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Homogenization of nonlinear partial differential equations

Silvia Jiménez
Louisiana State University and Agricultural and Mechanical College, silvia.jimenez.bolanos@gmail.com

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HOMOGENIZATION OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

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by

Silvia Jiménez

B.Sc. in Mathematics, Universidad de Costa Rica, Costa Rica, 2002
M.S. in Mathematics, Louisiana State University, USA, 2006
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“Caminante, son tus huellas el camino y nada más; 
Caminante, no hay camino, se hace camino al andar.
Al andar se hace el camino, y al volver la vista atrás
se ve la senda que nunca se ha de volver a pisar.”
Antonio Machado, “Proverbios y Cantares XXIX”.

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# Table of Contents

Acknowledgements  ........................................................................................................ ii

Abstract  ......................................................................................................................... v

Chapter 1 Introduction  .................................................................................................. 1

Chapter 2 Basic Ideas in Homogenization Theory  ..................................................... 3
  2.1 Motivation and Examples  ....................................................................................... 3
    2.1.1 1-Dimensional Example of Homogenization  ............................................... 5
    2.1.2 Homogenization in $\mathbb{R}^n$  .................................................................... 8
  2.2 Some Special Cases with Closed Form Expressions for the Homogenized Operator $b$  9
    2.2.1 The Hashin Structure (1962)  .................................................................... 9
    2.2.2 The Mortola-Steffé Structure and the Checkerboard Structure 10

Chapter 3 Dirichlet Boundary Value Problem  ......................................................... 12
  3.1 Description of the Problem  ................................................................................... 12
  3.2 Microgeometries Considered  ............................................................................... 13
  3.3 Notation and Preliminary Results  ...................................................................... 14
    3.3.1 Properties of $A$  .................................................................................. 14
    3.3.2 Dirichlet BVP and Homogenization Theorem  ............................................ 19
    3.3.3 Properties of $b$  .................................................................................. 21
    3.3.4 Properties of $p$  .................................................................................. 24

Chapter 4 Higher Order Integrability of the Homogenized Solution  ..................... 25
  4.1 Statement of the Theorem on the Higher Order Integrability of the Homogenized Solution 25
  4.2 Proof of Higher Order Integrability of the Homogenized Solution 26

Chapter 5 Corrector Theorem  .................................................................................... 31
  5.1 Statement of the Corrector Theorem  ................................................................... 31
  5.2 Some Properties of Correctors  .......................................................................... 32
  5.3 Proof of the Corrector Theorem  ...................................................................... 51
Chapter 6  Lower Bounds on Field Concentrations ...................................... 60
  6.1  Statement of the Lower Bound on the Amplification of the Macroscopic
       Field by the Microstructure ........................................... 60
  6.2  Young Measures ............................................................ 61
  6.3  Proof of Lower Bound on the Amplification of the Macroscopic Field
       by the Microstructure .................................................... 62

Chapter 7  Nonlinear Neutral Inclusions ................................................... 68
  7.1  Double Coated Nonlinear Neutral Inclusions ................................. 68
    7.1.1  Statement of the Problem and Result ................................ 68
  7.2  Calculations ................................................................. 69

Bibliography .............................................................................. 74

Vita ......................................................................................... 78
Abstract

This dissertation is concerned with properties of local fields inside composites made from two materials with different power law behavior. This simple constitutive model is frequently used to describe several phenomena ranging from plasticity to optical nonlinearities in dielectric media.

We provide the corrector theory for the strong approximation of fields inside composites made from two power-law materials with different exponents. The correctors are used to develop bounds on the local singularity strength for gradient fields inside microstructured media. The bounds are multiscale in nature and can be used to measure the amplification of applied macroscopic fields by the microstructure. These results are shown to hold for finely mixed periodic dispersions of inclusions and for layers.
Chapter 1

Introduction

We consider heterogeneous materials that have inhomogeneities on length scales that are much larger than the atomic scale but which are essentially homogeneous at macroscopic length scales. Heterogeneous materials such as fiber reinforced composites and polycrystalline metals and dielectrics appear in many physical contexts.

The determination of the macroscopic effective properties for problems in heat transfer, elasticity, and electromagnetics is an important problem. It is also equally important to understand the behavior of the local fields, such as higher moments of fields inside heterogeneous media. The presence of large local fields either electric or mechanical often precede the onset of material failure [KM86]. Heterogeneities can amplify the applied load and generate local fields with very high intensities. The goal of the analysis presented in this research is to develop tools for quantifying the effect of load transfer between length scales inside heterogeneous media. In this thesis, we provide methods for quantitatively measuring the excursions of local fields generated by applied loads. These local quantities are extremely useful for understanding the evolution of nonlinear phenomena such as plasticity or damage.

The research developed in this thesis investigates the properties of local fields inside mixtures of two nonlinear power law materials. This simple constitutive model is frequently used to describe several phenomena ranging from plasticity to optical nonlinearities in dielectric media. The main achievement of this thesis is that it develops the corrector theory necessary for the study of local fields inside mixtures of two power law materials. Further the corrector theory is applied to deliver new multiscale tools to bound the local singularity strength inside micro-structured media in terms of the macroscopic applied fields.

The thesis is organized as follows. In the next Chapter we provide background and motivate the theory of homogenization for a simple class of examples. In Chapter 3 we introduce the equilibrium problem for two phase nonlinear power law materials. Here the materials are assumed to have two different power law exponents and represent strongly nonlinear materials. In this thesis we consider two common two phase power law microstructures. The first is given by a periodic dispersion of particles embedded in a matrix and the second given by layered microstructures. In Chapter 4 we establish higher order integrability properties for the gradient of the solution of the equilibrium problem inside the material with the larger exponent. This is used in Chapter 5 where we develop the corrector theory necessary for the study of local fields inside mixtures of two power law materials. In Chapter 6 we provide lower bounds on the local field strength inside microstructured media in terms of the macro-
scopic applied fields. We conclude in Chapter 7 where we will introduce special neutrally conducting microstructures made with power law materials.
Chapter 2

Basic Ideas in Homogenization Theory

The theory of homogenization or averaging of partial differential equations dates back to the late sixties [Spa67], it has been very rapidly developed during the last two decades, and it is now established as a distinct discipline within mathematics.

Homogenization theory is concerned with the derivation of equations for averages of solutions of equations with rapidly varying coefficients. This problem arises in obtaining macroscopic, or “homogenized”, or “effective” equations for systems with a fine microscopic structure. The goal is to represent a complex, rapidly-varying medium with a slowly-varying medium in which the fine scale structure is averaged out in an appropriate way.

2.1 Motivation and Examples

Suppose we would like to know the stationary temperature distribution in an homogeneous body $\Omega \subset \mathbb{R}^3$ with an internal heat source $f$, heat conductivity $A$ and zero temperature on the boundary $\partial \Omega$. The model to describe this problem is given by the following boundary value problem: Find $u \in W_0^{1,p}(\Omega)$, $1 < p < \infty$, such that

$$-\text{div} (A \nabla u) = f \text{ on } \Omega,$$  

(2.1)

where $\Omega$ is a bounded open subset of $\mathbb{R}^n$, $f$ is a given function on $\Omega$, and $A : \mathbb{R}^n \to \mathbb{R}^n$ satisfies suitable continuity and monotonicity conditions that allows the existence and uniqueness of the solution of (2.1).

Now, suppose that we would like to be able to model the case when the underlying material is heterogeneous. Then we replace $A$ in (2.1) with a map $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ to obtain

$$\begin{cases} -\text{div} (A(x, \nabla u)) = f \text{ on } \Omega \\ u \in W_0^{1,p}(\Omega). \end{cases}$$  

(2.2)

Since (2.2) depends on $x$, this is much more difficult to handle than (2.1).

An interesting special case is a two-phase composite where one material is periodically distributed in the other. In this case, the underlying periodic inclusions are often microscopic.
with respect to $\Omega$. By periodicity, we can divide $\Omega$ into periodic cells, and we call the representative unit cell by $Y$ (the microstructure of a given periodic material can be described by several different period cells). This is described by maps of the form $A_\epsilon(x, \xi) = A\left(\frac{x}{\epsilon}, \xi\right)$, where $A(\cdot, \xi)$ is assumed to be $Y$-periodic and $\epsilon$ is the fineness of the periodic structure. Equation (2.2) becomes

$$\begin{align*}
-\text{div} \left( A_\epsilon \left( x, \nabla u_\epsilon \right) \right) &= -\text{div} \left( A \left( \frac{x}{\epsilon}, \nabla u_\epsilon \right) \right) = f \quad \text{on } \Omega \\
u_\epsilon &\in W^{1,p}_0(\Omega).
\end{align*}$$  

The function $u_\epsilon$ can be interpreted as the electric potential, magnetic potential, or the temperature and $A_\epsilon$ describes the physical properties of the different materials constituting the body (they are the dielectric coefficients, the magnetic permeability and the thermic conductivity coefficients, respectively).

Let $\{\epsilon_k\}_{k=0}^\infty$ be a sequence of positive real numbers such that $\epsilon_k \to 0$ as $k \to \infty$. In this way we get a sequence of problems, one for each value of $k$. The smaller $\epsilon_k$ gets, the finer the microstructure becomes.

It is natural to ask ourselves if there exists some type of convergence of the solutions $u_{\epsilon_k}$. If we assume convergence in an appropriate sense, that is $u_{\epsilon_k} \to u$, as $k \to \infty$, we could also ask if $u$ satisfies an equation of a similar type as the one $u_{\epsilon_k}$ satisfies

$$\begin{align*}
-\text{div} \left( b \left( x, \nabla u \right) \right) &= f, \quad \text{on } \Omega \\
u &\in W^{1,p}_0(\Omega),
\end{align*}$$

and if this is the case, how to find $b$.

For large values of $k$, the material behaves like a homogeneous material from a macroscopic point of view, even though the material is strongly heterogeneous at a microscopic level. This makes it reasonable to assume that $b$ should be independent of $x$, which means that $u$ satisfies a homogenized equation of the form

$$\begin{align*}
-\text{div} \left( b \nabla u \right) &= f, \quad \text{on } \Omega \\
u &\in W^{1,p}_0(\Omega).
\end{align*}$$

The “homogenized” $b$ represents the physical parameters of a homogeneous body, whose behavior is equivalent, from a “macroscopic” point of view, to the behavior of the material with the given periodic microstructure, described by (2.3).

Homogenization Theory deals with the questions mentioned above. Another approach to answer those questions is by using the fact that the state of the material $u$ can be often found as the solution of a minimization problem of the form

$$E_\epsilon = \min_{u \in W^{1,p}_0(\Omega)} \left\{ \int_\Omega g \left( \frac{x}{\epsilon}, \nabla u(x) \right) dx - \int_\Omega f u dx \right\},$$

where the local energy density function $g(\cdot, \xi)$ is periodic and is assumed to satisfy the so called natural growth conditions. The convergence of this type of integral functionals is
called Γ-convergence (Introduced by DeGiorgi [DG75]). From the theory of Γ-convergence it follows that $E_\epsilon \to E_{\text{hom}}$, as $\epsilon \to 0$, where

$$E_{\text{hom}} = \min_{u \in W^{1,p}_0(\Omega)} \left\{ \int_{\Omega} g_{\text{hom}}(\nabla u(x)) \, dx - \int_{\Omega} f u \, dx \right\}.$$  

Here the homogenized energy density function $g_{\text{hom}}$ is given by

$$g_{\text{hom}}(\xi) = \frac{1}{|\Omega|} \min_{u \in W^{1,p}_{\text{per}}(Y)} \int_Y g(x, \xi + \nabla u) \, dx,$$

where $W^{1,p}_{\text{per}}(Y)$ is the set of all functions $u \in W^{1,p}(Y)$ which are $Y$-periodic and have mean value zero. Again we note that the limit problem does not depend on $x$, that is, $g_{\text{hom}}$ is the energy density function of a homogeneous material.

To demonstrate some of the techniques and difficulties encountered in the homogenization procedure, we consider homogenization of the one dimensional Poisson equation. This simple example reveals the main difficulty.

### 2.1.1 1-Dimensional Example of Homogenization

Let $\Omega = (0, 1)$, $f \in H^{-1}(\Omega)$, and $A \in L^\infty(\Omega)$ be a measurable and periodic function with period 1 satisfying

$$0 < \beta_1 \leq A(x) \leq \beta_2 < \infty, \text{ for a.e. } x \in \mathbb{R}. \quad (2.5)$$

**Remark 2.1.** For example, consider a periodic mixture of two materials. Let $\chi_1$ be the the characteristic function of material 1 and $\chi_2$ be the the characteristic function of material 2, both periodic of periodicity 1. Let

$$A(y) = \beta_1 \chi_1(y) + \beta_2 \chi_2(y)$$

defined in $\Omega = (0, 1)$ and extend it to all $\mathbb{R}$ by periodicity, and call this extension by $A$ as well. Note that $A \in L^\infty(\Omega)$, because

$$\|A\|_{L^\infty(\Omega)} = \beta_2$$

and clearly it satisfies (2.5) for all $x \in \mathbb{R}$.

We define $A_{\epsilon_k}(x) = A \left( \frac{x}{\epsilon_k} \right)$. The weak form of (2.3) becomes

$$\begin{cases} \int_0^1 A_{\epsilon_k}(x) \partial_x u_{\epsilon_k}(x) \partial_x \phi(x) \, dx = \int_0^1 f(x) \phi(x) \, dx & \text{for every } \phi \in W^{1,2}_0(0, 1), \\ u_{\epsilon_k} \in W^{1,2}_0(0, 1), \end{cases} \quad (2.6)$$

and (2.3) becomes

$$\begin{cases} -\partial_x (A_{\epsilon_k}(x) \partial_x u_{\epsilon_k}(x)) = f(x) & \text{in } (0, 1), \\ u_{\epsilon_k} \in W^{1,2}_0(0, 1). \end{cases} \quad (2.7)$$
By a standard result in the existence theory of partial differential equations (using Lax-Milgram Lemma [Eva98]), there exists a unique solution of these problems for each \( k \). By choosing \( \phi = u_{\epsilon_k} \) in (2.6) and taking (2.5) into account, we obtain by Hölder’s inequality that 
\[
\beta_1 \| \partial_x u_{\epsilon_k} \|^2_{L^2(0,1)} \leq \int_0^1 A_{\epsilon_k}(x) |\partial_x u_{\epsilon_k}(x)|^2 \, dx
\]
\[
= \int_0^1 f(x) u_{\epsilon_k}(x) \, dx
\]
\[
\leq \|f\|_{H^{-1}(\Omega)} \|u_{\epsilon_k}\|_{W_0^{1,2}(\Omega)}.
\]

Recall that 
\[
\|u_{\epsilon_k}\|_{W_0^{1,2}(\Omega)}^2 = \|u_{\epsilon_k}\|_{L^2(\Omega)}^2 + \|\partial_x u_{\epsilon_k}\|_{L^2(\Omega)}^2.
\]

The Poincaré inequality for functions with zero boundary values states that there is a constant \( C \) only depending on \( \Omega = (0,1) \) such that 
\[
\|u_{\epsilon_k}\|_{L^2(\Omega)} \leq C \|\partial_x u_{\epsilon_k}\|_{L^2(\Omega)}.
\]
This implies that 
\[
\|u_{\epsilon_k}\|_{W_0^{1,2}(\Omega)}^2 \leq C,
\]
where \( C \) is a constant independent of \( k \). Since \( W_0^{1,2}(\Omega) \) is reflexive, there exists a subsequence, still denoted by \( \{u_{\epsilon_k}\} \), such that 
\[
u_{\epsilon_k} \rightharpoonup u_\ast \text{ in } W_0^{1,2}(\Omega).
\]

Since \( W_0^{1,2}(\Omega) \) is compactly embedded in \( L^2(\Omega) \), we have by the Rellich Embedding Theorem that 
\[
u_{\epsilon_k} \to u_\ast \text{ in } L^2(\Omega).
\]
In general, however, we only have that 
\[
\partial_x u_{\epsilon_k} \rightharpoonup \partial_x u_\ast \text{ in } L^2(\Omega).
\]
Since \( A \) is 1-periodic, we have that \( \{A_{\epsilon_k}\} \) converges weakly* \( L^\infty(\Omega) \), as \( k \to \infty \), to its arithmetic mean \( \langle A \rangle \), i.e., 
\[
A_{\epsilon_k} \rightharpoonup^* \langle A \rangle = \int_0^1 A(y) \, dy \text{ in } L^\infty(\Omega).
\]

From (2.6), (2.9), and (2.10), it could be reasonable to assume that, in the limit, we have 
\[
\begin{cases}
\int_0^1 \langle A \rangle \partial_x u_\ast(x) \partial \phi(x) \, dx = \int_0^1 f(x) \phi(x) \, dx \quad \text{for every } \phi \in W_0^{1,2}(0,1), \\
u_\ast \in W_0^{1,2}(0,1).
\end{cases}
\]
However, this is not true in general, since \( A_{\epsilon_k} \partial_x u_{\epsilon_k} \) is the product of two weakly converging sequences. This is the main difficulty in the limit process. To obtain the correct answer we
proceed in the following way: first we note that, according to (2.10) and (2.8), \(\{A_{\epsilon k} \partial_x u_{\epsilon_k}\}\) is bounded in \(L^2(\Omega)\) and that (2.6) implies that \(-\partial_x (A_{\epsilon_k}(x) \partial_x u_{\epsilon_k}) = f\). Hence there is a constant \(C\) independent of \(k\) such that

\[
\|A_{\epsilon k} \partial_x u_{\epsilon_k}\|_{W^{1,2}(\Omega)} \leq C.
\]

As before, since \(W^{1,2}(\Omega)\) is reflexive, there exists a subsequence, still denoted \(\{A_{\epsilon k} \partial_x u_{\epsilon_k}\}\) and a \(M^0 \in L^2(\Omega)\) such that

\[
A_{\epsilon k} \partial_x u_{\epsilon_k} \rightharpoonup M^0 \text{ in } L^2(\Omega).
\]

Since \(\{\frac{1}{A_{\epsilon_k}}\}\) converges to \(\langle \frac{1}{A} \rangle\) weakly* in \(L^\infty(\Omega)\) by periodicity (and hence weakly in \(L^2(\Omega)\)), we have

\[
\partial_x u_{\epsilon_k} = \left(\frac{1}{A_{\epsilon_k}}\right) (A_{\epsilon_k} \partial_x u_{\epsilon_k}) \rightharpoonup \langle \frac{1}{A} \rangle M^0, \text{ in } L^2(\Omega).
\]

(2.11)

Thus, by (2.9) and (2.11), we see that

\[
M^0 = \frac{1}{\langle \frac{1}{A} \rangle} \partial_x u_* = \langle A^{-1} \rangle^{-1} \partial_x u_*.
\]

Now, by passing to the limit in (2.6) we obtain that

\[
\int_0^1 b \partial_x u_* \partial_x \phi dx = \int_0^1 f(x) \phi(x) dx \quad \text{for every } \phi \in W^{1,2}_0(0,1),
\]

\[u_* \in W^{1,2}_0(0,1).
\]

where the homogenized operator is given by \(b = \frac{1}{\langle \frac{1}{A} \rangle} = \langle A^{-1} \rangle^{-1}\), the harmonic mean of \(A\); and since

\[
\frac{1}{\beta_2} \leq \langle \frac{1}{A} \rangle \leq \frac{1}{\beta_1},
\]

we conclude that the homogenized equation has a unique solution and thus that the whole sequence \(\{u_{\epsilon_k}\}\) converges.

**Remark 2.2.** For the example given in Remark 2.1, we obtain \(M^0 = h_\theta \partial_x u_*\), where

\[
h_\theta = \left(\theta_1 \beta_1^{-1} + \theta_2 \beta_2^{-1}\right)^{-1}
\]

and

\[
\theta_1 = \int_0^1 \chi_1(y) dy, \text{ and } \theta_2 = 1 - \theta_1.
\]

**Remark 2.3.** The corresponding homogenization problem for the one-dimensional Poisson equation

\[
\int_\Omega A_{\epsilon_k}(x) |\partial_x u_{\epsilon_k}|^{p-2} \partial_x u_{\epsilon_k} \partial_x \phi dx = \int_\Omega f(x) \phi(x) dx \quad \text{for every } \phi \in W^{1,p}_0(\Omega),
\]

\[u_{\epsilon_k} \in W^{1,p}_0(\Omega)
\]

gives the homogenized operator \(b = \langle A_\frac{1}{p} \rangle^{1-p}\).

7
In higher dimensions, the problem of passing to the limit is rather delicate and requires the introduction of new techniques. One of the main tools to overcome this difficulty is the Compensated Compactness method introduced by Murat and Tartar [MT97]. This method shows that under some additional assumptions, the product of two weakly convergent sequences in $L^2(\Omega)$ converges in the sense of distributions to the product of their limits.

2.1.2 Homogenization in $\mathbb{R}^n$

Assume that $A$ satisfies suitable structure conditions.

Remark 2.4. A common assumption is that $A(x, \xi)$ satisfies the conditions

\[
|A(x, \xi_1) - A(x, \xi_2)| \leq c_1 \lambda(x) (1 + |\xi_1| + |\xi_2|)^{p-1-\alpha} |\xi_1 - \xi_2|^{\alpha},
\]

\[
(A(x, \xi_1) - A(x, \xi_2), \xi_1 - \xi_2) \geq c_2 \lambda(x) (1 + |\xi_1| + |\xi_2|)^{p-\beta} |\xi_1 - \xi_2|^{\beta},
\]

for constants $c_1, c_2 > 0$, where $\alpha$ and $\beta$ satisfy $0 \leq \alpha \leq \min(1, p-1)$ and $\max(p, 2) \leq \beta < \infty$ (see, for example, [DMD90, FM87]).

For example, these conditions are satisfied by the $p$-Poisson operator

\[
A(x, \xi) = \lambda(x) |\xi|^{p-2} \xi,
\]

where $c_1, c_2$ can be chosen as $c_1 = \max \left( p-1, (2\sqrt{2})^{2-p} \right)$, and $c_2 = \min \left( p-1, (2\sqrt{2})^{2-p} \right)$.

We have the following homogenization theorem.

Theorem 2.5. Let $1 < p < \infty$ and $q$ its dual conjugate. The solutions $u_{\epsilon_k}$ of

\[
\begin{cases}
- \text{div} \left( A_{\epsilon_k}(x, \nabla u_{\epsilon_k}) \right) = f & \text{on } \Omega, \\
u_{\epsilon_k} \in W^{1,p}_0(\Omega)
\end{cases}
\]

satisfy

\[
u_{\epsilon_k} \rightharpoonup u \text{ in } W^{1,p}_0(\Omega),
\]

and

\[
A_{\epsilon_k}(x, \nabla u_{\epsilon_k}) \rightharpoonup b(\nabla u) \text{ in } L^q(\Omega, \mathbb{R}^n),
\]

as $k \to \infty$, where $u$ is the solution of the homogenized equation

\[
\begin{cases}
- \text{div} \left( b(\nabla u) \right) = f & \text{on } \Omega, \\
u \in W^{1,p}_0(\Omega),
\end{cases}
\]

where the homogenized operator $b : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

\[
b(\xi) = \frac{1}{|Y|} \int_Y A(x, \xi + \nabla \omega^\xi(x)) \, dx,
\]

and where $\omega^\xi$ is the solution of the local problem on $Y$

\[
\begin{cases}
\int_Y \left( A \left( x, \xi + \nabla \omega^\xi \right), \nabla \phi \right) \, dx = 0 & \text{for every } \phi \in W^{1,p}_{\text{per}}(Y), \\
\omega^\xi \in W^{1,p}_{\text{per}}(Y).
\end{cases}
\]

A common technique to prove this theorem is Tartar’s method of oscillating test functions related to the notion of compensated compactness mentioned above. Another technique is the two-scale convergence method. For a proof see [FM87].
2.2 Some Special Cases with Closed Form Expressions for the Homogenized Operator $b$

The homogenized operator $b$ in (2.13) depends on the solution of a cell problem (2.14), which means that the effective properties of a composite depend in a complicated way on the microstructure. We describe two special cases when we can get closed form expressions for $b$. We consider (2.3) with $p = 2$ and $A(x, \xi) = \lambda(x)\xi$ (linear), with $\xi \in \mathbb{R}^2$. In Chapter 7 of this thesis, we obtain similar results in an example that deals with nonlinear materials.

2.2.1 The Hashin Structure (1962)

We study a three-phase composite consisting of three isotropic materials (coated sphere assemblage), let us call them materials 1, 2, and 3, with conductivity

$$\lambda(x)I = [\sigma_1\chi_{\Omega_1}(x) + \sigma_2\chi_{\Omega_2}(x) + \sigma_3\chi_{\Omega_3}(x)]I$$

where $\chi_{\Omega_i}$ is the characteristic function for the set $\Omega_i$ and $I$ is the unit matrix.

Let the unit cell geometry be described by

$$\Omega_1 = \{x : |x| \leq r_1\}, \quad \Omega_2 = \left\{ x : r_1 \leq |x| \leq r_2 < \frac{1}{2} \right\},$$

$$\Omega_3 = \left\{ x : |x_1| < \frac{1}{2} \land |x| \geq r_2, \ i = 1, 2 \right\}.$$

In order to compute the homogenized coefficients (2.13), we need to solve the cell problem (2.14)

$$-\text{div} \left( \lambda(y)\nabla \phi^\xi(y) \right) = 0 \text{ on } Y,$$

where $\phi^\xi(y) = \xi \cdot y + \omega^\xi(y)$ and $\omega^\xi(y)$ is $Y$-periodic.

In the case $\xi = \bar{e}_1 = [1 \ 0]^T$, we look for a solution of the type

$$\phi^{e_1}(x) = \begin{cases} C_1 x_1, & x \in \Omega_1, \\ x_1 \left( C_2 + \frac{K_2}{|x|^2} \right), & x \in \Omega_2, \\ x_1, & x \in \Omega_3. \end{cases}$$

(2.16)

It is easily seen that (2.16) satisfies (2.15) on $Y$.

By physical reasons, the solution $\phi^\xi(x)$ as well as the flux $\lambda(x)\partial_n \phi^\xi$ must be continuous over the boundaries $\Omega_1 \cap \Omega_2$ and $\Omega_2 \cap \Omega_3$. This gives four equations to solve for the three unknowns $C_1$, $C_2$, and $K_2$. In order to get a consistent solution, we get that $\sigma_3$ must be

$$\sigma_3 = \sigma_2 \left( C_2 - \frac{K_2}{r_2^2} \right) = \frac{\sigma_2 \left( 1 + \frac{\sigma_1}{\sigma_2} + m_1 \left( \frac{\sigma_1}{\sigma_2} - 1 \right) \right)}{1 + \frac{\sigma_1}{\sigma_2} - m_1 \left( \frac{\sigma_1}{\sigma_2} - 1 \right)},$$

(2.17)
where \( m_1 = \frac{r_1^2}{r_2^2} \), the volume fraction of material 1 in material 2. Since we now know the solution \( \omega^{e_1}(y) = \phi^{e_1}(y) - e_1 \cdot y \) of the cell problem, we can compute the homogenized coefficients
\[
b(e_1) = \int_Y \lambda(x)(e_1 + \nabla \omega^{e_1}) dx = [\sigma_3 \, 0]^T
\]
and similarly
\[
b(e_2) = \int_Y \lambda(x)(e_2 + \nabla \omega^{e_2}) dx = [0 \, \sigma_3]^T.
\]

This means that we can put the coated disk consisting of material 1 coated by material 2 into the homogeneous isotropic material 3 without changing the effective properties (neutral inclusions). By filling the whole cell with such homothetically coated disks, we get an isotropic two-phase composite with conductivity \( \sigma_3 \).

For more details see [Mil02].

### 2.2.2 The Mortola-Steffé Structure and the Checkerboard Structure

Let \( Y = (0, 1)^2 \) and divide it into four equal parts
\[
Y_1 = \left(0, \frac{1}{2}\right) \times \left(\frac{1}{2}, 1\right), \quad Y_2 = \left(\frac{1}{2}, 1\right) \times \left(\frac{1}{2}, 1\right),
\]
\[
Y_3 = \left(0, \frac{1}{2}\right) \times \left(0, \frac{1}{2}\right), \quad Y_4 = \left(\frac{1}{2}, 1\right) \times \left(0, \frac{1}{2}\right).
\]

We study a four-phase composite consisting of four isotropic materials, let us call them materials 1, 2, 3, and 4, with conductivity
\[
\lambda(x)I = [\alpha \chi_{Y_1}(x) + \beta \chi_{Y_2}(x) + \gamma \chi_{Y_3}(x) + \delta \chi_{Y_4}(x)] I,
\]
where \( \chi_{Y_i}(x) \) is the characteristic function for the set \( Y_i \) and \( I \) is the unit matrix.

In 1985, Mortola and Steffé [MS85] conjectured that the homogenized conductivity coefficients of this structure are
\[
(\bar{\lambda}_{ij}) = \begin{pmatrix} \bar{\lambda}_{11} & 0 \\ 0 & \bar{\lambda}_{22} \end{pmatrix},
\]
where
\[
\bar{\lambda}_{11} = \sqrt{\frac{(\alpha \beta \gamma + \alpha \beta \delta + \alpha \gamma \delta + \beta \gamma \delta)}{\alpha + \beta + \gamma + \delta}} \frac{(\alpha + \gamma) \, (\beta + \delta)}{(\alpha + \beta) \, (\gamma + \delta)},
\]
\[
\bar{\lambda}_{22} = \sqrt{\frac{(\alpha \beta \gamma + \alpha \beta \delta + \alpha \gamma \delta + \beta \gamma \delta)}{\alpha + \beta + \gamma + \delta}} \frac{(\alpha + \beta) \, (\gamma + \delta)}{(\alpha + \gamma) \, (\beta + \delta)}.
\]
This conjecture was proven by Milton in 2000.
If we let $\delta = \alpha$ and $\gamma = \beta$, we get the so called checkerboard structure. We immediately see that the homogenized conductivity coefficients for the checkerboard structure are

$$\bar{\lambda}_{11} = \bar{\lambda}_{22} = \sqrt{\alpha \beta},$$

the geometric mean. This was proved already in 1970 by Dykhne, but Schulgasser (1977) showed that this was a corollary of Keller’s phase interchange identity from 1963.
Chapter 3

Dirichlet Boundary Value Problem

In this chapter, we study boundary value problems associated with fields inside composites made from two materials with different power law behavior.

3.1 Description of the Problem

The geometry of the composite is specified by the indicator function of the sets occupied by each of the materials. The indicator function of material one and two are denoted by \( \chi_1 \) and \( \chi_2 \), where \( \chi_1(y) = 1 \) in material one and is zero outside and \( \chi_2(y) = 1 - \chi_1(y) \). The constitutive law for the heterogeneous medium is described by \( A : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \)

\[
A(y, \xi) = \sigma(y) |\xi|^{p(y)-2} \xi \\
= \sigma_1 \chi_1(y) |\xi|^{p_1-2} \xi + \sigma_2 \chi_2(y) |\xi|^{p_2-2} \xi;
\]

where \( \sigma(y) = \chi_1(y) \sigma_1 + \chi_2(y) \sigma_2 \), and \( p(y) = \chi_1(y) p_1 + \chi_2(y) p_2 \), periodic in \( y \), with unit period cell \( Y = (0,1)^n \). This simple constitutive model is used in the mathematical description of many physical phenomena including plasticity [PCS97, PCW99, Suq93, Idi08], nonlinear dielectrics [GNP01, GK03, LK98, TW94a, TW94b], and fluid flow [Ruž00, AR06]. We study the problem of periodic homogenization associated with the solutions \( u_\epsilon \) to the problems

\[
-\text{div} \left( A \left( \frac{x}{\epsilon}, \nabla u_\epsilon \right) \right) = f \text{ on } \Omega, \ u_\epsilon \in W^{1,p_1}_0(\Omega),
\]

where \( \Omega \) is a bounded open subset of \( \mathbb{R}^n \), \( 2 \leq p_1 \leq p_2 \), \( f \in W^{-1,q_2}(\Omega) \), and \( 1/p_1 + 1/q_2 = 1 \). The differential operator appearing on the left hand side of (3.2) is commonly referred to as the \( p_\epsilon(x) \)-Laplacian. For the case at hand, the exponents \( p(x) \) and coefficients \( \sigma(x) \) are taken to be simple functions. Because the level sets associated with these functions can be quite general and irregular they are referred to as rough exponents and coefficients. In this context all solutions are understood in the usual weak sense [ZKO94].

One of the basic problems in homogenization theory is to understand the asymptotic behavior as \( \epsilon \rightarrow 0 \), of the solutions \( u_\epsilon \) to the problems (3.2). It was proved in [ZKO94] that \( \{u_\epsilon\}_{\epsilon>0} \) converges weakly in \( W^{1,p_1}(\Omega) \) to the solution \( u \) of the homogenized problem

\[
-\text{div} (b(\nabla u)) = f \text{ on } \Omega, \ u \in W^{1,p_1}_0(\Omega),
\]
where the monotone map $b : \mathbb{R}^n \to \mathbb{R}^n$ (independent of $f$ and $\Omega$) can be obtained by solving an auxiliary problem for the operator (3.2) on a periodicity cell.

The notion of homogenization is intimately tied to the $\Gamma$-convergence of a suitable family energy functionals $I_\epsilon$ as $\epsilon \to 0$ [DM93], [ZKO94]. Here the connection is natural in that the family of boundary value problems (3.3) correspond to the Euler equations of the associated energy functional $I_\epsilon$ and the solutions $u_\epsilon$ are their minimizers. The homogenized solution is precisely the minimizer of the $\Gamma$-limit of the sequence $\{I_\epsilon\}_{\epsilon > 0}$. The connections between $\Gamma$ limits and homogenization for the power-law materials studied here can be found in [ZKO94]. The explicit formula for the $\Gamma$-limit of the associated energy functionals for layered materials was obtained recently in [PS06].

The earlier work of [DMD90] provides the corrector theory for homogenization of monotone operators that in our case applies to composite materials made from constituents having the same power-law growth but with rough coefficients $\sigma(x)$. The corrector theory for monotone operators with uniform power law growth is developed further in [EP04] where it is used to extend multiscale finite element methods to nonlinear equations for stationary random media. Recent work considers the homogenization of $p_\epsilon(x)$-Laplacian boundary value problems for smooth exponential functions $p_\epsilon(x)$ uniformly converging to a limit function $p_0(x)$ [AAPP08]. There the convergence of the family of solutions for these homogenization problems is expressed in the topology of $L^{p_0}(\Omega)$ [AAPP08].

### 3.2 Microgeometries Considered

We carry out this investigation for two nonlinear power-law materials periodically distributed inside a domain $\Omega$. Here $\Omega$ is an open bounded subset of $\mathbb{R}^n$, which represents a sample of the material. The length scale of the microstructure relative to the domain is denoted by $\epsilon$. The periodic mixture is described as follows. We introduce the unit period cell $Y = (0,1)^n$ of the microstructure. Let $F$ be an open subset of $Y$ of material one, with smooth boundary $\partial F$, such that $\overline{F} \subset Y$. The function $\chi_1(y) = 1$ inside $F$ and 0 outside and $\chi_2(y) = 1 - \chi_1(y)$. We extend $\chi_1(y)$ and $\chi_2(y)$ by periodicity to $\mathbb{R}^n$ and the $\epsilon$-periodic mixture inside $\Omega$ is described by the oscillatory characteristic functions $\chi_1^\epsilon(x) = \chi_1(x/\epsilon)$ and $\chi_2^\epsilon(x) = \chi_2(x/\epsilon)$. Here we will consider the case where $F$ is given by a simply connected inclusion embedded inside a host material (see Figure 3.1). A distribution of such inclusions is commonly referred to as a periodic dispersion of inclusions.

![Figure 3.1: Unit cell: Dispersed Microstructure](image)

We also consider layered materials. For this case, the representative unit cell consists of a layer of material one, denoted by $R_1$, sandwiched between layers of material two, denoted
by $R_2$. The interior boundary of $R_1$ is denoted by $\Gamma$. Here $\chi_1(y) = 1$ for $y \in R_1$ and $0$ in $R_2$, and $\chi_2(y) = 1 - \chi_1(y)$ (see Figure 3.2).

Figure 3.2: Unit cell: Layered material

3.3 Notation and Preliminary Results

On the unit cell $Y$, the constitutive law for the nonlinear material is given by (3.1) with exponents $p_1$ and $p_2$ satisfying $2 \leq p_1 \leq p_2$. Their Hölder conjugates (or dual conjugates) are denoted by $q_2 = p_1/(p_1 - 1)$ and $q_1 = p_2/(p_2 - 1)$ respectively. For $i = 1, 2$, $W^{1,p_i}(Y)$ denotes the set of all functions $u \in W^{1,p_i}(Y)$ with mean value zero that have the same trace on the opposite faces of $Y$. Each function $u \in W^{1,p_i}(Y)$ can be extended by periodicity to a function of $W^{1,p_i}_{\text{per}}(\mathbb{R}^n)$.

The Euclidean norm and the scalar product in $\mathbb{R}^n$ are denoted by $|\cdot|$ and $(\cdot, \cdot)$, respectively. If $D \subset \mathbb{R}^n$, $|D|$ denotes the Lebesgue measure and $\chi_D(x)$ denotes its characteristic function.

The constitutive law for the $\epsilon$-periodic composite is described by $A_\epsilon(x, \xi) = A(x/\epsilon, \xi)$, for every $\epsilon > 0$, for every $x \in \Omega$, and for every $\xi \in \mathbb{R}^n$.

In the following, the letter $C$ will represent a generic positive constant independent of $\epsilon$, and it can take different values from line to line.

3.3.1 Properties of $A$

The function $A$, defined in (3.1), satisfies the following properties:

1. For all $\xi \in \mathbb{R}^n$, $A(\cdot, \xi)$ is $Y$-periodic and Lebesgue measurable.

2. $|A(y, 0)| = 0$ for all $y \in \mathbb{R}^n$.

3. Continuity: for almost every $y \in \mathbb{R}^n$ and for every $\xi_i \in \mathbb{R}^n$ ($i = 1, 2$) we have

$$|A(y, \xi_1) - A(y, \xi_2)| \leq C \left(\chi_1(y) |\xi_1 - \xi_2| (1 + |\xi_1| + |\xi_2|)^{p_1-2} + \chi_2(y) |\xi_1 - \xi_2| (1 + |\xi_1| + |\xi_2|)^{p_2-2}\right). \quad (3.4)$$

4. Monotonicity: for almost every $y \in \mathbb{R}^n$ and for every $\xi_i \in \mathbb{R}^n$ ($i = 1, 2$) we have

$$\langle A(y, \xi_1) - A(y, \xi_2), \xi_1 - \xi_2 \rangle \geq C (\chi_1(y) |\xi_1 - \xi_2|^{p_1} + \chi_2(y) |\xi_1 - \xi_2|^{p_2}). \quad (3.5)$$
Proof of (3.4): Continuity of $A$

Proof. By (3.1), we have

$$|A(y, \xi_1) - A(y, \xi_2)|$$

By (3.2), we have

$$|\sigma_1 \chi_1 (y) |\xi_1|^{p_1-2} \xi_1 + \sigma_2 \chi_2 (y) |\xi_1|^{p_2-2} \xi_1 - \sigma_1 \chi_1 (y) |\xi_2|^{p_1-2} \xi_2 - \sigma_2 \chi_2 (y) |\xi_2|^{p_2-2} \xi_2|$$

$$\leq |\sigma_1 \chi_1 (y) (|\xi_1|^{p_1-2} |\xi_1| - |\xi_2|^{p_1-2} |\xi_2|) + |\sigma_2 \chi_2 (y) (|\xi_1|^{p_2-2} |\xi_1| - |\xi_2|^{p_2-2} |\xi_2|)|$$

$$\leq C (\chi_1 (y) (|\xi_1|^{p_1-2} |\xi_1| - |\xi_2|^{p_1-2} |\xi_2|) + \chi_2 (y) (|\xi_1|^{p_2-2} |\xi_1| - |\xi_2|^{p_2-2} |\xi_2|))$$

Let us study the expression

$$||\xi_1|^{p_1-2} - |\xi_2|^{p_1-2}|, \text{ for } i = 1, 2.$$}

Observe that

$$||\xi_1|^{p_1-2} - |\xi_2|^{p_1-2}|^2$$

$$= |\xi_1|^{2(p_1-1)} + |\xi_2|^{2(p_1-1)} - 2 |\xi_1|^{p_1-2} |\xi_2|^{p_1-2} \xi_1 \cdot \xi_2$$

$$= |\xi_1|^{2(p_1-1)} + |\xi_2|^{2(p_1-1)} - 2 |\xi_1|^{p_1-1} |\xi_2|^{p_1-1} + 2 |\xi_1|^{p_1-1} |\xi_2|^{p_1-1} - 2 |\xi_1|^{p_1-2} |\xi_2|^{p_1-2} \xi_1 \cdot \xi_2$$

$$= (|\xi_1|^{p_1-1} - |\xi_2|^{p_1-1})^2 + 2 |\xi_1|^{p_1-2} |\xi_2|^{p_1-2} (|\xi_1| |\xi_2| - \xi_1 \cdot \xi_2)$$

$$= (|\xi_1|^{p_1-1} - |\xi_2|^{p_1-1})^2 + (|\xi_1|^{p_1-2} + |\xi_2|^{p_1-2})^2 (|\xi_1| + |\xi_2| - \xi_1 \cdot \xi_2)$$

By Remark 3.1, we have

$$\leq (|\xi_1|^{p_1-1} - |\xi_2|^{p_1-1})^2 + |\xi_1|^{p_1-2} |\xi_2|^{p_1-2} |\xi_1 - \xi_2|^2$$

By Remark 3.2, we have

$$\leq ||\xi_1|^{p_1-1} - |\xi_2|^{p_1-1}|^2 + (|\xi_1|^{p_1-2} + |\xi_2|^{p_1-2})^2 |\xi_1 - \xi_2|^2$$

And by Remark 3.3, we have

$$\leq (p_i - 1)^2 ((|\xi_1|^{p_1-2} + |\xi_2|^{p_1-2})^2 |\xi_1 - \xi_2| + ((|\xi_1|^{p_1-2} + |\xi_2|^{p_1-2})^2 |\xi_1 - \xi_2|$$

$$\leq C (|\xi_1|^{p_1-2} + |\xi_2|^{p_1-2})^2 |\xi_1 - \xi_2|^2.$$

Taking square root on both sides, we obtain

$$|\xi_1|^{p_1-2} \xi_1 - |\xi_2|^{p_2-2} \xi_2 \leq C (|\xi_1| + |\xi_2|) p_i-2 |\xi_1 - \xi_2|,$$

which proves (3.4).
Remark 3.1. Observe that
\[ 0 \leq (|\xi_1| - |\xi_2|)^2 \iff 0 \leq |\xi_1|^2 - 2|\xi_1||\xi_2| + |\xi_2|^2 \]
\[ \iff 2|\xi_1||\xi_2| \leq |\xi_1|^2 + |\xi_2|^2 \]
\[ \iff 2(|\xi_1||\xi_2| - 2\xi_1 \cdot \xi_2 \leq |\xi_1|^2 - 2\xi_1 \cdot \xi_2 + |\xi_2|^2 \]
\[ \iff 2(|\xi_1||\xi_2| - \xi_1 \cdot \xi_2) \leq |\xi_1 - \xi_2|^2. \]

Remark 3.2. For \( i = 1, 2 \), we have
\[ |\xi_i|^{p_i - 2} \leq (|\xi_i|^{p_i - 2} + |\xi_2|^{p_i - 2}). \]

Remark 3.3. Consider the function \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) defined by
\[ f(x) = x^{p_i - 1}, \]
where \( p_i \geq 2 \).
Since \( f \) is a convex function, it satisfies
\[ \begin{aligned}
&f'(|\xi_1|) (|\xi_2| - |\xi_1|) \leq f(|\xi_2|) - f(|\xi_1|), \\
&f'(|\xi_2|) (|\xi_1| - |\xi_2|) \leq f(|\xi_1|) - f(|\xi_2|);
\end{aligned} \]
or equivalently,
\[ \begin{aligned}
&(p_i - 1) |\xi_1|^{p_i - 2} (|\xi_2| - |\xi_1|) \leq |\xi_2|^{p_i - 1} - |\xi_1|^{p_i - 1}, \\
&(p_i - 1) |\xi_2|^{p_i - 2} (|\xi_1| - |\xi_2|) \leq |\xi_1|^{p_i - 1} - |\xi_2|^{p_i - 1}.
\end{aligned} \]
Then
\[ (p_i - 1) |\xi_2|^{p_i - 2} (|\xi_1| - |\xi_2|) \leq |\xi_1|^{p_i - 1} - |\xi_2|^{p_i - 1} \leq (p_i - 1) |\xi_1|^{p_i - 2} (|\xi_1| - |\xi_2|). \]
Therefore we have
\[ \| |\xi_1|^{p_i - 1} - |\xi_2|^{p_i - 1} \| \leq (p_i - 1) (|\xi_1|^{p_i - 2} + |\xi_2|^{p_i - 2}) \|\xi_1| - |\xi_2||. \]

Proof of (3.5): Monotonicity of \( A \)
Proof. By (3.1), we have
\[ (A(y, \xi_1) - A(y, \xi_2), \xi_1 - \xi_2) \]
\[ = (A(y, \xi_1), \xi_1) - (A(y, \xi_1), \xi_2) - (A(y, \xi_2), \xi_1) + (A(y, \xi_2), \xi_2) \]
\[ = \sigma_1 \chi_1 (y) (|\xi_1|^{p_1} + |\xi_2|^{p_2} - \sigma_2 \chi_2 (y) |\xi_1|^{p_1 - 2} \xi_1 \cdot \xi_2 - |\xi_2|^{p_2 - 2} \xi_1 \cdot \xi_2 - \sigma_2 \chi_2 (y) |\xi_1|^{p_1} + \sigma_2 \chi_2 (y) |\xi_2|^{p_2}) \]
\[ = \sigma_1 \chi_1 (y) \left[ |\xi_1|^{p_1} - |\xi_1|^{p_1 - 2} \xi_1 \cdot \xi_2 - |\xi_2|^{p_2 - 2} \xi_1 \cdot \xi_2 + |\xi_2|^{p_1} \right] \]
\[ + \sigma_2 \chi_2 (y) \left[ |\xi_2|^{p_2} - |\xi_1|^{p_1 - 2} \xi_1 \cdot \xi_2 - |\xi_2|^{p_2 - 2} \xi_1 \cdot \xi_2 + |\xi_2|^{p_2} \right] \]
\[ = \sigma_1 \chi_1 (y) \left[ |\xi_1|^{p_1} + |\xi_2|^{p_1} - \xi_1 \cdot \xi_2 \left( |\xi_1|^{p_1 - 2} - |\xi_2|^{p_1 - 2} \right) \right] \]
\[ + \sigma_2 \chi_2 (y) \left[ |\xi_1|^{p_2} + |\xi_2|^{p_2} - \xi_1 \cdot \xi_2 \left( |\xi_1|^{p_2 - 2} - |\xi_2|^{p_2 - 2} \right) \right] \]
Let us study the expression

$$|\xi_1|^{p_i} + |\xi_2|^{p_i} - \xi_1 \cdot \xi_2 \left( |\xi_1|^{p_i-2} + |\xi_2|^{p_i-2} \right), \text{ for } i = 1, 2.$$  

- If $|\xi_1| = |\xi_2|:

$$|\xi_1|^{p_i} + |\xi_2|^{p_i} - \xi_1 \cdot \xi_2 \left( |\xi_1|^{p_i-2} + |\xi_2|^{p_i-2} \right)
= |\xi_1|^{p_i} + |\xi_2|^{p_i} - 2|\xi_1|^{p_i-2} \xi_1 \cdot \xi_2
= |\xi_1|^{p_i-2} \left(|\xi_1|^2 + |\xi_2|^2 - 2\xi_1 \cdot \xi_2\right)
= |\xi_1|^{p_i-2} \left(|\xi_1|^2 - 2\xi_1 \cdot \xi_2 + |\xi_2|^2\right)
= |\xi_1|^{p_i-2} |\xi_1 - \xi_2|^2.$$ 

Since

$$\frac{1}{2} \left| \frac{\xi_1}{|\xi_1|} - \frac{\xi_2}{|\xi_2|} \right| = \frac{1}{2} \left| \frac{\xi_1}{|\xi_1|} - \frac{\xi_2}{|\xi_2|} \right| \leq 1,$$

then

$$|\xi_1| \geq \frac{1}{2} |\xi_1 - \xi_2|.$$ 

Therefore

$$|\xi_1|^{p_i} + |\xi_2|^{p_i} - \xi_1 \cdot \xi_2 \left( |\xi_1|^{p_i-2} + |\xi_2|^{p_i-2} \right) = |\xi_1|^{p_i-2} |\xi_1 - \xi_2|^2
\geq \left(\frac{1}{2}\right)^{p_i-2} \left|\xi_1 - \xi_2\right|^{p_i-2} \left|\xi_1 - \xi_2\right|^2
= 2^{2-p_i} |\xi_1 - \xi_2|^{p_i}.$$ 

- If $|\xi_1| > |\xi_2| > 0$, we can write

$$\xi_2 = \beta \xi_1 + \gamma \omega,$$

where $\omega \neq \bar{0}$ is a vector orthogonal to $\xi_1$, and $\beta, \gamma \in \mathbb{R}$ with $|\beta| < 1$.

Since

$$\xi_1 \cdot \xi_2 = \xi_1 \cdot (\beta \xi_1 + \gamma \omega) = \beta |\xi_1|^2,$$  

we obtain

$$|\xi_1|^{p_i} + |\xi_2|^{p_i} - \xi_1 \cdot \xi_2 \left( |\xi_1|^{p_i-2} + |\xi_2|^{p_i-2} \right) = |\xi_1|^{p_i} + |\xi_2|^{p_i} - \beta |\xi_1|^2 \left( |\xi_1|^{p_i-2} + |\xi_2|^{p_i-2} \right).$$ 

- For $\beta \leq 0$:

$$|\xi_1|^{p_i} + |\xi_2|^{p_i} - \beta |\xi_1|^2 \left( |\xi_1|^{p_i-2} + |\xi_2|^{p_i-2} \right) \geq |\xi_1|^{p_i} + |\xi_2|^{p_i}
\geq 2^{1-p_i} |\xi_1 - \xi_2|^{p_i}.$$
- For $0 < \beta < \frac{1}{4}$:

$$|\xi_1|^{p_1} + |\xi_2|^{p_2} - \beta |\xi_1|^2 (|\xi_1|^{p_1-2} + |\xi_2|^{p_2-2}) = |\xi_1|^{p_1} + |\xi_2|^{p_2} - \beta |\xi_1|^{p_1} - \beta |\xi_2|^{p_2-2} |\xi_1|^2$$

$$\geq |\xi_1|^{p_1} - 2\beta |\xi_1|^{p_1}$$

$$= |\xi_1|^{p_1} (1 - 2\beta)$$

$$> \frac{1}{2} |\xi_1|^{p_1}$$

$$= \frac{1}{4} |\xi_1|^{p_1} + \frac{1}{4} |\xi_1|^{p_1}$$

$$> \frac{|\xi_2|^{p_1}}{4} + \frac{|\xi_1|^{p_1}}{4}$$

$$\geq 2^{-(p_1+1)} |\xi_1 - \xi_2|^{p_1}.$$

- For $\frac{1}{4} \leq \beta < 1$:

$$|\xi_1|^{p_1} + |\xi_2|^{p_2} - (|\xi_1|^{p_1-2} + |\xi_2|^{p_2-2}) \xi_1 \cdot \xi_2$$

$$= |\xi_1|^{p_1} - |\xi_1|^{p_1-2} \xi_1 \cdot \xi_2 + |\xi_2|^{p_2} - |\xi_2|^{p_2-2} \xi_1 \cdot \xi_2$$

$$= |\xi_1|^{p_1-2} (|\xi_1|^2 - \xi_1 \cdot \xi_2) + |\xi_2|^{p_2-2} (|\xi_2|^2 - \xi_1 \cdot \xi_2)$$

$$\geq |\xi_2|^{p_2-2} (|\xi_1|^2 - \xi_1 \cdot \xi_2) + |\xi_2|^{p_2-2} (|\xi_2|^2 - \xi_1 \cdot \xi_2)$$

$$= |\xi_2|^{p_2-2} (|\xi_1|^2 - 2\xi_1 \cdot \xi_2 + |\xi_2|^2)$$

$$= |\xi_2|^{p_2-2} |\xi_1 - \xi_2|^2.$$

By (3.6), we have

$$\frac{|\xi_1|}{|\xi_2|} \leq \frac{1}{\beta} \leq 4.$$

Therefore

$$\frac{|\xi_1 - \xi_2|}{|\xi_2|} \leq \frac{|\xi_1|}{|\xi_2|} + 1 \leq 5;$$

obtaining this way

$$|\xi_2| \geq 5^{-1} |\xi_1 - \xi_2|.$$

Then

$$|\xi_1|^{p_1} + |\xi_2|^{p_2} - (|\xi_1|^{p_1-2} + |\xi_2|^{p_2-2}) \xi_1 \cdot \xi_2 = |\xi_2|^{p_2-2} |\xi_1 - \xi_2|^2$$

$$\geq 5^{2-p_1} |\xi_1 - \xi_2|^{p_2-2} |\xi_1 - \xi_2|^2$$

$$= 5^{2-p_1} |\xi_1 - \xi_2|^{p_2}.$$

Taking $C = \min \{5^{2-p_1}, 2^{-(p_1+1)}\}$, we have proved

$$|\xi_1|^{p_1} + |\xi_2|^{p_2} - \xi_1 \cdot \xi_2 (|\xi_1|^{p_1-2} + |\xi_2|^{p_2-2}) \geq C |\xi_1 - \xi_2|^{p_1},$$

for $p_i \geq 2$.

Therefore

$$(A(y, \xi_1) - A(y, \xi_2), \xi_1 - \xi_2) \geq C (\chi_1(y) |\xi_1 - \xi_2|^{p_1} + \chi_2(y) |\xi_1 - \xi_2|^{p_2}).$$

A different proof for the monotonicity and continuity of $A$ can be found in [Bys05].

\[\Box\]
3.3.2 Dirichlet BVP and Homogenization Theorem

We shall consider the following Dirichlet boundary value problem

\[
\begin{cases}
-\text{div} \left( A_\epsilon(x, \nabla u_\epsilon) \right) = f \text{ on } \Omega, \\
u_\epsilon \in W^{1,p}_0(\Omega); 
\end{cases}
\]  

(3.7)

where \( f \in W^{-1,q}_2(\Omega) \).

The following homogenization result holds.

Theorem 3.4 (Homogenization Theorem). As \( \epsilon \to 0 \), the solutions \( u_\epsilon \) of (3.7) converge weakly to \( u \) in \( W^{1,p}_1(\Omega) \), where \( u \) is the solution of

\[
\begin{cases}
-\text{div} \left( b(\nabla u) \right) = f \text{ on } \Omega, \\
u \in W^{1,p}_0(\Omega); 
\end{cases}
\]  

(3.8)

and the function \( b : \mathbb{R}^n \to \mathbb{R}^n \) (independent of \( f \) and \( \Omega \)) is defined for all \( \xi \in \mathbb{R}^n \) by

\[
b(\xi) = \int_Y A(y, p(y, \xi)) dy,
\]  

(3.9)

where \( p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) is defined by

\[
p(y, \xi) = \xi + \nabla v_\xi(y),
\]  

(3.10)

where \( v_\xi \) is the solution to the cell problem:

\[
\begin{cases}
\int_Y (A(y, \xi + \nabla v_\xi), \nabla w) dy = 0, \text{ for every } w \in W^{1,p}_\text{per}(Y), \\
v_\xi \in W^{1,p}_\text{per}(Y).
\end{cases}
\]  

(3.11)

For a proof of Theorem 3.4, see Chapter 15 of [ZKO94].

Lemma 3.5. The following a priori bound is satisfied

\[
\sup_{\epsilon > 0} \left( \int_\Omega \chi_1(x) |\nabla u_\epsilon(x)|^{p_1} dx + \int_\Omega \chi_2(x) |\nabla u_\epsilon(x)|^{p_2} dx \right) \leq C < \infty.
\]  

(3.12)

Proof. The weak formulation for (3.7) is given by

\[
\int_\Omega (A_\epsilon(x, \nabla u_\epsilon), \nabla \varphi) dx = \int_\Omega f(x) \varphi(x) dx,
\]

for all \( \varphi \in W^{1,p}_0(\Omega) \).

In particular, taking \( \varphi = u_\epsilon \) above, we obtain

\[
\int_\Omega (A_\epsilon(x, \nabla u_\epsilon), \nabla u_\epsilon) dx = \int_\Omega f(x) u_\epsilon(x) dx.
\]

19
The last equation can be rewritten as

\[
\sigma_1 \int_{\Omega} \chi_1^\epsilon(x) |\nabla u_\epsilon|^{p_1} dx + \sigma_2 \int_{\Omega} \chi_2^\epsilon(x) |\nabla u_\epsilon|^{p_2} dx = \int_{\Omega} f(x) u_\epsilon(x) dx. \tag{3.13}
\]

Applying Hölder’s inequality to the right hand side of (3.13), we obtain

\[
\int_{\Omega} f(x) u_\epsilon(x) dx = \int_{\Omega} \chi_1^\epsilon(x) f(x) u_\epsilon(x) dx + \int_{\Omega} \chi_1^\epsilon(x) f(x) u_\epsilon(x) dx \\
\leq \left( \int_{\Omega} |f(x)|^{q_2} dx \right)^{\frac{1}{q_2}} \left( \int_{\Omega} \chi_1^\epsilon(x) |u_\epsilon(x)|^{p_1} dx \right)^{\frac{1}{p_1}} \\
+ \left( \int_{\Omega} |f(x)|^{q_1} dx \right)^{\frac{1}{q_1}} \left( \int_{\Omega} \chi_2^\epsilon(x) |u_\epsilon(x)|^{p_2} dx \right)^{\frac{1}{p_2}}. \tag{3.14}
\]

If we combine (3.13) and (3.14), and we use the fact that \( f \in W^{-1, q_2}(\Omega) \), we get

\[
\sigma_1 \int_{\Omega} \chi_1^\epsilon(x) |\nabla u_\epsilon|^{p_1} dx + \sigma_2 \int_{\Omega} \chi_2^\epsilon(x) |\nabla u_\epsilon|^{p_2} dx \\
\leq C \left[ \left( \int_{\Omega} \chi_1^\epsilon(x) |u_\epsilon(x)|^{p_1} dx \right)^{\frac{1}{p_1}} + \left( \int_{\Omega} \chi_2^\epsilon(x) |u_\epsilon(x)|^{p_2} dx \right)^{\frac{1}{p_2}} \right].
\]

Poincaré’s inequality (see [Neč67]) gives

\[
\leq C \left[ \left( \int_{\Omega} \chi_1^\epsilon(x) |\nabla u_\epsilon|^{p_1} dx \right)^{\frac{1}{p_1}} + \left( \int_{\Omega} \chi_2^\epsilon(x) |\nabla u_\epsilon|^{p_2} dx \right)^{\frac{1}{p_2}} \right]
\]

and applying Young’s inequality, we obtain

\[
\leq C \left[ \frac{\delta^{p_1}}{p_1} \int_{\Omega} \chi_1^\epsilon(x) |\nabla u_\epsilon|^{p_1} dx + \frac{\delta^{-q_2}}{q_2} + \frac{\delta^{p_2}}{p_2} \int_{\Omega} \chi_2^\epsilon(x) |\nabla u_\epsilon|^{p_2} dx + \frac{\delta^{-q_1}}{q_1} \right].
\]

By rearranging the terms in the inequality, one gets

\[
\left( \sigma_1 - C \frac{\delta^{p_1}}{p_1} \right) \int_{\Omega} \chi_1^\epsilon(x) |\nabla u_\epsilon|^{p_1} dx + \left( \sigma_2 - C \frac{\delta^{p_2}}{p_2} \right) \int_{\Omega} \chi_2^\epsilon(x) |\nabla u_\epsilon|^{p_2} dx \\
\leq \frac{\delta^{-q_2}}{q_2} + \frac{\delta^{-q_1}}{q_1}.
\]

Therefore, by taking \( \delta \) small enough so that \( \min \left\{ \sigma_1 - C \frac{\delta^{p_1}}{p_1}, \sigma_2 - C \frac{\delta^{p_2}}{p_2} \right\} \) is positive, one obtains

\[
\int_{\Omega} \chi_1^\epsilon(x) |\nabla u_\epsilon|^{p_1} dx + \int_{\Omega} \chi_2^\epsilon(x) |\nabla u_\epsilon|^{p_2} dx \leq C,
\]

where \( C \) does not depend on \( \epsilon \). \qed
3.3.3 Properties of $b$

The function $b$, defined in (3.9), satisfies the following properties

1. Continuity: for every $\xi_1, \xi_2 \in \mathbb{R}^n$, we have

$$|b(\xi_1) - b(\xi_2)| \leq C \left[ |\xi_1 - \xi_2|^{\frac{1}{p_1 - 1}} (1 + |\xi_1|^{p_1} + |\xi_2|^{p_1} + |\xi_1|^{p_2} + |\xi_2|^{p_2})^{\frac{p_1 - 2}{p_1 - 1}} + |\xi_1 - \xi_2|^{\frac{1}{p_2 - 1}} (1 + |\xi_1|^{p_1} + |\xi_2|^{p_1} + |\xi_1|^{p_2} + |\xi_2|^{p_2})^{\frac{p_2 - 2}{p_2 - 1}} \right]$$

(3.15)

2. Monotonicity: for every $\xi_1, \xi_2 \in \mathbb{R}^n$, we have

$$(b(\xi_1) - b(\xi_2), \xi_1 - \xi_2) \geq C \left( \int_Y \chi_1(y) |p(y, \xi_1) - p(y, \xi_2)|^{p_1} dy + \int_Y \chi_2(y) |p(y, \xi_1) - p(y, \xi_2)|^{p_2} dy \right)$$

$$\geq 0.$$ 

Proof of Continuity of $b$ (3.15)

Proof. By (3.9), (3.4), we have

$$|b(\xi_1) - b(\xi_2)|$$

$$= \left| \int_Y A(y, p(y, \xi_1)) dy - \int_Y A(y, p(y, \xi_2)) dy \right|$$

$$\leq \int_Y |A(y, p(y, \xi_1)) - A(y, p(y, \xi_2))| dy$$

$$\leq C \left[ \int_Y \chi_1(y) |p(y, \xi_1) - p(y, \xi_2)| (1 + |p(y, \xi_1)| + |p(y, \xi_2)|)^{p_1 - 2} dy$$

$$+ \int_Y \chi_2(y) |p(y, \xi_1) - p(y, \xi_2)| (1 + |p(y, \xi_1)| + |p(y, \xi_2)|)^{p_2 - 2} dy \right]$$

Applying Hölder’s inequality in both integrals, we obtain

$$\leq C \left[ \left( \int_Y \chi_1(y) |p(y, \xi_1) - p(y, \xi_2)|^{p_1} dy \right)^{\frac{1}{p_1}}$$

$$\times \left( \int_Y \chi_1(y) (1 + |p(y, \xi_1)| + |p(y, \xi_2)|)^{q_1(p_1 - 2)} dy \right)^{\frac{1}{q_1}}$$

$$+ \left( \int_Y \chi_2(y) |p(y, \xi_1) - p(y, \xi_2)|^{p_2} dy \right)^{\frac{1}{p_2}}$$

$$\times \left( \int_Y \chi_2(y) (1 + |p(y, \xi_1)| + |p(y, \xi_2)|)^{q_2(p_2 - 2)} dy \right)^{\frac{1}{q_2}} \right]$$

21
By (3.5), we have
\[
\leq C \left[ \left( \int_Y \chi_1(y) (A(y, p(y, \xi_1)) - A(y, p(y, \xi_2)), p(y, \xi_1) - p(y, \xi_2)) \, dy \right)^{\frac{1}{p_1}} 
\times \left( \int_Y \chi_1(y)(1 + |p(y, \xi_1)| + |p(y, \xi_2)|)^{q_2(p_1-2)} \, dy \right)^{\frac{1}{q_2}} 
+ \left( \int_Y \chi_2(y) (A(y, p(y, \xi_1)) - A(y, p(y, \xi_2)), p(y, \xi_1) - p(y, \xi_2)) \, dy \right)^{\frac{1}{p_2}} 
\times \left( \int_Y \chi_2(y)(1 + |p(y, \xi_1)| + |p(y, \xi_2)|)^{q_1(p_2-2)} \, dy \right)^{\frac{1}{q_2}} \right]^{\frac{1}{p_1}} 
\]
\[
\leq C \left[ \left( \int_Y (A(y, p(y, \xi_1)) - A(y, p(y, \xi_2)), p(y, \xi_1) - p(y, \xi_2)) \, dy \right)^{\frac{1}{p_1}} 
\times \left( \int_Y \chi_1(y)(1 + |p(y, \xi_1)| + |p(y, \xi_2)|)^{q_2(p_1-2)} \, dy \right)^{\frac{1}{q_2}} 
+ \left( \int_Y (A(y, p(y, \xi_1)) - A(y, p(y, \xi_2)), p(y, \xi_1) - p(y, \xi_2)) \, dy \right)^{\frac{1}{p_2}} 
\times \left( \int_Y \chi_2(y)(1 + |p(y, \xi_1)| + |p(y, \xi_2)|)^{q_1(p_2-2)} \, dy \right)^{\frac{1}{q_2}} \right]^{\frac{1}{p_1}} 
\]
By (3.11), we get
\[
= C \left[ (b(\xi_1) - b(\xi_2), \xi_1 - \xi_2)^{\frac{1}{p_1}} \left( \int_Y \chi_1(y)(1 + |p(y, \xi_1)| + |p(y, \xi_2)|)^{q_2(p_1-2)} \, dy \right)^{\frac{1}{q_2}} 
+ (b(\xi_1) - b(\xi_2), \xi_1 - \xi_2)^{\frac{1}{p_2}} \left( \int_Y \chi_2(y)(1 + |p(y, \xi_1)| + |p(y, \xi_2)|)^{q_1(p_2-2)} \, dy \right)^{\frac{1}{q_2}} \right]^{\frac{1}{p_1}} 
\]
Applying the Cauchy-Schwarz inequality and Hölder’s inequality we have
\[
\leq C \left[ (b(\xi_1) - b(\xi_2)) \frac{1}{p_1} |\xi_1 - \xi_2|^{\frac{1}{p_2}} \left( \int_Y \chi_1(y)1^{p_1-1} \right)^{\frac{1}{p_1}} \left( \frac{1}{p_1-1} \right)^{\frac{1}{p_1-1}} \left( \frac{p_1-1}{p_1} \right)^{\frac{p_1}{p_1-1}} \right]^{\frac{1}{p_1}} 
\times \left( \int_Y \chi_1(y)(1 + |p(y, \xi_1)| + |p(y, \xi_2)|)^{\frac{p_1}{p_1-1}(p_1-1)} \, dy \right)^{\frac{1}{p_1-1}} 
\times \left( \int_Y \chi_2(y)(1 + |p(y, \xi_1)| + |p(y, \xi_2)|)^{\frac{p_2}{p_2-1}(p_2-1)} \, dy \right)^{\frac{1}{p_2-1}} 
\leq C \left[ (b(\xi_1) - b(\xi_2)) \frac{1}{p_2} |\xi_1 - \xi_2|^{\frac{1}{p_2}} \left( \int_Y \chi_2(y)1^{p_1-1} \right)^{\frac{1}{p_2}} \left( \frac{1}{p_2-1} \right)^{\frac{1}{p_2-1}} \right]^{\frac{1}{p_2}} 
\times \left( \int_Y \chi_2(y)(1 + |p(y, \xi_1)| + |p(y, \xi_2)|)^{\frac{p_2}{p_2-1}(p_2-1)} \, dy \right)^{\frac{1}{p_2-1}} 
\right]^{\frac{1}{p_1}}
Lemma 5.3 delivers

Applying Young’s inequality we obtain

\[
\leq C \left[ b(\xi_1) - b(\xi_2) \right]^\frac{1}{\theta_1} \left( \int_Y \chi_1(y) \left( 1 + |p(y, \xi_1)| + |p(y, \xi_2)| \right)^{p_1} \, dy \right]^{\frac{p_1-2}{p_1}} \\
+ b(\xi_1) - b(\xi_2) \right]^\frac{1}{\theta_2} \left( \int_Y \chi_2(y) \left( 1 + |p(y, \xi_1)| + |p(y, \xi_2)| \right)^{p_2} \, dy \right]^{\frac{p_2-2}{p_2}}
\]

Therefore

\[
\left[ 1 - C \left( \frac{\delta^{p_1}}{p_1} + \frac{\delta^{p_2}}{p_2} \right) \right] \left| b(\xi_1) - b(\xi_2) \right|
\leq C \left[ \frac{\delta^{-q_1} |\xi_1 - \xi_2|^{\theta_2^{p_2-1}}}{q_2} \left( 1 + |\xi_1|^{p_1} \theta_1 + |\xi_2|^{p_1} \theta_1 + |\xi_1|^{p_2} \theta_2 + |\xi_2|^{p_2} \theta_2 \right)^{\frac{p_2-2}{p_2}} \\
+ \frac{\delta^{-q_2} |\xi_1 - \xi_2|^{\theta_1^{p_1-1}}}{q_2} \left( 1 + |\xi_1|^{p_1} \theta_1 + |\xi_2|^{p_1} \theta_1 + |\xi_1|^{p_2} \theta_2 + |\xi_2|^{p_2} \theta_2 \right)^{\frac{p_1-2}{p_1}} \right]
\]

Taking \( \delta \) small enough, we obtain

\[
\left| b(\xi_1) - b(\xi_2) \right| 
\leq C \left[ \frac{\delta^{-q_1} |\xi_1 - \xi_2|^{\theta_2^{p_2-1}}}{q_2} \left( 1 + |\xi_1|^{p_1} \theta_1 + |\xi_2|^{p_1} \theta_1 + |\xi_1|^{p_2} \theta_2 + |\xi_2|^{p_2} \theta_2 \right)^{\frac{p_2-2}{p_2}} \\
+ \frac{\delta^{-q_2} |\xi_1 - \xi_2|^{\theta_1^{p_1-1}}}{q_2} \left( 1 + |\xi_1|^{p_1} \theta_1 + |\xi_2|^{p_1} \theta_1 + |\xi_1|^{p_2} \theta_2 + |\xi_2|^{p_2} \theta_2 \right)^{\frac{p_1-2}{p_1}} \right].
\]
Proof of (3.16): Monotonicity of $b$

Proof. Using (3.11) and (3.5), we have

\[
(b(\xi_2) - b(\xi_1), \xi_2 - \xi_1) = \left( \int_Y A(y, p(y, \xi_2))dy - \int_Y A(y, p(y, \xi_1))dy, \xi_2 - \xi_1 \right)
\]

\[
= \int_Y (A(y, p(y, \xi_2)) - A(y, p(y, \xi_1)), \xi_2 - \xi_1) dy
\]

\[
= \int_Y (A(y, p(y, \xi_2)) - A(y, p(y, \xi_1)), p(y, \xi_2) - p(y, \xi_1)) dy
\]

\[
\geq C \left( \int_Y \chi_1(y) |p(y, \xi_1) - p(y, \xi_2)|^{p_1} dy + \int_Y \chi_2(y) |p(y, \xi_1) - p(y, \xi_2)|^{p_2} dy \right) \geq 0.
\]

\[
\square
\]

3.3.4 Properties of $p$

Since the solution $v_\xi$ of (3.11) can be extended by periodicity to a function of $W^{1,p_1}_{loc}(\mathbb{R}^n)$, then (3.11) is equivalent to $-div(A(y, \xi + \nabla v_\xi(y))) = 0$ over $D(\mathbb{R}^n)$, i.e.,

\[
-div(A(y, p(y, \xi))) = 0 \text{ in } D'(\mathbb{R}^n) \text{ for every } \xi \in \mathbb{R}^n.
\] (3.17)

Moreover, by (3.11), we have

\[
\int_Y (A(y, p(y, \xi)), p(y, \xi)) dy = \int_Y (A(y, p(y, \xi)), \xi) dy = (b(\xi), \xi).
\] (3.18)

For $\epsilon > 0$, define $p_\epsilon : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

\[
p_\epsilon(x, \xi) = p \left( \frac{x}{\epsilon}, \xi \right) = \xi + \nabla v_\xi \left( \frac{x}{\epsilon} \right).
\] (3.19)

where $v_\xi$ is the unique solution of (3.11).

The functions $p$ and $p_\epsilon$ are easily seen to satisfy the following properties

$p(\cdot, \xi)$ is $Y$-periodic and $p_\epsilon(x, \xi)$ is $\epsilon$-periodic in $x$.

\[
\int_Y p(y, \xi) dy = \xi.
\] (3.20)

\[
p_\epsilon(\cdot, \xi) \to \xi \text{ in } L^{p_1}(\Omega; \mathbb{R}^n) \text{ as } \epsilon \to 0.
\] (3.21)

$p(y, 0) = 0$ for almost every $y$.

\[
A \left( \frac{\cdot}{\epsilon}, p_\epsilon(\cdot, \xi) \right) \to b(\xi) \text{ in } L^{p_2}(\Omega; \mathbb{R}^n), \text{ as } \epsilon \to 0.
\] (3.22)
Chapter 4

Higher Order Integrability of the Homogenized Solution

In this chapter, we display higher order integrability results for the field gradients inside dispersed microstructures and layered materials. For dispersions of inclusions, the included material is taken to have a lower power-law exponent than that of the host phase. For both of these cases it is shown that the homogenized solution lies in $W^{1,2}_{0}(\Omega)$. In the following chapters we will apply these facts to establish strong approximations for the sequences $\{\chi_\epsilon \nabla u_\epsilon\}_{\epsilon > 0}$ in $L^{p_2}(\Omega, \mathbb{R}^n)$. The approach taken here is variational and uses the homogenized Lagrangian associated with $b(\xi)$ defined in (3.9). The integrability of the homogenized solution $u$ of (3.8) is determined by the growth of the homogenized Lagrangian with respect to its argument.

4.1 Statement of the Theorem on the Higher Order Integrability of the Homogenized Solution

We now state the higher order integrability properties of the homogenized solution for periodic dispersions of inclusions and layered microgeometries.

**Theorem 4.1.** Given a periodic dispersion of inclusions or a layered material then the solution $u$ of (3.8) belongs to $W^{1,2}_{0}(\Omega)$.

Before we can prove this theorem we need some definitions.

**Definition 4.2.** Functions $f(x, \xi)$ depending on two variables $x, \xi \in \mathbb{R}^n$ will be referred to as Lagrangians.

**Definition 4.3.** If the Lagrangian $f(x, \xi), \xi \in \mathbb{R}^n, x \in \Omega \subset \mathbb{R}^n$ satisfies

$$-c_0 + c_1 |\xi|^p \leq f(x, \xi) \leq c_2 |\xi|^p + c_0$$

(4.1)

with $c_0 \geq 0, c_1, c_2 > 0$, and $p > 1$, then it is called standard.

A much wider class of Lagrangians which includes the standard ones, is specified by the estimate

$$-c_0 + c_1 |\xi|^{p_1} \leq f(x, \xi) \leq c_2 |\xi|^{p_2} + c_0$$

(4.2)
with $p_2 \geq p_1 > 1$. These are called nonstandard Lagrangians.

**Definition 4.4.** The conjugate of a nonstandard Lagrangian $f$, denoted $g = f^*$, is defined by
\[
g(x, \xi) = \sup_{\eta \in \mathbb{R}^n} \{ \xi \cdot \eta - f(x, \eta) \},
\]
(4.3)
and satisfies the estimate
\[
-c_0 + c_1^* |\xi|^{q_1} \leq g(x, \xi) \leq c_2^* |\xi|^{q_2} + c_0.
\]
(4.4)

### 4.2 Proof of Higher Order Integrability of the Homogenized Solution

To proceed, we introduce the local Lagrangian associated with power-law composites. The Lagrangian corresponding to the problem studied here is given by
\[
\tilde{f}(x, \xi) = q(x) |\xi|^{p(x)} = \frac{\sigma_1}{p_1} \chi_1(x) |\xi|^{p_1} + \frac{\sigma_2}{p_2} \chi_2(x) |\xi|^{p_2},
\]
(4.5)
with
\[
q(x) = \frac{\sigma_1}{p_1} \chi_1(x) + \frac{\sigma_2}{p_2} \chi_2(x);
\]
where $\xi \in \mathbb{R}^n$ and $x \in \Omega \subset \mathbb{R}^n$. Here $\nabla_{\xi} \tilde{f}(x, \xi) = A(x, \xi)$, where $A(x, \xi)$ is given by (3.1).

We consider the rescaled Lagrangian
\[
\tilde{f}_\epsilon(x, \xi) = \tilde{f} \left( \frac{x}{\epsilon}, \xi \right) = \frac{\sigma_1}{p_1} \chi_1^{\epsilon}(x) |\xi|^{p_1} + \frac{\sigma_2}{p_2} \chi_2^{\epsilon}(x) |\xi|^{p_2},
\]
(4.6)
where $\chi_i^{\epsilon}(x) = \chi_i(x/\epsilon)$, $i = 1, 2$, $\xi \in \mathbb{R}^n$, and $x \in \Omega \subset \mathbb{R}^n$.

The Dirichlet problem given by (3.7) is associated with the variational problem given by
\[
E_\epsilon^1(f) = \inf_{u \in W^{1,p_1}_0(\Omega)} \left\{ \int_\Omega \tilde{f}_\epsilon(x, \nabla u)dx - \langle f, u \rangle \right\},
\]
(4.7)
with $f \in W^{-1,q_2}(\Omega)$. Here (3.7) is the Euler equation for (4.7). However, we also consider
\[
E_\epsilon^2(f) = \inf_{u \in W^{1,p_2}_0(\Omega)} \left\{ \int_\Omega \tilde{f}_\epsilon(x, \nabla u)dx - \langle f, u \rangle \right\},
\]
(4.8)
with $f \in W^{-1,q_2}(\Omega)$ (See [Zhi92]). Here $\langle \cdot, \cdot \rangle$ is the duality pairing between $W^{1,p_1}_0(\Omega)$ and $W^{-1,q_2}(\Omega)$.

From [ZKO94], we have $\lim_{\epsilon \to 0} E_\epsilon^i = E_i$, for $i = 1, 2$, where
\[
E_i = \inf_{u \in W^{1,p_i}_0(\Omega)} \left\{ \int_\Omega \tilde{f}_i(x, \nabla u(x))dx - \langle f, u \rangle \right\}.
\]
(4.9)
In (4.9), \( \hat{f}_i(\xi) \) is given by
\[
\hat{f}_i(\xi) = \inf_{v \in W^{1,p}_\text{per}(Y)} \int_Y \tilde{f}(y, \xi + \nabla v(y)) \, dy
\]
and satisfies
\[
-c_0 + c_1 |\xi|^{p_1} \leq \hat{f}_i(\xi) \leq c_2 |\xi|^{p_2} + c_0.
\]

In general, (see [Zhi95]) Lavrentiev phenomenon can occur and \( E_1 < E_2 \). However, for periodic dispersed and layered microstructures, no Lavrentiev phenomenon occurs and we have the following Homogenization Theorem.

**Theorem 4.5.** Homogenization Theorem for periodic dispersed and layered microstructures. For periodic dispersed and layered microstructures, the homogenized Dirichlet problems satisfy \( E_1 = E_2 \), where \( \hat{f} = \hat{f}_1 = \hat{f}_2 \) and \( c_2 + c_1 |\xi|^{p_2} \leq \hat{f}(\xi) \). Moreover, \( \nabla_\xi \hat{f}(\xi) = b(\xi) \), where \( b \) is the homogenized operator (3.9).

**Proof.** Theorem 4.5 has been proved for dispersed periodic media in Chapter 14 of [ZKO94]. We prove Theorem 4.5 for layers following the steps outlined in [ZKO94].

We first show that \( \hat{f} = \hat{f}_1 = \hat{f}_2 \) holds for layered media. Then we show that the homogenized Lagrangian \( \hat{f} \) satisfies the standard estimate given by
\[
-c_0 + c_1 |\xi|^{p_2} \leq \hat{f}(\xi) \leq c_2 |\xi|^{p_2} + c_0
\]
with \( c_0 \geq 0 \), and \( c_1, c_2 > 0 \).

We introduce the space of functions \( W^{1,p}_\text{per}(R_2) \) that belong to \( W^{1,p_2}(R_2) \) and are periodic on \( \partial R_2 \cap \partial Y \).

**Lemma 4.6.** Any function \( v \in W^{1,p_2}(R_2) \) can be extended to \( R_1 \) in such a way that the extension \( \tilde{v}(y) \) belongs to \( W^{1,p_2}_\text{per}(Y) \) and \( \tilde{v}(y) = v(y) \) on \( R_2 \).

**Proof.** Let \( \varphi \) to be the solution of
\[
\begin{aligned}
\Delta_{p_2} \varphi &= 0, & & \text{on } R_1 \\
\varphi \text{ takes periodic boundary values on opposite faces of } \partial Y \cap \partial R_1 \\
\varphi \big|_1 &= v \big|_2, & & \text{on } \Gamma
\end{aligned}
\]
Here the subscript 1 indicates the trace on the \( R_1 \) side of \( \Gamma \) and 2 indicates the trace on the \( R_2 \) side of \( \Gamma \). For a proof of existence of the solution \( \varphi \) see [Eva82] or [Lew77].

The extension \( \tilde{v} \) is given by
\[
\tilde{v} = \begin{cases} 
v, & \text{in } R_2, \\
\varphi, & \text{on } R_1. \end{cases}
\]
It is clear that \( \hat{f}_1 \leq \hat{f}_2 \). To prove \( \hat{f}_1 = \hat{f}_2 \), it suffices to show that for every \( v \in W^1_{\text{per}}(Y) \) satisfying \( \int_Y \tilde{f}(y, \xi + \nabla v(y)) dy < \infty \) there exists a sequence \( v_\epsilon \in W^1_{\text{per}}(Y) \) such that

\[
\lim_{\epsilon \to 0} \int_Y \tilde{f}(y, \xi + \nabla v_\epsilon(y)) dy = \int_Y \tilde{f}(y, \xi + \nabla v(y)) dy.
\]

For \( v \) as above, let \( \tilde{v} \) be as in Lemma 4.6 and set \( z = v - \tilde{v} \). It is clear that \( z \in W^1_{\text{per}}(R_1) \), is periodic on opposite faces of \( \partial Y \cap \partial R_1 \), zero on \( \Gamma \) and we write

\[
\int_Y \tilde{f}(y, \xi + \nabla v(y)) dy = \int_{R_2} f_2(\xi + \nabla \tilde{v}(y)) dy + \int_{R_1} f_1(\xi + \nabla \tilde{v}(y) + \nabla z(y)) dy,
\]

where \( f_1(\xi) = c_1 |\xi|^{p_1} \) and \( f_2(\xi) = c_2 |\xi|^{p_2} \); i.e, \( f_1 \) and \( f_2 \) are standard Lagrangians satisfying (4.1) with exponents \( p_1 \) and \( p_2 \) respectively.

We can choose a sequence \( \{z_\epsilon\}_{\epsilon > 0} \subseteq C_0^\infty(R_1) \) such that \( z_\epsilon \) vanishes in \( R_2 \) and \( z_\epsilon \to z \) in \( W^1_{\text{per}}(R_1) \).

Define \( v_\epsilon \in W^1_{\text{per}}(Y) \) by

\[
v_\epsilon = \begin{cases} v & \text{in } R_2, \\ \tilde{v} + z_\epsilon & \text{in } R_1. \end{cases}
\]

Since \( v_\epsilon \to v \) in \( W^1_{\text{per}}(Y) \), we see that

\[
\lim_{\epsilon \to 0} \int_Y \tilde{f}(y, \xi + \nabla v_\epsilon(y)) dy = \int_Y \tilde{f}(y, \xi + \nabla v(y)) dy.
\]

Therefore \( \hat{f} = \hat{f}_1 = \hat{f}_2 \) for layered media.

We establish (4.12) by introducing the convex conjugate of \( \hat{f} \). We denote the convex dual of \( \hat{f}_1(\xi) \) by \( \hat{g}_1(\xi) \) and the convex dual of \( \hat{f} \) by \( \hat{g} \).

It is easily verified (see [Zhi92]) that

\[
\hat{g}_i(\xi) = \inf_{w \in \text{Sol}^\eta(Y)} \int_Y \tilde{g}(y, \xi + w(y)) dy \tag{4.13}
\]

and

\[
-c_0 + c_1^* |\xi|^{p_1} \leq \hat{g}_i(\xi) \leq c_2^* |\xi|^{p_2} + c_0. \tag{4.14}
\]

Here \( \text{Sol}^\eta(Y) \) are the solenoidal vector fields belonging to \( L^\eta(Y, \mathbb{R}^n) \) and having mean value zero

\[
\text{Sol}^\eta(Y) = \{ w \in L^\eta(Y; \mathbb{R}^n) : \text{div } w = 0, w \cdot n \text{ anti-periodic} \}.
\]
We will show that \( \hat{g} = \hat{g}_1 = \hat{g}_2 \) satisfies \( \hat{g}(\xi) \leq c_2 |\xi|^{q_1} + c_1 \), and apply duality to recover \( \hat{f}(\xi) \geq c^*_2 |\xi|^{p_2} + c^*_1 \).

To get the upper bound on \( \hat{g} \) we use the following lemma.

**Lemma 4.7.** There exists \( \tau \) with \( \text{div} \tau = 0 \) in \( Y \), such that \( \tau \cdot n \) is anti-periodic on the boundary of \( Y \), \( \tau = -\xi \) in \( R_1 \), and

\[
\int_Y |\tau(y)|^{q_1} \, dy \leq C |\xi|^{q_1}.
\]

**Proof.** Let the function \( \varphi \in W^{1,p_2}_*(R_2) \) be the solution of

\[
\begin{cases}
\nabla \varphi \cdot |\nabla \varphi|^{p_2-2} \cdot n \text{ is anti-periodic on } \partial R_2 \cap \partial Y; \\
\Delta_{p_2} \varphi = 0 \text{ in } R_2; \\
\nabla \varphi \cdot |\nabla \varphi|^{p_2-2} \cdot |n|_2 = -\xi \cdot |n|_1 \text{ on } \Gamma,
\end{cases}
\]

where the subscript 1 indicates the trace on the \( R_1 \) side of \( \Gamma \) and 2 indicates the trace on the \( R_2 \) side of \( \Gamma \). The Neumann problem given above is the stationarity condition for the energy

\[
\int_{R_2} |\nabla \varphi|^{p_2} \, dx - \int_{\Gamma} \phi \xi \cdot n \, dS \text{ when minimized over all } \phi \in W^{1,p_2}_*(R_2).
\]

The solution of the Neumann problem is unique up to a constant. Here the anti-periodic boundary condition on \( \nabla \varphi \cdot |\nabla \varphi|^{p_2-2} \cdot n \) is the natural boundary condition for the problem.

Now we define \( \tau \) according to

\[
\tau = \begin{cases}
-\xi; & \text{in } R_1 \\
\nabla \varphi \cdot |\nabla \varphi|^{p_2-2}; & \text{in } R_2
\end{cases}
\]

and it follows that

\[
|\tau|^{q_1} = \begin{cases}
|\xi|^{q_1}; & \text{in } R_1 \\
\left( (|\nabla \varphi|^{p_2-2})^2 \right)^{\frac{q_1}{2}} = (|\nabla \varphi|^{p_2-1})^{q_1} = |\nabla \varphi|^{p_2}; & \text{in } R_2.
\end{cases}
\]

Then, for \( \psi \in W^{1,p_2}_*(R_2) \) we have

\[
\int_{R_2} |\nabla \varphi|^{p_2-2} \nabla \varphi \cdot \nabla \psi \, dy = \int_{\Gamma} \psi |\nabla \varphi|^{p_2-2} \nabla \varphi \cdot ndS + \int_{\partial R_2 \cap \partial Y} \psi |\nabla \varphi|^{p_2-2} \nabla \varphi \cdot ndS
\]

\[
= -\int_{\Gamma} \psi \xi \cdot ndS.
\]
Set $\psi = \varphi$ in (4.16) and an application of Hölder’s inequality gives

$$
\int_{R_2} |\nabla \varphi|^{p_2} dx = - \int_\Gamma \varphi \xi \cdot ndS \\
= - \int_{R_2} \nabla (\varphi \xi) dx \\
= - \int_{R_2} (\nabla \varphi) \cdot \xi dx \\
\leq \left( \int_{R_2} |\nabla \varphi|^{p_2} dx \right)^{1/p_2} \left( \int_{R_2} |\xi|^{q_1} dx \right)^{1/q_1}.
$$

Then

$$
\int_{R_2} |\nabla \varphi|^{p_2} dx \leq \int_{R_2} |\xi|^{q_1} dx. \tag{4.17}
$$

Therefore, using (4.15) and (4.17), we have

$$
\int_Y |\tau(y)|^{q_1} dy \\
= \int_{R_1} |\tau(y)|^{q_1} dy + \int_{R_2} |\tau(y)|^{q_1} dy \\
= \int_{R_1} |\xi|^{q_1} dy + \int_{R_2} |\nabla \varphi(y)|^{p_2} dy \\
\leq C |\xi|^{q_1}.
$$

Taking $\hat{g}$ to be the conjugate of $\hat{f}$, and choosing $\tau$ in $Sol^{q_1}(Y)$ as in Lemma 4.7, we obtain

$$
\hat{g}(\xi) = \inf_{\tau \in Sol^{q_1}(Y)} \int_Y \tilde{g}(y, \xi + \tau) dy \\
\leq \int_Y \tilde{g}(y, \xi + \tau) dy \\
\leq \int_{R_1} \tilde{g}(y, 0) dy + \int_{R_2} \tilde{g}(y, \xi + \tau) dy \\
\leq c_1 + c_2 \int_{R_2} |\xi + \tau|^{q_1} dy \\
\leq c_1 + c_2 |\xi|^{q_1},
$$

and the left hand inequality in (4.12) follows from duality.

This concludes the proof of Theorem 4.5.

Collecting results we now prove Theorem 4.1. Indeed the minimizer of $E_1$ is precisely the solution $u$ of (3.8). Theorem 4.5 establishes the coercivity of $E_1$ over $W_0^{1,p_2}(\Omega)$, thus the solution $u$ lies in $W_0^{1,p_2}(\Omega)$.

30
Chapter 5

Corrector Theorem

In this chapter, we develop new strong convergence results that capture the asymptotic behavior of the gradients $\nabla u_\epsilon$, as $\epsilon$ tends to 0. Our approach delivers strong approximations for the gradients inside each phase $\chi_\epsilon^i \nabla u_\epsilon$, $i = 1, 2$.

Homogenization theory relates the average behavior seen at large length scales to the underlying heterogeneous structure. It allows one to approximate $\{ \nabla u_\epsilon \}_{\epsilon > 0}$ in terms of $\nabla u$, where $u$ is the solution of the homogenized problem (3.3). The homogenization result given in [ZKO94] shows that the average of the error incurred in this approximation of $\nabla u_\epsilon$ decays to 0. We present a new corrector result that delivers an approximation to $\nabla u_\epsilon$ up to an error that converges to zero strongly in the norm.

The corrector results are presented for layered materials and for dispersions of inclusions embedded inside a host medium. For the dispersed microstructures the included material is taken to have the lower power law exponent than that of the host phase. For both of these cases it is shown that the homogenized solution lies in $W^{1,p}_0(\Omega)$ (See Chapter 4). With this higher order integrability in hand, we provide an algorithm for building correctors and establish strong approximations for the sequences $\{ \chi_\epsilon^i \nabla u_\epsilon \}_{\epsilon > 0}$ in $L^p(\Omega; \mathbb{R}^n)$, see Theorem 5.2. When the host phase has a lower power-law exponent than the included phase one can only conclude that the homogenized solution lies in $W^{1,p}_0(\Omega)$ and the techniques developed here do not apply.

5.1 Statement of the Corrector Theorem

We now describe the family of correctors that provide a strong approximation of the sequence $\{ \chi_\epsilon^i \nabla u_\epsilon \}_{\epsilon > 0}$ in the $L^p(\Omega; \mathbb{R}^n)$ norm. We denote the rescaled period cell with side length $\epsilon > 0$ by $Y_\epsilon$ and write $Y_\epsilon^i = \epsilon i + Y_\epsilon$, where $i \in \mathbb{Z}^n$. In what follows it is convenient to define the index set $I_\epsilon = \{i \in \mathbb{Z}^n : Y_\epsilon^i \subset \Omega \}$. For $\varphi \in L^p(\Omega; \mathbb{R}^n)$, we define the local average operator $M_\epsilon$ associated with the partition $Y_\epsilon^i$, $i \in I_\epsilon$ by

$$M_\epsilon(\varphi)(x) = \begin{cases} \sum_{i \in I_\epsilon} \chi_{Y_\epsilon^i}(x) \frac{1}{|Y_\epsilon^i|} \int_{Y_\epsilon^i} \varphi(y) dy; & \text{if } x \in \bigcup_{i \in I_\epsilon} Y_\epsilon^i, \\ 0; & \text{if } x \in \Omega \setminus \bigcup_{i \in I_\epsilon} Y_\epsilon^i. \end{cases} \quad (5.1)$$
Remark 5.1. The family $M_\epsilon$ has the following properties

1. For $i = 1, 2$, $\|M_\epsilon(\varphi) - \varphi\|_{L^p(\Omega; \mathbb{R}^n)} \to 0$ as $\epsilon \to 0$. For a proof, see, for instance Chapter 8 of [Zaa58].

2. $M_\epsilon(\varphi) \to \varphi$ a.e. on $\Omega$. For a proof, see, for instance Chapter 8 of [Zaa58].

3. From Jensen’s inequality, we have
   \[ \|M_\epsilon(\varphi)\|_{L^p(\Omega; \mathbb{R}^n)} \leq \|\varphi\|_{L^p(\Omega; \mathbb{R}^n)}, \]
   for every $\varphi \in L^p(\Omega; \mathbb{R}^n)$ and $i = 1, 2$.

The strong approximation to the sequence \( \{\chi^\epsilon u_\epsilon\}_{\epsilon > 0} \) is given by the following corrector theorem.

Theorem 5.2 (Corrector Theorem). Let $f \in W^{-1,q_2}(\Omega)$, let $u_\epsilon$ be the solution to the problem (3.7), and let $u$ be the solution to problem (3.8). Then, for periodic dispersions of inclusions and layered materials and $i = 1, 2$, we have
\[
\int_\Omega \left| \chi^\epsilon_i(x)p_\epsilon(x, M_\epsilon(\nabla u)(x)) - \chi^\epsilon_i(x)\nabla u_\epsilon(x) \right|^{p_i} \, dx \to 0, \quad \text{as} \quad \epsilon \to 0. \tag{5.2}
\]

Before we can give the proof of this theorem, we need the results from the following section.

5.2 Some Properties of Correctors

In this section, we state and prove a priori bounds and convergence properties for the sequences $p_\epsilon$ defined in (3.19), $\nabla u_\epsilon$, and $A_\epsilon(x, p_\epsilon(x, \nabla u_\epsilon))$ that are used in the proof of Theorem 5.2.

In the following, the letter $C$ will represent a generic positive constant independent of $\epsilon$, and it can take different values from line to line.

Lemma 5.3. For every $\xi \in \mathbb{R}^n$, we have
\[
\int_Y \chi_1(y) |p(y, \xi)|^{p_1} \, dy + \int_Y \chi_2(y) |p(y, \xi)|^{p_2} \, dy \leq C \left( 1 + |\xi|^{p_1} \theta_1 + |\xi|^{p_2} \theta_2 \right), \tag{5.3}
\]
and by a change of variables, we obtain
\[
\int_{Y_\epsilon} \chi_1^\epsilon(x) |p_\epsilon(x, \xi)|^{p_1} \, dx + \int_{Y_\epsilon} \chi_2^\epsilon(x) |p_\epsilon(x, \xi)|^{p_2} \, dx \leq C \left( 1 + |\xi|^{p_1} \theta_1 + |\xi|^{p_2} \theta_2 \right) |Y_\epsilon|. \tag{5.4}
\]

Proof. Let $\xi \in \mathbb{R}^n$. By (3.5), we have that
\[
(A(y, p(y, \xi)), p(y, \xi)) \geq C \left( \chi_1(y) |p(y, \xi)|^{p_1} + \chi_2(y) |p(y, \xi)|^{p_2} \right).
\]
Integrating both sides over $Y$ and using (3.11), we get

$$\int_Y \chi_1(y) |p(y, \xi)|^{p_1} \, dy + \int_Y \chi_2(y) |p(y, \xi)|^{p_2} \, dy$$

$$\leq C \int_Y (A_y p(y, \xi), p(y, \xi)) \, dy$$

$$= C \int_Y (A_y p(y, \xi), \xi) \, dy$$

By Cauchy-Schwarz Inequality and (3.4), we have

$$\leq C \int_Y |A_y p(y, \xi)| \, |\xi| \, dy$$

$$\leq C \left[ \int_Y \chi_1(y) |p(y, \xi)| (1 + |p(y, \xi)|)^{p_1-2} |\xi| \, dy 
+ \int_Y \chi_2(y) |p(y, \xi)| (1 + |p(y, \xi)|)^{p_2-2} |\xi| \, dy \right]$$

$$\leq C \left[ \int_Y \chi_1(y)(1 + |p(y, \xi)|)^{p_1-1} |\xi| \, dy + \int_Y \chi_2(y)(1 + |p(y, \xi)|)^{p_2-1} |\xi| \, dy \right]$$

Using Young’s Inequality, we obtain

$$\leq C \left[ \frac{\delta^{q_2} \int_Y \chi_1(y)(1 + |p(y, \xi)|)^{p_1} \, dy}{q_2} + \frac{\delta^{-p_1} \int_Y \chi_1(y) |\xi|^{p_1} \, dy}{p_1} 
+ \frac{\delta^{q_1} \int_Y \chi_2(y)(1 + |p(y, \xi)|)^{p_2} \, dy}{q_1} + \frac{\delta^{-p_2} \int_Y \chi_2(y) |\xi|^{p_2} \, dy}{p_2} \right]$$

$$= C \left[ \delta^{q_2} \theta_1 + \delta^{q_2} \int_Y \chi_1(y) |p(y, \xi)|^{p_1} \, dy + \delta^{-p_1} |\xi|^{p_1} \theta_1 
+ \delta^{q_1} \int_Y \chi_2(y) |p(y, \xi)|^{p_2} \, dy + \delta^{-p_2} |\xi|^{p_2} \theta_2 \right]$$

$$\leq C \left[ \delta^{q_2} \theta_1 + \delta^{q_2} \int_Y \chi_1(y) |p(y, \xi)|^{p_1} \, dy + \delta^{-p_1} |\xi|^{p_1} \theta_1 + \delta^{q_1} \theta_2 
+ \delta^{q_1} \int_Y \chi_2(y) |p(y, \xi)|^{p_2} \, dy + \delta^{-p_2} |\xi|^{p_2} \theta_2 \right]$$

$$\leq C \left[ (\delta^{q_2} \theta_1 + \delta^{q_2} \theta_2) + \delta^{-p_1} |\xi|^{p_1} \theta_1 + \delta^{-p_2} |\xi|^{p_2} \theta_2 \right]$$

$$+ (\delta^{q_2} + \delta^{q_1}) \left( \int_Y \chi_1(y) |p(y, \xi)|^{p_1} \, dy + \int_Y \chi_2(y) |p(y, \xi)|^{p_2} \, dy \right).$$
Doing some algebraic manipulations, we obtain
\[
(1 - C(\delta^{p_2} + \delta^{q_1})) \left( \int_Y \chi_1(y) |p(y, \xi)|^{p_1} \, dy + \int_Y \chi_2(y) |p(y, \xi)|^{p_2} \, dy \right)
\leq C \left[ (\delta^{p_2} \theta_1 + \delta^{q_1} \theta_2) + (\delta^{-p_1} |\xi|^{p_1} \theta_1 + \delta^{-p_2} |\xi|^{p_2} \theta_2) \right]
\]

On choosing an appropriate \( \delta \), we finally obtain (5.3).

**Lemma 5.4.** For every \( \xi_1, \xi_2 \in \mathbb{R}^n \), we have
\[
\int_Y \chi_1(y) |p(y, \xi_1) - p(y, \xi_2)|^{p_1} \, dy + \int_Y \chi_2(y) |p(y, \xi_1) - p(y, \xi_2)|^{p_2} \, dy
\leq C \left[ (1 + |\xi_1|^{p_1} \theta_1 + |\xi_2|^{p_1} \theta_1) \left( |\xi_1 - \xi_2|^{p_1} \theta_1^{1 \over p_1 + 1} \right) \right.
\left. + (1 + |\xi_1|^{p_2} \theta_2 + |\xi_2|^{p_2} \theta_2) \left( |\xi_1 - \xi_2|^{p_2} \theta_2^{1 \over p_2 + 1} \right) \right],
\]

and by doing a change a variables, we obtain
\[
\int_{Y_\epsilon} \chi_1(x) |p_\epsilon(x, \xi_1) - p_\epsilon(x, \xi_2)|^{p_1} \, dx + \int_{Y_\epsilon} \chi_2(x) |p_\epsilon(x, \xi_1) - p_\epsilon(x, \xi_2)|^{p_2} \, dx
\leq C \left[ (1 + |\xi_1|^{p_1} \theta_1 + |\xi_2|^{p_1} \theta_1) \left( |\xi_1 - \xi_2|^{p_1} \theta_1^{1 \over p_1 + 1} \right) \right.
\left. + (1 + |\xi_1|^{p_2} \theta_2 + |\xi_2|^{p_2} \theta_2) \left( |\xi_1 - \xi_2|^{p_2} \theta_2^{1 \over p_2 + 1} \right) \right] |Y_\epsilon|.
\]

**Proof.** By (3.5), we have
\[
(A(y, p(y, \xi_1)) - A(y, p(y, \xi_2)), p(y, \xi_1) - p(y, \xi_2))
\geq C \left( \chi_1(y) |p(y, \xi_1) - p(y, \xi_2)|^{p_1} + \chi_2(y) |p(y, \xi_1) - p(y, \xi_2)|^{p_2} \right).
\]

Integrating over \( Y \) and using (3.11) and the Cauchy-Schwarz inequality, we get
\[
\int_Y \chi_1(y) |p(y, \xi_1) - p(y, \xi_2)|^{p_1} \, dy + \int_Y \chi_2(y) |p(y, \xi_1) - p(y, \xi_2)|^{p_2} \, dy
\leq C \int_Y (A(y, p(y, \xi_1)) - A(y, p(y, \xi_2)), p(y, \xi_1) - p(y, \xi_2)) \, dy
\leq C \int_Y (A(y, p(y, \xi_1)) - A(y, p(y, \xi_2)), \xi_1 - \xi_2) \, dy
\leq C \int_Y |A(y, p(y, \xi_1)) - A(y, p(y, \xi_2))| |\xi_1 - \xi_2| \, dy
\]

Using (3.4), we have
\[
\leq C \left[ \int_Y \chi_1(y) |p(y, \xi_1) - p(y, \xi_2)| \left( 1 + |p(y, \xi_1)| + |p(y, \xi_2)| \right)^{p_1 - 2} |\xi_1 - \xi_2| \, dy
\left. + \int_Y \chi_2(y) |p(y, \xi_1) - p(y, \xi_2)| \left( 1 + |p(y, \xi_1)| + |p(y, \xi_2)| \right)^{p_2 - 2} |\xi_1 - \xi_2| \, dy \right]
\]

34
Using Hölder’s inequality in the first expression with \( r_1 = \frac{p_1}{p_1 - 2} \), \( r_2 = p_1 \), and \( r_3 = p_1 \); and in the second expression with \( s_1 = \frac{p_2}{p_2 - 2} \), \( s_2 = p_2 \), and \( s_3 = p_2 \), we obtain

\[
\begin{align*}
&\leq C \left[ (\int_Y \chi_1(y) (1 + |p(y, \xi_1)| + |p(y, \xi_2)|)^{p_1} \, dy)^{\frac{p_1-2}{p_1}} \right. \\
&\quad \times \left. \left( \int_Y \chi_1(y) |p(y, \xi_1) - p(y, \xi_2)|^{p_1} \, dy \right)^{\frac{1}{p_1}} \left( \int_Y \chi_1(y) |\xi_1 - \xi_2|^{p_1} \, dy \right)^{\frac{1}{p_1}} \\
&\quad + \left. \left( \int_Y \chi_2(y) (1 + |p(y, \xi_1)| + |p(y, \xi_2)|)^{p_2} \, dy \right)^{\frac{p_2-2}{p_2}} \right] \\
&\leq C \left[ \left( \int_Y \chi_1(y) dy + \int_Y \chi_1(y) |p(y, \xi_1)|^{p_1} \, dy + \int_Y \chi_1(y) |p(y, \xi_2)|^{p_1} \, dy \right)^{\frac{p_1-2}{p_1}} \right. \\
&\quad \times \left. \left( \int_Y \chi_1(y) |p(y, \xi_1) - p(y, \xi_2)|^{p_1} \, dy \right)^{\frac{1}{p_1}} |\xi_1 - \xi_2|^{\frac{1}{p_1}} \\
&\quad + \left. \left( \int_Y \chi_2(y) dy + \int_Y \chi_2(y) |p(y, \xi_1)|^{p_2} \, dy + \int_Y \chi_2(y) |p(y, \xi_2)|^{p_2} \, dy \right)^{\frac{p_2-2}{p_2}} \right] \\
&\quad \times \left. \left( \int_Y \chi_2(y) |p(y, \xi_1) - p(y, \xi_2)|^{p_2} \, dy \right)^{\frac{1}{p_2}} |\xi_1 - \xi_2|^{\frac{1}{p_2}} \right]
\end{align*}
\]

Use Lemma 5.3 to get

\[
\begin{align*}
&\leq C \left[ (1 + |\xi_1|^{p_1} \theta_1 + |\xi_1|^{p_2} \theta_2 + |\xi_2|^{p_1} \theta_1 + |\xi_2|^{p_2} \theta_2)^{\frac{p_1-2}{p_1}} \right. \\
&\quad \times \left. |\xi_1 - \xi_2|^{\frac{1}{p_1}} \left( \int_Y \chi_1(y) |p(y, \xi_1) - p(y, \xi_2)|^{p_1} \, dy \right)^{\frac{1}{p_1}} \\
&\quad + \left. (1 + |\xi_1|^{p_1} \theta_1 + |\xi_1|^{p_2} \theta_2 + |\xi_2|^{p_1} \theta_1 + |\xi_2|^{p_2} \theta_2)^{\frac{p_2-2}{p_2}} \right] \\
&\quad \times \left. |\xi_1 - \xi_2|^{\frac{1}{p_2}} \left( \int_Y \chi_2(y) |p(y, \xi_1) - p(y, \xi_2)|^{p_2} \, dy \right)^{\frac{1}{p_2}} \right]
\end{align*}
\]

By Young’s Inequality, we obtain

\[
\begin{align*}
&\leq C \left[ \delta^{-q_2} (1 + |\xi_1|^{p_1} \theta_1 + |\xi_1|^{p_2} \theta_2 + |\xi_2|^{p_1} \theta_1 + |\xi_2|^{p_2} \theta_2)^{\frac{(p_1-2)q_2}{p_1}} |\xi_1 - \xi_2|^{q_2} \theta_1^{\frac{q_2}{p_1}} \\
&\quad + \delta^{-p_1} \int_Y \chi_1(y) |p(y, \xi_1) - p(y, \xi_2)|^{p_1} \, dy \right. \\
&\quad + \left. \delta^{-p_2} \int_Y \chi_2(y) |p(y, \xi_1) - p(y, \xi_2)|^{p_2} \, dy \right]
\end{align*}
\]
and let $\Psi$ be a simple function of the form
\[ \eta \]
with $k \in \mathbb{R}$.

Lemma 5.5. Let $\varphi$ be such that
\[
\sup_{\epsilon > 0} \left\{ \int_{\Omega} \chi_1^\epsilon(x) |\varphi(x)|^{p_1} \, dx + \int_{\Omega} \chi_2^\epsilon(x) |\varphi(x)|^{p_2} \, dx \right\} < \infty,
\]
and let $\Psi$ be a simple function of the form
\[
\Psi(x) = \sum_{j=0}^m \eta_j \chi_{\Omega_j}(x), \tag{5.7}
\]
with $\eta_j \in \mathbb{R} \setminus \{0\}$, $\Omega_j \subset \subset \Omega$, $|\partial \Omega_j| = 0$, $\Omega_j \cap \Omega_k = \emptyset$ for $j \neq k$ and $j, k = 1, \ldots, m$; and set $\eta_0 = 0$ and $\Omega_0 = \Omega \setminus \bigcup_{j=1}^m \Omega_j$. Then
\[
\limsup_{\epsilon \to 0} \left( \int_{\Omega} \chi_1^\epsilon(x) |p_\epsilon(x, M_\epsilon \varphi(x)) - p_\epsilon(x, \Psi(x))|^{p_1} \, dx 
+ \int_{\Omega} \chi_2^\epsilon(x) |p_\epsilon(x, M_\epsilon \varphi(x)) - p_\epsilon(x, \Psi(x))|^{p_2} \, dx \right)
\leq \limsup_{\epsilon \to 0} C \sum_{i=1}^2 \left( |\Omega| + \int_{\Omega} \chi_i^\epsilon(x) |\varphi(x)|^{p_1} \, dx + \int_{\Omega} \chi_i^\epsilon(x) |\varphi(x)|^{p_2} \, dx \right)
\]
\[ + \int_{\Omega} \chi_1^\epsilon(x) |\Psi(x)|^{p_1} dx + \int_{\Omega} \chi_2^\epsilon(x) |\Psi(x)|^{p_2} dx \]
\[ \times \left( \int_{\Omega} \chi_i^\epsilon(x) |\varphi(x) - \Psi(x)|^{p_i} dx \right)^{\frac{p_i - 2}{p_i - 1}} \right) \}\]

(5.8)

**Proof.** Let \( \Psi \) of the form (5.7). For every \( \epsilon > 0 \), let us denote by
\[ \Omega_\epsilon = \bigcup Y_i^\epsilon \text{ for } i \in I_\epsilon; \]
and for \( j = 0, 1, 2, \ldots, m \), we set
\[ I_i^j = \{ i \in I_\epsilon : Y_i^\epsilon \subseteq \Omega_j \}, \]
and
\[ J_i^j = \{ i \in I_\epsilon : Y_i^\epsilon \cap \Omega_j \neq \emptyset, Y_i^\epsilon \setminus \Omega_j \neq \emptyset \}. \]
Furthermore,
\[ E_i^j = \bigcup_{i \in I_i^j} Y_i^\epsilon, \]
and
\[ F_i^j = \bigcup_{i \in J_i^j} Y_i^\epsilon, \]
with
\[ |F_i^j| \to 0, \text{ as } \epsilon \to 0. \]

Set
\[ \xi_i^\epsilon = \frac{1}{|Y_i^\epsilon|} \int_{Y_i^\epsilon} \varphi(x) dx. \]

(5.12)

For \( \epsilon \) sufficiently small \( \Omega_j \) is contained in \( \Omega_\epsilon \), for all \( j \neq 0 \).
From (5.7), we have
\[ \int_{\Omega} \chi_1^\epsilon(x) |p_\epsilon(x, M_\epsilon \varphi) - p_\epsilon(x, \Psi)|^{p_1} dx + \int_{\Omega} \chi_2^\epsilon(x) |p_\epsilon(x, M_\epsilon \varphi) - p_\epsilon(x, \Psi)|^{p_2} dx \]
\[ = \int_{\Omega} \chi_1^\epsilon(x) \left| p_\epsilon(x, M_\epsilon \varphi) - p_\epsilon \left( x, \sum_{j=0}^{m} \eta_j \chi_{\Omega_j} \right) \right|^{p_1} dx \]
\[ + \int_{\Omega} \chi_2^\epsilon(x) \left| p_\epsilon(x, M_\epsilon \varphi) - p_\epsilon \left( x, \sum_{j=0}^{m} \eta_j \chi_{\Omega_j} \right) \right|^{p_2} dx \]
or equivalently,
\[ = \sum_{j=0}^{m} \int_{\Omega_j} \chi_1^\epsilon(x) |p_\epsilon(x, M_\epsilon \varphi) - p_\epsilon(x, \eta_j)|^{p_1} dx \]
\[ + \sum_{j=0}^{m} \int_{\Omega_j} \chi_2^\epsilon(x) |p_\epsilon(x, M_\epsilon \varphi) - p_\epsilon(x, \eta_j)|^{p_2} dx \]

37
Using the fact that $\Omega_j \subset E^j_\varepsilon \cup F^j_\varepsilon$ and (5.1), we have

$$\leq \sum_{j=0}^{m} \int_{E^j_\varepsilon} \chi^j_1(x) |p_\varepsilon(x, M_\varepsilon \varphi) - p_\varepsilon(x, \eta_j)|^{p_1} \, dx$$

$$+ \sum_{j=0}^{m} \int_{F^j_\varepsilon} \chi^j_1(x) |p_\varepsilon(x, M_\varepsilon \varphi) - p_\varepsilon(x, \eta_j)|^{p_1} \, dx$$

$$+ \sum_{j=0}^{m} \int_{E^j_\varepsilon} \chi^j_2(x) |p_\varepsilon(x, M_\varepsilon \varphi) - p_\varepsilon(x, \eta_j)|^{p_2} \, dx$$

$$+ \sum_{j=0}^{m} \int_{F^j_\varepsilon} \chi^j_2(x) |p_\varepsilon(x, M_\varepsilon \varphi) - p_\varepsilon(x, \eta_j)|^{p_2} \, dx$$

$$= \sum_{j=0}^{m} \int_{E^j_\varepsilon} \left( \sum_{i \in I_k} \chi^j_1(x) \chi^i_{Y_\varepsilon}(x) \xi^j_\varepsilon - p_\varepsilon(x, \eta_j) \right)^{p_1} \, dx$$

$$+ \sum_{j=0}^{m} \int_{F^j_\varepsilon} \left( \sum_{i \in I_k} \chi^j_1(x) \chi^i_{Y_\varepsilon}(x) \xi^j_\varepsilon - p_\varepsilon(x, \eta_j) \right)^{p_1} \, dx$$

$$+ \sum_{j=0}^{m} \int_{E^j_\varepsilon} \left( \sum_{i \in I_k} \chi^j_2(x) \chi^i_{Y_\varepsilon}(x) \xi^j_\varepsilon - p_\varepsilon(x, \eta_j) \right)^{p_2} \, dx$$

$$+ \sum_{j=0}^{m} \int_{F^j_\varepsilon} \left( \sum_{i \in I_k} \chi^j_2(x) \chi^i_{Y_\varepsilon}(x) \xi^j_\varepsilon - p_\varepsilon(x, \eta_j) \right)^{p_2} \, dx$$

Using (5.9), (5.10) and reorganizing the sums, we get

$$= \sum_{j=0}^{m} \left( \sum_{i \in I_k} \int_{Y^j_\varepsilon} \chi^j_1(x) |p_\varepsilon(x, \xi^j_\varepsilon) - p_\varepsilon(x, \eta_j)|^{p_1} \, dx \right)$$

$$+ \sum_{j=0}^{m} \left( \sum_{i \in I_k} \int_{Y^j_\varepsilon} \chi^j_1(x) |p_\varepsilon(x, \xi^j_\varepsilon) - p_\varepsilon(x, \eta_j)|^{p_1} \, dx \right)$$

$$+ \sum_{j=0}^{m} \left( \sum_{i \in I_k} \int_{Y^j_\varepsilon} \chi^j_2(x) |p_\varepsilon(x, \xi^j_\varepsilon) - p_\varepsilon(x, \eta_j)|^{p_2} \, dx \right)$$

$$+ \sum_{j=0}^{m} \left( \sum_{i \in I_k} \int_{Y^j_\varepsilon} \chi^j_2(x) |p_\varepsilon(x, \xi^j_\varepsilon) - p_\varepsilon(x, \eta_j)|^{p_2} \, dx \right)$$
\[
= \sum_{j=0}^{m} \left[ \sum_{i \in I_j} \left( \int_{Y_{i}^{\epsilon}} \chi_{1}^{i}(x) \left| p_{x}(x, \xi_{i}^{j}) - p_{x}(x, \eta_{j}) \right|^{p_{1}} dx \\
+ \int_{Y_{i}^{\epsilon}} \chi_{2}^{i}(x) \left| p_{x}(x, \xi_{i}^{j}) - p_{x}(x, \eta_{j}) \right|^{p_{2}} dx \right) \right]
+ \sum_{j=0}^{m} \left[ \sum_{i \in I_j} \left( \int_{Y_{i}^{\epsilon}} \chi_{1}^{i}(x) \left| p_{x}(x, \xi_{i}^{j}) - p_{x}(x, \eta_{j}) \right|^{p_{1}} dx \\
+ \int_{Y_{i}^{\epsilon}} \chi_{2}^{i}(x) \left| p_{x}(x, \xi_{i}^{j}) - p_{x}(x, \eta_{j}) \right|^{p_{2}} dx \right) \right]
\]

Using Lemma 5.4, we obtain

\[
\leq C \sum_{j=0}^{m} \left\{ \sum_{i \in I_j} \left[ (1 + |\xi_{i}^{j}|^{p_{1}} \theta_{1} + |\xi_{i}^{j}|^{p_{2}} \theta_{2} + |\eta_{j}|^{p_{1}} \theta_{1} + |\eta_{j}|^{p_{2}} \theta_{2})^{\frac{p_{1}-2}{p_{1}-1}} \\
\times |\xi_{i}^{j} - \eta_{j}|^{\frac{p_{1}-1}{p_{1}-2}} \theta_{1}^{\frac{1}{p_{1}-2}} |Y_{i}^{\epsilon}| \right] + (1 + |\xi_{i}^{j}|^{p_{1}} \theta_{1} + |\xi_{i}^{j}|^{p_{2}} \theta_{2} + |\eta_{j}|^{p_{1}} \theta_{1} + |\eta_{j}|^{p_{2}} \theta_{2})^{\frac{p_{2}-2}{p_{2}-1}} |\xi_{i}^{j} - \eta_{j}|^{\frac{p_{2}-1}{p_{2}-2}} \theta_{2}^{\frac{1}{p_{2}-1}} |Y_{i}^{\epsilon}| \right] \right\}
+ C \sum_{j=0}^{m} \left\{ \sum_{i \in I_j} \left[ (1 + |\xi_{i}^{j}|^{p_{1}} \theta_{1} + |\xi_{i}^{j}|^{p_{2}} \theta_{2} + |\eta_{j}|^{p_{1}} \theta_{1} + |\eta_{j}|^{p_{2}} \theta_{2})^{\frac{p_{1}-2}{p_{1}-1}} \\
\times |\xi_{i}^{j} - \eta_{j}|^{\frac{p_{1}-1}{p_{1}-2}} \theta_{1}^{\frac{1}{p_{1}-2}} |Y_{i}^{\epsilon}| \right] + (1 + |\xi_{i}^{j}|^{p_{1}} \theta_{1} + |\xi_{i}^{j}|^{p_{2}} \theta_{2} + |\eta_{j}|^{p_{1}} \theta_{1} + |\eta_{j}|^{p_{2}} \theta_{2})^{\frac{p_{2}-2}{p_{2}-1}} |\xi_{i}^{j} - \eta_{j}|^{\frac{p_{2}-1}{p_{2}-2}} \theta_{2}^{\frac{1}{p_{2}-1}} |Y_{i}^{\epsilon}| \right] \right\}
\leq C \sum_{j=0}^{m} \sum_{i \in I_j} \int_{Y_{i}^{\epsilon}} \left[ (1 + |\xi_{i}^{j}|^{p_{1}} \theta_{1} + |\xi_{i}^{j}|^{p_{2}} \theta_{2} + |\eta_{j}|^{p_{1}} \theta_{1} + |\eta_{j}|^{p_{2}} \theta_{2})^{\frac{p_{1}-2}{p_{1}-1}} |\xi_{i}^{j} - \eta_{j}|^{\frac{p_{1}-1}{p_{1}-2}} \theta_{1}^{\frac{1}{p_{1}-2}} \\
+ (1 + |\xi_{i}^{j}|^{p_{1}} \theta_{1} + |\xi_{i}^{j}|^{p_{2}} \theta_{2} + |\eta_{j}|^{p_{1}} \theta_{1} + |\eta_{j}|^{p_{2}} \theta_{2})^{\frac{p_{2}-2}{p_{2}-1}} |\xi_{i}^{j} - \eta_{j}|^{\frac{p_{2}-1}{p_{2}-2}} \theta_{2}^{\frac{1}{p_{2}-1}} \right] dx 
+ C \sum_{j=0}^{m} \sum_{i \in I_j} \int_{Y_{i}^{\epsilon}} \left[ (1 + |\xi_{i}^{j}|^{p_{1}} \theta_{1} + |\xi_{i}^{j}|^{p_{2}} \theta_{2} + |\eta_{j}|^{p_{1}} \theta_{1} + |\eta_{j}|^{p_{2}} \theta_{2})^{\frac{p_{1}-2}{p_{1}-1}} |\xi_{i}^{j} - \eta_{j}|^{\frac{p_{1}-1}{p_{1}-2}} \theta_{1}^{\frac{1}{p_{1}-2}} \\
+ (1 + |\xi_{i}^{j}|^{p_{1}} \theta_{1} + |\xi_{i}^{j}|^{p_{2}} \theta_{2} + |\eta_{j}|^{p_{1}} \theta_{1} + |\eta_{j}|^{p_{2}} \theta_{2})^{\frac{p_{2}-2}{p_{2}-1}} |\xi_{i}^{j} - \eta_{j}|^{\frac{p_{2}-1}{p_{2}-2}} \theta_{2}^{\frac{1}{p_{2}-1}} \right] dx
\]
Using (5.9), (5.10) again

\[
C \sum_{j=0}^{m} \left\{ \int_{E_{1}^{j}} \left[ \left( 1 + \left| \sum_{i \in I_{1}^{j}} \chi_{Y_{i}} \xi_{i}^{i} \right|^{p_{1}} \theta_{1} + \left| \sum_{i \in I_{2}^{j}} \chi_{Y_{i}} \xi_{i}^{i} \right|^{p_{2}} \theta_{2} + |\eta_{j}|^{p_{1}} \theta_{1} + |\eta_{j}|^{p_{2}} \theta_{2} \right) \right]^{\frac{p_{1}-2}{p_{1}-1}} \theta_{1}^{\frac{1}{p_{1}-1}} \right. \\
\times \left| \sum_{i \in I_{1}^{j}} \chi_{Y_{i}} \xi_{i}^{i} - \eta_{j} \right|^{\frac{p_{1}}{p_{1}-1}} \int_{E_{2}^{j}} \left| \sum_{i \in I_{2}^{j}} \chi_{Y_{i}} \xi_{i}^{i} - \eta_{j} \right|^{\frac{p_{2}}{p_{2}-1}} \theta_{2}^{\frac{1}{p_{2}-1}} \right. \\
\times \left( 1 + \left| \sum_{i \in I_{1}^{j}} \chi_{Y_{i}} \xi_{i}^{i} \right|^{p_{1}} \theta_{1} + \left| \sum_{i \in I_{2}^{j}} \chi_{Y_{i}} \xi_{i}^{i} \right|^{p_{2}} \theta_{2} + |\eta_{j}|^{p_{1}} \theta_{1} + |\eta_{j}|^{p_{2}} \theta_{2} \right) \left. \right\} \right. \\
+ C \sum_{j=0}^{m} \left\{ \int_{F_{1}^{j}} \left[ \left( 1 + \left| \sum_{i \in J_{1}^{j}} \chi_{Y_{i}} \xi_{i}^{i} \right|^{p_{1}} \theta_{1} + \left| \sum_{i \in J_{2}^{j}} \chi_{Y_{i}} \xi_{i}^{i} \right|^{p_{2}} \theta_{2} + |\eta_{j}|^{p_{1}} \theta_{1} + |\eta_{j}|^{p_{2}} \theta_{2} \right) \right]^{\frac{p_{1}-2}{p_{1}-1}} \theta_{1}^{\frac{1}{p_{1}-1}} \right. \\
\times \left| \sum_{i \in J_{1}^{j}} \chi_{Y_{i}} \xi_{i}^{i} - \eta_{j} \right|^{\frac{p_{1}}{p_{1}-1}} \int_{F_{2}^{j}} \left| \sum_{i \in J_{2}^{j}} \chi_{Y_{i}} \xi_{i}^{i} - \eta_{j} \right|^{\frac{p_{2}}{p_{2}-1}} \theta_{2}^{\frac{1}{p_{2}-1}} \right. \\
\times \left( 1 + \left| \sum_{i \in J_{1}^{j}} \chi_{Y_{i}} \xi_{i}^{i} \right|^{p_{1}} \theta_{1} + \left| \sum_{i \in J_{2}^{j}} \chi_{Y_{i}} \xi_{i}^{i} \right|^{p_{2}} \theta_{2} + |\eta_{j}|^{p_{1}} \theta_{1} + |\eta_{j}|^{p_{2}} \theta_{2} \right) \left. \right\} \right. \\
+ |\eta_{j}|^{p_{2}} \theta_{2} \right) \frac{1}{p_{1}-1} \theta_{1}^{\frac{1}{p_{1}-1}} \int_{E_{1}^{j}} \left| \sum_{i \in I_{1}^{j}} \chi_{Y_{i}} \xi_{i}^{i} - \eta_{j} \right|^{\frac{p_{1}}{p_{1}-1}} \int_{E_{2}^{j}} \left| \sum_{i \in I_{2}^{j}} \chi_{Y_{i}} \xi_{i}^{i} - \eta_{j} \right|^{\frac{p_{2}}{p_{2}-1}} \theta_{2}^{\frac{1}{p_{2}-1}} \right. \\
\times \left( 1 + \left| \sum_{i \in J_{1}^{j}} \chi_{Y_{i}} \xi_{i}^{i} \right|^{p_{1}} \theta_{1} + \left| \sum_{i \in J_{2}^{j}} \chi_{Y_{i}} \xi_{i}^{i} \right|^{p_{2}} \theta_{2} + |\eta_{j}|^{p_{1}} \theta_{1} + |\eta_{j}|^{p_{2}} \theta_{2} \right) \left. \right\} \right. \\
+ C \sum_{j=0}^{m} \left[ \left( \int_{E_{1}^{j}} \left( 1 + \left| \sum_{i \in I_{1}^{j}} \chi_{Y_{i}} \xi_{i}^{i} \right|^{p_{1}} \theta_{1} + \left| \sum_{i \in I_{2}^{j}} \chi_{Y_{i}} \xi_{i}^{i} \right|^{p_{2}} \theta_{2} + |\eta_{j}|^{p_{1}} \theta_{1} + |\eta_{j}|^{p_{2}} \theta_{2} \right) \right]^{\frac{p_{1}-2}{p_{1}-1}} \right. \\
\times \left( \int_{E_{1}^{j}} \left| \sum_{i \in I_{1}^{j}} \chi_{Y_{i}} \xi_{i}^{i} - \eta_{j} \right|^{p_{1}} \frac{1}{p_{1}-1} \right. \\
\times \left( \int_{E_{2}^{j}} \left| \sum_{i \in I_{2}^{j}} \chi_{Y_{i}} \xi_{i}^{i} - \eta_{j} \right|^{p_{2}} \frac{1}{p_{2}-1} \right. \\
\times \left( \int_{E_{1}^{j}} \left( 1 + \left| \sum_{i \in J_{1}^{j}} \chi_{Y_{i}} \xi_{i}^{i} \right|^{p_{1}} \theta_{1} + \left| \sum_{i \in J_{2}^{j}} \chi_{Y_{i}} \xi_{i}^{i} \right|^{p_{2}} \theta_{2} + |\eta_{j}|^{p_{1}} \theta_{1} + |\eta_{j}|^{p_{2}} \theta_{2} \right) \right) \left. \right\} \right. \\
+ C \sum_{j=0}^{m} \left[ \left( \int_{F_{1}^{j}} \left( 1 + \left| \sum_{i \in J_{1}^{j}} \chi_{Y_{i}} \xi_{i}^{i} \right|^{p_{1}} \theta_{1} + |\eta_{j}|^{p_{1}} \theta_{1} \right. \right. \\
\right]
Reorganizing and using (5.12), we have

\[
\leq C \sum_{j=0}^{m} \left[ \left( |E_{e,j}^i| + \int_{E_{e,j}^i} \sum_{i \in I_j^l} \langle \chi_{Y_j^i} \xi_{e}^i \rangle_{p_2} \theta_1 \, dx + \int_{E_{e,j}^i} \sum_{i \in I_j^l} \langle \chi_{Y_j^i} \xi_{e}^i \rangle_{p_2} \theta_2 \, dx \right)^{\frac{p_1-2}{p_1-1}} \times \left( \int_{E_{e,j}^i} \theta_1 \left| \sum_{i \in I_j^l} \langle \chi_{Y_j^i} \xi_{e}^i \rangle_{p_1} - \eta_j \right| \, dx \right)^{\frac{1}{p_1-1}} \\
+ \left( |E_{e,j}^i| + \int_{E_{e,j}^i} \sum_{i \in I_j^l} \langle \chi_{Y_j^i} \xi_{e}^i \rangle_{p_2} \theta_1 \, dx + \int_{E_{e,j}^i} \sum_{i \in I_j^l} \langle \chi_{Y_j^i} \xi_{e}^i \rangle_{p_2} \theta_2 \, dx \right)^{\frac{p_2-2}{p_2-1}} \times \left( \int_{E_{e,j}^i} \theta_2 \left| \sum_{i \in I_j^l} \langle \chi_{Y_j^i} \xi_{e}^i \rangle_{p_1} - \eta_j \right| \, dx \right)^{\frac{1}{p_2-1}} \right] \\
+ C \sum_{j=0}^{m} \left[ \left( |F_{e,j}^i| + \int_{F_{e,j}^i} \sum_{i \in I_j^l} \langle \chi_{Y_j^i} \xi_{e}^i \rangle_{p_1} \theta_1 \, dx + \int_{F_{e,j}^i} \sum_{i \in I_j^l} \langle \chi_{Y_j^i} \xi_{e}^i \rangle_{p_2} \theta_2 \, dx \right)^{\frac{p_1-2}{p_1-1}} \times \left( \int_{F_{e,j}^i} \theta_1 \left| \sum_{i \in I_j^l} \langle \chi_{Y_j^i} \xi_{e}^i \rangle_{p_1} - \eta_j \right| \, dx \right)^{\frac{1}{p_1-1}} \\
+ \left( |F_{e,j}^i| + \int_{F_{e,j}^i} \sum_{i \in I_j^l} \langle \chi_{Y_j^i} \xi_{e}^i \rangle_{p_2} \theta_1 \, dx + \int_{F_{e,j}^i} \sum_{i \in I_j^l} \langle \chi_{Y_j^i} \xi_{e}^i \rangle_{p_2} \theta_2 \, dx \right)^{\frac{p_2-2}{p_2-1}} \times \left( \int_{F_{e,j}^i} \theta_2 \left| \sum_{i \in I_j^l} \langle \chi_{Y_j^i} \xi_{e}^i \rangle_{p_1} - \eta_j \right| \, dx \right)^{\frac{1}{p_2-1}} \right] \right]
\]
\[
\leq C \sum_{j=0}^{m} \left[ \left( |E^j_\epsilon| + \sum_{i \in I^j_\epsilon} \int_{Y^j_\epsilon} |\xi^i_\epsilon|^{p_1} \theta_1 dx + \sum_{i \in I^j_\epsilon} \int_{Y^j_\epsilon} |\xi^i_\epsilon|^{p_2} \theta_2 dx \right) \right. \\
+ |\eta^j_\epsilon|^{p_1} \theta_1 |E^j_\epsilon| + |\eta^j_\epsilon|^{p_2} \theta_2 |E^j_\epsilon| \right) \frac{1}{p_1-1} \left( \sum_{i \in I^j_\epsilon} \int_{Y^j_\epsilon} \theta_1 |\xi^i_\epsilon - \eta^j_\epsilon|^p_1 dx \right) \\
+ \left. \left( |E^j_\epsilon| + \sum_{i \in I^j_\epsilon} \int_{Y^j_\epsilon} |\xi^i_\epsilon|^{p_1} \theta_1 dx + \sum_{i \in I^j_\epsilon} \int_{Y^j_\epsilon} |\xi^i_\epsilon|^{p_2} \theta_2 dx \right) \right. \\
+ |\eta^j_\epsilon|^{p_1} \theta_1 |E^j_\epsilon| + |\eta^j_\epsilon|^{p_2} \theta_2 |E^j_\epsilon| \right) \frac{1}{p_2-1} \left( \sum_{i \in I^j_\epsilon} \int_{Y^j_\epsilon} \theta_2 |\xi^i_\epsilon - \eta^j_\epsilon|^p_2 dx \right) \\
+ C \sum_{j=0}^{m} \left[ \left( |F^j_\epsilon| + \int_{F^j_\epsilon} \sum_{i \in J^j_\epsilon} \chi_{Y^j_\epsilon} \xi^i_\epsilon \right)^{p_1} \theta_1 dx + \int_{F^j_\epsilon} \sum_{i \in J^j_\epsilon} \chi_{Y^j_\epsilon} \xi^i_\epsilon \right)^{p_2} \theta_2 dx \\
+ |\eta^j_\epsilon|^{p_1} \theta_1 |F^j_\epsilon| + |\eta^j_\epsilon|^{p_2} \theta_2 |F^j_\epsilon| \right) \frac{1}{p_1-1} \left( \int_{F^j_\epsilon} \theta_1 \sum_{i \in J^j_\epsilon} \chi_{Y^j_\epsilon} \xi^i_\epsilon - \eta^j_\epsilon \right)^{p_1} dx \\
+ \left. \left( |F^j_\epsilon| + \int_{F^j_\epsilon} \sum_{i \in J^j_\epsilon} \chi_{Y^j_\epsilon} \xi^i_\epsilon \right)^{p_1} \theta_1 dx + \int_{F^j_\epsilon} \sum_{i \in J^j_\epsilon} \chi_{Y^j_\epsilon} \xi^i_\epsilon \right)^{p_2} \theta_2 dx \\
+ |\eta^j_\epsilon|^{p_1} \theta_1 |F^j_\epsilon| + |\eta^j_\epsilon|^{p_2} \theta_2 |F^j_\epsilon| \right) \frac{1}{p_2-1} \left( \int_{F^j_\epsilon} \theta_2 \sum_{i \in J^j_\epsilon} \chi_{Y^j_\epsilon} \xi^i_\epsilon - \eta^j_\epsilon \right)^{p_2} dx \right]
\]

By (5.1) and (5.9), we have

\[
\leq C \sum_{j=0}^{m} \left[ \left( |E^j_\epsilon| + \sum_{i \in I^j_\epsilon} |Y^j_\epsilon| \left| \xi^i_\epsilon \right|^{p_1} \theta_1 + \sum_{i \in I^j_\epsilon} |Y^j_\epsilon| \left| \xi^i_\epsilon \right|^{p_2} \theta_2 \right) \right. \\
+ |\eta^j_\epsilon|^{p_1} \theta_1 |E^j_\epsilon| + |\eta^j_\epsilon|^{p_2} \theta_2 |E^j_\epsilon| \right) \frac{1}{p_1-1} \left( \sum_{i \in I^j_\epsilon} |Y^j_\epsilon| \left| \theta_1 \left| \xi^i_\epsilon - \eta^j_\epsilon \right|^{p_1} \right) \\
+ \left. \left( |E^j_\epsilon| + \sum_{i \in I^j_\epsilon} |Y^j_\epsilon| \left| \xi^i_\epsilon \right|^{p_1} \theta_1 + \sum_{i \in I^j_\epsilon} |Y^j_\epsilon| \left| \xi^i_\epsilon \right|^{p_2} \theta_2 \right) \right. \\
+ |\eta^j_\epsilon|^{p_1} \theta_1 |E^j_\epsilon| + |\eta^j_\epsilon|^{p_2} \theta_2 |E^j_\epsilon| \right) \frac{1}{p_2-1} \left( \sum_{i \in I^j_\epsilon} |Y^j_\epsilon| \left| \theta_2 \left| \xi^i_\epsilon - \eta^j_\epsilon \right|^{p_2} \right) \right]
\]

42
\[
\begin{align*}
+ C \sum_{j=0}^m \left[ \left( |F^j_\varepsilon| + \int_{F^j_\varepsilon} |M_\varepsilon \varphi|^{p_1} \theta_1 \, dx + \int_{F^j_\varepsilon} |M_\varepsilon \varphi|^{p_2} \theta_2 \, dx \right) \right]^{\frac{1}{p_1-1}} \\
+ |\eta_j|^{p_1} \theta_1 |F^j_\varepsilon| + |\eta_j|^{p_2} \theta_2 |F^j_\varepsilon| \right) \frac{p_1-2}{p_1-1} \times \left( \int_{F^j_\varepsilon} \theta_2 \left| \sum_{i \in I^j_\varepsilon} \chi_{Y^j_\varepsilon} \xi_{i}^{j} \right| \eta_i \right)^{\frac{1}{p_2-1}} \\
+ \frac{1}{p_1} \left( |F^j_\varepsilon| + \int_{F^j_\varepsilon} |M_\varepsilon \varphi|^{p_1} \theta_1 \, dx + \int_{F^j_\varepsilon} |M_\varepsilon \varphi|^{p_2} \theta_2 \, dx \right) \right]^{\frac{1}{p_2-1}} \\
+ |\eta_j|^{p_1} \theta_1 |F^j_\varepsilon| + |\eta_j|^{p_2} \theta_2 |F^j_\varepsilon| \right) \frac{p_2-2}{p_2-1} \times \left( \int_{F^j_\varepsilon} \theta_2 \left| \sum_{i \in I^j_\varepsilon} \chi_{Y^j_\varepsilon} \xi_{i}^{j} \right| \eta_i \right)^{\frac{1}{p_2-1}} \\
+ \frac{1}{p_1} \left( |F^j_\varepsilon| + \int_{F^j_\varepsilon} |M_\varepsilon \varphi|^{p_1} \theta_1 \, dx + \int_{F^j_\varepsilon} |M_\varepsilon \varphi|^{p_2} \theta_2 \, dx \right) \right]^{\frac{1}{p_2-1}} \\
+ \frac{1}{p_1} \left( |F^j_\varepsilon| + \int_{F^j_\varepsilon} |M_\varepsilon \varphi|^{p_1} \theta_1 \, dx + \int_{F^j_\varepsilon} |M_\varepsilon \varphi|^{p_2} \theta_2 \, dx \right) \right]^{\frac{1}{p_2-1}} \\
\end{align*}
\]
\[
C \sum_{j=0}^{m} \left[ \left( E^j + \int_{E^j} \chi^e_1(x) \left| \sum_{i \in I^j} \chi_{Y^i}(x) \xi^i \right|^2 \right) dx + \int_{E^j} \chi^e_2(x) \left| \sum_{i \in I^j} \chi_{Y^i}(x) \xi^i \right|^2 \right]
\]

+ \left[ \left| \eta_j \right| \theta_1 \left| E^j \right| + \left| \eta_j \right| \theta_2 \left| E^j \right|^2 \right] \times \left( \int_{E^j} \chi^e_1(x) \left| \sum_{i \in I^j} \chi_{Y^i}(x) \xi^i - \eta_j \right| \right)

+ \left( \left| E^j \right| + \int_{E^j} \chi^e_1(x) \left| \sum_{i \in I^j} \chi_{Y^i}(x) \xi^i \right|^p \right) \times \left( \int_{E^j} \chi^e_2(x) \left| \sum_{i \in I^j} \chi_{Y^i}(x) \xi^i - \eta_j \right| \right)

+ \left[ \left| E^j \right| + \int_{E^j} \left| M \varphi \right|^p \theta_1 \right] + \left[ \left| E^j \right| + \int_{E^j} \left| M \varphi \right|^2 \theta_2 \right] \times \left( \int_{E^j} \theta_1 \left| \sum_{i \in I^j} \chi_{Y^i}(x) \xi^i - \eta_j \right| \right)

+ \left( \left| F^j \right| + \int_{F^j} \left| M \varphi \right|^p \theta_1 \right) + \left( \left| F^j \right| + \int_{F^j} \left| M \varphi \right|^2 \theta_2 \right) \times \left( \int_{F^j} \theta_2 \left| \sum_{i \in I^j} \chi_{Y^i}(x) \xi^i - \eta_j \right| \right)

By (5.1) and (5.7), we get

\[
\leq C \sum_{j=0}^{m} \left[ \left( E^j + \int_{E^j} \chi^e_1(x) \left| M \varphi \right|^p dx \right) + \int_{E^j} \chi^e_2(x) \left| M \varphi \right|^2 dx
\]

+ \int_{E^j} \chi^e_1(x) \left| \eta_j \right|^p dx + \int_{E^j} \chi^e_2(x) \left| \eta_j \right|^p \left. dx \right|^{p-2} \times \left( \int_{\Omega} \chi^e_1(x) \left| M \varphi - \Psi \right| \right)^{p-1}

+ \left( \left| E^j \right| + \int_{E^j} \chi^e_1(x) \left| M \varphi \right|^p dx \right) + \int_{E^j} \chi^e_2(x) \left| M \varphi \right|^2 dx

+ \int_{E^j} \chi^e_1(x) \left| \eta_j \right|^p \left. dx \right|^{p-2} \times \left( \int_{\Omega} \chi^e_2(x) \left| M \varphi - \Psi \right| \right)^{p-1}

+ C \sum_{j=0}^{m} \left[ \left( F^j + \int_{F^j} \left| M \varphi \right|^p \theta_1 \right) + \int_{F^j} \left| M \varphi \right|^2 \theta_2 \right] dx

\]

44
\[
+ |\eta_j|^{p_1} \theta_1 \left| F^j_\varepsilon \right| + |\eta_j|^{p_2} \theta_2 \left| F^j_\varepsilon \right| \frac{p_1 - 2}{p_1 - 1} \times \left( \int_{F^j_\varepsilon} \theta_1 \left| \sum_{i \in J^j} \chi_{Y^j_i} \xi^i_{\varepsilon} - \eta_j \right| \, dx \right) \frac{1}{p_1 - 1} \\
+ \left( \left| F^j_\varepsilon \right| + \int_{F^j_\varepsilon} |M_\varepsilon \varphi|^{p_1} \theta_1 \, dx + \int_{F^j_\varepsilon} |M_\varepsilon \varphi|^{p_2} \theta_2 \, dx \right) \\
+ |\eta_j|^{p_1} \theta_1 \left| F^j_\varepsilon \right| + |\eta_j|^{p_2} \theta_2 \left| F^j_\varepsilon \right| \frac{p_2 - 2}{p_2 - 1} \times \left( \int_{F^j_\varepsilon} \theta_2 \left| \sum_{i \in J^j} \chi_{Y^j_i} \xi^i_{\varepsilon} - \eta_j \right| \, dx \right) \frac{1}{p_2 - 1} \right] \\
\leq C \sum_{j=0}^m \left[ \left( |\Omega_j| + \int_{\Omega_j} \chi^j_1(x) |M_\varepsilon \varphi - \varphi + \varphi|^{p_1} \, dx + \int_{\Omega_j} \chi^j_2(x) |M_\varepsilon \varphi - \varphi + \varphi|^{p_2} \, dx \right) \\
+ \int_{\Omega_j} \chi^j_1(x) |\eta_j|^{p_1} \, dx + \int_{\Omega_j} \chi^j_2(x) |\eta_j|^{p_2} \, dx \right) \frac{p_1 - 2}{p_1 - 1} \times \left( \int_{\Omega_j} \chi^j_1(x) |M_\varepsilon \varphi - \varphi + \varphi - \Psi|^{p_1} \, dx \right) \frac{1}{p_1 - 1} \\
+ \left( |\Omega_j| + \int_{\Omega_j} \chi^j_1(x) |M_\varepsilon \varphi - \varphi + \varphi|^{p_1} \, dx + \int_{\Omega_j} \chi^j_2(x) |M_\varepsilon \varphi - \varphi + \varphi|^{p_2} \, dx \right) \\
+ \int_{\Omega_j} \chi^j_1(x) |\eta_j|^{p_1} \, dx + \int_{\Omega_j} \chi^j_2(x) |\eta_j|^{p_2} \, dx \right) \frac{p_2 - 2}{p_2 - 1} \times \left( \int_{\Omega_j} \chi^j_2(x) |M_\varepsilon \varphi - \varphi + \varphi - \Psi|^{p_2} \, dx \right) \frac{1}{p_2 - 1} \right] \\
+ C \sum_{j=0}^m \left[ \left( |F^j_\varepsilon| + \int_{F^j_\varepsilon} |M_\varepsilon \varphi|^{p_1} \theta_1 \, dx + \int_{F^j_\varepsilon} |M_\varepsilon \varphi|^{p_2} \theta_2 \, dx \right) \\
+ |\eta_j|^{p_1} \theta_1 \left| F^j_\varepsilon \right| + |\eta_j|^{p_2} \theta_2 \left| F^j_\varepsilon \right| \frac{p_1 - 2}{p_1 - 1} \times \left( \int_{F^j_\varepsilon} \theta_1 \left| \sum_{i \in J^j} \chi_{Y^j_i} \xi^i_{\varepsilon} - \eta_j \right| \, dx \right) \frac{1}{p_1 - 1} \\
+ \left( |F^j_\varepsilon| + \int_{F^j_\varepsilon} |M_\varepsilon \varphi|^{p_1} \theta_1 \, dx + \int_{F^j_\varepsilon} |M_\varepsilon \varphi|^{p_2} \theta_2 \, dx \right) \\
+ |\eta_j|^{p_1} \theta_1 \left| F^j_\varepsilon \right| + |\eta_j|^{p_2} \theta_2 \left| F^j_\varepsilon \right| \frac{p_2 - 2}{p_2 - 1} \times \left( \int_{F^j_\varepsilon} \theta_2 \left| \sum_{i \in J^j} \chi_{Y^j_i} \xi^i_{\varepsilon} - \eta_j \right| \, dx \right) \frac{1}{p_2 - 1} \right] 
\]
\[
\leq C \sum_{j=0}^{m} \left[ \left( \int_{\Omega_j} + \int_{\Omega_j} \chi_1^j(x) \right) M_\varepsilon \varphi - \varphi \right]^{p_1} dx + \sum_{j=0}^{m} \int_{\Omega_j} \chi_1^j(x) |\varphi|^{p_1} dx \\
+ \int_{\Omega_j} \chi_2^j(x) |M_\varepsilon \varphi - \varphi|^{p_2} dx + \int_{\Omega_j} \chi_2^j(x) |\varphi|^{p_2} dx + \int_{\Omega_j} \chi_1^j(x) |\eta_j|^{p_1} dx \\
+ \int_{\Omega_j} \chi_2^j(x) |\eta_j|^{p_2} dx \right]^{\frac{p_1-2}{p_1-1}} \times \left( \int_{\Omega_j} \chi_1^j(x) |M_\varepsilon \varphi - \varphi|^{p_1} dx + \int_{\Omega_j} \chi_1^j(x) |\varphi - \Psi|^{p_1} dx \right)^{\frac{1}{p_1-1}} \\
+ \left( \int_{\Omega_j} + \int_{\Omega_j} \chi_1^j(x) \right) M_\varepsilon \varphi - \varphi \right]^{p_1} dx + \int_{\Omega_j} \chi_1^j(x) |\varphi|^{p_1} dx + \int_{\Omega_j} \chi_2^j(x) M_\varepsilon \varphi - \varphi \right]^{p_2} dx \\
+ \int_{\Omega_j} \chi_2^j(x) |\varphi|^{p_2} dx + \int_{\Omega_j} \chi_1^j(x) |\eta_j|^{p_1} dx + \int_{\Omega_j} \chi_2^j(x) |\eta_j|^{p_2} dx \right]^{\frac{p_2-2}{p_2-1}} \times \left( \int_{\Omega_j} \chi_2^j(x) |M_\varepsilon \varphi - \varphi|^{p_2} dx + \int_{\Omega_j} \chi_2^j(x) |\varphi - \Psi|^{p_2} dx \right)^{\frac{1}{p_2-1}} \\
+ C \sum_{j=0}^{m} \left[ \left( \int_{F_i^j} |F_i^j| + \int_{F_i^j} |M_\varepsilon \varphi|^{p_1} \theta_1 dx + \int_{F_i^j} |M_\varepsilon \varphi|^{p_2} \theta_2 dx \right) \\
+ |\eta_j|^{p_1} \theta_1 \left| F_i^j \right| + |\eta_j|^{p_2} \theta_2 \left| F_i^j \right| \right]^{\frac{p_1-2}{p_1-1}} \times \left( \int_{F_i^j} \theta_1 \left| \sum_{i \in J_i} \chi Y_i^j \xi_i - \eta_j \right|^{p_1} dx \right)^{\frac{1}{p_1-1}} \\
+ \left( \int_{F_i^j} |F_i^j| + \int_{F_i^j} |M_\varepsilon \varphi|^{p_1} \theta_1 dx + \int_{F_i^j} |M_\varepsilon \varphi|^{p_2} \theta_2 dx \right) \\
+ |\eta_j|^{p_1} \theta_1 \left| F_i^j \right| + |\eta_j|^{p_2} \theta_2 \left| F_i^j \right| \right]^{\frac{p_2-2}{p_2-1}} \times \left( \int_{F_i^j} \theta_2 \left| \sum_{i \in J_i} \chi Y_i^j \xi_i - \eta_j \right|^{p_2} dx \right)^{\frac{1}{p_2-1}} \\
\right]
\]

Reorganizing the sums, we obtain

\[
\leq C \left[ \left( \sum_{j=0}^{m} |\Omega_j| + \sum_{j=0}^{m} \int_{\Omega_j} \chi_1^j(x) \right) M_\varepsilon \varphi - \varphi \right]^{p_1} dx + \sum_{j=0}^{m} \int_{\Omega_j} \chi_1^j(x) |\varphi|^{p_1} dx \\
+ \sum_{j=0}^{m} \int_{\Omega_j} \chi_2^j(x) |M_\varepsilon \varphi - \varphi|^{p_2} dx + \sum_{j=0}^{m} \int_{\Omega_j} \chi_2^j(x) |\varphi|^{p_2} dx + \sum_{j=0}^{m} \int_{\Omega_j} \chi_1^j(x) |\eta_j|^{p_1} dx \\
+ \sum_{j=0}^{m} \int_{\Omega_j} \chi_2^j(x) |\eta_j|^{p_2} dx \right]^{\frac{p_1-2}{p_1-1}} \times \left( \int_{\Omega_j} \chi_1^j(x) |M_\varepsilon \varphi - \varphi|^{p_1} dx \\
+ \int_{\Omega_j} \chi_1^j(x) |\varphi - \Psi|^{p_1} dx \right)^{\frac{1}{p_1-1}}
\]
\[ + \left( \sum_{j=0}^{m} |\Omega_j| + \sum_{j=0}^{m} \int_{\Omega_j} \chi_1^e(x) |M_\epsilon \varphi - \varphi|^p_1 \, dx + \sum_{j=0}^{m} \int_{\Omega_j} \chi_1^e(x) |\varphi|^p_1 \, dx \right) \\
+ \left( \sum_{j=0}^{m} \int_{\Omega_j} \chi_2^e(x) |M_\epsilon \varphi - \varphi|^p_2 \, dx + \sum_{j=0}^{m} \int_{\Omega_j} \chi_2^e(x) |\varphi|^p_2 \, dx + \sum_{j=0}^{m} \int_{\Omega_j} \chi_1^e(x) |\eta_j|^p_1 \, dx \right) \\
+ \sum_{j=0}^{m} \int_{\Omega_j} \chi_2^e(x) |\eta_j|^p_2 \, dx \right) \frac{p_2-2}{p_2-1} \\
\times \left( \int_{\Omega} \chi_2^e(x) |M_\epsilon \varphi - \varphi|^p_2 \, dx + \int_{\Omega} \chi_2^e(x) |\varphi - \Psi|^p_2 \, dx \right) \frac{1}{p_2-1} \right) \right] \\
+ C \sum_{j=0}^{m} \left[ \left( |F_j^2| + \int_{F_j^2} |M_\epsilon \varphi|^p_1 \, \theta_1 \, dx + \int_{F_j^2} |M_\epsilon \varphi|^p_2 \, \theta_2 \, dx \right) \frac{p_1-2}{p_1-1} \right] \left( \int_{F_j^2} \theta_1 \left| \sum_{i \in J_j^1} \chi_{2,2i} \xi_i \right|^p_1 \, dx \right) \frac{1}{p_1-1} \right) \right] \]
\leq C \left[ \left( |\Omega| + \int_{\Omega} \chi_1^e(x) |M_\epsilon \varphi - \varphi|^p_1 \, dx + \int_{\Omega} \chi_1^e(x) |\varphi|^p_1 \, dx + \int_{\Omega} \chi_2^e(x) |M_\epsilon \varphi - \varphi|^p_2 \, dx \right) \frac{p_1-2}{p_1-1} \right) \\
\times \left( \int_{\Omega} \chi_1^e(x) |M_\epsilon \varphi - \varphi|^p_1 \, dx + \int_{\Omega} \chi_1^e(x) |\varphi - \Psi|^p_1 \, dx \right) \frac{1}{p_1-1} \right) \\
+ \left( |\Omega| + \int_{\Omega} \chi_1^e(x) |M_\epsilon \varphi - \varphi|^p_1 \, dx + \int_{\Omega} \chi_1^e(x) |\varphi|^p_1 \, dx + \int_{\Omega} \chi_2^e(x) |M_\epsilon \varphi - \varphi|^p_2 \, dx \right) \frac{p_2-2}{p_2-1} \right) \left( \int_{\Omega} \chi_2^e(x) |M_\epsilon \varphi - \varphi|^p_2 \, dx + \int_{\Omega} \chi_2^e(x) |\varphi - \Psi|^p_2 \, dx \right) \frac{1}{p_2-1} \right] \\
+ C \sum_{j=0}^{m} \left[ \left( |F_j^2| + \int_{F_j^2} |M_\epsilon \varphi|^p_1 \, \theta_1 \, dx + \int_{F_j^2} |M_\epsilon \varphi|^p_2 \, \theta_2 \, dx \right) \right] \]
Also, by Property 1 of $M$,
\[ \text{Since } + \text{+} + \text{+} + | \partial \Omega_j | = 0 \text{ for } j \neq 0, \text{ then we have that } | F_j^\varepsilon | \to 0 \text{ as } \varepsilon \to 0 \text{ for every } j = 0, 1, 2, \ldots, m. \]

Also, by Property 1 of $M_\varepsilon$ in Remark 5.1, we have
\[ \int_\Omega \chi^\varepsilon(x) | M_\varepsilon \varphi - \varphi | \big|^{p_1} dx \to 0, \]
as $\epsilon \to 0$, for $i = 1, 2$.

Therefore, taking $\limsup$ as $\epsilon \to 0$ above, we obtain (5.8).

Lemma 5.6. If the microstructure is dispersed or layered, we have that

$$\sup_{\epsilon > 0} \left\{ \int_{\Omega} \chi_i^\epsilon(x) |p_\epsilon(x, M_\epsilon \nabla u)|^{p_i} \, dx \right\} \leq C < \infty, \text{ for } i = 1, 2.$$

Proof. Using (5.1), we have

$$\int_{\Omega} \chi_1^\epsilon(x) |p_\epsilon(x, M_\epsilon \nabla u)|^{p_1} \, dx + \int_{\Omega} \chi_2^\epsilon(x) |p_\epsilon(x, M_\epsilon \nabla u)|^{p_2} \, dx$$

$$= \int_{\Omega} \chi_1^\epsilon(x) |p_\epsilon(x, M_\epsilon \nabla u)|^{p_1} \, dx + \int_{\Omega} \chi_2^\epsilon(x) |p_\epsilon(x, M_\epsilon \nabla u)|^{p_2} \, dx$$

$$= \sum_{i \in I_\epsilon} \int_{Y_i^\epsilon} \chi_1^\epsilon(x) |p_\epsilon(x, \xi_i^\epsilon)|^{p_1} \, dx + \sum_{i \in I_\epsilon} \int_{Y_i^\epsilon} \chi_2^\epsilon(x) |p_\epsilon(x, \xi_i^\epsilon)|^{p_2} \, dx$$

By Lemma 5.3, Jensen’s inequality, and Theorem 4.1 we get

$$\leq C \sum_{i \in I_\epsilon} \left( 1 + |\xi_i^\epsilon|^{p_1} \theta_1 + |\xi_i^\epsilon|^{p_2} \theta_2 \right) |Y_i^\epsilon|$$

$$= C \sum_{i \in I_\epsilon} \left( |Y_i^\epsilon| + |\xi_i^\epsilon|^{p_1} \theta_1 |Y_i^\epsilon| + |\xi_i^\epsilon|^{p_2} \theta_2 |Y_i^\epsilon| \right)$$

$$\leq C \left( |\Omega| + \|M_\epsilon \nabla u\|_{L^{p_1}(\Omega)} + \|M_\epsilon \nabla u\|_{L^{p_2}(\Omega)} \right)$$

$$\leq C \left( |\Omega| + \|\nabla u\|_{L^{p_1}(\Omega)} + \|\nabla u\|_{L^{p_2}(\Omega)} \right)$$

$$< \infty \text{ (uniformly with respect to } \epsilon).$$

Lemma 5.7. For all $j = 0, \ldots, m$, we have that $\int_{\Omega_j} |(A_\epsilon(x, p_\epsilon(x, \eta_j)), \nabla u_\epsilon)| \, dx$ as well as

$$\int_{\Omega_j} |(A_\epsilon(x, \nabla u_\epsilon), p_\epsilon(x, \eta_j))| \, dx$$

are uniformly bounded with respect to $\epsilon$.

Proof. Using Hölder’s Inequality, (3.4), and (3.12), we obtain

$$\int_{\Omega_j} |(A_\epsilon(x, p_\epsilon(x, \eta_j)), \nabla u_\epsilon)| \, dx \leq \int_{\Omega_j} |A_\epsilon(x, p_\epsilon(x, \eta_j))| |\nabla u_\epsilon| \, dx$$

$$\leq C \left[ \left( \int_{\Omega_j} \chi_1^\epsilon(x) (1 + |p_\epsilon(x, \eta_j)|)^{p_1} \, dx \right)^{\frac{1}{q_2}} + \left( \int_{\Omega_j} \chi_2^\epsilon(x) (1 + |p_\epsilon(x, \eta_j)|)^{p_2} \, dx \right)^{\frac{1}{q_1}} \right]$$

$$\leq C, \text{ where } C \text{ does not depend on } \epsilon.$$
The proof of the uniform boundedness of $\int_{\Omega_j} |(A_\epsilon(x, \nabla u_\epsilon), p_\epsilon(x, \eta_j))| \, dx$ follows in the same manner.

Lemma 5.8. As $\epsilon \to 0$, up to a subsequence, $(A_\epsilon(\cdot, p_\epsilon(\cdot, \eta_j)), \nabla u_\epsilon(\cdot))$ converges weakly to a function $g_j \in L^1(\Omega_j; \mathbb{R})$, for all $j = 0, \ldots, m$. In a similar way, up to a subsequence, $(A_\epsilon(\cdot, \nabla u_\epsilon(\cdot)), p_\epsilon(\cdot, \eta_j))$ converges weakly to a function $h_j \in L^1(\Omega_j; \mathbb{R})$, for all $j = 0, \ldots, m$.

Proof. We prove the first statement of the lemma, the second statement follows in a similar way. The lemma follows from the Dunford-Pettis theorem (see [Dac89]). To apply this theorem we establish the following conditions given by:

1. $\int_{\Omega_j} |(A_\epsilon(x, p_\epsilon(x, \eta_j)), \nabla u_\epsilon)| \, dx$ is uniformly bounded with respect to $\epsilon$, and

2. For all $j = 0, \ldots, m$, $(A_\epsilon(\cdot, p_\epsilon(\cdot, \eta_j)), \nabla u_\epsilon)$ is equiintegrable.

The first condition is proved in Lemma 5.7. For the second condition, we have that $\chi_1^\epsilon(\cdot)|A_\epsilon(\cdot, p_\epsilon(\cdot, \eta_j))|^{q_2}$ and $\chi_2^\epsilon(\cdot)|A_\epsilon(\cdot, p_\epsilon(\cdot, \eta_j))|^{q_1}$ are equiintegrable (see for example Theorem 1.5 of [Dac89]).

By (3.12), for any $E \subset \Omega$, we have

$$\max_{i=1,2} \left\{ \sup_{\epsilon > 0} \left\{ \left( \int_E \chi_i^\epsilon(x)|\nabla u_\epsilon|^{p_i} \, dx \right)^{\frac{1}{p_i}} \right\} \right\} \leq C.$$ 

Let $\alpha > 0$ arbitrary and choose $\alpha_1 > 0$ and $\alpha_2 > 0$ such that $\alpha_1^{q_2} + \alpha_2^{q_1} < \alpha/C$.

For $\alpha_1$ and $\alpha_2$, there exist $\lambda(\alpha_1) > 0$ and $\lambda(\alpha_2) > 0$ such that for every $E \subset \Omega$ with $|E| < \min \{\lambda(\alpha_1), \lambda(\alpha_2)\}$,

$$\int_E \chi_1^\epsilon(x)|A_\epsilon(x, p_\epsilon(x, \eta_j))|^{q_2} \, dx < \alpha_1, \text{ and } \int_E \chi_2^\epsilon(x)|A_\epsilon(x, p_\epsilon(x, \eta_j))|^{q_1} \, dx < \alpha_2.$$

Take $\lambda = \lambda(\alpha) = \min \{\lambda(\alpha_1), \lambda(\alpha_2)\}$. Then, for all $E \subset \Omega$ with $|E| < \lambda(\alpha)$, we have

$$\int_E |(A_\epsilon(x, p_\epsilon(x, \eta_j)), \nabla u_\epsilon)| \, dx \leq \int_E |A_\epsilon(x, p_\epsilon(x, \eta_j))| |\nabla u_\epsilon| \, dx$$

$$\leq \left( \int_E \chi_1^\epsilon(x)|A_\epsilon(x, p_\epsilon(x, \eta_j))|^{q_2} \, dx \right)^{\frac{1}{q_2}} \left( \int_E \chi_1^\epsilon(x)|\nabla u_\epsilon|^{p_1} \, dx \right)^{\frac{1}{p_1}}$$

$$+ \left( \int_E \chi_2^\epsilon(x)|A_\epsilon(x, p_\epsilon(x, \eta_j))|^{q_1} \, dx \right)^{\frac{1}{q_1}} \left( \int_E \chi_2^\epsilon(x)|\nabla u_\epsilon|^{p_2} \, dx \right)^{\frac{1}{p_2}}$$

$$\leq C(\alpha_1^{q_2} + \alpha_2^{q_1}) < \alpha,$$

for every $\alpha > 0$, and so $(A_\epsilon(\cdot, p_\epsilon(\cdot, \eta_j)), \nabla u_\epsilon)$ is equiintegrable.
5.3 Proof of the Corrector Theorem

We are now in the position to give the proof of Theorem 5.2.

Proof. Let $u_\epsilon \in W^{1,p_1}_0(\Omega)$ the solutions of (3.7). By (3.5), we have that
\[
\int_\Omega \left[ \chi_1^\epsilon (x) \left| p_\epsilon (x, M_\epsilon \nabla u(x)) - \nabla u_\epsilon(x) \right|^{p_1} + \chi_2^\epsilon (x) \left| p_\epsilon (x, M_\epsilon \nabla u(x)) - \nabla u_\epsilon(x) \right|^{p_2} \right] dx \\
\leq C \int_\Omega (A_\epsilon (x, p_\epsilon (x, M_\epsilon \nabla u(x))) - A_\epsilon (x, \nabla u_\epsilon(x)), p_\epsilon (x, M_\epsilon \nabla u(x)) - \nabla u_\epsilon(x)) dx
\]

To prove Theorem 5.2, we show that
\[
\int_\Omega (A_\epsilon (x, p_\epsilon (x, M_\epsilon \nabla u)) - A_\epsilon (x, \nabla u_\epsilon(x)), p_\epsilon (x, M_\epsilon \nabla u)) dx \\
= \int_\Omega (A_\epsilon (x, p_\epsilon (x, M_\epsilon \nabla u)) \cdot p_\epsilon (x, M_\epsilon \nabla u)) dx - \int_\Omega (A_\epsilon (x, p_\epsilon (x, M_\epsilon \nabla u)) \cdot \nabla u_\epsilon(x)) dx \\
- \int_\Omega (A_\epsilon (x, \nabla u_\epsilon(x)) \cdot p_\epsilon (x, M_\epsilon \nabla u)) dx + \int_\Omega (A_\epsilon (x, \nabla u_\epsilon(x)) \cdot \nabla u_\epsilon(x)) dx
\]
goesto 0, as $\epsilon \to 0$. This is done in four steps.

In what follows, we use the following notation
\[
\xi_\epsilon^i = \frac{1}{|Y_\epsilon^i|} \int_{Y_\epsilon^i} \nabla u(x) dx.
\]

Step 1

Let us prove that
\[
\int_\Omega (A_\epsilon (x, p_\epsilon (x, M_\epsilon \nabla u)) \cdot p_\epsilon (x, M_\epsilon \nabla u)) dx \to \int_\Omega (b(\nabla u), \nabla u)) dx, \text{ as } \epsilon \to 0. \quad (5.13)
\]

Proof. From (3.18) and (5.1), we obtain
\[
\int_\Omega (A_\epsilon (x, p_\epsilon (x, M_\epsilon \nabla u))) \cdot p_\epsilon (x, M_\epsilon \nabla u)) dx \\
= \int_{\Omega_\epsilon} (A_\epsilon (x, p_\epsilon (x, M_\epsilon \nabla u))) \cdot p_\epsilon (x, M_\epsilon \nabla u)) dx \\
= \sum_{i \in I_\epsilon} \int_{Y_\epsilon^i} (A_\epsilon \left( \frac{x}{\epsilon} , p_\epsilon \left( \frac{x}{\epsilon}, \xi_\epsilon^i \right) \right) \cdot p_\epsilon \left( \frac{x}{\epsilon}, \xi_\epsilon^i \right)) dx \\
= e^n \sum_{i \in I_\epsilon} \int_{Y_\epsilon^i} (A_\epsilon \left( y, p_\epsilon (y, \xi_\epsilon^i) \right)) \cdot p_\epsilon \left( y, \xi_\epsilon^i \right) dy \\
= \sum_{i \in I_\epsilon} \int_{Y_\epsilon^i} \chi_{Y_\epsilon^i}(x) \left( b(\xi_\epsilon^i), \xi_\epsilon^i \right) dx \\
= \int_\Omega (b(M_\epsilon \nabla u(x)), M_\epsilon \nabla u(x)) dx.
\]
By (3.15) and the definition of $q_1$, we have

$$
\int_{\Omega} |b(M_\epsilon \nabla u(x)) - b(\nabla u(x))|^{q_1} \, dx
\leq C \left[ \left( \int_{\Omega} |M_\epsilon \nabla u(x) - \nabla u(x)|^{p_2} \, dx \right)^{\frac{1}{(p_2-1)(p_1-1)}} \right.
\times \left( \int_{\Omega} \left( 1 + |M_\epsilon \nabla u|^{p_1} + |\nabla u|^{p_1} + |M_\epsilon \nabla u|^{p_2} + |\nabla u|^{p_2} \right) \frac{p_2(p_2-2)}{(p_1-1)(p_2-1)-2} \, dx \right)^{\frac{(p_1-1)(p_2-1)-1}{(p_1-1)(p_2-1)}}
+ \left( \int_{\Omega} |M_\epsilon \nabla u(x) - \nabla u(x)|^{p_2} \, dx \right)^{\frac{1}{(p_2-1)^2}}
\times \left( \int_{\Omega} \left( 1 + |M_\epsilon \nabla u|^{p_1} + |\nabla u|^{p_1} + |M_\epsilon \nabla u|^{p_2} + |\nabla u|^{p_2} \right) \, dx \right)^{\frac{p_2(p_2-2)}{(p_2-1)^2}} \left. \right]
$$

By Hölder’s Inequality in the first integral with $p = (p_2-1)(p_1-1) > 1$ and in the second integral with $p = (p_2-1)^2 > 1$, we obtain

$$
\leq C \left[ \left( \int_{\Omega} |M_\epsilon \nabla u(x) - \nabla u(x)|^{p_2} \, dx \right)^{\frac{1}{(p_2-1)(p_1-1)}} \right.
\times \left( \int_{\Omega} \left( 1 + |M_\epsilon \nabla u|^{p_1} + |\nabla u|^{p_1} + |M_\epsilon \nabla u|^{p_2} + |\nabla u|^{p_2} \right) \frac{p_2(p_2-2)}{(p_1-1)(p_2-1)-2} \, dx \right)^{\frac{(p_1-1)(p_2-1)-1}{(p_1-1)(p_2-1)}}
+ \left( \int_{\Omega} |M_\epsilon \nabla u(x) - \nabla u(x)|^{p_2} \, dx \right)^{\frac{1}{(p_2-1)^2}}
\times \left( \int_{\Omega} \left( 1 + |M_\epsilon \nabla u|^{p_1} + |\nabla u|^{p_1} + |M_\epsilon \nabla u|^{p_2} + |\nabla u|^{p_2} \right) \, dx \right)^{\frac{p_2(p_2-2)}{(p_2-1)^2}} \left. \right]
$$

By Theorem 4.1 and Jensen’s inequality, we have

$$
\leq C \left[ \left( \int_{\Omega} |M_\epsilon \nabla u - \nabla u|^{p_2} \, dx \right)^{\frac{1}{(p_2-1)(p_1-1)}} \right. + \left( \int_{\Omega} |M_\epsilon \nabla u - \nabla u|^{p_2} \, dx \right)^{\frac{1}{(p_2-1)^2}} \left. \right]
$$

From Property 1 of $M_\epsilon$ in Remark 5.1, we obtain that

$$
b(M_\epsilon \nabla u) \to b(\nabla u) \text{ in } L^{q_1}(\Omega; \mathbb{R}^n), \text{ as } \epsilon \to 0. \quad (5.14)
$$

Now, (5.13) follows from (5.14) since $M_\epsilon \nabla u \to \nabla u$ in $L^{p_2}(\Omega; \mathbb{R}^n)$, so

$$
\int_{\Omega} (A_\epsilon(x, p_\epsilon(x, M_\epsilon \nabla u)), p_\epsilon(x, M_\epsilon \nabla u)) \, dx = \int_{\Omega} (b(M_\epsilon \nabla u(x)), M_\epsilon \nabla u(x)) \, dx
\to \int_{\Omega} (b(\nabla u(x)), \nabla u(x)) \, dx, \text{ as } \epsilon \to 0.
$$

\qed
Step 2

We now show that
\[ \int_\Omega (A_\epsilon (x, p_\epsilon (x, M_\epsilon \nabla u)), \nabla u_\epsilon) \, dx \rightarrow \int_\Omega (b(\nabla u), \nabla u) \, dx, \text{ as } \epsilon \rightarrow 0. \] (5.15)

**Proof.** Let \( \delta > 0. \) From Theorem 4.1 we have that \( \nabla u \in L^{p_2}(\Omega; \mathbb{R}^n) \) and there exists a simple function \( \Psi \) satisfying the assumptions of Lemma 5.5 such that
\[ \| \nabla u - \Psi \|_{L^{p_2}(\Omega; \mathbb{R}^n)} \leq \delta. \] (5.16)

Let us write
\[ \int_\Omega (A_\epsilon (x, p_\epsilon (x, M_\epsilon \nabla u)), \nabla u_\epsilon (x)) \, dx = \int_\Omega (A_\epsilon (x, p_\epsilon (x, \Psi)), \nabla u_\epsilon (x)) \, dx + \int_\Omega (A_\epsilon (x, p_\epsilon (x, M_\epsilon \nabla u)) - A_\epsilon (x, p_\epsilon (x, \Psi)), \nabla u_\epsilon (x)) \, dx. \]

We first show that
\[ \int_\Omega (A_\epsilon (x, p_\epsilon (x, \Psi)), \nabla u_\epsilon (x)) \, dx \rightarrow \int_\Omega (b(\Psi), \nabla u) \, dx, \text{ as } \epsilon \rightarrow 0. \]

We have
\[ \int_\Omega (A_\epsilon (x, p_\epsilon (x, \Psi)), \nabla u_\epsilon (x)) \, dx = \sum_{j=0}^{m} \int_{\Omega_j} (A_\epsilon (x, p_\epsilon (x, \eta_j)), \nabla u_\epsilon (x)) \, dx. \]

Now from (3.24), we have that \( A_\epsilon (\cdot, p_\epsilon (\cdot, \eta_j)) \rightharpoonup b(\eta_j) \in L^{q_2}(\Omega_j; \mathbb{R}^n), \) and by (3.17) we have that \( \int_{\Omega_j} (A_\epsilon (x, p_\epsilon (x, \eta_j)), \nabla \varphi) \, dx = 0, \) for \( \varphi \in W_0^{1, p_1}(\Omega_j). \)

Take \( \varphi = \delta u_\epsilon, \) with \( \delta \in C_0^\infty(\Omega_j) \) to get
\[ 0 = \int_{\Omega_j} (A_\epsilon (x, p_\epsilon (x, \eta_j)), (\nabla \delta) u_\epsilon) \, dx + \int_{\Omega_j} (A_\epsilon (x, p_\epsilon (x, \eta_j)), (\nabla u_\epsilon) \delta) \, dx. \]

Taking the limit as \( \epsilon \rightarrow 0, \) and using the fact that \( u^\epsilon \rightharpoonup u \) in \( W_0^{1, p_1}(\Omega) \) and (3.24), we have by Lemma 5.8 that
\[ \int_{\Omega_j} g_j \, dx = \lim_{\epsilon \rightarrow 0} \int_{\Omega_j} (A_\epsilon (x, p_\epsilon (x, \eta_j)), (\nabla u_\epsilon) \delta) \, dx \]
\[ = - \lim_{\epsilon \rightarrow 0} \int_{\Omega_j} (A_\epsilon (x, p_\epsilon (x, \eta_j)), (\nabla \delta) u_\epsilon) \, dx \]
\[ = - \int_{\Omega_j} (b(\eta_j), (\nabla \delta) u) \, dx \]
\[ = \int_{\Omega_j} (b(\eta_j), (\nabla u) \delta) \, dx. \]
Then \((A_\epsilon (\cdot, p_\epsilon (\cdot, \eta_j)), \nabla u_\epsilon (\cdot)) \to (b(\eta_j), \nabla u)\) in \(D'(\Omega_j)\), as \(\epsilon \to 0\).

Therefore, we may conclude that \(g_j = (b(\eta_j), \nabla u)\), so

\[
\sum_{j=0}^n \int_{\Omega_j} (A_\epsilon (x, p_\epsilon (x, \eta_j)), \nabla u_\epsilon) \, dx \to \sum_{j=0}^n \int_{\Omega_j} (b(\eta_j), \nabla u) \, dx, \text{ as } \epsilon \to 0.
\]

Thus, we get

\[
\int_{\Omega} (A_\epsilon (x, p_\epsilon (x, \Psi(x))), \nabla u_\epsilon(x)) \, dx \to \int_{\Omega} (b(\Psi(x)), \nabla u(x)) \, dx, \text{ as } \epsilon \to 0.
\]

On the other hand, let us estimate

\[
\int_{\Omega} (A_\epsilon (x, p_\epsilon (x, M_\epsilon \nabla u(x)))) - A_\epsilon (x, p_\epsilon (x, \Psi(x))) \, \nabla u_\epsilon(x) \, dx.
\]

By (3.4), we have

\[
\left| \int_{\Omega} (A_\epsilon (x, p_\epsilon (x, M_\epsilon \nabla u)) - A_\epsilon (x, p_\epsilon (x, \Psi))) \, \nabla u_\epsilon \, dx \right|
\]

\[
\leq \int_{\Omega} |A_\epsilon (x, p_\epsilon (x, M_\epsilon \nabla u)) - A_\epsilon (x, p_\epsilon (x, \Psi))| \, |\nabla u_\epsilon| \, dx
\]

\[
\leq C \int_{\Omega} \chi_1^\epsilon (x) |p_\epsilon (x, M_\epsilon \nabla u) - p_\epsilon (x, \Psi)| (1 + |p_\epsilon (x, M_\epsilon Du)| + |p_\epsilon (x, \Psi)|^{p_1-2} |\nabla u_\epsilon| \, dx
\]

\[
+ C \int_{\Omega} \chi^\epsilon_2 (x) |p_\epsilon (x, M_\epsilon \nabla u) - p_\epsilon (x, \Psi)| (1 + |p_\epsilon (x, M_\epsilon Du)| + |p_\epsilon (x, \Psi)|^{p_2-2} |\nabla u_\epsilon| \, dx
\]

Applying Hölder’s inequality we obtain

\[
\leq C \left( \int_{\Omega} \chi_1^\epsilon (x) |p_\epsilon (x, M_\epsilon \nabla u) - p_\epsilon (x, \Psi)|^{p_1} \, dx \right)^{\frac{1}{p_1}} \left( \int_{\Omega} \chi_1^\epsilon (x) |\nabla u_\epsilon|^{p_1} \, dx \right)^{\frac{1}{p_1}}
\]

\[
\times \left( \int_{\Omega} \chi^\epsilon_1 (x) (1 + |p_\epsilon (x, M_\epsilon \nabla u)| + |p_\epsilon (x, \Psi)|)^{p_1-2} \, dx \right)^{\frac{1}{p_1}}
\]

\[
+ C \left( \int_{\Omega} \chi_2^\epsilon (x) |p_\epsilon (x, M_\epsilon \nabla u) - p_\epsilon (x, \Psi)|^{p_2} \, dx \right)^{\frac{1}{p_2}} \left( \int_{\Omega} \chi_2^\epsilon (x) |\nabla u_\epsilon|^{p_2} \, dx \right)^{\frac{1}{p_2}}
\]

\[
\times \left( \int_{\Omega} \chi^\epsilon_2 (x) (1 + |p_\epsilon (x, M_\epsilon \nabla u)| + |p_\epsilon (x, \Psi)|)^{p_2-2} \, dx \right)^{\frac{1}{p_2}}
\]

\[
\leq C \left( \int_{\Omega} \chi_1^\epsilon (x) |p_\epsilon (x, M_\epsilon \nabla u) - p_\epsilon (x, \Psi)|^{p_1} \, dx \right)^{\frac{1}{p_1}} \left( \int_{\Omega} \chi_1^\epsilon (x) |\nabla u_\epsilon|^{p_1} \, dx \right)^{\frac{1}{p_1}}
\]

\[
\times \left( \int_{\Omega} \chi^\epsilon_1 (x) (1 + |p_\epsilon (x, M_\epsilon \nabla u)|^{p_1} + |p_\epsilon (x, \Psi)|^{p_1}) \, dx \right)^{\frac{1}{p_1}}
\]

\[
+ C \left( \int_{\Omega} \chi_2^\epsilon (x) |p_\epsilon (x, M_\epsilon \nabla u) - p_\epsilon (x, \Psi)|^{p_2} \, dx \right)^{\frac{1}{p_2}} \left( \int_{\Omega} \chi_2^\epsilon (x) |\nabla u_\epsilon|^{p_2} \, dx \right)^{\frac{1}{p_2}}
\]

\[
\times \left( \int_{\Omega} \chi^\epsilon_2 (x) (1 + |p_\epsilon (x, M_\epsilon \nabla u)|^{p_2} + |p_\epsilon (x, \Psi)|^{p_2}) \, dx \right)^{\frac{1}{p_2}}
\]
By (3.12), we get

\[
\leq C \left( \int_{\Omega} \chi_1^\varepsilon(x) |p_\varepsilon(x, M_\varepsilon \nabla u) - p_\varepsilon(x, \Psi)|^{p_1} \, dx \right)^{\frac{1}{p_1}} \\
\times \left( 1 + \int_{\Omega} \chi_1^\varepsilon(x) |p_\varepsilon(x, M_\varepsilon \nabla u)|^{p_1} \, dx + \int_{\Omega} \chi_1^\varepsilon(x) |p_\varepsilon(x, \Psi)|^{p_1} \, dx \right)^{\frac{p_1 - 2}{p_1}} \\
+ C \left( \int_{\Omega} \chi_2^\varepsilon(x) |p_\varepsilon(x, M_\varepsilon \nabla u) - p_\varepsilon(x, \Psi)|^{p_2} \, dx \right)^{\frac{1}{p_2}} \\
\times \left( 1 + \int_{\Omega} \chi_2^\varepsilon(x) |p_\varepsilon(x, M_\varepsilon \nabla u)|^{p_2} \, dx + \int_{\Omega} \chi_2^\varepsilon(x) |p_\varepsilon(x, \Psi)|^{p_2} \, dx \right)^{\frac{p_2 - 2}{p_2}}
\]

Using (5.6) and Lemma 5.3, we get

\[
\leq C \left[ \left( \int_{\Omega} \chi_1^\varepsilon(x) |p_\varepsilon(x, M_\varepsilon \nabla u) - p_\varepsilon(x, \Psi)|^{p_1} \, dx \right)^{\frac{1}{p_1}} \\
+ \left( \int_{\Omega} \chi_2^\varepsilon(x) |p_\varepsilon(x, M_\varepsilon \nabla u) - p_\varepsilon(x, \Psi)|^{p_2} \, dx \right)^{\frac{1}{p_2}} \right]
\]

Applying Lemma 5.5 and (5.16), we discover that

\[
\limsup_{\varepsilon \to 0} \left| \int_{\Omega} \left( A_\varepsilon(x, p_\varepsilon(x, M_\varepsilon \nabla u)) - A_\varepsilon(x, p_\varepsilon(x, \Psi)) \right) \, \nabla u_\varepsilon \, dx \right|
\leq \limsup_{\varepsilon \to 0} C \left[ \left( \int_{\Omega} \chi_1^\varepsilon(x) |p_\varepsilon(x, M_\varepsilon \nabla u) - p_\varepsilon(x, \Psi)|^{p_1} \, dx \right)^{\frac{1}{p_1}} \\
+ \left( \int_{\Omega} \chi_2^\varepsilon(x) |p_\varepsilon(x, M_\varepsilon \nabla u) - p_\varepsilon(x, \Psi)|^{p_2} \, dx \right)^{\frac{1}{p_2}} \right]
\leq \limsup_{\varepsilon \to 0} C \left\{ \left[ \left( \int_{\Omega} \chi_1^\varepsilon(x) |\nabla u - \Psi |^{p_1} \, dx \right)^{\frac{1}{p_1}} + \left( \int_{\Omega} \chi_2^\varepsilon(x) |\nabla u - \Psi |^{p_2} \, dx \right)^{\frac{1}{p_2}} \right]^{\frac{1}{p_1}} \\
+ \left[ \left( \int_{\Omega} \chi_1^\varepsilon(x) |\nabla u - \Psi |^{p_1} \, dx \right)^{\frac{1}{p_1}} + \left( \int_{\Omega} \chi_2^\varepsilon(x) |\nabla u - \Psi |^{p_2} \, dx \right)^{\frac{1}{p_2}} \right]^{\frac{1}{p_2}} \right\} 
\leq C \left[ \left( \delta_1^{q_1} + \delta_2^{q_2} \right)^{\frac{1}{p_1}} + \left( \delta_1^{q_1} + \delta_2^{q_2} \right)^{\frac{1}{p_2}} \right],
\]

(5.17)

where \( C \) does not depend on \( \delta \).

Since \( \delta \) is arbitrary we conclude that the limit on the left hand side of (5.17) is equal to 0.
Finally, using (3.15), Theorem 4.1, and Hölder’s inequality, we obtain

\[
\int_\Omega (b(\nabla u) - b(\Psi), \nabla u) \leq \int_\Omega |b(\nabla u) - b(\Psi)| |\nabla u| \, dx
\]

\[
\leq C \left( \int_\Omega |b(\nabla u) - b(\Psi)|^{q_1} \, dx \right)^{\frac{1}{q_1}} \left( \int_\Omega |\nabla u|^{p_2} \, dx \right)^{\frac{1}{p_2}}
\]

\[
\leq C \left( \int_\Omega |b(\nabla u) - b(\Psi)|^{q_1} \, dx \right)^{\frac{1}{q_1}}
\]

\[
\leq C \left\{ \int_\Omega \left[ |\nabla u|^{p_2} \left( 1 + |\nabla u|^{p_1} + |\nabla u|^{p_2} + |\Psi|^{p_1} + |\Psi|^{p_2} \right)^{\frac{p_2-2}{p_2-1}} \right] \, dx \right\}^{\frac{1}{q_1}}
\]

\[
\leq C \left\{ \int_\Omega \left[ |\nabla u - \Psi|^{\frac{p_2}{p_2-1}} \left( 1 + |\nabla u|^{p_1} + |\nabla u|^{p_2} + |\Psi|^{p_1} + |\Psi|^{p_2} \right)^{\frac{p_2-2}{p_2-1}} \right] \, dx \right\}^{\frac{1}{q_1}}
\]

\[
\leq C \left\{ \int_\Omega \left[ |\nabla u - \Psi|^{\frac{p_2}{p_2-1}} \left( 1 + |\nabla u|^{p_1} + |\nabla u|^{p_2} + |\Psi|^{p_1} + |\Psi|^{p_2} \right)^{\frac{p_2(p_2-2)}{p_2-1}} \right] \, dx \right\}^{\frac{1}{q_1}}
\]

Applying Hölder’s inequality and Theorem 4.1 again, we obtain

\[
\leq C \left[ \left( \int_\Omega |\nabla u(x) - \Psi(x)|^{p_2} \, dx \right)^{\frac{1}{p_2(p_2-1)}} \right. \times \left( \int_\Omega \left( 1 + |\nabla u|^{p_1} + |\nabla u|^{p_2} + |\Psi|^{p_1} + |\Psi|^{p_2} \right)^{\frac{p_2(p_2-1)}{(p_2-1)(p_2-1)-1}} \, dx \right) \]

\[
+ \left( \int_\Omega |\nabla u(x) - \Psi(x)|^{p_2} \, dx \right)^{\frac{1}{(p_2-1)^2}} \times \left( \int_\Omega \left( 1 + |\nabla u|^{p_1} + |\nabla u|^{p_2} + |\Psi|^{p_1} + |\Psi|^{p_2} \right) \, dx \right) \]

\[
\left. \times \left( \int_\Omega \left( 1 + |\nabla u|^{p_1} + |\nabla u|^{p_2} + |\Psi|^{p_1} + |\Psi|^{p_2} \right)^{\frac{(p_2-2)^2-1}{(p_2-1)^2}} \, dx \right) \right]^{\frac{1}{q_1}}
\]

\[
\leq C \left[ \left( \int_\Omega |\nabla u(x) - \Psi(x)|^{p_2} \, dx \right)^{\frac{1}{(p_2-1)(p_2-1)}} \right. + \left( \int_\Omega |\nabla u(x) - \Psi(x)|^{p_2} \, dx \right)^{\frac{1}{(p_2-1)^2}} \right]^{\frac{1}{q_1}}
\]

\[
\leq C \left[ \delta_{p_2-1}^{\frac{1}{p_1}} + \delta_{p_2-1}^{\frac{q_1}{p_2-1}} \right]^{\frac{1}{q_1}},
\]

where \( C \) does not depend on \( \delta \).

Now, since \( \delta \) is arbitrarily small we conclude the proof of Step 2. \( \square \)

**Step 3**

We will show that

\[
\int_\Omega (A_\epsilon(x, \nabla u_\epsilon), p_\epsilon(x, M_\epsilon \nabla u)) \, dx \rightarrow \int_\Omega (b(\nabla u), \nabla u) \, dx, \ \text{as} \ \epsilon \rightarrow 0.
\]

(5.18)
Proof. Let $\delta > 0$. As in the proof of Step 2, assume $\Psi$ is a simple function satisfying assumptions of Lemma 5.5 and such that $\|\nabla u - \Psi\|_{L^p(\Omega; \mathbb{R}^n)} < \delta$.

Let us write
\[
\int_\Omega (A_\epsilon (x, \nabla u_\epsilon) , p_\epsilon (x, M_\epsilon \nabla u)) \, dx
\]
\[= \int_\Omega (A_\epsilon (x, \nabla u_\epsilon), p_\epsilon (x, \Psi)) \, dx + \int_\Omega (A_\epsilon (x, \nabla u_\epsilon), p_\epsilon (x, M_\epsilon \nabla u) - p_\epsilon (x, \Psi)) \, dx.
\]

We first show that
\[
\int_\Omega (A_\epsilon (x, \nabla u_\epsilon(x)), p_\epsilon (x, \Psi(x))) \, dx \to \int_\Omega (b(\nabla u(x)), \Psi(x)) \, dx.
\]

We start by writing
\[
\int_\Omega (A_\epsilon (x, \nabla u_\epsilon(x)), p_\epsilon (x, \Psi(x))) \, dx = \sum_{j=0}^m \int_{\Omega_j} (A_\epsilon (x, \nabla u_\epsilon(x)), p_\epsilon (x, \eta_j)) \, dx.
\]

From Lemma 5.8, up to a subsequence, $(A_\epsilon (\cdot, \nabla u_\epsilon(\cdot)), p_\epsilon (\cdot, \eta_j))$ converges weakly to a function $h_j \in L^1(\Omega_j; \mathbb{R})$, as $\epsilon \to 0$.

By Theorem 3.4, we have $A_\epsilon (\cdot, \nabla u_\epsilon) \rightharpoonup b(\nabla u) \in L^{q^2} (\Omega; \mathbb{R}^n)$ and
\[-\text{div} (A_\epsilon (x, \nabla u_\epsilon)) = f = -\text{div} (b(\nabla u)).
\]

Also, from (3.22), $p_\epsilon$ satisfies $p_\epsilon (\cdot, \eta_j) \to \eta_j$ in $L^{p^2} (\Omega_j, \mathbb{R}^n)$.

Arguing as in Step 2, we find that $(A_\epsilon (x, \nabla u_\epsilon(x)), p_\epsilon (x, \eta_j)) \rightharpoonup (b(\nabla u(x)), \eta_j)$ in $D'(\Omega_j)$, as $\epsilon \to 0$.

Therefore, we may conclude that $h_j = (b(\nabla u), \eta_j)$, and hence,
\[
\sum_{j=0}^n \int_{\Omega_j} (A_\epsilon (x, \nabla u_\epsilon(x)), p_\epsilon (x, \eta_j)) \, dx \to \sum_{j=0}^n \int_{\Omega_j} (b(\nabla u(x)), \eta_j) \, dx, \text{ as } \epsilon \to 0.
\]

Thus, we get
\[
\int_\Omega (A_\epsilon (x, \nabla u_\epsilon(x)), p_\epsilon (x, \Psi(x))) \, dx \to \int_\Omega (b(\nabla u(x)), \Psi(x)) \, dx, \text{ as } \epsilon \to 0.
\]

Moreover, applying Cauchy-Schwarz inequality we have
\[
\left| \int_\Omega (A_\epsilon (x, \nabla u_\epsilon), p_\epsilon (x, M_\epsilon \nabla u) - p_\epsilon (x, \Psi)) \, dx \right|
\]
\[\leq \int_\Omega |A_\epsilon (x, \nabla u_\epsilon)||p_\epsilon (x, M_\epsilon \nabla u(x)) - p_\epsilon (x, \Psi(x))| \, dx.
\]
Hölder’s inequality delivers

\[
\leq \int |A_\epsilon (x, \nabla u_\epsilon (x))| |p_\epsilon (x, M_\epsilon \nabla u (x)) - p_\epsilon (x, \Psi (x))| \, dx \\
+ \int |A_\epsilon (x, \nabla u_\epsilon (x))| |p_\epsilon (x, M_\epsilon \nabla u (x)) - p_\epsilon (x, \Psi (x))| \, dx \\
\leq \left( \int |A_\epsilon (x, \nabla u_\epsilon (x))|^{q_2} \right)^{\frac{1}{q_2}} \left( \int |p_\epsilon (x, M_\epsilon \nabla u) - p_\epsilon (x, \Psi)|^{p_1} \, dx \right)^{\frac{1}{p_1}} \\
+ \left( \int |A_\epsilon (x, \nabla u_\epsilon (x))|^{q_1} \right)^{\frac{1}{q_1}} \left( \int |p_\epsilon (x, M_\epsilon \nabla u) - p_\epsilon (x, \Psi)|^{p_2} \, dx \right)^{\frac{1}{p_2}} \right]
\]

By (3.4) and (3.12) we get

\[
\leq C \left[ \left( \int |A_\epsilon (x, \nabla u_\epsilon (x))|^{q_2 (p_1 - 1)} \right)^{\frac{1}{q_2}} \left( \int |p_\epsilon (x, M_\epsilon \nabla u) - p_\epsilon (x, \Psi)|^{p_1} \, dx \right)^{\frac{1}{p_1}} \\
+ \left( \int |A_\epsilon (x, \nabla u_\epsilon (x))|^{q_1 (p_2 - 1)} \right)^{\frac{1}{q_1}} \left( \int |p_\epsilon (x, M_\epsilon \nabla u) - p_\epsilon (x, \Psi)|^{p_2} \, dx \right)^{\frac{1}{p_2}} \right]
\]

As in the proof of Step 2 we see that

\[
\limsup_{\epsilon \to 0} \left| \int (A_\epsilon (x, \nabla u_\epsilon), p_\epsilon (x, M_\epsilon \nabla u) - p_\epsilon (x, \Psi)) \, dx \right| \\
\leq C \limsup_{\epsilon \to 0} \left[ \left( \int |p_\epsilon (x, M_\epsilon \nabla u) - p_\epsilon (x, \Psi)|^{p_1} \, dx \right)^{\frac{1}{p_1}} \\
+ \left( \int |p_\epsilon (x, M_\epsilon \nabla u) - p_\epsilon (x, \Psi)|^{p_2} \, dx \right)^{\frac{1}{p_2}} \right]
\]

58
By Lemma 5.5, we get

\[ \leq C \left[ (\delta^{q_2} + \delta^{q_1})^{\frac{1}{p_1}} + (\delta^{q_2} + \delta^{q_1})^{\frac{1}{p_2}} \right], \]

where \( C \) does not depend on \( \delta \).

Hence, proceeding as in Step 2, we find that

\[
\limsup_{\epsilon \to 0} \left| \int_{\Omega} (A_{\epsilon}(x, \nabla u_{\epsilon}(x)), p_{\epsilon}(x, M_{\epsilon} \nabla u(x))) \, dx - \int_{\Omega} (b(\nabla u(x)), \nabla u(x)) \, dx \right|
\]

\[
= \limsup_{\epsilon \to 0} \left| \int_{\Omega} (A_{\epsilon}(x, \nabla u_{\epsilon}(x)), p_{\epsilon}(x, M_{\epsilon} \nabla u)) \, dx - \int_{\Omega} (A_{\epsilon}(x, \nabla u_{\epsilon}(x)), p_{\epsilon}(x, \Psi)) \, dx \right|
\]

\[
+ \int_{\Omega} (A_{\epsilon}(x, \nabla u_{\epsilon}(x)), p_{\epsilon}(x, \Psi)) \, dx - \int_{\Omega} (b(\nabla u), \Psi) \, dx
\]

\[
+ \int_{\Omega} (b(\nabla u), \Psi) \, dx - \int_{\Omega} (b(\nabla u), \nabla u) \, dx \right|
\]

\[
\leq \limsup_{\epsilon \to 0} \left| \int_{\Omega} (A_{\epsilon}(x, \nabla u_{\epsilon}(x)), p_{\epsilon}(x, M_{\epsilon} \nabla u)) \, dx - \int_{\Omega} (A_{\epsilon}(x, \nabla u_{\epsilon}(x)), p_{\epsilon}(x, \Psi)) \, dx \right|
\]

\[
+ \limsup_{\epsilon \to 0} \left| \int_{\Omega} (A_{\epsilon}(x, \nabla u_{\epsilon}(x)), p_{\epsilon}(x, \Psi)) \, dx - \int_{\Omega} (b(\nabla u), \Psi) \, dx \right|
\]

\[
+ \limsup_{\epsilon \to 0} \left| \int_{\Omega} (b(\nabla u), \Psi) \, dx - \int_{\Omega} (b(\nabla u), \nabla u) \, dx \right|
\]

\[
\leq C \left( (\delta^{q_2} + \delta^{q_1})^{\frac{1}{p_1}} + (\delta^{q_2} + \delta^{q_1})^{\frac{1}{p_2}} + 0 + \delta \|b(\nabla u)\|_{L^p(\Omega, \mathbb{R}^n)} \right),
\]

where \( C \) does not depend on \( \delta \). Now since \( \delta \) is arbitrarily small, the proof of Step 3 is complete. \( \square \)

**Step 4**

Finally, let us prove that

\[
\int_{\Omega} (A_{\epsilon}(x, \nabla u_{\epsilon}(x)), \nabla u_{\epsilon}(x)) \, dx \to \int_{\Omega} (b(\nabla u(x)), \nabla u(x)) \, dx, \text{ as } \epsilon \to 0. \quad (5.19)
\]

**Proof.** Since

\[
\int_{\Omega} (A_{\epsilon}(x, \nabla u_{\epsilon}(x)), \nabla u_{\epsilon}(x)) \, dx = \langle -\text{div} \, (A_{\epsilon}(x, \nabla u_{\epsilon}(x)), u_{\epsilon}) \rangle = \langle f, u_{\epsilon} \rangle, \quad (5.20)
\]

and

\[
\int_{\Omega} (b(\nabla u), \nabla u) \, dx = \langle -\text{div} \, (b(\nabla u)), u \rangle = \langle f, u \rangle, \quad (5.21)
\]

and \( u_{\epsilon} \to u \) in \( W^{1,p_1}(\Omega) \), the result follows immediately. \( \square \)

Finally, Theorem 5.2 follows from (5.13), (5.15), (5.18) and (5.19). \( \square \)
Chapter 6

Lower Bounds on Field Concentrations

In composites, failure initiation is a multiscale phenomenon. A load applied at the structural scale is often amplified by the microstructure creating local zones of high field concentration. The regions containing high fields are often the first to suffer damage during service. Therefore it is of relevance to assess the load transfer between macroscopic and microscopic length scales.

In this chapter, we bound the local singularity strength inside microstructured media in terms of the macroscopic applied fields.

The strong approximations in Chapter 5 are used to develop new tools that provide lower bounds on the local gradient field intensity inside microstructured media. These results provide a lower bound on the amplification of the macroscopic (average) gradient field by the microstructure. In [Lip06], similar lower bounds are established for field concentrations for mixtures of linear electrical conductors in the context of two scale convergence.

6.1 Statement of the Lower Bound on the Amplification of the Macroscopic Field by the Microstructure

We begin by presenting a general lower bound that holds for the composition of the sequence \{\chi_i^\epsilon \nabla u_\epsilon\}_{\epsilon>0} with any non-negative Carathéodory function.

Definition 6.1. A function \(\psi : \Omega \times \mathbb{R}^n \to \mathbb{R}\) is a Carathéodory function if \(\psi(x, \cdot)\) is continuous for almost every \(x \in \Omega\) and if \(\psi(\cdot, \lambda)\) is measurable in \(x\) for every \(\lambda \in \mathbb{R}^n\).

The lower bound on the sequence obtained by the composition of \(\psi(x, \cdot)\) with \(\chi_i^\epsilon(x) \nabla u_\epsilon(x)\) is given by

\[
\int_D \int_Y \psi(x, \chi_i^\epsilon(y)p(y, \nabla u(x))) \, dy \, dx \leq \liminf_{\epsilon \to 0} \int_D \psi(x, \chi_i^\epsilon(x) \nabla u_\epsilon(x)) \, dx, \quad (i = 1, 2).
\]
If the sequence \( \{ \psi(x, \chi^i_\varepsilon(x) \nabla u_\varepsilon(x)) \}_{\varepsilon > 0} \) is weakly convergent in \( L^1(\Omega) \), then the inequality becomes an equality.

**Remark 6.3.** As a direct consequence of Theorem 6.2, taking \( \psi(x, \lambda) = |\lambda|^q \) with \( q \geq 2 \), we display lower bounds on the \( L^q \) norm of the gradient fields inside each material that are given in terms of the correctors presented in Theorem 5.2

\[
\int_{D} \int_{Y} \chi^i(y) |p(y, \nabla u(x))|^q \, dy \, dx \leq \liminf_{\varepsilon \to 0} \int_{D} \chi^i_\varepsilon(x) |\nabla u_\varepsilon(x)|^q \, dx,
\]

for \( i = 1, 2 \).

This result is still valid for \( q = \infty \) if we have that \( \chi^i_\varepsilon(x) \nabla u_\varepsilon(x) \) belongs to \( L^\infty(\Omega; \mathbb{R}^n) \) (since \( \chi^i_\varepsilon(x) \nabla u_\varepsilon(x) \) belongs to \( L^p(\Omega; \mathbb{R}^n) \) by (3.12)) and if \( \chi^i(y)p(y, \nabla u(x)) \) belongs to \( L^\infty(Y \times \Omega; \mathbb{R}^n) \) (since \( \int_{D} \int_{Y} \chi^i_\varepsilon(x) \nabla u_\varepsilon(x) \, dx \leq C \int_{D} (1 + |\nabla u(x)|^{p_1} + |\nabla u(x)|^{p_2}) \, dx < \infty \), by Lemma 5.3 and Theorem 4.1).

It is clear from (6.1) that the \( L^q(Y \times \Omega; \mathbb{R}^n) \) integrability of \( p(y, \nabla u(x)) \) provides a lower bound on the \( L^q(\Omega; \mathbb{R}^n) \) integrability of \( \nabla u_\varepsilon \).

Theorem 6.2 together with (6.1) provide explicit lower bounds on the gradient field inside each material. It relates the local excursions of the gradient inside each phase \( \chi^i_\varepsilon \nabla u_\varepsilon \) to the average gradient \( \nabla u \) through the multiscale quantity given by the corrector \( p(y, \nabla u(x)) \).

### 6.2 Young Measures

We use the results from Theorem 5.2 and Young Measures to study the behavior of gradients of solutions of the Dirichlet problem (3.7). These tools allow us to bound nonlinear quantities of these gradients from below in terms of the local solution \( p \) and the gradient of the homogenized solution \( u \) as stated in Section 6.1.

Young Measures can be used as a tool to organize our ideas about oscillatory behavior and to deal in a consistent way with oscillations [Ped99]. Young Measures are a family of probability measures \( \nu = \{ \nu_x \}_{x \in \Omega} \) associated with a sequence of functions \( f^\varepsilon : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that the \( \text{supp}(\nu_x) \subset \mathbb{R}^n \) and they depend measurably on \( x \in \Omega \), which means that for any continuous function \( \varphi : \mathbb{R}^n \rightarrow \mathbb{R} \), the function

\[
\overline{\varphi}(x) = \int_{\mathbb{R}^n} \varphi(\lambda) d\nu_x(\lambda)
\]

is measurable. The fundamental property of this family of probability measures is that whenever \( \{ \varphi(f^\varepsilon) \}_{\varepsilon > 0} \) converges weakly* in \( L^\infty(\Omega) \) (or more generally weak in some \( L^p(\Omega) \)) the weak limit can be identified with the function \( \overline{\varphi} \) in (6.2):

\[
\lim_{\varepsilon \to 0} \int_{\Omega} \varphi(f^\varepsilon) g(x) \, dx = \int_{\Omega} g(x) \int_{\mathbb{R}^n} \varphi(\lambda) d\nu_x(\lambda) \, dx,
\]

for all \( g \in L^1(\Omega) \).
6.3 Proof of Lower Bound on the Amplification of the Macroscopic Field by the Microstructure

The sequence \( \{ \chi_i^\epsilon(x) \nabla u_i(x) \} \) has a Young Measure \( \nu_i = \{ \nu_i^\epsilon \}_{x \in \Omega} \) associated to it (see Theorem 6.2 and the discussion following in [Ped97]), for \( i = 1, 2 \).

As a consequence of Theorem 5.2 proved in Chapter 5, we have

\[
\left\| \chi_i^\epsilon(x) p \left( \frac{x}{\epsilon}, M_i(\nabla u)(x) \right) - \chi_i^\epsilon(x) \nabla u_i(x) \right\|_{L^p(\Omega; \mathbb{R}^n)} \rightarrow 0,
\]

as \( \epsilon \rightarrow 0 \) for \( i = 1, 2 \), which implies that the sequences

\[
\left\{ \chi_i^\epsilon(x) p \left( \frac{x}{\epsilon}, M_i(\nabla u)(x) \right) \right\}_{\epsilon > 0} \quad \text{and} \quad \left\{ \chi_i^\epsilon(x) \nabla u_i(x) \right\}_{\epsilon > 0}
\]

share the same Young Measure \( \nu_i = \{ \nu_i^\epsilon \}_{x \in \Omega} \) (see Lemma 6.3 of [Ped97]).

The next Lemma identifies the Young measure \( \nu_i \).

**Lemma 6.4.** For all \( \phi \in C_0(\mathbb{R}^n) \) and for all \( \zeta \in C_0^\infty(\mathbb{R}^n) \), we have

\[
\int_{\Omega} \zeta(x) \int_{\mathbb{R}^n} \phi(\lambda) d\nu_i^\epsilon(\lambda) dx = \int_{\Omega} \zeta(x) \int_{Y} \phi(\chi_i(y)p(y, \nabla u(x))) dy dx. \quad (6.4)
\]

(From discussion in Chapter 6 of [Ped97], this is enough to identify the Young measure \( \nu_i \))

**Proof.** To prove (6.4), we will show that given \( \phi \in C_0(\mathbb{R}^n) \) and \( \zeta \in C_0^\infty(\mathbb{R}^n) \)

\[
\lim_{\epsilon \rightarrow 0} \int_{\Omega} \zeta(x) \phi \left( \chi_i^\epsilon(x) p \left( \frac{x}{\epsilon}, M_i(\nabla u)(x) \right) \right) dx = \int_{\Omega} \zeta(x) \int_{Y} \phi(\chi_i(y)p(y, \nabla u(x))) dy dx. \quad (6.5)
\]

We consider the difference

\[
\left| \int_{\Omega} \zeta(x) \phi \left( \chi_i^\epsilon(x) p \left( \frac{x}{\epsilon}, M_i(\nabla u)(x) \right) \right) dx - \int_{\Omega} \zeta(x) \int_{Y} \phi(\chi_i(y)p(y, \nabla u(x))) dy dx \right|
\]

By (5.1), we have

\[
\leq \left| \sum_{i \in I} \int_{Y_i} \zeta(x) \phi \left( \chi_i^\epsilon \left( \frac{x}{\epsilon}, \xi_i \right) p \left( \frac{x}{\epsilon}, \xi_i \right) \right) dx - \int_{\Omega} \zeta(x) \int_{Y} \phi(\chi_i(y)p(y, \nabla u(x))) dy dx \right|
\]

\[
+ \left| \int_{\Omega \setminus \Omega_c} \zeta(x) \phi \left( \chi_i^\epsilon \left( \frac{x}{\epsilon}, 0 \right) \right) p \left( \frac{x}{\epsilon}, 0 \right) dx - \int_{\Omega \setminus \Omega_c} \zeta(x) \int_{Y} \phi(\chi_i(y)p(y, \nabla u(x))) dy dx \right|
\]

\[
\leq \sum_{i \in I} \int_{Y_i} \zeta(x) \phi \left( \chi_i^\epsilon \left( \frac{x}{\epsilon}, \xi_i \right) p \left( \frac{x}{\epsilon}, \xi_i \right) \right) dx - \int_{\Omega_c} \zeta(x) \int_{Y} \phi(\chi_i(y)p(y, \nabla u(x))) dy dx \]

\[
+ C |\Omega \setminus \Omega_c|. \quad (6.6)
\]
Note that the term $C |\Omega \setminus \Omega_i|$ goes to 0, as $\epsilon \to 0$. Now set $x_\epsilon^i$ to be the center of $Y_\epsilon^i$. On the first integral use the change of variables $x = x_\epsilon^i + \epsilon y$, where $y$ belongs to $Y$, and since $dx = e^n dy$, we get

$$\left| \sum_{i \in I_\epsilon} \int_{Y_\epsilon^i} \zeta(x) \phi \left( \chi_i \left( \frac{x}{\epsilon} \right) p \left( \frac{x}{\epsilon}, \xi^i \right) \right) dx - \sum_{i \in I_\epsilon} \int_{Y_\epsilon^i} \zeta(x) \int_Y \phi \left( \chi_i \left( y \right) p \left( y, \nabla u(x) \right) \right) dy dx \right|$$

$$= \left| \sum_{i \in I_\epsilon} e^n \int_Y \zeta \left( x_\epsilon^i + \epsilon y \right) \phi \left( \chi_i \left( y \right) p \left( y, \xi^i \right) \right) dy \right.$$  

$$- \sum_{i \in I_\epsilon} \int_{Y_\epsilon^i} \zeta \left( x_\epsilon^i \right) \phi \left( \chi_i \left( y \right) p \left( y, \nabla u(x) \right) \right) dy dx \right|$$

$$\leq \left| \sum_{i \in I_\epsilon} e^n \int_Y \zeta \left( x_\epsilon^i \right) \phi \left( \chi_i \left( y \right) p \left( y, \xi^i \right) \right) dy \right.$$  

$$- \sum_{i \in I_\epsilon} \int_{Y_\epsilon^i} \zeta \left( x_\epsilon^i \right) \phi \left( \chi_i \left( y \right) p \left( y, \nabla u(x) \right) \right) dy dx \right| + CO(\epsilon)$$

$$= \left| \sum_{i \in I_\epsilon} \int_Y \zeta \left( x_\epsilon^i \right) \phi \left( \chi_i \left( y \right) p \left( y, \xi^i \right) \right) dy dx \right.$$  

$$- \sum_{i \in I_\epsilon} \int_{Y_\epsilon^i} \zeta \left( x_\epsilon^i \right) \phi \left( \chi_i \left( y \right) p \left( y, \nabla u(x) \right) \right) dy dx \right| + CO(\epsilon)$$

$$= \left| \sum_{i \in I_\epsilon} \int_Y \zeta \left( x_\epsilon^i \right) \left[ \phi \left( \chi_i \left( y \right) p \left( y, \xi^i \right) \right) - \phi \left( \chi_i \left( y \right) p \left( y, \nabla u(x) \right) \right) \right] dy dx \right| + CO(\epsilon)$$
Let us use a Taylor’s expansion for $\zeta$ again

$$
\leq \sum_{i \in L} \int_{Y_i} \int_Y (\zeta(x) + CO(\epsilon)) \left[ \phi (x_i (y) p (y, \xi_i)) - \phi (x_i (y) p (y, \nabla u(x))) \right] dydx + CO(\epsilon)
$$

$$
\leq \sum_{i \in L} \int_{Y_i} \int_Y \zeta(x) \left[ \phi (x_i (y) p (y, \xi_i)) - \phi (x_i (y) p (y, \nabla u(x))) \right] dydx + CO(\epsilon)
$$

$$
= \left| \int_{\Omega_e} \zeta(x) \int_Y \left[ \phi (x_i (y) p (y, M_x \nabla u(x))) - \phi (x_i (y) p (y, \nabla u(x))) \right] dydx + CO(\epsilon)
$$

$$
\leq \left| \int_{\Omega_e} \left| \zeta(x) \int_Y \left[ \phi (x_i (y) p (y, M_x \nabla u(x))) - \phi (x_i (y) p (y, \nabla u(x))) \right] dydx + CO(\epsilon)
$$

Because of the uniform Lipschitz continuity of $\phi$, we get

$$
\leq C \left| \int_{\Omega_e} \left| \zeta(x) \int_Y |p (y, M_x \nabla u(x)) - p (y, \nabla u(x))| dydx + CO(\epsilon)
$$

By Hölder’s inequality we have

$$
\leq C \left| \int_{\Omega_e} \left| \zeta(x) \left[ \left( \int_Y \chi_i(y) |p (y, M_x \nabla u(x)) - p (y, \nabla u(x))|^{p_1} dy \right)^{1/p_1}
$$

$$
+ \left( \int_Y \chi_i(y) |p (y, M_x \nabla u(x)) - p (y, \nabla u(x))|^{p_2} dy \right)^{1/p_2} \right] dx + CO(\epsilon)
$$

Using Lemma 5.4, we obtain

$$
\leq C \left| \int_{\Omega_e} \left| \zeta(x) \left\{ \left[ (1 + |M_x \nabla u(x)|^{p_1} \theta_1 + |M_x \nabla u(x)|^{p_2} \theta_2 + |\nabla u(x)|^{p_1} \theta_1
$$

$$
+ |\nabla u(x)|^{p_2} \theta_2 \right)^{p_1 - 1} |M_x \nabla u(x) - \nabla u(x)|^{p_1 - 1} \theta_1^{1/p_1 - 1}
$$

$$
+ |M_x \nabla u(x) - \nabla u(x)|^{p_2 - 1} \theta_2^{1/p_2 - 1} (1 + |M_x \nabla u(x)|^{p_1} \theta_1 + |M_x \nabla u(x)|^{p_2} \theta_2
$$

$$
+ |\nabla u(x)|^{p_1} \theta_1 + |\nabla u(x)|^{p_2} \theta_2 \right)^{p_2 - 1} \right]^{1/p_1}
$$

$$
+ \left[ (1 + |M_x \nabla u(x)|^{p_1} \theta_1 + |M_x \nabla u(x)|^{p_2} \theta_2 + |\nabla u(x)|^{p_1} \theta_1
$$

$$
+ |\nabla u(x)|^{p_2} \theta_2 \right)^{p_1 - 1} |M_x \nabla u(x) - \nabla u(x)|^{p_1 - 1} \theta_1^{1/p_1 - 1}
$$

$$
+ |M_x \nabla u(x) - \nabla u(x)|^{p_2 - 1} \theta_2^{1/p_2 - 1} (1 + |M_x \nabla u(x)|^{p_1} \theta_1 + |M_x \nabla u(x)|^{p_2} \theta_2
$$

$$
+ |\nabla u(x)|^{p_1} \theta_1 + |\nabla u(x)|^{p_2} \theta_2 \right)^{p_2 - 1} \right]^{1/p_2}\right\} dx + CO(\epsilon)
$$
By Hölder’s inequality we get

\[
\leq C \left\{ \left( \int_{\Omega_\epsilon} |\zeta(x)|^{q_2} \, dx \right)^{1/q_2} \left[ \int_{\Omega_\epsilon} \left( |M_x \nabla u(x) - \nabla u(x)|^{\frac{p_1}{p_2-1}} \right) \, dx \right]^{1/p_2} \right. \\
\times \left( 1 + |M_x \nabla u(x)|^{p_1} + |M_x \nabla u(x)|^{p_2} + |\nabla u(x)|^{p_1} + |\nabla u(x)|^{p_2} \right)^{\frac{p_1-2}{p_2-1}} \\
+ \left| M_x \nabla u(x) - \nabla u(x) \right|^{\frac{p_2}{p_2-1}} \\
\left. \times \left( 1 + |M_x \nabla u(x)|^{p_1} + |M_x \nabla u(x)|^{p_2} + |\nabla u(x)|^{p_1} + |\nabla u(x)|^{p_2} \right)^{\frac{p_1-2}{p_2-1}} \right) \, dx \right\}^{1/p_2} \\
+ CO(\epsilon).
\]

From the fact that \( \zeta \in C_0^\infty(\mathbb{R}^n) \), we have

\[
\leq C \left\{ \left[ \int_{\Omega_\epsilon} |M_x \nabla u(x) - \nabla u(x)|^{\frac{p_1}{p_2-1}} \right. \\
\times \left( 1 + |M_x \nabla u(x)|^{p_1} + |M_x \nabla u(x)|^{p_2} + |\nabla u(x)|^{p_1} + |\nabla u(x)|^{p_2} \right)^{\frac{p_1-2}{p_2-1}} \, dx \\
+ \left[ \int_{\Omega_\epsilon} |M_x \nabla u(x) - \nabla u(x)|^{\frac{p_2}{p_2-1}} \right. \\
\times \left( 1 + |M_x \nabla u(x)|^{p_1} + |M_x \nabla u(x)|^{p_2} + |\nabla u(x)|^{p_1} + |\nabla u(x)|^{p_2} \right)^{\frac{p_1-2}{p_2-1}} \, dx \\
+ \left[ \int_{\Omega_\epsilon} |M_x \nabla u(x) - \nabla u(x)|^{\frac{p_2}{p_2-1}} \right. \\
\left. \times \left( 1 + |M_x \nabla u(x)|^{p_1} + |M_x \nabla u(x)|^{p_2} + |\nabla u(x)|^{p_1} + |\nabla u(x)|^{p_2} \right)^{\frac{p_1-2}{p_2-1}} \, dx \right\}^{1/p_2} \\
+ CO(\epsilon)
\]
Applying Hölder’s Inequality again, we get
\[
\leq C \left\{ \left( \int_{\Omega_\varepsilon} \left( 1 + |M_\varepsilon \nabla u(x)|^{p_1} + |M_\varepsilon \nabla u(x)|^{p_2} + |\nabla u(x)|^{p_1} \right) \right)^{\frac{p_1+2}{p_1-1}} \left( \int_{\Omega_\varepsilon} |M_\varepsilon \nabla u(x) - \nabla u(x)|^{p_1} dx \right) \right. \\
+ \left. \left( \int_{\Omega_\varepsilon} |M_\varepsilon \nabla u(x) - \nabla u(x)|^{p_2} dx \right)^{\frac{1}{p_2-1}} \left( \int_{\Omega_\varepsilon} (1 + |M_\varepsilon \nabla u(x)|^{p_1} + |M_\varepsilon \nabla u(x)|^{p_2} + |\nabla u(x)|^{p_1}) \right)^{\frac{p_2+2}{p_2-1}} \right\}^{1/p_1} \\
+ \left[ \left( \int_{\Omega_\varepsilon} (1 + |M_\varepsilon \nabla u(x)|^{p_1} + |M_\varepsilon \nabla u(x)|^{p_2}) \right)^{\frac{1}{p_1-1}} \left( \int_{\Omega_\varepsilon} (1 + |M_\varepsilon \nabla u(x)|^{p_1} + |M_\varepsilon \nabla u(x)|^{p_2} + |\nabla u(x)|^{p_1}) \right)^{\frac{p_2+2}{p_2-1}} \right] \right\}^{1/p_2} + CO(\varepsilon)
\]

From Jensen’s inequality and Theorem 4.1, we obtain
\[
\leq C \left\{ \left( \int_{\Omega_\varepsilon} |M_\varepsilon \nabla u(x) - \nabla u(x)|^{p_1} dx \right)^{\frac{1}{p_1-1}} \right. \\
+ \left. \left( \int_{\Omega_\varepsilon} |M_\varepsilon \nabla u(x) - \nabla u(x)|^{p_2} dx \right)^{\frac{1}{p_2-1}} \right\}^{1/p_1} \\
+ \left[ \left( \int_{\Omega_\varepsilon} |M_\varepsilon \nabla u(x) - \nabla u(x)|^{p_1} dx \right)^{\frac{1}{p_1-1}} \right] \left( \int_{\Omega_\varepsilon} (1 + |M_\varepsilon \nabla u(x)|^{p_1} + |M_\varepsilon \nabla u(x)|^{p_2} + |\nabla u(x)|^{p_1}) \right)^{\frac{p_2+2}{p_2-1}} \right\}^{1/p_2} + CO(\varepsilon).
\]

Finally, taking \( \varepsilon \to 0 \) together with Property 1 of \( M_\varepsilon \) in Remark 5.1, we obtain (6.5). \( \Box \)
Therefore, from Proposition 4.4 of [Ped99] and Lemma 6.4 we have

\[
\int_\Omega \zeta(x) \int_{\mathbb{R}^n} \phi(\lambda) d\nu^i_\varepsilon(\lambda) dx
\]

\[
= \int_\Omega \zeta(x) \int_Y \phi(\chi_i(y)p(y, \nabla u(x))) dy dx
\]

\[
= \lim_{\varepsilon \to 0} \int_\Omega \zeta(x) \phi \left( \chi^\varepsilon_i(x)p \left( \frac{x}{\varepsilon}, M_\varepsilon(\nabla u)(x) \right) \right) dx
\]

\[
\leq \lim_{\varepsilon \to 0} \int_\Omega \zeta(x) \phi \left( \chi^\varepsilon_i(x)\nabla u_\varepsilon(x) \right) dx,
\]

for all \( \phi \in C_0^0(\mathbb{R}^n) \) and for all \( \zeta \in C_0^\infty(\mathbb{R}^n) \).

The proof of Theorem 6.2 follows from Lemma 6.4 and Theorem 6.11 in [Ped97].
Chapter 7

Nonlinear Neutral Inclusions

A neutral inclusion, when inserted in a matrix containing a uniform applied electric field, does not disturb the field outside the inclusion. The problem of finding neutral inclusions goes back to 1953 when Mansfield found that certain reinforced holes, which he called “neutral holes”, could be cut out of a uniformly stressed plate without disturbing the surrounding stress field in the plate [Man53]. The analogous problem of a “neutral elastic inhomogeneity” in which the introduction of the inhomogeneity into an elastic body (of a different material), does not disturb the original stress field in the uncut body, was first studied by Ru [Ru98].

The well known Hashin coated sphere is an example of a neutral coated inclusion [Has62]. Neutral spherical inclusions have been studied in [TR95],[LV96],[Lip97a], [Lip97b] and [LT99]. For more information on neutral coated inclusions see [Mil02].

In this chapter, we consider the problem of constructing neutral inclusions from nonlinear materials. We study the design of (double coated nonlinear) neutral inclusions that do not disturb the prescribed uniform applied electric field in the surrounding body.

7.1 Double Coated Nonlinear Neutral Inclusions

7.1.1 Statement of the Problem and Result

For a particular three phase coated sphere (see Figure 7.1), we apply the linear field $\vec{E} \cdot \vec{x} = Ex_1$, (where $\vec{E} = Ee^1$, with $e^1 = (1, 0)$ and $\vec{x} = (x_1, x_2)$) as a boundary condition to
the exterior boundary of the outer coating, to find that \( u = Ex_1 \) solves

\[
\begin{cases}
\sigma \frac{\partial^2}{\partial r^2} \frac{x}{r} \Delta_{p_1} u = 0 \text{ (nonlinear) in the core,} \\
\Delta u = 0 \text{ (linear) in the middle coating,} \\
\sigma \Delta_{p_2} u = 0 \text{ (nonlinear) in the outer coating,}
\end{cases}
\tag{7.1}
\]

where \( \sigma > 0 \) and \( \Delta_p \) represents the \( p \)-Laplacian, together with the usual interface conditions (continuity of the electric potential and normal component of the current), for \( |E| = \sigma \frac{1}{r} \).

Our calculations in Section 7.2 show that we can replace the three-phase coated disk with a disk composed only of linear material of conductivity one. One could continue to add coated disks of various sizes (see Figure 7.2) without disturbing the prescribed uniform applied electric field surrounding the inclusions. In fact, we can fill the space with these coated disks. The space filling configuration of triple coated disks can be solved explicitly and the field inside it is precisely \( u = Ex_1 \) when \( |E| = \sigma \frac{1}{r} \).

This configuration of nonlinear materials dissipates energy the same as a linear material with thermal conductivity one.

![Space Filling Configuration of Coated Disks](image)

Figure 7.2: Space Filling Configuration of Coated Disks

### 7.2 Calculations

Let \( \bar{x} \) be the center of the disk (see Figure 7.1). Inside the disk, we ask that

\[
\begin{cases}
\sigma_1 \Delta_{p_1} u = 0 & 0 < |x - \bar{x}| < a \\
\sigma_3 \Delta u = 0 & a < |x - \bar{x}| < b \\
\sigma_2 \Delta_{p_2} u = 0 & b < |x - \bar{x}| < c,
\end{cases}
\tag{7.2}
\]

where \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) are positive constants.

Have \( r = |x - \bar{x}| \) and \( \hat{n} = \frac{x - \bar{x}}{|x - \bar{x}|} \).

The solution \( u \) of (7.2) is such that

\[
\begin{align*}
& u \text{ continuous across } |x - \bar{x}| = a, \quad \tag{7.3} \\
& u \text{ continuous across } |x - \bar{x}| = b, \quad \tag{7.4} \\
& u = Ex_1 \text{ at } |x - \bar{x}| = c. \quad \tag{7.5}
\end{align*}
\]
and satisfies the transmission conditions
\[ \sigma_1 \vec{n} \cdot |\nabla u|^{p_1-2} \nabla u = \sigma_3 \vec{n} \cdot \nabla u, \text{ across } |x - \bar{x}| = a, \] (7.6)
and
\[ \sigma_3 \vec{n} \cdot \nabla u = \sigma_2 \vec{n} \cdot |\nabla u|^{p_2-2} \nabla u, \text{ across } |x - \bar{x}| = b. \] (7.7)

We look for solution \( u \) of the form
\[
\begin{cases}
  u = c_1 r \cos \theta & \text{for } 0 < r < a, \\
  u = \frac{c_3}{r} \cos \theta + c_4 r \cos \theta & \text{for } a < r < b, \\
  u = c_2 r \cos \theta & \text{for } b < r < c.
\end{cases}
\] (7.8)

It is easily seen that (7.8) satisfies (7.2).

In what follows, we explain how the unknowns \( c_1, c_2, c_3, \) and \( c_4 \) are determined from (7.3), (7.4), (7.5), (7.6), and (7.7).

**First, we look at the conditions \( u \) must satisfy when \( r = a \):**

By (7.3), we have
\[
c_1 a \cos \theta = \frac{c_3}{a} \cos \theta + c_4 a \cos \theta
\]
\[
\Rightarrow c_1 a = \frac{c_3}{a} + c_4 a
\]
\[
\Rightarrow c_1 a^2 = c_3 + c_4 a^2,
\] (7.9)

and from (7.6), we obtain
\[
\sigma_1 \vec{e}_r \cdot |\nabla u|^{p_1-2} \nabla u = \sigma_3 \vec{e}_r \cdot \nabla u
\]
\[
\Rightarrow \sigma_1 \vec{e}_r \cdot |c_1|^{p_1-2} c_1 (\vec{e}_r \cos \theta - \vec{\theta} \sin \theta) = \sigma_3 \left( \frac{-c_3}{a^2} \cos \theta + c_4 \cos \theta \right)
\]
\[
\Rightarrow \sigma_1 (\text{sign}(c_1)) |c_1|^{p_1-1} \cos \theta = \frac{-\sigma_3}{a^2} c_3 \cos \theta + \sigma_3 c_4 \cos \theta
\]
\[
\Rightarrow a^2 \sigma_1 (\text{sign}(c_1)) |c_1|^{p_1-1} = -\sigma_3 c_3 + a^2 \sigma_3 c_4.
\] (7.10)

**Second, we look at the conditions \( u \) must satisfy when \( r = b \):**

By (7.4), we have
\[
\frac{c_3}{b} \cos \theta + c_4 b \cos \theta = c_2 b \cos \theta
\]
\[
\Rightarrow \frac{c_3}{b} + c_4 b = c_2 b
\]
\[
\Rightarrow c_3 + c_4 b^2 = c_2 b^2,
\] (7.11)

and from (7.7), we obtain
\[
\sigma_3 \vec{e}_r \cdot \nabla u = \sigma_2 \vec{e}_r \cdot |\nabla u|^{p_2-2} \nabla u
\]
\[
\Rightarrow \sigma_3 \left( \frac{-c_3}{b^2} \cos \theta + c_4 \cos \theta \right) = \sigma_2 (\text{sign}(c_2)) |c_2|^{p_2-1} \cos \theta
\]
\[
\Rightarrow -\sigma_3 c_3 + b^2 \sigma_3 c_4 = b^2 \sigma_2 (\text{sign}(c_2)) |c_2|^{p_2-1}.
\] (7.12)
Finally, we look at the conditions $u$ must satisfy when $r = c$:

By (7.5) we have

$$c_2 c \cos \theta = E c \cos \theta$$
$$\Rightarrow c_2 = E.$$  \hspace{1cm} (7.13)

Using (7.13), we can rewrite equations (7.11) and (7.12) in the following way:

$$c_3 + c_4 b^2 = E b^2,$$  \hspace{1cm} (7.14)

$$-\sigma_3 c_3 + b^2 \sigma_3 c_4 = b^2 \sigma_2 (\text{sign}(E)) |E|^{p_2 - 1}.$$  \hspace{1cm} (7.15)

Now we use (7.9), (7.10), (7.14), and (7.15) to determine the unknowns $c_1$, $c_3$, and $c_4$ ($c_2 = E$ by (7.13)).

From (7.15), we have

$$c_3 = b^2 c_4 - b^2 \frac{\sigma_2}{\sigma_3} (\text{sign}(E)) |E|^{p_2 - 1},$$  \hspace{1cm} (7.16)

and (7.14) implies that

$$c_3 = E b^2 - c_4 b^2.$$  \hspace{1cm} (7.17)

If we combine (7.16) and (7.17), we obtain

$$E b^2 - c_4 b^2 = b^2 c_4 - b^2 \frac{\sigma_2}{\sigma_3} (\text{sign}(E)) |E|^{p_2 - 1}$$
$$\Rightarrow 2c_4 b^2 = E b^2 + b^2 \frac{\sigma_2}{\sigma_3} (\text{sign}(E)) |E|^{p_2 - 1}$$
$$\Rightarrow c_4 = \frac{E + (\text{sign}(E)) |E|^{p_2 - 1} \left( \frac{\sigma_2}{\sigma_3} \right)}{2},$$  \hspace{1cm} (7.18)

and this way we find $c_4$.

Evaluating $c_4$ (given by (7.18)) in (7.17), we find $c_3$:

$$c_3 = E b^2 - \left( \frac{E + (\text{sign}(E)) |E|^{p_2 - 1} \left( \frac{\sigma_2}{\sigma_3} \right)}{2} \right) b^2$$
$$\Rightarrow c_3 = \frac{2E b^2 - E b^2 - (\text{sign}(E)) |E|^{p_2 - 1} \left( \frac{\sigma_2}{\sigma_3} \right) b^2}{2}$$
$$\Rightarrow c_3 = \frac{E b^2 - (\text{sign}(E)) |E|^{p_2 - 1} \left( \frac{\sigma_2}{\sigma_3} \right) b^2}{2}.$$  \hspace{1cm} (7.19)
Finally, to obtain \( c_1 \), we evaluate (7.18) and (7.19) in (7.9):

\[
c_1 a^2 = \left( \frac{E b^2 - \text{sign}(E) |E|^{p_2-1} \left( \frac{c_2}{\sigma_3} \right) b^2}{2} \right) + \left( \frac{E + \text{sign}(E) |E|^{p_2-1} \left( \frac{c_2}{\sigma_3} \right)}{2} \right) a^2
\]

\[\Rightarrow c_1 a^2 = \frac{E (a^2 + b^2) + \text{sign}(E) |E|^{p_2-1} \left( \frac{c_2}{\sigma_3} \right) (a^2 - b^2)}{2}\]

\[\Rightarrow c_1 = \frac{E \left( 1 + \left( \frac{b}{a} \right)^2 \right) + \text{sign}(E) |E|^{p_2-1} \left( \frac{c_2}{\sigma_3} \right) \left( 1 - \left( \frac{b}{a} \right)^2 \right)}{2}. \quad (7.20)\]

Therefore the values of \( c_1, c_2, c_3, \) and \( c_4 \) are given by (7.20), (7.13), (7.19), and (7.18), respectively. These values must also satisfy (7.10).

Note that in (7.10), using (7.19) and (7.18), we have

\[
a^2 \sigma_1 (\text{sign}(c_1)) |c_1|^{p_1-1} = -\sigma_3 c_3 + a^2 \sigma_3 c_4
\]

\[= -\sigma_3 \left( \frac{E b^2 - \text{sign}(E) |E|^{p_2-1} \left( \frac{c_2}{\sigma_3} \right) b^2}{2} \right) + a^2 \sigma_3 \left( \frac{E + \text{sign}(E) |E|^{p_2-1} \left( \frac{c_2}{\sigma_3} \right)}{2} \right)
\]

\[= -\sigma_3 E b^2 + \text{sign}(E) |E|^{p_2-1} \sigma_3 b^2 + \sigma_3 E a^2 + \text{sign}(E) |E|^{p_2-1} \sigma_2 a^2
\]

\[= \frac{\sigma_3 E (a^2 - b^2) + \text{sign}(E) |E|^{p_2-1} \sigma_2 (a^2 + b^2)}{2}.
\]

Therefore, using (7.20) above, we get

\[
\sigma_1 (\text{sign}(c_1)) \left( \frac{E \left( 1 + \left( \frac{b}{a} \right)^2 \right) + \text{sign}(E) |E|^{p_2-1} \left( \frac{c_2}{\sigma_3} \right) \left( 1 - \left( \frac{b}{a} \right)^2 \right)}{2} \right)^{p_1-1}
\]

\[= \frac{\sigma_3 E \left( 1 - \left( \frac{b}{a} \right)^2 \right) + \text{sign}(E) |E|^{p_2-1} \sigma_2 \left( 1 + \left( \frac{b}{a} \right)^2 \right)}{2}. \quad (7.21)\]

We have that (7.21) is verified if

\[
\begin{align*}
\sigma_3 &= 1 \\
\sigma_2 &= |E|^{2-p_2} \\
\sigma_1 &= |E|^{2-p_1}
\end{align*}
\]

If we set \( \sigma = \sigma_2 = |E|^{2-p_2} \), then \( \sigma_1 = \sigma^{2-p_1} \). This values of \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) correspond to the coefficients in (7.1). We also obtain that \( |E| = \sigma^{\frac{1}{p_2}} \).

Here it can be checked that \( c_1 = c_2 = c_4 = E \) and \( c_3 = 0 \). Therefore \( u = E r \cos \theta \) is the solution to (7.1).
At $r = c$, we have $u = E c \cos \theta$ and

$$\sigma_2 e_r^2 \cdot |\nabla u|^{p_2-2} \nabla u = |E|^{2-p_2} \ |c_2|^{p_2-1} \ (\text{sign}(c_2)) \cos \theta$$

$$= |E|^{2-p_2} \ |E|^{p_2-1} \ (\text{sign}(E)) \cos \theta$$

$$= |E| \ (\text{sign}(E)) \cos \theta$$

$$= h E \cos \theta,$$

with $h = 1$ which does not depend on $a$, $b$, or $c$, which means that it is independent of scale.
Bibliography


Vita

Silvia Elena Jiménez Bolaños was born in July 1980, in Alajuela, Costa Rica. In August 2002 she finished her undergraduate studies in mathematics at the Universidad de Costa Rica, Costa Rica. In August 2004 she came to Louisiana State University to pursue graduate studies in mathematics and earned a master of science in mathematics in 2006. She is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2010.