2013

# Multiplicity formulas for perverse coherent sheaves on the nilpotent cone

Myron Minn-Thu-Aye
*Louisiana State University and Agricultural and Mechanical College*, minnthuaye@gmail.com

# MULTIPLICITY FORMULAS FOR PERVERSE COHERENT SHEAVES ON THE NILPOTENT CONE

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
Myron Minn-Thu-Aye
B.A., Williams College, 2007
M.S., Louisiana State University, 2009
August 2013

# Acknowledgments

First and foremost, I must thank my advisor, Pramod Achar, for the incredible amount of patience, guidance, and inspiration he has provided during my graduate studies. He has taught me much of the mathematics I know, and this dissertation would not be possible without him.

I will be forever grateful for the love and support I received from family and friends throughout this entire process. I thank my my parents, sister, and brother-in-law for providing me with the confidence and determination to pursue my graduate studies. I thank my academic siblings Amber Russell and Laura Rider for their constant friendship, and for teaching me a lot about mathematics and hard work. I thank my fellow graduate students Susan Abernathy and Jesse Taylor for being the greatest friends a budding mathematician could ask for. Last, but certainly not least, I thank my fiancé Matt for supporting me through every stage of this difficult and rewarding endeavor. He has made all the time, thought, and effort that produced this dissertation worthwhile.

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# Abstract

Arinkin and Bezrukavnikov have given the construction of the category of equivariant perverse coherent sheaves on the nilpotent cone of a complex reductive algebraic group. Bezrukavnikov has shown that this category is in fact weakly quasi-hereditary with Andersen–Jantzen sheaves playing a role analogous to that of Verma modules in category $\mathcal{O}$ for a semi-simple Lie algebra. Our goal is to show that the category of perverse coherent sheaves possesses the added structure of a properly stratified category, and to use this structure to give an effective algorithm to compute multiplicities of simple objects in perverse coherent sheaves. The algorithm is developed by studying mixed constructible sheaves and Kazhdan–Lusztig polynomials on the affine Grassmannian, and their relation to certain complexes of coherent sheaves on the nilpotent cone.

# Chapter 1
# Introduction

The following briefly describes the construction of perverse coherent sheaves, and their role in the representation theory of algebraic groups. This provides motivation for the further study of these objects in this dissertation. We also describe work that has guided and inspired the results that follow. More detailed explanations of the appropriate background material can be found in Chapter 2.

Let $G$ be a complex reductive algebraic group, $\mathfrak{g}$ its Lie algebra, and $\mathcal{N} \subset \mathfrak{g}$ the variety consisting of nilpotent elements. Let $\mathcal{D} := \mathrm{D}^{\mathrm{b}}\mathrm{Coh}^{G}(\mathcal{N})$ be the bounded derived category of $G$-equivariant coherent sheaves on $\mathcal{N}$, whose objects are bounded chain complexes of coherent sheaves. Arinkin and Bezrukavnikov [AB] construct a perverse $t$-structure on $\mathcal{D}$ in analogy with the work of Beilinson, Bernstein, and Deligne [BBD] in the case of constructible sheaves. The heart of this $t$-structure, the abelian category of perverse coherent sheaves, is denoted $\mathcal{P}$. As for the case of constructible sheaves, there exists a middle extension functor, or IC functor, which can be used to explicitly construct the simple objects in $\mathcal{P}$. This construction places the simple perverse coherent sheaves in bijection with pairs $(O, L)$ where $O$ is a $G$-orbit on $\mathcal{N}$ and $L$ is an irreducible vector bundle on $O$.

An alternative description of this perverse coherent $t$-structure is provided by Bezrukavnikov [Be1], who shows that it is the $t$-structure associated to a quasi-exceptional set $\{\bar{\nabla}_\lambda \mid \lambda \in \Lambda_+\}$ where $\Lambda_+$ is the set of dominant weights of $G$. Furthermore, it is shown that $\mathcal{P}$ is in fact a weakly quasi-hereditary category, with sets of proper standard objects $\{\bar{\Delta}_\lambda\}$ and proper costandard objects $\{\bar{\nabla}_\lambda\}$, all in bijection with $\Lambda_+$. Each of these sets is in bijection with the simple perverse coherent sheaves. Therefore, this construction proves a bijection, originally conjectured by Lusztig, between the dominant weights and the pairs $(O, L)$ described above. Perverse coherent sheaves have further proven useful in determining the cohomology of tilting modules for quantum groups [Be2], and also provide a Langlands dual description of constructible sheaves on the affine flag variety [Be3].

In this paper, we build upon the weakly quasi-hereditary structure described in [Be1] by showing that $\mathcal{P}$ possesses the richer structure of a properly stratified category, in analogy with Frisk and Mazorchuk's notion of properly stratified algebras [FM]. In this setting, there exist standard objects $\Delta_\lambda$ and costandard objects $\nabla_\lambda$, which are also in bijection with the simple objects. This generalizes the definition of a true quasi-hereditary category, where the standard and proper standard objects coincide, as do the costandards and proper costandards. Although $\mathcal{P}$ itself does not contain enough projectives or injectives, we show that any properly stratified category has subcategories with enough projectives and injectives. Moreover, the projectives and injectives in these subcategories possess filtrations by the objects distinguished above. Any injective object has a filtration by costandards, which in turn have filtrations by proper costandards. Dual results hold relating the projectives, standards, and proper standards.

Our main result is an effective algorithm to compute the multiplicities of standard and costandard objects in injective and projective objects respectively. We are particularly interested in a graded version of $\mathcal{P}$ and our algorithm produces polynomials that encode both the multiplicities of objects in a filtration (given by the coefficients) and the grading degrees in which the objects appear (given by the exponents). A key step involves identifying computations involving coherent sheaves on $\mathcal{N}$ with computations involving constructible sheaves on the affine Grassmannian for the Langlands dual group $\check{G}$ via results of Arkhipov, Bezrukavnikov, and

Ginzburg [ABG], and Achar and Riche [AR2]. Here, we are able to exploit our understanding of the geometry of the affine Grassmannian and make use of Kazhdan–Lusztig polynomials to then construct the polynomials carrying the desired multiplicity information.

By a result analogous to BGG reciprocity, we also obtain the graded multiplicities of simple perverse coherent sheaves in proper standard and proper costandard objects. This allows us to compute Ext-algebras between proper standards or between proper costandards. We give examples of these computations in the cases $G = \mathrm{SL}_2(\mathbb{C})$ and $G = \mathrm{SL}_3(\mathbb{C})$.

It is known from [Be3] that there exists an equivalence $\mathcal{D} \simeq \mathrm{D}^{\mathrm{b}}\mathcal{P}$, which has also been proven in positive characteristic [A]. We give an explicit description of tilting objects in $\mathcal{P}$ and give partial results towards Frisk and Mazorchuk's version of Ringel duality for properly stratified categories in general and for $\mathcal{P}$ in particular. We are able to explain an equivalence between certain subcategories of $\mathcal{D}$ and $\mathrm{D}^{\mathrm{b}}\mathcal{P}$ and hope to generalize this to provide an alternative explanation of the above equivalence via Ringel duality.

# Chapter 2
# Background

In this chapter, we will review the concepts necessary for understanding our main results. The primary topics are the construction of the category of perverse coherent sheaves, and the theory of mixed constructible sheaves on the affine Grassmannian. We will also give a brief overview of properly stratified algebras, which have inspired our definition of a properly stratified category, and suggested the existence of Ringel duality for perverse coherent sheaves. A key result gives an equivalence of categories relating perverse coherent sheaves on the nilpotent cone with mixed sheaves on the affine Grassmannian. Combined with further results on the computation of Kazhdan–Lusztig polynomials on the affine Grassmannian, this will allow us to construct the desired algorithm describing multiplicities for perverse coherent sheaves.

### Perverse Coherent Sheaves

The construction of perverse coherent sheaves, originally known to Deligne, is explained by Arinkin and Bezrukavnikov in [AB], which contains the results found in this section. Those familiar with the classical definition of perverse sheaves will notice a clear analogy with the work of Beilinson, Bernstein, and Deligne [BBD] in the context of constructible sheaves. Although we restrict our attention to the case of equivariant coherent sheaves on a scheme, Arinkin and Bezrukavnikov work more generally with coherent sheaves on a stack.

First, we fix our notation. Let $X$ be a Noetherian scheme carrying the action of an algebraic group $G$ and let $X^{\mathrm{top}}$ be the underlying topological space of $X$ equipped with the Zariski topology. We will write $\mathrm{Coh}^G(X)$ for the category of coherent sheaves on $X$ and $\mathrm{D^b}(\mathrm{Coh}^G(X))$ for its bounded derived category, consisting of bounded chain complexes of coherent sheaves on $X$. We denote by $j : U \to X$ an open inclusion and $i : Z \to X$ a closed inclusion.

One of the main difficulties in defining a perverse $t$-structure on $\mathrm{D^b}(\mathrm{Coh}^G(X))$ is that the pullback functor $j^*$ does not have adjoints. In particular, we can define the functors $j_*$ and $j_!$ but if $\mathcal{F} \in \mathrm{D^b}(\mathrm{Coh}^G(U))$, then $j_*(\mathcal{F})$ is a quasi-coherent sheaf while $j_!(\mathcal{F})$ is a pro-coherent sheaf. These sheaf functors are important in the gluing of $t$-structures, but $j_*(\mathcal{F})$ and $j_!(\mathcal{F})$ can be replaced in this context by certain extensions of $\mathcal{F}$ to $X$.

**Lemma 2.1.** *Let $U \subset X$ be an open, $G$-equivariant subscheme. Then*

*(i) any $\mathcal{F} \in \mathrm{D^b}(\mathrm{Coh}^G(U))$ extends to $\widetilde{\mathcal{F}} \in \mathrm{D^b}(\mathrm{Coh}^G(X))$ such that $\widetilde{\mathcal{F}}|_U \simeq \mathcal{F}$.*

*(ii) for any morphism $f : \mathcal{F} \to \mathcal{G}$ in $\mathrm{D^b}(\mathrm{Coh}^G(U))$, there exist extensions $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ in $\mathrm{D^b}(\mathrm{Coh}^G(X))$ of $\mathcal{F}$ and $\mathcal{G}$, respectively, and a morphism $\widetilde{f} : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$ such that $\widetilde{f}|_U \simeq f$.*

*(iii) for any $\widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2 \in \mathrm{D^b}(\mathrm{Coh}^G(X))$ with an isomorphism $f : \widetilde{\mathcal{F}}_1|_U \to \widetilde{\mathcal{F}}_2|_U$, there exists $\widetilde{\mathcal{F}} \in \mathrm{D^b}(\mathrm{Coh}^G(X))$ and morphisms $f_1 : \widetilde{\mathcal{F}}_1 \to \widetilde{\mathcal{F}}$ and $f_2 : \widetilde{\mathcal{F}}_2 \to \widetilde{\mathcal{F}}$ such that $f_1|_U$ and $f_2|_U$ are isomorphisms and $f = (f_2|_U)^{-1} \circ (f_1|_U)$.*

In order to construct appropriate extensions of $\mathcal{F} \in \mathrm{D^b}(\mathrm{Coh}^G(U))$ to replace $j_*(\mathcal{F})$ and $j_!(\mathcal{F})$, we require certain conditions on our perversity $p$. Since we have a stratification of $X$ by $G$-orbits, our perversity is defined on the $G$-orbits in $X$. For any point $x \in X$, we set $p(x) = p(O)$ where $O \subset X$ is the orbit containing $x$.

**Definition 2.2.** A perversity $p$ is said to be monotone if $p(x') \geq p(x)$ if $x' \in \overline{\{x\}}$. It is said to be comonotone if the dual perversity $\bar{p}(x) := -\dim(x) - p(x)$ is monotone.

A perversity is said to be strictly monotone and comonotone if it satisfies the above conditions but where the inequality is made strict.

**Theorem 2.3.** *If $p$ is a perversity that is both monotone and comonotone, then the following subcategories define a $t$-structure on* $\mathrm{D^b}(\mathrm{Coh}^G(X))$*:*

$$\begin{aligned} \mathcal{D}^{p,\leq 0} &= \{\mathcal{F} \in \mathrm{D^b}(\mathrm{Coh}^G(X)) \mid i_x^!(\mathcal{F}) \in \mathcal{D}^{\leq p(x)}(\mathcal{O}_x) \textit{ for all } x \in X\} \\ \mathcal{D}^{p,\geq 0} &= \{\mathcal{F} \in \mathrm{D^b}(\mathrm{Coh}^G(X)) \mid i_x^*(\mathcal{F}) \in \mathcal{D}^{\geq p(x)}(\mathcal{O}_x) \textit{ for all } x \in X\} \end{aligned}$$

*where $i_x : \{x\} \to X$ is the inclusion of the point $x$, and $\mathcal{D}^{\leq p(x)}(\mathcal{O}_x)$ and $\mathcal{D}^{\geq p(x)}(\mathcal{O}_x)$ refer to the standard $t$-structure on the derived category of $\mathcal{O}_x$-modules.*

The heart of this $t$-structure is the category of $G$-equivariant perverse coherent sheaves on $X$, denoted $\mathrm{PCoh}^G(X)$.

If the perversity $p$ is strictly monotone and comonotone, then we are able to give an explicit description of the simple objects in $\mathrm{PCoh}^G(X)$.

**Theorem 2.4.** *For each locally closed $G$-orbit $O$, there exists a functor denoted* $\mathrm{IC}(O,-)$ *from* $\mathrm{PCoh}^G(O)$ *to* $\mathrm{PCoh}^G(X)$ *such that every simple perverse coherent sheaf in* $\mathrm{PCoh}^G(X)$ *is of the form* $\mathrm{IC}(O, L[p(O)])$ *for some irreducible vector bundle $L$ on $O$.*

We are specifically interested in the situation where $G$ acts on the variety of nilpotent elements in its own Lie algebra. Here, the middle perversity defined by $p(\mathcal{O}) = \frac{\dim(\mathcal{O})}{2}$ is strictly monotone and comonotone, and we are able to define the perverse coherent $t$-structure and give a description of the simple objects as above. An alternative construction of the perverse coherent $t$-structure is given by Bezrukavnikov [Be1] and described in Chapter 3.

**Properly Stratified Algebras and Ringel Duality**

Properly stratified algebras were introduced by Frisk and Mazorchuk [FM] as a natural generalization of quasi-hereditary algebras, which commonly appear in the representation theory of algebraic groups. It turns out that the category of equivariant perverse coherent sheaves on the nilpotent cone resembles the module category of a properly stratified algebra, and this structure is vital to our computation of multiplicities. Frisk and Mazorchuk's version of Ringel duality also suggests a possible explanation for a known equivalence of categories involving perverse coherent sheaves.

Let $A$ be a finite-dimensional associative algebra over an algebraically closed field and let $A$-mod represent the category of $A$-modules. Let $I$ be an ordered set of isomorphism classes of simple $A$-modules. We will fix a representative simple module $L_i$ for each $i \in I$, and denote its projective cover by $P_i$. For each $i \in I$, there exists a standard module $\Delta_i$, which is a certain quotient of $P_i$. Similarly, for each $i \in I$, there exists a proper standard module $\bar{\Delta}_i$, which is a certain quotient of $\Delta_i$. We refer the reader to [FM] for details.

**Definition 2.5.** The algebra $A$ is properly stratified if:

(i) the kernel of the surjection $P_i \to \Delta_i$ has a filtration whose subquotients are of the form $\Delta_j$ with $j < i$.

(ii) for each $i \in I$, the standard module $\Delta_i$ has a filtration whose subquotients are isomorphic to $\bar{\Delta}_i$.

**Definition 2.6.** The algebra $A$ is quasi-hereditary if it is properly stratified and $\Delta_i = \bar{\Delta}_i$ for each $i \in I$.

Analogous definitions can be made involving injective objects $I_i$, costandard objects $\nabla_i$ and proper costandard objects $\bar{\nabla}_i$.

Let $\mathcal{F}(\Delta)$, respectively $\mathcal{F}(\bar{\nabla})$, be the full subcategory of $A$-mod consisting of objects with a filtration whose subquotients are of the form $\Delta_i$, respectively $\bar{\nabla}_i$, with $i \in I$.

**Definition 2.7.** A tilting $A$-module is an object in the category $\mathcal{F}(\Delta) \cap \mathcal{F}(\bar{\nabla})$.

Tilting modules share properties in common with both projective and injective modules, and can be used to construct equivalences of categories. In the context of perverse coherent sheaves, we will use tilting objects to construct derived equivalences. The existence and classification of indecomposable tilting modules in $A$-mod can be found in [AHLU].

**Lemma 2.8.** *For each $i \in I$, there exists a unique indecomposable tilting module $T_i$ such that there exists a short exact sequence*

$$0 \to \Delta_i \to T_i \to C \to 0$$

*where $C$ is an object in $\mathcal{F}(\Delta)$.*

The characteristic tilting module in $A$-mod is defined to be $T = \bigoplus_{i \in I} T_i$.

**Definition 2.9.** Let $R = \mathrm{End}(T)$. Then the Ringel duality functor is the functor

$$\mathrm{Hom}(T, -) : A\text{−mod} \to R\text{−mod},$$

which takes any tilting $A$-module to a projective $R$-module.

In Chapters 3 and 4, we define analogous functors on the level of derived categories.

### Mixed Sheaves on the Affine Grassmannian

Our main result, an effective algorithm describing multiplicities in perverse coherent sheaves, relies heavily upon computations that are carried out in a category of mixed sheaves on the affine Grassmannian. In this section, we describe this particular category and its relationship to perverse coherent sheaves. Furthermore, we discuss the computation of Kazhdan–Lusztig polynomials for sheaves on the affine Grassmannian, a key component of our algorithm.

We begin by defining the notion of a mixed sheaf on the affine Grassmannian, then discuss the appropriate setting for our computations. We fix a finite field $\mathbb{F}_q$, and let $\mathfrak{O} = \mathbb{F}_q[[t]]$ and $\mathfrak{K} = \mathbb{F}_q((t))$

**Definition 2.10.** The affine Grassmannian for an algebraic group $G$ is defined as $\mathrm{Gr} := G(\mathfrak{K})/G(\mathfrak{O})$.

There is a $G(\mathfrak{O})$-action on Gr by left multiplication and the orbits of this action are labeled by the dominant weights of $\check{G}$, the Langlands dual group of $G$. The closure ordering on these orbits agrees with the usual ordering on the dominant weights.

Let $l$ be a prime different from char $\mathbb{F}_q$. We will consider constructible $\bar{\mathbb{Q}}_l$-sheaves on Gr, meaning sheaves whose restriction to each orbit is a locally constant sheaf.

**Definition 2.11.** A sheaf $\mathcal{F}$ on Gr is said to be pointwise pure if there exists $w \in \mathbb{Z}$ such that for all $n \geq 1$ and every $x \in \mathrm{Gr}(\mathbb{F}_{q^n})$, the eigenvalues of the Frobenius automorphism on the stalk $\mathcal{F}_x$ are algebraic numbers whose complex conjugates have absolute value $q^{\frac{nw}{2}}$.

**Definition 2.12.** A sheaf $\mathcal{F}$ on Gr is said to be mixed if it has a finite filtration whose subquotients are all pointwise pure.

Let $\mathrm{D}^{\mathrm{b}}_{\mathrm{m,c}}(\mathrm{Gr})$ be the bounded derived category of mixed sheaves on Gr with constructible cohomology. For our purposes, this category turns out to be "too large": we only know of an equivalence between a particular subcategory and a category of perverse coherent sheaves. It should be noted that other categories of mixed constructible sheaves defined in this way have also been found to be "too large" in some sense, for example in the work of Beilinson, Ginzburg, and Soergel [BGS]. Instead, we will work with the following category.

**Definition 2.13.** The category $\mathcal{D}^{\mathrm{mix}}_{G(\mathfrak{O})}(\mathrm{Gr})$ is the derived category of mixed constructible sheaves with respect to the $G(\mathfrak{O})$-orbits whose stalks carry a semisimple action of Frobenius with integral eigenvalues.

We will now precisely describe the relationship between mixed sheaves on the affine Grassmannian and perverse coherent sheaves, using a major result from [AR1] based upon the work of Arkhipov, Bezrukavnikov, and Ginzburg in [ABG]. First, we fix our notation for coherent sheaves. Let $G$ be a complex reductive algebraic group and $\mathcal{N} \subset \mathrm{Lie}(G)$ the variety of nilpotent elements. Define a $\mathbb{G}_m$-action on $\mathcal{N}$ by $(t, x) \mapsto t^2 x$, which commutes with the $G$-action on $\mathcal{N}$. Let $\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}^{G\times\mathbb{G}_m}(\mathcal{N}))$ be the bounded derived category of $G\times\mathbb{G}_m$-equivariant coherent sheaves on $\mathcal{N}$. The full subcategory of perfect complexes, which will be denoted $\mathrm{D}^{\mathrm{b}}_{\mathrm{free}}(\mathrm{Coh}^{G\times\mathbb{G}_m}(\mathcal{N}))$, consists of those complexes that have finite resolutions by free sheaves.

**Theorem 2.14.** *Let* Gr *be the affine Grassmannian for* $\check{G}$. *Then there exists an equivalence of categories*

$$\mathcal{D}^{\mathrm{mix}}_{\check{G}(\mathfrak{O})}(\mathrm{Gr}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{free}}(\mathrm{Coh}^{G\times\mathbb{G}_m}(\mathcal{N})).$$

This equivalence allows us to carry out computations involving mixed sheaves on the affine Grassmannian and translate these into computations involving perverse coherent sheaves on the nilpotent cone. Furthermore, results from [AR2] show that important sheaf functors preserve these stricter conditions on the Frobenius action, allowing us to employ these functors in our computations and maintain the correspondence with perverse coherent sheaves.

**Theorem 2.15.** *Let $\lambda, \mu \in \Lambda_+$ with $\mu < \lambda$. Let $i : \overline{\mathrm{Gr}_\mu} \to \overline{\mathrm{Gr}_\lambda}$ be the inclusion of the closure of one orbit into another and $j : U \to \overline{\mathrm{Gr}_\lambda}$ the complementary open inclusion. Then the functors $i^*$, $i^!$, $i_*$, $j^*$, $j_*$, and $j_!$ all preserve the semisimplicity of the Frobenius action and the integrality of the eigenvalues of Frobenius.*

Kazhdan–Lusztig polynomials are a vital ingredient in our main result as they give us multiplicity information for the IC sheaves, which are also labeled by dominant weights. We denote by $p_\lambda, \mu(t)$ the Kazhdan–Lusztig polynomial giving multiplicities for the restriction of $\mathrm{IC}_\lambda$ to $\mathrm{Gr}_\mu$. Let $\mathcal{M}_\lambda^\mu(t)$ represent Lusztig's $q$-analog of the weight multiplicity, where $\mathcal{M}_\lambda^\mu(1)$ gives the multiplicity of the weight $\mu$ in the irreducible $G$-representation of highest weight $\lambda$. The following theorem, which can be found in [G], gives us an incredibly useful description of Kazhdan–Lusztig polynomials in terms of the $\mathcal{M}_\lambda^\mu(t)$.

**Theorem 2.16.** *Assume $\mu \le \lambda$, otherwise $p_{\lambda,\mu}(t) = \mathcal{M}_\lambda^\mu(t) = 0$. Write $\lambda - \mu$ as a sum of simple roots $\sum_\alpha n_\alpha \alpha$ and let $\mathrm{ht}(\lambda - \mu) = \sum_\alpha n_\alpha$. Then*

$$p_{\lambda,\mu}(t) = t^{\mathrm{ht}(\lambda-\mu)} \mathcal{M}_\lambda^\mu(t^{-1}).$$

Via the above result and Broer's algorithm for Lusztig's $q$-analog of the weight multiplicity [Br, Procedure 1], we are able to compute multiplicities for mixed sheaves on the affine Grassmannian, then go on to construct the appropriate algorithm for multiplicities for perverse coherent sheaves on the nilpotent cone.

# Chapter 3
# Properly Stratified Categories

This chapter introduces the notion of an abstract properly stratified category. We proceed to show that such categories contain subcategories with enough injectives and that these injectives have finite filtrations by particular classes of objects. Our main goal will be to compute multiplicities of various objects in such filtrations. The existence of tilting objects in properly stratified categories will prove to be useful in these computations.

Throughout this section, let $\mathcal{A}$ be a finite length abelian category whose Hom spaces are vector spaces over some field $F$. Suppose $\mathcal{A}$ has an ordered set $\mathcal{I}$ of isomorphism classes of simple objects such that $\{j \in I \mid j < i\}$ is finite for all $i \in \mathcal{I}$. For each $i \in \mathcal{I}$, fix a representative simple object $L_i \in \mathcal{A}$ and let $\mathcal{A}_{\leq i}$, respectively $\mathcal{A}_{<i}$, be the Serre subcategory of $\mathcal{A}$ generated by $\{L_j \mid j \leq i, \text{ respectively } j < i\}$. We recall from [Be1] the definition of a weakly quasi-hereditary category.

**Definition 3.1.** The category $\mathcal{A}$ is weakly quasi-hereditary if there exist objects and maps $L_i \to \bar{\nabla}_i$ and $\bar{\Delta}_i \to L_i$ for each $i \in \mathcal{I}$ such that:

(i) $\mathrm{Coker}(L_i \to \bar{\nabla}_i) \in \mathcal{A}_{<i}$

(ii) $\mathrm{Hom}(L_j, \bar{\nabla}_i) = \mathrm{Ext}^1(L_j, \bar{\nabla}_i) = 0$ for $j < i$

(iii) $\mathrm{Ker}(\bar{\Delta}_i \to L_i) \in \mathcal{A}_{<i}$

(iv) $\mathrm{Hom}(\bar{\Delta}_i, L_j) = \mathrm{Ext}^1(\bar{\Delta}_i, L_j) = 0$ for $j < i$

The $\bar{\nabla}_i$ and $\bar{\Delta}_i$ will be called proper costandard and proper standard objects, respectively. We remark that $\mathcal{A}$ is quasi-hereditary if conditions (ii) and (iv) hold for $j \leq i$.

Inspired by Frisk and Mazorchuk's [FM] work on properly stratified algebras, we generalize the definition of a quasi-hereditary to define the notion of a properly stratified category.

**Definition 3.2.** The category $\mathcal{A}$ is properly stratified if there exist objects and maps $L_i \to \bar{\nabla}_i$, $L_i \to \nabla_i$, $\bar{\Delta}_i \to L_i$, and $\Delta_i \to L_i$ for each $i \in \mathcal{I}$ such that:

(i) $\mathrm{End}(L_i) = F$ for all $i \in \mathcal{I}$.

(ii) $\mathrm{Coker}(L_i \to \bar{\nabla}_i) \in \mathcal{A}_{<i}$ for all $i \in \mathcal{I}$ and $\mathrm{Hom}(L_j, \bar{\nabla}_i) = \mathrm{Ext}^1(L_j, \bar{\nabla}_i) = 0$ for $j < i$.

(iii) $\mathrm{Ker}(\bar{\Delta}_i \to L_i) \in \mathcal{A}_{<i}$ for all $i \in \mathcal{I}$ and $\mathrm{Hom}(\bar{\Delta}_i, L_j) = \mathrm{Ext}^1(\bar{\Delta}_i, L_j) = 0$ for $j < i$.

(iv) For each $i \in \mathcal{I}$, we have that $L_i \to \nabla_i$ is an injective hull and $\Delta_i \to L_i$ is a projective cover in $\mathcal{A}_{\leq i}$.

(v) For each $i \in \mathcal{I}$, the object $\nabla_i$, respectively $\Delta_i$, has a finite filtration whose subquotients are $\bar{\nabla}_i$, respectively $\bar{\Delta}_i$.

(vi) $\mathrm{Ext}^2(\bar{\Delta}_i, \nabla_j) = \mathrm{Ext}^2(\Delta_i, \bar{\nabla}_j) = 0$ for all $i, j \in \mathcal{I}$.

The $\nabla_i$ and $\Delta_i$ will be called costandard and standard objects, respectively. Note that a properly stratified category is a weakly quasi-hereditary category by conditions (ii) and (iii). Also, this definition coincides with the definition of a quasi-hereditary category if $\nabla_i = \bar{\nabla}_i$ and $\Delta_i = \bar{\Delta}_i$ for all $i \in \mathcal{I}$.

Following the work of Beilinson, Ginzburg, and Soergel [BGS], we prove the existence of injectives in particular subcategories of a properly stratified category and that the injectives possess finite filtrations by costandard objects. Dualizing the argument gives the analogous result for projectives and standard objects.

**Proposition 3.3** (cf. [BGS])**.** *If $\mathcal{A}$ is a properly stratified category, then for each $i \in \mathcal{I}$, the subcategory $\mathcal{A}_{\leq i}$ has enough injectives and each injective has a finite costandard filtration (a finite filtration whose subquotients are costandard objects).*

*Proof.* We proceed by induction. Assume that, for some $i \in \mathcal{I}$, the category $\mathcal{A}_{<i}$ has enough injectives and that each of these injectives has a finite costandard filtration. We will prove that $\mathcal{A}_{\leq i}$ also satisfies these properties.

First, we note that since $\mathcal{A}$ is properly stratified, the simple object $L_i$ has an injective hull, namely $\nabla_i$, in $\mathcal{A}_{\leq i}$. Now consider a simple object $L_j$ with $j < i$. By induction, we know that $L_j$ has an injective hull $I'_j$ in $\mathcal{A}_{<i}$ with a finite costandard filtration. Let $E = \mathrm{Ext}^1(\nabla_i, I'_j)$ and $\bar{E} = \mathrm{Ext}^1(\bar{\nabla}_i, I'_j)$. Since $\nabla_i$ has a filtration by $\bar{\nabla}_i$, we can choose a map $\nabla_i \twoheadrightarrow \bar{\nabla}_i$, which induces a map $\bar{E} \to E$. This in turn gives rise to an element of $\bar{E}^* \otimes E = \bar{E}^* \otimes \mathrm{Ext}^1(\nabla_i, I'_j) = \mathrm{Ext}^1(\bar{E} \otimes \nabla_i, I'_j)$. Let this element correspond to the extension $I_j$ in the short exact sequence

$$0 \longrightarrow I'_j \longrightarrow I_j \longrightarrow \bar{E} \otimes \nabla_i \longrightarrow 0.$$

We claim that $L_j \to I_j$ is the desired injective hull in $\mathcal{A}_{\leq i}$. Certainly, since $I'_j$ has a finite costandard filtration, we can see that $I_j$ does as well. We will prove that $I_j$ is the injective hull of $L_j$ by showing that, for $k \leq i$,

$$\mathrm{Hom}(L_k, I_j) \simeq \mathrm{Hom}(L_k, I'_j) = \begin{cases} F \text{ if } k = j \\ 0 \text{ if } k \neq j \end{cases} \tag{3.1}$$

$$\mathrm{Ext}^1(L_k, I_j) \simeq \mathrm{Ext}^1(L_k, I'_j) = 0. \tag{3.2}$$

Applying $\mathrm{Hom}(\bar{\nabla}_i, -)$ to the short exact sequence above gives a long exact sequence containing the map $f : \mathrm{Hom}(\bar{\nabla}_i, \bar{E} \otimes \nabla_i) \to \mathrm{Ext}^1(\bar{\nabla}_i, I'_j)$. In fact, we have the following commutative diagram:

$$\begin{tikzcd}
 & E \arrow[dl, "\mathrm{can}"'] \arrow[dr, "\mathrm{id}"] & & \\
\cdots \arrow[r] & \mathrm{Hom}(\bar{\nabla}_i, \bar{E} \otimes \nabla_i) \arrow[rr, "f"'] & & \mathrm{Ext}^1(\bar{\nabla}_i, I'_j) \arrow[r] & \cdots
\end{tikzcd}$$

Since $\mathrm{Hom}(\bar{\nabla}_i, \nabla_i) = F$, we have that can is an isomorphism, so $f$ is an isomorphism. Thus, in the long exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}(\bar{\nabla}_i, I'_j) \longrightarrow \mathrm{Hom}(\bar{\nabla}_i, I_j) \longrightarrow \mathrm{Hom}(\bar{\nabla}_i, \bar{E} \otimes \nabla_i) \\ &\xrightarrow{f} \mathrm{Ext}^1(\bar{\nabla}_i, I'_j) \longrightarrow \mathrm{Ext}^1(\bar{\nabla}_i, I_j) \longrightarrow \mathrm{Ext}^1(\bar{\nabla}_i, \bar{E} \otimes \nabla_i) \longrightarrow \cdots, \end{aligned}$$

we have $\mathrm{Hom}(\bar{\nabla}_i, I'_j) \simeq \mathrm{Hom}(\bar{\nabla}_i, I_j)$ and $\mathrm{Ext}^1(\bar{\nabla}_i, I_j) \to \mathrm{Ext}^1(\bar{\nabla}_i, \bar{E} \otimes \nabla_i)$ is an injection. Since $\nabla_i$ is injective in $\mathcal{A}_{\leq i}$, we have $\mathrm{Ext}^1(\bar{\nabla}_i, \bar{E} \otimes \nabla_i) = 0$, so $\mathrm{Ext}^1(\bar{\nabla}_i, I_j) = 0$ as well.

For $k \leq i$, applying $\mathrm{Hom}(L_k, -)$ to our original short exact sequence gives

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}(L_k, I'_j) \longrightarrow \mathrm{Hom}(L_k, I_j) \longrightarrow \mathrm{Hom}(L_k, E \otimes \nabla_i) \\ &\longrightarrow \mathrm{Ext}^1(L_k, I'_j) \longrightarrow \mathrm{Ext}^1(L_k, I_j) \longrightarrow \mathrm{Ext}^1(L_k, E \otimes \nabla_i) \longrightarrow \cdots \end{aligned}$$

If $k < i$, then by the injectivity of $\nabla_i$ we have $\mathrm{Hom}(L_k, E \otimes \nabla_i) = \mathrm{Ext}^1(L_k, E \otimes \nabla_i) = 0$. In this case, $L_k$ and $I'_j$ are objects in $\mathcal{A}_{<i}$ and $I'_j$ is the injective hull of $L_j$ in that category, so equations (3.1) and (3.2) hold for $k < i$. It only remains to be shown that

$$\mathrm{Hom}(L_i, I_j) = \mathrm{Ext}^1(L_i, I_j) = 0.$$

We begin by taking the short exact sequence $0 \to L_i \to \bar{\nabla}_i \to C \to 0$ and applying both $\mathrm{Hom}(-, I'_j)$ and $\mathrm{Hom}(-, I_j)$ to get

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathrm{Hom}(C, I'_j) & \longrightarrow & \mathrm{Hom}(\bar{\nabla}_i, I'_j) & \longrightarrow & \mathrm{Hom}(L_i, I'_j) & \longrightarrow & \mathrm{Ext}^1(C, I'_j) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathrm{Hom}(C, I_j) & \longrightarrow & \mathrm{Hom}(\bar{\nabla}_i, I_j) & \longrightarrow & \mathrm{Hom}(L_i, I_j) & \longrightarrow & \mathrm{Ext}^1(C, I_j) & \longrightarrow
\end{array}$$

The weakly quasi-hereditary property implies that the cokernel $C$ is in $\mathcal{A}_{<i}$. Then in the exact sequence $0 \to \mathrm{Hom}(C, I'_j) \to \mathrm{Hom}(C, I_j) \to \mathrm{Hom}(C, \bar{E} \otimes \nabla_i)$, the last term vanishes since $\nabla_i$ is the injective hull of $L_i$ in $\mathcal{A}_{\leq i}$. From our arguments above, we have that the second vertical map is an isomorphism and that $\mathrm{Ext}^1(C, I'_j) = \mathrm{Ext}^1(C, I_j) = 0$. Therefore, the third vertical map is an isomorphism and $\mathrm{Hom}(L_i, I_j) = \mathrm{Hom}(L_i, I'_j) = 0$.

Finally, we prove that $\mathrm{Ext}^1(L_i, I_j) = 0$. Applying $\mathrm{Hom}(-, I_j)$ to the short exact sequence $0 \to L_i \to \bar{\nabla}_i \to C \to 0$ gives the exact sequence

$$\mathrm{Ext}^1(\bar{\nabla}_i, I_j) \to \mathrm{Ext}^1(L_i, I_j) \to \mathrm{Ext}^2(C, L_j).$$

First, we have already seen that the term on the left vanishes.

To prove that $\mathrm{Ext}^2(C, I_j) = 0$, it suffices to show that $\mathrm{Ext}^2(L_k, I_j) = 0$ for all $k < i$ since $C \in \mathcal{A}_{<i}$. Applying $\mathrm{Hom}(-, I_j)$ to $0 \to K \to \bar{\Delta}_k \to L_k \to 0$ gives $\mathrm{Ext}^1(K, I_j) \to \mathrm{Ext}^2(L_k, I_j) \to \mathrm{Ext}^2(\bar{\Delta}_k, I_j)$. We know $K \in \mathcal{A}_{<k}$ so $\mathrm{Ext}^1(K, I_j) = 0$ while $\mathrm{Ext}^2(\bar{\Delta}_k, I_j) = 0$ since $I_j$ has a standard filtration and $\mathcal{A}$ is properly stratified. Thus, $\mathrm{Ext}^1(L_i, I_j) = 0$ and $I_j$ is the injective hull of $L_j$ in $\mathcal{A}_{\leq i}$. $\square$

Let $\mathcal{D} = \mathrm{D}^{\mathrm{b}}\mathcal{A}$, the bounded derived category of $\mathcal{A}$, whose objects are bounded chain complexes of objects in $\mathcal{A}$. We have that $\mathcal{D}$ is a triangulated category and for objects $X, Y \in \mathcal{D}$, we write $\mathrm{Hom}^k(X, Y) = \mathrm{Hom}(X, Y[k])$ for $k \in \mathbb{Z}$. For isomorphism classes $\mathcal{X}$ and $\mathcal{Y}$ of objects in $\mathcal{D}$, we introduce the notation $\mathcal{X} * \mathcal{Y}$, which represents the set of all objects $Z \in \mathcal{D}$ such that there exists a distinguished triangle $X \to Z \to Y \to$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

We are mainly concerned with the case where $\mathcal{D}$ possesses an autoequivalence denoted by $X \mapsto X\langle 1\rangle$ for any object $X \in \mathcal{D}$. This allows us to define graded Hom-spaces $\underline{\mathrm{Hom}}^k(X,Y)$ where $\underline{\mathrm{Hom}}^k(X,Y)_{-n} = \mathrm{Hom}^k(X, Y\langle n\rangle)$. In this setting, we define the subcategory $\mathcal{A}_{\leq i}$, respectively $\mathcal{A}_{<i}$, of $\mathcal{A}$ to be the Serre subcategory generated by the objects $L_j\langle n\rangle$ with $n \in \mathbb{Z}$ and $j \leq i$, respectively $j < i$. We obtain the following graded versions of Definition 3.2 and Proposition 3.3.

**Definition 3.4.** If $\mathcal{A}$ possesses objects and morphisms $L_i \to \bar{\nabla}_i$, $L_i \to \nabla_i$, $\bar{\Delta}_i \to L_i$, and $\Delta_i \to L_i$ for each $i \in \mathcal{I}$, we say that $\mathcal{A}$ is graded properly stratified if the following conditions hold:

(i) $\mathrm{End}(L_i) = F$ for all $i \in \mathcal{I}$.

(ii) $\mathrm{Coker}(L_i \to \bar{\nabla}_i) \in \mathcal{A}_{<i}$ for all $i \in \mathcal{I}$ and $\underline{\mathrm{Hom}}(L_j, \bar{\nabla}_i) = \underline{\mathrm{Ext}}^1(L_j, \bar{\nabla}_i) = 0$ for $j < i$.

(iii) $\mathrm{Ker}(\bar{\Delta}_i \to L_i) \in \mathcal{A}_{<i}$ for all $i \in \mathcal{I}$ and $\underline{\mathrm{Hom}}(\bar{\Delta}_i, L_j) = \underline{\mathrm{Ext}}^1(\bar{\Delta}_i, L_j) = 0$ for $j < i$.

(iv) For each $i \in \mathcal{I}$, we have that $L_i \to \nabla_i$ is an injective hull and $\Delta_i \to L_i$ is a projective cover in $\mathcal{A}_{\leq i}$.

(v) For each $i \in \mathcal{I}$, the object $\nabla_i$, respectively $\Delta_i$, has a finite filtration where each subquotient is of the form $\bar{\nabla}_i\langle n\rangle$, respectively $\bar{\Delta}_i\langle n\rangle$, where $n \in \mathbb{Z}$.

(vi) $\underline{\mathrm{Ext}}^2(\bar{\Delta}_i, \nabla_j) = \underline{\mathrm{Ext}}^2(\Delta_i, \bar{\nabla}_j) = 0$ for any $i, j \in \mathcal{I}$.

**Proposition 3.5.** *If $\mathcal{A}$ is graded properly stratified category, then for each $i \in \mathcal{I}$, the subcategory $\mathcal{A}_{\leq i}$ has enough injectives and each injective has a finite filtration whose subquotients are of the form $\nabla_j\langle n\rangle$ where $j \in \mathcal{I}$ and $n \in \mathbb{Z}$.*

From the definition of $\mathcal{D}$, we have that $\mathcal{A}$ is the heart of the standard $t$-structure. However, it is possible to provide an alternative construction of this $t$-structure, which we will describe now.

We define $\mathcal{D}_{\leq i}$, respectively $\mathcal{D}_{<i}$ and $\mathcal{D}_i$, to be the full triangulated subcategory of $\mathcal{D}$ generated by objects of the form $\bar{\nabla}_j\langle n\rangle$ where $j \leq i$, respectively $j < i$ and $j = i$, and $n \in \mathbb{Z}$. Similarly, we define the subcategory $\mathcal{D}^i$ to be the full triangulated subcategory generated by objects of the form $\bar{\Delta}_i\langle n\rangle$ for $n \in \mathbb{Z}$. For $i \in \mathcal{I}$, let $\iota : \mathcal{D}_{<i} \to \mathcal{D}_{\leq i}$ and $\Pi : \mathcal{D}_{\leq i} \to \mathcal{D}_{\leq i}/\mathcal{D}_{<i}$ be the inclusion and projection functor, respectively.

We wish to realize $\mathcal{A}$ as the heart of the $t$-structure associated to a dualizable quasi-exceptional set. It turns out to be the case that the the proper costandard objects form such a set in $\mathcal{D}$ with the proper standard objects forming the dual set.

**Definition 3.6.** An ordered set $\{\bar{\nabla}_i \mid i \in I\}$ of objects in $\mathcal{D}$ forms a quasi-exceptional set if

(i) $\underline{\mathrm{Hom}}^k(\bar{\nabla}_i, \bar{\nabla}_j) = 0$ for $i < j$ and $k \in \mathbb{Z}$.

(ii) $\underline{\mathrm{Hom}}^k(\bar{\nabla}_i, \bar{\nabla}_i) = 0$ for $i \in \mathcal{I}$ and $k < 0$.

A quasi-exceptional set $\{\bar{\nabla}_i \mid i \in \mathcal{I}\}$ is said to be dualizable if there exists a set of objects $\{\bar{\Delta}_i \mid i \in \mathcal{I}\}$ satisfying $\underline{\mathrm{Hom}}^k(\bar{\Delta}_i, \bar{\nabla}_j) = 0$ for $i > j$ and $k \in \mathbb{Z}$ and $\Pi(\bar{\Delta}_i) \simeq \Pi(\bar{\nabla}_i)$ for all $i \in \mathcal{I}$.

We claim that the proper costandard objects form a dualizable quasi-exceptional set in $\mathcal{D}$ whose dual set is the set of proper standard objects. We first discuss the case where the set $\mathcal{I}$ of isomorphism classes of simple objects in $\mathcal{A}$ is finite, then develop the general case.

**Lemma 3.7.** *If $\mathcal{A}$ is graded properly stratified with a finite set $\mathcal{I}$ of isomorphism classes of simple objects, the proper costandard objects $\{\bar{\nabla}_i \mid i \in \mathcal{I}\}$ form a dualizable quasi-exceptional set in $\mathcal{D}$ with dual set the proper standard objects $\{\bar{\Delta}_i \mid i \in \mathcal{I}\}$*

*Proof.* We begin by showing that $\underline{\mathrm{Hom}}^k(\Delta_i, \nabla_j) = 0$ for $i < j$ and $k > 0$. We proceed by induction on $k$. Let $P$ be the projective cover of $\Delta_i$ in $\mathcal{A}$. In the short exact sequence $0 \to K \to P \to \Delta_i \to 0$, we know that $P$ has a finite standard filtration so $K$ must also have a finite standard filtration. Applying $\underline{\mathrm{Hom}}(-, \nabla_j)$ gives

$$\cdots \longrightarrow \underline{\mathrm{Hom}}^{k-1}(K, \nabla_j) \longrightarrow \underline{\mathrm{Hom}}^k(\Delta_i, \nabla_j) \longrightarrow \underline{\mathrm{Hom}}^k(P, \nabla_j) \longrightarrow \cdots .$$

The term on the right vanishes since $P$ is projective and the term on the left vanishes by induction. Therefore, $\underline{\mathrm{Hom}}^k(\Delta_i, \nabla_j) = 0$ for $i < j$ and $k > 0$.

Since $\Delta_i$ has a finite filtration by twists of $\bar{\Delta}_i$, we can write

$$\underline{\mathrm{Hom}}^k(\Delta_i, \nabla_j) = \underline{\mathrm{Hom}}^k(\bar{\Delta}_i, \nabla_j\langle n_1\rangle) * \cdots * \underline{\mathrm{Hom}}^k(\bar{\Delta}_i, \nabla_j\langle n_m\rangle)$$

for some $n_1, \ldots, n_m \in \mathbb{Z}$. Similarly, $\nabla_j$ has a finite filtration by $\bar{\nabla}_j$, so we can write

$$\underline{\mathrm{Hom}}^k(\bar{\Delta}_i, \nabla_j) = \underline{\mathrm{Hom}}^k(\bar{\Delta}_i, \bar{\nabla}_j\langle n_1\rangle) * \cdots * \underline{\mathrm{Hom}}^k(\bar{\Delta}_i, \bar{\nabla}_j\langle n_l\rangle)$$

for some $n_1, \ldots, n_l \in \mathbb{Z}$. Then, by [A, Lemma 2.2], we have $\underline{\mathrm{Hom}}^k(\bar{\Delta}_i, \bar{\nabla}_j) = 0$ for $i < j$ and $k > 0$. If $i < j$, then $\bar{\Delta}_i \in \mathcal{A}_{\leq i} \subset \mathcal{A}_{<j}$ and $\underline{\mathrm{Hom}}(\bar{\Delta}_i, \bar{\nabla}_j) = 0$. We complete our proof by showing that $\underline{\mathrm{Hom}}^k(L_i, \bar{\nabla}_j) = 0$ for $i < j$. Again, we proceed by induction on $k$. Consider the short exact sequence $0 \to K \to \bar{\Delta}_i \to L_i \to 0$, where $K \in \mathcal{A}_{<i}$. Applying $\underline{\mathrm{Hom}}(-, \bar{\nabla}_j)$ gives

$$\cdots \longrightarrow \underline{\mathrm{Hom}}^{k-1}(K, \bar{\nabla}_j) \longrightarrow \underline{\mathrm{Hom}}^k(L_i, \bar{\nabla}_j) \longrightarrow \underline{\mathrm{Hom}}^k(\bar{\Delta}_i, \bar{\nabla}_j) \longrightarrow \cdots .$$

We already know that the term on the right vanishes and the term on the left vanishes by induction. Therefore, we have that $\{\bar{\nabla}_i \mid i \in \mathcal{I}\}$ is a quasi-exceptional set in $\mathcal{D}$. $\square$

**Lemma 3.8.** *If $\mathcal{A}$ is graded properly stratified with a finite set $\mathcal{I}$ of isomorphism classes of simple objects, then $\mathrm{D}^{\mathrm{b}}\mathcal{A}_{\leq i} \to \mathrm{D}^{\mathrm{b}}\mathcal{A}$ is fully faithful.*

*Proof.* By [BBD, Proposition 3.1.16], it is enough to show that $\mathrm{Hom}^m_{\mathcal{D}_{\leq i}}(P, L_j) = 0$ for any projective $P \in \mathcal{A}_{\leq i}$ and $j \leq i$. We know that $P$ has a finite filtration by objects of the form $\Delta_k$ where $k \leq i$. We have

$$\mathrm{Hom}^m_{\mathcal{D}_{\leq i}}(\Delta_k, \bar{\nabla}_j) = \mathrm{Ext}^m_{\mathcal{A}}(\Delta_k, \bar{\nabla}_j) = 0$$

since $\mathcal{D}_{\leq i}$ is a full subcategory of $\mathcal{D} = \mathrm{D}^{\mathrm{b}}\mathcal{A}$. From the short exact sequence $0 \to L_j \to \bar{\nabla}_j \to Q \to 0$ we get the exact sequence

$$\mathrm{Hom}^{m-1}(\Delta_k, Q) \to \mathrm{Hom}^m(\Delta_k, L_j) \to \mathrm{Hom}^m(\Delta_k, \bar{\nabla}_j).$$

Thus, we get that the middle term vanishes, as desired, and the functor above is fully faitfhul. $\square$

**Lemma 3.9.** *If $\mathcal{A}$ is graded properly stratified with an arbitrary set $\mathcal{I}$ of isomorphism classes of simple objects, not necessarily finite, then $\mathrm{D}^{\mathrm{b}}\mathcal{A}_{\leq i} \to \mathrm{D}^{\mathrm{b}}\mathcal{A}$ is fully faithful.*

*Proof.* We claim that $\mathrm{Ext}^k_{\mathcal{A}_{\leq i}}(X, Y) \to \mathrm{Ext}^k_{\mathcal{A}}(X, Y)$ is an isomorphism for all objects $X, Y \in \mathcal{A}_{\leq i}$ and any $k$. Since $\mathcal{A}_{\leq i}$ is a Serre subcategory of $\mathcal{A}$, we have an isomorphism for $k = 1$. By [BBD, Remark 3.1.17], it is enough to show that the map is surjective.

Let $f \in \mathrm{Ext}^k_{\mathcal{A}}(X, Y)$. This gives rise to an extension

$$Y \to Z_1 \to \cdots \to Z_k \to X.$$

Since $k$ is finite, there exists some $j \geq i$ such that $Z_1, \ldots, Z_k \in \mathcal{A}_{\leq j}$. Since $\mathcal{A}_{\leq j}$ has finitely many isomorphism classes of simple objects, we have

$$\mathrm{Ext}^k_{\mathcal{A}_{\leq i}}(X, Y) \tilde{\to} \mathrm{Ext}^k_{\mathcal{A}_{\leq j}}(X, Y) \to \mathrm{Ext}^k_{\mathcal{A}}(X, Y).$$

Since $f$ is in the image of the composition, we have that $\mathrm{Ext}^k_{\mathcal{A}_{\leq i}} \to \mathrm{Ext}^k_{\mathcal{A}}(X, Y)$ is surjective. This completes our proof. □

**Corollary 3.10.** *Lemma 3.7 holds for graded properly stratified categories with an arbitrary set of isomorphism classes of simple objects, not necessarily finite.*

**Proposition 3.11** ([Be1])**.** *If $\mathcal{D} = \mathrm{D}^{\mathrm{b}}\mathcal{A}$, where $\mathcal{A}$ is properly stratified, has an autoequivalence $X \mapsto X\langle 1\rangle$ and $\{\bar{\nabla}_i \mid i \in \mathcal{I}\}$ is a dualizable quasi-exceptional set with dual set $\{\bar{\Delta}_i \mid i \in \mathcal{I}\}$, then the subcategories*

$$\begin{aligned} \mathcal{D}^{\leq 0} &= \{X \in \mathcal{D} \mid \underline{\mathrm{Hom}}(X, \nabla_i[d]) = 0 \text{ for } d < 0\} \\ \mathcal{D}^{\leq 0} &= \{X \in \mathcal{D} \mid \underline{\mathrm{Hom}}(\Delta_i[d], X) = 0 \text{ for } d > 0\} \end{aligned}$$

*define a $t$-structure on $\mathcal{D}$ whose heart is $\mathcal{A}$.*

The $t$-structure in Proposition 3.11 is constructed by induction and gives rise to $t$-structures on the subcategories $\mathcal{D}_{\leq i}$, $\mathcal{D}_{<i}$ and $\mathcal{D}_i$. The hearts of the $t$-structures on $\mathcal{D}_{\leq i}$ and $\mathcal{D}_{<i}$ coincide with $\mathcal{A}_{\leq i}$ and $\mathcal{A}_{<i}$, respectively. We denote the heart of the $t$-structure on $\mathcal{D}_i$ by $\mathcal{A}_i$. The simple objects of $\mathcal{A}_i$ are all of the form $\bar{\nabla}_i\langle n\rangle$ where $n \in \mathbb{Z}$.

From [Be1], we have the following facts that allow us to apply the gluing of $t$-structures from [BBD].

**Proposition 3.12** ([Be1]). *(i) The functor* $\Pi$ *induces equivalences* $\mathcal{D}_i \tilde{\to} \mathcal{D}_{\leq i}/\mathcal{D}_{<i}$ *and* $\mathcal{D}^i \tilde{\to} \mathcal{D}_{\leq i}/\mathcal{D}_{<i}$*.*

*(ii)* $\Pi$ *has a left adjoint* $\Pi^l : \mathcal{D}_{\leq i}/\mathcal{D}_{<i} \to \mathcal{D}^i$ *and a right adjoint* $\Pi^r : \mathcal{D}_{\leq i}/\mathcal{D}_{<i} \to \mathcal{D}_i$*, which provide the inverse equivalences to those in (i).*

*(iii) The inclusion functor* $\iota$ *has left adjoint* $\iota^l$ *and a right adjoint* $\iota^r$

From the construction of the adjoint functors and the properties described above, we obtain the following useful description of the interaction between these functors and the properly stratified structure.

**Lemma 3.13.** *The functors* $\Pi$ *and* $\iota$ *preserve standard, costandard, proper standard, and proper costandard filtrations.* $\Pi^l$ *and* $\iota^l$ *preserve standard and proper standard filtrations while the functors* $\Pi^r$ *and* $\iota^r$ *preserve costandard and proper costandard filtrations.*

Frisk and Mazorchuk [FM] describe the role of tilting modules in the structure of the module category of a properly stratified algebra. We introduce tilting objects in the abstract setting of a properly stratified category and use them to obtain a derived equivalence in analogy with Frisk and Mazorchuk's version of Ringel duality. These constructions make sense in the graded setting as well. We let $\mathcal{F}(\diamondsuit)$ be the set of all objects in a properly stratified category $\mathcal{A}$ with a finite filtration by objects of the form $\diamondsuit_i$ where $i \in \mathcal{I}$ and $\diamondsuit$ is one of $\Delta$, $\nabla$, $\bar{\Delta}$ or $\bar{\nabla}$.

**Definition 3.14.** Let $\mathcal{A}$ be a properly stratified category. An object in $\mathcal{A}$ is tilting if it is contained in $\mathcal{F}(\Delta) \cap \mathcal{F}(\bar{\nabla})$.

In our discussion of tilting objects, it will be useful to provide characterizations of objects that have finite standard filtrations or finite proper costandard filtrations.

**Lemma 3.15.** *An object $X \in \mathcal{A}$ is in $\mathcal{F}(\Delta)$ if and only if* $\mathrm{Ext}^1(X, \bar{\nabla}_i) = 0$ *for all* $i \in \mathcal{I}$.

*Proof.* Let $X$ be in $\mathcal{A}$. We carry out induction on $i$ such that $X \in \mathcal{A}_{\leq i}$. Consider the distinguished triangle

$$\Pi^l\Pi(X) \to X \to \iota\iota^l(X) \to .$$

The term on the left is filtered by $\bar{\nabla}_i$ since $\Pi^l$ maps into the heart of the $t$-structure on $\mathcal{D}^i$. Applying $\mathrm{Hom}(-, \bar{\nabla}_j)$ for any $j \in \mathcal{I}$ gives

$$\mathrm{Hom}(\Pi^l\Pi(X), \bar{\nabla}_j) \to \mathrm{Hom}^1(\iota\iota^l(X), \bar{\nabla}_j) \to \mathrm{Hom}^1(X, \bar{\nabla}_j).$$

The first and third terms vanish, which give $\mathrm{Hom}^1(\iota\iota^r(X), \bar{\nabla}_j) = 0$ for all $j \in \mathcal{I}$. We know that $X$ and $\Pi^l\Pi(X)$ are both in the heart of the $t$-structure on $\mathcal{D}_{\leq i}$ and that $\iota\iota^l(X)$ may have nontrivial cohomology in degrees 0 and $-1$. However, if $H^{-1}(\iota^l(X)) \neq 0$, that would imply that $0 \neq \mathrm{Ext}^1(\iota^l(X), I) = \mathrm{Ext}^1(X, i(I))$ where $I \in \mathcal{A}_{<i}$ is injective. This contradicts our assumption since an injective $I$ is filtered by proper costandard objects and this filtration is preserved by $\iota$. Therefore, $\iota\iota^l(X)$ is contained in $\mathcal{A}_{<i}$ and has a standard filtration by induction.

We already know from the quasi-hereditary property that $\mathrm{Ext}^1(\Pi^l\Pi(X), L_j) = 0$ for $j < i$. We also have

$$\mathrm{Ext}^1(\Pi^l\Pi(X), \bar{\nabla}_i) = \mathrm{Ext}^1(\Pi(X), \Pi(\bar{\nabla}_i)) = \mathrm{Ext}^1(X, \Pi^r\Pi(\bar{\nabla}_i)) = 0$$

by assumption since $\Pi^r\Pi(\bar{\nabla}_i) \simeq \bar{\nabla}_i$. The short exact sequence $0 \to L_i \to \bar{\nabla}_i \to Q \to 0$ gives rise to the exact sequence

$$\mathrm{Hom}(\Pi^l\Pi(X), Q) \to \mathrm{Ext}^1(\Pi^l\Pi(X), L_i) \to \mathrm{Ext}^1(\Pi^l\Pi(X), \bar{\nabla}_i)$$

so we have that the middle term vanishes. Thus, $\Pi^l\Pi(X)$ is projective and has a finite standard filtration. Therefore, $X$ must also have a standard filtration, as desired. □

**Theorem 3.16.** *Let $\mathcal{A}$ be a graded properly stratified category with $I$ the set of isomorphism classes of simple objects. Let $i, j \in \mathcal{I}$ and assume that $j \leq i$. Let $\Pi_{ij}$ : $\mathcal{D}_{\leq i} \to \mathcal{D}_{\leq i}/\mathcal{D}_{\leq j}$ be the projection functor. If $T'$ is a tilting object in the quotient category $\mathcal{A}_{\leq i}/\mathcal{A}_{\leq j}$, then there exists an indecomposable tilting object $T \in \mathcal{A}_{\leq i}/\mathcal{A}_{<i}$ such that $\Pi_{ij}(T) = T'$.*

*Proof.* Let $\Pi = \Pi_{ij}$ and let $\Pi^l$ be its left adjoint. Since $T'$ is tilting, it possesses a finite standard filtration, which is preserved by the functor $\Pi^l$. In particular, $\Pi^l(T')$ is filtered by objects of the form $\Delta_k$ where $j < k \leq i$. Let $E = \mathrm{Ext}^1(\Delta_k, \Pi^l(T'))$. Then the canonical element in $E^* \otimes E = \mathrm{Ext}^1(E \otimes \Delta_k, \Pi^l(T'))$ gives rise to the short exact sequence

$$0 \to \Pi^l(T') \to T \to E \otimes \Delta_k \to 0.$$

We claim that $T$ is the desired tilting object in $\mathcal{A}_{\leq i}/\mathcal{A}_{<j}$. By definition, we can see that $T$ is filtered by standard objects.

Applying $\mathrm{Hom}(\Delta_k, -)$ with $j \leq k \leq i$ to this short exact sequence gives the exact sequence

$$\mathrm{Ext}^1(\Delta_k, \Pi^l(T')) \to \mathrm{Ext}^1(\Delta_k, T) \to \mathrm{Ext}^1(\Delta_k, E^{\otimes}\Delta_j).$$

Since $j \leq k \leq i$, by Lemma 3.9, we have that

$$\mathrm{Ext}^1_{\mathcal{A}_{\leq i}}(\Delta_k, E \otimes \Delta_j) \simeq \mathrm{Ext}^1_{\mathcal{A}_{\leq k}}(\Delta_k, E \otimes \Delta_j) = 0$$

by the projectivity of $\Delta_k$ in $\mathcal{A}_{\leq k}$.

Now assume $k > j$. If we write $\Delta_k = \Pi^l(\Delta'_k)$ for some standard objects $\Delta'_k \in \mathcal{A}_{\leq i}/\mathcal{A}_{\leq j}$, then we get

$$\mathrm{Ext}^1(\Pi^l(\Delta'_k), \Pi^l(T')) \simeq \mathrm{Ext}^1(\Delta_k, \Pi(\Pi^l(T'))) = 0$$

since $\Pi(\Pi^l(T')) \simeq T'$, which is tilting. Thus, the term on the left vanishes. If $k = j$, then notice that $\Delta_j$ is already tilting in $\mathcal{A}_{\leq i}/\mathcal{A}_{<j}$ as $\bar{\Delta}_j$ and $\bar{\nabla}_j$ coincide in this category. Therefore, we have that $\mathrm{Ext}^1(\Delta_k, T)$ vanishes for all $j \leq k \leq i$, which proves that $T$ has a finite proper costandard filtration. Thus, $T$ is a tilting object in $\mathcal{A}_{\leq i}/\mathcal{A}_{<j}$. $\square$

In $\mathcal{A}_{\leq i}$, we will call the object $\mathcal{T} = \bigoplus_{j \leq i} T_j$ the characteristic tilting module. Let $R = \mathrm{End}(\mathcal{T})^{\mathrm{op}}$. Then we have a functor

$$\varphi = \mathrm{RHom}(\mathcal{T}, -) : \mathrm{D}^{\mathrm{b}}\mathcal{A}_{\leq i} \to \mathrm{D}^{\mathrm{b}}R\text{-mod}$$

from the bounded derived category of $\mathcal{A}$ to the bounded derived category of finitely-generated $R$-modules. Notice that $\varphi(\mathcal{T}) = R$, a projective $R$-module. Since the direct summands of a projective module are themselves projective, the functor $\varphi$ bears resemblance to Ringel duality as it takes tilting objects to projectives.

Let $\mathrm{Tilt}_{\leq i}$ be the full subcategory of $\mathcal{A}_{\leq i}$ consisting of tilting objects and let $\mathscr{P}(R)$ be the full subcategory of $R$-mod consisting of projective modules. Since $\mathrm{Tilt}_{\leq i}$ is the full subcategory of $\mathcal{A}_{\leq i}$ generated by direct summands of $\mathcal{T}$, we have an equivalence $\mathrm{Tilt}_{\leq i} \xrightarrow{\sim} \mathscr{P}(R)$ by [ARS, §II.2, Proposition 2.1(c)]. This gives rise to the following diagram, where $\mathrm{K}^{\mathrm{b}}(-)$ denotes the bounded homotopy category and the vertical morphisms are the natural maps:

$$\begin{array}{ccc} \mathrm{K^b Tilt}_{\leq i} & \xrightarrow{\sim} & \mathrm{K^b}\mathscr{P}(R) \\ \downarrow & & \downarrow \\ \mathrm{D^b}\mathcal{A}_{\leq i} & & \mathrm{D^b}R\text{-mod} \end{array}$$

The image of $\mathrm{K^b Tilt}_{\leq i}$ in $\mathrm{D^b}\mathcal{A}_{\leq i}$ is the full subcategory consisting of objects that have finite resolutions by tilting objects, denoted $\mathrm{D^b_{ft}}\mathcal{A}_{\leq i}$. Similarly, the image of $\mathrm{K^b}\mathscr{P}(R)$ in $\mathrm{D^b}R$-mod is the full subcategory consisting of objects with a finite projective resolution, denoted $\mathrm{D^b_{fp}}R$-mod. Therefore, we get the following.

**Theorem 3.17.** *The functor $\varphi$ restricts to give $\mathrm{D^b_{ft}}\mathcal{A} \xrightarrow{\sim} \mathrm{D^b_{fp}}R$-mod.*

*Proof.* It suffices to show the natural functors $\mathrm{K^b Tilt}_{\leq i} \to \mathrm{D^b}\mathcal{A}_{\leq i}$ and $\mathrm{K^b}\mathscr{P}(R) \to \mathrm{D^b}R$-mod are fully faithful. Notice that $\mathrm{K^b Tilt}_{\leq i}$ is generated as a triangulated category by the set $\{T_j \mid j \leq i\}$. For any $T, T' \in \{T_j \mid j \leq i\}$, we have

$$\mathrm{Hom}_{\mathrm{K^b Tilt}_{\leq i}}(T, T') = \mathrm{Hom}_{\mathcal{A}_{\leq i}}(T, T')$$

since the inclusion of the full subcategory $\mathrm{Tilt}_{\leq i}$ into $\mathcal{A}_{\leq i}$ is fully faithful. Now consider the map

$$\mathrm{Hom}^k_{\mathrm{K^b Tilt}_{\leq i}}(T, T') \to \mathrm{Ext}^k_{\mathcal{A}_{\leq i}}(T, T')$$

where $k \neq 0$. The Hom-group on the left vanishes since we are considering chain maps between objects concentrated in different degrees. The Ext-group on the right vanishes for $k < 0$ since $T$ and $T'$ are in the heart of the $t$-structure. For $k > 0$, we already have $\mathrm{Ext}^k$-vanishing since $T$ has a finite filtration by standards and $T'$ has a finite filtration by proper costandards. Therefore, the functor $\mathrm{K^b Tilt}_{\leq i} \to \mathrm{D^b}\mathcal{A}_{\leq i}$ is fully faithful by dévissage.

We can show that $\mathrm{K^b}\mathscr{P}(R) \to \mathrm{D^b}R$-mod is fully faithful by a similar argument, where we get higher Ext-vanishing because the objects involved are projective. $\square$

# Chapter 4
# Perverse Coherent Sheaves on the Nilpotent Cone

Let $G$ be a reductive algebraic group over a field $\mathbb{K}$, which is either $\mathbb{C}$ or an algebraically closed field of good characteristic. Let $\Lambda$ be the set of weights of $G$, and let $\Lambda_+ \subset \Lambda$ be the set of dominant weights such that $B$ is a negative Borel with respect to $\Lambda_+$.

We begin by fixing notation for the representation theory of $G$. For more background and further details, we refer the reader to [J]. We denote by $V_\lambda$, respectively $M_\lambda$ and $N_\lambda$, the irreducible $G$-module, respectively Weyl module and dual Weyl module, of highest weight $\lambda \in \Lambda_+$. The category of finite-dimensional $G$-modules is in fact quasi-hereditary with the Weyl modules and dual Weyl modules playing the roles of the standard and costandard objects, respectively. In this context, following established conventions, we refer to a standard filtration as a Weyl filtration and a costandard filtration as a good filtration. A tilting $G$-module is an object with both a Weyl filtration and a good filtration and we denote the indecomposable tilting module of highest weight $\lambda$ by $T_\lambda$. Note that for $\mathbb{K} = \mathbb{C}$, the objects $V_\lambda$, $M_\lambda$, $N_\lambda$ and $T_\lambda$ coincide for each $\lambda \in \Lambda_+$.

Now let $\mathfrak{g}$ be the Lie algebra of $G$ and let $\mathcal{N} \subset \mathfrak{g}$ be the subvariety formed by the nilpotent elements. Let $\mathrm{D^bCoh}^G(\mathcal{N})$ be the bounded derived category of $G$-equivariant coherent sheaves on $\mathcal{N}$. In the case $\mathbb{K} = \mathbb{C}$, Bezrukavnikov [Be1] constructs the abelian subcategory of perverse coherent sheaves in via the $t$-structure associated to a quasi-exceptional set and shows that it is weakly quasi-hereditary. Achar [A] proves the same result in the graded setting in positive characteristic.

Our aim is to show that the category of perverse coherent sheaves is in fact properly stratified.

To describe the objects that give rise to the weakly quasi-hereditary structure of $\mathrm{D^bCoh}^G(\mathcal{N})$, we recall the Springer resolution $\mu : \widetilde{\mathcal{N}} \to \mathcal{N}$, where $\widetilde{\mathcal{N}} = T^*(G/B)$, the cotangent bundle of the flag variety. If we let $U \subset B$ be the unipotent radical and $\mathfrak{u}$ its Lie algebra, we can also write $\widetilde{\mathcal{N}} = G \times^B \mathfrak{u}$. We denote by $\pi : \widetilde{\mathcal{N}} \to G/B$ the projection map. Each $\lambda \in \Lambda$ gives rise to a line bundle $\mathcal{L}_\lambda$ on $G/B$, which we can view as a coherent sheaf. We set

$$A_\lambda := R\mu_* \pi^* \mathcal{L}_\lambda.$$

For $\lambda \in \Lambda_+$, the $A_\lambda$ are called Andersen–Jantzen sheaves and are in fact coherent sheaves rather than complexes of sheaves [AJ]. The Andersen–Jantzen sheaves form the set of proper costandard objects in $\mathrm{D^bCoh}^G(\mathcal{N})$ and generate $\mathrm{D^bCoh}^G(\mathcal{N})$ as a triangulated category [Be1].

To obtain a grading, we introduce an action of $\mathbb{G}_m$ on $\mathcal{N}$ by $(t, x) \mapsto t^2x$, which commutes with the $G$-action and gives rise to a $G \times \mathbb{G}_m$-action on $\mathcal{N}$. Let $\mathcal{D} = \mathrm{D^bCoh}^{G\times\mathbb{G}_m}(\mathcal{N})$ be the bounded derived category of $G \times \mathbb{G}_m$-equivariant coherent sheaves on $\mathcal{N}$. Given an object $X \in \mathcal{D}$, we denote a twist of the $\mathbb{G}_m$ action by $X\langle n\rangle$ where $n \in \mathbb{Z}$. With these conventions, the ring $\mathcal{O}_\mathcal{N}$ is graded in even, non-positive degrees. In the construction of the objects $A_\lambda$, we view the corresponding line bundle $\mathcal{L}_\lambda$ as having a trivial $\mathbb{G}_m$-action, which allows us to consider $A_\lambda$ as an object in $\mathcal{D}$.

The Serre–Grothendieck duality functor on $\mathcal{D}$ is defined as $S := \mathrm{Hom}(-, \mathcal{O}_\mathcal{N})$. We follow the conventions used in [A] so that the functor $S$ provides a correspondence between the proper standard and proper costandard labeled by the same dominant weight. Let $w_0$ be the longest element in the Weyl group $W$ of $G$. For

$\lambda \in \Lambda_+$, let $\delta_\lambda$ be the length of the shortest element $w \in W$ satisfying $w\lambda = w_0\lambda$. We set

$$\begin{aligned} \bar{\nabla}_\lambda &= A_\lambda\langle -\delta_\lambda\rangle \\ \bar{\Delta}_\lambda &= A_{w_0\lambda}\langle \delta_\lambda\rangle. \end{aligned}$$

The set $\{\bar{\nabla}_\lambda \mid \lambda \in \Lambda_+\}$ is a quasi-exceptional set in $\mathcal{D}$ with dual set $\{\bar{\Delta}_\lambda \mid \lambda \in \Lambda_+\}$.

We now fix the notation necessary to define apropriate standard and costandard objects in $\mathcal{D}$. Let $\mathcal{D}_{\leq\lambda}$, respectively $\mathcal{D}_{<\lambda}$ and $\mathcal{D}_\lambda$, be the full triangulated subcategory of $\mathcal{D}$ generated by objects of the form $\bar{\nabla}_\mu\langle n\rangle$ with $n \in \mathbb{Z}$ and $\mu \leq \lambda$, respectively $\mu < \lambda$ and $\mu = \lambda$. The categories $\mathcal{D}^{\leq\lambda}$, $\mathcal{D}^{<\lambda}$, and $\mathcal{D}^\lambda$ are defined similarly, but generated by objects of the form $\bar{\Delta}_\mu\langle n\rangle$. From Proposition 3.11, we know that the category $\mathcal{D}$, respectively $\mathcal{D}_{\leq\lambda}$, $\mathcal{D}_{<\lambda}$, and $\mathcal{D}_\lambda$, admits a $t$-structure associated to a quasi-exceptional set with heart $\mathcal{P}$, respectively $\mathcal{P}_{\leq\lambda}$, $\mathcal{P}_{<\lambda}$, and $\mathcal{P}_\lambda$. Since we know that $\mathcal{P}$ is weakly quasi-hereditary with proper costandard objects $\{\bar{\nabla}_\lambda \mid \lambda \in \Lambda_+\}$, we can write a set of representatives of isomorphism classes of simple objects in $\mathcal{P}$ as $\{L_\lambda \mid \lambda \in \Lambda_+\}$, in bijection with the dominant weights.

For $\lambda \in \Lambda_+$, let $\Pi_\lambda : \mathcal{D}_{\leq\lambda} \to \mathcal{D}_{\leq\lambda}/\mathcal{D}_{<\lambda}$ be the projection functor with right adjoint $\Pi_\lambda^r : \mathcal{D}_{\leq\lambda}/\mathcal{D}_{<\lambda} \to \mathcal{D}_\lambda$ and left adjoint $\Pi_\lambda^l : \mathcal{D}_{\leq\lambda}/\mathcal{D}_{<\lambda} \to \mathcal{D}^\lambda$, as described in Proposition 3.12.

The construction of standard and costandard objects in $\mathcal{P}$ relies upon understanding certain classes of free sheaves in this category. We can view any $G$-module $V$ as a $G \times \mathbb{G}_m$-equivariant sheaf on a point, then denote its pull back to $\mathcal{N}$ by $V \otimes \mathcal{O}_\mathcal{N}$. For $\lambda \in \Lambda_+$, consider $M_\lambda \otimes \mathcal{O}_\mathcal{N}$, the pullback of the Weyl module of heighest weight $\lambda$. The following lemma will be useful in explaining the relationship between these free sheaves and the standard and costandard objects. These statements are proved in characteristic 0 in [AR2].

**Lemma 4.1** (cf. [AR2]). *(i) For $\lambda, \mu \in \Lambda_+$, we have* $\mathrm{Hom}^1(M_\lambda \otimes \mathcal{O}_{\mathcal{N}}, \bar{\nabla}_\mu) = 0$.

*(ii)* $\Pi(M_\lambda \otimes \mathcal{O}_{\mathcal{N}})$ *is the projective cover of* $\Pi(A_\lambda)$ *in* $\Pi(\mathcal{P}_\lambda)$.

*Proof.* (i) Let $a$ be the map to a point $a : \mathcal{N} \to \mathrm{pt}$. We have $\mathrm{Coh}^G(\mathrm{pt}) \simeq \mathrm{Rep}(G)$ and $\mathrm{Coh}^{G\times\mathbb{G}_m}(\mathrm{pt}) = \mathrm{gr}\,\mathrm{Rep}(G)$. For $\lambda \in \Lambda_+$, we construct the free sheaf $M_\lambda \otimes \mathcal{O}_{\mathcal{N}}$ by taking a $G \times \mathbb{G}_m$-representation $M_\lambda \in \mathrm{Coh}^{G\times\mathbb{G}_m}(\mathrm{pt})$ and setting $M_\lambda \otimes \mathcal{O}_{\mathcal{N}} = a^* M_\lambda$. By adjunction, we get

$$\mathrm{Hom}^1_{\mathcal{N}}(a^* M_\lambda, \bar{\nabla}_\mu) \simeq \mathrm{Hom}^1_{\mathrm{pt}}(M_\lambda, a_* \bar{\nabla}_\mu) = \mathrm{Hom}^1_{\mathrm{pt}}(M_\lambda, \Gamma(\bar{\nabla}_\mu))$$

where $\Gamma$ is the global sections functor. A non-elementary fact from [BK, Corollary 5.1.13] is that $\Gamma(\bar{\nabla}_\mu) = \Gamma(A_\mu\langle -\delta_m u\rangle)$ has a good filtration in each graded degree in the category of graded $G$-modules. Since $\mathrm{gr}\,\mathrm{Rep}(G)$ is quasi-hereditary, we have that $\mathrm{Hom}^1_{\mathcal{N}}(M_\lambda \otimes \mathcal{O}_{\mathcal{N}}, \bar{\nabla}_\mu)$ vanishes.

(ii) By adjunction and Proposition 3.12, we have that

$$\mathrm{Hom}^i(\Pi(M_\lambda \otimes \mathcal{O}_{\mathcal{N}}), \Pi(A_\lambda)) \simeq \mathrm{Hom}^i(M_\lambda \otimes \mathcal{O}_{\mathcal{N}}, \Pi^r \Pi(A_\lambda)) \simeq \mathrm{Hom}^i(M_\lambda \otimes \mathcal{O}_{\mathcal{N}}, A_\lambda),$$

which vanishes for $i = 1$ by the argument in (i). We know that $\Pi(A_\lambda)$ is the unique simple object in $\Pi(\mathcal{P}_\lambda)$, up to shifts in grading, so we have that $\Pi(M_\lambda \otimes \mathcal{O}_{\mathcal{N}})$ is projective in $\Pi(\mathcal{P}_\lambda)$. In the case $i = 0$, we have

$$\mathrm{Hom}(\Pi(M_\lambda \otimes \mathcal{O}_{\mathcal{N}}), \Pi(A_\lambda)) \simeq \mathrm{Hom}_{\mathrm{pt}}(M_\lambda, \Gamma(A_\lambda)) = \mathbb{K}.$$

□

We now define the standard objects $\{\Delta_\lambda\}$ and costandard objects $\{\nabla_\lambda\}$ in $\mathcal{P}$.

**Definition 4.2.** For $\lambda \in \Lambda_+$, let

$$\begin{aligned} \Delta_\lambda &= \Pi^l \Pi(M_\lambda \otimes \mathcal{O}_{\mathcal{N}})\langle \delta_\lambda \rangle \\ \nabla_\lambda &= \Pi^r \Pi(M_\lambda \otimes \mathcal{O}_{\mathcal{N}})\langle -\delta_\lambda \rangle \end{aligned}$$

**Theorem 4.3.** *$\mathcal{P}$ is a properly stratified category whose standard and costandard objects are $\{\Delta_\lambda \mid \lambda \in \Lambda_+\}$ and $\{\nabla_\lambda \mid \lambda \in \Lambda_+\}$, respectively. Given $\lambda \in \Lambda_+$, the object $\nabla_\lambda$ is an injective hull of $L_\lambda$ and has a finite filtration by the objects $\bar{\nabla}_\lambda$.*

*Proof.* We need to show that in $\mathcal{P}_{\leq\lambda}$, the object $\Delta_\lambda$ is a projective cover of $L_\lambda$ with a finite proper standard filtration, and that $\nabla_\lambda$ is an injective hull of $L_\lambda$ with a finite proper costandard filtration, for all $\lambda \in \Lambda_+$.

By definition, $\Delta_\lambda \in \mathcal{D}_\lambda$. Since $\Pi$ and $\Pi^l$ preserve proper standard filtrations, $\Delta_\lambda$ is in fact an object in $\mathcal{P}^\lambda$, a finite length category whose simple objects are all shifts of $\bar{\Delta}_\lambda$. Therefore, $\Delta_\lambda$ has a finite filtration by $\bar{\nabla}_\lambda$, as desired.

From Lemma 4.1, we know that $\Pi(M_\lambda \otimes \mathcal{O}_\mathcal{N})\langle\delta_\lambda\rangle \to \Pi(\bar{\nabla}_\lambda)$ is a projective cover in $\mathcal{P}_{\leq\lambda}/\mathcal{P}_{<\lambda}$. By the equivalence of categories $D_\lambda \to \mathcal{D}_{\leq\lambda}/\mathcal{D}_{<\lambda}$ and the induced $t$-structures on these categories,

$$\Delta_\lambda = \Pi^l\Pi(M_\lambda \otimes \mathcal{O}_\mathcal{N})\langle\delta_\lambda\rangle \to \Pi^l\Pi(\bar{\nabla}_\lambda) \simeq \bar{\Delta}_\lambda$$

is a projective cover in $\mathcal{P}^\lambda$. Recalling the Serre–Grothendieck duality functor $S$, we obtain the following injective hull in $\mathcal{P}_\lambda$:

$$\bar{\nabla}_\lambda = S(\bar{\Delta}_\lambda) \to S\Pi^l\Pi(M_\lambda \otimes \mathcal{O}_\mathcal{N})\langle\delta_\lambda\rangle = \nabla_\lambda.$$

We have that $\bar{\nabla}_\lambda$ is the unique simple socle of $\nabla_\lambda$ in $\mathcal{P}_\lambda$ and, from the weakly quasi-hereditary structure of $\mathcal{P}$, that $L_\lambda$ is the unique simple socle of $\bar{\nabla}_\lambda$ in $\mathcal{P}$. Therefore, $L_\lambda \to \nabla_\lambda$ is our desired injective hull in $\mathcal{P}_{\leq\lambda}$. $\square$

Notice that the objects $M_\lambda \otimes \mathcal{O}_\mathcal{N}$ with $\lambda \in \Lambda_+$ are free $\mathcal{O}_\mathcal{N}$-modules, so they are free, and thus projective, in $\mathcal{D}$. Since $\mathcal{P}$ is properly stratified and each $M_\lambda \otimes \mathcal{O}_\mathcal{N}$ is an object in $\mathcal{P}_{\leq\lambda}$, every $M_\lambda \otimes \mathcal{O}_\mathcal{N}$ certainly has finite length.

**Lemma 4.4.** *For $\lambda \in \Lambda_+$, the free sheaf $M_\lambda \otimes \mathcal{O}_\mathcal{N}$ has a finite standard filtration.*

*Proof.* This statement is a consequence of the arguments and results in Lemma 4.1(i) and Lemma 3.15. □

The Serre–Grothendieck dual of a free sheaf $M_\lambda \otimes \mathcal{O}_\mathcal{N}$ is a free sheaf of the form $N_\lambda \otimes \mathcal{O}_\mathcal{N}$, which possesses a finite costandard filtration since Serre–Grothendieck duality exchanges standard and costandard objects. Since the indecomposable tilting modules $T_\lambda$ in $\mathrm{Rep}(G)$ have both a Weyl filtration and good filtration, we have that the object $T_\lambda \otimes \mathcal{O}_\mathcal{N}$ for $\lambda \in \Lambda_+$ is a tilting object in $\mathcal{P}_{\leq \mu}$ for any $\mu \geq \lambda$.

Let $\mathrm{Tilt}_{\leq\lambda}$ be the full subcategory of $\mathcal{P}_{\leq\lambda}$ consisting of tilting objects and let $\mathrm{Proj}_{\leq\lambda}$ be the full subcategory of $\mathcal{D}_{\leq\lambda}$ consisting of projective objects. The $T_\mu \otimes \mathcal{O}_\mathcal{N}$ with $\mu \leq \lambda$ are indecomposable tilting objects in $\mathcal{P}_{\leq\lambda}$ and projective objects in $\mathcal{D}_{\leq\lambda}$. Therefore, $\mathrm{Tilt}_{\leq\lambda} = \mathrm{Proj}_{\leq\lambda}$. Therefore, for any $\lambda \in \Lambda_+$ we have the following:

$$\begin{tikzcd}
\mathrm{K^b Tilt}_{\leq\lambda} \arrow[r] \arrow[d] & \mathrm{K^b Proj}_{\leq\lambda} \arrow[d] \\
\mathrm{D^b}\mathcal{P}_{\leq\lambda} & \mathcal{D}_{\leq\lambda}
\end{tikzcd}$$

The image of the first vertical map is $\mathrm{D^b_{ft}}\mathcal{P}_{\leq\lambda}$, the full subcategory of $\mathrm{D^b}\mathcal{P}_{\leq\lambda}$ consisting of objects with a finite tilting resolution. The image of the second vertical map is $\mathcal{D}_{\leq\lambda,\mathrm{free}}$, the full subcategory of $\mathcal{D}_{\leq\lambda}$ consisting of objects called perfect complexes, which have a finite resolution by free sheaves. Therefore, we get the following as a particular case of Theorem 3.17.

**Theorem 4.5.** *There is an equivalence of categories $\mathrm{D^b_{ft}}\mathcal{P}_{\leq\lambda} \xrightarrow{\sim} \mathcal{D}_{\leq\lambda,\mathrm{free}}$.*

# Chapter 5
# Mixed Sheaves on the Affine Grassmannian

From now on, we assume that we are working over $\mathbb{K} = \mathbb{C}$. Our main result provides an effective algorithm to compute multiplicities of simple objects in perverse coherent sheaves. Since $\mathcal{P}$ is properly stratified, our goal is to use the multiplicities of costandards and standards in injectives and projectives, respectively, in the categories $\mathcal{P}_{\leq\lambda}$ to determine the multiplicities of simple objects in perverse coherent sheaves. The principal tool in these computations is an equivalence between coherent sheaves on $\mathcal{N}$ and constructible sheaves on the affine Grassmannian for $\check{G}$, the Langlands dual group of $G$.

Fix a prime $p$, let $\mathfrak{O} = \mathbb{F}_p[[t]]$ and let $\mathfrak{K} = \mathbb{F}_p((t))$. Then the affine Grassmannian for $\check{G}$ is $\mathrm{Gr} = \check{G}(\mathfrak{K})/\check{G}(\mathfrak{K})$. The orbits of the $\check{G}(\mathfrak{O})$-action on Gr are in bijection with $\Lambda_+$ so we denote the orbits $\mathrm{Gr}_\lambda$ where $\lambda \in \Lambda_+$. The closure ordering on these orbits coincides with our original ordering on the dominant weights.

Let $\mathrm{D}^{\mathrm{mix}}_{\check{G}(\mathfrak{O})}(\mathrm{Gr})$ be the bounded derived category of mixed sheaves on the affine Grassmannian constructible with respect to the $\check{G}(\mathfrak{O})$-orbits where the Frobenius action is semisimple with integral eigenvalues. It is important to note that this is different from what one might normally think of as the category of mixed perverse sheaves on the affine Grassmannian. We carry out our computations in the Grothendieck group $K\big(\mathrm{D}^{\mathrm{mix}}_{\check{G}(\mathfrak{O})}(\mathrm{Gr})\big)$ using the following conventions: $X\langle n\rangle = X(-\frac{n}{2})$, a Tate twist of $-\frac{n}{2}$, and $[X\langle n\rangle] = t^{\frac{n}{2}}[X]$.

Let $\mathcal{D}_{\mathrm{free}}$ be the full triangulated subcategory of $\mathcal{D}$ consisting of the perfect complexes. A consequence of work by Arkhipov, Bezrukavnikov, and Ginzburg

[ABG] is the following equivalence:

$$\mathrm{D}^{\mathrm{mix}}_{\check{G}(\mathfrak{O})}(\mathrm{Gr}) \xrightarrow{\sim} \mathcal{D}_{\mathrm{free}}.$$

Therefore, we wish to characterize the objects in $\mathcal{D}_{\mathrm{free}}$.

**Proposition 5.1.** *Let $X$ be an object in $\mathcal{D}_{\leq\lambda}$. Then $X$ is a perfect complex if and only if there exists $N \in \mathbb{N}$ such that $\mathrm{Hom}(X, \mathcal{G}[n]) = 0$ for all $\mathcal{G} \in \mathcal{P}_{\leq\lambda}$ and $n > N$.*

*Proof.* Suppose that $X \in \mathcal{D}_{\leq\lambda}$ is perfect. Then $X$ has a finite resolution by free sheaves, say of length $N$. For any $\mathcal{G} \in \mathcal{P}_{\leq\lambda}$, let $L$ represent the length of $\mathcal{G}$. Then $\mathrm{Hom}(X, \mathcal{G}[n]) = \mathrm{Ext}^n(X, \mathcal{G}) = 0$ for $n > N + L$.

Now suppose that there exists $N \in \mathbb{N}$ such that $0 = \mathrm{Hom}(X, \mathcal{G}[n]) = \mathrm{Ext}^n(X, \mathcal{G})$ for all $\mathcal{G} \in \mathcal{P}_{\leq\lambda}$ and all $n > N$. Again, for $\mathcal{G} \in \mathcal{P}_{\leq\lambda}$, let $L$ be the length of $G$. Consider a resolution of $X$ of the form

$$0 \to M_{N+L} \to P_{N+L-1} \to P_{N+L-2} \to \ldots \to P_0 \to X \to 0$$

where $P_i$ is free, and thus projective, for $1 \leq i \leq N+L-1$. We claim that $M_N + L$ is projective. By dimension shifting, we have $\mathrm{Ext}^1(M_{N+L}, \mathcal{G}) \simeq \mathrm{Ext}^{N+L+1}(X, \mathcal{G}) = 0$ so $M_N$ is projective. Since $M_N$ is a graded projective module over a graded local ring, it is in fact free. Therefore, $X$ has a finite resolution by free sheaves and is a perfect complex. □

In particular, we wish to understand how perfect complexes behave under the functor $\Pi$ and its adjoints.

**Definition 5.2.** An object $X$ in $\mathcal{D}_{\leq\lambda}/\mathcal{D}_{<\lambda} \simeq \mathcal{D}_\lambda$ is perfect if there exists $N \in \mathbb{N}$ such that $\mathrm{Hom}(X, \mathcal{G}[n]) = 0$ for all $\mathcal{G} \in \mathcal{P}_\lambda$ and $n > N$.

**Proposition 5.3.** *An object $X$ in $\mathcal{D}_{\leq\lambda}/\mathcal{D}< \lambda \simeq \mathcal{D}_\lambda$ is perfect if and only if there exists a perfect complex $Y$ in $\mathcal{D}_{\leq\lambda}$ such that $\Pi(Y) \simeq X$.*

*Proof.* Suppose $X$ is perfect in $\mathcal{D}_{\leq\lambda}/\mathcal{D}_{<\lambda}$, so $\mathrm{Hom}(X, \mathcal{G}[n]) = 0$ for any $\mathcal{G} \in \mathcal{P}_\lambda$ and high enough values of $n$. Let $Y = \Pi^l(X)$. Then we have

$$\mathrm{Hom}(Y, \mathcal{G}[n]) = \mathrm{Hom}(\Pi^l(X), \mathcal{G}[n]) \simeq \mathrm{Hom}(X, \Pi(\mathcal{G}[n])) = 0$$

for any $\mathcal{G} \in \mathcal{P}_{\leq\lambda}$ since $X$ is perfect and $\Pi$ is $t$-exact. Therefore, $\Pi^l(X)$ is perfect. We also have that $\Pi\Pi^l(X) \simeq X$.

Now suppose that $X = \Pi(Y)$ for some perfect complex $Y$ in $\mathcal{D}_{\leq\lambda}$. Now we have

$$\mathrm{Hom}(X, \Pi(A_\lambda[n])) = \mathrm{Hom}(\Pi(Y), \Pi(A_\lambda[n])) \simeq \mathrm{Hom}(Y, A_\lambda[n]) = 0.$$

The isomorphism is due to the fact that the Andersen–Jantzen sheaves form a quasi-exceptional set and the vanishing is because $Y$ is perfect. □

**Proposition 5.4.** *The functors* $\Pi$*,* $\Pi^r$ *and* $\Pi^l$ *preserve perfect complexes.*

*Proof.* Let $X \in \mathcal{D}_{\leq\lambda}$ be a perfect complex. Then there exists $N \in \mathbb{N}$ such that for $n > N$, we have

$$\mathrm{Hom}(\Pi(X), \Pi(\bar{\nabla}_\lambda)[n]) \simeq \mathrm{Hom}(X, \bar{\nabla}_\lambda[n]) = 0$$

where the isomorphism is by [Be1, Lemma 2(c)]. Since $\Pi(\bar{\nabla}_\lambda)$ is the unique simple object in $\mathcal{P}_\lambda$, we have that $\Pi(X)$ is a perfect complex.

Now let $Y \in \mathcal{D}_{\leq\lambda}/\mathcal{D}_{<\lambda}$ be a perfect complex. Then there exists $N \in \mathbb{N}$ such that for $n > N$ and $\mathcal{G} \in \mathcal{P}_{\leq\lambda}$, we have

$$\mathrm{Hom}(\Pi^l(Y), \mathcal{G}[n]) \simeq \mathrm{Hom}\,(Y, \Pi(\mathcal{G})[n]) = 0$$

since $Y$ is perfect and $\Pi$ is $t$-exact. Thus, $\Pi^l(Y)$ is a perfect complex.

Finally, let $Y \in \mathcal{D}_{\leq\lambda}/\mathcal{D}_{<\lambda}$ be a perfect complex. Then there exists $N \in \mathbb{N}$ such that for $n > N$ and $\mathcal{G} \in \mathcal{P}_{\leq\lambda}$, we have

$$\mathrm{Hom}(\mathcal{G}[n], \Pi^r(Y)) \simeq \mathrm{Hom}(\Pi(\mathcal{G})[n], Y) = 0$$

since $Y$ is the dual of a perfect complex. Therefore, $S\Pi^r(Y)$ is perfect, as is its dual $\Pi^r(Y)$. $\square$

It is known that under the equivalence between perfect complexes and mixed perverse sheaves, the free sheaf $V_\lambda \otimes \mathcal{O}_{\mathcal{N}}$ corresponds to $\mathrm{IC}_\lambda = \mathrm{IC}(\mathrm{Gr}_\lambda)$, which we set to be $\underline{\bar{\mathbb{Q}}_l}_{\overline{\mathrm{Gr}_\lambda}}\langle -n_\lambda \rangle$ where $n_\lambda = \dim \overline{\mathrm{Gr}_\lambda}$, making $\mathrm{IC}_\lambda$ a pure object of weight 0. If we let $j_\lambda : \mathrm{Gr}_\lambda \to \overline{\mathrm{Gr}_\lambda}$ be the inclusion, Achar and Riche [AR2] prove the existence of the functors $j_\lambda^*$, $j_{\lambda *}$ and $j_{\lambda !}$, which we can use to for computations on the affine Grassmannian that will allow us to study objects in $\mathcal{D}$. We have the following commutative diagram

$$\begin{tikzcd}
\mathrm{D}^{\mathrm{mix}}_{\check{G}(\mathfrak{O})}(\overline{\mathrm{Gr}_\lambda}) \arrow[r, "\sim"] \arrow[d, "\Pi"'] & (\mathcal{D}_{\leq \lambda})_{\mathrm{free}} \arrow[d, "j_\lambda^*"] \\
\mathrm{D}^{\mathrm{mix}}_{\check{G}(\mathfrak{O})}(\mathrm{Gr}_\lambda) \arrow[r, "\sim"] & (\mathcal{D}_{< \lambda})_{\mathrm{free}}
\end{tikzcd}$$

and by uniqueness of adjoints, the functors $\Pi_\lambda$, $\Pi_\lambda^r$ and $\Pi_\lambda^l$ correspond to $j_\lambda^*$, $j_{\lambda *}$, and $j_{\lambda !}$ respectively. Thus, for each dominant weight $\lambda$, the object $\nabla_\lambda = \Pi^r\Pi(V_\lambda \otimes \mathcal{O}_{\mathcal{N}})\langle -\delta_\lambda \rangle \in \mathcal{D}$ corresponds to $j_{\lambda *} j_\lambda^* \mathrm{IC}_\lambda \langle -\delta_\lambda \rangle = j_{\lambda *} j_\lambda^* \underline{\bar{\mathbb{Q}}_l}_{\overline{\mathrm{Gr}_\lambda}}\langle -n_\lambda - \delta_\lambda \rangle \in \mathrm{D}^{\mathrm{mix}}_{\check{G}(\mathfrak{O})}(\mathrm{Gr})$.

If $X \in \mathrm{D}^{\mathrm{mix}}_{\check{G}(\mathfrak{O})}(\mathrm{Gr})$ has a finite filtration by objects of the form $\nabla_\lambda$, then we can write the class $[X] \in \mathrm{K}(\mathrm{D}^{\mathrm{mix}}_{\check{G}(\mathfrak{O})}(\mathrm{Gr}))$ as the sum

$$[X] = \sum_{\lambda \in \Lambda_+} r_\lambda(t)[\nabla_\lambda]$$

where $r_\lambda(t) \in \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$. We set $[X : \nabla_\lambda] = r_\lambda(t)$ for each $\lambda \in \Lambda_+$.

We begin by computing $\left[\mathrm{RHom}(\nabla_\lambda, \nabla_\mu)\right]$ for $\lambda, \mu \in \Lambda_+$. Fix the isomorphism class $\left[\underline{\bar{\mathbb{Q}}_l}_{\mathrm{Gr}_\mu}\langle -n_\mu \rangle\right]$ as the basis element in $K\big(\mathrm{D}^{\mathrm{mix}}_{\check{G}(\mathfrak{O})}(\mathrm{Gr}_\mu)\big)$ for each $\mu \in \Lambda_+$. Then we can write

$$\begin{aligned}\big[\mathrm{RHom}(\nabla_\lambda,\nabla_\mu)\big] &= \big[\mathrm{RHom}(j_{\lambda*}j_\lambda^*\mathrm{IC}_\lambda\,,\,j_{\mu*}j_\mu^*\mathrm{IC}_\mu)\big]\\ &= \big[\mathrm{RHom}(j_\mu^*j_{\lambda*}j_\lambda^*\mathrm{IC}_\lambda\,,\,j_\mu^*\mathrm{IC}_\mu)\big].\end{aligned}$$

We have

$$\begin{aligned}\big[j_\mu^*j_{\lambda*}j_\lambda^*\mathrm{IC}_\lambda\big] &= q_1(t)\big[\underline{\bar{\mathbb{Q}}_l}_{\mathrm{Gr}_\mu}\langle -n_\mu\rangle\big]\\ \big[j_\mu^*\mathrm{IC}_\mu\big] &= q_2(t)\big[\underline{\bar{\mathbb{Q}}_l}_{\mathrm{Gr}_\mu}\langle -n_\mu\rangle\big]\end{aligned}$$

for some $q_1(t)$ and $q_2(t)$ in $\mathbb{Z}\big[t^{\frac{1}{2}},t^{-\frac{1}{2}}\big]$. Letting $q_{\lambda,\mu}(t)=q_1(t^{-1})q_2(t)$ gives

$$\begin{aligned}\big[\mathrm{RHom}(\nabla_\lambda,\nabla_\mu)\big] &= q_{\lambda,\mu}(t)\big[\mathrm{RHom}(\underline{\bar{\mathbb{Q}}_l}_{\mathrm{Gr}_\mu}\langle -n_\mu\rangle,\underline{\bar{\mathbb{Q}}_l}_{\mathrm{Gr}_\mu}\langle -n_\mu\rangle)\big]\\ &= q_{\lambda,\mu}(t)\big[H^\bullet(\mathrm{Gr}_\mu)\big].\end{aligned}$$

To determine the polynomials $q_{\lambda,\mu}(t)$, we must also compute $\big[\mathrm{RHom}(\mathrm{IC}_\lambda,\nabla_\mu)\big]$. For any $\lambda,\mu\in\Lambda_+$, let $p_{\lambda,\mu}(t)$ be the corresponding Kazhdan–Lusztig polynomial. Then we can express the restriction of $\mathrm{IC}_\lambda$ to $\mathrm{Gr}_\mu$ in $K\big(\mathrm{D}^{\mathrm{mix}}_{\check{G}(\mathfrak{O})}(\mathrm{Gr}_\mu)\big)$ as

$$\big[\mathrm{IC}_\lambda|_{\mathrm{Gr}_\mu}\big]=p_{\lambda,\mu}(t)\big[\underline{\bar{\mathbb{Q}}_l}_{\mathrm{Gr}_\mu}\langle -n_\mu\rangle\big].$$

Using adjunction properties together with the correspondence between $\nabla_\lambda$ and $\mathrm{IC}_\lambda$ for all $\lambda\in\Lambda_+$ gives

$$\begin{aligned}\big[\mathrm{RHom}(\mathrm{IC}_\lambda,\nabla_\mu)\big] &= \big[\mathrm{RHom}(\mathrm{IC}_\lambda,j_{\mu*}j_\mu^*\underline{\bar{\mathbb{Q}}_l}_{\overline{\mathrm{Gr}_\mu}}\langle -n_\mu-\delta_\mu\rangle)\big]\\ &= \big[\mathrm{RHom}(j_\mu^*\mathrm{IC}_\lambda,j_\mu^*\underline{\bar{\mathbb{Q}}_l}_{\overline{\mathrm{Gr}_\mu}}\langle -n_\mu-\delta_\mu\rangle)\big]\\ &= \big[\mathrm{RHom}(\mathrm{IC}_\lambda|_{\mathrm{Gr}_\mu},\underline{\bar{\mathbb{Q}}_l}_{\mathrm{Gr}_\mu}\langle -n_\mu-\delta_\mu\rangle)\big]\\ &= t^{-\frac{\delta_\mu}{2}}p_{\lambda,\mu}(t^{-1})\big[\mathrm{RHom}(\underline{\bar{\mathbb{Q}}_l}_{\mathrm{Gr}_\mu},\underline{\bar{\mathbb{Q}}_l}_{\mathrm{Gr}_\mu})\big]\\ &= t^{-\frac{\delta_\mu}{2}}p_{\lambda,\mu}(t^{-1})\big[H^\bullet(\mathrm{Gr}_\mu)\big].\end{aligned}$$

Now we have the equation

$$\sum_{\mu\le\nu\le\lambda}t^{\frac{\delta_\nu}{2}}p_{\lambda,\nu}(t)q_{\nu,\mu}(t)=t^{-\frac{\delta_\mu}{2}}p_{\lambda,\mu}(t^{-1}).$$

Since $\{\bar{\nabla}_\lambda \mid \lambda \in \Lambda_+\}$ forms a quasi-exceptional set and each costandard is filtered by proper costandards, we know that $\mathrm{RHom}(\nabla_\lambda, \nabla_\mu) = 0$ for $\lambda < \mu$. Therefore, we have $p_{\lambda,\mu}(t^{-1}) = q_{\lambda,\mu}(t) = 0$ if $\lambda < \mu$. We also have that $p_{\lambda,\mu}(t^) = q_{\lambda,\mu}(t) = 1$ if $\lambda = \mu$.

**Theorem 5.5.** *The equation above can be solved recursively as follows:*

$$q_{\lambda,\mu}(t) = t^{-\frac{\delta_\lambda}{2}} \left( t^{-\frac{\delta_\mu}{2}} p_{\lambda,\mu}(t^{-1}) - \sum_{\mu \le \nu < \lambda} t^{\frac{\delta_\nu}{2}} p_{\lambda,\nu}(t) q_{\nu,\mu}(t) \right).$$

For dominant weights $\mu < \lambda$, let $L_\mu$ have injective hulls $I'_\mu$ in $\mathcal{D}_{<\lambda}$ and $I_\mu$ in $\mathcal{D}_{\le\lambda}$, as constructed in Proposition 3.3. The construction of injectives is inductive and $I_\mu$ arises as an extension $0 \to I'_\mu \to I_\mu \to \mathrm{Ext}^1(\bar{\nabla}_\lambda, I'_\mu) \otimes \nabla_\lambda$. We wish to compute

$$r_{\lambda,\mu}(t) = \big[I_\mu : \nabla_\lambda\big] = \big[\mathrm{Ext}^1(\bar{\nabla}_\lambda, I'_\mu)\big].$$

By taking the expression in $K\big(\mathcal{D}_{\le\lambda}\big)$ corresponding to the extension above and applying $\mathrm{RHom}(\nabla_\lambda, -)$, we get

$$\begin{aligned}
r_{\lambda,\mu}(t)\big[\nabla_\lambda\big] + \big[I'_\mu\big] &= \big[I_\mu\big] \\
r_{\lambda,\mu}(t)\big[\mathrm{RHom}(\bar{\nabla}_\lambda, \nabla_\lambda)\big] + \big[\mathrm{RHom}(\bar{\nabla}_\lambda, I'_\mu)\big] &= \big[\mathrm{RHom}(\bar{\nabla}_\lambda, I_\mu)\big] \\
r_{\lambda,\mu}(t) \cdot 1 + \big[\mathrm{RHom}(\bar{\nabla}_\lambda, I'_\mu)\big] &= \big[\mathrm{RHom}(\bar{\nabla}_\lambda, I_\mu)\big] \\
\big[\mathrm{RHom}(\bar{\nabla}_\lambda, I'_\mu)\big] &= \big[\mathrm{RHom}(\bar{\nabla}_\lambda, I_\mu)\big] - r_{\lambda,\mu}(t)
\end{aligned}$$

To proceed with the computation, we need to write $[\mathrm{RHom}(\bar{\nabla}_\lambda, I_\mu)]$ in terms of $r_{\lambda,\mu}(t)$. In $\mathcal{D}_{\le\lambda}$, for any $\mu, \nu \le \lambda$, we have $\dim \mathrm{RHom}(L_\mu, I_\nu) = \dim \mathrm{RHom}(P_\mu, L_\nu) = 1$ if and only if $\mu = \nu$. Then for any $X \in \mathcal{D}_{\le\lambda}$ and any $\mu \le \lambda$,

$$\begin{aligned}
\big[\mathrm{RHom}(X, I_\mu)\big] &= \big[X : L_\mu\big]\big|_{t^{-1}} \\
\big[\mathrm{RHom}(P_\mu, X)\big] &= \big[X : L_\mu\big].
\end{aligned}$$

**Lemma 5.6.** *Let* $\lambda, \mu \in \Lambda_+$ *with* $\lambda \geq \mu$. *Then* $[\mathrm{RHom}(\bar{\nabla}_\lambda, I_\mu)] = r_{\lambda,\mu}(t^{-1})$.

*Proof.* First, we have that

$$\big[\mathrm{RHom}(\bar{\nabla}_\lambda, I_\mu)\big] = \big[\bar{\nabla}_\lambda : L_\mu\big]\big|_{t^{-1}}.$$

Since each $\nabla_\lambda$ is filtered by $\bar{\nabla}_\lambda$ and $\bar{\nabla}_\lambda \to \nabla_\lambda$ is an injective hull in $\mathcal{P}_\lambda$ we have

$$\begin{aligned} [\nabla_\lambda : L_\mu]\big|_{t^{-1}} &= [\nabla_\lambda : \bar{\nabla}_\lambda]\big|_{t^{-1}}[\bar{\nabla}_\lambda : L_\mu]\big|_{t^{-1}} \\ &= [\mathrm{RHom}(\nabla_\lambda, \nabla_\lambda)]\big|_{t^{-1}}[\bar{\nabla}_\lambda : L_\mu]\big|_{t^{-1}}. \end{aligned}$$

On the other hand, we can write

$$\begin{aligned} [\nabla_\lambda : L_\mu]\big|_{t^{-1}} &= [\mathrm{RHom}(P_\mu, \nabla_\lambda)]\big|_{t^{-1}} \\ &= \big([P_\mu : L_\lambda]\big|_{t^{-1}}\big)\big|_{t^{-1}} \\ &= [P_\mu : L_\lambda] \end{aligned}$$

since $L_\lambda \to \nabla_\lambda$ is an injective hull in $\mathcal{P}_{\leq\lambda}$. Now, we know that $[\bar{\Delta}_\lambda : L_\lambda] = 1$ and that $r_{\lambda,\mu}(t^{-1}) = [P_\mu : \Delta_\lambda]$ since the construction of projectives is dual to the construction of injectives. Therefore, we have

$$\begin{aligned} [\nabla_\lambda : L_\mu]\big|_{t^{-1}} &= [P_\mu : \bar{\Delta}_\lambda] \\ &= [P_\mu : \Delta_\lambda][\Delta_\lambda : \bar{\Delta}_\lambda] \\ &= r_{\lambda,\mu}(t^{-1})[\mathrm{RHom}(\Delta_\lambda, \Delta_\lambda)] \\ &= r_{\lambda,\mu}(t^{-1})[\mathrm{RHom}(\nabla_\lambda, \nabla_\lambda)]\big|_{t^{-1}} \end{aligned}$$

Thus, $r_{\lambda,\mu}(t^{-1}) = [\bar{\nabla}_\lambda : L_\mu]\big|_{t^{-1}} = [\mathrm{RHom}(\bar{\nabla}_\lambda, I_\mu)]$, as desired. $\square$

By our earlier computation and Lemma 5.6, we have

$$\big[\mathrm{RHom}(\bar{\nabla}_\lambda, I'_\mu)\big] = r_{\lambda,\mu}(t^{-1}) - r_{\lambda,\mu}(t).$$

We are able to compute $\left[\mathrm{RHom}(\bar{\nabla}_\lambda, I'_\mu)\right]$ by induction, since $I'_\mu$ has a finite costandard filtration, and from our knowledge of $\left[\mathrm{RHom}(\nabla_\lambda, \nabla_\mu)\right]$ for $\lambda, \mu \in \Lambda_+$. Since $\nabla_\lambda$ is filtered by $\bar{\nabla}_\lambda$, we can write

$$[\mathrm{RHom}(\nabla_\lambda, \nabla_\mu)] = \left([\nabla_\lambda : \bar{\nabla}_\lambda]\big|_{t^{-1}}\right) [\mathrm{RHom}(\bar{\nabla}_\lambda, \nabla_\mu)].$$

We claim that $r_{\lambda,\mu}(t)$ contains only non-negative powers of $t$, which would allow us to determine $r_{\lambda,\mu}(t)$ from the above expression. To prove our claim, we make use of results from [Be2] concerning equivariant coherent sheaves on the Springer resolution $\widetilde{\mathcal{N}}$ and their relation to those on the nilpotent cone.

Recall that we can write $\widetilde{\mathcal{N}} = G \times^B \mathfrak{u}$ where $\mathfrak{u}$ is the Lie algebra of the unipotent radical. The $\mathbb{G}_m$-action originally defined on $\mathcal{N}$ can also be applied to $\mathfrak{u}$ and we get a $G \times \mathbb{G}_m$-action on $\widetilde{\mathcal{N}}$. Let $\mathrm{D^bCoh}^{G\times\mathbb{G}_m}(\widetilde{\mathcal{N}})$ be the bounded derived category of $G \times \mathbb{G}_m$-equivariant coherent sheaves on the Springer resolution. In [Be2], Bezrukavnikov describes an exceptional set in $\mathrm{D^bCoh}^{G\times\mathbb{G}_m}(\widetilde{\mathcal{N}})$ that gives rise to what is known as the exotic $t$-structure on this category. The heart of the exotic $t$-structure, denoted $\mathcal{E}$, is quasi-hereditary and thus properly stratified. Recall that in such an instance, the costandard objects coincide with the proper costandard objects, and the same is true for the standards and proper standards. In $\mathcal{E}$, the sets of isomorphism classes of simple objects $\{\widetilde{L}_\lambda\}$, costandard objects $\{\widetilde{\nabla}_\lambda\}$, and standard objects $\{\widetilde{\Delta}_\lambda\}$ are all in bijection with $\Lambda$, the set of weights of $G$. We restate the following results from [Be2], describing the relationship between $\mathcal{E}$, the heart of the exotic $t$-structure on $\mathrm{D^bCoh}^{G\times\mathbb{G}_m}(\widetilde{\mathcal{N}})$, and $\mathcal{P}$, the heart of the perverse coherent $t$-structure on $\mathcal{D}$.

**Proposition 5.7.** *(i) The functor $\mu_*$ from $\mathrm{D^bCoh}^{G\times\mathbb{G}_m}(\widetilde{\mathcal{N}})$ to $\mathcal{D}$ is $t$-exact with respect to the exotic and perverse coherent $t$-structures, respectively.*

*(ii) For $\lambda \in -\Lambda_+$, we have $\mu_*(\widetilde{L}_\lambda) = L_{w_0\lambda}$, and for $\lambda \notin \Lambda_+$, we have $\mu_*(\widetilde{L}_\lambda) = 0$.*

To prove our claim concerning the multiplicity polynomial $r_{\lambda,\mu}(t)$, we require an Ext-vanishing condition between particular objects with particular shifts in grading. We begin by re-stating Lemma 9 from [Be2], also known as the Positivity Lemma. Recall that $\delta_\lambda$ is the length of the shortest element $w$ of the Weyl group such that $w\lambda = w_0\lambda$.

**Lemma 5.8** ([Be2])**.** *In $\mathcal{E}$, for any $\lambda, \mu \in \Lambda$, we have*

$$\begin{aligned} \mathrm{Ext}^i(\widetilde{L}_\mu\langle\delta_\mu\rangle, \widetilde{\nabla}_\lambda\langle\delta_\lambda\rangle\langle n\rangle) &= 0 \text{ if } i > -n \\ \mathrm{Ext}^i(\widetilde{\Delta}_\lambda\langle\delta_\lambda\rangle\langle n\rangle, \widetilde{L}_\mu\langle\delta_\mu\rangle) &= 0 \text{ if } i > n \end{aligned}$$

As a consequence, we obtain the following two propositions, which imply the desired property for $r_{\lambda,\mu}(t)$.

**Proposition 5.9.** *The heart $\mathcal{E}$ of the exotic t-structure on $\mathrm{D^b}(\mathrm{Coh}^{G\times\mathbb{G}_m}\widetilde{\mathcal{N}})$ is mixed. In particular, for any $\lambda, \mu \in \Lambda$,*

$$\mathrm{Ext}^1(\widetilde{L}_\mu\langle\delta_\mu\rangle, \widetilde{L}_\lambda\langle\delta_\lambda\rangle\langle n\rangle) = 0$$

*for $n \geq 0$.*

*Proof.* First, consider the case $\mu \geq \lambda$. Since $\mathcal{E}$ is quasi-hereditary, there exists a short exact sequence $0 \to \widetilde{L}_\lambda\langle\delta_\lambda\rangle\langle n\rangle \to \widetilde{\nabla}_\lambda\langle\delta_\lambda\rangle\langle n\rangle \to C\langle n\rangle \to 0$. This gives rise to the long exact sequence

$$\cdots \to \mathrm{Hom}(\widetilde{L}_\mu\langle\delta_\mu\rangle, C\langle n\rangle) \to \mathrm{Ext}^1(\widetilde{L}_\mu\langle\delta_\mu\rangle, \widetilde{L}_\lambda\langle\delta_\lambda\rangle\langle n\rangle) \to \cdots$$

and $\mathrm{Hom}(\widetilde{L}_\mu\langle\delta_\mu\rangle, C\langle n\rangle) = 0$ because $C \in \mathcal{E}_{<\lambda}$ and $\mu \geq \lambda$. By Lemma 5.8, in the long exact sequence above, the preceding term $\mathrm{Hom}(\widetilde{L}_\mu\langle\delta_\mu\rangle, \widetilde{\nabla}_\lambda\delta_\lambda\rangle\langle n\rangle)$ vanishes for $n > 0$ and the following term $\mathrm{Ext}^1(\widetilde{L}_\mu\langle\delta_\mu\rangle, \widetilde{\nabla}_\lambda\delta_\lambda\rangle\langle n\rangle)$ vanishes for $n \geq 0$. We know that $\mathrm{Hom}(\widetilde{L}_\mu\langle\delta_\mu\rangle, \widetilde{\nabla}_\lambda\langle\delta_\lambda\rangle\langle n\rangle)$ also vanishes for $\mu > \lambda$ and any value of $n$ so now consider the case $\mu = \lambda$ and $n = 0$. Given an element of $\mathrm{Hom}(\widetilde{L}_\lambda\langle\delta_\lambda\rangle, \widetilde{\nabla}_\lambda\langle\delta_\lambda\rangle)$, we compose with $\widetilde{\nabla}_\lambda\langle\delta_\lambda\rangle\langle n\rangle \to C\langle n\rangle$ to obtain an element in $\mathrm{Hom}(\widetilde{L}_\lambda\langle\delta_\lambda\rangle, C\langle n\rangle)$. Since $C \in \mathcal{E}_{<\lambda}$, this composition is 0 and the map

$$\mathrm{Hom}(\widetilde{L}_\lambda\langle\delta_\lambda\rangle, \widetilde{\nabla}_\lambda\langle\delta_\lambda\rangle) \to \mathrm{Hom}(\widetilde{L}_\lambda\langle\delta_\lambda\rangle, C\langle n\rangle)$$

is the zero map. Therefore, we have the desired result for the case $\mu \geq \lambda$.

Now let $\mu < \lambda$ and consider $0 \to K\langle n\rangle \to \widetilde{\Delta}_\mu\langle\delta_\mu\rangle\langle n\rangle \to \widetilde{L}_\mu\langle\delta_\mu\rangle\langle n\rangle \to 0$. The corresponding long exact sequence contains

$$\cdots \to \mathrm{Hom}(K\langle n\rangle, \widetilde{L}_\lambda\langle\delta_\lambda\rangle) \to \mathrm{Ext}^1(\widetilde{L}_\mu\langle\delta_\mu\rangle\langle n\rangle, \widetilde{L}_\lambda\langle\delta_\lambda\rangle) \to \cdots$$

and $\mathrm{Hom}(K\langle n\rangle, \widetilde{L}_\lambda\langle\delta_\lambda\rangle) = 0$ since $K \in \mathcal{E}_{<\mu}$ and $\mu < \lambda$. By Lemma 5.8, in this long exact sequence, the preceding term $\mathrm{Hom}(\widetilde{\Delta}_\mu\langle\delta_\mu\rangle\langle n\rangle, \widetilde{L}_\lambda\langle\delta_\lambda\rangle)$ vanishes for $n < 0$ and the following term $\mathrm{Ext}^1(\widetilde{\Delta}_\mu\langle\delta_\mu\rangle\langle n\rangle, \widetilde{L}_\lambda\langle\delta_\lambda\rangle)$ vanishes for $n \leq 0$. We also know that $\mathrm{Hom}(\widetilde{\Delta}_\mu\langle\delta_\mu\rangle\langle n\rangle, \widetilde{L}_\lambda\langle\delta_\lambda\rangle)$ vanishes for any value of $n$ if $\mu < \lambda$ by the properties of quasi-hereditary categories. This completes our proof. □

We are now able to return to the setting of the nilpotent cone, and show that our multiplicity polynomials satisfy the desired property.

**Proposition 5.10.** *The heart $\mathcal{P}$ of the perverse coherent t-structure on $\mathcal{D}$ is mixed. In particular, for any $\lambda, \mu \in \lambda_+$,*

$$\mathrm{Ext}^1(L_\lambda, L_\mu\langle n\rangle) = 0$$

*for $n \geq 0$.*

*Proof.* Since $\mathcal{E}$ is mixed, every object $\mathcal{F} \in \mathcal{E}$ has an increasing filtration $W_\bullet\mathcal{F}$, known as the weight filtration (see [BGS]), where for each $i \in \mathbb{Z}$ the $i$th subquotient $W_i\mathcal{F}/W_{i-1}\mathcal{F}$ is semi-simple with direct summands of the form $\widetilde{L}_\lambda\langle i\rangle$ with $\lambda \in \Lambda$.

Any costandard object $\widetilde{\nabla}_\lambda$ in $\mathcal{E}$ has $\widetilde{L}_\lambda$ as its unique simple socle so its weight filtration must consist of subquotients whose direct summands are of the form $\widetilde{L}_\mu\langle n\rangle$ with $\mu \in \Lambda$ and $n \geq 0$.

We claim that for $\lambda \in \Lambda_+$, we have $\mu_*(\widetilde{\nabla}_\lambda) \simeq A_\lambda\langle 2\delta_\lambda\rangle$. To prove this claim, we proceed by induction on $\delta_\lambda$. In the base case, suppose that $w_0\lambda = s\lambda$ for some simple reflection $s \in W$. By Proposition 7(b) and Lemma 8 in [Be2], we have that

$$\mu_*(\widetilde{\nabla}_\lambda) \simeq \mu_*(\widetilde{\nabla}_{s\lambda})\langle 2\rangle = A_\lambda\langle 2\rangle.$$

Then every simple subquotient of $\mu_*(\widetilde{\nabla}_\lambda)$ is of the form $L_\mu\langle n\rangle$ with $n \geq 0$. Thus, in the category $\mathcal{P}$, for any $\lambda, \mu \in \Lambda_+$, we have $[\nabla_\mu : L_\lambda\langle n\rangle] = 0$ for $n \leq 0$. Therefore, $\operatorname{Ext}^1(L_\lambda\langle n\rangle, L_\mu) = \operatorname{Ext}^1(L_\lambda, L_\mu\langle -n\rangle) = 0$ if $n \leq 0$ and $\operatorname{Ext}^1(L_\lambda, L_\mu\langle n\rangle) = 0$ if $n \geq 0$. $\square$

**Corollary 5.11.** *For $\lambda \in \Lambda_+$, every composition factor of an injective object in $\mathcal{P}_{\leq\lambda}$ is of the form $L_\mu\langle n\rangle$ for some dominant weight $\mu \leq \lambda$ and $n \geq 0$. In particular, only non-negative exponents appear in $r_{\lambda,\mu}(t) = \left[I_\mu : \nabla_\lambda\right]$.*

*Proof.* Since $\mathcal{P}_{\leq\lambda}$ is mixed, and an injective hull $I_\mu \in \mathcal{P}_{\leq\lambda}$ possesses a unique simple socle $L_\mu$, its weight filtration consists of subquotients $L_\nu\langle n\rangle$ with $\nu \leq \mu$ and $n \geq 0$, as desired. $\square$

# Chapter 6
# Examples

Let $G = \mathrm{PGL}_2(\mathbb{C})$ so its Langlands dual group is $\check{G} = \mathrm{SL}_2(\mathbb{C})$. In this case, $\Lambda_+ = 2\mathbb{Z}_{\geq 0}$. We know that $\nabla_0$ is the injective hull of $L_0$ in $\mathcal{P}_0$. In the category $\mathcal{P}_{\leq 2}$, let $I_0$ be the injective hull of $L_0$. We will compute $r_{2,0}(t) = [I_0 : \nabla_2]$. From the computation described above, we have the following:

$$r_{2,0}(t^{-1}) - r_{2,0}(t) = [\mathrm{RHom}(\bar{\nabla}_2, \nabla_0)].$$

Consider the equation

$$\begin{aligned} [\mathrm{RHom}(\nabla_2, \nabla_0)] &= \left([\nabla_2 : \bar{\nabla}_2]\big|_{t^{-1}}\right)[\mathrm{RHom}(\bar{\nabla}_2, \nabla_0)] \\ &= (1 + t^{-1})[\mathrm{RHom}(\bar{\nabla}_2, \nabla_0)]. \end{aligned}$$

On the other hand, we can apply Theorem 5.5 to get

$$\begin{aligned} [\mathrm{RHom}(\nabla_2, \nabla_0)] &= q_{2,0}(t)[H^\bullet(\mathrm{Gr}_0)] \\ &= \left(t^{-\frac{\delta_2}{2}}\left(t^{-\frac{\delta_0}{2}} p_{2,0}(t^{-1}) - t^{\frac{\delta_0}{2}} p_{2,0}(t) q_{0,0}(t)\right)\right) \cdot 1 \end{aligned}$$

and we know that $\delta_0 = 0$, $\delta_2 = 1$, $q_{0,0}(t) = 1$ and $p_{2,0}(t) = t$. Therefore, we have

$$\begin{aligned} [\mathrm{RHom}(\nabla_2, \nabla_0)] &= t^{-\frac{\delta_2}{2}}\left(t^{-\frac{\delta_0}{2}} p_{2,0}(t^{-1}) - t^{\frac{\delta_0}{2}} p_{2,0}(t) q_{0,0}(t)\right) \\ &= t^{-\frac{1}{2}}(1 \cdot t^{-1} - 1 \cdot t \cdot 1) \\ &= t^{-\frac{3}{2}} - t^{\frac{1}{2}} \end{aligned}$$

By comparing with our earlier computation, we have

$$\begin{aligned} (1 + t^{-1})[\mathrm{RHom}(\bar{\nabla}_2, \nabla_0)] &= t^{-\frac{3}{2}} - t^{\frac{1}{2}} \\ (1 + t^{-1})[\mathrm{RHom}(\bar{\nabla}_2, \nabla_0)] &= t^{\frac{1}{2}}(t^{-2} - 1) \\ [\mathrm{RHom}(\bar{\nabla}_2, \nabla_0)] &= t^{-\frac{1}{2}} - t^{\frac{1}{2}} \end{aligned}$$

so $[I_0 : \nabla_2] = r_{2,0}(t) = t^{\frac{1}{2}}$. Therefore, we can write $[I_0] \in K(\mathrm{D}^{\mathrm{mix}}_{\check{G}(\mathfrak{O})}(\overline{\mathrm{Gr}_2}))$ as $[I_0] = [\nabla_0] + t^{\frac{1}{2}}[\nabla_2] = [\nabla_0] + [\nabla_2\langle 1\rangle]$.

Now consider the case where $G = \mathrm{PGL}_3(\mathbb{C})$, which has Langlands dual group $\check{G} = \mathrm{SL}_3(C)$. In this case, the set of dominant weights is

$$\Lambda_+ = \{(a, b, c) \in \mathbb{Z}^3 \mid a + b + c = 0,\ a \geq b \geq c\},$$

which is in the span of the fundamental weight basis for $\check{G}$ given by $\omega_1 = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ and $\omega_2 = (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$. In this case, the Weyl group is $W = S_3$ generated by two simple reflections. We also have that for any $\lambda = (a, b, c) \in \Lambda_+$,

$$[H^\bullet(\mathrm{Gr}_\lambda)] = \begin{cases} [H^\bullet(G/B)] = 1 + 2t + 2t^2 + t^3 \text{ if } a > b > c \\ [H^\bullet(\mathbb{C}P^2)] = 1 + t + t^2 \text{ if } a = b > c \text{ or } a > b = c \\ [H^\bullet(\mathrm{pt})] = 1 \text{ if } a = b = c = 0 \end{cases}$$

Now let $\lambda = \omega_1 + \omega_2 = (1, 0, -1)$ and $\mu = (0, 0, 0)$. We will compute $[I_\mu : \nabla_\lambda]$ in $\mathcal{P}_{\leq\lambda}$ where $I_\mu$ is in the injective hull of $L_\mu$. We let $I'_\mu$ be the injective hull of $L_\mu$ in $\mathcal{P}_{<\lambda}$ and in this case $I'_\mu = \nabla_\mu$.

We require the following values in our computation:

$$\begin{aligned} \delta_\lambda &= 3 \\ \delta_\mu &= 0 \\ [H^\bullet \mathrm{Gr}_\lambda] &= 1 + 2t + 2t^2 + t^3 \\ [H^\bullet \mathrm{Gr}_\mu] &= 1 \\ p_{\lambda,\mu}(t) &= t + t^2 \end{aligned}$$

From Theorem 5.5, we have

$$\begin{aligned} q_{\lambda,\mu}(t) &= t^{-\frac{\delta_\lambda}{2}}\left(t^{-\frac{\delta_\mu}{2}}p_{\lambda,\mu}(t^{-1}) - t^{\frac{\delta_\mu}{2}}p_{\lambda,\mu}(t)q_(\mu,\mu)(t)\right) \\ &= t^{-\frac{3}{2}}\left(t^0(t^{-1}+t^{-2}) - t^0(t+t^2)\cdot 1\right) \\ &= t^{-\frac{3}{2}}(t^{-2}+t^{-1}-t-t^2) \end{aligned}$$

so

$$[\mathrm{RHom}(\nabla_\lambda, \nabla_\mu)] = q_{\lambda,\mu}(t)[H^\bullet(\mathrm{Gr}_\mu)] = t^{-\frac{3}{2}}(t^{-2}+t^{-1}-t-t^2).$$

By comparing this with the equation

$$[\mathrm{RHom}(\nabla_\lambda, \nabla_\mu)] = \left([\nabla_\lambda : \bar{\nabla}_\lambda]\big|_{t^{-1}}\right)[\mathrm{RHom}(\bar{\nabla}_\lambda, \nabla_\mu)],$$

we get

$$\begin{aligned} (1+2t^{-1}+2t^{-2}+t^{-3})[\mathrm{RHom}(\bar{\nabla}_\lambda, \nabla_\mu)] &= t^{-\frac{3}{2}}(t^{-2}+t^{-1}-t-t^2) \\ (1+2t^{-1}+2t^{-2}+t^{-3})[\mathrm{RHom}(\bar{\nabla}_\lambda, \nabla_\mu)] &= t^{-\frac{3}{2}}(t-t^2)(1+2t^{-1}+2t^{-2}+t^{-3}) \\ [\mathrm{RHom}(\bar{\nabla}_\lambda, \nabla_\mu)] &= t^{-\frac{1}{2}} - t^{\frac{1}{2}}. \end{aligned}$$

Therefore, we have that $[I_\mu : \nabla_\lambda] = t^{\frac{1}{2}}$ in $\mathcal{P}_{\leq\lambda}$. Therefore, in $K(\mathcal{P}_{\leq\lambda})$, we have $[I_\mu] = [\nabla_\mu] + t^{\frac{1}{2}}[\nabla_\lambda] = [\nabla_\mu] + [\nabla_\lambda\langle 1\rangle]$.

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# Vita

Myron Minn-Thu-Aye was born in Hong Kong in 1985. He completed his undergraduate studies at Williams College in June 2007, earning a Bachelor of Arts in mathematics and computer science. He began his graduate studies at Louisiana State University in August 2007, and earned a Master of Science in mathematics in May 2009. He is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2013.