\rangle + (1-\gamma)\langle x, (A_s^T\Gamma_0 + \Gamma_0 A_s)x\rangle \\
&= -\gamma s(x) - (1-\gamma)\langle x, C^T C x\rangle
\end{aligned} \tag{3.37}$$

If the system is observable, it is well known that the function $-\langle x, C^T Cx\rangle$ is non positive and cannot be zero over any time interval. When $s < 0$, $\gamma = 0$, the term $\gamma s(x)$ vanishes from Eq. 3.37. When $s > 0$, $\gamma = 1$ and $\dot{V}$ is negative. It is clear we can find a single lyapunov function $V(x)$ such that $\dot{V}$ is negative for any $x \in \Re^n$. Hence we can state that the system is asymptotically stable, provided there is no sliding. $\square$

It should be noted from theorem 3.3 that the presence of sliding does not ensure stability. During sliding motion the structure of the system is neither the primary nor the alternate structures. Hence it is important to study the existence of sliding along each of the switching subspaces.

3.3 Geometry of the Switching Region

In this section we analyze the switching function $s(x)$ in detail and study the existence of sliding motion. Our results are based on simple geometric argument. For a matrix L, we denote its range space, that is subspace spanned by columns of L, as $r[L]$. The null space of L which is the subspace $\{x : Lx = 0\}$ is denoted by $n[L]$. For any subspace V, the orthogonal complement is denoted by $V^\perp$.

We have introduced the switching function

$$s(x) \quad = \quad \langle x, Sx\rangle \tag{3.38}$$

where

$$S \quad = \quad -(A_u^T\Gamma_0 + \Gamma_0 A_u + C^T C) \tag{3.39}$$

Since Γ_0 satisfies the Lyapunov equation Eq. 3.11, the matrix S can also be expressed as

$$\begin{aligned} S \quad &= \quad (A_s^T - A_u^T)\Gamma_0 + \Gamma_0(A_s - A_u) \\ &= \quad (K_u - K_s)^T B^T\Gamma_0 + \Gamma_0 B(K_u - K_s) \end{aligned} \tag{3.40}$$

Hence, the switching function can be written as

$$s(x) = \langle B^T\Gamma_0 x, (K_u - K_s)x\rangle \tag{3.41}$$

Let

$$\alpha_1(x) = B^T\Gamma_0 x, \quad \alpha_2(x) = (K_u - K_s)x \tag{3.42}$$

In a single input case $B^T\Gamma_0 x$ and $(K_u - K_s)x$ are scalars and $s(x) = \alpha_1(x)\alpha_2(x)$. If $B^T\Gamma_0$ and $K_u - K_s$ are linearly dependent, we can state the following Lemma 2.

Lemma 2. *If $B^T\Gamma_0$ and $(K_u - K_s)$ are linearly dependent then, $s(x)$ never changes sign.*

Proof. We know that

$$\alpha_1(x) = B^T\Gamma_0 x, \quad \alpha_2(x) = (K_u - K_s)x \tag{3.43}$$

If α_1 and α_2 are linearly dependent, we have a nonzero constant μ such that $\alpha_1 = \mu\alpha_2$. Then,

$$s(x) = \alpha_1(x)\alpha_2(x) = \mu(\alpha(x))^2 \tag{3.44}$$

From Eq. 3.44, for any $x \in \Re^n$, the sign of $s(x)$ depends on the sign of μ which is a constant. Hence the sign of $s(x)$ never changes and switching cannot occur. $\square$

In view of Lemma 2, we assume that $B^T\Gamma_0$ and $(K_u - K_s)$ are linearly independent. Hence the switching surface $\{x : s(x) = 0\}$ is union of the two regions (Fig. 3.2)

$$\{x : s(x) = 0\} = \{x : (K_u - K_s)x = 0\} \bigcup \{x : B^T\Gamma_0 x = 0\} \tag{3.45}$$

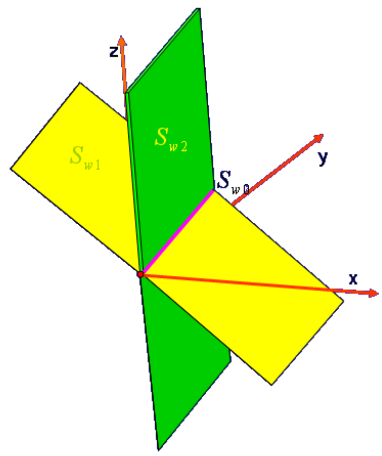

FIGURE 3.2. Switching regions

This switching surfaces can be divided into following regions

$$S_{w1} = \{x : \alpha_1(x) = 0, \quad \alpha_2(x) \neq 0\} \tag{3.46}$$

$$S_{w2} = \{x : \alpha_1(x) \neq 0, \quad \alpha_2(x) = 0\} \tag{3.47}$$

$$S_{w0} = \{x : \alpha_1(x) = 0, \quad \alpha_2(x) = 0\} \tag{3.48}$$

The region S_{w0} is the null space of the switching matrix S. The gradient of the switching function which is $g_s(x) = Sx$, will be zero along S_{w0}. These switching regions divide the complete state space in four regions (Fig. 3.3).

$$R_1 = \{x : \Gamma_0 Bx > 0\} \quad \text{and} \quad \{x : (K_u - K_s)^T x > 0\} \tag{3.49}$$

$$R_2 = \{x : \Gamma_0 Bx < 0\} \quad \text{and} \quad \{x : (K_u - K_s)^T x > 0\} \tag{3.50}$$

$$R_3 = \{x : \Gamma_0 Bx > 0\} \quad \text{and} \quad \{x : (K_u - K_s)^T x < 0\} \tag{3.51}$$

$$R_4 = \{x : \Gamma_0 Bx < 0\} \quad \text{and} \quad \{x : (K_u - K_s)^T x < 0\} \tag{3.52}$$

From the derived control law we can see that the alternate feedback is used in regions R_1 and R_4 and primary feedback is used in regions R_2 and R_4.

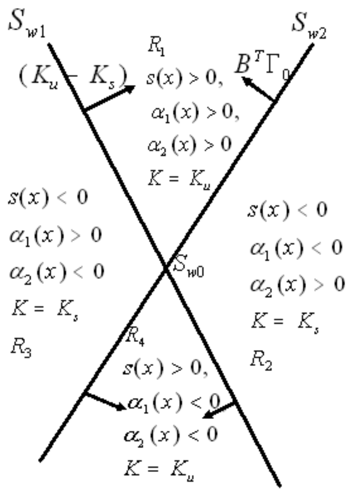

FIGURE 3.3. Visualization of the complete geometry of the switching region

We can define two velocity vectors at any state

$$\dot{x}_u = A_u x$$
$$\dot{x}_s = A_s x \tag{3.53}$$

Since $A_u = A - BK_u$ and $A_s = A - BK_s$,we have

$$\dot{x}_s = \dot{x}_u + B(K_u - K_s)x \tag{3.54}$$

The derivative with respect to time of the switching function

$$\frac{ds(x)}{dt} = \langle \dot{x}, Sx \rangle + \langle x, S\dot{x} \rangle \tag{3.55}$$

For Eq. 3.53 and Eq. 3.55, we have the time derivative of the switching function in both cases to be

$$\dot{s}_u = \langle A_u x, Sx \rangle + \langle x, SA_u x \rangle \tag{3.56}$$

$$\dot{s}_u = \langle A_u x, Sx \rangle + \langle x, SA_s x \rangle \tag{3.57}$$

From Eq. 3.54 we have

$$\begin{aligned}\dot{s}_s &= \dot{s}_u + \langle B(K_u - K_s), Sx\rangle \\ &= \dot{s}_u(x) + \langle (K_u - K_s)B(K_u - K_s)x, B^T\Gamma_0 x\rangle + B(K_u - K_s)x, \Gamma_0 B(K_u - K_s)x\rangle\end{aligned} \tag{3.58}$$

From Eq. 3.58 we can state the follow

- For any state, $z_0 \in n[K_u - K_s] \bigcap n[B^T\Gamma_0]$, $\dot{s}_u(z_0) = \dot{s}_s(z_0) = 0$.
- For any state, $z_2 \in n[\Gamma_0 B]$, the time derivative of the switching function when using primary feedback and alternate feedbacks are related as

$$z_2 \in n[\Gamma_0 B] \Rightarrow \dot{s}_s(z_2) > \dot{s}_u(z_2) \tag{3.59}$$

- For any state, $z_1 \in n[(K_u - K_s)^T]$, the time derivative of the switching function using primary and alternate feedbacks are related as

$$z_1 \in n[(K_u - K_s)^T] \Rightarrow \dot{s}_s(z_1) \quad = \quad \dot{s}_u(z_1) \tag{3.60}$$

3.3.1 Conditions for Sliding to Exist

Before studying the existence of sliding, it is important to know the conditions that could cause the sliding motion. Hence we give an overview of the necessary conditions for sliding to exist in this section. Utkin [38], stated that sliding exists on a switching plane $s = 0$ if the state trajectories are directed towards the plane as shown in Fig. 3.4. That is

$$\lim_{s\to 0^+} \dot{s} < 0 \quad \text{and} \quad \lim_{s\to 0^-} \dot{s} > 0 \tag{3.61}$$

From the above discussion we have the following proposition.

Lemma 3. *Sliding exists if the following conditions is satisfied along the switching surface.*

$$\dot{s}_u < 0 \quad and \quad \dot{s}_s > 0 \tag{3.62}$$

Proof. For our case, we use K_u when $s(x) > 0$ and K_s when $s(x) < 0$. Hence, we have

$$\lim_{s \to 0^+} \dot{s} \;=\; \langle A_u x, Sx \rangle, \quad \lim_{s \to 0^-} \dot{s} = \langle A_s x, Sx \rangle \tag{3.63}$$

The condition for the existence of sliding can be explained as follows

If when using the strategy K_u, the trajectories approach the switching subspace; i.e., $\dot{s}(x) < 0$, then when using the strategy K_s, the trajectories should approach the subspace and $\dot{s}(x) > 0$ (Fig. 3.4).

We can derive a similar condition for each of the switching surfaces S_{w1} and S_{w2}. Let the switching surfaces be represented by $\alpha_1(x) = \langle x, \Gamma_0 B \rangle$ and $\alpha_2(x) = \langle x, (K_u - K_s) \rangle$. The condition for sliding to exist along a switching surface S_{wi}, for $i = \{1, 2\}$ is

$$\dot{\alpha}_{is} > 0 \quad \text{and} \quad \dot{\alpha}_{iu} < 0 \tag{3.64}$$

where

$$\begin{aligned} \dot{\alpha}_{1s} &= \langle A_s x, \Gamma_0 B \rangle, \quad \dot{\alpha}_{1u} = \langle A_u x, \Gamma_0 B \rangle \\ \dot{\alpha}_{2s} &= \langle A_s x, (K_u - K_s) \rangle, \quad \dot{\alpha}_{1u} = \langle A_u x, (K_u - K_s) \rangle \end{aligned}$$

□

3.3.2 Sliding on S_{w1}

Along the subspace, S_{w1}, the gradient $g_s(x) = Sx$ is non zero and is normal to the subspace as shown in Fig. 3.5. It is clear that a sufficient condition for sliding

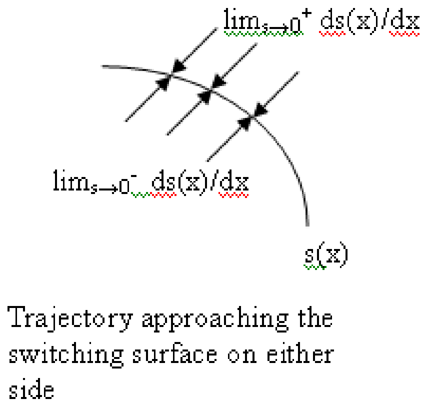

FIGURE 3.4. Condition for sliding to exist

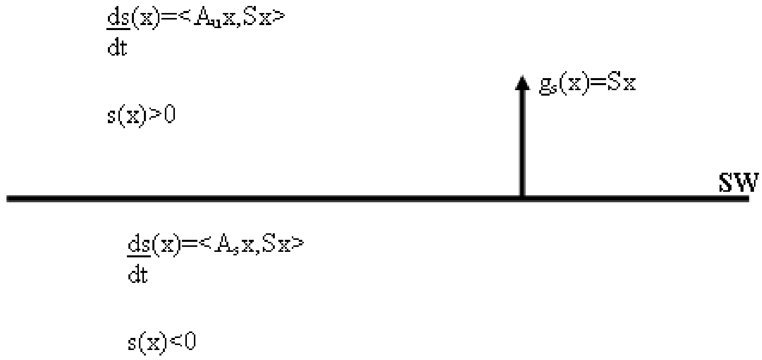

FIGURE 3.5. Geometry of the switching regions S_{w1} and S_{w2}

mode not to occur is that the trajectories "should not push into the subspace at the same time (Fig. 3.4)." The condition can be stated as follows

If when using the strategy K_u, the trajectories approach the switching subspace; i.e., $\dot{s}(x) < 0$, then when using the strategy K_s, the trajectories should move away from the subspace and again $\dot{s}(x) < 0$. If when using K_u the trajectories move away from the subspace ($\dot{s}(x) > 0$) then the trajectories should move into the subspace when strategy K_s is used or $\dot{s}(x) > 0$. Thus in this region, the states satisfy

$$(K_u - K_s)x = 0 \Leftrightarrow K_u x = K_s x \tag{3.65}$$

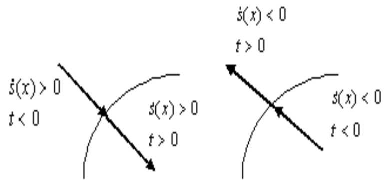

FIGURE 3.6. Switching in subspace S_{w1}

The following lemma immediately follows from the relation Eq. 3.65

Lemma 4. *Sliding does not exist in the region S_{w1}.*

Proof. In region S_{w1}, the control effort for using the primary feedback and alternate feedback is the same. This implies that trajectories posses the same slope before and after switching

$$\begin{aligned}
\dot{x}_u &= A_u x \\
&= (A - BK_u)x \\
&= (A - BK_s)x \\
&= A_s x \\
&= \dot{x}_s \qquad (3.66)
\end{aligned}$$

Since the slope of the trajectories before switching and after switching is the same, we cannot have trajectories pushing into or away from the switching surface at the same time as in Fig. 3.6. Hence we cannot have sliding in this region. $\square$

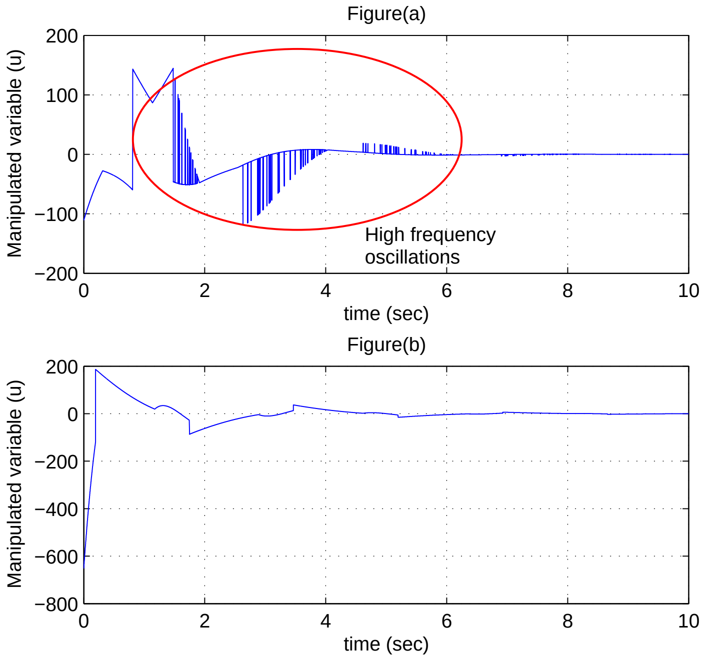

FIGURE 3.7. Control input of a fourth order plant (Helicopter) for a randomly chosen initial condition in the switching region S_{w2}: (a)Unstable alternate feedback - sliding is present; (b) Stable alternate feedback - no sliding

3.3.3 Switching on S_{w0} and S_{w2}

The switching regions S_{w0} and S_{w2} are more critical. In the region S_{w0}, the switching function $s = \langle x, Sx \rangle = 0$. In the region S_{w2}, control efforts are different for the two control strategies. We do not have a generalized proof for the non-existence of sliding along these surfaces. Infact, sliding has been detected present in the region S_{w2}. For example, Fig. 3.7 shows the control inputs of a fourth order plant (see section 4.3) for a randomly selected initial condition in region S_{w2}. Fig 3.7(a) shows the control input when the alternate feedback was unstable. Fig. 3.7(b) shows the control input, for the same initial condition and a stable alternate feedback designed by choosing lesser weight on the control cost. This example illustrates that chattering in the region S_{w2} might be absent if a stable alternate feedback was chosen.

We have simulated a fourth order system (see section 4.3) for about 1000 initial conditions, selected at random, on surfaces S_{w0} and S_{w2} respectively. Dual vector approach was used to select these initial condition (for details see Appendix A). Conditions given in Lemma 3 of section 3.3.1 were used to detect the presence of sliding on any switching surface. For example, taking $\lambda_3 = 9$ and $z = \begin{pmatrix} -5.37 & 2.14 & -0.28 & 7.83 \end{pmatrix}$ in Eq. 5.6, we get a point in the region S_{w0}.

$$x_{Sw0} = \begin{pmatrix} -23.34 & -19.56 & 10.02 & 4.54 \end{pmatrix} \tag{3.67}$$

According to Eq. 3.64, the condition for presence of sliding in region S_{w0} is

$$\dot{\alpha}_{1s}\dot{\alpha}_{1u} < 0 \quad \text{and} \quad \dot{\alpha}_{2s}\dot{\alpha}_{2u} < 0 \tag{3.68}$$

That is

$$\begin{aligned} \langle \Gamma_0 B, A_s x \rangle . \langle \Gamma_0 B, A_u x \rangle < 0 \quad \text{and} \\ \langle (K_u - K_s)^T, A_u x \rangle . \langle (K_u - K_s)^T, A_s x \rangle < 0 \end{aligned} \tag{3.69}$$

For the above obtained initial condition x_{Sw0}, we have $\dot{\alpha}_{1s}\dot{\alpha}_{1u} = 277$ and $\dot{\alpha}_{2s}\dot{\alpha}_{2u} = 43595$. This shows that sliding is not present at x_{Sw0}. Similar analysis of other points in S_{w0} showed that sliding was absent in this region. However, sliding was detected in the region S_{w2}.

There are many methods to eliminate chattering, which are studied in [17], [31]. The easiest way of removing chattering is to use lowpass filter. This lowpass filter blocks the high frequency oscillations of the manipulated variable. Thus the actual input to the plant does not have any high frequency switching. The transfer function of the filter is given by

$$G(s) = \frac{1}{Ts + 1} \tag{3.70}$$

A higher value of T will reduce the oscillations of the manipulated variable, while a smaller value would speed up the transient response. The value of T should be chosen such that the filter eliminates the chattering without, significantly effecting the transient response. The value of T should be chosen very carefully as it can make the system unstable in some cases.

Chapter 4
Results and Discussion

In this section, we give examples of single input single output LTI systems to illustrate the established theory about PBSC. In the present work, all the primary controllers are LQ regulators. The unstable alternate feedback was designed by reflecting the primary poles about the imaginary axis. The inexpensive controller was a LQ regulator with less weight on the control cost. It should be noted that sliding was present when PBSC was used with an unstable alternate feedback and chattering elimination techniques were not used in some of these examples.

4.1 Second Order System

The control strategy discussed in previous chapters has been tested by simulating a double integrator system controlled by performance based switching controller. We considered two choices of alternate feedbacks, namely- inexpensive controller and unstable controller. Here the primary controller is an LQR with the design parameters Q and R chosen as C^TC and 1 respectively. The unstable alternate controller is designed by reflecting the primary poles about the imaginary axis. The inexpensive controller is designed by decreasing the weight on the control cost to $R = 0.01$. Thus, the primary controller places the poles at $p_{1,2} = -\frac{1}{\sqrt{2}} \pm \frac{j}{\sqrt{2}}$ while the unstable controller places them at $p_{1,2} = \frac{1}{\sqrt{2}} \pm \frac{j}{\sqrt{2}}$. The "inexpensive" controller places them at $p_{1,2} = -2.2361 \pm 2.2361j$. For this example, it can be proved that sliding does not exist when using unstable alternate control.

4.1.1 Unstable Alternate Feedback

The stable and unstable structures are given below

$$A_s = \begin{bmatrix} -1.4142 & -1 \\ 1 & 0 \end{bmatrix} \tag{4.1}$$

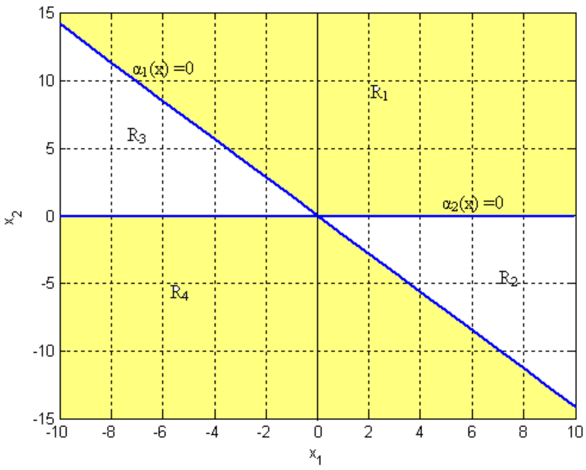

FIGURE 4.1. The switching surfaces α_1 and α_2 and the regions R_1, R_2, R_3, R_4 for a second order system

$$A_u = \begin{bmatrix} 1.4142 & -1 \\ 1 & 0 \end{bmatrix} \tag{4.2}$$

The solution of lypaunov equation is given by

$$\Gamma_0 = \begin{bmatrix} 0.3536 & 0.5 \\ 0.5 & 1.06 \end{bmatrix} \qquad s(\xi) = -(\xi_2(1.4142\xi_1 + \xi_2)) \tag{4.3}$$

The switching matrix, computed from the Eqn. 3.33 is

$$S = \begin{bmatrix} -2 & -1.4142 \\ -1.4142 & 0 \end{bmatrix} \tag{4.4}$$

The switching function is $s(x_1, x_2) = (x_2^2 + 1.4142x_1x_2)$. The two switching functions are $\alpha_1(x) = 1.4142x_1 + x_2 = 0$ and $\alpha_2(x) = x_2 = 0$. The switching surfaces and the four regions are shown in Fig. 4.1

Switching Along S_{w1}

For the stable system we have

$$\begin{aligned} \dot{x}_{1s} &= 0 \\ \dot{x}_{2s} &= -x_{1s} \end{aligned} \tag{4.5}$$

Similarly for the unstable system we have

$$\begin{aligned} \dot{x}_{1u} &= 0 \\ \dot{x}_{2u} &= -x_{1u} \end{aligned} \tag{4.6}$$

Hence the gradient at the switching surface is same for both the structures . Both the strategies move towards the switching line before switching and move away from switching surface after switching. Hence we can say that sliding cannot occur along this surface.

Switching Along S_{w2}. Considering the primary feedback we have

$$\begin{aligned} \dot{x}_{1s} &= x_{2s} \\ \dot{x}_{2s} &= -1.4142x_{2s} - x_{1s} = -2.1214x_{2s} \end{aligned} \tag{4.7}$$

Considering the alternate feedback, we have

$$\begin{aligned} \dot{x}_{1u} &= x_{2u} \\ \dot{x}_{2u} &= 1.4142x_{2s} - x_{1s} = 0.707x_{2s} \end{aligned} \tag{4.8}$$

The time derivative of the switching function $\alpha_1(x) = 0$ is given by

$$\dot{\alpha}_{1c}(x_c) = \langle \Gamma_0 B, \dot{x}_c \rangle = \langle \Gamma_0 B, A_c x_c \rangle \tag{4.9}$$

where $c = \{s, u\}$. When using primary feedback, the time derivative is $\dot{\alpha}_{1s}(x_s) = 0.25x_{2s}$. When unstable alternate feedback is used, the time derivative is $\dot{\alpha}_{1u}(x_u) = 1.25x_{2u}$. Now, consider a system trajectory moving from region R_3 to R_1 . In these regions, since $x_{2u}, x_{2s} > 0$ (Fig. 4.1) we have $\dot{\alpha}_{1s}(x_s), \dot{\alpha}_{1u}(x_u) > 0$. Since $\dot{\alpha}_{1s}(x_s)$ and $\dot{\alpha}_{1u}(x_u)$ are of same sign, condition for sliding given in Eq. 3.64 is not satisfied and sliding does not exist.

Similarly, if a system trajectory moving from region R_2 to R_4 is considered, we would have $\dot{\alpha}_{1s}(x_s), \dot{\alpha}_{1u}(x_u) < 0$. $\dot{\alpha}_{1s}(x_s)$ and $\dot{\alpha}_{1u}(x_u)$ are of same sign and sliding is absent. From this we can say that sliding cannot exist along the switching surface S_{w2}

Switching Along S_{w0}. Along the null space $1.4142x_1 + x_2 = 0$ and $x_2 = 0$ that is $(x_1, x_2) = (0, 0)$. The null space in second order case is the origin. Since $\dot{x} = 0$ in this region, trajectories reaching S_{w0} stay in this region for the rest of time. Hence switching does not occur in this region. Thus we prove that no sliding exists in the second order case.

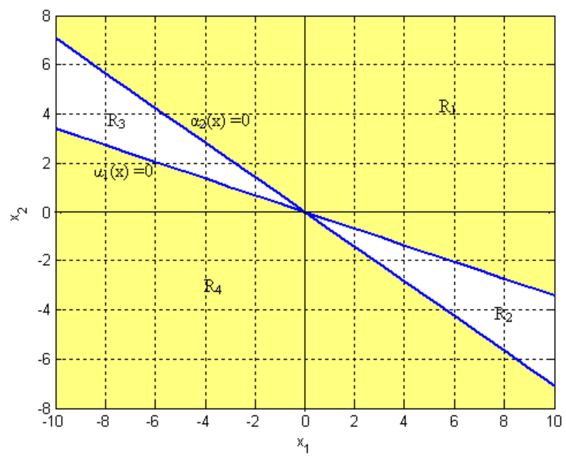

FIGURE 4.2. The switching surfaces α_1 and α_2 and the regions R_1, R_2, R_3, R_4 for a second order system

4.1.2 Stable Alternate Feedback

When a stable alternate feedback is considered we have

$$\begin{aligned} \dot{x}_{1u} &= x_{2u} \\ \dot{x}_{2u} &= 4.4721x_{2u} - 10x_{1u} = 2.599x_{2u} \end{aligned} \tag{4.10}$$

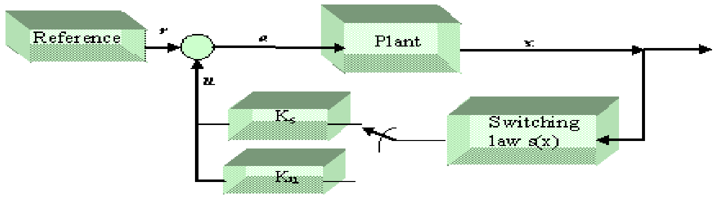

FIGURE 4.3. General implementation for performance based switching control

The switching function is $s(x_1, x_2) = (9x_1^2 + 9.4218x_1x_2 + 2.1623x_2^2)$. The two switching functions are $\alpha_1(x) = 1.4142x_1 + x_2 = 0$ and $\alpha_2(x) = 2.9507x_1 + x_2 = 0$. The switching surfaces and the four regions are shown in Fig. 4.2.

Since we can prove that there is no sliding along S_{w1} and S_{w0}, we will consider only the switching surface S_{w2}. Proceeding as in the previous section, the time derivative of the switching function $\alpha_1(x) = 0$

$$\dot{\alpha}_{1c}(x_c) = \langle \Gamma_0 B, \dot{x}_c \rangle = \langle \Gamma_0 B, A_c x_c \rangle \tag{4.11}$$

where $c = \{s, u\}$. When using primary feedback, the time derivative is $\dot{\alpha}_{1s}(x_s) = 0.25x_{2s}$. When the inexpensive alternate feedback is used, the time derivative is $\dot{\alpha}_{1u}(x_u) = 1.42x_{2u}$. Now, consider a system trajectory moving from region R_4 to R_3 . In these regions, since $x_{2u}, x_{2s} > 0$ (Fig. 4.2) we have $\dot{\alpha}_{1s}(x_s), \dot{\alpha}_{1u}(x_u) > 0$. Since $\dot{\alpha}_{1s}(x_s)$ and $\dot{\alpha}_{1u}(x_u)$ are of same sign, condition for sliding given in Eq. 3.64 is not satisfied and sliding does not exist. Similarly, if a system trajectory moving from region R_1 to R_2 is considered, we would have $\dot{\alpha}_{1s}(x_s), \dot{\alpha}_{1u}(x_u) < 0$. $\dot{\alpha}_{1s}(x_s)$ and $\dot{\alpha}_{1u}(x_u)$ are of same sign and sliding is absent. Similar explanation can be given for system trajectories moving from R_2 to R_4. This shows that sliding cannot

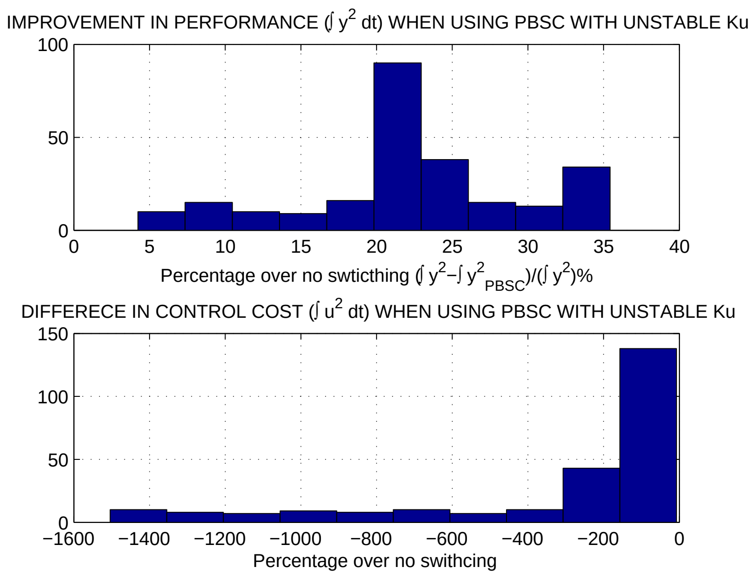

FIGURE 4.4. Improvement in performance and difference in control cost of the PBSC with unstable alternate feedback over the state feedback control

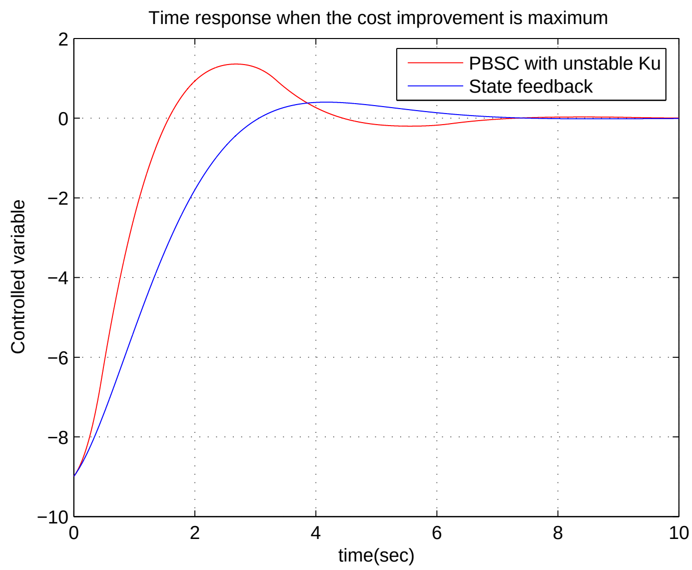

FIGURE 4.5. Time response when the improvement is maximum

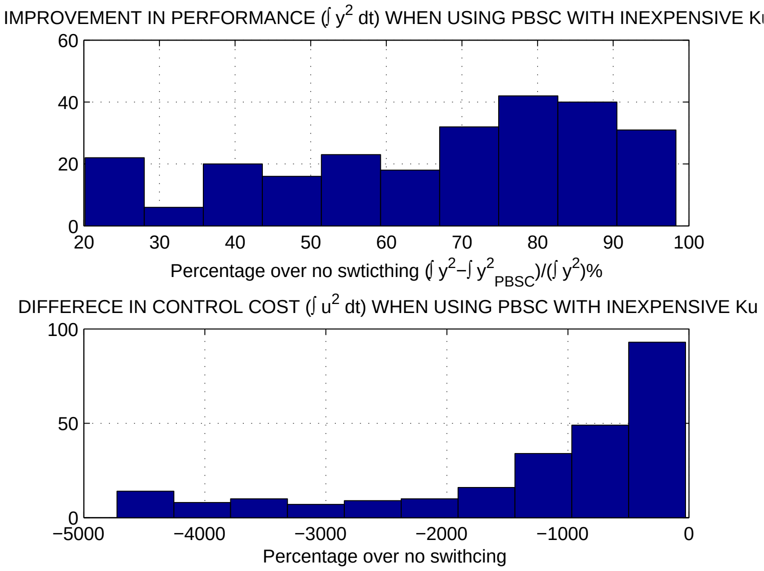

FIGURE 4.6. Improvement in performance of the PBSC with inexpensive feedback over the state feedback

exist along the switching surface S_{w2}. The basic construction of the system can be shown in the Fig. 4.3.

4.1.3 Improvement in Performance

Average improvement in performance was estimated by simulating the model for different initial conditions. The initial conditions were selected at random in the square $-10 < x_1, x_2 < 10$. About 250 initial conditions were considered. The percentage improvement in performance was calculated as

$$\begin{aligned} \text{Percentage improvement in performance} & \\ \text{over the state feedback control system} &= \frac{P_s - P_{PBSC}}{P_s} * 100 \\ \text{where} & \\ \text{Performance of the PBSC system } P_{PBSC} &= \int_0^\infty \|y_{PBSC}\|^2 dt \\ \text{Performance of the state feedback control system } P_s &= \int_0^\infty \|y_s\|^2 dt \end{aligned}$$

The average improvement is 23% with a standard deviation of 7.3%. Fig. 4.4 shows the plot of improvement in performance and difference in control cost. In the figure

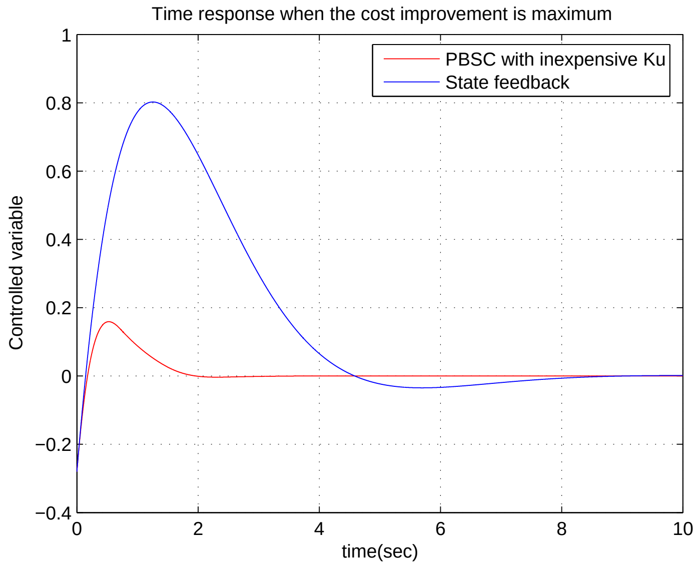

FIGURE 4.7. Time response when the improvement is maximum

y indicates the output of the system with state feedback control strategy and y_{PBSC} indicate the output of the system with PBSC strategy. The time response when the improvement in performance is maximum is shown in Fig. 4.5. Fig 4.6 shows the improvement in performance and difference in control cost when the alternate feedback is designed using inexpensive control. In this case the mean improvement is 66% with a standard deviation of 21%. Fig. 4.7 shows the time response when the improvement is maximum. The maximum improvement is about 98% when compared to 35% for the PBSC with unstable feedback. However, the control cost is high in the case of PBSC with alternate inexpensive control. In the present case, PBSC with inexpensive control gave better improvement in performance when compared to PBSC with unstable feedback.

4.2 Third Order System

The proposed control strategy was implemented for a third order system. The third order plant considered had three poles at the origin given by the transfer function

$$G(s) = \frac{1}{s^3}$$

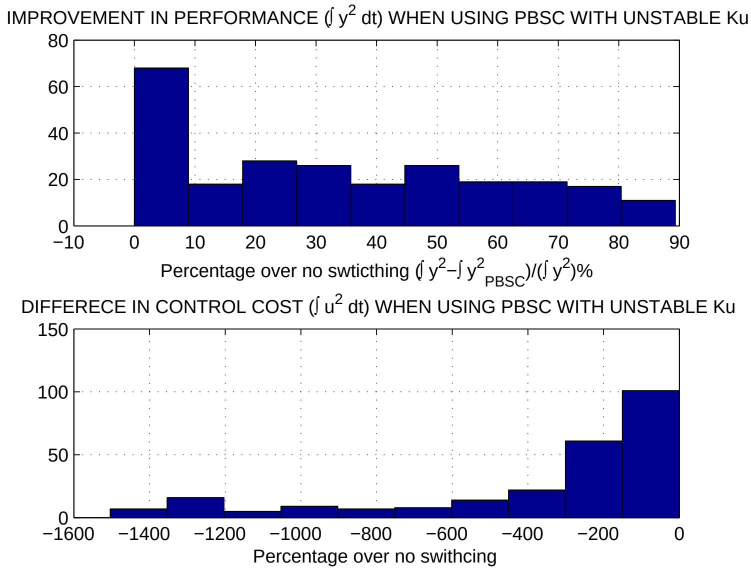

FIGURE 4.8. Improvement in performance and difference in control cost of the PBSC with unstable alternate feedback over the state feedback control

Here the primary controller is an LQR with the design parameters Q and R chosen as C^TC and 1 respectively. The unstable alternate controller is designed by reflecting the primary poles about the imaginary axis. The inexpensive controller is designed by decreasing the weight on the control cost to $R = 0.01$. Thus, the primary controller places the poles at $p_{1,2,3} = \{-0.5, -0.5 + j0.86, -0.5 - j0.86\}$ while the unstable alternate controller places them at $p_{1,2,3} = \{0.5, 0.5+j0.86, 0.5-j0.86\}$. The "inexpensive" controller places the poles at $p_{1,2,3} = \{-2, 15, -1.0772+j1.8658, -1.0772 - j1.8658\}$.

4.2.1 Improvement in Performance

The average improvement in performance was found out by simulating the third order system for about 250 initial conditions. The initial conditions were selected at random in the cube $-10 < x_1, x_2, x_3 < 10$. The improvement in performance and difference in control cost for each simulation is presented in Fig. 4.8. In the figure y indicates the output of the system with state feedback control strategy and y_{PBSC} indicate the output of the system with PBSC strategy. It was observed

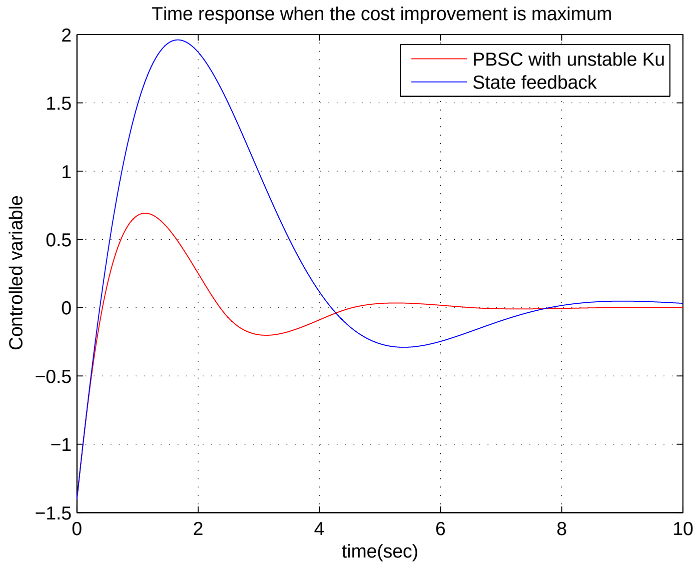

FIGURE 4.9. Time response when the improvement in performance is maximum

that the average improvement was 33% with a standard deviation of about 26%. Maximum improvement was about 89%. Fig. 4.9 show the time response of the system starting from initial condition with maximum improvement. The Fig. 4.10 shows the improvement in performance and the difference in control cost when using the alternate feedback designed using inexpensive control. In this case the mean improvement was 57% with a standard deviation of 25%. The maximum improvement was about 98% the time response of which is shown in Fig. 4.11.

4.3 Fourth Order System

In this section we present the simulations of a single rotor helicopter model near hover. It is a fourth order system obtained from [3]. The systems has one pole at the origin, two unstable poles and two unstable zeros (Eq. 4.12).

$$p_{1,2,3,4} = 0, \quad -0.6789, \quad 0.1224 + j0.3792, \quad 0.1224 - j0.3792 \qquad (4.12)$$

$$z_{1,2} = 0.2505 + j2.4914, \quad 0.2505 - j2.4914. \qquad (4.13)$$

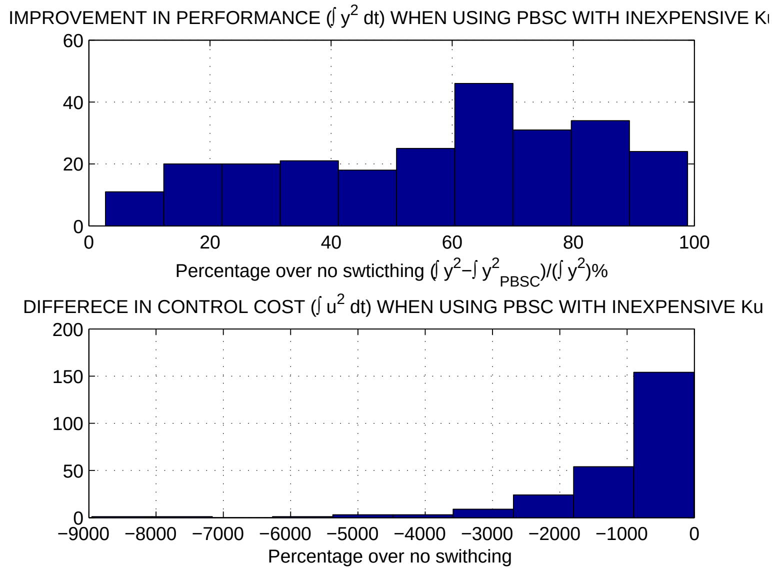

FIGURE 4.10. Improvement in performance and difference in control cost of the PBSC with inexpensive alternate feedback over the state feedback control

The A, B, C and D matrices are

$$A = \begin{pmatrix} -0.4341 & 0.0074 & -0.1078 & 0 \\ 1.0000 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.14}$$

$$C = \begin{pmatrix} 0 & 1.0000 & -0.5010 & 6.2698 \end{pmatrix} \quad D = 0 \tag{4.15}$$

Here the primary controller is an LQR with the design parameters Q and R chosen as C^TC and 1 respectively. The unstable alternate controller is designed by reflecting the primary poles about the imaginary axis. The inexpensive controller is designed by decreasing the weight on the control cost to $R = 0.01$. Thus, the primary controller placed the poles at $p_{1,2,3,4} = -1.5357 - j0.7475, \quad -1.5357 + j0.7475, \quad -0.475 - j1.3869, \quad -0.475 + j1.3869$ and the unstable alternate controller places them at $p_{1,2,3,4} = 1.5357 - j0.7475, \quad 1.5357 + j0.7475, \quad 0.475 -$

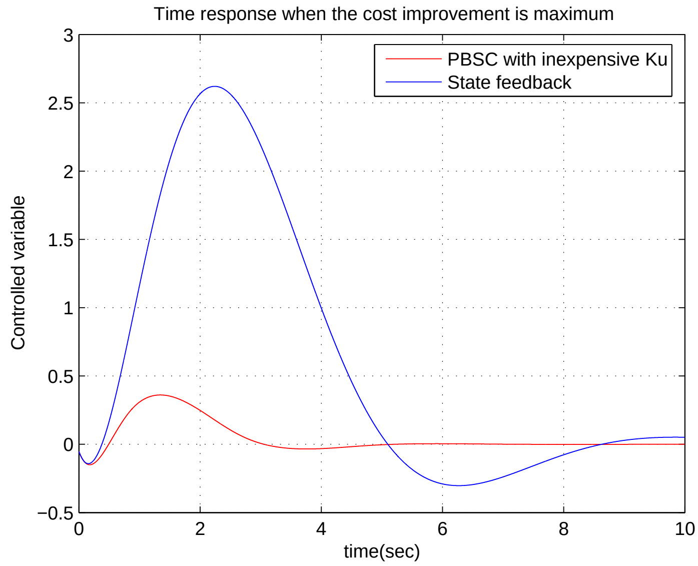

FIGURE 4.11. Time response when the performance is maximum

$j1.3869, \quad 0.475 + j1.3869$. The inexpensive controller places the poles at $p_{1,2,3,4} = -3 - j2, \quad -3 + j2, \quad -0.45 - j2.18, \quad -0.45 + j2.18$

4.3.1 Improvement in Performance

The improvement in performance was found out by simulating the system for about 250 random initial conditions. Fig. 4.12 shows the improvement in performance and difference in control cost. In the figure y indicates the output of the system with state feedback control strategy and y_{PBSC} indicate the output of the system with PBSC strategy. The mean improvement was found to be 25% with a standard deviation of about 23%. The maximum improvement is found to be 93%. The time response when the improvement is maximum is given in Fig. 4.13.

Fig. 4.14 shows the control input for an arbitrary initial condition. It is clear from Fig. 4.14 that sliding exists when alternative feedback is unstable. A lowpass filter was used to remove the high frequency oscillations. When the time constant of the filter was chosen to be 0.5, the system goes unstable (Fig. 4.13). For a filter time constant of 0.01, amplitude of oscillations decreased but sliding was still present (Fig. 4.14). From Fig. 4.17 it is clear that the the use of filter with low time

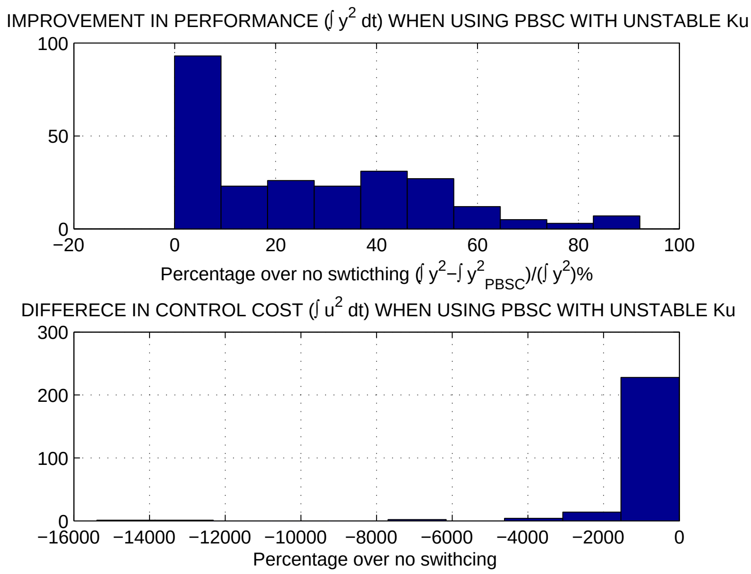

FIGURE 4.12. Improvement in performance and difference in control cost of the PBSC with inexpensive alternate feedback over the state feedback control

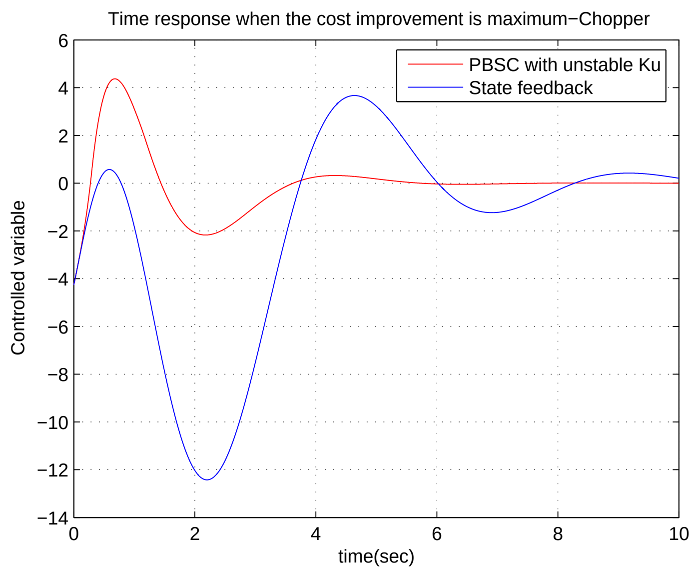

FIGURE 4.13. Time response when the performance is maximum

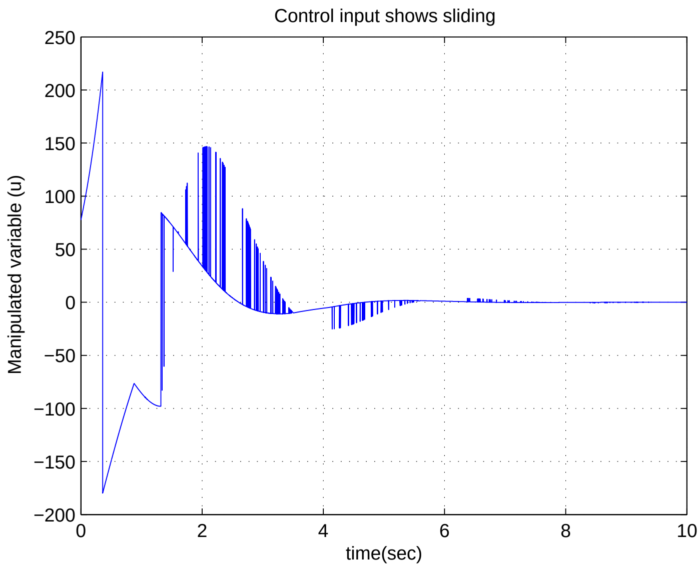

FIGURE 4.14. Control input when filter is not used- Sliding is present

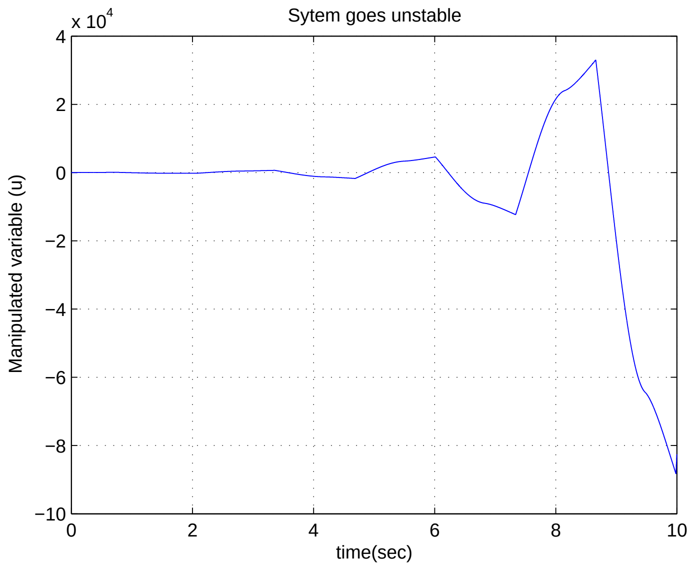

FIGURE 4.15. Control input when filter (T=0.5) is used: system goes unstable

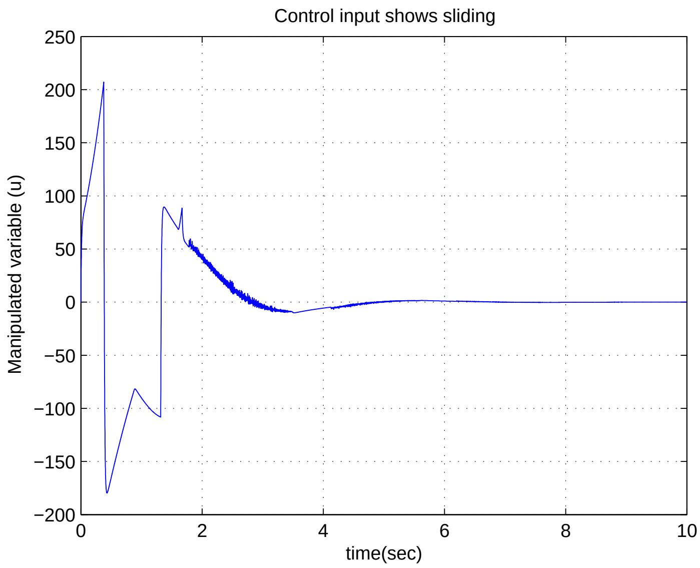

FIGURE 4.16. Control input when filter(T=0.01) is used- Sliding is present

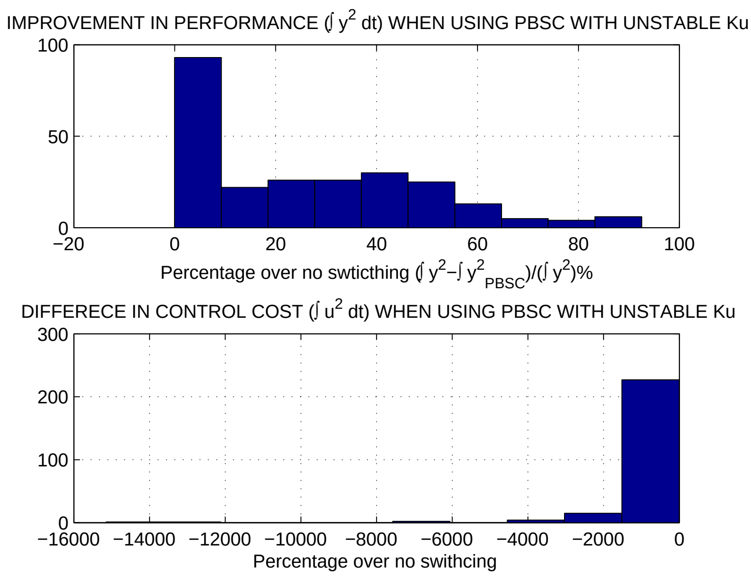

FIGURE 4.17. Improvement performance when the time constant of the filter is (T=0.01)

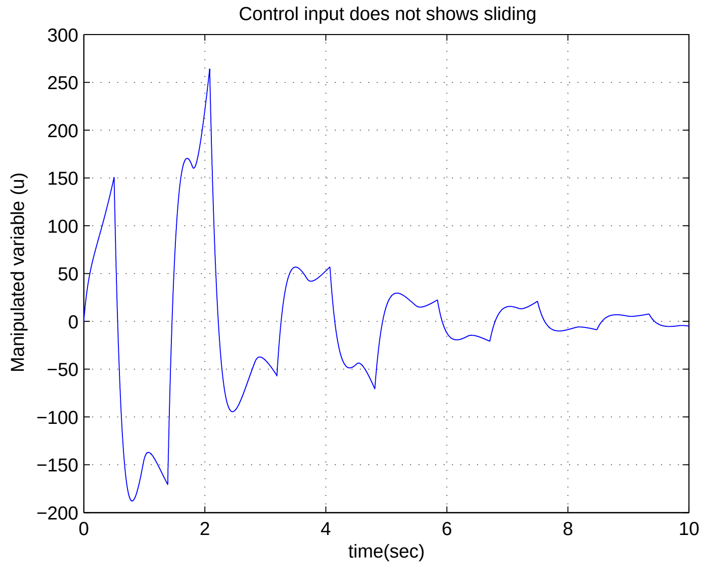

FIGURE 4.18. Control input when filter (T=0.1) is used- Sliding is not present

constant does not effect the response of the system significantly. When the time constant of the filter was increased to 0.1, sliding was reduced significantly (Fig. 4.18). But the response of the system for some initial conditions might degrade as shown in Fig. 4.19. Hence the time constant of the system should be chosen carefully.

Similarly, the improvement in performance and the difference in control cost when the PBSC uses inexpensive controller as alternate feedback are shown in Fig. 4.20. The mean improvement was about 55% and the standard deviation was about 23%. Fig. 4.20 shows the control input for the same initial condition as in Fig. 4.14. This supports the premise that sliding may not be present when inexpensive controller is used as alternative controller. The time response when the improvement is maximum (about 90%) is shown in Fig. 4.21. From this example it is clear the proposed strategy improves the response even in the presence of zeros. It can also be stated that use of inexpensive control as alternate control strategy gives a better improvement in response compared to that obtained when

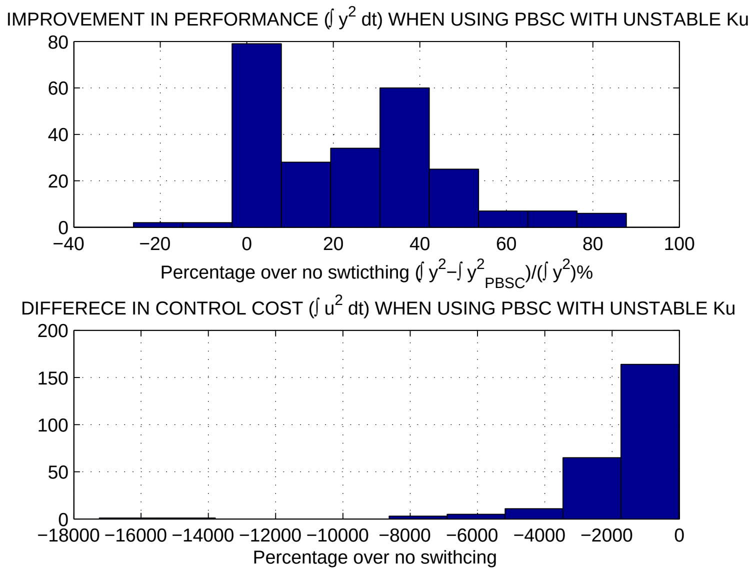

FIGURE 4.19. Improvement in performance when the time constant of the filter is (T=0.1)

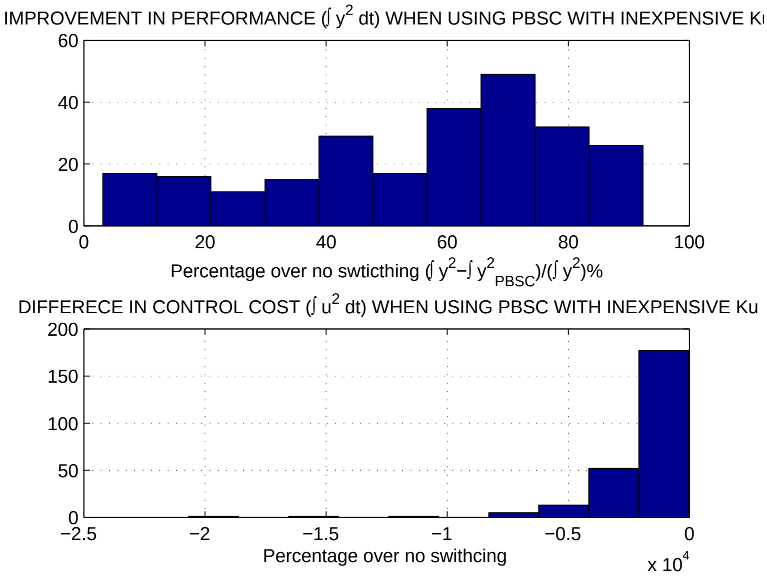

FIGURE 4.20. Improvement in performance and difference in control cost of the PBSC with inexpensive alternate feedback over the state feedback control

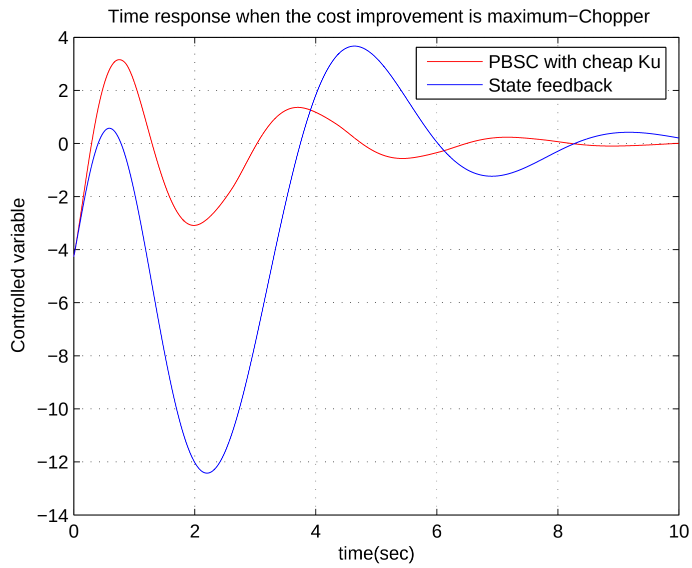

FIGURE 4.21. Time response when the performance is maximum

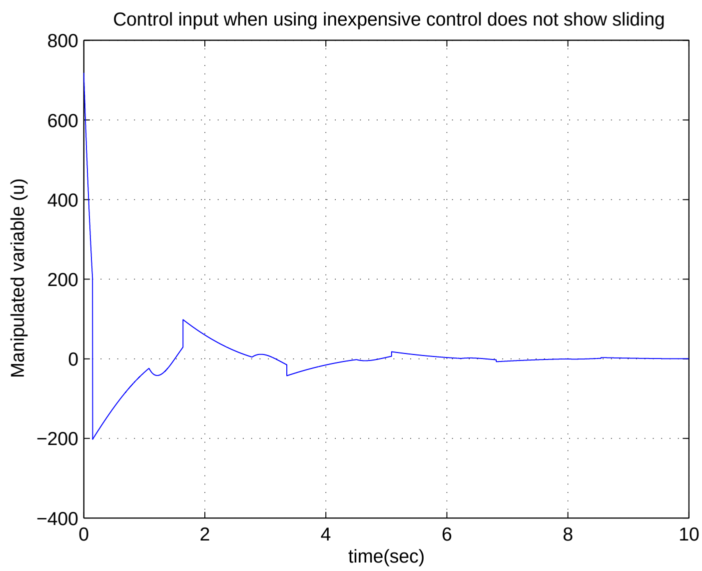

FIGURE 4.22. Control input for an initial condition showing no sliding

the alternate control strategy was unstable. But the control cost was comparatively high.

4.4 Helicopter Model With Zero Alternate Feedback

As a special case, we consider the fourth order helicopter model with zero alternate control. That is, primary feedback K_s is used when $s < 0$. When $s > 0$, the plant is allowed to operate open loop. The primary feedback pole placement is same as in the previous case. Fig 4.23 shows the improvement when the system was simulated for about 250 initial conditions. In the figure y indicates the output of the system with state feedback control strategy and y_{PBSC} indicate the output of the system with PBSC strategy. Similarly, u indicates the control input of the system with state feedback control strategy and u_{PBSC} indicate the control input to the system with PBSC strategy. The mean improvement was about 18.5% with a standard deviation of 7.5%. The maximum improvement was about 41% for which the time response is shown in Fig. 4.24. The difference in control cost, given in Fig. 4.23, shows that the cost for using PBSC was high. The mean difference in control cost was around -114% with an standard deviation of 62%. This was significantly lesser than the case when non zero alternate feedback is used (Fig. 4.12 and Fig. 4.20). This shows that if zero alternate control was used, the performance was improved at a lesser control cost when compared to control cost when alternate feedback was non zero.

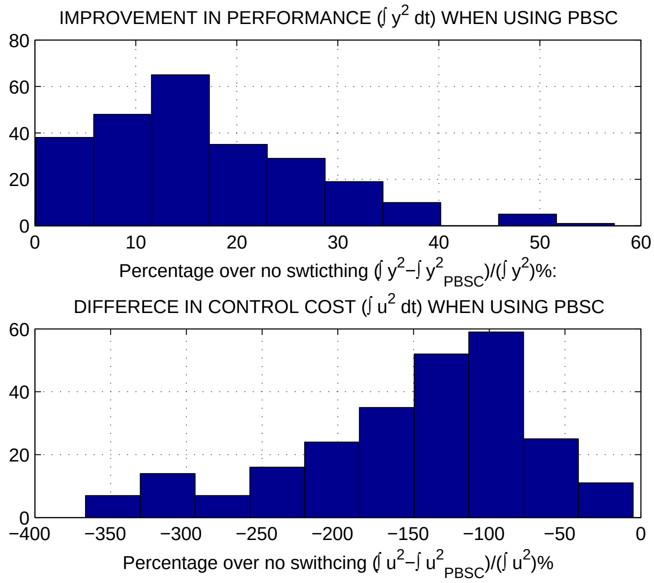

FIGURE 4.23. Performance and control cost for different random initial conditions (zero alternate feedback)

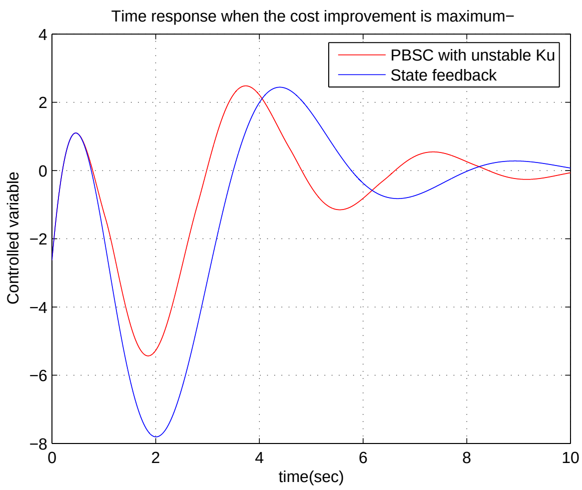

FIGURE 4.24. Time response when the improvement is maximum (zero alternate feedback)

Chapter 5
Conclusion and Future Work

This section gives the summary of work so far done in performance based switching control for single input systems. We also mention some of the unexplored aspects of the present method.

5.1 Conclusion

In this research, we have developed a new control strategy that improves the output performance of a single input system by switching between two control structures. It is necessary that one of the structures is asymptotically stable. The alternate structure can be stable or unstable. A switching law that uses the best features of both the control structures was derived. Since, switching between the two structures is guided by the performance of the controlled variable, this control strategy is called performance based switching control. It was proved that the proposed control strategy makes the system asymptotically stable, provided sliding was absent.

Geometric analysis of this switching law showed that the switching region was the union of three subregions S_{w0}, S_{w1} and S_{w2}. It was proved that sliding was absent on the switching surface S_{w1}. We were not able to provide a generalized proof to prove that sliding is absent on the switching surfaces S_{w0} and S_{w2}. However, simulation results indicated that sliding might depend on the choice of alternate feedback. To illustrate this, a fourth order plant was simulated for about 1000 initial conditions, chosen at random on the surfaces S_{w0} and S_{w2}. The simulation results showed that sliding was present in the region S_{w2}, when unstable alternate controller was used. When a stable alternate controller, with inexpensive control was used, sliding was not detected. No sliding was detected in the region S_{w0} in any case. This shows that the best choice of alternate controller would be a stable linear quadratic regulator with less weight on the control cost.

The proposed control was implemented for systems up to order four. In case of the double integrator plant, we were able to prove that sliding is absent, irrespective of the choice of the alternate control. The improvement in performance is the percentage difference in performance index obtained when using the performance based switching control and its equivalent obtained when using state feedback control. A helicopter model near hover was controlled using the proposed control strategy. The improvement in performance was better when an inexpensive alternate control is used.

We also controlled the helicopter model using the performance based switching control, with zero alternate feedback. Here the mean improvement was about 18% with a standard deviation of 7.5. In addition, the cost of control was lesser when compared to the case when the same strategy was implemented with non zero alternate control. This yields a fuel efficient control that uses the system instability to its advantage.

Higher control cost could be major drawback of this performance based switching control. The proposed control could find applications where system performance rather than the control cost is the primary concern.

5.2 Future Work

In the present work, we lay the foundation of a new switching based control strategy. There are many unexplored aspect of this method some of which are mentioned here.

- Firstly, there is no analytical proof to rule out the existence of sliding in the proposed method. In fact, simulation results show sliding when the alternative feedback chosen was unstable. However, sliding was not detected when alternative feedback chosen was stable. There is still need to prove the above result analytically.

- All the examples considered in the present work are all single input systems. Development of a similar control strategy for MIMO system is yet to be studied.

- In the present research, the number for primary and alternate feedbacks considered was one each. This number can be more than one, in which case the control law would be complex and would need further analysis.

- To this stage, we have assumed that switching between subsystems is ideal. That is, there are no delays in switching. However most of the systems have some amount of switching delay. The behavior of the system with time delays should be studied.

- Usually, the actuators have saturations to limit the control effort. The effect of saturation on control effort should also be considered.

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Appendix : Dual Vector Approach

In this section we derive the equation that describe the states in the switching regions in terms of dual vectors. For convenience let us replace the vectors $(K_u - K_s)^T$ and $\Gamma_0 B$ by p and q respectively. Their dual vectors will be represented by $p^\dagger$ and $q^\dagger$ respectively. Let

$$p^\dagger = \psi_1 p + \psi_2 q \tag{5.1}$$

$$q^\dagger = \beta_1 p + \beta_2 q \tag{5.2}$$

Let us introduce a matrix G of dimensions $(n, 2)$ whose columns are vector p and q and a permutation matrix R_p of dimension $(2, 2)$. The switching matrix S can be expressed as

$$S \quad = \quad G R_p G^T \quad \text{where} \quad G = \begin{pmatrix} p & q \end{pmatrix} \quad \text{and} \quad R_p = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5.3}$$

From properties of dual vectors, we have $\langle p, p^\dagger \rangle = \langle q, q^\dagger \rangle = 1$ and $\langle p, q^\dagger \rangle = \langle q, p^\dagger \rangle = 0$. From these relations we can find $\psi_{1,2}$ and $\beta_{1,2}$ as

$$\begin{pmatrix} \psi_1 & \psi_2 \end{pmatrix} = \begin{pmatrix} \langle p, p \rangle & \langle p, q \rangle \\ \langle q, p \rangle & \langle q, q \rangle \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5.4}$$

$$\begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix} = \begin{pmatrix} \langle p, p \rangle & \langle p, q \rangle \\ \langle q, p \rangle & \langle q, q \rangle \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.5}$$

A vector "close" to the switching region can be expressed as

$$x \quad = \quad \lambda_1 p^\dagger + \lambda_2 q^\dagger + \lambda_3 P_f z \tag{5.6}$$

where λ_1, λ_2 and λ_3 are arbitrary constants and z is an arbitrary vector. P_f is a matrix given by

$$P_f \quad = \quad I - G^T (G G^T)^{-1} G \tag{5.7}$$

where G is given in Eq. 5.3. When $\lambda_1 = 0$, varying λ_2 and z, Eq. 5.6 would give all the points on the subspace S_{w1}. Similarly, when $\lambda_2 = 0$, varying λ_1 and z in Eq. 5.6, we would obtain the points on region S_{w2}. When $\lambda_1 = \lambda_2 = 0$, we can obtain all the points in the region S_{w0} by varying z in Eq. 5.6.

Vita

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