2003

Book embeddings of graphs

Robin Leigh Blankenship
Louisiana State University and Agricultural and Mechanical College, rblank1@lsu.edu

BOOK EMBEDDINGS OF GRAPHS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Arts and Science College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
Robin L. Blankenship
B.S. in Math., East Tennessee State University, 1992
M.A. in Math., University of North Carolina, Wilmington, 1994
M.S. in Math., Louisiana State University, 1997
August, 2003

Acknowledgments

This dissertation would not be possible without several contributions. It is a pleasure to thank Dr. Oporowski for his patient attention and assistance. He always paid close attention to my interests and he was a model advisor for my dissertation work. His knowledge and expertise were invaluable in the course of my studies, and I feel very lucky to have worked with him. It is also a pleasure to thank Dr. Oxley for providing me with both monetary support and friendly advice. A special thanks to everyone in the LSU Math Department for providing me with a positive study environment. Thanks so much to Dr. Neubrander for joining my committee at the last possible moment and for watching my progress throughout my stay here. The list of professors who have befriended me and encouraged me is so long, and I want everyone to know that I sincerely appreciate all of their effort and attention. Thanks also to the staff of the LSU Math Department for their smiles and continued assistance.

This dissertation is dedicated to my family and friends for their support and encouragement, especially my mother and father, who have always believed in me and my dreams. To Robert, for being an inspiration, giving me constant reassurance and especially for helping with the graphics. To my brother Jarrett, you're next!

Table of Contents

Abstract

We use a structural theorem of Robertson and Seymour to show that for every minor-closed class of graphs, other than the class of all graphs, there is a number k such that every member of the class can be embedded in a book with k pages. Book embeddings of graphs with relation to surfaces, apex vertices, clique-sums and r-rings are combined into a single book embedding of a graph in the minor-closed class.

The effects of subdividing a complete graph and a complete bipartite graph with respect to book thickness are studied. We prove that if $n \geq 3$, then the book thickness of K_n is $\lceil \frac{n}{2} \rceil$. We also prove that for each m and B, there exists an integer N such that for all $n \geq N$, the book thickness of the graph obtained from subdividing each edge of K_n exactly m times has book thickness at least B. Additionally, there are corresponding theorems for complete bipartite graphs.

1. Introduction

A graph $G = (V, E)$ is a pair consisting of a set V of vertices, and a set E of edges, where each edge $e = (u, v)$ is adjacent to exactly two vertices u and v. The graphs we consider in this dissertation are commonly known in the literature as simple, undirected graphs. Alternatively, a graph can be defined as a topological space where each vertex is a point, each edge is homeomorphic to an open interval and where a vertex is incident to an edge if it is in the closure of the edge. Many interesting questions in graph theory are due to the fact that graphs can be viewed as both a combinatorial construct and a topological space.

A *book* consists of a set of *pages* (half-planes) whose boundaries are glued together on a *spine* (line). It is natural to ask which graphs can be embedded in which books.

When a graph is embedded in a surface, topologically speaking, there is an injective continuous function between the graph and the surface. In this context, the subject of book embeddings is uninteresting, because every graph can be embedded in 3 pages (see Theorem 6.31 in Chapter 6 for more details.) The questions become much more interesting when embedding is considered in a more restrictive sense. Embed the vertices of G in the spine of the book, and then place the edges in the pages so that (1) every edge lies in exactly one page, and (2) no two edges cross in a given page. Condition (2) is the classic view of embedding a graph in the topological sense. Condition (1) is the restriction on the embedding that makes the question of book embeddings interesting. For the remainder of this dissertation, the *book embedding* will be understood in this more restrictive sense.

The fewest number of pages needed to embed a graph on a book is called the *book thickness* of the graph. A *complete graph* K_n is a graph on n vertices such

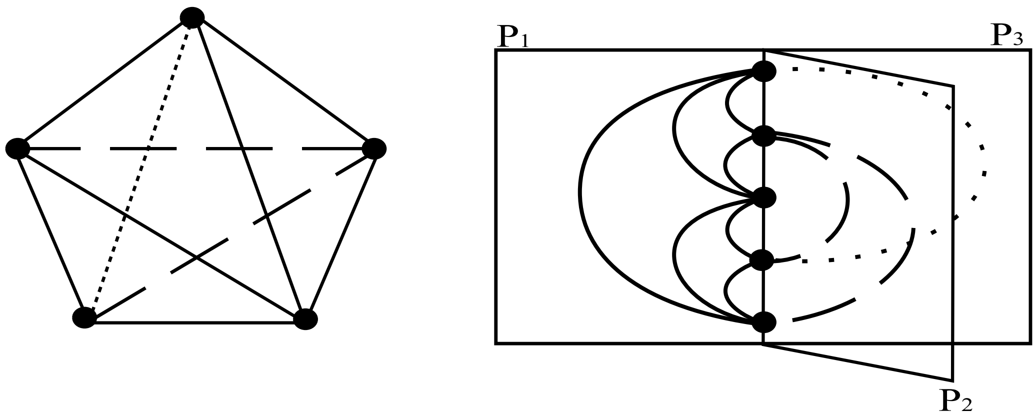

FIGURE 1.1. Embedding K_5 in a three-page book.

that all possible edges between two vertices exists in the graph. Note that K_n is commonly referred to as a *clique* on n vertices. A *subclique* is a clique that is a subgraph of a clique. In Figure 1.1, an embedding of the complete graph K_5 in a book with three pages is given.

Just like graphs, books can be considered combinatorially. Next, notation is developed concerning the book thickness of a graph in the combinatorial context.

Definition 1. *If G is a graph and $\sigma : V(G) \to \mathbb{R}$ is an injection, then (G, σ) is an* ordered graph *and σ is the* ordering function.

Although the edges of G are not directed, we will adopt the convention that if (u, v) is an edge of (G, σ), then $\sigma(u) < \sigma(v)$.

Definition 2. *If (G, σ) is an ordered graph and $\{(u, v), (u', v')\} \subseteq E(G)$, then we say (u, v) and (u', v') are* locked *when $\sigma(u) < \sigma(u') < \sigma(v) < \sigma(v')$ or $\sigma(u') < \sigma(u) < \sigma(v') < \sigma(v)$. If K and K' are subcliques of G, and there are two locked edges $(u, v) \in E(K)$ and $(u', v') \in E(K')$, then we say K and K' are* locked. *When two edges or two cliques are not locked, we say they are* nested.

In Figure 1.2, locked edges are depicted on the left, and nested edges are depicted on the right.

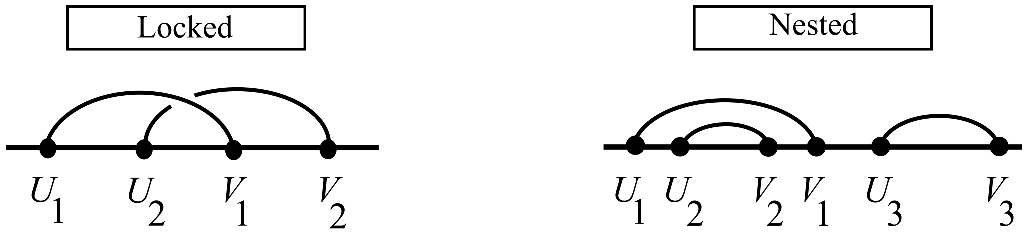

FIGURE 1.2. Defining nested and locked edges.

Definition 3. *Let* (G, σ) *be an ordered graph and let* $\mathbb{P}$ *be a set of pages. If there is a page assignment* $\pi : E(G) \to \mathbb{P}$ *so that* $\pi(e) \neq \pi(f)$ *whenever* e *and* f *lock, we say that* (G, σ, π) *is an* embedded ordered graph.

Compare this definition with the description given earlier. The page assignment π embeds an edge in exactly one page and no two edges cross in a given page.

Definition 4. *The* thickness *of* (G, σ, π) *is the cardinality of the range of* π*.*

Definition 5. *The* thickness *of* (G, σ) *is the smallest thickness of an embedded ordered graph* (G, σ, π) *where the minimum is taken over all possible page assignments* π*.*

Definition 6. *The* book thickness *of* G*, denoted* $BT(G)$*, is the smallest thickness of an embedded ordered graph* (G, σ, π) *where the minimum is taken over all possible page assignments* π *and ordering functions* σ*.*

A *surface* is a compact 2-manifold without boundary. An *open* 2*-cell embedding* of a graph in a surface is one in which every face is homeomorphic to an open disk. Surfaces play a particularly important role in the study of graphs. There are many theorems concerning the embedding of graphs on surfaces. Also, note that theorems about embedding in surfaces do not immediately apply to embeddings in books, because the neighborhood of a point in the spine of a book is not locally homeomorphic to an open disk.

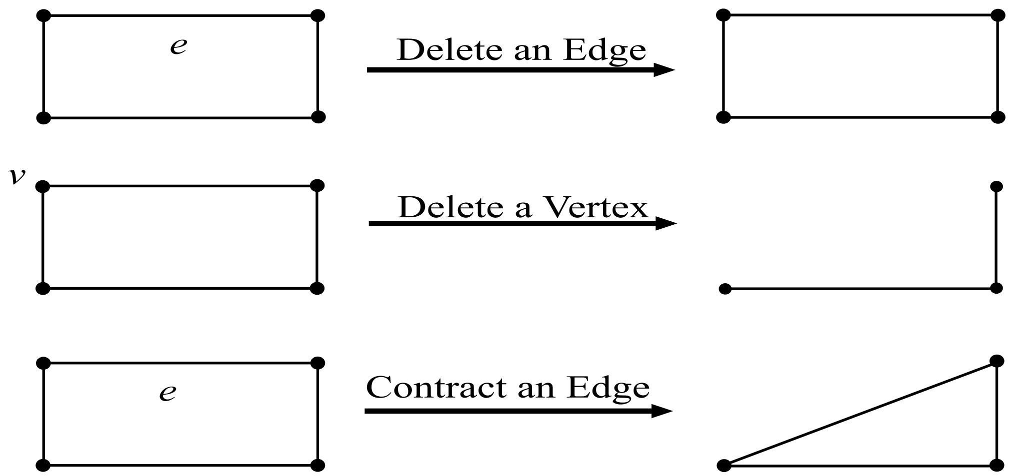

FIGURE 1.3. Defining the operations for taking a minor of a graph.

Heath and Istrail proved that for any fixed surface there is a function depending only on the genus of the surface that provides an upper bound on the book thickness of any graph embedded in that surface [6]. It will be used extensively in this dissertation.

Theorem 1.1. *There is a function ζ such that if G is a graph embedded in a surface of genus g, then the book thickness of G is at most $\zeta(g)$. Moreover, $\zeta(g)$ is $O(g)$.*

We generalize this result to larger classes of graphs in the course of this dissertation.

Definition 7. *A graph H is a minor of a graph G if it is obtained from G by a sequence operations, each of which is an edge contraction, an edge deletion, or a vertex deletion.*

In Figure 1.3, the operations of taking minors are demonstrated on a cycle of length 4.

Definition 8. *A class of graphs is minor-closed if every minor of every member of the class is also in the class.*

Not all minor-closed classes of graphs have a description which arises from a surface. An example is the class of all graphs which have no minor isomorphic to the well-known Petersen graph. The subject of book embeddings is quite new and differs from the previously studied surface embeddings because, in addition to the fact that books are not compact manifolds, also the class of graphs embedable in a book with B pages is not a minor-closed class.

To see this, consider that for each n, there is a subdivision of a clique K_n which has book thickness at most 3. Refer to Chapter 6 for details about the effects of subdividing a graph on the book thickness of the graph. Additionally, if $n \geq 4$, we prove in Theorem 6.32 that $\lceil \frac{n}{2} \rceil$ is a lower bound on the book thickness of K_n. Since G is a minor of any subdivision of G, the class of all graphs which can be embedded in a book with B pages is not a minor-closed class.

The study of book embeddings of graphs seems to originate around 1971 with Evan and Itai's paper [4], which emphasizes the applicable nature of these embeddings. It is natural to question which graphs can be embedded in a book with B pages. Various mathematicians have studied the properties of book embeddings, yet not much progress has been made towards a characterization of all graphs embedable in a book with B pages. It is easy to characterize which graphs are embedable in a book with one or two pages. An *outerplanar* graph is a planar graph that can be drawn so that all its vertices lie in the boundary of the infinite face. A graph has book thickness one if and only if it is outerplanar. In Figure 1.4, an outerplanar graph is depicted on the left, and an embedding of the graph is depicted on the right. One of the edges has an X on it. Think of cutting the outerplanar graph on the X and opening it up so that the vertices in the boundary of the infinite face lie in the spine of the book in exactly the same order. A graph has book thickness two if and only if it is a subgraph of a planar Hamiltonian graph.

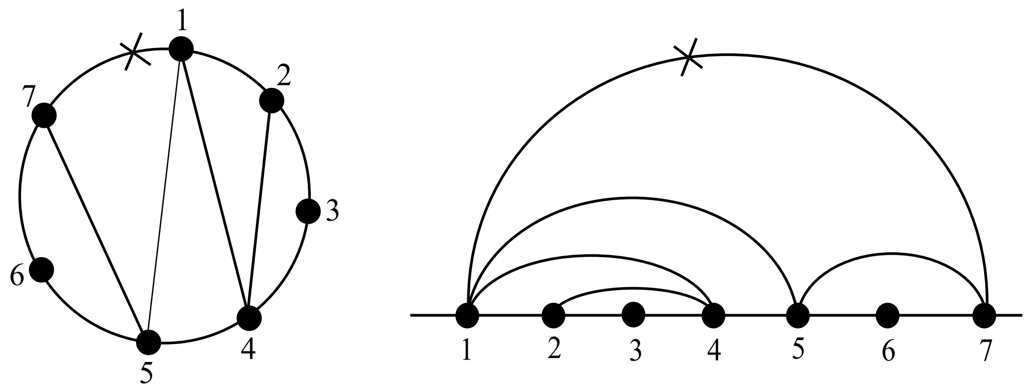

FIGURE 1.4. Embedding an outerplanar graph in a book.

For any given number $n \geq 3$, it is very difficult to characterize which graphs have book thickness at most n. Some progress has been made in this direction. A planar graph has book thickness less than or equal to four [12]. Yannakakis' constructive proof yields an algorithm to embed any planar graph in four pages. It is interesting to note that it is quite difficult to construct an example of a planar graph which actually requires four pages for its embedding. If a graph G is embedable in a torus, then G has book thickness less than or equal to seven [4]. As stated in Theorem 1.1, it has been shown that the book thickness of a graph can be bounded from above by a number depending only on the genus of the minimum surface in which it can be embedded [6].

The next theorem is the main result of this dissertation, and the majority of this dissertation is spent developing notation and proving this result.

Theorem 1.2. *For every minor-closed class of graphs, other than the class of all graphs, there is a number k such that every member of the class can be embedded in a book with k pages.*

The one chapter which does not contribute to the proof of this theorem is Chapter 6. In Chapter 6, we study the effects of subdividing a complete graph and a complete bipartite graph with respect to book thickness. As mentioned earlier, we prove in Theorem 6.32 that if $n \geq 3$, then the book thickness of K_n is $\lceil \frac{n}{2} \rceil$.

In Theorem 6.37 we prove that for each m and B, there exists an integer N, such that for all $n \geq N$, the book thickness of the graph obtained from subdividing each edge of K_n exactly m times has book thickness at least B. Even though it is a corollary to Theorem 6.37, the proof of the case where $m = 1$ is both efficient and elegant, and it is given in Proposition 6.35. The proof of Theorem 6.37 is much more difficult and significantly longer. There are corresponding theorems for complete bipartite graphs.

2. Tree Width and Book Embeddings

Consider a graph G which embeds in a surface with the exception of a bounded number of disks inside of which is an area of local non-planarity. These disks are called *r-rounds* and are defined later in Definition 11. Providing a book embedding of the graph G which is compatible with conditions favorable to the inclusion of the r-rounds requires a significant amount of detail.

Definition 9. *Given a graph G, a T-*decomposition* of G is a pair (T, X), where T is a graph, and $X = \{X_t\}_{v \in V(T)}$ is a collection of subsets of $V(G)$, called* bags *such that the following are satisfied:*

1. $\bigcup_{t \in V(T)} X_t = V(G)$;
2. *For every edge (x, y) of G, there is a $t \in V(T)$ such that $\{x, y\} \subseteq X_t$; and*
3. *For every vertex $x \in V(G)$, the subgraph of T induced by $\{t \in V(T) : x \in X_t\}$ is connected.*

The width *of (T, X) is $\max\{|X_t| - 1 : X_t \in X\}$. If T is a tree, then (T, X) is a* tree-decomposition. *The* tree-width *of a graph G, denoted* $\mathrm{tw}(G)$, *is the smallest integer w such that G has a tree-decomposition of width w. A graph is a* k-tree *if it has tree-width at most k. A graph is a* partial k-tree *if it is a subgraph of a k-tree.*

Definition 10. *Let P_n be the path on vertices (in order) $t_1, t_2, \ldots, t_n$. Given a positive integer r, an* r-ring with perimeter $(t_1, t_2, \ldots, t_n)$ *is a graph R on the vertex set $\{t_1, t_2, \ldots, t_n\}$ such that there is a collection of* bags $X = \{X_t\}_{t \in V(T)}$ *for which:*

1. *(P_n, X) is a P_n-decomposition of R of width at most $r - 1$,*

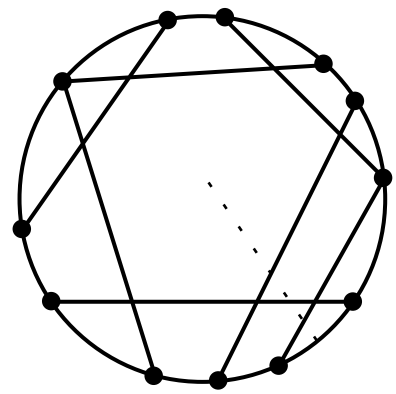

FIGURE 2.5. The dotted line drawn from the center to the boundary of this 2-ring crosses at most 3 edges.

2. $t_i \in X_{t_i}$ *for each* $1 \leq i \leq n$.

Definition 11. *Let* C_n *be the circuit on vertices (in cyclic order)* $t_1, t_2, \ldots, t_n$. *Given a positive integer* r*, an* r-round with perimeter $(t_1, t_2, \ldots, t_n)$ *is a graph* R *on the vertex set* $\{t_1, t_2, \ldots, t_n\}$ *such that there is a collection of* bags $X = \{X_t\}_{t \in V(T)}$ *for which:*

1. (C_n, X) *is a* C_n*-decomposition of* R *of width at most* $r - 1$*,*

2. $t_i \in X_{t_i}$ *for each* $1 \leq i \leq n$.

Notice that if $X = \{X_t\}_{t \in V(T)}$ is the collection of bags of an r-ring with perimeter $(t_1, t_2, \ldots, t_n)$, then X is also the collection of bags of an r-round with the same perimeter. If an r-round R has perimeter $(t_1, t_2, \ldots, t_n)$, then the *boundary edges* of R, denoted E^b, are the edges $(t_{i,i+1})$ for $1 \leq i \leq n$ where index arithmetic is performed modulo n. The edges of the r-round which are not in its boundary are called the *interior edges*.

An easier way to see the bound on the complexity of an r-round is demonstrated in 2.5. If a line is drawn from the center to the boundary of an r-ring, it will cross at most $r + 1$ edges.

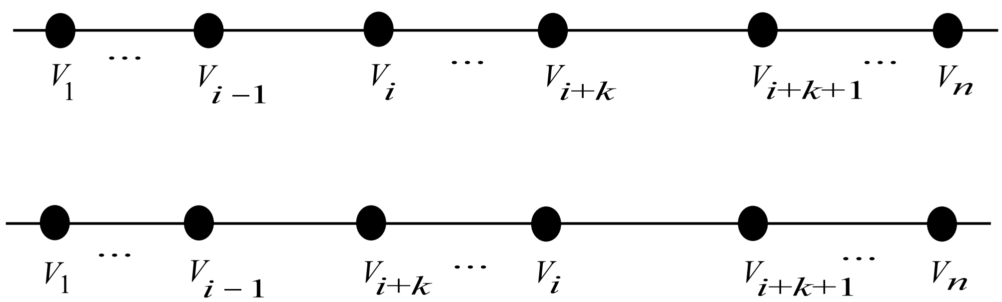

FIGURE 2.6. Reversing an interval.

Lemma 2.3. *Let $\overline{v} = (v_1, v_2, \ldots, v_n)$ and let $\overline{w}$ be the sequence obtained from $\overline{v}$ by reversing the segment $v_i, v_{i+1}, \ldots, v_{i+k},$, that is, let*

$$\overline{w} = (v_1, \ldots v_{i-1}, v_{i+k}, v_{i+k-1}, \ldots, v_{i+1}, v_i, v_{i+k+1}, \ldots, v_n).$$

Then an r-round with perimeter $\overline{v}$ is a $3r$-round with perimeter $\overline{w}$.

Proof. Suppose G is an r-round with perimeter $\overline{v}$, let P denote the path on the elements of $\overline{v}$ in the order listed, and let X be a set $\{X_{v_t}\}_{t=1}^{n}$ of bags such that (P, X) is a P-decomposition of G of width at most $r-1$ and $v_t \in X_{v_t}$ for each t. Let $(v'_1, v'_2, \ldots, v'_n) = \overline{w}$, and let P' be the path on the elements of $\overline{w}$ in the order listed. For each t in $\{1, 2, \ldots, n\}$, let $X'_{v'_t} = X_{v_t} \cup X_{v_i} \cup X_{v_{i+k}}$, and let $X' = \{X'_{v'_t}\}_{t=1}^{n}$. Since (P, X) is a P-decomposition of G, it is clear that (P', X') satisfies (1) and (2) of Definition 9. To see that (P', X') also satisfies (3) of Definition 9, let $v'_{s'}$ be a vertex of P', and let s be the number for which $v_s = v'_{s'}$. Let P_{v_s} be the subgraph of P induced by the vertices of v_t for which X_{v_t} contains v_s, and, similarly, let $P'_{v'_{s'}}$ be the subgraph of P' induced by the vertices v'_t for which $X'_{v'_t}$ contains $v'_{s'}$. Since (P, X) is a P-decomposition of G, the graph P_{v_s} is connected. If P_{v_s} is contained in one of $P[v_1, v_{i-1}]$, $P[v_i, v_{i+k}]$, or $P[v_{i+k+1}, v_n]$, then $P'_{v'_{s'}}$ equals P_{v_s}, and hence is connected. Otherwise, $P'_{v'_{s'}}$ can be expressed as the union of two overlapping subpaths P_{v_s} and $P'[v'_i, v'_{i+k}]$, and hence is connected as well. We conclude that (P', X') satisfies (3) of Definition 9, and therefore is a P'-decomposition of G.

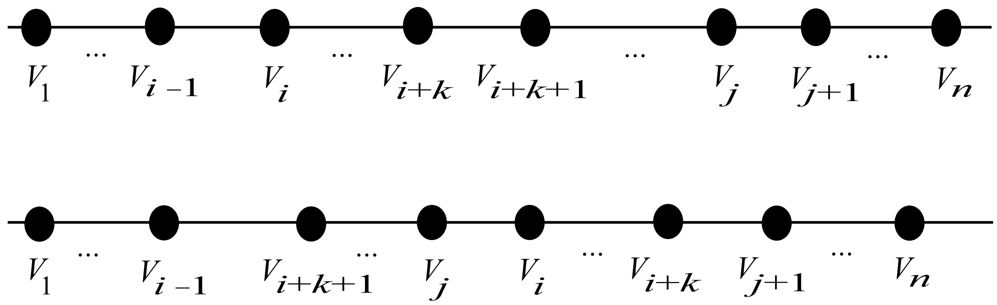

FIGURE 2.7. Moving an interval forward.

Moreover, $|X'_{v'_t}| \leq |X_{v_t}| + |X_{v_i}| + |X_{v_{i+k}}| \leq 3r$ for each t in $\{1, 2, \ldots, n\}$, and so the width of (P', X') is at most $3r - 1$. Thus, property (1) of Definition 11 is satisfied. Property (2) of Definition 11 is also satisfied because $v_t \in X_{v_t} \subseteq X'_{v'_t}$ for each t in $\{1, 2, \ldots, n\}$. □

Lemma 2.4. *Let $\overline{v} = (v_1, v_2, \ldots, v_n)$ and let $\overline{w}$ be the sequence obtained from $\overline{v}$ by moving the segment $v_i, v_{i+1}, \ldots, v_{i+k}$ forward $j - i$ places, that is, let*

$$\overline{w} = (v_1, \ldots, v_{i-1}, v_{i+k+1}, \ldots, v_j, v_i, \ldots, v_{i+k}, v_{j+1}, \ldots, v_n)$$

for some $i + k \leq j \leq n - 1$. Then an r-round with perimeter $\overline{v}$ is a $4r$-round with perimeter $\overline{w}$.

Proof. Suppose G is an r-round with perimeter $\overline{v}$, let P denote the path on the elements of $\overline{v}$ in the order listed, and let X be a set $\{X_{v_t}\}_{t=1}^n$ of bags such that (P, X) is a P-decomposition of G of width at most $r - 1$ and $v_t \in X_{v_t}$ for each t. Let $(v'_1, v'_2, \ldots, v'_n) = \overline{w}$, and let P' be the path on the elements of $\overline{w}$ in the order listed. For each t in $\{1, 2, \ldots, n\}$, let $X'_{v'_t} = X_{v_t} \cup X_{v_i} \cup X_{v_{i+k}} \cup X_{v_j}$, and let $X' = \{X'_{v'_t}\}_{t=1}^n$.

Since (P, X) is a P-decomposition of G, it is clear that (P', X') satisfies (1) and (2) of Definition 9. To see that (P', X') also satisfies (3) of Definition 9, let $v'_{s'}$ be a vertex of P', and let s be the number for which $v_s = v'_{s'}$. Let P_{v_s} be the subgraph of P induced by the vertices of v_t for which X_{v_t} contains v_s, and, similarly, let $P'_{v'_{s'}}$ be the subgraph of P' induced by the vertices v'_t for which $X'_{v'_t}$ contains $v'_{s'}$. Since (P, X) is a P-decomposition of G, the graph P_{v_s} is connected. If P_{v_s} is contained in one of $P[v_1, v_{i-1}]$, $P[v_i, v_{i+k}]$, $P[v_{i+k+1}, v_j]$, or $P[v_{j+1}, v_n]$, then $P'_{v'_{s'}}$ equals P_{v_s}, and hence is connected. Otherwise, there are two cases. First, $P'_{v'_{s'}}$ could be expressed as the union of two overlapping subpaths P_{v_s} and $P'[v'_i, v'_{i+k}]$, and hence is connected as well. Second, $P'_{v'_{s'}}$ could be expressed as the union of two overlapping subpaths P_{v_s} and $P'[v'_{i+k+1}, v'_j]$, and hence is connected. We conclude that (P', X') satisfies (3) of Definition 9, and therefore is a P'-decomposition of G. Moreover, $|X'_{v'_t}| \leq |X_{v_t}| + |X_{v_i}| + |X_{v_{i+k}}| + |X_{v_j}| \leq 4r$ for each t in $\{1, 2, \ldots, n\}$, and so the width of (P', X') is at most $4r - 1$. Thus, property (1) of Definition 11 is satisfied. Property (2) of Definition 11 is also satisfied because $v_t \in X_{v_t} \subseteq X'_{v'_t}$ for each t in $\{1, 2, \ldots, n\}$. □

Lemma 2.5. *Let $\overline{v} = (v_1, v_2, \ldots, v_n)$ and let $\overline{w}$ be the sequence obtained from $\overline{v}$ by moving the segment $v_i, v_{i+1}, \ldots, v_{i+k}$, backward $i - j$ places, that is, let*

$$\overline{w} = (v_1, \ldots, v_j, v_i, \ldots, v_{i+k}, v_{j+1}, \ldots, v_{i-1}, v_{i+k+1}, \ldots, v_n)$$

for some $1 \leq j \leq i - 2$. Then an r-round with perimeter $\overline{v}$ is a $4r$-round with perimeter $\overline{w_B}$.

We omit the proof of Lemma 2.5, since it is very similar to the proof of Lemma 2.4.

An n-tree can be decomposed as a sequence of graphs where $G_0 = K_{n+1}$ and G_k is formed from G_{k-1} by connecting a vertex to a clique of order n in G_{k-1}.

Define *layer* L_0 to be the vertices of an initial K_n on which the n-tree G is built. Let layer L_1 be the set of vertices which have their n neighbors in L_0. Let layer L_2 consist of those vertices which have one neighbor in L_1, and the other $n - 1$

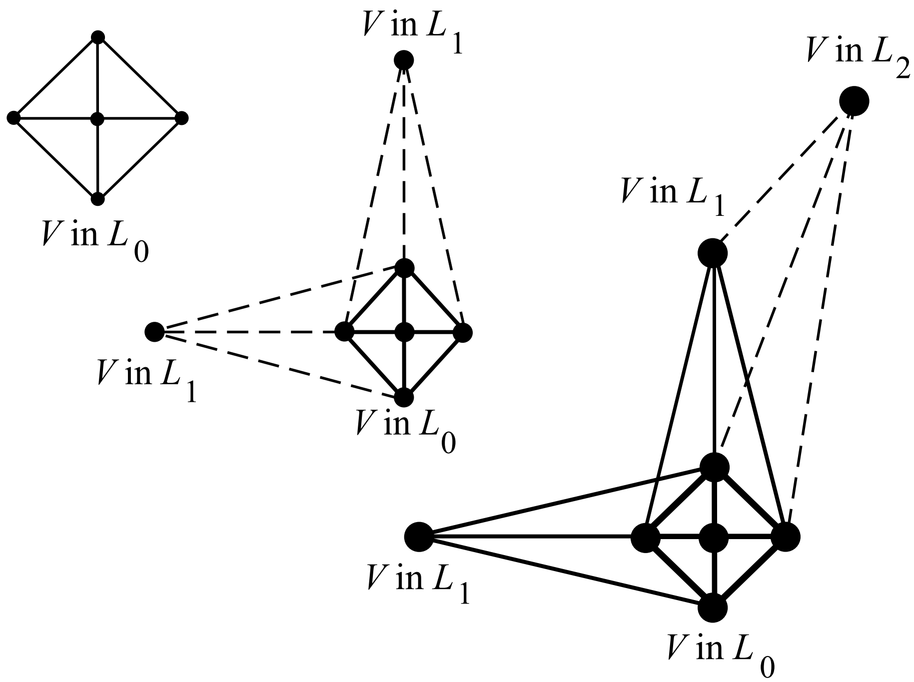

FIGURE 2.8. A 3-tree.

neighbors in L_0. Let layer L_3 consist of those vertices which have one neighbor in L_1, one neighbor in L_2, and the other $n-2$ neighbors in L_0. If $k < n$, let layer L_k consist of those vertices which have one neighbor in each of L_{k-1}, L_{k-2}, ..., L_1, and the other $n-k$ vertices in L_0. If $k \geq n$, let layer L_k consist of those vertices which have one neighbor in each of L_{k-1}, L_{k-2}, ..., L_{k-n}.

Note that the index of a layer is equal to the length of a longest path from a vertex in that layer to a vertex in L_0 that travels through consecutively through layers of smaller index. By construction of an n-tree, if $w_{k-1}, \ldots, w_{k-n}$ are neighbors of a vertex v, then they lie in layers $L_{k-1}, \ldots, L_{k-n}$ and must induce a K_n. Two vertices in a layer L_i are not connected by an edge if $i \geq 1$.

Define an *ascending edge* through an n-tree as follows. Beginning with a vertex $v_i \in L_i$, proceed along any edge leading to $v_{i+1} \in L_{i+1}$. An *ascending path* is a path of connected ascending edges. A *rooted ascending path* is an ascending path which begins with a vertex $v_0 \in L_0$. Similarly, define a *descending* path.

Define a *depth-first search (DFS)* on an n-tree as follows. Beginning with a vertex $v_0 \in L_0$, investigate any rooted ascending path. An order on the vertices is automatically assigned. Vertices visited earlier are called *older* and vertices visited later are called *younger*. Continue along the rooted ascending path until a vertex is reached with the highest possible index. Define *backtracking* as follows. Descend along the vertices already reached until a vertex v is found in a level with the largest index such that there is an ascending path not yet explored beginning with v. Explore all possible ascending paths via backtracking.

Lemma 2.6. *An n-tree has chromatic number $n+1$.*

Proof. The vertices of L_0 form a clique of order n, so each vertex must receive a distinct color. All vertices in L_1 must receive color $n+1$ because each one is connected to every vertex in layer L_0 which utilize the first n colors. Each time a vertex v_i in layer L_i is connected to a clique of order n, there is one unused color to assign to v_i. □

The next theorem provides information about the book thickness of an ordered graph. It will be used to relate book thickness to tree-width.

Theorem 2.7. *If G is an $(r-1)$-round with perimeter $(t_1, t_2, \ldots, t_n)$ and σ is an ordering function that agrees with the order of vertices in the perimeter of G, then $BT(G, \sigma)$ is at most $r+1$.*

Proof. Let G be an $(r-1)$-round with perimeter $(t_1, t_2, \ldots, t_n)$, and (G, σ) be the ordered graph where the ordering function σ agrees with the order of the vertices on the perimeter. Let $X = \{X_{v_t}\}_{v_t \in V(T)}$ be the collection of bags of an $(r-1)$-

round R with perimeter $(t_1, t_2, \ldots, t_n)$. Then the collection of bags needed for R^b is $X = \{X_{v_t} \cup v_{t+1}\}_{v_t \in V(T)}$. Thus, R^b is an r-round.

If $\text{tw}(G) \leq r$, then G is a partial n-tree. Without loss of generality, we may assume G is an n-tree, which has a vertex coloring using $(r+1)$ colors by Lemma 2.6. Designate the $r+1$ pages of a book by the $r+1$ colors. Since the perimeter of G is a Hamiltonian path through the vertices of G, it forms a simple depth-first search. Order the vertices of G in the spine as prescribed by σ.

We will now embed the edges of G according to the vertex coloring assured by Lemma 2.6. For ease of notation, let $\sigma(v_i) < \sigma(v_j)$ mean $i < j$. Assign an edge (v_i, v_j) to the page denoted by the color of the left endpoint v_i. We need to show this is a book embedding. Assume it is not. Then there is a page on which two edges cross, say (v_1, v_2) and (w_1, w_2). Note this means the color of v_1 is the same as the color w_1. Then w_1 was reached before v_2 in DFS, since we always embed to the right of the most recently embedded vertex. In this case, w_1 will be reached before v_1 when backtracking. Since w_2 has not yet been embedded, by DFS we have $\sigma(w_2) > \sigma(v_1)$. Backtracking again yields $\sigma(v_2) > \sigma(w_2)$. This is a contradiction of the assumption that edges (v_1, v_2) and (w_1, w_2) crossed on a single page. Therefore, $\text{BT}(G, \sigma)$ is at most $r+1$. □

Corollary 2.8. *For any graph G, $BT(G) \leq tw(G) + 2$.*

The inequality in Corollary 2.8 can be reduced to $\text{tw}(G)+1$ by modifying the proof of Theorem 2.7. The depth-first search which achieves this reduction is as follows. Consider the spine of the book as a real number line, so that $v_1 < v_2$ means vertex $\sigma(v_1) < \sigma(v_2)$ on the spine. Consider ordering the vertices of the r-tree as prescribed by a depth-first search. First embed L_0. Since $L_0 = K_r$, by symmetry it does not matter in which order these vertices are placed on the spine, but once

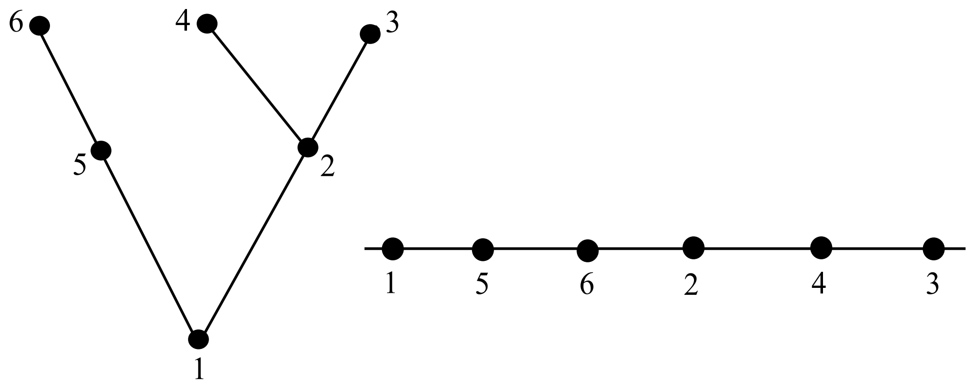

FIGURE 2.9. Depth-First Search

it is embedded, the spine induces an ordering on the vertices, say $\sigma(v_1) < \sigma(v_2) < \ldots < \sigma(v_n)$. Beginning with v_1, embed the vertices of an ascending path emanating from v_1 in order of appearance, so that $\sigma(v_n) < \sigma(v)$ for any v in the path.

Continue to embed vertices of the paths reached via backtracking in order of appearance between the most recently embedded vertex v and the vertex immediately to the right of vertex v, until all such paths are exhausted. Proceed to v_2, and repeat the process. Make an exhaustive search the vertices of L_0 in order of their appearance. Note that backtracking causes a nesting of the paths which emanate from vertices of a single path of $v_i \in L_0$, with the beginning vertex of such paths in layers with successively larger indices.

If an edge leads to a previously embedded vertex, then DFS has already searched and embedded the vertices in any path containing that vertex, so backtracking the moment we hit a previously embedded vertex will not cause us to miss any vertices of G. Also note that there is a path from a vertex $v \in L_0$ to a vertex $w \in L_k$ since w is connected to layers $L_{k-1}, \ldots, L_{k-n}$, so there is a vertex in L_{k-n} which is connected to $L_{k-n-1}, \ldots, L_{k-2n}$, and so on, so that in $\lceil \frac{k}{n} \rceil$ steps we must reach L_0. So if every path is searched, we will have reached every vertex exactly once each, and the embedding of the vertices will be complete.

3. Rounds

3.1 Notation

We develop notation in this section to allow the presentation of Robertson and Seymour's Structure Theorem [10]. Elements of the set V in the next lemma are commonly referred to as *apex vertices*.

Lemma 3.9. *Let G be a graph with book thickness B and V be a subset of $V(G)$. Then $BT(G) \leq BT(G - V) + k$.*

Proof. Take a book embedding of $G-V$ where $V = \{v_i\}_{i=1}^{k}$ and create an additional page $P(v_i)$ for each v_i in V. Embed all of the edges adjacent to v_i in page $P(v_i)$. Hence, $\mathrm{BT}(G) \leq \mathrm{BT}(G - V) + k$. □

A *circuit* C in a surface Σ is a subset of Σ that is homeomorphic to the unit circle. Define $\Sigma \backslash C$ to be the surface, with boundary, formed by cutting Σ along C. Then $\Sigma \backslash C$ has either one or two components. If $\Sigma \backslash C$ has one component, then C is called *nonseparating*. If $\Sigma \backslash C$ has two components, then C is called *separating*. If C is separating and one of the components of $\Sigma \backslash C$ is homeomorphic to an open 2-cell, then C is *trivial*. All circuits which are not trivial are said to be *essential*.

Representativity of an embedding is a measure of how "densely" a graph is embedded in a surface. It was developed by Robertson and Seymour [11]. Assume the surface $\Sigma(\Psi)$ is not a sphere. Then the *representativity* of Ψ is defined to be $\varrho(\Psi) = \min\{|C \cap G(\Psi)| : C \text{ is an essential circuit of } \Sigma(\Psi)\}$. By elementary topology, it is enough to use essential circuits which pass through only vertices and faces to calculate $\varrho(\Psi)$.

Let $\mathcal{C}$ be a minor-closed class of graphs other than the class of all graphs. A graph $H \in \mathcal{C}$ has a decomposition into graphs H_i, see Figure 3.12, where each graph H_i is

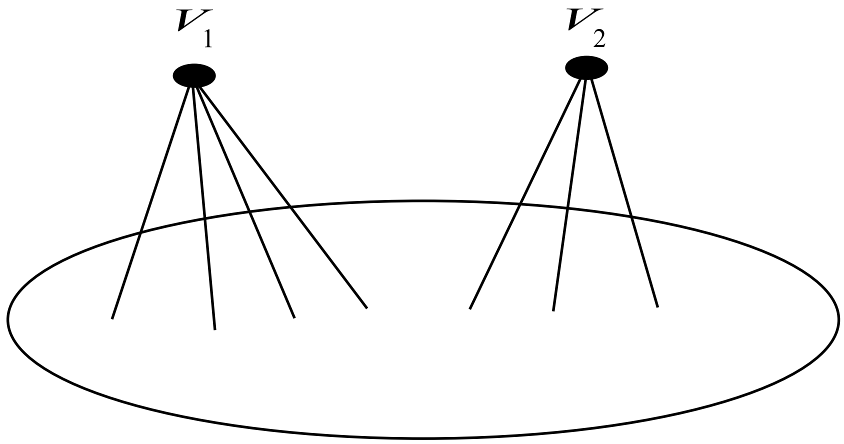

FIGURE 3.10. Apex vertices

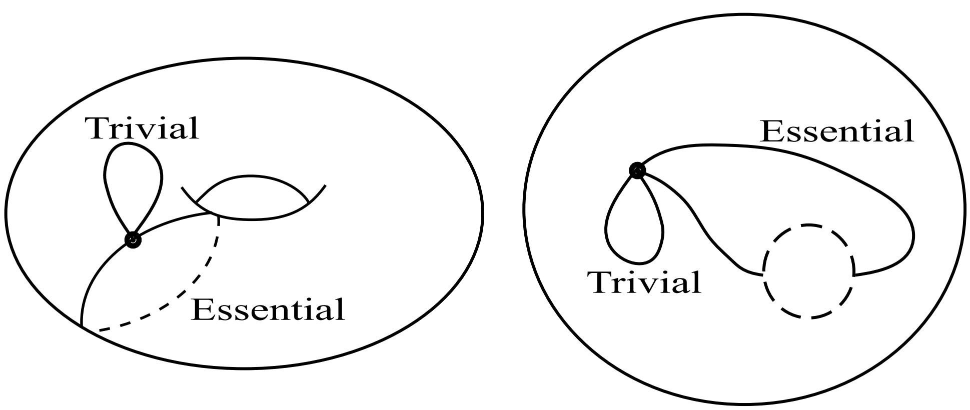

FIGURE 3.11. Trivial and essential circuits.

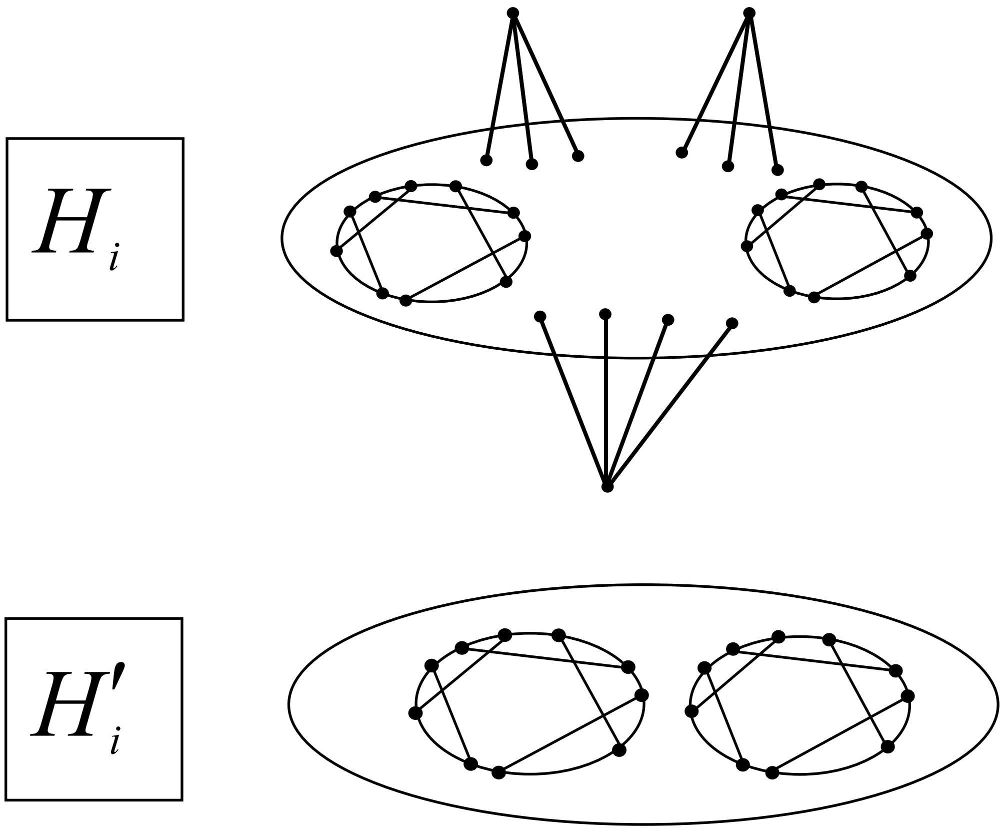

FIGURE 3.12. H_i and H'_i

"almost" embedded in a surface Σ_i of genus g_i. Moreover, H is obtained by clique-summing the graphs H_i together so that they have an underlying tree structure. Let V_i be a set of apex vertices of H_i and let $H'_i = H_i - V_i$. See Figure 3.12. Let R_i be a set of r-rounds of H_i and let $E'(R_i)$ denote the set of all cap edges of each r-round in R_i. Recall the definition of apex vertices and r-rounds given on pages 17 and 9. Denote $H''_i = H'_i - E'(R_i)$.

Now H''_i is embedded in Σ_i. The decomposition of H into the pieces H_i can be chosen so that either Σ_i is a sphere or the embedding of H''_i in Σ_i has high representativity. Let $\varrho(\mathcal{C})$ be a lower bound on the representativity of the embedding of each H''_i in Σ_i.

Let $\mathcal{H}''(g, \varrho)$ denote the sest of graphs that have an open 2-cell embedding on a surface (orientable or non-orientable) of genus at most g that have representativity at least ρ when $g \neq 0$. Denote the sest of graphs $\mathcal{H}'$ containing k subgraphs $R_1, R_2, \ldots, R_k$, where $0 \leq k \leq R$ and each R_i is an r_i-round with $r_i \leq \delta$ such that the deletion of all interior edges of all R_i's results in an element of $\mathcal{H}''(g, \varrho)$. Let $\mathcal{H}(g, \varrho, \delta, R, w)$ denote the set of graphs H such that the deletion of at most w vertices from H results in an element of $\mathcal{H}'(g, \varrho, \delta, R, w)$.

Robertson and Seymour's Structure Theorem [10] provides the framework for the proof of Theorem 1.2.

Theorem 3.10. *If $\mathcal{C}$ is a minor-closed class of graphs, other than the class of all graphs, and ϱ is a non-negative integer, then there are integers g, δ, R and w that depend only on $\mathcal{C}$ and ϱ such that every member of $\mathcal{C}$ can be obtained by repeated clique-summing of elements of $\mathcal{H}(g, \varrho, \delta, R, w)$.*

The features of this theorem are discussed extensively in the following sections. Begin with a graph that is a member of a minor-closed class of graphs. This graph is decomposed into pieces which are then summed together. The pieces are "almost" embedded in a surface. The proof of Theorem 1.2 involves providing an appropriate book embedding through thorough examination of the details in Theorem 3.10.

3.2 Developments of Heath and Istrail

Certain aspects of Heath and Istrail's work [7] need to be described so that they can be used later. The rotational system developed by Gross and Tucker [5] is used by Heath and Istrail to provide a combinatorial description of an embedding.

Definition 12. *A* rotation *at a vertex v is an ordered list, unique up to a cyclic permutation, of the edges incident to v.*

Definition 13. *A* rotation system *on a graph G is an assignment of a rotation to each vertex and a designation of orientation type for each edge.*

Theorem 3.11. *Every rotation system on a graph G defines (up to equivalence of embeddings) a unique locally oriented graph embedding $G \to \Sigma$. Conversely, every locally oriented graph embedding $G \to \Sigma$ defines a rotation system for G.*

Definition 14. *A planar-nonplanar decomposition of a graph $G = (V, E)$ is given by (R, P) where R is a rotation of G representing a surface embedding, $P = (V, E(P))$ is a planar subgraph of G, and $E_N = E(G) - E(P)$ which satisfies these properties*

1. *the subrotation R_P induces a planar embedding of P;*

2. *there exists a face F_0 of the planar embedding such that every edge in $e \in E_N$ is incident to two vertices on the boundary of F_0;*

3. *$E(P)$ is maximal, that is, no edge of E_N can be added to P without violating property (1) or (2).*

Definition 15. *An edge e is* essentially nonplanar *with respect to P if e cannot be embedded in the plane with P without violating Definition 14.*

Note that if $e = (u, v)$ and $e' = (u', v')$ are essentially nonplanar edges, then they necessarily have both endpoints on the boundary of P. Traversal of the boundary of a planar graph defines a directed cycle (which, in general, is not simple).

Definition 16. *A directed subpath of the traversal of the boundary of a planar graph is called a* trace. *If $T = v_1 \to v_2 \to \ldots \to v_t$ is a trace, then the* inverse trace *is $T^{-1} = v_t \to v_{t-1} \to \ldots \to v_1$.*

In general, given a planar-nonplanar decomposition (R, P) of a graph G, the next aim is to partition the essentially nonplanar edges into equivalence classes.

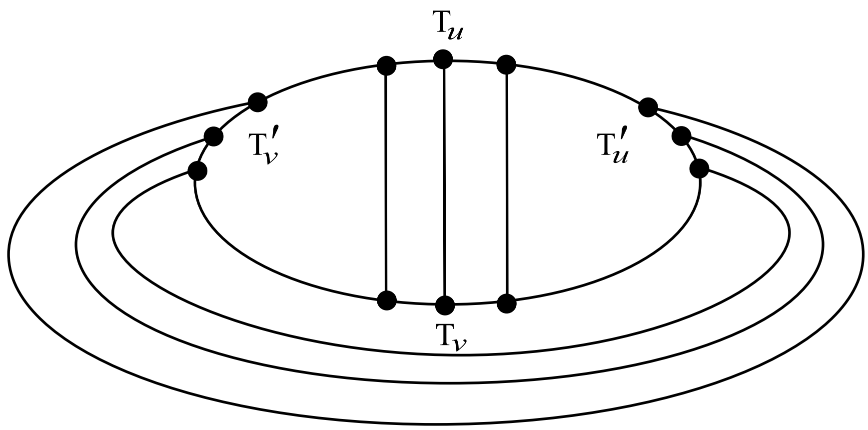

FIGURE 3.13. Homotopy classes and their traces.

Suppose $e = (u, v)$ and $e' = (u', v')$ are essentially nonplanar edges and that they are part of the boundary of the same face F of the embedding of G.

Definition 17. *If $e = (u, v)$ and $e' = (u', v')$ are essentially nonplanar edges, then e and e' are* homotopic *with respect to the boundary of F if*

1. *e and e' are the only edges of E_N on the boundary of F;*

2. *there are disjoint traces $T_u = u \to \ldots \to u'$ and $T_v = v \to \ldots \to v'$ such that T_u and T_v lie on the boundary of F.*

Note that if $e = (u, v)$ and $e' = (u', v')$ are homotopic, then the entire boundary of F consists of edges e and e', and traces T_u and T_v. The notion of homotopy in this paper is related to the notion of homotopy in topology in the sense that if one shrinks the planar part to a point, then two nonplanar edges are homotopic in our sense if and only if they are homotopic in the topological sense. To see this, consider that T_u and T_v lie on the boundary of P. Shrinking the planar part P to a point also shrinks T_u and T_v to a point. Also, e to e' are on the boundary of the face F, which bounds a disk. Then there is a continuous deformation taking e to e' across the disk bounded by F. The homotopy relationship is defined to be the reflexive, symmetric and transitive closure on $E - E(P)$. Each equivalence class is a *homotopy class*. The next lemma translates the transitive aspect of the homotopy relationship into the language of traces.

Lemma 3.12. *If G is a graph embedded in a surface Σ, the planar part of a planar-nonplanar decomposition of G is P and if C is a homotopy class, then the elements of C can be ordered $e_1, e_2, \ldots, e_k$ where e_i has endpoints (u_i, v_i) and two traces T_1 and T_2 where*

1. *for each $1 \leq i \leq k-1$, the edge e_i is homotopic to the edge e_{i+1} with corresponding traces T_{u_i} and T_{v_i};*

2. *T_1 is the concatenation of $T_{u_1}, T_{u_2}, \ldots, T_{u_{k-1}}$ and T_2 is the concatenation of $T_{v_1}, T_{v_2}, \ldots, T_{v_{k-1}}$.*

Proof. Let G be a graph embedded in a surface Σ and P be the planar part of a planar-nonplanar decomposition. Let C be a homotopy class and let the elements of C be ordered as given in Lemma 3.12.

If $1 \leq i \leq k-1$ and $\{(u_i, v_i), (u_{i+1}, v_{i+1})\} \subseteq C$, then (u_i, v_i) is homotopic to (u_{i+1}, v_{i+1}) and there are corresponding traces $T_{u_i} = u_i \rightarrow \ldots \rightarrow u_{i+1}$ and $T_{v_i} = v_i \rightarrow \ldots \rightarrow v_{i+1}$ such that T_{u_i} and T_{v_i} lie on the boundary of P and the elements of $\{(u_i, v_i), (u_{i+1}, v_{i+1}), T_{u_i}, T_{v_i}\}$ form the boundary of a face of G.

Suppose (u_{i+1}, v_{i+1}) is homotopic to (u_{i+2}, v_{i+2}). Then there are corresponding traces $T_{u_{i+1}} = u_{i+1} \rightarrow \ldots \rightarrow u_{i+2}$ and $T_{v_{i+1}} = v_{i+1} \rightarrow \ldots \rightarrow v_{i+2}$ such that $T_{u_{i+1}}$ and $T_{v_{i+1}}$ lie on the boundary of P and $\{(u_{i+1}, v_{i+1}), (u_{i+2}, v_{i+2}), T_{u_{i+1}}, T_{v_{i+1}}\}$ form the boundary of a face of G.

The concatenation of T_{u_i} and $T_{u_{i+1}}$ yields $u_i \rightarrow \ldots \rightarrow u_{i+1} \rightarrow \ldots \rightarrow u_{i+2}$, so there is a trace from u_i to u_{i+1} on the boundary of P. The concatenation of T_{v_i} and $T_{v_{i+1}}$ yields $v_i \rightarrow \ldots \rightarrow v_{i+1} \rightarrow \ldots \rightarrow v_{i+2}$, so there is a trace from v_i to v_{i+1} on the boundary of P. Since this is true for any i, the conclusion follows. □

The most important property of the homotopy classes with respect to the planar-nonplanar decomposition is that Heath and Istrail provide an upper bound on the number of homotopy classes in [7], are given in the next two theorems.

Theorem 3.13. *If $G = (V, E)$ has an open 2-cell embedding in an orientable surface of genus g, where $g \geq 1$, then any planar-nonplanar decomposition of G has at most $6g - 3$ homotopy classes.*

Theorem 3.14. *If $G = (V, E)$ has an open 2-cell embedding in a nonorientable surface of genus g, where $g \geq 1$, then any planar-nonplanar decomposition of G has at most $\max(1, 3g - 3)$ homotopy classes.*

Denote the bound on the number of homotopy classes given in Theorem 3.13 and Theorem 3.14 by γ. Note that we are dealing primarily with nonorientable surfaces. If there is an r-round, then the surface we must deal with is non-orientable. If there is not an r-round, then Heath and Istrail's embedding [7] along with the additional apex vertices described on page 17 would be sufficient to provide a reasonable book embedding.

3.3 Triangulating the Graph

Recall that R_i is a set of r-rounds of H_i and $E'(R_i)$ denotes the set of all cap edges of each r-round in R_i. Let F_i be the face of the embedding of H_i'' in Σ_i which contains the vertices of a ring R_i. In the next definition, we insert edges so that the only vertices in the boundary of F_i are vertices of R_i.

Definition 18. *If H_i'' is embedded in Σ_i and the vertices of R_i are ordered $(v_1, v_2, \ldots, v_k)$ and are in the boundary of face F_i, then the edges (v_i, v_{i+1}), where the index arithmetic is performed modulo k, are called the* boundary edges*. Denote the graph obtained by inserting the boundary edges, if they do not exist in H_i'', by $(H_i'')^B$.*

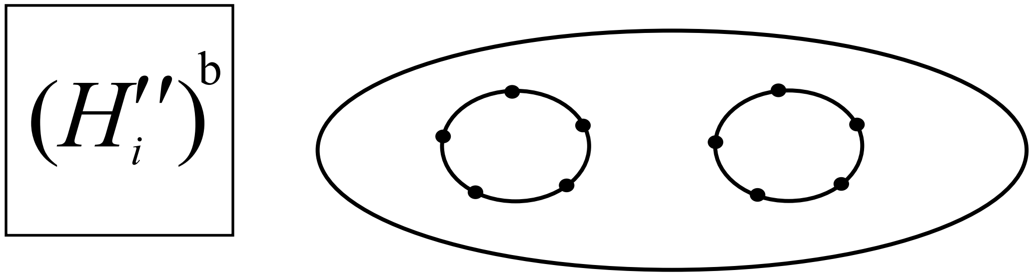

FIGURE 3.14. Delete the interior edges of the r-rounds and inserting their boundary edges.

Lemma 3.15. *The graph $(H_i'')^B$ has an embedding in Σ_i.*

Proof. The graph H_i'' has an open 2-cell embedding in Σ_i with representativity greater than or equal to 3. So each face is bounded by a cycle. The vertices which compose the perimeter of an r-round lie on such a cycle. Since the order of these vertices as they appear on the cycle is the same as they appear on the perimeter of the r-round, inserting the boundary edges will not violate planarity. □

Our next goal is to triangulate $(H_i'')^B$. This occurs in two distinct stages. The first of these two stages is described in Definition 19, Definition 20 and Lemma 3.16.

Definition 19. *Let $F \in \{F_i\}_{i=1}^n$ be a face of a graph $(H_i'')^B$, and boundary of the face is a cycle of vertices in the order $(v_1, v_2, \ldots, v_k, v_1)$. If k is odd, the* cap edges *are $(v_a, v_{a+\frac{k-1}{2}})$ for $1 \leq a \leq k$ where the index arithmetic is performed modulo k. If k is even,* cap edges *are $(v_a, v_{a+\frac{k}{2}})$ for $1 \leq a \leq \frac{k}{2}$ and $(v_a, v_{a+\frac{k}{2}+1})$ for $1 \leq a \leq \frac{k}{2}$. Denote the graph obtained by inserting the cap edges, if they do not exist in $(H_i'')^B$, by $(H_i'')^{bc}$.*

Definition 20. *If Σ_i is a surface and F is a face of an embedding of $(H_i'')^B$ in Σ_i, then* cap *the face by removing it from the surface and identifying the boundary of a Möbius band with the boundary of the face. Say the surface Σ_i is* augmented *by*

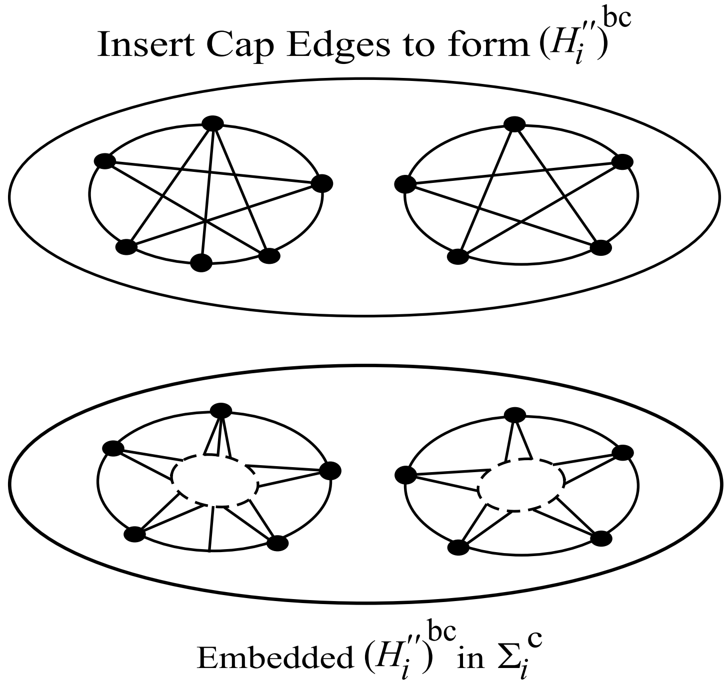

FIGURE 3.15. Capping the r-rounds and embedding the result in a surface.

a cross cap. If Σ_i is a surface and $\{F_i\}_{i=1}^n$ are some of the faces of an embedding of $(H_i'')^B$ in Σ_i, then Σ_i^c is obtained from Σ_i by augmenting Σ_i with n cross caps.

Lemma 3.16. *The graph $(H_i'')^{bc}$ has an embedding in Σ_i^c.*

Proof. The boundary edges and the cap edges can be embedded using the cap so that the faces created by the embedding are as follows. If the number of vertices on the boundary of the face is odd, arrange the edges inserted by capping F so that the faces are $(v_a, v_{a+\frac{k-1}{2}}, v_{a+\frac{k-1}{2}+1})$ for $1 \leq a \leq k$, where arithmetic on the index is performed modulo k. If the number of vertices on the boundary of the face is even, then arrange the edges inserted by capping F so that the faces are $(v_a, v_{a+\frac{k}{2}}, v_{a+\frac{k}{2}+1})$ for $1 \leq a \leq \frac{k}{2}$ and $(v_a, v_{a+1}, v_{a+1+\frac{k}{2}})$ for $1 \leq a \leq \frac{k}{2}$. See Figure 3.15.

□

Consider the boundaries of the faces as well as the edges inserted by Definition 19. If F_i is a face and F_i^c is the graph resulting from capping the face, then let

$E(F_i^c)$ designate the edges inserted by Definition 19. If H_i'' contained an edge before the capping process, it is now considered an edge of $E(F_i^c)$.

The next triangulation procedure is presented in [7].

Definition 21. *Consider any non-triangular face F of $(H_i'')^{bc}$. Add a vertex v in the face. Add an edge from v to each vertex on the boundary of F. Denote the result by $(H_i'')^{bct}$. Note that $(H_i'')^{bct}$ is embedded in Σ_i^c.*

Lemma 3.17. *No vertex w occurs multiple times on the boundary of F, and, therefore, no multiple edges are created by the triangulation process.*

Proof. The graph H_i'' has an embedding in Σ_i where Σ_i is a sphere, or H_i'' has an embedding in Σ_i with representativity greater than 3. A face of this embedding bounds a disk, and thus does not have any vertex appearing multiple times on the cycle which bounds it. The faces created in the capping process of Definition 19 also do not have any vertex appearing multiple times on the cycle which bounds it. These faces are explicitly listed in Lemma 3.16. Lastly, Heath's triangulation process does not create any multiple edges.

No vertex w occurs multiple times on the cycle which bounds a face F. Therefore, no multiple edges are created by the triangulation process. □

Now we have a graph $(H_i'')^{bct}$ embedded in Σ_i^c such that every face is a triangle. Denote the subgraph of $(H_i'')^{bct}$ which does not include the cap edges by $(H_i'')^{bt}$.

3.4 A Decomposition Algorithm

In the following discussion we will only allow edge choices from $(H_i'')^{bt}$. A planar graph P will be constructed incrementally until it contains all of the vertices of $(H_i'')^{bt}$ and some of the edges. All remaining edges will have both endpoints on the boundary of P. The algorithm will proceed until P is maximal.

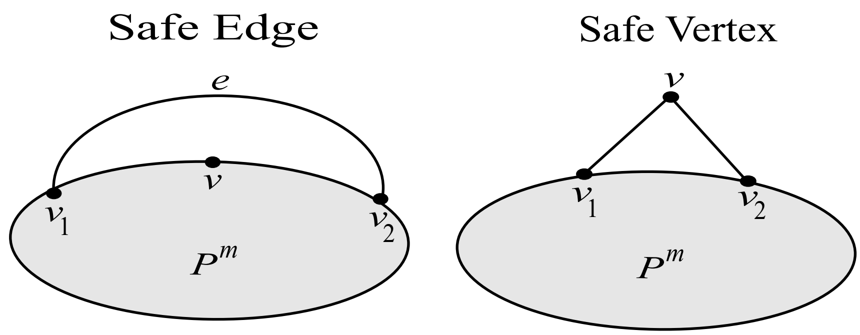

FIGURE 3.16. Defining a safe vertex and safe edge.

One triangle is chosen as the initial part P^0 and faces are added to the planar part incrementally as possible. After m vertex choices of the algorithm, $P^m = (V(P^m), E(P^m))$ will represent the planar part of the decomposition. The set $E - E(P)$ consists of the essentially nonplanar edges, which necessarily have both endpoints on the boundary of P.

Definition 22. *If $v_i \to v_j \to v_k$ is a trace on the boundary of P^m with no edge of $E - E(P^m)$ incident to v_j, then (v_i, v_j) is called a* safe edge *with respect to the boundary of P^m.*

Definition 23. *If $v_i \to v_j$ is a trace on the boundary of P^m, there is a vertex v_k not in the planar part P^m and (v_i, v_j, v_k) is a face of P, then v_k is a* safe vertex *with respect to the trace $v_i \to v_j$.*

In general, the algorithm proceeds iteratively to construct P^{m+1} from P^m by choosing a safe vertex and fill in safe edges until it is not possible to do so. Then the algorithm will choose an unsafe vertex incident to a boundary vertex of P^m. The algorithm also ages the edges, vertices and blocks of P^m. Those added later are newer, those added earlier are older. This aging process is used explicitly in the discussion on choosing an unsafe vertex.

The key difference between Heath's algorithm and this algorithm is the avoidance of cap edges of an r-round until the end of the algorithm. It is important never to choose an edge of $E'(R_i)$ until the final stage of the algorithm because avoiding these choices will force the vertices of the boundaries of the round to lie on the boundary of the planar graph at the completion of the algorithm.

Algorithm 3.4.1. A Planar-Nonplanar Decomposition Algorithm

While $V(P^m) \neq V(G)$ *and* $E(G) \setminus E(\{F_i^T\}) \neq E - E(P^m)$ *is not maximal*

Do

If $\exists$ *safe vertex* v_k *with respect to* $v_i \rightarrow v_j$

Then *(*add safe vertex*)*

$V(P^m) \leftarrow V(P^{m-1}) \ \cup \ \{v_j\}$

$E(P^m) \leftarrow E(P^{m-1}) \ \cup \ \{(v_i, v_j), (v_j, v_k)\}$

Else *(*start a new block*)*

$w' \leftarrow$ *newest vertex in* $V(P^{m-1})$ *incident to a vertex in* $V - V(P^m)$

$w \leftarrow$ *vertex in* $V(G) - V(P^{m-1})$ *incident to* w' *(*see text below*)*

$V(P^m) \leftarrow V(P^{m-1}) \cup \{w\}$

$E(P^m) \leftarrow E(P^{m-1}) \cup \{(w, w')\}$

While $\exists$ *safe edge* $(v_i, v_k) \in E - E(P^m) \setminus E(\{F_i^T\})$

Do

$E(P^m) \leftarrow E(P^{m-1}) \cup \{(v_i, v_k)\}$ *(*add safe edge*)*

EndDo

EndDo

While $\exists$ *safe edge* $(v_i, v_k) \in E(\{F_i^T\})$

Do

$E(P^m) \leftarrow E(P^{m-1}) \cup \{(v_i, v_k)\}$ *(*add safe cap edge*)*

EndDo

Now let us describe the selection of w. See Figure 3.17. Let (x, w') be the newest edge on the boundary of P^{m-1} that is incident to w'. Then there must be a triangle (x, w', z) exterior to P^{m-1}. Since z is unsafe, z is necessarily on the boundary of P^{m-1}. Also, (x, z) and (w', z) are essentially nonplanar. Examine the edges incident to w' which are not in $E(\{F_i^T\})$. Start with the edge (x, w') and sweep rotationally about w' in the direction of z. Let (w, w') be the first edge encountered such that $w \in V(G)\backslash V(P^{m-1})$. Let (w', y) be the last essentially nonplanar edge encountered before (w, w'). Let y' be the next vertex incident to w' after encountering w. Notice $(w', y') \in E - E(P^{m-1})$. If it were not, w would be a safe vertex. Now, (w', y, w) is a triangle. Once (w', w) is added to P^{m-1} it is true that (w, y) becomes essentially nonplanar and will be homotopic to (w', y). Also, w is newer than y; thus the homotopy class will be extended by edges incident to y and never by edges incident to w. This means that w (not y) will have the role of w' in future executions of the algorithm. If $y' \in V(P^{m-1})$, then (w', y') is already essentially nonplanar. Therefore, (w, y') also becomes essentially nonplanar and homotopic to (w', y'). In this case, w is newer than y' and the homotopy class of (w', y') must necessarily be extended by edges incident to y' (not w).

Theorem 3.18. *Given an embedding of $(H_i'')^{bct}$ in Σ_i^c, denote the faces arising from the deletion of the r-round cap edges of the embedding of H_i'' in Σ_i by $\{F_i\}_{i=1}^n$. If the number of vertices of face F_i is odd, no more than one vertex from the boundary of F_i is removed from the boundary of P. If the number of vertices of face F_i is even, no more than two vertices from the perimeter of F_i are removed from the boundary*

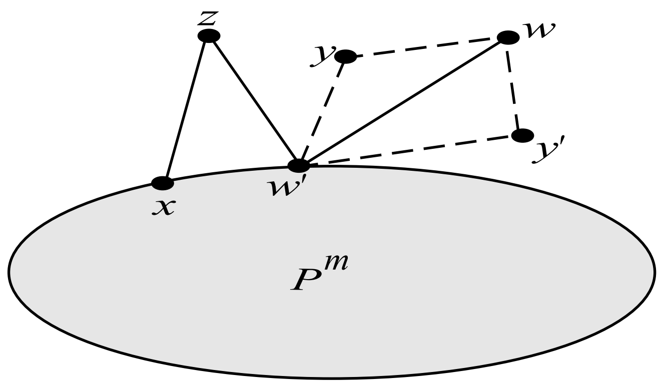

FIGURE 3.17. The selection of an unsafe vertex w in the planar-nonplanar decomposition algorithm.

of P. Moreover, the remaining vertices of the perimeter of F_i are partitioned into two intervals which are traces on the boundary of P.

Proof. Given an embedding of $(H_i'')^{bct}$ in Σ_i^c, and the faces $\{F_i\}_{i=1}^n$ arising from the deletion of the r-round cap edges $E'(R_i)$ of the embedding of $(H_i'')^{bt}$ in Σ_i, apply the Planar-Nonplanar Decomposition Algorithm. Consider the last step of the algorithm where cap edges of an r-round have a chance of being absorbed into P^m.

In the last step of the algorithm, the cap edges $E'(R_i)$ for each i are searched and included in P^m if they are safe. This has the effect of maximizing P. Up to this step, all vertices on the boundary of each r-round lie on the boundary of P^m. The reason they are still on the boundary is because they each are incident to at least one edge in E_N.

If R is an r-round of the set of r-rounds R_i, consider the graph consisting of the vertices of R, the boundary edges of R and the edges inserted by the capping of R. Note that the vertices of R together with the boundary edges of R form a face of the embedding of $(H_i'')^{bt}$ in Σ_i. Denote the cap edges of R by $E'(R)$.

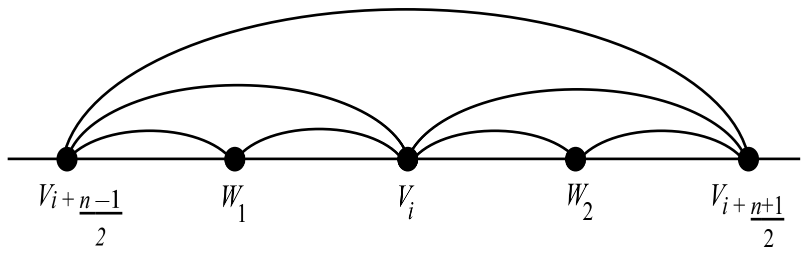

FIGURE 3.18. Removing a vertex from the boundary of the planar part of the decomposition in the case that an r-round has an odd number of vertices on its boundary.

We first consider the case when R contains an odd number n of vertices. Suppose that the addition of a safe edge from $E'(R)$ removes a vertex v_i from the boundary of P^m. Then there are two traces $v_i \to w_1 \to v_{i+\frac{n-1}{2}}$ and $v_i \to w_2 \to v_{i+\frac{n+1}{2}}$ in the boundary of P^m where the boundary vertices of the r-round are $\{v_i\}_{i=1}^{\frac{n+1}{2}}$ and $\{w_1, w_2\}$ are vertices of $(H_i'')^{bct}$ which are not on the boundary of the r-round. Then edges $(v_i, v_{i+\frac{n-1}{2}})$ and $(v_i, v_{i+\frac{n+1}{2}})$ are safe. Their addition to P^m extends the planar part P^m of the decomposition and results in a trace $v_{i+\frac{n-1}{2}} \to v_i \to v_{i+\frac{n+1}{2}}$ on the boundary of $P^{m'}$ for some $m \leq m'$. So the edge $(v_{i+\frac{n-1}{2}}, v_{i+\frac{n+1}{2}})$ is safe with respect to the boundary of $P^{m'}$. Thus, there is an m'' such that $m \leq m' \leq m''$ where $(v_{i+\frac{n-1}{2}}, v_{i+\frac{n+1}{2}})$ is on the boundary of $P^{m''}$. Repeat the above argument for another vertex v_j where $i \neq j$. See Figure 3.18.

It remains to show that no other cap edge is safe. There are two cases. If the graph without the cap edges $E'(R)$ is embedded in a sphere, then the algorithm cannot remove a third vertex v_k from the boundary of the planar part of the decomposition.

To see this, consider the following vertices are on the boundary of R: v_i, $v_{i+\frac{n}{2}-1}$, $v_{i+\frac{n}{2}}$, v_j, $v_{j+\frac{n}{2}-1}$, $v_{j+\frac{n}{2}}$, v_k, and $v_{k+\frac{n}{2}}$. Consider the edges $(v_i, v_{i+\frac{n}{2}})$, $(v_i, v_{i+\frac{n}{2}-1})$, $(v_j, v_{j+\frac{n}{2}})$, $(v_j, v_{j+\frac{n}{2}-1})$ where i, j and k are distinct. If these edges were absorbed then $(v_k, v_{k+\frac{n}{2}})$ could not be absorbed because P^m is planar and therefore has no subdivision of $K_{3,3}$. In this case, the construction of a subdivision of $K_{3,3}$ would

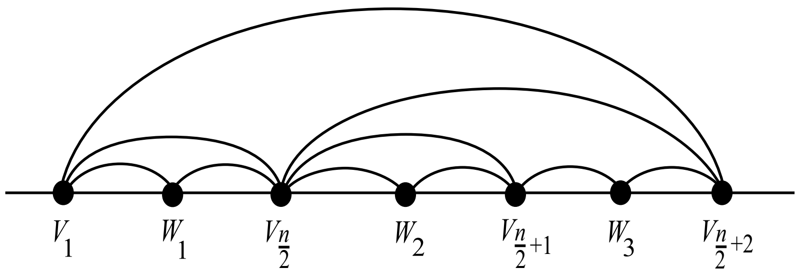

FIGURE 3.19. Removing two vertices from the boundary of the planar part of the decomposition in the case that an r-round has an even number of vertices on its boundary.

consist of edges $(v_i, v_{i+\frac{n}{2}})$, $(v_j, v_{j+\frac{n}{2}})$ and $(v_k, v_{k+\frac{n}{2}})$, and traces $v_i \rightarrow v_j$, $v_j \rightarrow v_k$, $v_k \rightarrow v_{i+\frac{n}{2}}$, $v_{i+\frac{n}{2}} \rightarrow v_{j+\frac{n}{2}}$, $v_{j+\frac{n}{2}} \rightarrow v_{k+\frac{n}{2}}$ and $v_{k+\frac{n}{2}} \rightarrow v_i$. Therefore, no more than two vertices from the boundary of r-round with an odd number of vertices can be removed from the boundary of P. Give each of these vertices its own page to embed any edges incident with them.

If the graph without the cap edges is not embedded in a sphere, then the trace $v_j \rightarrow w_3 \rightarrow v_k$ where $i \neq j$ and $k = j + \frac{n-1}{2}$ or $k = j + \frac{n+1}{2}$ cannot exist, because it violates representativity. The reason is one of the following two circuits must be nontrivial, and both of the circuits are short: $v_i \rightarrow w_1 \rightarrow v_{i+\frac{n-1}{2}} \rightarrow v_i$ or $v_j \rightarrow w_3 \rightarrow v_{j+\frac{n-1}{2}} \rightarrow v_j$.

In this case, no more than one vertex from an r-round $R \in R_i$ with an odd number of vertices on its boundary can be removed from the boundary of P. Again, give this vertex its own page to embed any edges incident to it.

Suppose the boundary of R contains an even number n of vertices. Consider the possibility that a cap edge of $E'(R)$ is safe with respect to the boundary of P. One of the vertices on the boundary of the face has three incident edges. Mimicking the previous argument, at most two vertices from the boundary of each face can be removed from the boundary of P. See Figure 3.19. Give each such vertex its own page. If the number of r-rounds is n, then at most $2n$ pages needed.

At the completion of the algorithm, a planar-nonplanar decomposition of $(H_i'')^{bct}$ has been constructed. Each vertex removed from the boundary of P receives its own page for embedding edges incident to it. The remaining edges of $E'(R)$ are necessarily incident to vertices on the boundary of P. Recall the number of homotopy classes given in Theorem 3.13 and Theorem 3.14 is denoted by γ. The remaining edges are essentially nonplanar and are partitioned into γ homotopy classes by [7]. Each homotopy class is defined by two traces on the boundary of P. Note that the cap edges of an r-round are homotopically equivalent, and thus they belong to one homotopy class. Thus, the perimeter of each r-round $R \in R_i$ is partitioned into 2 distinct intervals, where at most 2 vertices are exceptional in the sense that they are removed from the boundary of the planar part P. Moreover, these intervals form a trace on the boundary of the planar part P of the decomposition. □

Later each exceptional vertex will receive its own page for embedding edges incident to it, and the 2 intervals will place the remaining vertices of the perimeter of the r-round in order in the spine of the book.

Lemma 3.19. *This algorithm produces a planar-nonplanar decomposition.*

Proof. If a graph $(H_i'')^{bct}$ with an embedding in Σ_i^c is described by a rotation R and P is a subgraph of $(H_i'')^{bct}$, then there is a subrotation R' representing an embedding of P in Σ_i^c. Obtain R' from R by simply deleting the vertices of $(H_i'')^{bct} \backslash P$ from the directed edge form listing of the vertices of $(H_i'')^{bct}$.

Since $(H_i'')^{bt}$ is connected and the boundary of each designated face is a cycle in G, the algorithm will eventually choose every vertex (either in a safe or unsafe way). Thus, $V((H_i'')^{bt}) = V(P)$.

The algorithm also requires P to be maximal before it will be completed. The maximality of P is discussed in Theorem 3.18. It remains to show that every edge not in P is incident to two vertices on the boundary of a single face of P, namely the boundary of P. The only possibility of removing a vertex from the boundary of P is the inclusion of a safe edge. By definition, an edge (v_i, v_k) is safe when there is a trace $v_i \rightarrow v_j \rightarrow v_k$ on the boundary of P and the vertex v_j is not incident to any edge not in P. Therefore, vertices incident to edges not in P are always attached to the boundary of P. The algorithm produces a planar-nonplanar decomposition. □

4. Clique Summing

This notation is generalized from the work of Dittman [3]. The next definition describes a function which assigns to each edge e in E_0 a label $s(e)$ and a direction where $u(e)$ is the tail of e and $v(e)$ is the head of e.

Definition 24. *Let S be a set, G be a graph, and H be a subgraph of G. Let $u(e)$ and $v(e)$ denote the endpoints of e. Then a* directed labeling *of G is a function $L_G : E(H) \to S \times (V(H) \times V(H))$ where $e \mapsto (s(e), (u(e), v(e)))$, and $s(e) = s(f)$ implies $e = f$. If the domain of L_G is the empty set, then G is said to be* unlabeled*.*

Definition 25. *Let H and K be two disjoint graphs with directed labelings L_H and L_K. Let $E(H)$ and $E(K)$ be labeled subsets of H and K that induce cliques of the same order. A function $\alpha : E(H) \to E(K)$ is an* identification function *if $s(E(H)) = s(E(\alpha(H)))$ for $h \in E(H)$ and $\alpha(h) \in E(K)$, then $s(h) = s(\alpha(h))$ implies $h = \alpha(h)$.*

This is a bijective correspondence that uniquely identifies the clique $E'(H)$ to $E'(K)$. In Theorem 3.10, a k-sum is used to identify two graphs. Here, however, a more restrictive operation called a *clique sum* is used. The next step is to find a clique of the same order in two graphs, label them, and then identify them.

Definition 26. *The* clique-sum *of two graphs H and K (with respect to cliques L_H and L_K), denoted $(H, L_H) \oplus (K, L_K)$, is a graph defined as follows. For each $h \in E(H)$ and $\alpha(h) \in E(K)$, identify h and $\alpha(h)$ head-to-head and tail-to-tail. Some subset of identified edges can then be deleted.*

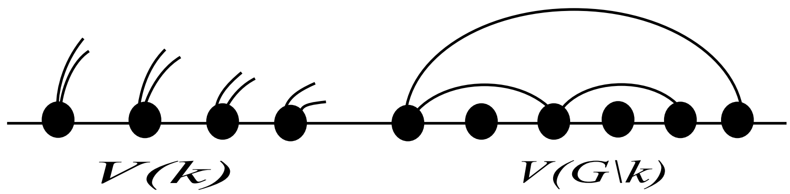

FIGURE 4.20. Move the vertices of the clique in front of the vertices of the remainder of the graph.

Recall the definition of an n-tree from page 8. This uses a special case of clique-summing. An n-tree is formed by summing two cliques of order $n+1$ together on a clique of order n.

Note that if K and K' both have order two, then the definition of locked cliques is equivalent to the definition of locked edges given in Definition 2. If two edges are locked, they will require different pages in the book embedding. In order to form a single book embedding from the book embeddings of two graphs which are clique summed together, special attention needs to be given to the clique involved in the sum.

Definition 27. *If (G, σ) is an ordered graph and K is a complete subgraph of G, then σ_K is a K-*rooted ordering function *compatible with σ when the following hold:*

1. $\sigma_K(u) < \sigma_K(v)$ *whenever* $u \in V(K)$ *and* $v \notin V(K)$
2. $\sigma_K(u) < \sigma_K(v)$ *whenever* $\{u, v\} \in V(K)$, *and* $\sigma(u) < \sigma(v)$
3. $\sigma_K(u) < \sigma_K(v)$ *whenever* $\{u, v\} \in V(G \setminus K)$, *and* $\sigma(u) < \sigma(v)$

Definition 28. *If (G, σ) is an ordered graph, K is a complete subgraph of G and σ_K is a K-rooted ordering function compatible with σ, then (G, σ_K) is a K-*rooted graph.

Definition 29. *The* thickness *of the K-rooted graph (G, σ_K) is the smallest thickness of an embedded ordered graph (G, σ_K, π) where the minimum is taken over all possible page assignments π.*

The next lemma describes the relationship between the book thickness of G and the thickness of (G, σ_K).

Lemma 4.20. *If (G, σ) is a K-rooted graph and σ_K is a K-rooted ordering function compatible with σ, then the thickness of (G, σ_K) is at most $3BT(G, \sigma)$.*

Proof. Suppose (G, σ, π) is an embedded ordered graph where $\mathrm{BT}(G) = B$. Let K be a complete subgraph of G. Consider a K-rooted ordering function σ_K compatible with σ. It remains to define a page assignment π_K from π which embeds the edges in the pages of the book. If (u, v) is an edge of G from cases (2) or (3) of Definition 27, then $\pi_K(u, v) = \pi(u, v)$.

Add a new page $P(u)$ for each vertex $u \in V(K_n)$. If (u, v) is an edge of G from case (1) of Definition 27, then $\pi_K(u, v) = P(u)$. Recall that an algorithm for embedding clique K_n in a book with $\lceil \frac{n}{2} \rceil$ pages was given in [1]. Together $\mathrm{BT}(G) = B$ and $K_n \subseteq G$ imply $n \leq 2B$. Therefore $|V(K_n)| \leq 2B$, and hence the number of added pages is at most $2B$.

Edges from cases (2) and (3) of Definition 27 require B pages and edges from case (3) of Definition 27 require $2B$ pages. Hence, the thickness of (G, σ_K) is at most $3\mathrm{BT}(G)$. □

Later it will be important to know exactly which clique K is involved in a particular sum. A single vertex v may be in the vertex set of several different maximal cliques, and an edge (u, v) will be embedded on a page depending both on u and on a clique K.

Definition 30. *If G is a graph, then we define* QueZoo(G) *to be the set of all subgraphs K of G such that K is a maximal clique in G.*

Definition 31. *The* clique-graph of (G, σ)*, denoted $Que(G, \sigma)$, is constructed as follows. If $K \subseteq G$ is a maximal clique in G, then $K \in V(Que(G, \sigma))$. If K and K' are in $V(Que(G, \sigma))$ and K and K' are locked, then $(K, K') \in E(Que(G, \sigma))$.*

Note that $QueZoo(G) = V(Que(G, \sigma))$. The next goal is to provide a proper vertex coloring of $Que(G, \sigma)$. The following definitions provide the notation needed to do this. For the remainder of this section we will use a particular embedding function π. We say (G, σ, π) is *neatly embedded* if $e = (v_i, v_j)$ is an edge of K implies $\pi(e) = P(v_i)$. In general, we will abbreviate $\pi(v_i, v_j) = \pi(v_i)$.

Recall that the spine of the book is considered as a real line so a lexicographic ordering of the edges of a clique can be specified.

Definition 32. *Let (G, σ, π) be a neatly embedded ordered graph and suppose K is an element of $QueZoo(G)$ with n vertices. Then define the* clique color *of K to be $QueHue(K) = (\pi(e_1), \pi(e_2), \ldots, \pi(e_{\binom{n}{2}}))$, where the edges $e_1, e_2, \ldots, e_{\binom{n}{2}}$ are listed in the lexicographic order induced by σ.*

Definition 33. *If G is a graph, then let $QueHueZoo(G, \sigma, \pi) = \{QueHue(K) : K \in QueZoo(G)\}$.*

Lemma 4.21. *If $QueHue(K) = QueHue(K')$, then K and K' are nested.*

Proof. Let (G, σ, π) be a neatly embedded graph. Suppose K and K' are maximal cliques of (G, σ, π) and $QueHue(K) = QueHue(K')$. If K and K' are of order 1, then they could not lock because they have no edges. If K and K' are of order 2 and $QueHue(K) = QueHue(K')$, then their edges are nested on a single page.

Suppose K and K' are of order 3 and $QueHue(K) = QueHue(K')$. Suppose the vertices $\{u, v, w\}$ of K are ordered $\sigma(u) < \sigma(v) < \sigma(w)$ and the vertices $\{u', v', w'\}$ of K' are ordered $\sigma(u') < \sigma(v') < \sigma(w')$.

Now $QueHue(K) = QueHue(K')$, so $\pi(u, v) = \pi(u', v')$. Thus, either $\sigma(u) < \sigma(u') < \sigma(v') < \sigma(v)$ or $\sigma(u') < \sigma(u) < \sigma(v) < \sigma(w)$. Also, $\pi(u, w) = \pi(u', w')$. Thus, either $\sigma(u) < \sigma(u') < \sigma(w') < \sigma(w)$ or $\sigma(u') < \sigma(u) < \sigma(w) < \sigma(w')$. Also, $\pi(v, w) = \pi(v', w')$. Thus, either $\sigma(v) < \sigma(v') < \sigma(w') < \sigma(w)$ or $\sigma(v') < \sigma(v) < \sigma(w) < \sigma(w')$. Any combination of these conditions imply K and K' are nested.

We may assume K and K' have at least 4 vertices. Suppose $\{u, v, w, x\}$ are vertices of K where $\sigma(u) < \sigma(v) < \sigma(w) < \sigma(x)$ and $\{u', v', w', x'\}$ are vertices of K' where $\sigma(u') < \sigma(v') < \sigma(w') < \sigma(x')$. Additionally assume that (a, b) and (a', b') are in the same lexicographic position in the ordering of the edges of each clique for $\{a, b\} \subseteq \{u, v, w, x\}$ and for $\{a', b'\} \subseteq \{u', v', w', x'\}$.

So $\pi(u) = \pi(u')$, $\pi(v) = \pi(v')$, $\pi(w) = \pi(w')$ and $\pi(x) = \pi(x')$.

In the following discussion, there are 256 conceivable cases. Many of these cases cannot occur because there is a contradiction of the sort $\sigma(v) < \sigma(v')$ and $\sigma(v') < \sigma(v)$. Taking this into consideration, there are still many cases left to check. We investigate one case in detail and leave the other cases to be similarly analyzed by the reader. To assist the reader in this process, discussion of which cases arise is intermixed with determining which cases can be eliminated. With respect to the enumeration which follows, we discuss case $\{1a, 2c, 3a, 4c, 5c\}$.

Since $\pi(u) = \pi(u')$, the edges (u, v) and (u', v') are embedded on the same page. Therefore, (u, v) and (u', v') are nested. Four cases arise.

1. (a) $\sigma(u) < \sigma(u') < \sigma(v') < \sigma(v)$

   (b) $\sigma(u) < \sigma(v) < \sigma(u') < \sigma(v')$

(c) $\sigma(u') < \sigma(v') < \sigma(u) < \sigma(v)$

(d) $\sigma(u') < \sigma(u) < \sigma(v) < \sigma(v')$

Since $\pi(v) = \pi(v')$, the edges (v, w) and (v', w') are embedded on the same page. Therefore, (v, w) and (v', w') are nested. Four cases arise.

2. (a) $\sigma(v) < \sigma(v') < \sigma(w') < \sigma(w)$

   (b) $\sigma(v) < \sigma(w) < \sigma(v') < \sigma(w')$

   (c) $\sigma(v') < \sigma(w') < \sigma(v) < \sigma(w)$

   (d) $\sigma(v') < \sigma(v) < \sigma(w) < \sigma(w')$

This yields sixteen cases altogether. The following cases $\{1a, 2a\}$, $\{1a, 2b\}$, $\{1b, 2c\}$, $\{1b, 2d\}$, $\{1c, 2a\}$, $\{1c, 2b\}$, $\{1d, 2c\}$, $\{1d, 2d\}$ do not occur because $\sigma(v) < \sigma(v')$ and $\sigma(v') < \sigma(v)$ is a contradiction.

Of the remaining eight cases, consider case $\{1a, 2c\}$. This means $\sigma(u) < \sigma(u') < \sigma(v') < \sigma(v)$ and $\sigma(v') < \sigma(w') < \sigma(v) < \sigma(w)$. It is also true that since $\pi(u) = \pi(u')$, the edges (u, w) and (u', w') are embedded on the same page. Therefore, (u, w) and (u', w') are nested. Four cases arise.

3. (a) $\sigma(u) < \sigma(u') < \sigma(w') < \sigma(w)$

   (b) $\sigma(u) < \sigma(w) < \sigma(u') < \sigma(w')$

   (c) $\sigma(u') < \sigma(w') < \sigma(u) < \sigma(w)$

   (d) $\sigma(u') < \sigma(u) < \sigma(w) < \sigma(w')$

Case $3a$ does not occur because $\sigma(w) < \sigma(w')$ and $\sigma(w') < \sigma(w)$ is a contradiction. Cases $3b$ and $3c$ do not occur because it $\sigma(u) < \sigma(u')$ and $\sigma(u') < \sigma(u)$ is a contradiction. Putting together the inequalities in case $\{1a, 2c, 3d\}$ yields the inequality $\sigma(u) < \sigma(u') < \sigma(v') < \sigma(w') < \sigma(v) < \sigma(w) < \sigma(x)$.

Since $\pi(v) = \pi(v')$, the edges (v, x) and (v', x') are embedded on the same page. Therefore, (v, x) and (v', x') are nested. Four cases arise.

4. (a) $\sigma(v) < \sigma(v') < \sigma(x') < \sigma(x)$

   (b) $\sigma(v) < \sigma(x) < \sigma(v') < \sigma(x')$

   (c) $\sigma(v') < \sigma(x') < \sigma(v) < \sigma(x)$

   (d) $\sigma(v') < \sigma(v) < \sigma(x) < \sigma(x')$

Cases $\{1a, 2c, 3a, 4a\}$ and $\{1a, 2c, 3a, 4b\}$ do not occur because $\sigma(v) < \sigma(v')$ and $\sigma(v') < \sigma(v)$ is a contradiction.

Since $\pi(w) = \pi(w')$, the edges (w, x) and (w', x') are embedded on the same page. Therefore, (w, x) and (w', x') are nested. Four cases arise.

5. (a) $\sigma(w) < \sigma(w') < \sigma(x') < \sigma(x)$

   (b) $\sigma(w) < \sigma(x) < \sigma(w') < \sigma(x')$

   (c) $\sigma(w') < \sigma(x') < \sigma(w) < \sigma(x)$

   (d) $\sigma(w') < \sigma(w) < \sigma(x) < \sigma(x')$

Cases $\{1a, 2c, 3a, 4c, 5a\}$ and $\{1a, 2c, 3a, 4d, 5b\}$ do not occur because $\sigma(w) < \sigma(w')$ and $\sigma(w') < \sigma(w)$ is a contradiction.

Consider case $\{1a, 2c, 3a, 4c, 5c\}$. It yields three inequalities:

1. $\sigma(v') < \sigma(x') < \sigma(v) < \sigma(x)$

2. $\sigma(w') < \sigma(x') < \sigma(w) < \sigma(x)$

3. $\sigma(u) < \sigma(u') < \sigma(v') < \sigma(w') < \sigma(v) < \sigma(w) < \sigma(x)$

Use (2) and (3) to obtain the inequality $\sigma(u) < \sigma(u') < \sigma(v') < \sigma(w') < \sigma(x') < \sigma(w) < \sigma(x)$. Now $\sigma(v) < \sigma(w)$, also $\sigma(x') < \sigma(v) < \sigma(x)$ by (1) and

$\sigma(x') < \sigma(w) < \sigma(x)$ by (2). The resulting inequality is $\sigma(u) < \sigma(u') < \sigma(v') < \sigma(w') < \sigma(x') < \sigma(v) < \sigma(w) < \sigma(x)$.

Thus, the vertices $\{u', v', w', x'\}$ of K' are nested between the two vertices u and v of K. Each of the cases is similar to this one. Take each case over all combinations of four vertices of K and K' to conclude the two cliques are nested with respect to each other. □

The next lemma properly colors the clique-graph $Que(G, \sigma)$ by assigning to each vertex K the clique color $QueHue(K)$.

Lemma 4.22. *If $BT(G, \sigma, \pi)$ is a neatly embedded ordered graph, then there is a proper vertex coloring of $Que(G, \sigma)$ with $|QueHueZoo(G, \sigma, \pi)|$ colors.*

Proof. Suppose $QueHue(K) = QueHue(K')$. Then K and K' are nested by Lemma 4.21. Hence (K, K') is not an edge of $Que(G, \sigma)$. Therefore the vertex coloring of $Que(G, \sigma)$ by $QueHueZoo$ is proper. □

Lemma 4.23. *If (G, σ_K, π) is a neatly embedded K-rooted ordered graph with book thickness B, then the number of clique colors in $QueHueZoo(G, \sigma_K, \pi)$ is at most $B^{\binom{n}{2}}$ where $n \leq \lceil \frac{2}{3} \rceil B$.*

Proof. Let (G, σ_K, π) be a neatly embedded K-rooted ordered graph with book thickness B. Recall that a K-rooted ordering function has the property $\sigma_K(u) < \sigma_K(v)$ whenever $u \in V(K)$ and $v \notin V(K)$. Let $\mathbb{P}$ be a collection of no more than B pages. Recall that the edges of K may be lexicographically ordered by the order on the vertices of K given by the ordering function σ_K, and assign to each entry of $QueHue(K)$ the page each edge of K was embedded in by π. Specifically, if $(u, v) \in E(K)$ and $\pi(u, v) = \pi(u)$, then $QueHue(K) = (\pi(e_1), \pi(e_2), \ldots, \pi(e_{\binom{n}{2}}))$. Since $\pi(v) \in \mathbb{P}$ and $|\mathbb{P}| \leq$ B, there are B choices for each edge (u, v).

Now $\mathrm{BT}(G, \sigma_K) = B$ means $\mathrm{BT}(G, \sigma) \leq \frac{1}{3}\mathrm{B}$ by Lemma 4.20. If $K \subseteq G$ is a clique on n vertices, then $\mathrm{BT}(K_n \leq \lceil \frac{n}{2} \rceil)$. Therefore, $n \leq \lceil \frac{2}{3} \rceil \mathrm{B}$. Also, K has $\binom{n}{2}$ edges, and hence $QueHue(K)$ is a sequence of length $\binom{n}{2}$. Therefore, the total number of clique colors in $QueHueZoo$ is at most $\mathrm{B}^{\binom{n}{2}}$ where $n \leq \lceil \frac{2}{3} \rceil \mathrm{B}$.

□

Definition 34. *If T is a tree and $\mathcal{G} = \{G_t\}_{t \in V(T)}$ is a collection of graphs, then a* clique-sum tree *is a graph $G(\mathcal{G}, T)$ where edge $(v_i, v_j) \in E(T)$ if G_i is clique-summed with G_j.*

The following theorem is taken from [2].

Theorem 4.24. *For every graph K, there is an integer $k_V = k_V(K)$ such that every graph with no K-minor has a vertex partition into two graphs with tree-width at most k_V.*

Robertson and Seymour have proved that if $\mathcal{C}$ is a minor-closed class of graphs, other than the class of all graphs, then $\mathcal{C}$ can be characterized by a finite list of excluded minors $\{H_i\}_{i=1}^{n}$. Apply Theorem 4.24 to each excluded minor H_i individually and obtain the constants $k_V(H_i)$. Let $k_V(\{H_i\}_{i=1}^{n}) = \max\{k_V(H_i)\}$ where the maximum is taken over all i.

Corollary 4.25. *For every class of graphs, other than the class of all graphs, there is an integer n depending only on the class such that all members of the class have chromatic number at most n.*

Proof. Suppose G is a graph with no H_i minor. Then by Theorem 4.24, G can be decomposed into two graphs G_1 and G_2 where $tw(G_1) \leq k_V(H_i)$ and $tw(G_1) \leq k_V(H_i)$. Thus, $tw(G) \leq 2k_V(H_i)$. So, by Corollary 2.8, we have the book thickness of G is at most $2(k_V(H_i) + 2)$. □

Suppose $\mathcal{G}$ is a collection of graphs where BT(G) $\leq$ B for all G $\in \mathcal{G}$.

Theorem 4.26. *If $G(\mathcal{G}, T)$ is a clique-sum tree, and $BT(G_i) \leq B$ for every $G_i \in \mathcal{G}$, then $BT(G(\mathcal{G}, T)) \leq \chi \times (3B)^{\binom{n}{2}} + B$ where $n \leq \lceil \frac{2}{3} \rceil B$ and χ is the chromatic number of $G(\mathcal{G}, T)$.*

Proof. Conduct a depth-first search on T. It will be used to create σ which will order the vertices of G(T) in the spine. Recall that BT(G_i) $\leq$ B for all $G_i \in G$. For each graph G_i, let σ^i denote the ordering function of an optimal book embedding of G_i. Let σ^i_K be a K-rooted ordering function for G_i.

Choose a root graph $G_0 \in \mathcal{G}$. Notice G_0 is a vertex in the underlying tree T. Place the vertices of G_0 in the spine in the order prescribed by σ^0. This ordering function induces a lexicographic ordering on the set of cliques in G_0. So investigate $V(G_0)$ lexicographically until a clique K_0 is found which has the smallest lexicographic order on $V(K_0)$ with respect to σ^0, where K_0 is involved in a clique-sum with another graph, $G_1 \in \mathcal{G}$. Specifically, $K_0 \subseteq G_0$ and $K_0 \subseteq G_1$, where $K_0 \subseteq G_0 \oplus G_1$ in G(T).

Next consider the vertices of G_1 which have not yet been embedded in the spine. These are the vertices of $G'_1 = G_1 \setminus K$. Although BT(G_1) $= B$ gives an order on the vertices of G_1 via σ^1, the vertices of K_0 have already been ordered with respect to σ^0. Place the vertices of $G'_1 = G_1 \setminus K_0$ in the spine according to the order prescribed by $\sigma^1_{K_0}$ in between the last vertex of K_0 and the next vertex of $V(G_0)$ with respect to the order prescribed by σ^0. By Lemma 4.20, BT(G_1) $\leq 3B$. Rename all of the vertices of $G_0 \oplus G_1$ and let $\sigma^0 \oplus \sigma^1_{K_0}$ denote the permutation equivalent to this ordering.

If K is a maximal clique, then it was a member of $QueZoo$(G(T)) and received a clique color $QueHue(K)$. If K is not a maximal clique, then choose any maximal clique K' such that $K \subseteq K'$. Assign $QueHue(K')$ to be $QueHue(K)$. If $QueHue(K_0)$ with respect to σ^0 is the same as $QueHue(K_0)$ with respect to σ^1, then the proper coloring of the clique-graph of $G_0 \oplus G_1$ does not need to be adjusted. If $QueHue(K_0)$ with respect to σ^0 is not the same as the $QueHue(K_0)$ with respect to σ^1, then rearrange the clique colors of $Que(G_1)$ until it is. Then the two graphs may be summed so that a proper coloring is induced on $Que(G_0 \oplus G_1)$.

Next investigate the vertices ordered with respect to $\sigma^0 \oplus \sigma^1_{K_0}$ and choose a clique K_1 with the smallest lexicographic order. Sum on the next graph $G_2 \in \mathcal{G}$ via the depth-first search of the underlying tree T. Now $K_1 \subseteq (G_0 \oplus G_1)$ and $K \subseteq G_2$. Repeat the process described above. Since (G_2, σ_2) is an ordered graph, let $\sigma^2_{K_1}$ be the K_1-rooted ordering of G_2. Identify the last vertex of K_1, and the next vertex in the spine belonging to $G_0 \oplus G_1$. Place the vertices of G_2 which have not already been embedded by $G_0 \oplus G_1$ between these two vertices. Adjust the clique colors of $Que(G_2)$ so the coloring of K_1 with respect to $Que(G_1)$ matches the coloring of K_1 with respect to $Que(G_2)$ if necessary. Rename all the vertices in the spine so that the next clique to be involved in a clique-sum can be lexicographically chosen with respect to vertex order. Denote this ordering $(\sigma^0 \oplus \sigma^1_{K_0}) \oplus \sigma^2_{K_1}$.

Continue by induction. Suppose $G = (((G_0 \oplus G_1) \oplus G_2) \oplus \ldots) \oplus G_{i-1})$ and consider $G \oplus G_i)$. Because $\mathrm{BT}(G_i) \leq$ B, there is an ordering of the vertices of G_i denoted $\sigma = (((\sigma^0 \oplus \sigma^1_{K_0}) \oplus \sigma^2_{K_1}) \oplus \ldots) \oplus \sigma^{i-1}_{K_{i-2}})$. Identify the lexicographically smallest clique K_{i-1} involved in the clique sum of G with G_i. The vertices of K_{i-1} have already been ordered by σ since $K_{i-1} \subseteq G$. Rearrange the colors in $QueHueZoo(G_n)$ so that the clique color is consistent with respect to both $Que(G, \sigma)$ and $Que(G_i, \sigma^i)$. The same set of clique colors is kept at every stage of the induction. Identify the last vertex of K_{i-1} with respect to the current permutation σ. Place the vertices

$V(G \oplus G_i \setminus K_{i-1})$ in between the last vertex of K_{i-1} and the next vertex according to σ. Rename the vertices in the spine and denote the permutation $\sigma \oplus \sigma^i_{K_{i-1}}$.

The clique-graph $Que(G)$ is properly colored with $|QueHueZoo|$ clique colors by Lemma 4.22. Recall these colors were obtained by lexicographically ordering their edges and looking at the pages on which they were embedding in an optimal book embedding guaranteed by Lemma 4.20. Also, if K and K' are distinct cliques and $QueHue(K) = QueHue(K')$, then all of the vertices of K lie in between two consecutive vertices of K', or vice versa by Lemma 4.21. Denote the final ordering of the vertices of G(T) by σ_G.

The edges of G(T) need page assignments. If $(u, v) \in E(G_i)$, then $\pi(u, v) = \pi_i(u, v)$. Suppose $(u, v) \in E(G_i)$ and $(w, x) \in E(G_j)$ where $i \neq j$. Suppose $\pi(u, v) = \pi(w, x)$. The edges of G_i are nested with respect to the edges of G_j with respect to σ_G because they do not overlap on any clique involved in the summing process. So (u, v) and (w, x) can lie on the same page in the book embedding. All of the edges in all of the graphs G_i that do not have an endpoint involved in a clique sum are placed in B pages by their original ordering σ^i.

Consider the edges of G(T) that do have an endpoint in a clique involved in a clique-sum forming G(T). Suppose (u, v) is such an edge. Then $u \in V(K)$ for some clique K involved in a clique-sum. Note that v need not necessarily be involved in the vertex set of a clique used in summing. There are $\chi \times (3B)^{\binom{n}{2}}$ colors where $n \leq \lceil \frac{2}{3} \rceil$B by Lemma 4.23 and Lemma 4.20. Also, u received a color $\chi(u)$ from χ choices from the proper vertex coloring of G(T) by [2]. Let $\pi(u, v)$ be the page assigned to the color pair $(\chi(u), QueHue(K))$, where K is the lexicographically smallest clique for which $(u, v) \in E(K)$. Note the clique color of K may have been adjusted in the course of the proof. It remains to be shown this edge assignment produces a book embedding.

Suppose $(u, v) \in E(K)$ and $(u', v') \in E(K')$ were embedded on the same page. Since $\chi(u) = \chi(u')$, it is true that (u, u') is not an edge in G(T), and hence could not be in either clique K or K'. Thus K and K' are distinct cliques. Since $QueHue(K) = QueHue(K')$, it is true that K and K' are nested by Lemma 4.20. Thus, the vertex set $V(K) \setminus V(K')$ lies entirely within the interval created by two distinct consecutive vertices of $V(K') \setminus V(K)$, or vice versa. So either $\sigma_G(u) < \sigma_G(u') < \sigma_G(v') < \sigma_G(v)$ or $\sigma_G(u') < \sigma_G(u) < \sigma_G(v) < \sigma_G(v')$. Since (u, v) and (u', v') are nested, the book embedding is completed. $\square$

5. The Main Theorem Revisited

We return to Theorem 1.2, which states that for every minor-closed class of graphs, other than the class of all graphs, there is a number k such that every member of the class can be embedded in a book with k pages.

Proof of Theorem 1.2. Let $\mathcal{C}$ be a minor-closed class of graphs other than the class of all graphs, and let H be a member of $\mathcal{C}$. Let $H = \bigoplus H_i$ be the decomposition guaranteed by Robertson and Seymour [10]. Recall that each graph H_i is "almost" embedded in a surface Σ_i of genus g_i. Let V_i be a set of apex vertices of H_i and let $H_i' = H_i - V_i$. Note that there are at most $V(\mathcal{C})$ vertices in each V_i. Let R_i be a set of r-rounds of H_i and let $E'(R_i)$ denote the set of all cap edges of each r-round in R_i. Note that there are at most $R(\mathcal{C})$ r-rounds in each R_i. Moreover, the depth of any one of these rings is at most $\rho(\mathcal{C})$. The definitions of each of these components were previously discussed on pages 17 and on 9. Denote $H_i'' = H_i' - E'(R_i)$. Now H_i'' can be embedded in a surface Σ_i. The decomposition of H into the pieces H_i can be chosen so that either Σ_i is a sphere or the embedding of H_i'' in Σ_i has representativity at least $\varrho(\mathcal{C})$ for each i. Specifically, choose $\varrho(\mathcal{C}) = 3$.

Now, H_i'' is embedded in surface Σ_i of genus $g_i \leq g(\mathcal{C})$. However, the planar-nonplanar decomposition algorithm is applied to $(H_i'')^{bct}$, which is embedded in the surface Σ_i^c which has genus g_i^c. Because H_i'' is a subgraph of $(H_i'')^{bct}$, by [7], H_i'' requires no more than $\zeta(g_i^c)$ pages, where $\zeta(g_i^c) = O(g_i^c)$.

By Theorem 3.18, no more than $4|R_i|$ pages are needed to embed the cap edges $E'(R_i)$ of all the r-rounds in R_i, and $|R_i| \leq R(\mathcal{C})$. Thus, the book thickness of H_i' is at most $\zeta(g_i^c) + 4R(\mathcal{C})$. By Lemma 3.9, $|V_i| \leq V(\mathcal{C})$ pages are needed to embed the apex vertices. Thus, the book thickness of H_i is at most $\zeta(g_i^c) + 4R(\mathcal{C}) + V(\mathcal{C})$.

By Theorem 4.26, the book thickness of the clique-sum tree $H = \bigoplus H_i$ is no more than

$$k = \chi \times \left(3[\zeta(g_i^c) + 4R(\mathfrak{C}) + V(\mathcal{C})]\right)^{\binom{n}{2}} + \zeta(g_i^c) + 4R(\mathfrak{C}) + V(\mathcal{C}))$$

where $n \leq \lceil \frac{2}{3} \rceil (\zeta(g_i^c) + 4R(\mathfrak{C}) + V(\mathcal{C})$.

□

6. Subdivisions and Book Embeddings

This section is devoted to the study of subdivisions of a complete graph K_n and a complete bipartite graph $K_{n,n}$ with regard to book thickness. An edge of a graph is *subdivided* if it is replaced by a path of length at least 2 that has the same endpoints. A *subdivision* of a graph G is a graph resulting from subdividing some edges of G. If G is a graph, denote the graph obtained by subdividing every edge of G exactly n times by $sub_n(G)$. This is equivalent to replacing every edge of G by a path of length $n+1$.

Recall from Definition 2 that (u, v) and (u', v') are locked when $\sigma(u) < \sigma(u') < \sigma(v) < \sigma(v')$ or $\sigma(u') < \sigma(u) < \sigma(v') < \sigma(v)$. When two edges are not locked, we said they were nested. In Figure 1.2, locked edges are depicted on the left, and nested edges are depicted on the right. Now we want to look at the nested edges and differentiate between two types of nesting.

Definition 35. *If (G, σ) is an ordered graph, two edges (u, v) and (u', v') are* nested in *when $\sigma(u) < \sigma(u') < \sigma(v') < \sigma(v)$ or $\sigma(u') < \sigma(u) < \sigma(v) < \sigma(v')$. Two edges (u, v) and (u', v') are* nested out *when $\sigma(u) < \sigma(v) < \sigma(u') < \sigma(v')$ or $\sigma(u') < \sigma(v') < \sigma(u) < \sigma(v)$.*

In Figure 1.2, edges (u_1, v_1) and (u_2, v_2) are nested in, where as edges (u_1, v_1) and (u_3, v_3) are nested out.

Proposition 6.27. *If G is a simple, outerplane graph with $|V(G)| = n$ where $n \geq 2$, then $|E(G)| \leq 2n - 3$.*

Proof. If $n = 2$ or 3, the conclusion follows immediately. Assume $n \geq 4$. Suppose $G = (V, E)$ is an outerplane graph with $|V(G)| = n$ and $|E(G)| \leq 2n - 3$.

Because G is an outerplane graph, the boundary of the infinite face must be a cycle C. There are n vertices and n edges in C. Triangulate the interior of C to obtain a maximal configuration; maximal in the sense that no additional edges can be added to the graph without violating the outerplanarity of the graph. Denote this graph by $G' = (V, E')$. Note that $|E(G')| = 2n - 3$.

Consider constructing an outerplane graph $G'' = (V'', E'')$ where $|V(G)| = n+1$. Subdivide an edge in C. This increases the number of edges by 1. Now there is a face of G'' that is bounded by a cycle of length 4. Another edge can be added to the interior of this face without violating the outerplanarity of the graph. The resulting graph is maximal and has $2n - 1 = 2(n + 1) - 3$ edges. Induction is complete and the conclusion follows. □

Recall from page 5 that a graph has book thickness one if and only if it is outerplanar.

Corollary 6.28. *If G is a simple graph with n vertices and $BT(G) = 1$, then G has at most $2n - 3$ edges.*

Corollary 6.29. *If G is a simple graph with $|V(G)| = n$ where $n \geq 2$ that can be embedded on B pages, then $|E(G)| \leq n + B(n - 3)$.*

Proof. The n edges between consecutive vertices of the embedding, including an edge between the first and last vertices on the spine, can be placed on any page of the book without interfering with any other edges embedded on that page. So, at most $n - 3$ edges that are not between consecutive vertices can be embedded in a single page. Therefore, at most $B(n - 3)$ edges that are not between consecutive vertices can be embedded in B pages, and so $|E(G)| \leq n + B(n - 3)$. □

Proposition 6.30. *If K_n is a complete graph on n vertices, then there is a subdivision of K_n with book thickness at most* 3.

Proof. Let K_n be a complete graph on n vertices and let B be a 3-page book. Place the vertices of K_n on the spine of the B in any order. If (v_i, v_j) is an edge of K_n where $i < j$, subdivide this edge twice and denote the path $P(v_i, v_j) = (v_i, v_{ij}^1, v_{ij}^2, v_j)$. Place each vertex v_{ij}^1 next to v_i on the spine of B and each vertex v_{ij}^2 next to v_j on the spine of B. This is done in sequence for $1 \leq i < j \leq n$. All edges of the form (v_i, v_{ij}^1) and (v_{ij}^2, v_j) are assigned to one page of B.

It remains to embed a matching of v_{ij}^1 and v_{ij}^2 for every $1 \leq i < j \leq n$. Note that this matching can be embedded in the plane formed by the remaining two pages of B. Moreover, the edges of the matching can be easily arranged so that each intersects the spine at a finite number of points. The edge (v_{ij}^1, v_{ij}^2) will receive one new subdivision each time it needs to cross the spine of the book. Therefore, there is a subdivision of K_n with book thickness atmost 3.

□

Note that every graph is a subgraph of a clique K_n for some value of n, so the previous proposition can be generalized to the following.

Theorem 6.31. *If G is a graph on n vertices, then there is a subdivision of G with book thickness at most* 3.

An algorithm for embedding clique K_n in a book with $\lceil \frac{n}{2} \rceil$ pages was given in [1], providing us with an upper bound on the book thickness of K_n. In the next theorem, we combine this result with a counting argument which provides a lower bound on the book thickness of K_n to conclude that the $\mathrm{BT}(K_n) = \lceil \frac{n}{2} \rceil$.

Theorem 6.32. *If $n \geq 4$, then $BT(K_n) = \lceil \frac{n}{2} \rceil$.*

Proof. Consider that K_n has

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

edges, which need to be embedded. Set the number of edges in the graph to be less than or equal to the maximum number of edges embedable in a book with B pages,

$$\frac{n(n-1)}{2} \leq n + B(n-3),$$

and solve for B. In doing so, we obtain the following:

$$B(n-3) \geq \frac{n(n-1) - 2n}{2} = \frac{n(n-3)}{2}.$$

Hence, $B \geq \frac{n}{2}$. Since B is an integer, $B \geq \lceil \frac{n}{2} \rceil$. The book thickness of K_n is at most $B \leq \lceil \frac{n}{2} \rceil$ by [1]. Thus, the number of pages B needed to embed K_n is $\lceil \frac{n}{2} \rceil$. □

Let us now show that the previous counting argument will not work to prove that the BT($sub_1(K_n)$) is large. The number of edges in $sub_1(K_n)$ is twice the number of edges in K_n, so

$$|E(sub_1(K_n))| = 2\binom{n}{2} = n(n-1).$$

However, the number of vertices has grown significantly. Since every edge of the clique K_n receives exactly one subdivision, the number of vertices in $sub_1(K_n)$ is

$$|V(sub_1(K_n))| = n + \binom{n}{2} = \frac{n^2+n}{2}.$$

By Corollary 6.29, the number of edges of $sub_1(K_n)$ that can be embedded in a B page book is at most

$$\frac{n(n+1)}{2} + B[\frac{n(n+1)}{2} - 3].$$

Again, set the number of edges in $sub_1(K_n)$ less than or equal to the maximum number of edges which can be embedded on B pages, and solve for B to obtain

$$B \geq \frac{n^2 - 3n}{n^2 + n - 6}.$$

Since B is simply larger than 1, no conclusion on the number of pages needed to embed $sub_1(K_n)$ in a book can be made by this counting argument.

Next we demonstrate that despite the failure of the above counting argument to demonstrate it, the number of pages needed to embed $sub_1(K_n)$ in a book is large when n is large. The proof of this will rely on a bound given in the following variation of Ramsey's Theorem [9].

Theorem 6.33. *There is a function ρ, called the* Ramsey Function*, such that for any m and c, if $n \geq \rho(m,c)$ and K_n is edge-colored by c colors, then it will contain K_m as a monochromatic subgraph.*

The following is a corresponding version of Ramsey's Theorem for complete bipartite graphs.

Theorem 6.34. *There is a function ρ', such that for any m and c, if $n \geq \rho'(m,c)$ and $K_{n,n}$ is edge-colored by c colors, then it contains $K_{m,m}$ as a monochromatic subgraph.*

Consider subdividing every edge of a complete graph exactly once. We will prove that the book thickness of $sub_1(K_N)$ is large when N is large.

Proposition 6.35. *For each B, there exists an integer $N = \rho(B)$ such that for all $n \geq N$, the book thickness of $sub_1(K_n)$ is greater than B.*

Proof. Let $N = \rho(\binom{B}{2}, 5)$. Suppose there is an B-page book embedding of $G = sub_1(K_n)$ where $n \geq N$. Let $\pi : E(G) \to \mathbb{P}$ be the one-to-one correspondence of edges to pages of the book embedding.

Let $\{v_i\}_{k=1}^{n}$ denote the vertices of K_n. Let $P_{ij} = (v_i, w_{ij}, v_j)$ be the path from v_i to v_j with $i < j$ in $sub_1(K_n)$. The path P_{ij} receives the color $\{\pi(e), \pi(e')\}$ where $e = (v_i, w_{ij})$ and $e' = (w_{ij}, v_j)$ are the two edges in the path P_{ij}. There are B choices for each of $\pi(e)$ and $\pi(e')$, so the path P_{ij} is assigned one of $\binom{B}{2}$ colors.

Assign each edge (v_i, v_j) of K_n the color $\{\pi(e), \pi(e')\}$ where e and e' are the two edges in the path P_{ij} in $sub_1(K_n)$. Applying Theorem 6.33 to K_n, colored with $\binom{B}{2}$ colors, we can conclude there must exist a monochromatic K_5. This implies a bichromatic $sub_1(K_5)$, which is impossible since $sub_1(K_5)$ is nonplanar by [8], and therefore has book thickness more than two. This is the contradiction. Therefore, the book thickness of $sub_1(K_n)$ is greater than B. □

It is also true that the book thickness of $sub_1(K_{n,n})$ is large when n is large.

Proposition 6.36. *For each B, there exists an integer $N = \rho'(B)$ such that for all $n \geq N$, the book thickness of $sub_1(K_{n,n})$ is greater than B.*

Proof. Let $N = \rho'(\binom{B}{2}, 3)$. Suppose there is an B-page book embedding of $G = sub_1(K_{n,n})$ where $n \geq N$. Let $\pi : E(G) \to \mathbb{P}$ be the one-to-one correspondence of edges to pages of the book embedding. Let $\{v_i\}_{k=1}^{2n}$ denote the vertices of $K_{n,n}$. Because $K_{n,n}$ is a complete bipartite graph, its vertices can be partitioned into two sets $V_1 = \{v_i\}_{k=1}^{n}$ and $V_2 = \{v_i\}_{k=n+1}^{2n}$. Assume $1 \leq i \leq n$ and $n+1 \leq j \leq 2n$. Let $P_{ij} = (v_i, w_{ij}, v_j)$ be the path from v_i to v_j with $i < j$ in $sub_1(K_{n,n})$. The path P_{ij} receives the color $\{\pi(e), \pi(e')\}$ where $e = (v_i, w_{ij})$ and $e' = (w_{ij}, v_j)$ are the two edges in the path P_{ij}. There are B choices for both $\pi(e)$ and $\pi(e')$, so the path P_{ij} is assigned one of $\binom{B}{2}$ colors.

Assign each edge (v_i, v_j) of $K_{n,n}$ the color $\{\pi(e), \pi(e')\}$ where e and e' are the two edges in the path P_{ij} in $sub_1(K_{n,n})$. Applying Theorem 6.34 to $K_{n,n}$, colored with $\binom{B}{2}$ colors, we can conclude there must exist a monochromatic $K_{3,3}$. This implies a bichromatic $sub_1(K_{3,3})$, which is impossible since $sub_1(K_{3,3})$ is nonplanar by [8], and therefore has book thickness more than two. This is the contradiction. Therefore, the book thickness of $sub_1(K_{n,n})$ is greater than B. □

Theorem 6.37. *For each m and B, there exists an integer N such that for all $n \geq N$, we have $BT(sub_m(K_n)) > B$.*

Proof. The following functions, applied recursively, will be needed. We will commonly refer to $g_k(m, B)$ as g_k in the following discussions.

$$g_1(m, B) = \rho(2^{m+1}B^{m+1}, g_2(m, B)), \quad (6.1)$$

$$g_k(m, B) = 2^{\binom{g_{k+1}(m,B)}{2}} \text{ for each } k \text{ such that } 2 \leq k \leq m, \quad (6.2)$$

$$g_{m+1}(m, B) = 5, \quad (6.3)$$

$$g(m, B) = g_1 \circ g_2 \circ ... \circ g_{m+1}(m, B), \quad (6.4)$$

$$g(i, m, B) = g_i \circ g_{i+1} \circ ... \circ g_{m+1}(m, B). \quad (6.5)$$

We will start with an embedding of $sub_m(K_{g(m,B)})$ into B pages, and then apply a sequence of Ramsey type arguments to conclude that a subdivision of K_5 is embedded on only two pages, which is clearly impossible since K_5 is nonplanar. The proof is quite long, and so, in order to aid in its comprehension, we will emphasize the key points as lemmas.

Suppose $G = sub_m(K_N)$ where $N = g(m, B)$ is embedded in a book with B pages. Let $V(K_N) = \{v_i\}_{i=1}^N$. Let w_{ij}^k where $1 \leq k \leq m$ be the ordered set of m subdivision vertices interior to the path P_{ij} from v_i to v_j where $1 \leq i < j \leq N$ in $sub_m(K_n)$. Let $v_i = w_{ij}^0$ and $v_j = w_{ij}^{m+1}$. Thus P_{ij} can be represented as $(v_i, w_{ij}^1, \ldots, w_{ij}^k, \ldots, w_{ij}^m, v_j)$.

The first step is to color each edge of G with respect to the page it is embedded in and use the sequence of colors of edges to give each path a color. Let $\pi : E(G) \to \mathbb{P}$

where $\mathbb{P}$ is the set of B pages in the book. The page that an edge e is embedded on is $\pi(e)$. Assign to P_{ij} the color $(\pi(e^1), ..., \pi(e^{m+1}))$ where $e^k = (w_{ij}^k, w_{ij}^{k+1})$ is the k^{th} edge from v_i to v_j in the path P_{ij} where $i < j$. Because $\pi(e^k)$ is chosen from B colors, there are B^{m+1} possible color choices for each path P_{ij} in $sub_m(K_n)$.

Begin with any vertex on the spine and traverse the spine in a consistent direction. If $e^k = (w_{ij}^k, w_{ij}^{k+1})$ is an edge and w_{ij}^k follows w_{ij}^{k+1} on the spine, we say that e^k is a *left edge*. If $\sigma(w_{ij}^k) < w_{ij}^{k+1}$, we say that e^k is a *right edge*. Let $X : E(G) \to \{left,\ right\}$. Assign to P_{ij} the color $(X(e^1), ..., X(e^{m+1}))$. Therefore there are 2^{m+1} possible color choices for each P_{ij} in $sub_m(K_N)$.

The final step in coloring a path is to combine the sequence of colors and the sequence of directions. Each edge is given a color $(\pi(e^k), X(e^k))$ which represents the page it is embedded in and the direction it has with respect to the order of its endpoints. Thus each path is given a color $\big((B^1, X^1), \ldots, (B^{m+1}, X^{m+1})\big)$ and there are no more than $2^{m+1}B^{m+1}$ path colors.

Lemma 6.38. *If $sub_m(K_{g_2})$ is a subgraph of $G = sub_m(K_{g_1})$ where $g_1(m, B) = \rho(2^{m+1}B^{m+1}, g_2)$, then $sub_m(K_{g_2})$ inherits an embedding in a book with $m+1$ pages from $sub_m(K_{g_1})$ where all of the edges embedded on a particular page have the same direction.*

Proof. Apply Theorem 6.33 to $G = sub_m(K_{g_1})$ colored with $g_1(m, B)$ colors to conclude there is a monochromatic subgraph $sub_m(K_{g_2})$. In this subgraph every path P_{ij} has the same sequence of pairs $\big((B^1, X^1), \ldots, (B^{m+1}, X^{m+1})\big)$. This represents an embedding of $sub_m(K_{g_2})$ in $m + 1$ pages, denoted $\mathcal{P} = \{P^i\}_{i=1}^m$, and includes the direction of every edge with respect to this embedding.

In particular, on any given page, every edge has the same direction with respect to its endpoints as labeled by the paths. This completes the proof of Lemma 6.38. □

Now, we continue the proof of Theorem 6.37.

Definition 36. *If* $e^k = (w_{ij}^k, w_{ij}^{k+1})$ *is the* k^{th} *edge in the path* P_{ij}*, then* w_{ij}^k *is called the* originating vertex *and* w_{ij}^{k+1} *is the* terminating vertex.

Definition 37. *If* $i < j$ *and* P_{ij} *is a path originating at* v_i *and terminating at* v_j*, then denote the set of all edges on page* $\pi(e^k)$ *of paths beginning with the originating vertex* v_i *by* $org^k(v_i)$ *for* $1 \leq k \leq m+1$*. Likewise, denote the set of edges on page* $\pi(e^k)$ *of paths ending with terminating vertex* v_j *by* $term^k(v_j)$*.*

We will commonly refer to a portion of the spine of a book as a subinterval of the spine. If w_{ix} is the vertex in the path originating with vertex v_i where $\sigma(v_i) \leq \sigma(v)$ for all $v \in \text{org}(v_i)$, and w_{iy} is the vertex in the path originating with vertex v_i where $\sigma(v_i) \geq \sigma(v)$ for all $v \in \text{org}(v_i)$, and w_{iy} is the vertex in the path originating with vertex v_i, then consider the subset of the spine between w_{ix} and w_{iy}.

Consider $\text{org}^k(v_i)$ as an subinterval of the spine determined by the vertex v_i where $\sigma(v_i) \leq \sigma(v)$ for all $v \in \text{org}(v_i)$ and the vertex v_i' where $\sigma(v) \leq \sigma(v_i')$ for all $v \in \text{org}(v_i)$. Define the subinterval determined by $\text{term}^k(v_j)$ in the same way. We desire $\text{org}^k(v_i)$ to lie in distinct subintervals of the spine of the book for each i.

If $\text{org}^k(v_i)$ and $\text{org}^k(v_i')$ do not lie in distinct subintervals of the spine of the book, we say their vertices are *mixed*.

Consider the page B^1. The first edge of every path P_{ij} traversed from v_i to v_j with $i < j$ lies on this page. The next lemma is the first step in an induction process.

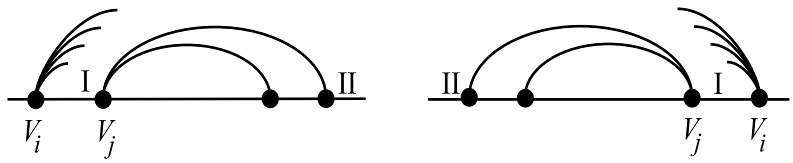

FIGURE 6.21. Possible locations for $\text{term}^1(v_i)$ in right and left cases.

Lemma 6.39. *There is an integer* $g_2(m, B) = 2^{\binom{g_3}{2}}$ *such that* $sub_m(K_{g_2})$ *inherits a* restricted embedding *in a book with* $m+1$ *pages from* $sub_m(K_{g_1})$ *where all of the edges embedded on a particular page have the same direction and the subintervals determined by* $term^1(v_i)$ *are distinct.*

Proof. Consider the first page B^1. By Lemma 6.38, we can assume all edges in this page are either left or right edges. Suppose all edges are right edges. Compare v_i to v_j where $i < j$, so that $\sigma(v_i) < \sigma(v_j)$ on the spine of the embedding. An arbitrary edge on page B^1 is the first edge of some path P_{ij}, and hence has the form (v_i, w_{ij}^1) for some j with $i < j$. See Figure 6.21 for an example.

Fix i and consider w_{ij}^1 for all such choices j. Then the set of terminating vertices w_{ij}^1 of paths which originate with v_i can lie in one of two subintervals of the spine of the book, namely preceding or following v_i. At least half of these terminating vertices w_{ij}^1 must lie in subinterval of the spine of the book preceding or following v_i.

Consider the subinterval of the spine of the book which contains less than or equal to half of the vertices w_{ij}^1 for the fixed i and all j with $i < j$. Follow the paths P_{ij} to the associated terminating vertices v_j. Delete these terminals v_j. We will commonly refer to the set of deleted vertices the *minority set* even though it is possible that exactly half of the vertices may be in it. The vertices remaining will be referred to as the *majority set*. If the majority of the set of terminating vertices on page B^1 precede v_i, then the nested out case arises. If the majority set of terminating vertices on page B^1 follow v_i, then the nested in case arises. For each such comparison, it is possible to lose up to half the size of the subdivided complete graph existing at that stage. Since there is a subgraph $sub_m(K_{g_3})$ of $sub_m(K_n)$, and every path P_{ij} for $i < j$ was subjected to a comparison, we made $\binom{g_3}{2}$ comparisons altogether.

Hence, there is an integer $g_2(m, B) = 2^{\binom{g_3}{2}}$ such that $sub_m(K_{g_2})$ inherits a *restricted embedding* in a book with $m + 1$ pages from $sub_m(K_{g_1})$ where all of the edges embedded on a particular page have the same direction and the subintervals determined by $\mathrm{term}^1(v_i)$ are distinct.

Consider B^1, with left edges. Thus v_j follows v_i on the spine. Again the terminating vertices of edges on page B^1 of paths which originate with v_j can lie in one of two subintervals of the spine of the book. At least half of these terminating vertices w_{ij}^1 must either all follow or all precede v_i. Follow the paths P_{ij} for fixed i and all j with $i < j$ of the set containing the minority of w_{ij}^1 to their associated terminating vertices v_j at the end of the paths. Delete the paths with these terminals v_j. If the majority set of terminating vertices precedes v_i, then the nested out case arises. If the majority set of terminating vertices on page B^1 follows v_i, then the nested in case arises. For each such comparison, we may lose up to half the order of the complete graph that was subdivided. Thus there is a subgraph $sub_m(K_{g_3})$ of $sub_m(K_n)$, and every path P_{ij} for $i < j$ was subjected to a comparison, so $\binom{g_3}{2}$ comparisons were made altogether.

Compare all the vertices v_i with the vertices v_j one at a time, where $i < j$. The final result is a subdivided complete graph on n_3 vertices for which $\mathrm{term}^1(v_i)$ lie in distinct subintervals of the spine of the book on the spine for all i. This completes the proof of Lemma 6.39. □

Now, we continue the proof of Theorem 6.37. The first step of induction is completed with Lemma 6.39. Because the subdivided complete graph on n_3 vertices has the property $\text{term}^1(v_i)$ lie in distinct subintervals of the spine of the book on the spine for all i, it also follows that the subdivided complete graph on n_3 vertices has the property that $\text{org}^2(v_i)$ lie in distinct subintervals of the spine of the book on the spine for all i.

For the purposes of induction, suppose there exists an integer g_{k-2} such that there is a restricted embedding of $sub_m(K_{n_{k-1}})$ in at most $m+1$ pages. All of the edges embedded on a page B^{k-1} have the same direction. Additionally, terminating vertices $\text{term}^{k-1}(v_i)$ lie in distinct subintervals of the spine of the book on the spine for all i. Now we begin the final stage of induction.

Lemma 6.40. *There exists an integer g_k for $2 \leq k \leq m$ such that for $k = 1, \ldots, m-2$ we have a restricted embedding of $sub_m(K_{n_{k-1}})$ in at most $m+1$ pages. All of the edges embedded on a particular page have the same direction. Additionally, the subintervals of the spine of the book of terminating vertices $\text{term}^k(v_i)$ lie in distinct subintervals of the spine of the book on the spine for all i.*

Proof. Consider all edges on page B^k. So if $e = (w_{ij}^{k-1}, w_{ij}^k)$, then $\pi(e^k) = B^k$ and $X(e^k) = \text{right}$. We know $\text{org}^k(v_i)$ lie in distinct subintervals of the spine of the book since they were $\text{term}^{k-1}(v_i)$ on page B^{k-1}. We wish to compare $\text{org}^k(v_i)$ to $\text{org}^k(v_j)$ where the subinterval of the spine of the book of $\text{org}^k(v_i)$ precedes the subinterval of the spine of the book containing $\text{org}^k(v_i)$.

Suppose $\text{org}^k(v_i)$ precedes $\text{org}^k(v_j)$. All edges are directed right, so consider the subinterval of the spine of the book in which $\text{term}^k(v_j)$ lies. Designate subinterval preceding $\text{term}^k(v_j)$ as I, and the subinterval following $\text{term}^k(v_j)$ as II.

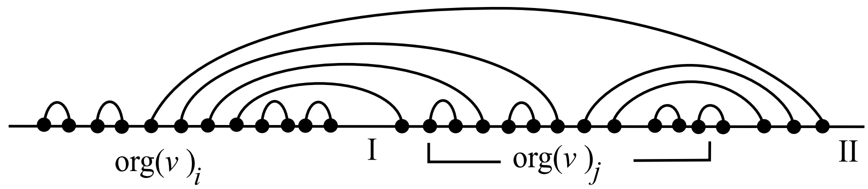

FIGURE 6.22. Interval $\text{term}^k(v_i)$ compared to $\text{term}^k(v_j)$.

If at most half of the $\text{term}^k(v_i)$ are mixed with $\text{term}^k(v_j)$, then follow them to their associated terminals and delete the undesirable paths. See Figure 6.22. At this time, the $\text{term}^k(v_i)$ and $\text{term}^k(v_j)$ are not mixed. Now consider whether most of $\text{term}^k(v_i)$ lie in subinterval I or II. Follow the paths P_{ij} for fixed i and $i < j$ of the set containing the minority of w_{ij}^k to their associated terminals v_j at the end of the paths. Delete these terminals v_j. If the majority set of terminating vertices on page B^k lay in subinterval I, then the nested out case arises. If the majority set of terminating vertices on page B^k lay in subinterval II, then the nested in case arises. For each such comparison, we may lose up to half the size of the subdivided complete graph we currently have. Since we have a subgraph $sub_m(K_{g_k})$ of $sub_m(K_n)$, and every path P_{ij} for $i < j$ was subjected to a comparison, we made $\binom{g_k}{2}$ comparisons altogether.

Hence, there exists an integer g_k for $2 \leq k \leq m$ such that for $k = 1, \ldots, m-2$ we have a restricted embedding of $sub_m(K_{n_{k-1}})$ in at most $m+1$ pages. All of the edges embedded on a particular page have the same direction. Additionally, the subintervals of the spine of the book of terminating vertices $\text{term}^k(v_i)$ lie in distinct subintervals of the spine of the book on the spine for all i.

If at least half of the $\text{org}^k(v_j)$ mix with the $\text{term}^k(v_i)$, then the deletion of v_i locally forces non-overlapping subintervals of $\text{term}^k(v_j)$. See Figure 6.23. The problem which could occur is the deletion of too many vertices v_i via too many repetitions of this case. If this happens, notice the terminating vertices of non-consecutive

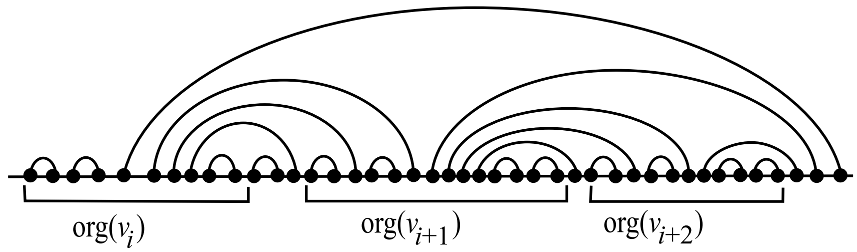

FIGURE 6.23. Case III, subcase II, argument B, right edges.

originating vertex subintervals do not mix. Delete every other originating vertex v_i maintains a complete graph at least half the size of the current one. Consider the location of the term$^k(v_i)$. It precedes or follows term$^k(v_j)$. Choose the majority case, and delete the terminating vertices at the end of the paths of the minority case, thus achieving a mix of nested out and nested in arrangements. There exists an integer g_k for $2 \leq k \leq m$ such that for $k = 1, \ldots, m-2$ we have a restricted embedding of $sub_m(K_{n_{k-1}})$ in at most $m+1$ pages. All of the edges embedded on a particular page have the same direction. Additionally, the subintervals of the spine of the book of terminating vertices term$^k(v_i)$ lie in distinct subintervals of the spine of the book on the spine for all i. This completes the proof of Lemma 6.40. □

Now, we continue the proof of Theorem 6.37. By Lemma 6.39 and Lemma 6.40, we have obtained a highly structured book embedding of a subdivided complete graph of at least size $sub_1(K_5)$. Each edge of every path from any v_i to any v_j with $i < j$ is embedded in a particular page and in a particular direction. The subintervals of terminating vertices term(v_i) are distinct for pages B^1 through B^m. Furthermore, the subintervals of terminating vertices org$^{m+1}(v_i)$ are distinct.

Consider the embedding of $sub_m(K_5)$. On the last page of the embedding, a matching must be made between org$^{m+1}(v_i)$ and v_j for $i < j$. These matching edges are the last edges of each of the paths P_{ij} in $sub_m(K_5)$. Now $sub_1(K_{2,3})$ has

$K_{2,3}$ as a minor, and $K_{2,3}$ is not an outerplanar graph, therefore it cannot be drawn on the last page of the embedding. This is a contradiction which completes the proof of Theorem 6.37.

□

Similarly, there is the corresponding theorem for complete bipartite graphs.

Proposition 6.41. *For each m and B, there exists an integer N such that for all $n \geq N$, we have $BT(sub_m(K_{n,n})) > B$.*

The proof of this is similar, and therefore, we omit it.

7. Conclusion

We investigated book embeddings of subdivided cliques. Although it was previously known that the $\mathrm{BT}(K_n) \leq \lceil \frac{n}{2} \rceil$ for $n \geq 4$, we still provided a counting argument demonstrating a lower bound to be $\frac{n}{2}$. A similar counting argument was shown not to work on $sub_1(K_n)$. We proved that for every n there is a subdivision of a clique K_n which has book thickness at most 3. The proof that $\mathrm{BT}(sub_1(K_n))$ requires a large book for embedding relied on Ramsey's Theorem. It assigned colors to the edges of a clique with respect to the embedding of the edges of $sub_1(K_n)$ on a book with R pages for the purpose of deriving a contradiction. A similar proof worked for a complete bipartite graph.

We proved a bounded number of subdivisions does not significantly reduce the book thickness of a clique. We obtain a highly structured embedding of a subdivided clique via Ramsey-type arguments, and a recursive look at each of the pages of the embedding. The contradiction came from the impossible one page embedding of a matching which included the nonplanar graph $K_{2,3}$ on the last page of the embedding. A similar proof worked for a complete bipartite graph.

We proved that any member of a minor closed class of graphs, other than the class of all graphs, can be embedded in a book with thickness that depends only on the class. Separate arguments concerning surface embeddings, apex vertices, clique-summing, tree-width and r-rounds came together to prove the theorem. The r-rounds needed most of the work, which relied heavily on Heath and Istrail's work on book embeddings.

Finally, we give a few open problems for readers interested in this subject. Given a book with n pages, characterize the graphs can be embedded in it. Given an arbitrary graph, what book provides the optimal embedding? A description of clique-like graphs requiring large books for embedding would be helpful in completing the question of book embeddings of an arbitrary graph.

References

[1] F. R. K. Chung, F. T. Leighton, and A. L. Rosenberg, *Embedding graphs in books: a layout problem with application to VLSI design*, SIAM J. Alg. Disc. Meth. **8** (1987), 33–58.

[2] M. Devos, G. Ding, B. Oporowski, B. Reed, D. P. Sanders, P. Seymour, and D. Vertigan, *Excluding any graph as a minor allows a low tree-width* 2 *coloring.*, submitted.

[3] J. Dittman, *Unavoidable minors of graphs of large type.*, Ph.D. thesis, LSU-Baton Rouge, 1997.

[4] S. Even and A. Itai, *Queues, stacks and graphs*, Theory of Machines and Computations (1971), 71–76.

[5] J. Gross and T. Tucker (eds.), *Topological graph theory*, Wiley-Interscience, New York, 1987.

[6] L. Heath and S. Istrail, *The pagenumber of genus g graph is* $O(g)$, J. Assoc. Computing Machinery **39** (1992), 479–501.

[7] L. S. Heath, *Algorithms for embedding graphs in books*, Ph.D. thesis, UNC-Chapel Hill, 1986.

[8] K. Kuratowski, *Sur le problème des courbes gauches en topologie*, Fund. Math. **15** (1930), 271–283.

[9] J. Nesetril and V. Rodl (eds.), *Mathematics of Ramsey theory*, Springer Verlag, Berlin, 1990.

[10] N. Robertson and P. D. Seymour, *Graph-minors I–XX*, preprints.

[11] ______, *Graph minors. VIII. A Kuratowski theorem for general surfaces*, J. Combin. Theory Ser. B **48** (1990), 255–288.

[12] M. Yannakakis, *Four pages are necessary and sufficient for planar graphs*, Proc. 18th Ann. ACM Symp. on Theory of Comp. (1986), 104–108.

Vita

Robin Blankenship was born on August 25, 1971, in Richlands, Virginia. She finished her undergraduate studies at East Tennessee State University at Johnson City May 1992. She earned a master of arts degree in mathematics from University of North Carolina at Wilmington in December 1994. She earned a master of science degree in mathematics from Louisiana State University in May 1997. She is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2003.