1982

Transient Inelastic Thermal Stresses in a Solid Cylinder With Temperature Dependent Properties.

Habib Pour-mohamadian

Louisiana State University and Agricultural & Mechanical College

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WITH TEMPERATURE DEPENDENT PROPERTIES

The Louisiana State University and Agricultural and Mechanical Col. Ph.D. 1982

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TRANSIENT INELASTIC THERMAL STRESSES
IN A SOLID CYLINDER WITH TEMPERATURE DEPENDENT PROPERTIES

A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy

in The Department of Mechanical Engineering

by Habib Pour-Mohamadian
B.S., University of Texas at Austin, 1976
M.S., Louisiana State University, 1978
May 1982
DEDICATION

To My Mother
ACKNOWLEDGMENT

I wish to express my deep appreciation to Dr. Mehdy Sabbaghian, Professor of Mechanical Engineering for his interest, encouragement, and guidance throughout the years of graduate work at Louisiana State University.

I want to thank my wife, Fatimeh, for her understanding, love and patience, without which, this work could not have been possible.

I am thankful to Professor A. J. Mcphate, Professor D. E. Thompson, Professor M. Zohdi, and Professor J. Retherford for their continued encouragement and valuable advice.
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<tr>
<td>a</td>
<td>Radius of a solid cylinder</td>
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<tr>
<td>C</td>
<td>Specific heat coefficient</td>
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<tr>
<td>e</td>
<td>Mean normal strain</td>
</tr>
<tr>
<td>E</td>
<td>Young's Modulus</td>
</tr>
<tr>
<td>F_r</td>
<td>Radial body force</td>
</tr>
<tr>
<td>F_θ</td>
<td>Tangential body force</td>
</tr>
<tr>
<td>g</td>
<td>Elastic-plastic parameter</td>
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<tr>
<td>h</td>
<td>Convective heat transfer coefficient</td>
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<tr>
<td>H'</td>
<td>Slope of the curve of equivalent stress versus equivalent strain</td>
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<td>J_0(r)</td>
<td>Bessel's function of the first kind and of zero order</td>
</tr>
<tr>
<td>J_1(r)</td>
<td>Bessel's function of the first kind and of first order</td>
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<tr>
<td>k</td>
<td>Coefficient of thermal conductivity</td>
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<tr>
<td>S_e</td>
<td>Dimensionless equivalent stress</td>
</tr>
<tr>
<td>S_r</td>
<td>Dimensionless radial stress</td>
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<tr>
<td>S_θ</td>
<td>Dimensionless tangential stress</td>
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<td>S_z</td>
<td>Dimensionless axial stress</td>
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<td>S_y</td>
<td>Yield strength</td>
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<td>t</td>
<td>Time</td>
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<td>T</td>
<td>Temperature</td>
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<td>T̃</td>
<td>Dimensionless temperature</td>
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<td>T_0</td>
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<td>T_∞</td>
<td>Medium temperature</td>
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<td>u</td>
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\[ Y_0(r) \] Neumann's Bessel function of the second kind and of zero order
\[ \alpha \] Coefficient of thermal expansion
\[ \chi \] Thermal diffusivity
\[ \varepsilon_r \] Radial strain
\[ \varepsilon_\theta \] Tangential strain
\[ \varepsilon_z \] Axial strain
\[ \mu \] Roots of the Bessel equation
\[ \nu \] Tangential displacement
\[ \nu \] Poisson's ratio
\[ \rho \] Dimensionless radius
\[ \rho \] Density
\[ \sigma_d \] Stress deviation
\[ \sigma_e \] Equivalent stress
\[ \sigma_m \] Mean stress
\[ \sigma_r \] Radial stress
\[ \sigma_\theta \] Tangential stress
\[ \sigma_z \] Axial stress
\[ \phi \] Stress function
\[ \tau \] Dimensionless time
ABSTRACT

Transient temperature distribution in an infinitely long solid cylinder initially at uniform temperature $T_0$, subjected to an axisymmetric thermal boundary condition, (immersed in an infinite medium of zero temperature) has been obtained. The mechanical properties of the material are considered to be temperature dependent. Using the available experimental data, an equation for each mechanical property as a function of temperature is obtained. Then, using a numerical method the corresponding transient and residual thermal stresses in a long solid cooling cylinder subjected to a transient axisymmetric temperature distributions are calculated.

The residual stresses obtained indicate the same pattern with good agreement between the experimental and computed results. Comparing with other theoretical results, the computed stresses in this investigation is far more accurate.
CHAPTER I

INTRODUCTION

Thermal-stress problems in recent years have become an important chapter in the history of applied mathematics and mechanics. Thermal stresses play a significant role in both metal-processing and machine design. This importance has been introduced by the increasingly large number of engineering applications of materials for the design of parts operating at high temperatures and pressures in such industries as aerospace, nuclear reactors, metal-processing and chemical plants.

Physically, when a solid is subjected to a force, the atomic lattice will adjust itself to oppose the applied force and maintain equilibrium on a macroscopic scale. The atomic adjustment is observed as deformation when the lattice remains continuous. The response of this deformation to the applied stress varies with magnitude and state of stress, temperature, and the rate of deformation. Elastic deformation results when the strain appears and disappears simultaneously with the application and removal of the stress.

A material that undergoes stress beyond a certain limiting stress, designated as its elastic limit, will break (fracture) if it is an ideally elastic (or brittle) material, or yield if it is a ductile material. When a material is subjected to stresses that are larger than its elastic limit, the distances between atoms increase to such an extent that atomic bonds are broken, and dislocation occurs in the atomic lattice of individual crystals. Since these place changes are
changes of positions of atoms relative to one another, the result is a much larger overall deformation of the material, and the action is not reversible when external force is relaxed or removed. This nonrecoverable deformation is called plastic deformation. Figure (1-1) shows an idealized stress-strain diagram for a ductile material [1].

Practically all engineering materials are polycrystalline in nature, i.e., they are composed of many crystals, usually randomly oriented. In such materials, the mechanism that produces plastic deformation is more complicated because of the effect of neighboring crystals and the interference of the crystalline or grain boundaries. The effect of the grain boundaries is to increase the elastic strength of a material because the boundaries interfere with the slip across the individual crystals. It follows, then, that a polycrystalline metals has higher strength than a single crystal, and that fine-grained metal have higher strength than coarse-grained metals because of the larger number of grain boundaries in the fine-grained metals [2].

The increase in stress required to produce an increase in strain in the plastic region is called strain hardening, or work hardening, a phenomenon familiar to anyone who has ever bent a wire hanger or paper clip and then tried to bend it back to its original shape. The more a material is plastically deformed, the more difficult it becomes to further plastically deform the material. The slope of the stress-strain curve in the strain hardening region, $\frac{d\sigma}{d\varepsilon}$, is usually not constant, but varies with strain. In general, $\frac{d\sigma}{d\varepsilon}$ varies from $\frac{1}{400}$ to about $\frac{1}{100}$ of elastic modulus $E$. There are no generalized laws of plastic deformation

* Numbers in brackets designate the corresponding references at the end of the text.
(corresponding to Hooke's law) that apply for all materials [3].

Figure 1-2 illustrates the strain-hardening effect in a polycrystalline metal as observed during a typical cyclic loading-unloading tensile test on a specimen of mild (low-carbon) steel.

First, consider how yielding starts and continues in such a specimen. It is reasonable to assume that yielding will start in a region of high stress, because a truly uniformly distributed stress is extremely difficult to achieve in a real member of an engineering materials. Thus a concentration of stress may be caused by the method of loading, the geometry of the member, material imperfections, such as inclusions of foreign matter or voids, etc. In a tension test such as that illustrated in Figure (1-2), the highest stress concentration will probably occur near the grips of the testing machine. Slip will first occur in those crystals whose slip planes are most favorably oriented in this area of high stress. When slip occurs, a rotation of crystal with respect to the axis of loading will also take place. The rotation will bring other slip planes into a more favorable position, and since the length of a slip line may be a thousand times the distance between atoms, other crystals in the neighborhood are now subjected to more stress and begin yielding. This process repeats itself until all fibers are in the plastic range. What we actually observe while the specimen yields is the cumulative effect of a great many macroscopic, microscopic, and submicroscopic (atomic) actions [2].

The basic equations which govern the linear theories of heat conduction and thermoelasticity have long been established. A fairly comprehensive and up-to-date bibliography on this subject, which has experienced an intensified interest during the past decade, may be found
Fig. 1—1. Stress Strain Diagram
in a specialized monograph by Parkus [4]; some further references are cited by Nowacki [5].

Inelastic stresses, perhaps mostly due to the complexity of the problems, are not well established. A literature survey indicates that the problems of elastoplastic cylinders subjected to mechanical loading, such as internal and/or external pressures and axial loads, have been solved by several investigators [6] to [10], but these consider somewhat different problems from that treated here and use different methods. In [6] and [7], the methods presented are directed toward steady-state temperature distribution in hollow cylinders subjected to internal pressure. In [8] and [9], the inelastic effect is creep; i.e., viscoelasticity.

Weiner and Huddleston [25] use an elastic, perfectly plastic material model using the Tresca yield condition and do not include the temperature dependence of yield stress. Landaw, Weiner, and Zwicky investigated on thermal stresses in a viscoelastic-plastic plate with temperature dependent yield stress [10]. Very little work, however, has been reported in the literature on the elastoplastic solutions for cylinders subjected to transient thermal loading. The few publications are generally limited to the elastic-perfectly plastic cases.

The practical importance of theoretical investigations of macroscopic transient deformation and thermal stresses in the plastic range becomes clear when one considers the application of various quenching techniques and the resulting thermal hardening and residual stresses. It is well known, based on experience, that uncontrolled quenching of the machine parts may result in initiation of cracks and sometimes causing changes in their dimensions beyond dimensional tolerance. The
residual stresses due to quenching also influence the fatigue durability of machine parts.

Two basic phenomena are responsible for the development of stresses during the thermal hardening of free bodies:

a) non-uniform thermal expansion and
b) non-uniform changes of specific volume due to phase transformations in the solid state.

Since the level of transient and residual equivalent stress are of order of up to 10 \( S_y \) (\( S_y \) is the yield strength in pure tension for annealed steel) [11], the adequate and reasonably accurate theory for the stress analysis is the one which incorporates the elasticity as well as the plastic flow developed in the course of quenching of machine parts. The development of an appropriate theory to predict the residual stresses in a cylinder subjected to transient temperature is the aim of this dissertation.

**Problem Statement**

Consider a solid cylinder of finite length with radius \( a \), having traction free surfaces. It is subjected to a known axially symmetric transient temperature distribution which does not vary along the axis when cooling the cylinder rapidly; The transient temperature distribution is non-uniform, hence inducing the thermal contraction which in turn produces strains in the solid cylinder. These strains are the results of thermal, elastic, and plastic deformations of the cylinder. The stresses resulting from these strains are referred to as the transient and residual thermal stresses. Determination of the distribution of these stresses are a part of the subject of this dissertation.
Requirements of a Thermoelastoplastic Solution

The conditions that must be fulfilled by a satisfactory thermoelastoplastic solution of a solid cylinder subjected to a transient temperature are as follows:

1) Strain compatibility condition

\[ \frac{d\varepsilon_\theta}{dr} = \frac{1}{r} [ \varepsilon_r - \varepsilon_\theta ] \]  \hspace{1cm} (1-1)

2) Equation of equilibrium

\[ \frac{d\sigma_r}{dr} = \frac{1}{r} (\sigma_\theta - \sigma_r) \] \hspace{1cm} (1-2)

3) Boundary conditions

\[ \sigma_r = 0 \text{ at } r = a \] \hspace{1cm} (1-3)

4) Plane sections remain plane

5) All stresses and strains are continuous across the elastic-plastic boundary. Stresses and strains that are continuous across the elastic-plastic interface can be obtained provided the same compressibility is assumed for stress-strain relations in the elastic region as for the elastic strain in the plastic region.

6) Stress-Strain relations

a) In an elastic region The Duhamel-Neuman's stress-strain-temperature relation must be satisfied.

b) In an inelastic region, the Prandtl-Reuss work hardening stress-strain flow rules should be satisfied.

7) Compressibility of material: The total strain for a volume element that has been inelastically deformed consists of an elastic strain and an inelastic strain; thus

\[ \varepsilon = \varepsilon^e + \varepsilon^p \] \hspace{1cm} (1-5)
where $\varepsilon^e$ and $\varepsilon^d$ are elastic and inelastic strains, respectively. Experiments [12] have shown that the inelastic components of strain do not contribute to volume change. In other words, the volume change is always elastic, hence

$$E_{de} = (1-2u)ds$$  \hspace{1cm} (1-6)$$

where $e$ and $s$ are mean normal strain and mean normal stress, respectively. If the stresses and strains satisfy Equation (1-6), the material is said to be compressible. A material is said to be incompressible if

$$de = 0$$  \hspace{1cm} (1-7)$$

To reduce the complexity of mathematical work, the elastic volume changes are usually neglected. In this investigation, however, the compressibility of the material will be considered.

8) The two widely accepted yield criteria are the Tresca maximum shearing stress yield criterion and the von Mises' yield criterion. Here the von Mises' yield criterion will be used.

$$2S_y^2 \left( \sigma_2 - \sigma_1 \right)^2 + \left( \sigma_3 - \sigma_1 \right)^2 + \left( \sigma_3 - \sigma_2 \right)^2$$  \hspace{1cm} (1-8)$$

**Assumptions**

The development of a theory which predicts the transient and residual thermal stresses in a solid cylinder subjected to a transient temperature distribution is based on the following assumptions:

a) Deformation is small, and the radius of cylinder for the purpose of application of boundary condition is the same as the radius of the unstrained cylinder.
b) Material is homogeneous and isotropic.

c) Material is compressible.

d) Material properties such as yield strength, modulus of elasticity, and thermal coefficient of expansion are temperature-dependent.

e) Prandtl-Reuss work-hardening flow rule is valid.

f) von Mises' yield criterion is valid.

g) Plane strain condition is valid.

h) The temperature distribution is axisymmetric.

i) At the surface, the cylinder is exposed to an instantaneous uniform change in temperature.

j) The treatment is quasi-static in the conventional sense, therefore inertia forces are neglected in the thermoelastoplastic equations.

**Approach**

After establishing all the requirements of a thermoelastoplastic solution to the specific problem of a solid cylinder, and obtaining the transient temperature distribution, a general relation for stresses in the radial, tangential and axial direction is obtained. A numerical procedure is used in this investigation and the theory and results are discussed in the following chapters.

First the transient temperature distribution using the classical heat conduction approach is determined. Temperature-dependent relation for mechanical properties of a specific material are obtained. Then a mathematical relation for strain due to thermoelastoplastic in radial and tangential directions are found. Having introduced a stress function, the stress rates in terms of strain and the continuity conditions are introduced. The stress behavior of the cooling cylinder subjected to this transient temperature distribution is described by a single
non-linear differential equation of the stress function. This equation is solved using the Runge-Kutta integration technique. The flow-chart for the integration process using a digital computer can be found in appendices. The numerical data thus obtained compare favorably with available experimental stress data.
Chapter II

BASIC CONCEPTS AND EQUATIONS

In this chapter the general concept and equations of thermoelasticity are discussed. The stress-strain relation, equilibrium and compatibility equations in polar coordinates are indicated. Distortion Energy Criterion along with mechanical properties of materials at high temperature are discussed.

Thermoelastic Stress-Strain Relations

The total strains in the elastic range at any point in a heated body consist of two parts, elastic and thermal strains. The elastic strains in the case of a multiaxial loading, considering the requirement of continuity of the body, are related to stresses by the generalized Hooke's law of linear isothermal elasticity. The elastic constitutive equations are [5]

\[
\varepsilon_r = \frac{1}{E} [\sigma_r - u(\sigma_\theta + \sigma_z)] + \alpha T \\
\varepsilon_\theta = \frac{1}{E} [\sigma_\theta - u(\sigma_r + \sigma_z)] + \alpha T \\
\varepsilon_z = \frac{1}{E} [\sigma_z - u(\sigma_r + \sigma_\theta)] + \alpha T
\]  

\tag{2-1}

In the case of long cylinders, which are the subject of discussion of this dissertation, plain strain conditions exist. This condition can be applied when the z dimension of the body becomes very large compared with the other dimensions. Solving Equations (2-1) for stresses one would obtain for the case of \( \varepsilon_z = 0 \)
\[ \sigma_r = \frac{E}{(1+v)(1-2v)} [(1-v)\varepsilon_r + \nu \varepsilon_\theta - (1+v)\alpha T] \]

\[ \sigma_\theta = \frac{E}{(1+v)(1-2v)} [(1-v)\varepsilon_\theta + \nu \varepsilon_r - (1+v)\alpha T] \] (2-2)

\[ \sigma_z = \nu(\sigma_r + \sigma_\theta) - E\alpha T \]

On the other hand if the cylinder is free to expand axially, that is if \( \varepsilon_z = \) constant, the axial stress \( \sigma_z \) becomes [6]

\[ \sigma_z = \sigma_r + \sigma_\theta \] (2-3)

**Equilibrium Equation**

The general form of the equilibrium equation for a cylinder, in the absence of temperature gradient, can be obtained by considering the force and stresses acting on an element as shown in Figure (2-1). The summation of forces in the radial direction taking unit thickness becomes:

\[ \Sigma F_r = 0 \]

\[ (\sigma_r + \frac{\partial \sigma_r}{\partial r} \, dr)(r+dr)d\theta dz - \sigma_r r d\theta dz - 2\sigma_\theta dr \sin \frac{d\theta}{2} \, dz \]

\[ + (\tau_{\theta r} + \frac{\partial \tau_{\theta r}}{\partial \theta} \, d\theta)dr - \tau_{\theta r} dr + F_r = 0 \]

Replacing \( \sin \frac{d\theta}{2} \approx \frac{d\theta}{2} \) and dropping higher terms one obtains

\[ \frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + F_r = 0 \] (2-4)

Similarly \( \Sigma F_\theta = 0 \) results in

\[ \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + 2\tau_{r\theta} + F_\theta = 0 \] (2-5)
Fig. 2-1. Stresses Acting on an Element
where $F_r$ and $F_\theta$ are components of the body force (including inertia forces) in radial and tangential directions respectively. Neglecting the body forces and considering the axisymmetric case, in which all derivatives with respect to $\theta$ vanish, the two equilibrium equations reduce to

\begin{align}
\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} &= 0 \quad (2-6) \\
\text{and} \quad \frac{dr_r}{dr} + \frac{2r_r}{r} &= 0 \quad (2-7)
\end{align}

Compatibility Equations

Compatibility equations derived based on the physical fact that the various strains cannot vary in a random manner; there must be relations among them that insure that no discontinuities are introduced in a body as the various elements are displaced relative to one another.

The radial, tangential and shearing strains are as follow \[7\]

\begin{align}
\varepsilon_r &= \frac{\partial u}{\partial r} \quad (2-8) \\
\varepsilon_\theta &= \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} \quad (2-9) \\
\text{and} \quad \varepsilon_{r\theta} &= \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \quad (2-10)
\end{align}

differentiating Equ. (2-8) one finds

\begin{equation}
\frac{\partial \varepsilon_\theta}{\partial r} + \frac{1}{r} (\varepsilon_\theta - \varepsilon_r) = 0 \quad (2-11)
\end{equation}

Equation (2-11) is known as the Compatibility Equation.
Distortion Energy Theory or
the von Mises Yield Criterion

Numerous criteria of failure are available. Distortion energy theory is considered the most accurate theory of failure. Knowing the tensile yield strength of a material, this theory predicts ductile yielding under combined loading. Where the stress involved is triaxial, this theory takes into account the influence of the third principal stress. Its validity is limited to materials having similar strengths in tension and compression.

This theory first was proposed by M. T. Hueber, of Poland in 1904 [8]. Later, it was further developed and explained by R. von Mises in 1913 [9] and H. Hencky in 1925 [10]. It is generally referred to as the von Mises-Hencky theory.

The von Mises-Hencky theory assumes that failure takes place when the internal energy of distortion is equal to that at which failure occurs in a simple tension test and is given by [8]

\[
2S_y^2 = (\sigma_2 - \sigma_1)^2 + (\sigma_3 - \sigma_1)^2 + (\sigma_3 - \sigma_2)^2
\]

Where \( S_y \) is the yield strength and \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) are principal stresses.

Mechanical Properties at High Temperatures

As a general rule, "high temperatures" for metals and alloys are temperature above about one-fourth the melting point of the base metal, in degrees absolute.

As expected, the effects of high temperatures on mechanical properties of materials are generally opposite to those of low temperatures. For any particular metal, however, metallurgical characteristics give
rise to detailed variations in these general trends. For example, figure (2-2) shows that because of strain aging, the maximum tensile strength (and minimum ductility) of carbon steel occurs at about 500°F. In fact, steels in general retain substantially their normal tensile strength up to 600 to 700°F. In similar fashion, ordinary cast irons retain nearly a constant tensile strength up to about 570°F, above which the tensile strength falls off to about half at 1000°F. Figure (2-2) shows that the yield point and proportional limit of carbon steel decrease continuously with increased temperature and that the modulus of elasticity decreases to about 20x10^6 psi at 930°F. Similar trends are observed with metals in general [11].

Increasing the temperature usually causes the coefficient of thermal expansion to increase and the thermal conductivity to decrease. These factors, combined with elastic modulus and ductility, largely control the resistance of materials to sudden temperature change (thermal shock).
Fig. 2.2 Short-time Tensile Properties of Normalized 0.37% Carbon Steel at Elevated Temperature [8]
Chapter III

PLASTIC STRESS-STRAIN RELATIONS

The relation between stress and strain is linear and related by Hooke's Law in the elastic range, but the relation will generally be nonlinear in the plastic range. These behaviors are evident from the uniaxial stress-strain curve. Although the strains are uniquely determined by stress in the elastic range, i.e., for a given set of stresses one can compute the strains directly using Hooke's Law without any regard as to how this stress state was attained, in the plastic range the strains are in general not uniquely determined by the stresses but depend on the whole history of loading or how the stress state was reached.

From the experimental analysis of different materials in the plastic range it has been found that the plastic strains are dependent upon the loading path. It becomes necessary, in general, to compute the differentials or increments of plastic strain throughout the loading history and then obtain the total strains by integration or summation. However, there is at least one important class of loading paths for which the plastic strains are independent of the loading path and depend only on the final state of stress. These are called radial or proportional loading paths, in which all the stresses increase in the same ratio [6].

The Levy-Mises and Prandtl-Reuss Equations

In 1870 Saint-Venant introduced the first relations for stress-
strain in the plastic range [12]. Saint-Venant proposed that the principal axes of the strain-increment, but not the total strain, will coincide with the axes of principal stress. A more general relationship relating the increments of total strain to the stress deviations were given by Levy in 1871 [13]. Levy's work was unknown until the same equations were independently suggested by von Mises in 1913 [9]. These equations are called the Levy-Mises equations and they are widely used as the basis of plasticity theory.

These equations, in cylindrical coordinates, are written as

\[ \frac{d\varepsilon_r}{\sigma_r} = \frac{d\varepsilon_\theta}{\sigma_\theta} = \frac{d\varepsilon_z}{\sigma_z} = \frac{d\gamma_\theta z}{\tau_\theta z} = \frac{d\gamma_z r}{\tau_z r} = \frac{d\gamma_r \theta}{\tau_r \theta} \]  \hspace{1cm} (3-1)

Or, in tensor notation

\[ d\varepsilon_{ij} = \sigma_{ij}^d d\lambda \]  \hspace{1cm} (3-2)

where \( \sigma_{ij}^d \) is the stress deviator tensor and \( d\lambda \) is a nonnegative constant which may vary throughout the loading history. In these equations the total strain increments are assumed to be equal to the plastic strain increments, hence the equations are strictly applicable only to a fictitious material in which the elastic strains are zero. The general form of Equation (3-1) taking into account the effect of elastic component of strain was carried out by Prandtl [14] for the plane problem, and in complete form by Reuss [15]. These equations are known as Prandtl-Reuss equations.

Reuss assumed that plastic strain increment is, at any instant of loading, proportional to the instantaneous stress deviation; i.e.
Equation (3-2) is equivalent to the combined statements that the principal axes of stress and that of plastic strain-increment coincide. Also it states that the increments of plastic strain depend on the current values of the deviatoric stress state, not on the stress increment required to reach this state.

To determine the actual magnitudes of plastic strain increments a yield criterion is required. Considering the principal directions

\[
\frac{d\varepsilon^P_1}{\sigma_{d1}} = \frac{d\varepsilon^P_2}{\sigma_{d2}} = \frac{d\varepsilon^P_3}{\sigma_{d3}} = d\lambda
\]

Equations (3-2) can be written in terms of the actual stresses as

\[
\frac{d\varepsilon^P_r}{d\lambda} = \frac{2}{3} d\lambda \left[ \sigma_r - \frac{1}{2} (\sigma_\theta + \sigma_z) \right]
\]

\[
\frac{d\varepsilon^P_\theta}{d\lambda} = \frac{2}{3} d\lambda \left[ \sigma_\theta - \frac{1}{2} (\sigma_r + \sigma_z) \right]
\]

\[
\frac{d\varepsilon^P_z}{d\lambda} = \frac{2}{3} d\lambda \left[ \sigma_z - \frac{1}{2} (\sigma_r + \sigma_\theta) \right]
\]

\[
d\varepsilon^P_{r\theta} = d\lambda \gamma_{r\theta}
\]

\[
d\varepsilon^P_{rz} = d\lambda \gamma_{rz}
\]
To determine \( d \lambda \) using Equations (3-2) and (3-4) one can show that

\[
d \lambda = \frac{d \gamma_o^p}{\tau_{\text{oct}}}
\]

where \( \tau_{\text{oct}} \) is called the octahedral shear stress and is equal to

\[
\tau_{\text{oct}} = \frac{1}{3} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{\frac{1}{2}}
\]

and \( d \gamma_o^p \) is the increment of octahedral plastic shear strain defined by

\[
(d \gamma_o^p)^2 = \frac{1}{9} \left[ (d\varepsilon_{\text{r}}^p - d\varepsilon_{\text{z}}^p)^2 + (d\varepsilon_{\theta}^p - d\varepsilon_{\theta z}^p)^2 + (d\varepsilon_{\text{z}}^p - d\varepsilon_{\text{zr}}^p)^2
\]

\[
+ 6(d\varepsilon_{\text{r}}^p)^2 + 6(d\varepsilon_{\text{z}}^p)^2 + 6(d\varepsilon_{\text{zr}}^p)^2
\]

Defining \( \sigma_e \) as an equivalent or effective stress and \( d\varepsilon_p \) as an equivalent or effective plastic strain increment as

\[
\sigma_e = \sqrt{\frac{2}{3}} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 + 6(\tau_{\text{r}} + \tau_{\text{z}} + \tau_{\text{zr}}) \right]^{\frac{1}{2}}
\]

\[
= \frac{3}{\sqrt{2}} \tau_{\text{oct}} \tag{3-6}
\]

and

\[
d\varepsilon_p = \sqrt{\frac{2}{3}} \left[ (d\varepsilon_{\text{r}}^p)^2 - (d\varepsilon_{\theta}^p)^2 + (d\varepsilon_{\theta z}^p)^2 + (d\varepsilon_{\text{z}}^p) - (d\varepsilon_{\text{z}}^p) - (d\varepsilon_{\text{zr}}^p)^2
\]

\[
+ 6(d\varepsilon_{\text{r}}^p)^2 + 6(d\varepsilon_{\theta z}^p)^2 + (d\varepsilon_{\text{zr}}^p)^2 \right]^{\frac{1}{2}}
\]

\[
= \sqrt{2} d \gamma_o^p \tag{3-7}
\]
respectively. The constant $d\lambda$ in terms of equivalent stress and strain using Equations (3-5), (3-6) and (3-7) can be obtained, therefore

$$d\lambda = \frac{3}{2} \frac{d\varepsilon^p}{\sigma_e}$$  \hfill (3-8)

and the stress-strain relation (3-4) becomes

$$\frac{\sigma_{ij}}{d\varepsilon_{ij}} = \frac{3}{2} \frac{d\varepsilon^p}{\sigma_e} \sigma_{ij}$$ \hfill (3-9)

**Plastic Work**

In general, the work done on an element during straining is

$$d\omega = \sigma_{ij} \, dA \, ds$$

or work per unit volume is

$$d\omega = \sigma_{ij} \, d\varepsilon_{ij}$$

$$= \sigma_{ij} \left( d\varepsilon^e_{ij} + d\varepsilon^p_{ij} \right) \hfill (3-10)$$

$$= d\omega^e + d\omega^p$$

where $d\omega^e = \sigma_{ij} d\varepsilon^e_{ij}$ is elastic energy and is recoverable, but the energy in plastic range, $d\omega^p$, is not recoverable. Hence the plastic work per unit volume in terms of principal stresses is

$$d\omega^p = \sigma_{d1} \, d\varepsilon^p_{1} + \sigma_{d2} \, d\varepsilon^p_{2} + \sigma_{d3} \, d\varepsilon^p_{3}$$ \hfill (3-11)

where $\sigma_{di}$ is the stress deviation.

Equation (3-11) can be shown in terms of equivalent stress $\sigma_e$, and equivalent strain increment, $d\varepsilon_p$, as

$$d\omega^p = \sigma_e \, d\varepsilon_p$$ \hfill (3-12)
substituting Equation (3-12) into (3-9) one finds

\[
d\varepsilon_{ij}^p = \frac{3}{2} \frac{d\varepsilon^p}{\varepsilon^p} \sigma_{ij}
\]  

(3-13)

A material in the plastic region work hardening or strain hardening due to plastic flow which takes place. To measure this work hardening, two work-hardening hypotheses, known as the two measures of work hardening, have been proposed [17]. One of these hypotheses will be considered here because it is simpler to use.

This hypothesis uses the equivalent plastic strain, \( \varepsilon_p \), as a measure of work hardening;

\[
\varepsilon_p = \int d\varepsilon_p
\]  

(3-14)

where \( d\varepsilon_p \) is given by Equation (3-7). Assuming the yield function is a function of the equivalent plastic strain [15] gives

\[
F(S_y) = H(\varepsilon_p)
\]  

(3-15)

where the functional relationship can be found by experiment. If the equivalent stress is used for the yield function, then

\[
\sigma_e = H(\varepsilon_p)
\]  

(3-16)

For the case of the Prandtl-Reuss equations and the von Mises yield criterion, Equation (3-16) is used. In actual application, the experimental relationship given by Equation (3-16) is taken from the uniaxial tensile stress-strain curve, as shown in Figure (3-1). The abscissa and ordinate of the uniaxial stress-strain curve are replaced by \( \varepsilon_p = \int d\varepsilon_p \) and \( \sigma_e \), respectively. In terms of the slope of this curve, Equation (3-16) can be written as
Fig. 3-1 Relation between Equivalent Stress and Equivalent plastic Strain [15]
\[ \frac{d\varepsilon_{ij}^p}{d\varepsilon_{ij}} = \frac{3}{2} \frac{d\sigma_e}{d\varepsilon_{ij}} \sigma_{diij} \] (3-17)

where

\[ H^* = \frac{d\sigma_e}{d\varepsilon_p} \] (3-18)

is the slope of the equivalent stress/plastic curve and \( \sigma_{diij} \) is the stress deviation. For more detail derivation of Equation (3-17) reader is referred to reference [18].

**Phase-Transformation Problem**

Phase transformations occur in the heat-treatment process. It causes volume changes and therefore in any plasticity analysis dealing with high temperature this effect must be considered. Since in phase transformations the yield strength changes as the metal phase transforms from one form to another as a result of the temperature gradient, an understanding of phase transformations and a relation to describe the dependency of yield strength on temperature is essential.

Consider an element of eutectoid steel at the initial (time \( t=0 \)) temperature \( T_0 \) and assume that \( T_0 \) is greater than the eutectoid temperature \( T_A \). At the temperature \( T_0 \) the element has one-phase structure which is given the name austenite (solid solution of carbon in \( \gamma \)-iron). Depending upon the rate of cooling, or more precisely, depending on history of temperature up to the given instant, the austenite in the temperature range \( M_S < T < T_A \) may remain untransformed or transformed partially (or totally) into new structures which are called pearlite (eutectoid mixture of ferrite and cementite \( \text{Fe}_3\text{C} \)) and bainite [2]. When specifying the overall mechanical properties of an element, one does not distinguish the mechanical and thermal properties of pearlite from that
of bainite. One treats it as a one structure component conventionally termed pearlite. Due to this assumption a steel element may be at most two constituents at temperature range $M_s < T < T_A$ (here $M_s$ is the temperature at which the martensitic transformation starts). At temperature range $M_f < T < M_s$ ($M_f$ is the temperature at which the martensitic transformation is finished) the pearlite does not form and the rest remaining austenite is transformed into new phase-martensite (intermediate unstable structure between the normal phases of iron e.g. $\gamma$-iron and $\alpha$-iron). At the room temperature, in an element there may exists some amount of so called "retained austenite". It thus follows that at temperature range $T_r < T < M_s$ there may exist at most three different constituents in steel element, namely: martensite (M), pearlite (P) and austenite (A) Figure (3-2). The type of structure depend upon the history of temperature [4].

A steel element becomes the composite in the course of quenching and the above mentioned transformations result in remarkable increase of the volume and changes in plastic properties of an element.
### Fig. 3–2 Schematic Representation of the Change of Structure during Quenching

<table>
<thead>
<tr>
<th>Time $t$</th>
<th>$0 \leq t &lt; t_A$</th>
<th>$t_A \leq t &lt; t_S$</th>
<th>$t_S \leq t &lt; t_f$</th>
<th>$t_f \leq t \leq t_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature $T$</td>
<td>$T_0 \geq T &gt; T_A$</td>
<td>$T_A \geq T &gt; M_S$</td>
<td>$M_S \geq T &gt; M_f$</td>
<td>$M_f \geq T \geq T_r$</td>
</tr>
</tbody>
</table>

**CONSTITUENTS**

- **AUSTENITE** (A)
- **PEARLITE** (P)
- **MARTENSITE** (M)
CHAPTER IV

TRANSIENT TEMPERATURE DISTRIBUTION

In this chapter the temperature distribution in a solid cylinder subjected to an initial temperature $T_0$, immersed into a media of zero temperature is obtained. The first law of thermodynamics in differential form is used to obtain the transient temperature for a homogeneous, isotropic solid cylinder. The temperature field is assumed to be axi-symmetric and constant along the cylinder axis. The solution is accomplished by using the method of separation of variables.

First Law of Thermodynamics

The first law of thermodynamics in differential form for a homogeneous, isotropic material, stationary fluid with no heat generation can be written as [26]

$$\frac{\partial T}{\partial t} = \alpha^* \nabla^2 T$$  \hspace{1cm} (4-1)

where $\alpha^* = \frac{K}{\rho C}$, and $\nabla^2$ is the Laplacian Operator. In cylindrical coordinates Equation (4-1) is written as:

$$\frac{1}{\alpha^*} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}$$  \hspace{1cm} (4-2)

Equation (4-2) is the most common form of the first law of thermodynamics in the heat conduction theory which describes the temperature as the only dependent variable.
Transient Temperature Distribution

Consider an infinitely long solid cylinder of homogeneous and isotropic material with radius a. The temperature field is assumed to be axisymmetric and constant along the axial directions, Figure 4.1. Hence the governing conduction equation becomes:

\[
\frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r}
\]  \hspace{1cm} (4-3)

In order to solve the fundamental equation in any coordinate system for any physical system, the constant(s) of integration and the various eigenvalues must be evaluated. To do this, the initial and boundary conditions are used. The cylinder is assumed to be at some initial temperature, \(T_0\). At time \(t\), the surface is dropped to zero temperature. The boundary and initial conditions assuming the heat transfer coefficient is large are:

At \(0 < r < a\) \hspace{1cm} T = T_0 \hspace{1cm} t = 0 \hspace{1cm} (4-4)

At \(r = a\) \hspace{1cm} T = 0 \hspace{1cm} 0 < t < \infty \hspace{1cm} (4-5)

Assumption of a solution of the form of

\[T(r,t) = S(t)R(r)\]  \hspace{1cm} (4-6)

results in

\[
\frac{1}{\alpha} \frac{S'}{S} = \frac{R''}{R} + \frac{1}{r} \frac{R'}{R}
\]  \hspace{1cm} (4-7)

where \(R = R(r)\) and \(S = S(t)\). The left-hand-side of Equation (4-7) is a function of \(t\) only, and the right-hand side is a function of \(r\) only.
Fig. 4-1. Long Solid Cylinder Subjected to Transient Temperature
This equality is possible only if both sides are equal to the same constant, say $-\mu^2$. Then Equation (4-7) becomes

$$\frac{1}{\alpha} \frac{S'}{S} = \frac{R''}{R} + \frac{1}{r} \frac{R'}{R} = -\mu^2 \quad (4-8)$$

where the sign of $\mu^2$ is chosen to be negative in order to ensure that temperature will decay with time.

The differential equation for the function $S$ is obtained from Equation (4-1) as

$$\frac{dS}{dt} + \alpha \mu^2 S = 0 \quad \text{for } S > 0 \quad (4-9)$$

The solution of Equation (4-9) for function $S(t)$ is taken as

$$S(t) = A_n e^{-\alpha \mu^2 t} \quad (4-10)$$

The differential equation for the function $R$ is obtained from Equation (4-8) as:

$$\frac{d}{dr} \left( r \frac{dR}{dr} \right) + \mu^2 r R = 0 \quad 0 \leq R \leq a \quad (4-11)$$

The solution of Equation (4-11) is

$$R(r) = C_1 J_0(\mu r) + C_2 Y_0(\mu r) \quad (4-12)$$

where $J_0$ and $Y_0$ are Bessel function of zero order and Neumann's Bessel function respectively. $C_1$ and $C_2$ are constants.

Applying the boundary condition, as $r$ approaches zero the Neumann's Bessel function $Y_0(\mu r)$ approaches infinity and this is not physically possible hence $C_2$ must be zero and
Using the second boundary condition \( R(a) = 0 \) the characteristic functions

\[ R_n(r) = C_n J_0(\mu_n r) \]  

(4-14)

and the zeros of \( J_0(\mu_n a) = 0 \) are the characteristic values. Now, according to Equation (4-6), the function

\[ T_n(r,t) = R(t)S(t) = C_n e^{-\alpha \mu_n^2 t} J_0(\mu_n r) \]  

(4-15)

is a solution, and these solutions are valid for each consecutive value of \( \mu_n, n = 1, 2, 3, \ldots \). Therefore, the complete solution for the temperature \( T(r,t) \) is the linear sum of all the individual solutions expressed as

\[ T(r,t) = \sum_{n=1}^{\infty} C_n e^{-\alpha \mu_n^2 t} J_0(\mu_n r) \]  

(4-16)

where \( C_n = A C_n \). The unknown expansion coefficients \( C_n \) are to be determined by applying the initial condition for the problem and by making use of the orthogonality property of the eigenfunctions as described below.

If Equation (4-16) is the solution of the above heat-conduction problem, it should satisfy the initial condition (4-4); that is,

\[ T(r,0) = T_0 \]

\[ T_0 = \sum_{n=1}^{\infty} C_n J_0(\mu_n r) \text{ in } 0 \leq r \leq a \]  

(4-17)
Equation (4-17) is the Fourier-Bessel series expansion of $T_o = T(r, o)$. To determine $C_n$'s, one may multiply both sides of Equation (4-17) by $rJ_0(\mu_n r)$ integrate it over the region from $r = 0$ to $r = a$, and obtain [27]

$$
\int_0^a T_o rJ_0(\mu_n r)dr = \sum_{n=1}^{\infty} C_n \int_0^a J_0(\mu_n r)rJ_0(\mu_n r)
$$

(4-18)

In view of the orthogonality property of the Bessel function, all the terms in the summation on the right-hand side of Equation (4-18) vanish except the term $\mu_m = \mu_n$. Hence

$$
\int_0^a T_o rJ_0(\mu_n r)dr = C_n \int_0^a rJ_0^2(\mu_n r)dr
$$

where integrating by parts

$$
\int_0^a T_o rJ_0(\mu_n r)dr = T_o \frac{a}{\mu_n} J_1(\mu_n a) - \int_0^a \frac{a}{\mu_n} J_1(\mu_n r)dr
$$

and

$$
\int_0^a rJ_0^2(\mu_n r)dr = \frac{a^2}{2} [J_0^2(\mu_n a) + J_1^2(\mu_n a)]
$$

since $J_0(\mu a) = 0$

$$
T_o \frac{a}{\mu_n} J_1(\mu_n a) = C_n a^2 \frac{J_1^2}{2}(\mu_n a)
$$

$$
C_n = \frac{2 T_o}{\mu_n a J_1(\mu_n a)}
$$

(4-19)
Thus introducing Equation (4-19) into Equation (4-17), one obtains the transient temperature of the solid cylinder as

\[ T(r,t) = 2T_0 \sum_{n=1}^{\infty} e^{-\alpha \mu_n^2 t} \frac{J_0(\mu_n r)}{(\mu_n a) J_1(\mu_n)} \]  

(4-20)

Rearranging Equation (4-20),

\[ T(r,t) = 2T_0 \sum_{n=1}^{\infty} e^{-\alpha \mu_n^2 t} \frac{J_0(\mu_n r)}{\frac{\mu_n a}{J_1(\mu_n)}} \]  

(4-21)

Introducing the nondimensional variables in order to facilitate the computation,

\[ T^* = \frac{T}{T_0}, \quad \rho = \frac{r}{a}, \quad \tau = \frac{\alpha t}{a^2} \]  

(4-22)

Using Equation (4-22) into (4-21) the nondimensional transient temperature distribution is written as:

\[ T^* = 2 \sum_{n=1}^{\infty} e^{-\mu_n^2 \tau} \frac{J_0(\mu_n \rho)}{\frac{\mu_n}{J_1(\mu_n)}} \]  

(4-23)

Using the tabulated data for \( J_0(\mu_n \rho) \) and \( J_1(\mu_n) \) [28] a computer program was written (see Appendix A). Figure (4-1) represents the distribution of temperature in a steel cylinder. The assumption for initial and boundary condition here is that of cylinder being kept at uniform temperature \( T_0 \) prior to \( t = 0 \). At \( t = 0 \) the cylindrical surface is exposed and thereafter maintained at \( T = 0 \). The temperature distributions along the radius, for various values of \( \tau \), are represented by
Fig. 4-2 Transient Temperature Distribution in a Solid Cooling Cylinder
curves. It can be seen from Equation (4-21) that the temperature distribution for cylinders of various diameters is the same if the duration of heating \( t \) is proportional to the square of the diameter. From Figure (4-2) the average temperature of the whole cylinder and also of an inner portion of the cylinder of radius \( r \) can be calculated.

Considering the transient temperature distribution in a cooling cylinder it is evident from Figure (4-2) that as the time elapse the temperature gradient approaches zero. In thermal stress analysis this temperature distribution will be used to evaluate the resulting thermal stresses in a cooling solid cylinder.
CHAPTER V

ELASTIC THERMAL STRESSES IN A SOLID CYLINDER

This chapter deals with the derivations of the equations for thermal stresses due to the temperature distribution obtained in the previous chapter. The calculation for the stresses in a solid circular cylindrical body of radius $a$ and length $l$, are given for the case in which the temperature distribution is a function of the radial distance $r$ and time $t$. The cylindrical surface is free of tractions. If the ratio $l/a$ is large compared to unity and if axial displacements are prevented, the problem is one of PLANE STRAIN [13].

Derivation of Thermal Stresses

Consider the case of plane strain in which the temperature is a function of $r$ and $t$. The radial, tangential, and axial stresses in terms of strain are given by Equation (2-2)

Making use of compatibility Equations (2-8), (2-9), (2-10) and substituting Equation (2-2) into equilibrium Equations (2-4) and (2-5) results in a second order ordinary differential equation for $u$, the radial displacement [29]

$$\frac{d^2u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} = \frac{1+\nu}{1-\nu} \alpha \frac{dT}{dr}$$  \hspace{1cm} (5-1)

The solution of this differential equation is given by

$$u = \frac{1+\nu}{1-\nu} \alpha \int_{r_0}^{r} Trdr + C_1r + \frac{C_2}{r}$$  \hspace{1cm} (5-2)
where \( C_1 \) and \( C_2 \) are the constants of integration and must be determined by the boundary conditions. Substituting Equation (5-2) into Equations (2-8) and (2-9), one obtains the strains as follows

\[
\varepsilon_r = - \frac{1+u}{1-u} \frac{a}{r^2} \int_0^r Tr \, dr + \frac{1+u}{1-u} \frac{a}{r} Tr + C_1 - \frac{C_2}{r^2} \quad (5-3)
\]

\[
\varepsilon_\theta = - \frac{1+u}{1-u} \frac{a}{r^2} \int_0^r Tr \, dr + C_1 + \frac{C_2}{r^2} \quad (5-4)
\]

Equations (5-3) and (5-4) substituted into Equation (2-2) with the boundary conditions to evaluate \( C_1 \) and \( C_2 \), give the radial and tangential stresses due to a temperature gradient,

\[
\sigma_r = \frac{E\alpha}{r^2(1-u)} \left[ \frac{r^2}{2} \int_0^a Tr \, dr - \int_0^r Tr \, dr \right] \quad (5-5)
\]

\[
\sigma_\theta = \frac{E\alpha}{r^2(1-u)} \left[ \frac{r^2}{2} \int_0^a Tr \, dr + \int_0^r Tr \, dr - Tr^2 \right] \quad (5-6)
\]

and from Equation (2-3)

\[
\sigma_z = \frac{E\alpha}{1-u} \left[ \frac{2}{a^2} \int_0^a Tr \, dr - T \right] \quad (5-7)
\]

Consider a long solid cylinder at a uniform initial temperature of \( T_0 \). At \( t = 0 \) and thereafter, the lateral surface of the cylinder is subjected to and maintained at zero temperature. The distribution of temperature at any time \( t \) was obtained in Chapter III as:
\[ T(r, t) = 2T_0 \sum_{n=1}^{\infty} e^{-\frac{\mu_n^2 t}{a^2}} \frac{J_o(\mu_n \frac{r}{a})}{\mu_n J_1(\mu_n)} \]  
\[ (5-8) \]

Substituting Equation (5-8) into Equation (5-5) and considering that
\[ \int_0^r J_o(\mu_n \frac{r}{a}) r dr = \frac{\mu_n}{\mu_n} J_1(\mu_n) \]  
\[ (5-9) \]

one finds that [18]
\[ \sigma_r = \frac{2\alpha E T_0}{1 - \nu} \sum_{n=1}^{\infty} e^{-\frac{\mu_n^2 t}{a^2}} \left[ \frac{1}{\mu_n^2} - \frac{1}{\mu_n^2} \frac{a}{J_1(\mu_n)} \right] \]  
\[ (5-10) \]

In the same manner, substituting Equation (5-8) into Equation (5-6), one obtains
\[ \sigma_\theta = \frac{2\alpha E T_0}{1 - \nu} \sum_{n=1}^{\infty} e^{-\frac{\mu_n^2 t}{a^2}} \left[ \frac{1}{\mu_n^2} + \frac{1}{\mu_n^2} \frac{a}{r J_1(\mu_n)} \right] \]  
\[ - \frac{J_o(\mu_n a)^r}{\mu_n J_1(\mu_n)} \]  
\[ (5-11) \]

Substitution of Equation (5-8) into (5-7) leads to
\[ \sigma_z = \frac{2\alpha E T_0}{1 - \nu} \sum_{n=1}^{\infty} e^{-\frac{\mu_n^2 t}{a^2}} \left[ \frac{2}{\mu_n^2} - \frac{J_o(\mu_n \frac{r}{a})}{\mu_n J_1(\mu_n)} \right] \]  
\[ (5-12) \]

Upon substitution of \( \rho = \frac{r}{a}, \tau = \frac{\alpha t}{a^2} \) the thermal stresses are written as
Equation (5-13) through (5-15) give the stress distribution in radial, tangential, and axial direction of a solid cylinder subjected to a transient temperature distribution in elastic range, respectively.

It must be mentioned that for $t<0.005$ the convergence of Equations (5-13) through (5-15) is unpleasantly slow. Since the tangential stresses near the surface change very rapidly for small values of the time [31].
CHAPTER VI

INELASTIC THERMAL STRESSES IN A SOLID CYLINDER

In the preceding chapters the thermoelastic stress-strain relations and the thermal stresses due to transient temperature distribution were developed. Although the concept of elastic behavior is an idealization which does not describe exactly any real solid, it is found [6] that the behavior of large classes of materials at relatively low temperature and stress levels is very nearly in accordance with the elastic theory.

At higher temperature and for higher stress levels, however, the divergence between the behavior of real solids and that of the ideal elastic solid increases and the elastic idealization becomes inadequate; the behavior of the real solid then become inelastic. In order to predict mathematically the inelastic behavior of a solid under given thermal and loading conditions, it is necessary to generalize the stress-strain relationship in an appropriate manner.

In this chapter the effects of elastic, plastic and thermal strains that have previously been defined will be superimposed and considering the effect of transient temperature distribution, a complete stress-strain rate will be derived.

Elasto-plastic Thermal Stress-Strain Relationship

By superposing the effects of elastic, thermal, and plastic strain, the total thermal strain can be written as

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\[ \varepsilon = \varepsilon^E + \varepsilon^P + \varepsilon^{th} \quad (6-1) \]

where \( \varepsilon^E \), \( \varepsilon^P \), and \( \varepsilon^{th} \) are elastic, plastic and thermal strains respectively. An expression for each of these terms has been developed previously which will be utilized here.

The yield points on the stress-strain curve is the boundary which separates the elastic and plastic regions. In the plastic region the deformation is permanent and strain will not vanish if the load or temperature gradient is removed. Using distorsion energy as the yield criterion one can write, in cylindrical coordinates

\[ (\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_r)^2 = 2S_y^2 \quad (6-2) \]

Defining the mean normal stress as

\[ \sigma_m = \frac{1}{3} [\sigma_r + \sigma_\theta + \sigma_z] \quad (6-3) \]

The stress deviation is obtained by subtracting the mean normal stress from each principal stresses. Hence the three stress deviations are

\[ \sigma_{rd} = \sigma_r - \sigma_m = \frac{1}{3} [2\sigma_r - \sigma_\theta - \sigma_z] \]

\[ \sigma_{\theta d} = \sigma_\theta - \sigma_m = \frac{1}{3} [2\sigma_\theta - \sigma_r - \sigma_z] \quad (6-4) \]

\[ \sigma_{zd} = \sigma_z - \sigma_m = \frac{1}{3} [2\sigma_z - \sigma_r - \sigma_\theta] \]

Considering the effects of transient temperature the time rate of stress deviation for the three principal stresses are

\[ \dot{\sigma}_{rd} = \frac{1}{3} [2\dot{\sigma}_r - \dot{\sigma}_\theta - \dot{\sigma}_z] \]
\[ \dot{\sigma}_{\theta d} = \frac{1}{3} [2\dot{\sigma}_{\theta} - \dot{\sigma}_{r} - \dot{\sigma}_{z}] \quad (6-5) \]

\[ \dot{\sigma}_{zd} = \frac{1}{3} [2\dot{\sigma}_{z} - \dot{\sigma}_{r} - \dot{\sigma}_{\theta}] \]

Introducing \( \sigma_e \) as equivalent stress defined by

\[ \sigma_e = \frac{1}{\sqrt{2}} \left[ (\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_r)^2 \right]^{\frac{1}{2}} \quad (6-6) \]

Adding and subtracting \( \sigma_m \) to each bracket in Equation (6-6)

\[ \sigma_e = \frac{1}{\sqrt{2}} \left\{ \left[ (\sigma_r - \sigma_m)^2 - (\sigma_\theta - \sigma_m)^2 \right] + \left[ (\sigma_\theta - \sigma_m)^2 - (\sigma_z - \sigma_m)^2 \right] + \left[ (\sigma_z - \sigma_m)^2 - (\sigma_r - \sigma_m)^2 \right] \right\} \quad (6-7) \]

Using Equation (6-4), Equation (6-7) can be written as

\[ \sigma_e = \frac{1}{\sqrt{2}} \left[ (\sigma_{rd} - \sigma_{\theta d})^2 + (\sigma_{\theta d} - \sigma_{zd})^2 + (\sigma_{zd} - \sigma_{rd})^2 \right]^{\frac{1}{2}} \]

Upon expansion and simplification one obtains

\[ \sigma_e = \left[ \sigma_{rd}^2 + \sigma_{\theta d}^2 + \sigma_{zd}^2 - (\sigma_{rd}\sigma_{\theta d} + \sigma_{\theta d}\sigma_{zd} + \sigma_{zd}\sigma_{rd}) \right]^{\frac{1}{2}} \quad (6-8) \]

Taking the time derivative of Equation (6-8) results in
\[ \dot{\sigma}_E = \frac{1}{2\sigma_e} \left[ \dot{\sigma}_r (2\sigma_{rd} - \sigma_{\theta d} - \sigma_{zd}) ight. \\
+ \dot{\sigma}_{\theta d} (2\sigma_{\theta d} - \sigma_{rd} - \sigma_{zd}) \\
+ \dot{\sigma}_{zd} (2\sigma_{zd} - \sigma_{rd} - \sigma_{\theta d}) \] 

(6-9)

Substituting Equation (6-5) into (6-9) and simplifying, one obtains

\[ \dot{\sigma}_E = \frac{1}{2\sigma_e} \left[ \dot{\sigma}_r (2\sigma_{rd} - \sigma_{\theta d} - \sigma_{zd}) ight. \\
+ \dot{\sigma}_{\theta d} (2\sigma_{\theta d} - \sigma_{rd} - \sigma_{zd}) \\
+ \dot{\sigma}_{zd} (2\sigma_{zd} - \sigma_{rd} - \sigma_{\theta d}) \] 

(6-10)

In the post-yielding state, the simplified flow rule for isotropic material gives the principal plastic strains by Equation (3-16)

\[ \frac{d\varepsilon_P^{ij}}{dt} = \frac{3}{2} \frac{d\sigma_e}{H' \sigma_e} \sigma_{dij} \] 

(3-16)

Taking the time derivative of (3-16) one obtains

\[ \dot{\varepsilon}_P^{ij} = \frac{3}{2} \frac{\sigma_{dij} \dot{\sigma}_e}{H' \sigma_e} i = r, \theta, z \] 

(6-11)

Where \( H' \) is the slope of the curve of the equivalent stress vs. the equivalent strain [25].

Substituting Equation (6-10) into Equation (6-11) results in

\[ \dot{\varepsilon}_P^r = \frac{3\sigma_{rd}}{4H' \sigma_e^2} \dot{\sigma}_r (2\sigma_{rd} - \sigma_{\theta d} - \sigma_{zd}) \]
\[ \varepsilon_{r} = \varepsilon_{r}^{E} + \varepsilon_{r}^{th} + \varepsilon_{r}^{p} \]  

\[ \varepsilon_{\theta} = \varepsilon_{\theta}^{E} + \varepsilon_{\theta}^{th} + \varepsilon_{\theta}^{p} \]  

\[ \varepsilon_{z} = \varepsilon_{z}^{E} + \varepsilon_{z}^{th} + \varepsilon_{z}^{p} \]

The total strain rates due to both thermal expansion and stress change are

\[ \varepsilon_{i} = \varepsilon_{i}^{E} + \varepsilon_{i}^{th} + \varepsilon_{i}^{p} \]  

Taking the time derivative of Equation (2-4) we have

\[ \varepsilon_{r}^{E} + \varepsilon_{r}^{th} = \frac{1}{E} [ \sigma_{r} - u(\sigma_{\theta} + \sigma_{z}) ] + \alpha \dot{T} \]

\[ \sigma_{\theta}^{E} + \sigma_{\theta}^{th} = \frac{1}{E} [ \sigma_{\theta} - u(\sigma_{r} + \sigma_{z}) ] + \alpha \dot{T} \]  

\[ \sigma_{z}^{E} + \sigma_{z}^{th} = \frac{1}{E} [ \sigma_{z} - u(\sigma_{r} + \sigma_{\theta}) ] + \alpha \dot{T} \]
Substituting Equations (6-12) and (6-14) into Equation (6-13) the total strain rates in three principal directions become

\[
\dot{\varepsilon}_r = \frac{1}{E}[\dot{\sigma}_r - u(\dot{\sigma}_\theta + \dot{\sigma}_z)] + \frac{3\sigma_{rd}}{4H's_e^2}[\dot{\sigma}_r (2\sigma_{rd} - \sigma_{\theta d} - \sigma_{zd})
\]
\[
+ \dot{\sigma}_\theta (2\sigma_{\theta d} - \sigma_{rd} - \sigma_{zd}) + \dot{\sigma}_z (2\sigma_{zd} - \sigma_{rd} - \sigma_{\theta d})] + \alpha T
\]

\[
\dot{\varepsilon}_\theta = \frac{1}{E}[\dot{\sigma}_\theta - u(\dot{\sigma}_r + \dot{\sigma}_z)] + \frac{3\sigma_{\theta d}}{4H's_e^2}[\dot{\sigma}_\theta (2\sigma_{\theta d} - \sigma_{\theta d} - \sigma_{zd})
\]
\[
+ \dot{\sigma}_\theta (2\sigma_{\theta d} - \sigma_{rd} - \sigma_{zd}) + \dot{\sigma}_z (2\sigma_{zd} - \sigma_{rd} - \sigma_{\theta d})] + \alpha T
\]

\[
\dot{\varepsilon}_z = \frac{1}{E}[\dot{\sigma}_z - u(\dot{\sigma}_r + \dot{\sigma}_\theta)] + \frac{3\sigma_{zd}}{4H's_e^2}[\dot{\sigma}_z (2\sigma_{zd} - \sigma_{\theta d} - \sigma_{zd})
\]
\[
+ \dot{\sigma}_\theta (2\sigma_{\theta d} - \sigma_{rd} - \sigma_{zd}) + \dot{\sigma}_z (2\sigma_{zd} - \sigma_{rd} - \sigma_{\theta d})] + \alpha T
\]

Introducing the following nondimensional variables for convenience

\[
\dot{\varepsilon}_r \equiv \frac{T}{T_0}, \quad S_r = \frac{\sigma_r}{S_y}; \quad S_\theta = \frac{\sigma_\theta}{S_y}; \quad S_z = \frac{\sigma_z}{S_y}
\]

\[
S_e = \frac{\sigma_e}{S_y}; \quad S_d = \frac{\sigma_d}{S_y}; \quad S_r' = \frac{\dot{\sigma}_r}{S_y}; \quad S_\theta' = \frac{\dot{\sigma}_\theta}{S_y}
\]

\[
(6-16)
\]

Where \( S_y \) is the yield strength of the material.

Using Equation (6-16) into Equation (6-15) one obtains

\[
\frac{\dot{\varepsilon}_r - \alpha T}{\sigma T_0} = \frac{S_y}{EdT_0}[\dot{S}_r - u(\dot{S}_\theta + \dot{S}_z)]
\]
Defining

\[ \frac{\dot{\varepsilon}_r - \alpha T}{\alpha T_0} = \frac{\dot{\sigma}_r - \alpha T_0}{\alpha T_0} = \frac{\dot{\varepsilon}_r - \dot{\varepsilon}_T}{\alpha T_0} = \frac{\dot{\varepsilon}_r}{\alpha T_0} \]

After substitution and simplification the strain rates are:

\[ \dot{\varepsilon}_r = \frac{S}{E} [\dot{\varepsilon}_r - \nu(\dot{\varepsilon}_\theta + \dot{\varepsilon}_z)] + \frac{3S_{rd}S_{\gamma}g}{4H'S_e^2} [\dot{S}_r(2S_{rd} - S_{\theta d} - S_{zd}) + \dot{S}_\theta(2S_{\theta d} - S_{rd} - S_{zd}) + \dot{S}_z(2S_{zd} - S_{rd} - S_{\theta d})] \] (6-19)

Similarly

\[ \dot{\varepsilon}_\theta = \frac{S}{E} [\dot{S}_\theta - \nu(\dot{S}_r + \dot{S}_z)] + \frac{3S_{\theta d}S_{\gamma}g}{4H'S_e^2} [\dot{S}_r(2S_{rd} - S_{\theta d} - S_{zd}) + \dot{S}_\theta(2S_{\theta d} - S_{rd} - S_{zd}) + \dot{S}_z(2S_{zd} - S_{rd} - S_{\theta d})] \] (6-20)

Where \( g \) is introduced and may assume two values [19]:

\[ g(r,t) = 0 \text{ if } S_e < 1 \] (6-21)

\[ g(r,t) = 1 \text{ if } S_e \geq 1 \]

If the material is still within the elastic range, Hooke's law governs and \( g = 0 \), which induces the plastic rate expression to vanish. This situation will exist when equivalent stress \( \sigma_e \), is less than the
yield strength, $S_y^*$, of the material. Outside the elastic range, the plastic strain rate must be accounted for. This situation arises whenever equivalent stress is greater than or equal to the yield strength of the material at the specified temperature. The general relation for the yield strength at any temperature will be specified for each material which is considered. Also a temperature dependent modulus of elasticity and thermal coefficient of expansion for each material will be obtained from the data available.

Writing the strain rate in terms of stress rate we get

$$\dot{\varepsilon}_r = \frac{S_y}{E} + \frac{3S_{rd}S_{yg}}{4H'S_e^2}(2S_{rd} - S_{0d} - S_{zd})] \dot{S}_r$$

$$+ \left[ - \frac{uS_y}{E} + \frac{3S_{rd}S_{yg}}{4H'S_e^2}(2S_{0d} - S_{rd} - S_{zd})\right] \dot{S}_\theta$$

$$+ \left[ - \frac{uS_y}{E} + \frac{3S_{rd}S_{yg}}{4H'S_e^2}(2S_{zd} - S_{rd} - S_{0d})\right] \dot{S}_z$$

Defining $P = \frac{3S_y}{4H'}$ gives

$$C_1 = \left[ \frac{S_y}{E} + \frac{PS_{rd}}{S_e^2}(2S_{rd} - S_{0d} - S_{zd})\right]$$

$$C_2 = \left[ - \frac{uS_y}{E} + \frac{PS_{rd}}{S_e^2}(2S_{0d} - S_{rd} - S_{zd})\right]$$

$$C_3 = \left[ - \frac{uS_y}{E} + \frac{PS_{rd}}{S_e^2}(2S_{zd} - S_{rd} - S_{0d})\right]$$
Equation (6-17) can be written as

\[ \dot{r} = C_1 \dot{r} + C_2 \dot{\theta} + C_3 \dot{z} \]  

(6-22)

In a similar manner Equation (6-21) is expanded using

\[ C_4 = \left[ -\frac{uS_y}{E} + \frac{PS \delta d g}{S_e^2} (2S_{rd} - S_{\theta d} - S_{zd}) \right] \]

\[ C_5 = \left[ \frac{S_y}{E} + \frac{PS \delta d g}{S_e^2} (2S_{\theta d} - S_{rd} - S_{zd}) \right] \]

\[ C_6 = \left[ -\frac{uS_y}{E} + \frac{PS \delta d g}{S_e^2} (2S_{zd} - S_{rd} - S_{\theta d}) \right] \]

To obtain

\[ \dot{\theta} = C_4 \dot{r} + C_5 \dot{\theta} + C_6 \dot{z} \]  

(6-23)

Using the equilibrium equation derived in Chapter II and

\[ S_r = \frac{\sigma_r}{S_y}, \quad \rho = \frac{r}{a}, \quad S_\theta = \frac{\sigma_\theta}{S_y} \]

where \( a \) is the radius of the solid cylinder, Equation (2-6) can be written as

\[ \frac{dS_r}{d\rho} + \frac{S_r - S_\theta}{\rho} = 0 \]

Taking the time derivative results in

\[ \frac{d\dot{S}_r}{d\rho} + \frac{\dot{S}_r - \dot{S}_\theta}{\rho} = 0 \]  

(6-24)
Introducing the stress function $\phi$ defined as

$$S_r = \frac{\phi}{\rho}, \quad S_\theta = \frac{\partial \phi}{\partial \rho}$$

the stress time rate in terms of stress function becomes

$$\dot{S}_r = \frac{\dot{\phi}}{\rho}, \quad \dot{S}_\theta = \frac{\partial \dot{\phi}}{\partial \rho} \quad (6-25)$$

Substituting Equation (6-25) into Equation (6-22) and (6-23) one obtains

$$\dot{e}_r = C_1 \frac{\dot{\phi}}{\rho} + C_2 \frac{\partial \dot{\phi}}{\partial \rho} + C_3 S_z \quad (6-26)$$

$$\dot{e}_\theta = C_4 \frac{\dot{\phi}}{\rho} + C_5 \frac{\partial \dot{\phi}}{\partial \rho} + C_6 S_z \quad (6-27)$$

To eliminate $\dot{S}_z$ from Equation (6-25) and (6-27) multiply them by $C_6$ and $C_3$ respectively

$$C_6 \dot{e}_r = C_1 C_6 \frac{\dot{\phi}}{\rho} + C_2 C_6 \frac{\partial \dot{\phi}}{\partial \rho} + C_3 C_6 \dot{S}_z$$

$$C_3 \dot{e}_\theta = C_4 C_3 \frac{\dot{\phi}}{\rho} + C_5 C_3 \frac{\partial \dot{\phi}}{\partial \rho} + C_3 C_6 \dot{S}_z$$

Subtracting above the equations and solving for $\frac{\partial \dot{\phi}}{\partial \rho}$ one obtains

$$\frac{\partial \dot{\phi}}{\partial \rho} = \frac{1}{C_2 C_6 - C_3 C_5} \left[ (C_4 C_3 - C_1 C_6) \frac{\dot{\phi}}{\rho} + C_6 \dot{e}_r - C_3 \dot{e}_\theta \right] \quad (6-28)$$

Equation (6-28) is a first order differential equation of the general form

$$\frac{dy}{dx} + M(x)y = N(x)$$
Upon integration the radial and tangential stress rates may be obtained. The stress rate in the axial direction can be obtained from Equation (6-26) as

\[ \dot{\tau}_z = \frac{1}{C_3} \left[ \dot{\varepsilon}_r - C_1 \dot{\tau}_r - S_2 \dot{\theta} \right] \quad (6-29) \]

In order to find the complete form of the right hand side of Equations (6-28) and (6-29), it is necessary to determine the plastic strain rates appearing on the right hand sides of these equations. An expression for the total strain rate in the plastic range can not be determined analytically because of the interwoven variables involved. To overcome this difficulty, an expression for the total strain rate in the elastic range will be found.

Considering the total strain rate in the elastic range evaluated at the yield point as the initial strain rate in the plastic range, the successive values of strain rate will be added or subtracting by a constant value until the condition of no resultant traction on a plane perpendicular to the axis of the cylinder is satisfied. That is, [13]

\[ \int_0^\rho \dot{\varepsilon}_{\rho \rho} \rho d\rho = 0 \quad (6-30) \]

Also, compatibility condition requires that

\[ \frac{d\varepsilon_\theta}{d\rho} + \frac{1}{\rho} (\varepsilon_\theta - \varepsilon_r) = 0 \quad (6-31) \]
Using Equation (6-30) and (6-31) insure the correct values for strain rate at any level.

**Total Strain Rate in Elastic Range**

In the elastic range, total strains at any point in a heated body consists of two parts, elastic and thermal strains. Total strain in the three principal axis due to temperature effects and the normal strains are

\[
\varepsilon_r = \frac{1}{E}[\sigma_r - u(\sigma_\theta + \sigma_z)] + \alpha T
\]

\[
\varepsilon_\theta = \frac{1}{E}[\sigma_\theta - u(\sigma_r + \sigma_z)] + \alpha T \tag{6-31}
\]

\[
\varepsilon_z = \frac{1}{E}[\sigma_z - u(\sigma_r + \sigma_\theta)] + \alpha T
\]

Adding Equations (6-31) and then solving for \( \varepsilon_z \) one obtains

\[
\varepsilon_z = 3\alpha T - \varepsilon_r - \varepsilon_\theta + \frac{1 - 2\nu}{E}[\sigma_r + \sigma_\theta + \sigma_z] \tag{6-32}
\]

From the plane strain assumption

\[
\frac{d\varepsilon_z}{dr} = 0 \tag{6-33}
\]

Taking the derivative of Equation (6-32) with respect to \( r \)

\[
\frac{\partial \varepsilon_z}{\partial r} = \frac{\partial}{\partial r}(3\alpha T) - \frac{\partial}{\partial r}(\varepsilon_r + \varepsilon_\theta) + \frac{1 - 2\nu}{E} \frac{\partial}{\partial r}(\sigma_r + \sigma_\theta + \sigma_z) \tag{6-34}
\]

Substituting Equation (6-23) into Equation (6-34)

\[
\frac{\partial}{\partial r}(\varepsilon_r + \varepsilon_\theta) = 3\alpha \frac{\partial T}{\partial r} + \frac{1 - 2\nu}{E}(\sigma_r + \sigma_\theta + \sigma_z) \tag{6-35}
\]
For the case that the cylinder is unrestrained and is free to expand axially, but under the plane strain condition

\[ \sigma_z = \sigma_r + \sigma_\theta \]  

(6-36)

Hence utilizing Equation (6-36) in Equation (6-35)

\[ \frac{\partial}{\partial r}(\varepsilon_r + \varepsilon_\theta) = 3\alpha \frac{\partial T}{\partial r} + 2 \left(1 - 2\nu\right) \frac{\partial \sigma_z}{\partial r} \]  

(6-37)

The compatibility equation was derived in Chapter II and for our particular case we had

\[ \frac{\partial \varepsilon_\theta}{\partial r} + \frac{1}{r}(\varepsilon_\theta - \varepsilon_r) = 0 \]  

(2-11)

Differentiating Equation (2-11)

\[ r \frac{\partial \varepsilon_\theta}{\partial r} + 2 \frac{\partial \varepsilon_\theta}{\partial r} - \frac{\partial \varepsilon_r}{\partial r} = 0 \]  

(6-38)

Equation (6-37) can be written as

\[ \frac{\partial \varepsilon_r}{\partial r} = 3\alpha \frac{\partial T}{\partial r} - \frac{\partial \varepsilon_\theta}{\partial r} + 2\left(1 - 2\nu\right) \frac{\partial}{\partial r}(\sigma_z) \]  

(6-39)

Substituting Equation (6-37) into Equation (6-39)

\[ r \frac{\partial \varepsilon_\theta}{\partial r} + 2 \frac{\partial \varepsilon_\theta}{\partial r} = 3\alpha \frac{\partial T}{\partial r} - \frac{\partial \varepsilon_\theta}{\partial r} + 2\left(1 - 2\nu\right) \frac{\partial}{\partial r}(\sigma_z) \]

or
To solve Equation (6-40) let

\[ y = \frac{2}{\beta r} \],

so that \( \frac{\partial y}{\partial r} = \frac{2}{\beta r^2} \) (6-41)

Substituting Equation (6-41) into (6-40) results in a linear equation of first order of

\[
\frac{\partial y}{\partial r} + \frac{3}{r} y = -\frac{3\alpha}{r} \frac{\partial T}{\partial r} + \frac{2(1 - 2\nu)}{E} \frac{1}{r} \frac{\partial}{\partial r} \sigma_z
\]

This is of the general form

\[
\frac{dy}{dx} + p(x)y = q(x)
\]

\( q(x) \), the integration factor is given by [24]

\[ q(x) = \exp\left[\int p(x)dx\right] \]

Comparing with the general form we have

\[ p(x) = \frac{3}{r} \]

then

\[ q(r) = \exp\left[\int \frac{3}{r} dr\right] = e^{3\ln r} = r^3 \] (6-43)

Multiplying both sides of Equation (6-42) by Equation (6-43)

\[
r^3 \frac{\partial y}{\partial r} + \frac{3}{r} r^2 = 3r^2 \alpha \frac{\partial T}{\partial r} + \frac{2r^2(1 - 2\nu)}{E} \frac{\partial}{\partial r} \sigma_z
\] (6-44)
Integrating both sides of Equation (6-44), remembering that the integral of the left side is just the $y$ time integration factor, one finds

$$r^3 y = \int er^2 \alpha \frac{\partial T}{\partial r} \, dr + \int \frac{2r^2 (1-2\nu)}{E} \frac{\partial}{\partial r} \sigma_z \, dr + c$$

Substituting back

$$\frac{\partial \varepsilon_0}{\partial r} = y$$

$$r^3 \frac{\partial \varepsilon_0}{\partial r} = 3\alpha \int \frac{\partial T}{\partial r} r^2 \, dr + \frac{2(1-2\nu)}{E} \int r^2 \frac{\partial \sigma_z}{\partial r} \, dr + c \quad (6-45)$$

Applying the boundary condition

$$\frac{\partial \varepsilon_0}{\partial r} = 0 \quad (6-46)$$

and using Equation (6-45) in Equation (6-46) results in

$$c = 0$$

The Equation (6-45) becomes

$$\frac{\partial \varepsilon_0}{\partial r} = \frac{3\alpha}{r^3} \int \frac{\partial T}{\partial r} r^2 \, dr + \frac{2(1-2\nu)}{E r} \int r^2 \frac{\partial \sigma_z}{\partial r} \, dr \quad (6-47)$$

Substituting the dummy variable $\xi$ into Equation (6-47) and integrating

$$\varepsilon_0 = 3\alpha \int \frac{\xi}{\xi^3} \int \frac{\partial T}{\partial \xi} \xi^2 d\xi d\xi$$

$$+ \frac{2(1-2\nu)}{E} \int \frac{\xi}{\xi^3} \int \xi^2 \frac{\partial \sigma_z}{\partial \xi} d\xi d\xi + Q(t) \quad (6-48)$$
Q(t) represents an undetermined factor of integration which may be a function of time.

Repeating Equation (4-23) for the temperature distribution

\[
\Phi_I(\rho, \tau) = 2 \sum_{n=1}^{\infty} e^{-\mu_n^2 \tau} J_0(\mu_n \rho) / \mu_n J_1(\mu_n) \quad (4-23)
\]

Equations (6-47) and (6-48) can be written with previously defined nondimensional parameters,

\[
\frac{\theta}{T_0}; \rho = \frac{r}{a}; \eta = \frac{\theta}{T_0}
\]

as

\[
\rho^3 \frac{\partial \varepsilon_\theta}{\partial \rho} = 3\alpha T_0 \int_0^r \left[ \frac{\partial \Phi_I}{\partial \rho} \rho^2 d\rho + \frac{2(1 - 2u)}{E} \int_0^\rho \frac{\partial}{\partial \rho} \sigma_z \rho^2 d\rho \right]
\]

and

\[
\varepsilon_\theta = 3\alpha T_0 \int_0^r \left[ \int_0^\rho \frac{\partial \Phi_I}{\partial \eta} \eta^2 d\eta d\eta \right]
\]

\[
+ \frac{2(1 - 2u)}{E} \int_0^r \left[ \int_0^\rho \frac{\partial}{\partial \eta} \sigma_z \eta^2 d\eta d\eta \right] + Q(\tau)
\]

The solution of (6-50) requires integration and differentiation of Bessel functions, where the following identity equations will be utilized

\[
\frac{d}{dx}[J_n(u)] = [J_{n-1}(u) - \frac{n}{u} J_n(u)] \frac{du}{dx}
\]

\[
\int xJ_0(u) dx = \frac{1}{du/dx}[xJ_1(u)]
\]

\[
\int J_1(u) dx = - \frac{J_0(u)}{du/dx}
\]
Taking the derivative of Equation (4-23) with respect to $p$

$$\frac{\partial \hat{T}}{\partial p} = 2 \sum_{n=1}^{\infty} \frac{e^{-\mu_n^2r}}{\mu_n J_1(\mu_n)} \left[ -\mu_n J_1(\mu_n p) \right]$$  \hspace{1cm} (6-51)

Elastic thermal stresses were found by Equations (5-13), (5-14) and (5-15)

$$\sigma_z = \frac{2\alpha E T_0}{1 - \nu} \sum_{n=1}^{\infty} \frac{e^{-\mu_n^2r}}{\mu_n^2 J_1(\mu_n)} \left[ 2J_1(\mu_n) - \mu_n J_0(\mu_n p) \right]$$  \hspace{1cm} (5-15)

$$\frac{\partial \sigma_z}{\partial p} = \frac{2\alpha E T_0}{1 - \nu} \sum_{n=1}^{\infty} \frac{e^{-\mu_n^2r}}{\mu_n^2 J_1(\mu_n)} \left[ \mu_n^2 J_1(\mu_n p) \right]$$  \hspace{1cm} (6-52)

Substituting Equations (6-51) and (6-52) into Equation (6-50) results

$$\frac{\varepsilon_{\theta}}{\alpha T_0} = -6 \int_0^1 \int_0^1 \int_0^1 \sum_{n=1}^{\infty} \frac{e^{-\mu_n^2r}}{J_1(\mu_n)} \left[ J_1(\mu_n \eta) \right] \eta d\eta d\eta$$

$$+ \frac{4(1 - 2\nu)}{1 - \nu} \int_0^1 \int_0^1 \int_0^1 \sum_{n=1}^{\infty} \frac{e^{-\mu_n^2r}}{J_1(\mu_n)} J_1(\mu_n \eta) \eta^2 d\eta$$

Since the term $\sum e^{-\mu_n^2r}$ is not dependent on $\eta$ let

$$S = 2 \sum_{n=1}^{\infty} e^{-\mu_n^2r}$$

$$\frac{\varepsilon_{\theta}}{\alpha T_0} = -3S \int_0^1 \int_0^1 \int_0^1 \eta^2 J_1(\mu_n \eta) d\eta d\eta$$
\[ + \frac{2(1 - 2\mu)S}{1 - \mu} \int \int \int J_0(\mu, \eta) \eta^2 d\eta d\mu + Q(\tau) \]

\[ \frac{\varepsilon_0}{\omega T_0} = S(-3 + 2(1 - 2\mu)) \int \int \int J_1(\mu, \eta) \eta^2 d\eta + Q(\tau) \]

\[ \frac{\varepsilon_0}{\omega T_0} = -S \left( \frac{1 + \mu}{1 - \mu} \right) \int \int \int J_1(\mu, \eta) \eta^2 d\eta + Q(\tau) \]

\[ u = \eta^2 \quad \text{du} = 2\eta d\eta \]

\[ dv = J_1(\mu, \eta) \quad v = -\frac{1}{\mu} J_1(\mu, \eta) \]

Integrating by parts results in

\[ \int \int \int J_1(\mu, \eta) \eta^2 d\eta = -\frac{J_0(\mu, \eta)}{\mu} \eta^2 + \frac{2}{\mu} \int \int \int J_0(\mu, \eta) \eta d\eta \]

where

\[ \frac{2}{\mu} \int \int \int J_0(\mu, \eta) \eta \eta d\eta = \frac{2}{\mu} \left[ \eta \frac{\eta}{\mu} - \frac{J_1(\mu, \eta)}{\mu} \right] = \frac{2\eta J_1(\mu, \eta)}{\mu^2} \]

Integrating the second integral of the first part

\[ \frac{\varepsilon_0}{\omega T_0} = -\frac{S(1 + \mu)}{1 - \mu} \int \int \int \frac{J_0(\mu, \eta)}{\eta} \eta^2 + \frac{J_1(\mu, \eta)}{\mu^2} d\eta \]

\[ = -\frac{1}{\mu} \int \int \int J_0(\mu, \eta) \frac{d\eta}{\eta} + \frac{2}{\mu} \int \int \int J_1(\mu, \eta) d\eta \]

(6-53)
Evaluating each part separately

\[ u = J_1(\mu_n \eta) \]

\[ \rho \int_0^1 J_1(\mu_n \eta) \frac{d\eta}{\eta^2} \quad du = \left[ J_0(\mu_n \eta) \right] \frac{1}{\mu_n \eta} J_1(\mu_n \eta) ] \mu_n d\eta \]

\[ dv = \frac{d\eta}{\eta^2} \quad v = -\frac{1}{\eta} \]

\[ \rho \int_0^1 J_1(\mu_n \eta) \frac{d\eta}{\eta^2} = -\frac{J_1(\mu_n \rho)}{\rho} - \rho \int_0^1 \frac{1}{\eta} [ J_0(\mu_n \eta) - \frac{1}{\mu_n \eta} J_1(\mu_n \eta) ] \mu_n d\eta \]

\[ = -\frac{J_1(\mu_n \eta)}{\eta} + \rho \int_0^1 \mu_n J_0(\mu_n \eta) \frac{d\eta}{\eta} - \rho \int_0^1 J_1(\mu_n \eta) \frac{d\eta}{\eta^2} \]

Rearrange

\[ 2 \rho \int_0^1 J_1(\mu_n \eta) \frac{d\eta}{\eta^2} = -\frac{J_1(\mu_n \rho)}{\rho} + \rho \int_0^1 \mu_n J_0(\mu_n \eta) \frac{d\eta}{\eta} \quad (b) \]

Substituting (b) into (a)

\[ \frac{\varepsilon_\theta}{\alpha To} = -\frac{S(1 + u)}{(1 - u)} \left[ -\frac{1}{\mu_n} \rho \int_0^1 J_0(\mu_n \eta) \frac{d\eta}{\eta} + \frac{1}{\mu_n} \right] \frac{J_0(\mu_n \rho)}{\rho} \]

\[ + \rho \int_0^1 \mu_n J_0(\mu_n \eta) \frac{d\eta}{\eta} + \frac{\varepsilon_\theta}{\alpha To} = \frac{S(1 + u)}{(1 - u)\mu_n^2} J_1 \frac{(\mu_n \rho)}{\rho} \quad (6-54) \]

Substituting S into (6-54)

\[ \frac{\varepsilon_\theta}{\alpha To} = \frac{2(1 + u)}{(1 - u)} \sum_{n=1}^{\infty} \frac{\varepsilon}{\mu_n^2} \frac{J_1(\mu_n)}{\mu_n^2 \mu_n J_1(\mu_n)} + Q(\tau) \quad (6-55) \]
Define

\[ \frac{e_\theta}{\alpha T_0} = \frac{e_\theta}{\alpha T_0} - \frac{e_r}{\alpha T_0} = \frac{e_r}{\alpha T_0} - \frac{\tau}{T} \]  

(6-56)

Substitute (6-55) and (4-23) into (6-56)

\[ \frac{e_\theta}{\alpha T_0} = \frac{2(1 + \nu)}{(1 - \nu)} \sum_{n=1}^{\infty} \frac{e_n}{\mu_n^2} \frac{J_1(\mu_n \rho)}{J_1(\mu_n \rho)} \]

\[ + Q(\tau) - 2 \sum_{n=1}^{\infty} \frac{e_n}{\mu_n^2} \frac{J_0(\mu_n \rho)}{J_1(\mu_n \rho)} \]

\[ e_\theta = \alpha T_0 \{ 2 \sum_{n=1}^{\infty} \frac{e_n}{\mu_n^2} \frac{1}{J_1(\mu_n \rho)} \frac{(1 + \nu)J_1(\mu_n \rho)}{\mu_n (1 - \nu)\rho} \]

\[ - J_0(\mu_n \rho) \} + Q(\tau) \} \]  

(6-57)

From the compatibility Equation (2-11)

\[ \frac{\partial e_\theta}{\partial r} + \frac{1}{r}(e_\theta - e_r) = 0 \]  

(2-17)

or in non-dimensional form

\[ e_r = e_\theta + \rho \frac{\partial e_\theta}{\partial \rho} \]  

(6-58)

From Equation (6-49) we have

\[ \rho^2 \frac{\partial e_\theta}{\partial \rho} = 3\alpha T_0 \int_0^\rho \frac{\partial^2}{\partial \rho^2} \rho^2 \, d\rho + 2(1 - 2\nu) \int_0^\rho \frac{\partial}{\partial \rho} \sigma_2 \rho^2 \, d\rho \]  

(6-49)

Upon substituting Equation (6-51) and (6-52) into Equation (6-49) one obtains
\[ \rho^3 \frac{\partial \varepsilon_\theta}{\partial \rho} = 3\alpha T_0 \int_0 \left[ \sum_{n=1}^{\infty} \frac{e^{-\mu_n^2 \tau}}{J_1(\mu_n)} J_1(\mu_n \rho) \right] \rho^2 \, d\rho \]

\[ + \frac{2(1 - 2u)}{E} \int_0 \left[ \sum_{n=1}^{\infty} \frac{e^{-\mu_n^2 \tau}}{J_1(\mu_n)} J_1(\mu_n \rho) \right] \rho^2 \, d\rho \]

\[ \rho^3 \frac{\partial \varepsilon_\theta}{\partial \rho} = 2\alpha T_0 \sum_{n=1}^{\infty} \frac{e^{-\mu_n^2 \tau}}{J_1(\mu_n)} \int_0 \left( J_1(\mu_n \rho) \rho^2 \, d\rho \right) \left[ -3 + \frac{2(1 - 2u)}{1 - u} \right] \]

Using the relation

\[ \int_0 J_1(\mu_n \rho) \rho^2 \, d\rho = \frac{J_0(\mu_n \rho)}{\mu_n} \rho^2 \]

results in

\[ \rho^3 \frac{\partial \varepsilon_\theta}{\partial \rho} = - \frac{2\alpha T_0 (1 + u)}{1 - u} \sum_{n=1}^{\infty} \frac{e^{-\mu_n^2 \tau}}{J_1(\mu_n)} \left[ - \frac{J_0(\mu_n \rho)}{\mu_n} \rho^2 + 2 \rho \frac{J_1(\mu_n \rho)}{\mu_n^2} \right] \]

Dividing through by \( \rho^3 \) gives

\[ \frac{\partial \varepsilon_\theta}{\partial \rho} = - \frac{2\alpha T_0 (1 + u)}{\rho(1 - u)} \sum_{n=1}^{\infty} \frac{e^{-\mu_n^2 \tau}}{\mu_n J_1(\mu_n)} \left[ - \frac{J_0(\mu_n \rho)}{\mu_n} + 2 \frac{J_1(\mu_n \rho)}{\mu_n^2} \right] \]

(6-59)

Substituting Equations (6-56) and (6-59) into Equation (6-58) results in
\[
\frac{e_r}{\alpha T \sigma} - \frac{\tau}{\alpha T \sigma} = \frac{3}{2} \frac{\tau}{\alpha T \sigma} - \frac{\tau}{\alpha T \sigma} + 2(1 + \nu) \left( \frac{1}{1 - \nu} \right) \Sigma \frac{e^{-\mu_n^2 \tau}}{\mu_n^2 J_1(\mu_n)} J_0(\mu_n \rho)
\]

\[
- 4(1 + \nu) \left( \frac{1}{1 - \nu} \right) \rho \Sigma \frac{e^{-\mu_n^2 \tau}}{\mu_n^2 J_1(\mu_n)} J_1(\mu_n \rho)
\]

\[
\frac{e_r}{\alpha T \sigma} = \frac{2(1 + \nu)}{1 - \nu} \Sigma \frac{e^{-\mu_n^2 \tau}}{\mu_n^2 J_1(\mu_n)} J_1(\mu_n \rho) - 2 \Sigma \frac{e^{-\mu_n^2 \tau}}{\mu_n^2 J_1(\mu_n)} J_0(\mu_n \rho)
\]

\[
+ \frac{2(1 + \nu)}{1 - \nu} \Sigma \frac{e^{-\mu_n^2 \tau}}{\mu_n^2 J_1(\mu_n)} J_0(\mu_n \rho)
\]

\[
- \frac{4}{1 - \nu} \Sigma \frac{e^{-\mu_n^2 \tau}}{\mu_n^2 J_1(\mu_n)} J_1(\mu_n \rho) + Q(\tau)
\]

\[
\frac{e_r}{\alpha T \sigma} = 2 \Sigma \frac{e^{-\mu_n^2 \tau}}{\mu_n^2 J_1(\mu_n)} J_0(\mu_n \rho) \left[ \frac{1 + \nu}{1 - \nu} - 1 \right]
\]

\[
+ 2 \Sigma \frac{e^{-\mu_n^2 \tau}}{\mu_n^2 J_1(\mu_n)} \left[ \frac{(1 + \nu)}{(1 - \nu)} - \frac{2(1 + \nu)}{1 - \nu} \right] + Q(\tau)
\]

\[
e_r = \alpha T \sigma \left\{ 2 \Sigma \frac{e^{-\mu_n^2 \tau}}{\mu_n^2 J_1(\mu_n)} \left[ \frac{2\nu}{1 - \nu} J_0(\mu_n \rho) \right] \frac{1 + \nu}{1 - \nu} \right\} + Q(\tau)
\]

- \( \left( \frac{1 + \nu}{1 - \nu} \right) \frac{J_1(\mu_n \rho)}{\mu_n \rho} \) + Q(\tau)}
Axial Strain

The strain component in the z direction can be found by using Equations (6-32) and (6-36) which results in

\[ \varepsilon_z = 3\alpha T - \varepsilon_r - \varepsilon_\theta + \frac{1 - 2\nu}{E} \sigma_z \]  

(6-61)

Substituting

\[ \varepsilon_r = \varepsilon_r + \alpha T; \varepsilon_\theta = \varepsilon_\theta + \alpha T; \varepsilon_z = \varepsilon_z + \alpha T \]

into Equation (6-61)

\[ (\varepsilon_z + \alpha T) = 3\alpha T - (\varepsilon_r + \alpha T) - (\varepsilon_\theta + \alpha T) + \frac{1 - 2\nu}{E\alpha T} \sigma_z \]

\[ \frac{\varepsilon_z}{\alpha T} = \frac{\varepsilon_r}{\alpha T} - \frac{\varepsilon_\theta}{\alpha T} + 2 \left( \frac{1 - 2\nu}{E\alpha T} \right) \sigma_z \]  

(6-62)

Substituting Equations (5-12), (6-57) and (6-60) into Equation (6-62) one obtains

\[ \frac{\varepsilon_z}{\alpha T} = -2 \sum \frac{\mu_n}{J_n(\mu_n)} \left[ \frac{2\nu}{1 - \nu} J_0(\mu_n \rho) - \frac{1 + \nu}{1 - \nu} \frac{J_1(\mu_n \rho)}{\mu_n \rho} \right] - Q(\tau) \]

\[-2 \sum \frac{\mu_n}{J_n(\mu_n)} \left[ -J_0(\mu_n \rho) + \frac{1 + \nu}{1 - \nu} \frac{J_1(\mu_n \rho)}{\mu_n \rho} \right] - Q(\tau) \]

\[ + 4 \sum \frac{\mu_n}{J_n(\mu_n)} \left[ \frac{1 - 2\nu}{(1 - \nu)} J_0(\mu_n \rho) + 2 \left( \frac{1 - 2\nu}{1 - \nu} \right) \frac{J_1(\mu_n \rho)}{\mu_n} \right] \]
From equations (6-59), (6-60) and (6-63) the term \( Q(t) \) is an undetermined function of time which has to be defined. To evaluate this function the equilibrium equations and boundary conditions must be satisfied. The total strain in \( z \) direction is

\[
\varepsilon_z = e_z + \alpha T
\]

which can be written as

\[
\frac{\varepsilon_z}{\alpha T_0} = \frac{e_z}{\alpha T_0} + \frac{\alpha T_0}{T}
\]

(6-64)

Substituting Equations (6-63) and (4-23) into (6-64) results in

\[
\frac{\varepsilon_z}{\alpha T_0} = -2 \sum \frac{e}{\mu^2 J_1(\mu \rho)} J_0(\mu \rho) - \frac{8(1 - 2u)}{1 - u} \sum \frac{e}{\mu^2 J_1(\mu \rho)} J_1(\mu \rho)
\]
\[ -2Q(\tau) + 2 \sum \frac{-\mu_n^2 r}{\mu_n J_1(\mu_n)} J_0(\mu_n \rho) \]

\[ \varepsilon_z = \frac{8(1-2\upsilon)}{(1-\upsilon)} \sum \frac{-\mu_n^2 r}{\mu_n^2 J_1(\mu_n)} - 2Q(\tau) \quad (6-65) \]

On the other hand the total strain in the z direction due to temperature effects and the normal strain in the elastic range is

\[ \varepsilon_z = \frac{1}{E_o \alpha_o} \left[ \sigma_z - \upsilon (\sigma_{\tau} + \sigma_\theta) \right] + \dot{\varepsilon} \]

Using Equation (6-36)

\[ \varepsilon_z = \frac{(1-\upsilon)}{E_o \alpha_o} \sigma_z + \dot{\varepsilon} \quad (6-66) \]

Using Equations (5-12) and (4-23) into Equation (6-66) results in

\[ \varepsilon_z = 2 \sum \frac{-\mu_n^2 r}{\mu_n J_1(\mu_n)} [2J_1(\mu_n) - \mu_n J_0(\mu_n \rho)] \]

\[ + 2 \sum \frac{-\mu_n^2 r}{\mu_n^2 J_1(\mu_n)} J_0(\mu_n \rho) \]

\[ \varepsilon_z = 4 \sum \frac{-\mu_n^2 r}{\mu_n^2 J_1(\mu_n)} J_1(\mu_n) \quad (6-67) \]

Equating Equations (6-65) and (6-67)
\[
\frac{8(1 - 2\nu)}{1 - \nu} \sum \frac{-\mu_n^2 \tau}{\mu_n^2 J_1(\mu_n)} - 2Q(\tau) = 4 \sum \frac{-\mu_n^2 \tau}{\mu_n^2}
\]

Solving for \(Q(\tau)\)

\[
Q(\tau) = -2 \sum \frac{-\mu_n^2 \tau}{\mu_n^2} + \frac{4(1 - 2\nu)}{(1 - \nu)} \sum \frac{-\mu_n^2 \tau}{\mu_n^2} J_1(\mu_n \rho)
\]

\[
Q(\tau) = 2 \sum \frac{-\mu_n^2 \tau}{\mu_n^2} [-1 + \frac{4(1 - 2\nu)}{(1 - \nu)} J_1(\mu_n \rho)]
\]  \(\text{(6-68)}\)

The complete form of thermoelastic strain are obtained by introducing Equation (6-68) into Equations (6-57), (6-60), and (6-63).

The results after simplification are

\[
e_r = \alpha T \sum \frac{-\mu_n^2 \tau}{(1 - \nu) \mu_n^2} [J_1(\mu_n \rho)(-\frac{(1 + \nu)}{\rho J_1(\mu_n)}} + 4(1 - 2\nu))
\]

\[
+ 2\mu_n \frac{J_0(\mu_n \rho)}{J_1(\mu_n)} - (1 - \nu)]
\]  \(\text{(6-69)}\)

\[
e_\theta = \alpha T \sum \frac{-\mu_n^2 \tau}{(1 - \nu) \mu_n^2} [J_1(\mu_n \rho)(\frac{1 + \nu}{\rho J_1(\mu_n)}} + 4(1 - 2\nu))
\]

\[
- \frac{(1 - \nu) \mu_n J_0(\mu_n)}{J_1(\mu_n)} - (1 - \nu)]
\]  \(\text{(6-70)}\)
\[ e_z = a \sum_{n=1}^{\infty} \frac{\mu_n^2}{(1 - \nu)} \sum_{n=1}^{\infty} \left[ J_1(\mu_n) \left( \frac{4(1 - 2\nu)}{J_1(\mu_n)} - 4(1 - 2\nu) \right) \right. \\
\left. - \frac{(1 - \nu)\mu_n J_0(\mu_n)}{J_1(\mu_n)} + (1 - \nu) \right] \]  

Figures (6.1) and (6-2) shows the radial and tangential strain distribution for various time.
Fig. 6-1 Elastic Radial Strain Distribution
Fig. 6-2 Elastic Tangential Strain Distribution
CHAPTER VII

NUMERICAL EVALUATION OF INELASTIC THERMAL STRESSES

The theory developed in Chapter VI along with the transient temperature distribution equation (4-21) can be used to determine the thermo-elastoplastic stress distributions in a cooling solid cylinder as the time elapses. In this chapter, the numerical procedure will be explained.

Numerical Procedure

First the transient temperature distribution for various time interval using Equation (4-21)

\[
T(r,t) = 2 \sum_{n=1}^{\infty} e^{2 \frac{2t}{\mu_n}} J_0 (\frac{r a}{\mu_n}) \frac{\mu_n}{J_1 (\mu_n)}
\]

(4-21)

is determined. Since the mechanical properties of a material are temperature dependent, a relation for each mechanical property as a function of temperature, using available experimental graphs, is obtained. With the aid of the transient temperature distribution, the mechanical properties at any time increment and any radial position in the solid cylinder are calculated and stored in an array.

To integrate Equation (6-28)

\[
\frac{\partial \phi}{\partial \rho} = \frac{1}{c_2 c_6 - c_3 c_5} \left[ (c_4 \epsilon_3 - c_1 \epsilon_6) \frac{\dot{\phi}}{\rho} + c_6 \dot{e}_r - c_3 \dot{e}_\theta \right]
\]

(6-28)
one will have to evaluate, as a first step, the constants parameters $C_1$ through $C_6$. To evaluate $C_1$ through $C_6$ when the material is in the plastic range, the stresses in radial, tangential, and axial directions are required. For the first time increment, i.e., $0 \leq \tau \leq 0.005$, it is assumed that the entire cylinder is in the elastic range. Hence, the dimensionless stress $S_r$, $S_\theta$, and $S_z$ are calculated from Equations (5-13), (5-14) and (5-15)

\[
S_r = \frac{2\alpha E_0}{S_y(1-v)} \sum_{n=1}^{\infty} \frac{-\mu_n^2}{\mu_n^2 J_1(\mu_n)} \left[ J_1(\mu_n) - \frac{1}{\rho} J_1(\mu_n \rho) \right] \tag{5-13}
\]

\[
S_\theta = \frac{2\alpha E_0}{S_y(1-v)} \sum_{n=1}^{\infty} \frac{-\mu_n^2}{\mu_n^2 J_1(\mu_n)} \left[ J_1(\mu_n) + \frac{1}{\rho} J_1(\mu_n \rho) - \mu_n J_0(\mu_n \rho) \right] \tag{5-14}
\]

\[
S_z = \frac{2\alpha E_0}{S_y(1-v)} \sum_{n=1}^{\infty} \frac{-\mu_n^2}{\mu_n^2 J_1(\mu_n)} \left[ 2J_1(\mu_n) - \mu_n J_0(\mu_n \rho) \right] \tag{5-15}
\]

Having these stress calculated stress deviations and equivalent stress are found using Equation (6-4) and (6-8).

\[
\sigma_{rd} = \sigma_r - \sigma_m = \frac{1}{3} \left[ 2\sigma_r - \sigma_\theta - \sigma_z \right]
\]

\[
\sigma_{\theta d} = \sigma_\theta - \sigma_m = \frac{1}{3} \left[ 2\sigma_\theta - \sigma_r - \sigma_z \right] \tag{6-4}
\]

\[
\sigma_{zd} = \sigma_z - \sigma_m = \frac{1}{3} \left[ 2\sigma_z - \sigma_r - \sigma_\theta \right]
\]

\[
S_e = \frac{1}{\sqrt{2}} \left[ (S_{rd} - S_{\theta d})^2 + (S_{\theta d} - S_{zd})^2 + (S_{zd} - S_{rd})^2 \right]^{\frac{1}{2}} \tag{6-8}
\]
Having the deviatoric stresses, equivalent stress, and physical properties of the material, the values of \( C_1 \) through \( C_6 \) can be obtained using

\[
C_1 = \frac{S_Y}{E} + \frac{P S_{rd}^g}{S_e^2} \left( 2S_{rd} - S_{\theta d} - S_{zd} \right)
\]

\[
C_2 = \frac{\nu S_Y}{E} + \frac{P S_{rd}^g}{S_e^2} \left( 2S_{\theta d} - S_{rd} - S_{zd} \right)
\]

\[
C_3 = \frac{\nu S_Y}{E} + \frac{P S_{zd}^g}{S_e^2} \left( 2S_{zd} - S_{rd} - S_{\theta d} \right)
\]

\[
C_4 = \frac{\nu S_Y}{E} + \frac{P S_{\theta d}^g}{S_e^2} \left( 2S_{\theta d} - S_{rd} - S_{zd} \right)
\]

\[
C_5 = \frac{S_Y}{E} + \frac{P S_{\theta d}^g}{S_e^2} \left( 2S_{\theta d} - S_{rd} - S_{zd} \right)
\]

\[
C_6 = \frac{\nu S_Y}{E} + \frac{P S_{\theta d}^g}{S_e^2} \left( 2S_{zd} - S_{rd} - S_{\theta d} \right)
\]

where \( P = \frac{3S_Y}{4H} \)

To integrate Equation (6-28), the thermoelastoplastic strain rate \( \dot{e}_r \) and \( \dot{e}_\theta \) must also be known. A mathematical expression for these strain rates are found to be difficult to obtain because of the lack of
knowledge of exact behavior of plastic strain. To circumvent this difficulty, the relations for thermoelastic strains are obtained using equations (6-69) and (6-70).

\[
e_r = \alpha T_0 \sum_{n=1}^{\infty} \frac{e^{-\mu_n^2 \tau}}{(1 - \nu)\mu_n^2} \left[ J_1(\mu_n \rho) (\frac{1 + \nu}{\rho J_1(\mu_n^2)} + 4(1 - 2\nu))ight] + 2\mu_n \frac{J_0(\mu_n \rho)}{J_1(\mu_n^2)} - (1 - \nu) \right] 
\]

\[
e_\theta = \alpha T_0 \sum_{n=1}^{\infty} \frac{e^{-\mu_n^2 \tau}}{(1 - \nu)\mu_n^2} \left[ J_1(\mu_n \rho) \left(\frac{1 + \nu}{\rho J_1(\mu_n^2)} + 4(1 - 2\nu)\right) - \frac{(1 - \nu)\mu_n J_0(\mu_n \rho)}{J_1(\mu_n^2)} - (1 - \nu) \right] 
\]

Equations (6-69) and (6-70) are used to compute the successive radial and tangential strain rates \( \dot{e}_r \) and \( \dot{e}_\theta \) values at a number of levels along radius \( \rho \). The values of strain rate along with the physical properties of materials as a function of temperature are again stored in the computer. Since these strain rates are thermoelastic strain rates they can not be used in Equation (6-28) which requires the total strain rates, elastic as well as the plastic strain rates. But these strains are used as initial values and the exact values of total strain are obtained utilizing the compatibility equations and the boundary conditions. To do this numerically, first using Equation (6-69) and (6-70), the strain rates \( \dot{e}_r \) and \( \dot{e}_\theta \) are evaluated, then an increment, delta, was added to the strain rates for the plastic strain rate portion. The strain rates are assumed to be the total strain rates.
Having evaluated all of the parameters in Equation (6-28) one can numerically integrate it. Runge-kutta integration method [34] may be applied and using equation (6-25)

\[ S_r = \frac{\phi}{\rho}, \quad S_\theta = \frac{\partial \phi}{\partial \rho} \]  

(6-25)

the stress rates in radial and tangential directions can be evaluated. Axial stress rate \( \dot{S}_z \), can be evaluated using Equation (6-29)

\[ \dot{S}_z = \frac{1}{C_3}[\varepsilon_r - C_1 \dot{\varepsilon}_r - C_2 \dot{\varepsilon}_\theta] \]  

(6-29)

From these stress rates, stresses are found for \( 0.0 < t < 0.005 \). For the next time interval (\( t > 0.005 \)) the stresses evaluated for previous time range are used to evaluate Equations (6-4) and (6-8) and then following the same procedure, the stresses in the second time interval are obtained.

To check the accuracy of these calculated stresses, 1 - One has to determine whether the material under the given condition of temperature gradient and the time interval, is in elastic or plastic range. By means of Equation (6-8) equivalent stress \( S_e \) is found in that time interval and compared the yield strength of the material. If \( S_e \) is less than \( S_y \), the strain for the plastic range are omitted in Equations (6-19) and (6-20) by letting \( g \) be equal to zero. On the other hand if \( S_e \) is greater or equal to \( S_y \), \( g \) is set equal to one. Mathematically stated

\[ g = 0 \text{ for } S_e < S_y \]

\[ g = 1 \text{ for } S_e \geq S_y \]
The stress Equation (6-28) is integrated a second time, if the equivalent stress is less than yield strength using \( g = 0 \), and the new values are obtained for \( C_1 \) through \( C_6 \).

2 - The results will be checked by Equations (6-30) and (6-31)

\[
\rho \int_0^\rho S_2 \rho d\rho = 0 \tag{6-30}
\]

\[
\frac{d\varepsilon_\theta}{dp} + \frac{1}{\rho} (\varepsilon_\theta - \varepsilon_r) = 0 \tag{6-31}
\]

If these two equations are satisfied the stress evaluated in this time interval will be accepted as the correct values, otherwise, through an iterative process the trial value, \( \Delta \), added to the strain rates will be modified. This process will have to be continued until the conditions set by Equation (6-30) and Equation (6-31) are satisfied.

The computer program is arranged such that \( \Delta \) will be added to the strain rates. If this addition results in an improvement toward satisfying Equations (6-30) and (6-31) it will keep on adding it until reaching the desired accuracy, otherwise - \( \Delta \) will be tried. In any case, the \( \Delta \) itself will be changed to smaller value in order to achieve a more accurate result. The accuracy criterion for both condition is \( X_{i+1} - X_i < 10^{-6} \).

After achieving the satisfactory results for the first time increment, the physical properties for the new time interval are obtained and the procedure is repeated in a similar fashion as described before. The convergence value of delta for each time interval is determined.
It must be mentioned that the temperature dependency of the yield stress is a very important factor in the calculation. If the effect of temperature on yield stress is not taken into account the result is substantially different compared with the case when temperature effect is considered. Comparing with the experimental data, as will be discussed in Chapter 8, the results are much more accurate when the temperature effect on yield stress is taken into account. Other physical properties expressed as function of temperature contribute to the accuracy of the results, but perhaps not as strongly as that of yield stress.

Input data are arranged such that, the effect of each parameter can be checked, and compared. In doing so, one can determine which parameter has a more profound effect.

Finally to assure the validity of the computer program as a whole, the program was first run for the pure elastic case, setting \( g \) equal to zero in the strain rates. The results were compared with the elastic stress Equations (5-13) and (5-14). It was found that the results were accurate up to six significant figures.

In Appendix B the flow-chart of the general thermo-elastoplastic stress solution, using the numerical method discussed in this chapter may be found.
CHAPTER VIII

SOLUTION FOR PARTICULAR CASES

In this chapter the numerical solution described in the previous chapter is applied to some specific cases. Since the residual stresses are of particular interest for comparison with experimental data, in most cases the stresses for time equal to infinity (\( t = \infty \)) are presented graphically.

Steels of various chemical compositions can be used for the analysis. The one which is considered here contained 0.05 percent carbon, 0.35 percent manganese, 0.13 percent silicon, and 16.9 percent nickel. Phase transformation for this composition is such that during the cooling process, austenite is transformed almost completely to martensite [32].

The temperature-dependent properties of this steel were obtained using available experimental charts [33]. The least square method was employed to obtain the following relation [34] for thermal coefficients of expansion, modulus of elasticity and yield strength respectively.

\[
\alpha = 6.5 \times 10^{-6} \left[ 1 + 0.22308 \times 10^{-3} T \right] \rho^{0.1} \quad (8-1)
\]
\[
E = 30 \times 10^{-6} \left[ 1 - 0.3463 \times 10^{-6} T^2 \right] \text{psi} \quad (8-2)
\]
\[
S_y = 60 \times 10^3 \left[ 1 - 0.00047583 T \right] \text{psi} \quad (8-3)
\]

Before presenting the results it is convenient to introduce the following dimensionless quantities:
\[
\tilde{\sigma}_r = \frac{\sigma_r}{(2EaT_0)} \quad \rho = \frac{r}{a} 
\]

\[
\tilde{\sigma}_\theta = \frac{\sigma_\theta}{(2EaT_0)} 
\]

\[
\tilde{\sigma}_z = \frac{\sigma_z}{(2EaT_0)} 
\]

The effects of plasticity can be seen when one compares the stress distribution for a purely thermoplastic condition with that of thermoelasto-plastic case. Figures (8-1), (8-2), and (8-3) show the stress distribution in radial, tangential, and axial directions for the pure thermoelastic condition. The numerical values obtained are for a solid cylinder having an initial uniform temperature of 1620°F (900°C) immersed in a media of zero temperature. The stress distribution for various time periods are shown. The radial stress is compressive throughout the cross-sectional area with the maximum values occurring at the center, reducing to zero at the surface of the cylinder. As the time elapses the stresses increase and then decrease. At time about \( \tau = 10 \), radial stresses almost vanish. Tangential thermal stress distribution is somewhat different. At the core of the cylinder tangential stress is compressive but increases with becoming zero and then tensile, reaching its maximum value at the surface of the cylinder. Again as the time passes the stresses change and at about \( \tau = 10 \), the steady state condition is obtained and tangential stresses vanish.

Next, the case of thermoelastoplastic is considered, where the effect of plasticity is taken into account. Figures, (8-4), (8-5), and (8-6) show the stress distribution in radial, tangential, and axial direction respectively. Radial stresses at \( \tau = 0.005 \) is exactly the same as thermoelastic case, but as time elapses the radial stresses increase until steady state condition is reached. In this case the
radial stresses remain unchanged beyond \( \tau = 5 \). This time is referred to as \( \tau = \infty \) on the graphs and the stress are called residual stress. A comparison of Figures (8-1) and (8-4) reveal that for the time range of \( 0.005 < \tau < 0.08 \), the radial thermoelastic and thermoplastic are the same for the radius range of \( 0.0 < \rho < 0.5 \). This indicates that the cylinder is still in the elastic state at the center up to about \( \rho = 0.5 \), but beyond that the material has passed the elastic range. As time elapses the entire cylinder becomes plastic and the stress distribution pattern becomes completely different than that of the thermoelastic case. The radial stress is compressive and maximum at the core of the cylinder, increasing to zero at the cylinder surface. At the early stage of cooling the radial stress slowly decreases reaching a maximum compressive value and as the time elapses the residual radial stress are established. Another distinguished difference between Figures (8-1) and (8-4) is that in thermoelastic case the as time elapses radial stress increases until about \( \tau = 0.08 \), after that as time passes radial stress decrease to zero at \( \tau = \infty \). But for the thermoelastoplastic case, as time elapses the radial stress increases continuously until steady state condition is reached and the residual radial stress established.

Figure 8-5 shows the tangential thermoelastoplastic stress distribution. Tangential stresses are compressive at the core and stay compressive to about \( \rho \approx 0.5 \), then changes to tensile stresses. Again comparing the tangential stress distribution for the thermoelastic and thermoelastoplastic case it can be seen from Figures (8-2) and (8-5) that tangential stress distribution drastically changes especially close to the surface of the cylinder. At small values of \( \tau \) and \( \rho < 0.8 \) the stress distribution resembles the thermoelastic distribution but
around \( \rho \approx 0.8 \) the stresses decrease. Thermoelastoplastic tangential stress distribution is drastically changed especially close to the surface of the cylinder. Again as the time increases the stress changes and at about \( t_{10} \), tangential stresses reach the steady state condition of residual tangential stress.

It is important to mention that at first the yield strength was considered to be temperature-independent, and this assumption resulted in a drastic decrease of tangential stresses at about \( \rho \approx 0.8 \) such that the tangential stresses became compressive. Since the results were erroneous compared to the available experimental result, a temperature-dependent property relation was obtained (Equations (8-1), (8-2), and (8-3)) and the results are as shown in Figure (8-5).

**Residual Stress for Various Yield Strength**

Obviously one of the important questions that may be asked is how the variation in yield strength, going from one composition to another composition of the same materials will effect the residual stresses. To answer this question, different values of yield strength, corresponding to different material compositions, are used. The radial and tangential stress distributions are shown in Figures (8-7) and (8-8). It is evident from these figures that as the yield strength of the material increases the residual stresses will decrease, as one may expect. This indicates that the high strength materials are less susceptible to the plasticity effect than relatively weak materials. In the case of tangential stress distribution, Figure (8-8), the region over which the stress is compressive along the radius of the cylinder is increased as the yield strength increases.
Effect of Elastic Compressibility on Residual Stress

The compressibility of the material in this investigation is taken into account. The total strain for a volume element that has been inelastically deformed consists of an elastic strain and an inelastic strain. Experiments [12] have shown that the inelastic components of strain do not contribute to volume change. In other words, the volume change is always elastic, hence

\[ E_{de} = (1-2u)\text{d}s \quad (8-1) \]

where \( s \) and \( e \) are mean normal stress and mean normal strain, respectively. \( E \) is modulus of elasticity and \( v \) is poisson's ratio. If the stresses and strain satisfy Equation (8-7), the material is said to be compressible. On the other hand, the material is said to be incompressible if

\[ de = 0 \quad (8-8) \]

Figures (8-9), (8-10), and (8-11) show the residual stress \( \tau = \omega \) in radial, tangential, and axial direction considering the effect of compressibility of the material. As is evident from graphs the compressibility of the material does not have a significant effect on the stress distribution. For the compressible case a Possion's ratio of 0.3 was used whereas for the incompressible case Poisson's ratio of 0.5 was applied. One can conclude that the assumption of incompressibility for the material does not significantly alter the stress distribution but rather reduces the complexity of mathematical work.

Residual Stress for Various Elastic-Plastic Slope Ratios

One of the advantages of the method which was developed for this
investigation, compared to other approaches, is that one does not have to assume that the material is elastic-perfect plastic. One can realize this advantage by noting that the strain rate for the plastic range is given by Equation (6-11) as

$$\dot{\varepsilon}_i^p = \frac{3}{2} \frac{\sigma_i \sigma_e}{H' \sigma_e} \quad i = r, \theta, z$$

(6-11)

where $H'$ is the slope of stress-strain curve in the plastic region. By changing the $H'$, stress distribution can be obtained for different materials which have different slope in their plastic region. Using available stress-strain curves the stress distribution for different materials several different elastic-plastic slope ratios $E/H'$ are obtained and the results are plotted on Figure (8-12) and (8-13). Figure (8-12) shows the radial residual stress distribution for various elastic-plastic slope ratio. It can be seen that as the slope in plastic range decreases the residual stresses increases.

**Comparison With Experiment**

Buhler and Scheil [35] have presented experimental results for the residual stresses in water-quenched solid cylinders, 1.9 inch diameter and 13.8 inch long. Steels of various chemical compositions were used in their experiments. The one which was selected for the comparison was quenched at 1620°F. The composition of the steel is 0.05 percent C, 0.035 percent Mn, 0.13 percent Si, and 16.9 percent Ni. Weiner and Huddleston [32] using the properties of this same steel have theoretically determined the residual stresses. Figures (8-14), (8-15), and (8-16) show the residual stresses in radial, tangential, and axial directions respectively. For the sake of comparison, in these figures
the experimental residual stress data given by Buhler and Scheil [35] are plotted. As it is evident by these graphs the computed residual stress data show good agreement with experimental results. For tangential residual stress Figure (8-15) indicates the same pattern and fair agreement between the experimental and computed results. Figure (8-16) shows the comparison of the axial residual stress between the experimental and computed results which again they have the same pattern. Also plotted in Figures (8-14), (8-15), and (8-16) are the theoretical results obtained by Weiner and Huddleson [33] for a cooling cylinder made of elastic, perfectly plastic material having the same chemical composition and properties as the one used in this investigation to compute the residual stresses. As is evident from the same figures, the results of the computed residual stresses based on the theory developed in this investigation are more in agreement with the experimental than these of Weiner and Huddleson.
Fig. 8-1. Thermoelastic Radial Stress Distribution
Fig. 8-2. Thermoelastic Tangential Stress Distribution
Fig. 8-3 Thermoelastic Axial Stress Distribution
Fig. 8-4 Thermoelastoplastic Radial Stress Distribution
Fig. 8–5 Thermoelastoplastic Tangential Stress Distribution
Fig. 8-6 Thermoelastoplastic Axial Stress Distribution
Fig. 8-7 Radial Residual Stress for Various Yield Strength
Fig. 8-8 Tangential Residual Stress for Various Yield Strength
Fig. 8-9 Effect of Compressibility on Radial Residual Stress
Fig. 8-10 Effect of Compressibility on Tangential Residual Stress
Fig. 8-11 Effect of Compressibility on Axial Residual Stress
Fig. 8-12 Radial Residual Stress for Various Elastic—Plastic Slope Ratio
Fig. 8-13 Tangential Residual Stress for Various Elastic-Plastic Slope Ratio
Fig. 8–14 Comparison of Measured and Calculated Radial Residual Stress
Fig. 8-15 Comparison of Measured and Calculated Tangential Residual Stress
Fig. 8-16 Comparison of Measured and Calculated Axial Residual Stress
General Conclusions

The transient and residual stress distribution for a cooling solid cylinder subjected to a transient temperature distribution were determined. The effect of plasticity was investigated and the stresses were compared with the pure elastic case. It was found that as time elapses the stress distribution drastically changes when plasticity is taken into account, and finally as the time approaches infinity the stress distribution becomes constant and these stresses are referred to residual stresses. The temperature-dependent mechanical properties were an important factor in obtaining the stress distribution especially the yield strength. Compressibility of the material did not have very much effect on the result, hence using a Poisson's ratio of $\frac{1}{2}$ might be advisable since it reduces the mathematical complexity significantly. From the plotted graphs it is evident that the tangential stresses are higher in magnitude than radial stresses and it was revealed that high residual thermal stresses are near the cylinder surface.

The length of the execution time for a typical complete run depended on the value of delta which is assuming for the plastic strain contribution as given in Equation (6-11), initially introduced into the program. The average execution time in this investigation was sixty seconds.

Recommendation

Further analysis and study might be made in this problem by broadening the temperature distribution for a media of small convective heat transfer. The accuracy of solution would be increased by using a relation for the slope at plastic range $H'$, rather than using a constant value. Also in temperature dependent properties consideration, thermal
properties such as convective heat transfer coefficient, $h$, and thermal conductivity, $k$, have to be expressed as a function of temperature. In the numerical procedure the accuracy of solution would be increased by using additional time increments and smaller radial increments in the program.
REFERENCES


APPENDIX A

TRANSIENT TEMPERATURE DISTRIBUTION FLOW CHART

START

READ $N = 1, 30$

$\mu_n, \mu_n^2, J_1(\mu_n), J_0(\mu_n\rho), J_1(\mu_n\rho), \rho$

$I = 1$

IF $N = 30$ THEN $N = N + 1$, ELSE $J = 1$

$T = 0.0$

$N = 1$

$\text{TEM} = e^{-\mu_n^2 \tau J_0(\mu_n\rho)} \frac{1}{\mu_n J_1(\mu_n)}$

$\text{TEM} = 2 \times \text{TEM}$

WRITE $\rho, r, \text{TEM}$

IF $I = 10$ THEN $I = 11$

YES

STOP
APPENDIX B

FLOW CHART FOR THERMOELASTOPLASTIC STRESS

START

READ

\[ \mu_n, J_1(\mu_n), J_0(\mu_n\rho), J_1(\mu_n\rho), \text{DEL, } T_0, \nu, GG, r, \rho, H^*, L \]

CALL TEMP

1

2

I = 1

J = 1

CALCULATE, \( S_y, E, \alpha \)

\[
S_r = \frac{2\alpha E T_0}{S_y(1-\nu)} \sum_{n=1}^{30} \frac{-\mu_n^2 \tau}{\mu_n^2 J_1(\mu_n)} \left[ J_1(\mu_n) - \frac{1}{\rho} J_1(\mu_n\rho) \right]
\]

\[
S_\theta = \frac{2\alpha E T_0}{S_y(1-\nu)} \sum_{n=1}^{30} \frac{-\mu_n^2 \tau}{\mu_n^2 J_1(\mu_n)} \left[ J_1(\mu_n) + \frac{1}{\rho} J_1(\mu_n\rho) - \mu_n J_0(\mu_n\rho) \right]
\]

\[
S_z = \frac{2\alpha E T_0}{S_y(1-\nu)} \sum_{n=1}^{30} \frac{-\mu_n^2 \tau}{\mu_n^2 J_1(\mu_n)} \left[ 2J_1(\mu_n) - \mu_n J_0(\mu_n\rho) \right]
\]
\[ e_r = \alpha \text{To}\{2 \sum_{n=1}^{30} \frac{J_1(\mu_n \rho)}{(1 - \nu)\mu_n^2} \left[ J_1(\mu_n \rho) \left( \frac{1 + \nu}{\rho J_1(\mu_n)} + 4(1 - 2\nu) \right) + 2\nu \mu_n \frac{J_0(\mu_n \rho)}{J_1(\mu_n)} - (1 - \nu) \right] \} \]

\[ e_\theta = \alpha \text{To}\{2 \sum_{n=1}^{30} \frac{J_1(\mu_n \rho)}{(1 - \nu)\mu_n^2} \left[ J_1(\mu_n \rho) \left( \frac{1 + \nu}{\rho J_1(\mu_n)} + 4(1 - 2\nu) \right) \right. \]
\[ - \frac{(1 - \nu)\mu_n J_0(\mu_n)}{J_1(\mu_n)} - (1 - \nu) \} \]
\[ M = I - 1 \]

\[ \Delta = r(I) - r(M) \]

\[ \dot{e}_r = \frac{[e_r(I) - e_r(M)]}{\Delta} \]

\[ \dot{e}_\theta = \frac{[e_\theta(I) - e_\theta(M)]}{\Delta} \]

\[ I \leq 10 \]

\[ I = 2 \]

\[ J = J + 1 \]

\[ J \leq 11 \]

\[ I = 2 \]

\[ \text{DELL} = \text{DELL} \]

\[ \text{ITER} = 1 \]

\[ \text{MM} = 1 \]

\[ J = 1 \to 11 \]
\[ S_{rd} = \frac{1}{3} (2S_r - S_\theta - S_z) \]
\[ S_{\theta d} = \frac{1}{3} (2S_\theta - S_r - S_\theta) \]
\[ S_{zd} = \frac{1}{3} (2S_z - S_r - S_\theta) \]

\[ S_e = \sqrt{\frac{1}{2} \left( (S_{rd} - S_{\theta d})^2 + (S_{\theta d} - S_{zd})^2 + (S_{zd} - S_{rd})^2 \right)} \]

\[ S_e < 1 \] YES \quad G = 0.0

\[ S_e \geq 1 \] NO \quad G = 1.0

\[ P = \frac{3S_y}{4H} \]
\[ C_1 = \frac{S_y}{E} + \frac{PS_{rd}G}{S_e^2} (2S_{rd} - S_{\theta d} - S_{zd}) \]
\[ C_2 = \frac{vS_y}{E} + \frac{PS_{rd}G}{S_e^2} (2S_{\theta d} - S_{rd} - S_{zd}) \]
\[ C_3 = \frac{vS_y}{E} + \frac{PS_{rd}G}{S_e^2} (2S_z - S_{rd} - S_{\theta d}) \]
\[ C_4 = \frac{vS_y}{E} + \frac{PS_{\theta d}G}{S_e^2} (2S_{\theta d} - S_{rd} - S_{zd}) \]
\[ C_5 = \frac{vS_y}{E} + \frac{PS_{\theta d}G}{S_e^2} (2S_{zd} - S_{rd} - S_{\theta d}) \]
\[ C_6 = \frac{vS_y}{E} + \frac{PS_{\theta d}G}{S_e^2} (2S_{zd} - S_{rd} - S_{\theta d}) \]
\[
\frac{\partial \Phi}{\partial \rho} = \frac{1}{c_2 c_6 - c_3 c_5} \left[ (c_4 c_3 - c_1 c_6) \frac{\partial \Phi}{\partial \rho} + c_6 \dot{\Phi}_r - c_3 \dot{\Phi}_\theta \right]
\]

\[
\dot{\Phi}_r = \frac{\Phi}{\rho}, \quad \dot{\Phi}_\theta = \frac{\partial \Phi}{\partial \rho}
\]

\[
\dot{\Phi}_z = \frac{1}{c_3} (\dot{\Phi}_r - c_1 \dot{\Phi}_r - s_2 \dot{\Phi}_\theta)
\]

\[
A = \int_{0}^{\rho} \dot{\Phi}_2 \rho d\rho
\]

\[
B = \frac{\partial \Phi_\theta}{\partial \rho} + \frac{1}{\rho} (\dot{\Phi}_\theta - \dot{\Phi}_r)
\]

WRITE

A, B, DEL, ITERATION, \( \rho \)

A&B \leq 10^{-6} \quad \text{NO} \quad 6

YES

\[
\begin{align*}
\dot{S}_r &= \dot{\Phi}_r \Delta r \\
\dot{S}_\theta &= \dot{\Phi}_\theta \Delta r \\
\dot{S}_z &= \dot{\Phi}_z \Delta r
\end{align*}
\]

ITER = 1

J = J + 1

DEL = DEL
VITA

Habib Pour-Mohamadian was born on September 2, 1946 in Kashan, Iran. He received his Baccalaureate of Science degree in Mechanical Engineering in August 1976 from the University of Texas at Austin. In January 1977 he enrolled in the graduate school of Louisiana State University and on August 1978 finished the requirements for the Master of Science in Mechanical Engineering. Since then, he has been working toward his Doctor of Philosophy degree in Mechanical Engineering where he is now a candidate.
EXAMINATION AND THESIS REPORT

Candidate: Habib Pour-Mohamadian

Major Field: Mechanical Engineering

Title of Thesis: Transient Inelastic Thermal Stresses in a Solid Cylinder with Temperature Dependent Properties

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Date of Examination:

April 30, 1982