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Group automorphisms and the decomposition of Plancherel measures

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Abstract

In this paper, a natural action of the automorphisms of a group on the space of irreducible unitary representations is used to decompose the Plancherel measure on the dual space as an integral of measures on homogeneous spaces. Explicit decompositions are obtained for the cases of free 2 and 3-step nilpotent Lie groups. These results are obtained using direct integral decompositions, induced representations, the Mackey Machine, and measure theory on homogeneous spaces.

Introduction

In [1], Chin-Te Chu showed that by using rotations and translations for free 2-step nilpotent Lie groups, a canonical description of the Plancherel measure can be derived without the use of a strong Malcev basis. To do this, Chu showed that the dual, except for a set of measure zero, is a single orbit (i.e., a homogeneous space) under the action of rotations and translations. Since Plancherel measure can easily be shown to be relatively invariant and relatively invariant measures on a homogeneous space only differ by constants, Chu identified Plancherel measure on the dual with the correct relatively invariant measure on the conull orbit. Chu used Kirillov theory and the associated non-canonical Plancherel formula (i.e. basis dependent) to establish the validity of this identification.

Counting dimensions shows that, for $k > 2$, free k-step nilpotent Lie groups have, in most situations, dual spaces which were too large to be essentially one orbit. This leads to the natural question as to how the Plancherel measure decomposes over the space of orbits. The goal of this paper is to illustrate how the natural action of automorphisms (not just rotations and translations) of a group can be used to decompose the Plancherel measure on the dual into an integral of relatively invariant measures on homogeneous spaces.

In Chapter 1, information on direct integrals, induced representations, and the Plancherel theorem is discussed. We use this information to obtain a result relating the support of the Plancherel measure of a group with the support of the Plancherel measure on a closed normal subgroup. This result is precisely what allows us to restrict our consideration to a conull subset of orbits instead of the full dual space.

The machinery of G. W. Mackey is applied to determine this conull subset of the dual space from a conull subset of the dual of a closed normal subgroup. The key to this analysis is obtaining a clear picture of the orbits and a cross section of the orbit space. In this paper, we consider only free nilpotent Lie groups since they have large automorphism groups.

In Chapter 2 free nilpotent Lie groups and their automorphisms are discussed. The automorphism group for the free k-step nilpotent Lie group on n generators is determined in terms of linear maps on the linear span of the generators. The program then is to analyze the action of the automorphism group on the dual and show one can restrict to those irreducible representations that have the largest orbits.

In Chapter 3, we consider the free 2-step nilpotent Lie groups and obtain canonical descriptions of their Plancherel measures. In the case where there is an even number of generators, the Plancherel measure is identified with a relatively invariant measure on the space of symplectic forms. When the number of generators is odd, the Plancherel measure is essentially determined by a relatively invariant measure on the space of alternating forms with nullity 1. In each case the dual is essentially a homogeneous space of the general linear group.

In Chapter 4, we consider the free 3-step nilpotent Lie groups. In the case of two generators, the dual space is again essentially one orbit. Thus Plancherel measure was identified with a relatively invariant measure on a homogenous space. When there is more than two generators, we need to consider a continuum of orbits. The case of three generators is considered in detail and Plancherel measure is explicitly identified with an integral of relatively invariant measures on homogeneous spaces.

The main tools used in Chapters 3 and 4 are direct integrals, induced representations, Fourier analysis, Hilbert-Schmidt and trace class operators, and measure theory on homogeneous spaces. Information on these topics can be found in most texts on Abstract Harmonic Analysis. The interested reader is referred to [5] or [6].

Preliminaries

1.1 Introduction

The purpose of this chapter is to introduce necessary background materials and to prove a result that relates the support of the Plancherel measure of a group with the support of the Plancherel measure of a closed normal subgroup. Some knowledge of measure theory and basic representation theory is assumed. In the following, two unitary representations will be called equivalent if there is a unitary isomorphism which intertwines them. A unitary representation and its equivalence class will normally be denoted by the same symbol. If a distinction needs to be made, brackets will be used to denote the equivalence class.

Let X be a topological space and $\mathcal{M}$ be a σ-algebra of subsets of X. The pair $(X, \mathcal{M})$ is a standard Borel space if $\mathcal{M}$ is the Borel σ-algebra of a topology generated by a complete separable metric on X. In this paper, groups will always be assumed to be second countable, locally compact, and Hausdorff. The unitary dual of a group G is the space consisting of equivalence classes of irreducible unitary representations of G. It will be denoted by $\hat{G}$. A unitary representation π is primary if the center of $\mathcal{C}(\pi)$ consists of scalar multiples of the identity. A group G is called type I if every primary representation of G is a multiple of an irreducible representation. In our situation, G being type I is equivalent to $\hat{G}$ being a standard Borel space (see [6], Theorem 7.6). In particular, simply connected nilpotent Lie groups are type I.

1.2 Direct Integrals

A *Hilbert bundle* over a standard Borel space S is a Borel space $\mathcal{H}$ and an onto Borel map $p : \mathcal{H} \to S$ such that the following conditions hold:

(1) For each s in S, $\mathcal{H}_s := p^{-1}(s)$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_s$.

(2) There exists a sequence, $\{f_n\}_{n=1}^{\infty}$, of Borel maps from S into $\mathcal{H}$ such that

(*a*) for all s, $p(f_n(s)) = s$;

(*b*) the linear span of $\{f_n(s)\}_{n=1}^{\infty}$ is dense in $\mathcal{H}_s$ for each s;

(*c*) for each m and n, the function $s \mapsto \langle f_m(s), f_n(s) \rangle_s$ is Borel; and

(*d*) a function F from a Borel space X into $\mathcal{H}$ is Borel if and only if the functions $x \mapsto p(F(x))$ and $x \mapsto \langle F(x), f_n(p(F(x))) \rangle_{p(F(x))}$ are Borel.

A *Borel section* of $\mathcal{H}$ is a map $f : S \to \mathcal{H}$ such that $p(f(s)) = s$ and $s \mapsto \langle f(s), f_n(s) \rangle_s$ is Borel for each n. Two Hilbert bundles $\mathcal{H}$ and $\mathcal{K}$ over S are isomorphic over S if there exists a field $s \mapsto U(s)$ of unitary maps $U(s) : \mathcal{H}_s \to \mathcal{K}_s$, such that $s \mapsto U(s)f(s)$ is a Borel section of $\mathcal{K}$ if and only if f is a Borel section of $\mathcal{H}$.

The *direct integral* $\int^{\oplus} \mathcal{H}_s \, d\mu(s)$ of the Hilbert bundle $\mathcal{H}$ with respect to μ, a σ-finite measure on S, is the space of Borel sections f such that $||f||^2 = \int ||f(s)||_s^2 \, d\mu(s) < \infty$. It is a Hilbert space and its inner product is given by $\langle f, g \rangle = \int \langle f(s), g(s) \rangle_s \, d\mu(s)$. If ν is another σ-finite measure on S that is equivalent to μ then the map $f \mapsto \sqrt{[\frac{d\mu}{d\nu}]}\, f$, where $[\frac{d\mu}{d\nu}]$ is the Radon-Nikodym derivative, is a unitary isomorphism from $\int^{\oplus} \mathcal{H}_s \, d\mu(s)$ onto $\int^{\oplus} \mathcal{H}_s \, d\nu(s)$. Thus, up to unitary isomorphism, the direct integral depends only on the equivalence class of the measure used.

Example 1.1. *Let $\mathcal{H}$ be the Hilbert bundle in which $\mathcal{H}_s$ is a fixed Hilbert space, $\mathcal{H}_0$, for each s in S. Then the sequence $\{f_n\}_{n=1}^{\infty}$ may be chosen to consist of the constant functions $f_n(s) = e_n$ where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}_0$. Since weak measurability is equivalent to measurability for Hilbert spaces, this choice satisfies the requirements of the definition. Then the direct integral of $\mathcal{H}$ with respect to μ is $L^2(S, \mathcal{H}_0)$. Indeed, the square integrability conditions on each space are the same and f is a Borel section of $\mathcal{H} \iff s \mapsto \langle f(s), f_n(s)\rangle_s$ is Borel for each $n \iff s \mapsto \langle f(s), e_n\rangle_0$ is Borel for each $n \iff s \mapsto \langle f(s), v\rangle_0$ is Borel for each v in $\mathcal{H}_0 \iff f$ is a Borel function from S into $\mathcal{H}_0$.*

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert bundles over S. A *Borel field* of bounded operators from $\mathcal{H}$ to $\mathcal{K}$ is a map $A : s \mapsto A(s)$ where each $A(s)$ is a bounded operator from $\mathcal{H}_s$ into $\mathcal{K}_s$ such that $Af : s \mapsto A(s)f(s)$ is a Borel section of S into $\mathcal{K}$ whenever f is a Borel section of S into $\mathcal{H}$. If the essential supremum of the operator norms $||A(s)||_s$ with respect to μ is finite then the *direct integral of* A, denoted $\int^{\oplus} A(s)d\mu(s)$, is a bounded linear operator from $\int^{\oplus} \mathcal{H}_s d\mu(s)$ into $\int^{\oplus} \mathcal{K}_s d\mu(s)$. It is defined by $[\int^{\oplus} A(s)d\mu(s)f](s) = A(s)f(s)$.

Suppose π_s is a unitary representation of G on $\mathcal{H}_s$ for each s in S and $s \mapsto \pi_s(g)$ is a Borel field of operators for each g and μ is a σ-finite measure on S. Then, since $||\pi_s(g)|| = 1$, the direct integral $\pi(g) = \int^{\oplus} \pi_s(g)\, d\mu(s)$ is defined for g. The mapping $g \mapsto \pi(g)$ is called the *direct integral of the representations* π_s. It is the representation of G on $\int^{\oplus} \mathcal{H}_s\, d\mu(s)$ defined by $[\pi(g)f](s) = \pi_s(g)f(s)$. Since each π_s is unitary, it follows that π is unitary.

The following two propositions establish properties of direct integrals which will be useful in the sequel.

Proposition 1.2. *Let G be a locally compact group, H a closed subgroup, $k \in \{\infty, 1, 2, ...\}$, and $s \mapsto \pi_s$ be a Borel field of unitary representations of G. Then $\int^{\oplus} \pi_s \, d\mu(s)|_H = \int^{\oplus} \pi_s|_H \, d\mu(s)$ and $k \int^{\oplus} \pi_s \, d\mu(s) \simeq \int^{\oplus} k\pi_s \, d\mu(s)$.*

Proof. Let $\pi = \int^{\oplus} \pi_s \, d\mu(s)$. Both $\pi|_H$ and $\int^{\oplus} \pi_s|_H \, d\mu(s)$ act on $\int^{\oplus} \mathcal{H}_s \, d\mu(s)$. Since $[\pi|_H(h)f](x) = \pi_x(h)f(x) = [\int^{\oplus} \pi_s|_H(h) \, d\mu(s) \, f](x)$, the first result is true.

The Hilbert space for $\int^{\oplus} k\pi_s \, d\mu(s)$ is

$$\int^{\oplus} k\mathcal{H}_s \, d\mu(s) = \{f : s \mapsto (f^1(s), ..., f^k(s)) \in k\mathcal{H}_s | \int \sum_{i=1}^{k} ||f^i(s)||_s^2 d\mu(s) < \infty\}.$$

Set $\mathcal{H}_\pi = \int^{\oplus} \mathcal{H}_s \, d\mu(s)$. Then the Hilbert space for $k\pi$ is $k\mathcal{H}_\pi = \{(F_1, ..., F_k) | F_i \in \mathcal{H}_\pi\}$. Let $\{f_n\}_{n=1}^{\infty}$ satisfy part (2) of the definition of the Hilbert bundle $\mathcal{H}$. If $f_0 \equiv 0$ then $\{f_n\}_{n=0}^{\infty}$ also satisfies part (2) of the definition. Set $f_{n_1, n_2, ..., n_k} = (f_{n_1}, f_{n_2}, ..., f_{n_k})$. Then $\{f_{n_1, n_2, ..., n_k}\}_{n_i=0}^{\infty}$ satisfies part (2) of the definition for $\{k\mathcal{H}_s\}$. Furthermore, f is a Borel section of $\{k\mathcal{H}_s\}$ $\iff$ $s \mapsto \langle f(s), f_{n_1, n_2, ..., n_k}(s)\rangle_{k\mathcal{H}_s} = \sum_{i=1}^{k} \langle f^i(s), f_{n_i}(s)\rangle_s$ is Borel for each $(n_1, n_2, ..., n_k)$ $\iff$ for $i = 1, ..., k$, $s \mapsto \langle f^i(s), f_n(s)\rangle_s$ is Borel for each n (choose $n_i = n$ and $n_j = 0 \; \forall \; j \neq i$) $\iff$ for $i = 1, ..., k$, f^i is a Borel section of $\mathcal{H}$. Thus there is a bijection $f \leftrightarrow (f^1, ..., f^k)$ between Borel sections of $\{k\mathcal{H}_s\}$ and the k-tuples of Borel sections of $\mathcal{H}$. Since

$$\begin{aligned} ||(f^1, ..., f^k)||_{k\mathcal{H}_\pi}^2 &= \sum_{i=1}^{k} ||f^i||_{\mathcal{H}_\pi}^2 \\ &= \sum_{i=1}^{k} \int ||f^i(s)||_s^2 \, d\mu(s) \\ &= \int \sum_{i=1}^{k} ||f^i(s)||_s^2 \, d\mu(s) \\ &= ||f||_{\int^{\oplus} k\mathcal{H}_s d\mu(s)}^2, \end{aligned}$$

$f \in \int^{\oplus} k\mathcal{H}_s\, d\mu(s) \iff (f_1, ..., f_k) \in k\mathcal{H}_\pi$. Thus $Uf := (f^1, ..., f^k)$ is a unitary map from $\int^{\oplus} k\mathcal{H}_s\, d\mu(s)$ onto $k\mathcal{H}_\pi$. Since

$$\begin{aligned}
[U^{-1}k\pi(g)Uf](s) &= [U^{-1}k\pi(g)(f^1, ..., f^k)](s) \\
&= U^{-1}(s \mapsto (\pi_s(g)f^1(s), ..., \pi_s(g)f^k(s)))(s) \\
&= k\pi_s(g)f(s) \\
&= [\int^{\oplus} k\pi_s(g)\, d\mu(s) f](s)
\end{aligned}$$

for all $g \in G$, the representations $k\pi$ and $\int^{\oplus} k\pi_s\, d\mu(s)$ are equivalent. □

Proposition 1.3. *If $k : S \to \{\infty, 1, 2, ...\}$ is Borel and π is any unitary representation, there is some $K \in \{\infty, 1, 2, ...\}$ such that $\int^{\oplus} k(s)\pi\, d\mu(s)$ is equivalent to $K\pi$.*

Proof. Let $S_n = \{s \in S \,|\, k(s) = n\}$. Then $S = \bigcup_{n=1}^{\infty} S_n$ and $\int_S^{\oplus} k(s)\pi d\mu(s) \simeq \sum_{n=1}^{\infty} \int_{S_n}^{\oplus} n\pi d\mu(s)$. Since the sum of multiples of a representation is a multiple of the representation, it suffices to prove the result for constant k. Since $\int^{\oplus} k\pi d\mu(s) \simeq k\int^{\oplus} \pi d\mu(s)$, it suffices to show that $\int^{\oplus} \pi d\mu(s)$ is a multiple of π. This direct integral acts on $\int^{\oplus} \mathcal{H}_\pi d\mu(s) = L^2(S, \mathcal{H}_\pi)$. Define $B : L^2(S) \times \mathcal{H}_\pi \to L^2(S, \mathcal{H}_\pi)$ by $B(f, v)(s) = f(s)v$. Note: B is bilinear and

$$||B(f, v)||^2 = \int ||f(s)v||^2_{\mathcal{H}_\pi} d\mu(s) = ||v||^2_{\mathcal{H}_\pi} \int |f(s)|^2 d\mu(s) = ||v||^2_{\mathcal{H}_\pi} ||f||_2^2. \quad (1.1)$$

Thus there is a unique continuous bilinear $T : L^2(S) \otimes \mathcal{H}_\pi \to L^2(S, \mathcal{H}_\pi)$ such that $T(f \otimes v) = B(f, v)$. Let $\{E_i\}$ be a countable neighborhood base of S and let $\{v_j\}$ be a orthonormal basis of $\mathcal{H}_\pi$. Then $\{\mathbf{1}_{E_i} v_j\}$ generates a countable dense subset of $L^2(S, \mathcal{H}_\pi)$, $\{\mathbf{1}_{E_i} \otimes v_j\}$ generates a countable dense subset of $L^2(S) \otimes \mathcal{H}_\pi$, $T(\mathbf{1}_{E_i} \otimes v_j) = \mathbf{1}_{E_i} v_j$, and $\mathbf{1}_{E_i} v_j = 0$ implies $\mathbf{1}_{E_i} \otimes v_j = 0$. Also (1.1) and the orthogonality of the v_j imply $||T(\sum \alpha_{i,j} \mathbf{1}_{E_i} \otimes v_j)||^2 = \sum \alpha_{i,j}^2 ||T(\mathbf{1}_{E_i} \otimes v_j)||^2 =$

$\sum \alpha_{i,j}^2 ||\mathbf{1}_{E_i} \otimes v_j||^2 = ||\sum \alpha_{i,j}\mathbf{1}_{E_i} \otimes v_j||^2$. Since T is continuous and Hilbert spaces are complete, T is unitary and $L^2(S, \mathcal{H}_\pi) \cong L^2(S) \otimes \mathcal{H}_\pi$. Since π acts on $\mathcal{H}_\pi$ the action of $\int^{\oplus} \pi d\mu(s)$ on $L^2(S) \otimes \mathcal{H}_\pi$ is $[\int^{\oplus} \pi(g) d\mu(s)(f \otimes v)](s) = \pi(g)(f \otimes v)(s) = \pi(g)(f(s) \otimes v) = f(s) \otimes \pi(g)v$. If $\{f_i\}$ is any orthonormal basis of $L^2(S)$ then $f_i \otimes \mathcal{H}_\pi$ are orthogonal subspaces of $L^2(S) \otimes \mathcal{H}_\pi$ on which the direct integral is equivalent to π. Thus $\int^{\oplus} \pi d\mu(s)$ is equivalent to a multiple of π. $\square$

The direct integral decomposition is a particularly important tool in the representation theory of type I groups. Indeed, the following theorem shows that any unitary representation can be expressed in an essentially unique way as a direct integral of multiples of irreducible representations when G is type I.

Theorem 1.4. ([5], Theorem VI.14) *Let G be a second countable, locally compact, Hausdorff, type I group and π be a unitary representation of G on a separable Hilbert space. Then there exist a finite measure μ on $\hat{G}$, a Borel map $n : \hat{G} \to \{\infty, 1, 2, ...\}$, a Hilbert bundle $\gamma \mapsto \mathcal{H}_\gamma$ over $\hat{G}$, and a Borel field of irreducible unitary representations $\gamma \mapsto \pi_\gamma$ where π_γ is in γ for almost every γ such that*

$$\pi \cong \int^{\oplus} n(\gamma)\, \pi_\gamma d\mu(\gamma).$$

Furthermore, the measure class of μ is unique and $\gamma \mapsto n(\gamma)$ is unique for μ almost every γ.

1.3 Induced Representations

Let G be a locally compact Hausdorff group, H be a closed subgroup, and π be a unitary representation of H. There is a method of constructing a unitary representation of G from π. This representation, which will be denoted $\mathrm{ind}_H^G(\pi)$, is called the *unitary representation induced* by π. In the following construction it will be assumed that $H\backslash G$ admits a G-invariant measure μ. All cases considered in this paper satisfy this assumption. While the inducing construction can be done more generally, it is simplified by this assumption.

Let $\mathcal{H}_\pi$ denote the Hilbert space of π and let $\mathcal{H}$ denote the space of Borel functions $f : G \to \mathcal{H}_\pi$ that satisfy

$$(i) f(hg) = \pi(h)f(g) \;\forall\; g \in G,\, h \in H \text{ and}$$
$$(ii) \int_{H\backslash G} ||f(g)||^2 d\mu(Hg) < \infty.$$

Then $\mathcal{H}$ is a Hilbert space with inner product $\langle f, f'\rangle_{\mathcal{H}} = \int_{H\backslash G}\langle f(x), f'(x)\rangle_\pi \, d\mu(Hx)$. Notice that since π is unitary and each element of $\mathcal{H}$ satisfies (i), the quantity $\langle f(x), f'(x)\rangle_{\mathcal{H}}$ is constant on cosets and so the inner product is well-defined. Let the right translate of a function f be defined by $R_g f(x) = f(xg)$. Then right translation, $f \mapsto R_g f$, is an action of G on $\mathcal{H}$. It is unitary by the G-invariance of μ. Strong operator continuity follows from the fact that the uniformly continuous elements in $\mathcal{H}$ are norm dense (see [6], pg. 152). Thus $g \mapsto R_g$ is a unitary representation of G on $\mathcal{H}$. This is the induced representation $\mathrm{ind}_H^G(\pi)$.

Example 1.5. *Let* $\mathbf{1}$ *be the trivial representation on* $\{e\}$*. Then* $ind_{\{e\}}^G(\mathbf{1})$ *acts on* $L^2(G)$ *by* $ind_{\{e\}}^G(\mathbf{1})(g)f(x) = f(xg)$*. Thus* $ind_{\{e\}}^G(\mathbf{1})$ *is the right regular representation of* G *on* $L^2(G)$*.*

The following are facts concerning induced representations:

Proposition 1.6. ([6], Proposition 6.9) *If* π *and* ψ *are equivalent unitary representations of* H*, then* $ind_H^G(\pi)$ *and* $ind_H^G(\psi)$ *are equivalent. Thus if* π *is in* $\hat{H}$*,* $ind_H^G(\pi)$ *is well-defined.*

Theorem 1.7 (Induction in Stages). ([5], Theorem V.2) *Suppose* H *is a closed subgroup of* G*,* K *is a closed subgroup of* H*, and* π *is a unitary representation of* K*. Then* $ind_H^G(ind_K^H(\pi))$ *is equivalent to* $ind_K^G(\pi)$*.*

The next proposition establishes another useful property of direct integrals.

Proposition 1.8. *Let G be a locally compact group, H a closed subgroup, and $s \mapsto \pi_s$ be a Borel field of unitary representations of H. Then the representations $ind_H^G \int^{\oplus} (\pi_s)\, d\mu(s)$ and $\int^{\oplus} ind_H^G(\pi_s)\, d\mu(s)$ are equivalent.*

Proof. Let $\pi = \int^{\oplus} \pi_s\, d\mu(s)$. Set $\gamma_s = \mathrm{ind}_H^G(\pi_s)$, $\gamma = \int^{\oplus} \gamma_s\, d\mu(s)$, and $\tilde{\pi} = \mathrm{ind}_H^G(\pi)$. Let $\mathcal{H}_s$ be the Hilbert space of π_s, $\mathcal{K}_s$ be the Hilbert space of γ_s, $\widetilde{\mathcal{H}}$ be the Hilbert space of $\tilde{\pi}$, $\mathcal{K}$ be the Hilbert space of γ, and ν be a σ-finite G-invariant measure on $H\backslash G$. Then

$$\mathcal{K}_s = \{f : G \to \mathcal{H}_s \,|\, f(hg) = \pi_s(h) f(g) \text{ and } \int_{H\backslash G} ||f||^2_{\mathcal{H}_s}\, d\nu < \infty\}$$

and $[\gamma_s(g) f](x) = f(xg)$;

$$\mathcal{K} = \{F : s \mapsto F(s) \in \mathcal{K}_s \,|\, \int_S ||F(s)||^2_{\mathcal{K}_s}\, d\mu(s) < \infty\}$$

and $[\gamma(g)F](s) = \gamma_s(g)F(s)$; and

$$\widetilde{\mathcal{H}} = \{f : G \to \int^{\oplus} \mathcal{H}_s d\mu(s) \,|\, f(hg) = \pi(h) f(g) \text{ and } \int_{H\backslash G} ||f(g)||^2_{\int^{\oplus} \mathcal{H}_s d\mu(s)}\, d\nu(s) < \infty\}$$

with $[\tilde{\pi}(g) f](x) = f(xg)$.

Since $\varphi : F \mapsto [(s,g) \mapsto F(s)(g)]$ and $\psi : f \mapsto [(s,g) \mapsto f(g)(s)]$ are Borel isomorphisms from $\mathcal{M}(S, \mu, \mathcal{M}(G, m, \mathcal{H}))$ to $\mathcal{M}(S \times G, \mu \times m, \mathcal{H})$ and from $\mathcal{M}(G, m, \mathcal{M}(S, \mu, \mathcal{H}))$ to $\mathcal{M}(S \times G, \mu \times m, \mathcal{H})$, respectively ([5], Corollary I.16), it follows that $\varphi F(s,g) = F(s)(g)$ for m-a.e. g, for μ-a.e. s and $\psi f(s,g) = f(g)(s)$ for μ-a.e. s, for m-a.e. g. Let $U = \psi^{-1} \cdot \varphi$. Then U is a Borel isomorphism from

$\mathcal{M}(S, \mu, \mathcal{M}(G, m, \mathcal{H}))$ to $\mathcal{M}(G, m, \mathcal{M}(S, \mu, \mathcal{H}))$ and

$$\begin{aligned}
||UF||^2_{\tilde{\mathcal{H}}} &= \iint ||UF(g)(s)||^2_s d\mu(s)\, d\nu(Hg) \\
&= \int ||\varphi F(s, g)||^2_s d(\mu \times \nu)(s, Hg) \\
&= \iint ||F(s)(g)||^2_s d\nu(Hg)\, d\mu(s) \\
&= ||F||^2_{\mathcal{K}}.
\end{aligned}$$

Since $F(s)(hg) = \pi_s(h)F(s)(g) \iff \varphi F(s, hg) = F(s)(hg) = \pi_s(h)F(s)(g) = \pi_s(h)\varphi F(s, g)$ for m-a.e. g, for μ-a.e. $s \iff UF(hg)(s) = \varphi F(s, hg) = \pi_s(h)\varphi F(s, g)$ $= \pi_s(h)UF(g)(s)$ for μ-a.e. s, for m-a.e. $g \iff UF(hg) = \pi(h)UF(g)$, we see that $UF \in \tilde{\mathcal{H}} \iff F \in \mathcal{K}$. Thus $U|_{\mathcal{K}}$ is unitary from $\mathcal{K}$ onto $\tilde{\mathcal{H}}$.

Since $UF(xg)(s) = \varphi F(s, xg)$ for μ-a.e. s, for m-a.e. x, and $\varphi F(s, xg) = F(s)(xg)$ for m-a.e. x, for μ-a.e. s, we have

$$U : (s \mapsto R_g F(s)) \mapsto (x \mapsto UF(xg)). \tag{1.2}$$

Note that $[\tilde{\pi}(g)UF](x) = UF(xg)$ and $\gamma(g)F(s)(x) = \gamma_s(g)F(s)(x) = F(s)(xg) = [R_g F(s)](x) \implies \gamma(g)F(s) = R_g F(s)$. Then (1.12) is equivalent to $U : (s \mapsto \gamma(g)F(s)) \mapsto (x \mapsto \tilde{\pi}(g)UF(x))$. Thus $U^{-1}\tilde{\pi}(g)UF = \gamma(g)F$ since U is injective. $\square$

Let $\mathcal{U}$ be the unitary group of some separable Hilbert space. A Borel function $\rho : G \to \mathcal{U}$ is called a *multiplier representation* (or a projective representation) of G if $\rho(e) = I$, $\rho(x)\rho(y) = \sigma(x, y)\rho(xy)$ for some $\sigma : G \times G \to \{z \in \mathbb{C} : |z| = 1\}$. The function σ is called the *multiplier* of ρ. It is a Borel function which satisfies $\sigma(x, e) = 1 = \sigma(e, x)$ and $\sigma(xy, z)\sigma(x, y) = \sigma(x, yz)\sigma(y, z)$ for all x, y, z in G.

Let N be a normal subgroup of G. There is a natural right action of G on $\hat{N}$ given by $\pi \cdot g(n) = \pi(gng^{-1})$. Set $G^\pi = \{g \in G \,|\, \pi \cdot g = \pi\}$. Then G^π is a closed subgroup of G called the *stabilizer* of π. Since π is a unitary representation of N, $\pi \cdot n = \pi$ for each n in N. Thus $N \subset G^{\pi}$. The space of G orbits in $\hat{N}$ will be denoted by $\hat{N}/G$.

Proposition 1.9. ([2], Lemma 2.1.3) *Let π be an irreducible unitary representation of N. Then for every g in G, the representations* $\mathrm{ind}_N^G(\pi \cdot g)$ *and* $\mathrm{ind}_N^G(\pi)$ *are equivalent.*

Theorem 1.10. ([5], Theorem V.9) *Suppose π is an irreducible unitary representation of N such that for each g in G the representation $\pi \cdot g$ is equivalent to π. Then π extends to a multiplier representation $\tilde{\pi}$ of G such that $\tilde{\pi}(g)\tilde{\pi}(n)\tilde{\pi}(g^{-1}) = \pi(gng^{-1})$ for every n in N and g in G. Moreover, the multiplier for $\tilde{\pi}$ may be chosen to be constant on cosets.*

In particular, an irreducible unitary representation will always extend to a multiplier representation of its stabilizer.

In general, induced representations are not irreducible. There is a procedure, known as the Mackey machine, for determining irreducible representations of G in terms of the multiplier representations of a closed normal subgroup N and the multiplier representations of various subgroups of G/N. For a large class of groups, it provides a complete list of the irreducible representations up to equivalence. The procedure is as follows.

Let π be an irreducible representation of N and $\tilde{\pi}$ be its extension to a multiplier representation of the stabilizer $G^{[\pi]}$. If σ is the multiplier of $\tilde{\pi}$ and ρ is an irreducible $\overline{\sigma}$-multiplier representation of $G^{[\pi]}/N$, then $\tilde{\pi} \times \rho(h) = \tilde{\pi}(h) \otimes \rho(Nh)$ defines an irreducible representation of $G^{[\pi]}$. Thus $\mathrm{ind}_{G^{[\pi]}}^G(\tilde{\pi} \times \rho)$ is a representation of G and, as it turns out, is irreducible (see [5], Theorem V.15). The following celebrated theorem (due to G.W. Mackey) implies that this procedure, in many situations, determines all of the irreducible unitary representations of G up to unitary equivalence.

Theorem 1.11. ([5], Theorem V.16) *Suppose $\hat{N}$ is a standard Borel space and the orbit space $\hat{N}/G$ is countably separated. Then if γ is an irreducible unitary representation of G, there exists a unique orbit $[\pi]\cdot G$ in $\hat{N}/G$ and an irreducible $\overline{\sigma}$-multiplier representation ρ of $G^{[\pi]}/N$ (which is unique up to equivalence) such that γ is equivalent to $ind_{G^{[\pi]}}^{G}(\tilde{\pi}\times\rho)$ where $\tilde{\pi}$ is an extension of π to a σ-multiplier representation of $G^{[\pi]}$.*

The next theorem is a measure theoretic result which is closely related to Fubini's theorem. It allows us to decompose a σ-finite measure into an integral of σ-finite measures.

Theorem 1.12. ([5], Theorem I.27) *Let ϕ be a Borel mapping from a standard Borel space X into a standard Borel space Y. Suppose μ is a σ-finite measure on X and ν is a σ-finite measure on Y that is equivalent to $\phi_*\mu$. Then there exists a map $y\mapsto\mu_y$ from Y into the set of σ-finite measures on X such that*

(1) *$y\mapsto\mu_y(E)$ is Borel for each Borel set E*

(2) *$\mu(E)=\int\mu_y(E)d\nu(y)$ for each Borel set E*

(3) *the fuction $y\mapsto\mu_y$ is unique for almost every y and*

(4) *$\mu_y(X-\phi^{-1}(y))=0$ for almost every y.*

This decomposition is called a *disintegration* of μ over the fibers of ϕ and is denoted by $\mu=\int\mu_y d\nu$.

The final theorem of this section is also due to G. W. Mackey. It gives the restriction of an induced representation as a direct integral of induced representations.

Theorem 1.13 (Mackey's Subgroup Theorem). ([5], Theorem V.7) *Suppose H and K are closed subgroups of G, π is unitary representations of H, $H\backslash G/K$ is a standard Borel space, and μ is a quasi-invariant measure on $H\backslash G$. Let p be the natural projection of $H\backslash G$ onto $H\backslash G/K$ and let ν be a σ-finite measure on $H\backslash G/K$ equivalent to $p_*\mu$. If the disintegration of μ is $\int_{H\backslash G/K} \mu_y d\nu(y)$, where each μ_y is a K quasi-invariant σ-finite measure on $p^{-1}(y)$, then $ind_H^G(\pi)\,|_K \cong \int_{H\backslash G/K}^{\oplus} ind_{g^{-1}Hg\cap K}^K(\pi\cdot g)d\nu(HgK)$ where $\pi \cdot g$ is the representation on $g^{-1}Hg \cap K$ defined by $\pi \cdot g(x) = \pi(gxg^{-1})$.*

1.4 The Plancherel Theorem

Theorem 1.14 (The Plancherel Theorem). ([5], Theorem VII.23) *Let G be a second countable, unimodular, locally compact, Hausdorff, type I group with Haar measure m. Then there is a unique σ-finite measure μ on $\hat{G}$ such that*

$$\int_G |f(x)|^2 dm(x) = \int_{\hat{G}} Tr(\pi(f)\pi(f)^*)d\mu(\pi)$$

for f in $L^1(G) \cap L^2(G)$. Moreover, if R is the right regular representation of G on $L^2(G)$ and $[\pi] \mapsto \pi$ is a Borel section of $\hat{G}$ then there is a unitary isomorphism U of $L^2(G)$ onto $\int^{\oplus} n(\pi)\mathcal{H}_\pi \, d\mu([\pi])$, where $\mathcal{H}_\pi$ is the Hilbert space of π and $n(\pi)$ is its dimension, such that $URU^* = \int^{\oplus} n(\pi)\pi \, d\mu([\pi])$.

Let G act on a measurable space X. A measure m on X is called relatively invariant with respect to the action of G if for every g there is a positive number $\Delta(g)$ such that $m(g \cdot E) = \Delta(g)m(E)$ for any measurable subset E of X. For example, if $G = \mathrm{GL}(X)$ where X is a vector space then Lebesgue measure, m, is relatively invariant. Indeed, $m(A \cdot E) = |A|\, m(E)$ for all A in G and E measurable. The map $g \mapsto \Delta(g)$ is a homomorphism from G to $\mathbb{R}^+$ (see [6]). Thus $\Delta(K) = \{1\}$ for any compact subgroup K of G. When $G = \mathrm{GL}(X)$, to determine Δ it suffices to determine Δ on diagonal matrices since $G = UDU$ where U is the unitary matrices (a compact subgroup) in G and D is the diagonal matrices in G.

For A an automorphism of G let $A \cdot \pi(x) := \pi(A^{-1}x)$. Then, if $dm(Ax) = \Delta(A)\, dm(x)$, the Plancherel theorem implies that

$$\begin{aligned}
\Delta(A) \int Tr(\pi(f)\pi(f)^*) d\mu(\pi) &= \Delta(A) \int |f(x)|^2 dm(x) \\
&= \int |f(x)|^2 \Delta(A)\, dm(x) \\
&= \int |f(x)|^2 dm(Ax) \\
&= \int |f(A^{-1}x)|^2 dm(x) \\
&= \int |f \cdot A^{-1}(x)|^2 dm(x) \\
&= \int Tr(\pi(f \cdot A^{-1})\pi(f \cdot A^{-1})^*) d\mu(\pi)
\end{aligned}$$

Since

$$\begin{aligned}
\pi(f \cdot A^{-1})(x) &= \int \pi(x) f(A^{-1}x) dm(x) \\
&= \int A^{-1} \cdot \pi(x) f(x) dm(Ax) \\
&= \Delta(A)\, A^{-1} \cdot \pi(f)(x),
\end{aligned}$$

it follows that

$$\begin{aligned}
\int Tr(\pi(f)\pi(f)^*)\Delta(A) d\mu(\pi) &= \int Tr(\Delta(A)^2\, A^{-1} \cdot \pi(f)\, A^{-1} \cdot \pi(f)^*)\, d\mu(\pi) \\
&= \int Tr(\pi(f)\pi(f)^*)\Delta(A)^2 d\mu(A \cdot \pi).
\end{aligned}$$

Thus $\Delta(A) d\mu(\pi) = \Delta(A)^2 d\mu(A \cdot \pi)$ or $d\mu(A \cdot \pi) = \Delta(A)^{-1} d\mu(\pi)$. Thus the following result has been shown.

Proposition 1.15. *If the Haar measure on G is relatively invariant with respect to $Aut(G)$ with relative invariance factor Δ, then Plancherel measure on $\hat{G}$ is relatively invariant with respect to $Aut(G)$ with relative invariance factor Δ^{-1}.*

Let R and R_N be the right regular representations of G and N, respectively, and let μ and η be the Plancherel measures on $\hat{G}$ and $\hat{N}$, respectively. Then Example 1.5, Induction in Stages, and the Plancherel theorem imply that

$$\begin{aligned}\int^{\oplus} n(\gamma)\gamma\, d\mu([\gamma]) \simeq R \simeq \mathrm{ind}_e^G(1) \simeq \mathrm{ind}_N^G(\mathrm{ind}_e^N(1)) \\ \simeq \mathrm{ind}_N^G(R_N) \simeq \mathrm{ind}_N^G \int^{\oplus} m(\pi)\pi\, d\eta([\pi])\end{aligned}$$

where $n(\gamma)$ and $m(\pi)$ are the dimensions of γ and π, respectively. Thus restricting to N yields

$$\int^{\oplus} n(\gamma)\gamma\, d\mu([\gamma])|_N \simeq \mathrm{ind}_N^G \int^{\oplus} m(\pi)\pi\, d\eta([\pi])|_N$$

which, by Proposition 1.2, may be written as

$$\int^{\oplus} n(\gamma)\gamma\,|_N\, d\mu([\gamma]) \simeq \int^{\oplus} m(\pi)\mathrm{ind}_N^G\pi\,|_N\, d\eta([\pi]) \tag{1.3}$$

Each side of (1.3) is a unitary representation of N and, as such, can be expressed as a direct integral over $\hat{N}$. Doing so will yield a useful relationship between μ and η. The left side will be considered first.

Let ϕ be a Borel mapping from $\hat{G}$ to $\hat{N}/G$ which maps an equivalence class to the unique orbit to which it corresponds via Theorem 1.11. Suppose ν is a σ-finite measure on $\hat{N}/G$ that is equivalent to $\phi_*\mu$. Then μ disintegrates over $\hat{N}/G$ as $\int \mu_O\, d\nu(O)$ and the left side of (1.3) can be written as

$$\int_{\hat{N}/G}^{\oplus}\int_{\phi^{-1}(O)}^{\oplus} n(\gamma)\gamma|_N d\mu_O(\gamma)\, d\nu(O). \tag{1.4}$$

In a measurable way, choose for each orbit $O \in \hat{N}/G$ a representation $\pi_O \in O$. Then Theorem 1.11 implies that for every $\gamma \in \phi^{-1}(O)$ there is a multiplier representation ρ_γ of $G^{[\pi_O]}/N$ such that γ is equivalent to $\mathrm{ind}_{G^{[\pi_O]}}^G(\tilde{\pi}_O \times \rho_\gamma)$. Thus $\gamma|_N \simeq ind_{G^{[\pi_O]}}^G(\tilde{\pi}_O \times \rho_\gamma)|_N$ which, by Mackey's Subgroup Theorem, is equivalent to

$\int^{\oplus}_{G^{[\pi_O]}\backslash G} d(\rho_\gamma)(\pi_O \cdot g) d\xi_O(G^{[\pi_O]}g)$ where $d(\rho_\gamma)$ is the dimension of ρ_γ, $(\pi_O \cdot g)(x) = \pi_\gamma(gxg^{-1})$, and ξ_O is a σ-finite quasi-invariant measure on $G^{[\pi_O]}\backslash G$. Thus

$$\int^{\oplus}_{\phi^{-1}(O)} n(\gamma)\gamma|_N \, d\mu_O([\gamma]) \simeq \int^{\oplus}_{\phi^{-1}(O)} n(\gamma) \int^{\oplus}_{G^{[\pi_O]}\backslash G} d(\rho_\gamma)(\pi_O \cdot g) \, d\xi_O(G^{[\pi_O]}g) \, d\mu_O([\gamma]).$$

Since $d(\rho_\gamma)$ is constant on $G^{[\pi_O]}\backslash G$, Proposition 1.2 says this is equivalent to

$$\int^{\oplus}_{\phi^{-1}(O)} n(\gamma)d(\rho_\gamma) \int^{\oplus}_{G^{[\pi_O]}\backslash G} (\pi_O \cdot g) \, d\xi_O(G^{[\pi_O]}g) \, d\mu_O([\gamma]).$$

Since $\int^{\oplus}_{G^{[\pi_O]}\backslash G} (\pi_O \cdot g) \, d\xi_O(G^{[\pi_O]}g)$ is constant for a fixed orbit, Proposition 1.3 says that there is an $n_O \in \{\infty, 1, 2, ...\}$ such that

$$\int^{\oplus}_{\phi^{-1}(O)} n(\gamma)d(\rho_\gamma) \int^{\oplus}_{G^{[\pi_O]}\backslash G} (\pi_O \cdot g) d\xi_O(G^{[\pi_O]}g) d\mu_O([\gamma]) \simeq n_O \int^{\oplus}_{G^{[\pi_O]}\backslash G} (\pi_O \cdot g) d\xi_O(G^{[\pi_O]}g).$$

Thus the left side of (1.3) is equivalent to

$$\int^{\oplus}_{\hat{N}/G} n_O \int^{\oplus}_{G^{[\pi_O]}\backslash G} (\pi_O \cdot g) d\xi_O(G^{[\pi_O]}g) d\nu(O)$$

which, in turn, is equivalent to

$$\int^{\oplus}_{\hat{N}/G} \int^{\oplus}_{G^{[\pi_O]}\backslash G} n_O(\pi_O \cdot g) d\xi_O(G^{[\pi_O]}g) d\nu(O).$$

Since the orbit O is isomorphic to $G^{[\pi_O]}\backslash G$,

$$\begin{aligned} \int_{\hat{N}/G} \int_{G^{[\pi_O]}\backslash G} n_O(g \cdot \pi_O) d\xi_O(G^{[\pi_O]}g) d\nu(O) &\simeq \int_{\hat{N}/G} \int_O n_O \pi \, d\xi_O(\pi) d\nu(O) \\ &\simeq \int_{\hat{N}} n_\pi \pi \, d\chi(\pi) \end{aligned}$$

where $n_\pi = n_{G \cdot [\pi]}$ and $\chi = \int_{\hat{N}/G} \xi_O d\nu(O)$.

Now consider the right side of (1.3): $\int^{\oplus} m(\pi) \operatorname{ind}_N^G(\pi)|_N d\eta([\pi])$. Let p be the canonical projection from $\hat{N}$ onto $\hat{N}/G$ and ς be a σ-finite measure on $\hat{N}/G$ that is equivalent to $p_*\eta$. Then η disintegrates over $\hat{N}/G$ as $\int \eta_O \, d\varsigma(O)$ and the right side of (1.3) can be written as $\int^{\oplus}_{\hat{N}/G} \int^{\oplus}_{p^{-1}(O)} m(\pi) \operatorname{ind}_N^G(\pi)|_N d\eta_O(\pi) \, d\varsigma(O)$. Note: $p^{-1}(O) = O$. Mackey's Subgroup Theorem says that $\operatorname{ind}_N^G \pi|_N \simeq \int_{N\backslash G} (\pi \cdot g) d\zeta(Ng)$

where ζ is a σ-finite (left) Haar measure on $N\backslash G$. If $\pi \in O$ then there is a coset Ng' such that $\pi_O \cdot g' = \pi$ and $m(\pi) = m(\pi_O)$. Thus for $\pi \in O$,

$$\begin{aligned} m(\pi) \int_{N\backslash G}^{\oplus} (\pi \cdot g) d\zeta(Ng) &\simeq m(\pi_O) \int_{N\backslash G}^{\oplus} (\pi_O \cdot g'g) d\zeta(Ng) \\ &\simeq m(\pi_O) \int_{N\backslash G}^{\oplus} (\pi_O \cdot g) d\zeta(N(g')^{-1}g) \\ &\simeq m(\pi_O) \int_{N\backslash G}^{\oplus} (\pi_O \cdot g) d\zeta(Ng). \end{aligned}$$

Thus

$$\int_O^{\oplus} m(\pi) \ \mathrm{ind}_N^G(\pi)|_N d\eta_O(\pi) \simeq \int_O^{\oplus} m(\pi_O) \int_{N\backslash G}^{\oplus} (\pi_O \cdot g) d\zeta(Ng) d\eta_O(\pi)$$

which, by Proposition 1.3, is equivalent to $m_O \int_{N\backslash G} (\pi_O \cdot g) d\zeta(Ng)$ for some $m_O \in \{\infty, 1, 2, ...\}$.

Let q be the canonical projection from $N\backslash G$ onto $G^{[\pi_O]}\backslash G$. Since ζ is Haar measure on $N\backslash G$, $q_*\zeta$ is quasi-invariant on $G^{[\pi_O]}\backslash G$. Since ζ is σ-finite there exists a finite measure $\tilde{\zeta}$ on $N\backslash G$ that is equivalent to ζ. This implies that $q_*\tilde{\zeta} \sim q_*\zeta$. Thus $q_*\tilde{\zeta}$ is quasi-invariant. Since $\tilde{\zeta}$ is finite, $q_*\tilde{\zeta}$ is finite and, therefore, σ-finite. Since all σ-finite, quasi-invariant measures on $G^{[\pi_O]}\backslash G$ are equivalent, $q_*\tilde{\zeta} \sim \xi_O$ and so $q_*\zeta \sim \xi_O$.

Thus ζ disintegrates over $G^{[\pi_O]}\backslash G$ as $\int \zeta_{G^{[\pi_O]}g} d\xi_O$. Then

$$\int_{N\backslash G}^{\oplus} (\pi_O \cdot g) d\zeta(Ng) \simeq \int_{G^{[\pi_O]}\backslash G}^{\oplus} \int_{q^{-1}(G^{[\pi_O]}g)}^{\oplus} (\pi_O \cdot g') d\zeta_{G^{[\pi_O]}g}(Ng') d\xi_O(G^{[\pi_O]}g).$$

Since $Ng' \in G^{[\pi_O]}g$ implies that $\pi \cdot g' \simeq \pi \cdot g$,

$$\int_{q^{-1}(G^{[\pi_O]}g)}^{\oplus} (\pi_O \cdot g') d\zeta_{G^{[\pi_O]}g}(Ng') \simeq \int_{q^{-1}(G^{[\pi_O]}g)}^{\oplus} (\pi_O \cdot g) d\zeta_{G^{[\pi_O]}g}(Ng')$$

which, by Proposition 1.3, is equivalent to $k_{G^{[\pi_O]}g}(\pi \cdot g)$ for some $k_{G^{[\pi_O]}g} \in \{\infty, 1, 2, ...\}$.

Thus the right side of (1.3) is equivalent to

$$\int_{\hat{N}/G}^{\oplus} m_O \int_{G^{[\pi_O]}\backslash G}^{\oplus} k_{G^{[\pi_O]}g} (\pi_O \cdot g) d\xi_O(G^{[\pi_O]}g) d\varsigma(O)$$

which, in turn, is equivalent to

$$\int_{\hat{N}/G}^{\oplus} \int_{G^{[\pi_O]}\backslash G}^{\oplus} m_O k_{G^{[\pi_O]}g}(\pi_O \cdot g) d\xi_O(G^{[\pi_O]}g) d\varsigma(O).$$

Since the orbit O is isomorphic to $G^{[\pi_O]}\backslash G$, this may be written as

$$\int_{\hat{N}/G}^{\oplus} \int_{O}^{\oplus} m_O k_\pi \pi d\xi_O(\pi) d\varsigma(O)$$

where $\pi = \pi_O \cdot g$ and $k_\pi = k_{G^{[\pi_O]}g}$. Taking $m_\pi = m_{[\pi]\cdot G} k_\pi$, and $\tilde{\chi} = \int_{\hat{N}/G} \xi_O d\varsigma(O)$, this may be expressed as

$$\int_{\hat{N}}^{\oplus} m_\pi \pi d\tilde{\chi}(\pi).$$

Thus (1.3) implies that

$$\int_{\hat{N}}^{\oplus} n_\pi \pi d\chi(\pi) \simeq \int_{\hat{N}}^{\oplus} m_\pi \pi d\tilde{\chi}(\pi).$$

By the uniqueness of the direct integral decomposition, this implies that $n_\pi = m_\pi$ for almost every π and $\chi \sim \tilde{\chi}$. Thus $\chi(p^{-1}(E)) = 0$ is equivalent to $\tilde{\chi}(p^{-1}(E)) = 0$ for any Borel subset E of $\hat{N}/G$. Since $\phi_*\mu \sim \nu \sim p_*\chi$ and $p_*\tilde{\chi} \sim \varsigma \sim p_*\eta$, the following result has been shown.

Theorem 1.16. *Suppose G is a locally compact, unimodular, type I group and N is a normal subgroup. Let μ and η denote the Plancherel measures of $\hat{G}$ and $\hat{N}$, respectively. Let $\phi : \hat{G} \to \hat{N}/G$ be defined by $\phi[ind(\pi \times \rho)] = [\pi] \cdot G$ and let $p : \hat{N} \to \hat{N}/G$ be the canonical projection. Then $\phi_*\mu$ and $p_*\eta$ are equivalent measures on the orbit space $\hat{N}/G$.*

Corollary 1.17. *Suppose S is a G-invariant null subset of $\hat{N}$. Then the set of irreducible representations of G induced by the representations of N in S has Plancherel measure zero, i.e., $\mu(\phi^{-1}(p(S))) = 0$.*

Proof. The G-invariance of S implies that $S = p^{-1}(p(S))$. Thus $0 = \eta(S) = \eta(p^{-1}(p(S))) = p_*\eta(p(S))$ implies $0 = \phi_*\mu(p(S)) = \mu(\phi^{-1}(p(S)))$. $\square$

Theorem 1.18. *Let G be a type I, unimodular, locally compact group and $\mathcal{A}$ be a subgroup of* $\mathrm{Aut}(G)$. *Suppose that $\hat{G}_0$ is a conull, $\mathcal{A}$-invariant subset of $\hat{G}$ and the orbit space $\hat{G}_0/\mathcal{A}$ is standard. Then the Plancherel measure μ decomposes over the orbits as $\mu = \int \mu_O d\nu(O)$ where μ_O is a relatively invariant measure on the homogeneous space $\mathcal{A}/\mathcal{A}^{\pi_O}$.*

Proof. Since G is type I, $\hat{G}$ is a standard Borel space. Let $\phi : \hat{G} \to \hat{G}/\mathcal{A}$ be the projection $[\pi] \mapsto \mathcal{A} \cdot [\pi]$. Then $\phi|_{\hat{G}_0}$ is a Borel map between standard spaces $\hat{G}_0$ and $\hat{G}_0/\mathcal{A}$. If $\tilde{\mu}$ is any finite measure equivalent to μ then $\phi_*\tilde{\mu}$ is a σ-finite measure on $\hat{G}_0/\mathcal{A}$ equivalent to $\phi_*\mu$. By Theorem 1.12, μ decomposes over the fibers of ϕ as $\mu = \int \mu_O d\nu(O)$ where $\nu = \phi_*\tilde{\mu}$ and $\mu_O(\hat{G}_0 - \phi^{-1}(O)) = 0$. Each orbit O is isomorphic to the homogeneous space $\mathcal{A}/\mathcal{A}^{\pi_O}$ where π_O is any representation on O. Since μ is relatively invariant, $\int \mu_O(A \cdot E)d\nu(O) = \mu(A \cdot E) = \Delta^{-1}(A)\mu(E) = \Delta^{-1}(A)\int \mu_O(E)d\nu(O) = \int \Delta^{-1}(A)\mu_O(E)d\nu(O)$ for $A \in \mathcal{A}$ and E Borel. Thus for a.e. $O \in \hat{G}_0/\mathcal{A}$, $\mu_O(A \cdot E) = \Delta^{-1}\mu_O(E)$ for $A \in \mathcal{A}$ and E Borel. $\square$

Free Nilpotent Lie Groups

2.1 Definitions and Notation

Let V be an n dimensional real vector space and $T(\mathrm{V})$ be the tensor algebra (i.e., the free associative algebra) over V. The tensor algebra may be turned into a Lie algebra by defining a Lie bracket $[X,Y] = XY - YX$. The free Lie algebra over V is then the Lie subalgebra of $T(\mathrm{V})$ generated by V. It will be denoted $\mathfrak{f}_n$. Define $\mathrm{V}^{(1)} = \mathrm{V}$ and $\mathrm{V}^{(i+1)} = [\mathrm{V}, \mathrm{V}^{(i)}]$. Then $\mathfrak{f}_n = \bigoplus_i \mathrm{V}^{(i)}$. The free k-step nilpotent Lie algebra on n generators is defined to be $\mathfrak{f}_{n,k} = \mathfrak{f}_n / \bigoplus_{i>k} \mathrm{V}^{(i)}$.

The free k-step nilpotent Lie group on n generators is then the simply connected Lie group with Lie algebra $\mathfrak{g} = \mathfrak{f}_{n,k}$. It will be denoted by G. Since the exponential map is a diffeomorphism for simply connected nilpotents, the group elements may be identified with their Lie algebra coordinates, i.e, the exponential map may be taken to be the identity. Multiplication in G is then given by the Campbell-Baker-Hausdorff formula (CBH): $X * Y = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \ldots$

Let $g = (v_1, v_2, \ldots, v_k)$ where $v_i \in \mathrm{V}^{(i)}$. Then CBH shows that $dm(g) = dv_1\, dv_2 \ldots dv_k$ where dv_i is a Lebesgue measure on $\mathrm{V}^{(i)}$ is a Haar measure on G.

2.2 The Automorphism Group

If ϕ is any homomorphism on G then there exists a linear map $d\phi$ on $\mathfrak{g}$ such that $\phi(\exp(X)) = \exp(d\phi(X)$. Since the exponential map is taken to be the identity, $\phi(X) = d\phi(X)$. Thus if A is an automorphism of G, $A = dA$ and, therefore, A is linear. The criterion

$$A(gg') = AgAg' \tag{2.5}$$

is equivalent to

$$A(tX * tY) = A(tX) * A(tY)$$

for all $t > 0$. Using CBH and the linearity of A, this equation may be written as

$$tA(X) + tA(Y) + t^2 A[X,Y] + O(t^3) = tA(X) + tA(Y) + t^2[AX, AY] + O(t^3)$$

or $A[X,Y] = [AX, AY] + \frac{O(t^3)}{t^2}$. Taking the limit as t goes to zero implies

$$A[X,Y] = [AX, AY]. \tag{2.6}$$

Clearly, (2.6), CBH, and linearity imply (2.5). Thus the following has been shown.

Lemma 2.19. *A is an automorphism of G if and only if A is linear, invertible, and satisfies $A[X,Y] = [AX, AY]$ for all X,Y in $\mathfrak{g}$.*

Lemma 2.20. *Let $l : \mathrm{V} \to \mathfrak{g}$ be linear. Then l extends uniquely to a Lie algebra homomorphism $L : \mathfrak{g} \to \mathfrak{g}$.*

Proof. By the Birkhoff Embedding Theorem ([2], Theorem 1.1.11), $\mathfrak{g}$ can be imbedded into $\mathfrak{gl}(n)$ for n sufficiently large. Thus l can be considered to be a map from V into $\mathfrak{gl}(n)$. Since l is linear and $\mathfrak{gl}(n)$ is an associative algebra, there exists a unique algebra homomorphism $\tilde{l} : T(\mathrm{V}) \to \mathfrak{gl}(n)$ that extends l. Turn both algebras into Lie algebras by associating to each the standard Lie bracket $[X,Y] = XY - YX$. Then $\tilde{l}$ is a Lie algebra homomorphism. Since V maps into $\mathfrak{g}$, the Lie algebra generated by V must map into $\mathfrak{g}$. That is, $\mathfrak{f}_n$ maps into $\mathfrak{g}$. Since $\tilde{l}$ is an algebra homomorphism, tensors of degree i map to tensors of degree i. Thus tensors of degree higher than k map to 0 in $\mathfrak{g}$. So $\tilde{l}$ factors to a unique Lie algebra homomorphism $L : \mathfrak{g} \to \mathfrak{g}$. □

Proposition 2.21. *There is a one-to-one correspondence between the automorphisms of G and the k-tuples $(l_1, ..., l_k)$ where $l_1 \in \mathrm{GL(V)}$ and for $i = 2, ..., k$ the maps $l_i : \mathrm{V} \to \mathrm{V}^{(i)}$ are linear.*

Proof. Suppose $A \in \operatorname{Aut}(G)$. By Lemma 2.18, A is linear, invertible, and satisfies (2.6). Write $A = (A_1, ..., A_k)$ where $A_i : \mathfrak{g} \to \mathrm{V}^{(i)}$. Set $l_i = A_i|_{\mathrm{V}}$. The linearity of A implies that each A_i (and, therefore, each l_i) is linear. The criterion $A[X, Y] = [AX, AY]$ is equivalent to

$$A_i[X, Y] = [AX, AY]_i = \sum_{\alpha+\beta=i} [A_\alpha X, A_\beta Y] \tag{2.7}$$

for $i = 1, ..., k$. Since any element in $W = \bigoplus_{i>1} \mathrm{V}^{(i)}$ can be expressed as a sum of elements of the form $[X, Y]$, equation (2.7) implies that $A_i|_W$ is determined by the maps $A_1, ..., A_{i-1}$. Write $A_i = A_i|_{\mathrm{V}} + A_i|_W$. Taking $i = 1$, equation (1.3) reads $A_1[X, Y] = [AX, AY]_1 = 0$. This implies that $A_1|_W = 0$. Thus A_1 is determined by l_1. Since A is surjective, A_1 is surjective. Since $A_1|_W = 0$, $l_1 = A_1|_{\mathrm{V}}$ must be surjective. Since V is finite dimensional, l_1 is invertible. Since l_1 and l_1^{-1} both map V linearly into $\mathrm{V} \subset \mathfrak{g}$, they each extend to Lie algebra homomorphisms ($\tilde{l_1}$ and $\widetilde{l_1^{-1}}$, respectively). Since $\tilde{l_1} \cdot \widetilde{l_1^{-1}}(v) = l_1 \cdot l_1^{-1}(v) = v$ for $v \in \mathrm{V}$, $\tilde{l_1} \cdot \widetilde{l_1^{-1}}$ is the identity on V. Since V generates and $\tilde{l_1} \cdot \widetilde{l_1^{-1}}$ is a homomorphism, $\tilde{l_1} \cdot \widetilde{l_1^{-1}}$ is the identity on $\mathfrak{g}$. Therefore $\tilde{l_1}$ is invertible. This implies $\tilde{l_1}$ is injective. Thus $\tilde{l_1}|_{\mathrm{V}^{(i)}}$ is injective for $i = 1, ..., k$.

Assume that for each $i < n$ the map A_i is determined by $l_1, ..., l_i$, $A_i|_{\mathrm{V}^{(i)}} = \tilde{l_1}|_{\mathrm{V}^{(i)}}$, and that $A_i|_{\mathrm{V}^{(i+1)} \oplus ... \oplus V^{(k)}} = 0$. Then $A_n = A_n|_{\mathrm{V}} + A_n|_W$ implies that A_n is determined by $l_1, ..., l_n$. Let $[X, Y] \in \mathrm{V}^{(n)}$. Then $X \in \mathrm{V}^{(i)}$ and $Y \in \mathrm{V}^{(j)}$ for some $i, j < n$ such that $i + j = n$. Thus $A[X, Y] = [A_i X, A_j Y] = [\tilde{l_1}|_{\mathrm{V}^{(i)}} X, \tilde{l_1}|_{\mathrm{V}^{(j)}} Y] = [\tilde{l_1} X, \tilde{l_1} Y] = \tilde{l_1}[X, Y] = \tilde{l_1}|_{\mathrm{V}^{(n)}}[X, Y]$. Since any element of $\mathrm{V}^{(n)}$ can be written as a sum of such brackets and A_n is linear, $A_n|_{\mathrm{V}^{(n)}} = \tilde{l_1}|_{\mathrm{V}^{(n)}}$. Let $[X, Y] \in \mathrm{V}^{(n+1)} \oplus ... \oplus \mathrm{V}^{(k)}$ and let x_i and y_j be the first non-zero components of X and Y, respectively. Then $i + j > n$. By the inductive hypothesis, $A_\alpha X = 0$ if $\alpha < i$ and $A_\beta Y = 0$ if $\beta < j$. Let $\alpha + \beta = n$. Then $\alpha \geq i$ implies $\beta < (\alpha - i) + \beta < j$ and $\beta \geq j$ implies

$\alpha < \alpha + (\beta - j) < i$. Thus $A[X,Y] = \sum_{\alpha+\beta=n}[A_\alpha X, A_\beta Y] = 0$. Since any element of $\mathrm{V}^{(n+1)} \oplus ... \oplus \mathrm{V}^{(k)}$ can be written as a sum of such brackets and A_n is linear, $A_n|_{\mathrm{V}^{(n+1)}\oplus...\oplus\mathrm{V}^{(k)}} = 0$. By induction, A_i is determined by l_1,...,l_i, $A_i|_{\mathrm{V}^{(i)}} = \tilde{l}_1|_{\mathrm{V}^{(i)}}$, and $A_i|_{V_{i+1}\oplus...\oplus V_k} = 0$ for $i = 1, ..., k$. So A is determined by $(l_1, ..., l_k)$.

Now suppose the k-tuple $(l_1, ..., l_k)$ is as in the statement of the proposition. Then $l : v \mapsto (l_1 v, ..., l_k v)$ is linear from V into $\mathfrak{g}$. By Lemma 2.18, l extends to a Lie algebra homomorphism $A : \mathfrak{g} \to \mathfrak{g}$. Clearly A is linear and satisfies (2.7). If A is invertible then, by Lemma 2.19, it is an automorphism. Since G is finite dimensional, it suffices to show that A is injective.

Suppose $AX = 0$. Then $0 = A_1 X = l_1(x_1) = \tilde{l}_1|_{\mathrm{V}} X$. Since l_1 is invertible, $x_1 = 0$. Assume that $x_i = 0$ for $i < n$. Then $0 = A_n X = A_n|_{\mathrm{V}^{(n)}} X + A_n|_{\mathrm{V}^{(n+1)}\oplus...\oplus\mathrm{V}^{(k)}} X = \tilde{l}_1|_{\mathrm{V}^{(n)}} x_n$. Since $\tilde{l}_1|_{\mathrm{V}^{(n)}}$ is injective, $x_n = 0$. By induction, $x_i = 0$ for $i = 1, ..., k$. Thus $X = 0$ and A is injective. □

Thus GL(V) may be considered as a subgroup of Aut(G). We now consider the action of this subgroup on $\hat{G}$.

Suppose $A \in \mathrm{GL(V)}$, N is a normal subgroup of G, $\pi \in \hat{N}$, and ρ is an irreducible multiplier representation of G^π/N. Suppose $\gamma_{\tilde{\pi}\times\rho} = \mathrm{ind}_{G^\pi}^{G}(\tilde{\pi}\times\rho)$. Then $\gamma_{\tilde{\pi}\times\rho}$ acts on the Hilbert space $\mathcal{H}_{\tilde{\pi}\times\rho}$ by right translation, i.e., $\gamma_{\tilde{\pi}\times\rho}(g)h(g') = h(g'g)$. If $A \cdot N \subseteq N$ and $A \cdot G^\pi \subseteq G^\pi$ (as is the case if A is diagonal), then $(A \cdot \tilde{\pi} \times \rho)(x) = \tilde{\pi} \times \rho(A^{-1} \cdot x)$ is well-defined. Let $h \in \mathcal{H}_{\tilde{\pi}\times\rho}$. Since

$$\begin{aligned} h \circ A^{-1}(ng) &= h(A^{-1} \cdot ng) \\ &= h(A^{-1} \cdot n\, A^{-1} \cdot g) \\ &= \tilde{\pi} \times \rho(A^{-1} \cdot n) h(A^{-1} \cdot g) \\ &= (A \cdot \tilde{\pi} \times \rho)(n)(h \circ A^{-1})(g) \end{aligned}$$

and $\int ||h \circ A^{-1}(g)||^2 d\mu(G^\pi g) = \int ||h(g)||^2 d\mu(G^\pi A \cdot g) = \int ||h(g)||^2 \Delta(A) d\mu(G^\pi g)$, $h \circ A^{-1} \in \mathcal{H}_{A \cdot \tilde{\pi} \times \rho}$. Since

$$\begin{aligned}
A \cdot \gamma_{\tilde{\pi} \times \rho}(g)(h \circ A^{-1})(g') &= \gamma_{\tilde{\pi} \times \rho}(A^{-1} \cdot g) h(A^{-1} \cdot g') \\
&= h(A^{-1} \cdot g' \, A^{-1} \cdot g) \\
&= h(A^{-1} \cdot g' g) \\
&= h \circ A^{-1}(g' g) \\
&= \gamma_{A \cdot \tilde{\pi} \times \rho}(g)(h \circ A^{-1})(g'),
\end{aligned}$$

we see that $A \cdot \gamma_{\tilde{\pi} \times \rho} = \gamma_{A \cdot \tilde{\pi} \times \rho}$.

Free Two Step Nilpotent Lie Groups

3.1 Definitions and Notation

Let V be an n dimensional real vector space. The free 2-step nilpotent Lie group G is the simply connected Lie group with Lie algebra $\mathfrak{g} = \mathfrak{f}_{n,2}$. Multiplication in G expressed in terms of exponential coordinates is $(v, w) \cdot (v', w') = (v + v', w + w' + \frac{1}{2} v \wedge v')$. Let $N = \exp(\mathrm{V} \wedge \mathrm{V})$. Then N is a normal, central, abelian subgroup of G of dimension $k = \frac{n(n-1)}{2}$. Since N is abelian, $\hat{N} \cong (\mathrm{V} \bigwedge \mathrm{V})^* \cong \mathbb{R}^k$ and Plancherel measure is a multiple of Lebesgue measure. Since N is central, the action of G on $\hat{N}$ is trivial. Thus the stabilizer G^λ is G for all λ in $\hat{N}$ and the orbit space $\hat{N}/G$ is $\hat{N}$.

We consider the action of GL(V) as elements of Aut(G) on G. We use the notation $A \cdot (v, w) = (Av, \tilde{A}w)$ where $\tilde{A}$ is defined by $\tilde{A}(v \wedge v') = Av \wedge Av'$ and linearity. Then the Haar measure $dm(g) = dv\, dw$ is relatively invariant with respect to the action of GL(V) with factor $\Delta(A) = |A||\tilde{A}|$. Since Δ is a homomorphism, we may assume that A is diagonal. Let a_i be the diagonal entries of A. Then $\tilde{A}(e_i \wedge e_j) = a_i a_j (e_i \wedge e_j)$, i.e., $\tilde{A}$ is diagonal. Thus $|\tilde{A}| = \prod_{j=2}^{n} \prod_{i<j} a_i a_j = a_1^{n-1} a_2^{n-1} ... a_n^{n-1} = |A|^{n-1}$. So $\Delta(A) = |A|^n$. By Proposition 1.15, the Plancherel measure on the dual is relatively invariant with factor $\Delta^{-1}(A) = |A|^{-n}$.

3.2 Bilinear Skew-Symmetric Maps

Proposition 3.22. *Let $B : \mathrm{V} \times \mathrm{V} \to \mathbb{R}$ be bilinear and skew-symmetric. Then the rank of B is even and there is a basis $\{x_1, ..., x_m, y_1, ..., y_m, z_1, ...z_l\}$ of* V *such that $B(x_i, y_j) = \delta_{i,j}$, $B(x_i, x_j) = 0 = B(y_i, y_j)$, and $z_i \in rad(B)$.*

Proof. The proof is by a modified Gram-Schmidt process.

Let $\{e_i\}$ be a basis of V. Check the values $B(e_1, e_i)$ for $i = 1, 2, ..., n$. If all values are zero then $B(e_1, v) = 0$ for all $v \in$ V. Thus $e_1 \in$ rad B. Set $z_1 = e_1$ and start over with $\{e_2, ..., e_n\}$. If for some $i > 1$, say i_0, $B(e_1, e_{i_0}) \neq 0$ then set $x_1 = e_1$ and $y_1 = \frac{e_{i_0}}{B(e_1,e_{i_0})}$. Then $B(x_1, y_1) = 1$. By reordering the basis it may be assumed that $i_0 = 2$. For $i = 3, ..., n$ replace e_i by $e'_i = e_i - B(x_1, e_i)y_1 + B(x_1, e_i)x_1$. The set $\{x_1, y_1, e'_3, ..., e'_n\}$ is a basis and $B(x_1, e'_i) = 0 = B(y_1, e'_i)$. Thus $B(x_1, v) = 0 = B(y_1, v)$ for every $v \in \langle e'_3, ..., e'_n\rangle$. Start over with $\{e'_3, ..., e'_n\}$.

Each time, either a $z_i \in$ rad B or a pair $x_i, y_i \notin$ rad B is produced. This process ends since n is finite. The result being a basis consisting of some number, say m, of paired elements x_i, y_i and some number, say l, of elements z_i in the radical. Thus the rank of B is $2m$ and the radical has dimension l. Also $B(x_i, y_j) = 0$ when $i \neq j$ since either y_j or x_i is in the span of the e'_i. □

Corollary 3.23. *Any two skew-symmetric forms are equivalent by a change of basis if and only if their radicals are the same dimension.*

Proof. Let B and B' be two skew-symmetric forms with the same dimension radicals. By Proposition 3.21, there are bases $\{x_1, ..., x_m, y_1, ..., y_m, z_1, ..., z_l\}$ and $\{x'_1, ..., x'_m, y'_1, ..., y'_m, z'_1, ..., z'_l\}$ where the $z_i \in$ rad B and the $z'_i \in$ rad B'. Let A be defined by $Ax_i = x'_i$, $Ay_i = y'_i$, and $Az_i = z'_i$. Then A is a change of basis and $B(v_1, v_2) = B'(Av_1, Av_2)$. Thus B and B' are equivalent. It is clear that B and B' cannot be equivalent without having radicals that are the same dimension. □

If T is a change of basis from the standard basis S to the basis S' in the statement of Proposition 3.21 then the matrix of the bilinear form with respect to the standard basis is

$$[B] = [T]^t \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} [T].$$

Both bases determine Lebesgue measures on V which differ by a positive constant. The constant is the Jacobian of the transition map from the symplectic basis to the standard basis. If B is symplectic then $det[B] = (det[T])^2$. So the constant equals $\sqrt{\det[B]} = \sqrt{\det[B(e_i, e_j)]}$.

Now assume B has a one dimensional radical $\langle z \rangle$. Let W be the span of $\{x_i, y_i\}_{i=1}^m$. Then the restriction $B|_W$ is symplectic and $[B|_W] = A^t \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} A$ where A is the transition matrix from $\{x_i, y_i\}$ to $\{e_1, ..., e_{2m}\}$. Let U be a unitary transformation such that $U(z) \in \langle e_n \rangle$. Then the map with matrix $\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \cdot [U]$ is a change of basis from $\{x_i, y_i, z\}$ to $\{e_i\}$. So the constant equals $\det A \cdot \det[U] = \det A = \sqrt{\det[B|_W]} = \sqrt{\det[B(e_i, e_j)]_{i,j \neq n}}$.

Let B be a symplectic form on V $\times$ V and let $X = \operatorname{span}\{x_i\}$ and $Y = \operatorname{span}\{y_i\}$ where the x_i and y_i are as in Proposition 3.21. Then $y \mapsto B(\cdot, y)$ is a isomorphism between Y and X^*. Since X is a linear space, $X^* \simeq \hat{X}$. Thus $\hat{X}$ may be identified with Y. Therefore the Fourier transform $F : L^1(X) \to L^1(Y)$ may be defined by $(Ff)(y) = \int f(x) e^{-iB(x,y)} dx$ where dx is the normalized Lebesgue on X determined by the x_i. The inversion formula is then $(F^{-1}f)(x) = \int f(y) e^{iB(x,y)} dy$ where dy is the normalized Lebesgue measure on Y determined by the y_i. This Fourier transform extends to linear isometry from $L^2(X)$ to $L^2(Y)$ just as the usual Fourier transform does. Similarly, the isomorphism $x \mapsto B(x, \cdot)$ between X and Y^* allows us to define the Fourier transform $F : L^1(Y) \to L^1(X)$ which extends to a linear isometry from $L^2(Y)$ to $L^2(X)$.

3.3 The Dual

The dual of G can be determined using Mackey induction in the following manner. Each λ in $\hat{N}$ extends to a multiplier representation π_λ of $G^\lambda = G$ defined by

$\pi_\lambda(v,w) = e^{i\lambda(w)}I$. Since

$$\begin{aligned}
\pi_\lambda((v,w)\cdot(v',w')) &= \pi_\lambda(v+v', w+w'+\frac{1}{2}v\wedge v') \\
&= e^{i\lambda(w+w'+\frac{1}{2}v\wedge v')}I \\
&= e^{\frac{i}{2}\lambda(v\wedge v')}e^{i\lambda(w)}e^{i\lambda(w')}I \\
&= e^{\frac{i}{2}\lambda(v\wedge v')}\pi_\lambda(v,w)\pi_\lambda(v',w')
\end{aligned}$$

the multiplier for π_λ is $\sigma_\lambda(v,v') = e^{\frac{-i}{2}\lambda(v\wedge v')}$. Next all irreducible $\overline{\sigma}_\lambda$-multiplier representations of $G^\lambda/N = G/N \cong \exp(\mathrm{V})$ must be determined. Once this is done, any $\pi \in \hat{G}$ is equivalent to $\pi_\lambda \times \rho = \mathrm{ind}_G^G(\pi_\lambda \times \rho)$ for some $\lambda \in \hat{N}$ and some irreducible $\overline{\sigma}_\lambda$-multiplier representation ρ.

For λ in $\hat{N}$ let B_λ be the bilinear skew-symmetric map from $\mathrm{V}\times\mathrm{V}$ to $\mathbb{R}$ defined by $B_\lambda(v,v') = \lambda(v\wedge v')$. Fix λ and let $\{x_1, x_2, ..., x_i, y_1, y_2, ..., y_i, z_1, z_2, ..., z_l\}$ be a basis of V as in Proposition 3.21. Then $\mathrm{V} = X+Y+Z$ where $X = \mathrm{span}\{x_1, x_2, ..., x_i\}$, $Y = \mathrm{span}\{y_1, y_2, ..., y_i\}$, and $Z = \mathrm{span}\{z_1, ..., z_l\}$. For $\nu \in Z^*$, the formula

$$(\pi_{\nu,\lambda}(x+y+z)f)(x') = e^{i\nu(z)}e^{\frac{i}{2}B_\lambda(x+2x',y)}f(x'+x)$$

defines a $\overline{\sigma}_\lambda$-multiplier representation of V on $L^2(X,\mathbb{C})$. Indeed,

$$\begin{aligned}
(\pi_{\nu,\lambda}(v)\pi_{\nu,\lambda}(v')f)(x_0) &= (\pi_{\nu,\lambda}(x+y+z)\pi_{\nu,\lambda}(x'+y'+z')f)(x_0) \\
&= e^{i\nu(z)}e^{\frac{i}{2}B_\lambda(x+2x_0,y)}(\pi_{\nu,\lambda}(x'+y'+z')f)(x_0+x) \\
&= e^{i\nu(z)}e^{i\nu(z')}e^{\frac{i}{2}B_\lambda(x+2x_0,y)}e^{\frac{i}{2}B_\lambda(x'+2x_0+2x,y')}f(x_0+x+x') \\
&= e^{i\nu(z+z')}e^{\frac{i}{2}B_\lambda(x,y')}e^{\frac{-i}{2}B_\lambda(x',y)}e^{\frac{i}{2}B_\lambda(x+x'+2x_0,y+y')}f(x_0+x+x') \\
&= e^{i\nu(z+z')}e^{\frac{i}{2}B_\lambda(x,y')}e^{\frac{i}{2}B_\lambda(x+y+z,x'+y'+z')}f(x_0+x+x') \\
&= \overline{\sigma}_\lambda(x+y+z,x'+y'+z)[\pi_{\nu,\lambda}(x+x'+y+y'+z+z')f](x_0) \\
&= \overline{\sigma}_\lambda(v,v')[\pi_{\nu,\lambda}(v+v')f](x_0).
\end{aligned}$$

The $\pi_{\nu,\lambda}$ are also irreducible since $[\pi_{\nu,\lambda}(x)f](x') = f(x'+x)$, $[\pi_{\nu,\lambda}(y)f](x') = e^{iB_\lambda(x',y)}f(x')$, and there are no proper closed subspaces of $L^2(X)$ invariant under all translation and multiplication operators. These are, as the next proposition shows, the only $\overline{\sigma}_\lambda$-multiplier representations up to unitary equivalence.

Proposition 3.24. *There is a bijection ($\nu \mapsto \pi_{\nu,\lambda}$) between the linear dual of rad(B_λ) and the unitary equivalence classes of irreducible $\overline{\sigma}_\lambda$-multiplier representations of* V*.*

Proof. Let π be any irreducible unitary $\overline{\sigma}_\lambda$-multiplier representation of V. Since $\overline{\sigma}_\lambda(z,\cdot) = 1 = \overline{\sigma}_\lambda(\cdot,z)$ for all $z \in Z$, $\pi|_Z$ is a unitary representation and $\pi(z)$ commutes with $\pi(v)$. Since π is irreducible, $\pi(z)$ must be a multiple of the identity. Thus $\pi(z) = e^{i\nu(z)}I$ for a unique $\nu \in Z^*$. Let $\pi_0 = \pi_{\nu,\lambda}$.

Since $\overline{\sigma}_\lambda(y,y') = 1\ \forall\ y,y' \in Y$, $\pi|_Y$ is a unitary representation of Y. Thus for some finite measure μ and some $c : \hat{Y} \to \{\infty,1,2,...\}$ (which is a.e. unique), $\pi|_Y(y) \simeq \int_{\hat{Y}\simeq X}^{\oplus} c(x)e^{i\lambda(x,y)}d\mu(x)$. (This direct integral acts on $\int_X^{\oplus} c(x)\mathbb{C}d\mu(x)$.) Since

$$\begin{aligned}
\pi|_Y(y) &\simeq \pi(x')\pi|_Y(y)\pi(-x') \\
&= \overline{\sigma}_\lambda(x',y)\overline{\sigma}_\lambda(x'+y,-x')\pi|_Y(y) \\
&= e^{iB_\lambda(x',y)}\int_X^{\oplus} c(x)e^{iB_\lambda(x,y)}d\mu(x) \\
&= \int_X^{\oplus} c(x)e^{iB_\lambda(x+x',y)}d\mu(x) \\
&= \int_X^{\oplus} c(x-x')e^{iB_\lambda(x,y)}d\mu(x-x')
\end{aligned}$$

for any $x' \in X$, the uniqueness of the direct integral decomposition implies $c(x)$ must be constant almost everywhere and $\mu \sim m$ (Lebesgue measure on X). Without loss, $c(x)$ may be redefined to be constant and $d\mu$ may be replaced by dm. Thus $\pi|_Y(y) \simeq c\int_X^{\oplus} e^{iB_\lambda(x,y)}dm(x)$ on $L^2(X,\mathbb{C}^c)$. The action of the above direct integral representation on $f \in L^2(X, \mathbb{C}^c)$ is $e^{iB_\lambda(x,y)}f(x)$, i.e., the direct integral is $c\pi_0(y)$. In other words, $\pi|_Y \simeq c\pi_0|_Y$.

Using a unitary equivalence, it may be assumed that π acts on $L^2(X, \mathbb{C}^c)$ and $\pi|_Y = c\pi_0|_Y$. Then $\pi(z) = e^{i\nu(z)}I = c\pi_0(z)$ and

$$\begin{aligned}
\pi(x)(c\pi_0)^{-1}(x)(c\pi_0)(y) &= \overline{\sigma}_\lambda(-x,y)(\overline{\sigma}_\lambda(y,-x))^{-1}\pi(x)(c\pi_0)(y)(c\pi_0)(-x) \\
&= \overline{\sigma}_\lambda(x,y)\overline{\sigma}_\lambda(y,x)^{-1}\overline{\sigma}_\lambda(-x,y)\overline{\sigma}_\lambda(y,-x)^{-1}(c\pi_0)(y)\pi(x)(c\pi_0)(-x) \\
&= (c\pi_0)(y)\pi(x)(c\pi_0)^{-1}(x).
\end{aligned}$$

This implies that $\pi(x)(c\pi_0)^{-1}(x)$ commutes with the Von Neumann algebra of $c\pi_{\nu,\lambda}|_Y$. This is the algebra $\int_X^\oplus D\,dm$ where D is the algebra of diagonal $c \times c$ matrices. Since all $c\times c$ matrices commute with diagonals, $D' = M$ the algebra of $c\times c$ matrices. By ([5], Theorem III.16), $\pi(x)(c\pi_0)^{-1}(x) \in (\int_X^\oplus D\,dm)' = \int_X^\oplus D'\,dm = \int_X^\oplus M\,dm$. Thus $\pi(x)(c\pi_0)^{-1}(x) = \int_X^\oplus M(x',x)dm(x')$ for some Borel field $M : X \to \mathcal{M}(X, \mathcal{U}(\mathbb{C}^c))$. Alternatively,

$$\pi(x) = [\int_X^\oplus M(x',x)dm(x')](c\pi_0)(x).$$

Thus for a.e. x',

$$\begin{aligned}
[\pi(x)f](x') &= [(\int_X^\oplus M(x',x)dm(x'))(c\pi_0)(x)f](x') \\
&= M(x',x)[c\pi_0(x)f](x') \\
&= M(x',x)f(x'+x)
\end{aligned}$$

Now

$$\begin{aligned}
[\pi(x_1)\pi(x_2)f](x') &= M(x',x_1)[\pi(x_2)f](x'+x_1) \\
&= M(x',x_1)M(x'+x_1,x_2)f(x'+x_1+x_2)
\end{aligned}$$

and $[\pi(x_1+x_2)f](x') = M(x', x_1+x_2)f(x'+x_1+x_2)$. Since π is a unitary representation, $\pi(x_1)\pi(x_2) = \pi(x_1+x_2)$. Thus for every x_1 and x_2, M must satisfy

$$M(x', x_1)M(x'+x_1, x_2) = M(x', x_1+x_2)$$

for a.e. x'. By ([5], Corollary I.16), there is a Borel function $\tilde{M} : X \times X \to \mathcal{U}(\mathbb{C}^c)$ such that for a.e. x $\tilde{M}(x', x) = M(x', x)$ for a.e. x'. Then $\tilde{M}$ must satisfy

$$\tilde{M}(x', x_1)\tilde{M}(x'+x_1, x_2) = \tilde{M}(x', x_1+x_2)$$

for a.e. x', x_1, x_2. By ([5], Theorem IV.9), there is a ψ equal to $\tilde{M}$ a.e. on $X \times X$ such that $\psi(x', x_1)\psi(x'+x_1, x_2) = \psi(x', x_1+x_2)$ for all x',x_1, and x_2. Thus the unitary representation $x \mapsto \int_X^\oplus \psi(x', x)dm(x')(c\pi_0)(x)$ defined by ψ (see [5], section V.2) is a.e. equal to π. Since measurable homomorphisms which agree a.e. are in fact equal, $\pi(x) = [\int_X^\oplus \psi(x', x)dm(x')](c\pi_0)(x)$. Set $Uf(x) = \psi(0, x)f(x)$. Then U is unitary and

$$\begin{aligned}
[U^{-1}(c\pi_0)(x)Uf](x') &= \psi(0, x')^{-1}Uf(x'+x) \\
&= \psi(0, x')^{-1}\psi(0, x'+x)f(x'+x) = [\psi(x', x)(c\pi_0)(x)f](x') \\
&= [\pi(x)f](x').
\end{aligned}$$

Additionally,

$$U^{-1}\pi(z)Uf = U^{-1}e^{i\nu(z)}Uf = e^{i\nu(z)}U^{-1}Uf = e^{i\nu(z)}f = \pi(z)f$$

and

$$[U^{-1}(c\pi_0)(y)Uf](x') = \psi(0, x')^{-1}e^{iB_\lambda(x', y)}\psi(0, x')f(x') = [(c\pi_0)(y)f](x').$$

Thus there is a unitary operator that intertwines $c\pi_0|_X$ and $\pi|_X$ and leaves $\pi|_Y$ and $\pi|_Z$ invariant. Thus $\pi \simeq c\pi_0$. Since π and π_0 are both irreducible, $c = 1$. Thus $\pi \cong \pi_{\nu,\lambda}$. $\square$

The next theorem is a direct consequence of Proposition 3.23 and Induction.

Theorem 3.25. *Let π be an irreducible unitary representation of G. Then there is a direct sum decomposition $X+Y+Z$ of V, a unique λ in N^*, and a unique ν in Z^* such that π can be taken to have the form*

$$[\pi(x,y,z,w)f](x') = e^{i\lambda(w)}e^{i\nu(z)}e^{\frac{i}{2}\lambda[x+2x',y]}f(x'+x)$$

where f is in $L^2(X)$.

3.4 A Conull Orbit

Set $\hat{N}_d = \{\lambda \in \hat{N} \mid \mathrm{rad}(B_\lambda) \text{ has dimension } d\}$. Since any two skew-symmetric forms are equivalent by a change of basis if and only if their radicals are the same dimension, $\hat{N}_d$ is a single GL(V) orbit for each d. Let $\{e_1, ...e_n\}$ be a basis of V. Set $p(\lambda) = \sum_{k,l}(\det[B_\lambda(e_i,e_j)]_{k,l})^2$ where $[B_\lambda(e_i,e_j)]_{k,l}$ is the matrix obtained by deleting the kth row and lth column from $[B_\lambda(e_i,e_j)]$. Then p is polynomial in the entries of $[B_\lambda] \in \mathbb{R}^{n^2}$. If $\{w_1, w_2, ..., w_k\}$ is the basis of N determined by $\{e_1, ..., e_n\}$ then $\{w_1^*, ..., w_k^*\}$ is a basis for $\hat{N}$. Thus $\lambda = \sum \lambda_i w_i^*$ for some $\lambda_1, \lambda_2, ..., \lambda_k \in \mathbb{R}$ and λ may be identified with $(\lambda_1, ..., \lambda_k)$. Then $B_\lambda(e_i,e_j) = \sum w_i^*(e_i \wedge e_j)\lambda_i$. Thus p is polynomial in λ.

Proposition 3.26. *For n even, the orbit $\hat{N}_0$ is conull in $\hat{N}$. For n odd, the orbit $\hat{N}_1$ is conull in $\hat{N}$.*

Proof. The condition $p(\lambda) \neq 0$ is equivalent to $\det[B_\lambda(e_i,e_j)]_{k,l} \neq 0$ for some k, l. This, in turn, is equivalent to rank of the matrix of B_λ equaling either n or $n-1$. The rank of any skew-symmetric form must be even. Thus when n is even $p(\lambda) \neq 0$ if and only if λ is in $\hat{N}_0$. And when n is odd $p(\lambda) \neq 0$ if and only if λ is in $\hat{N}_1$. Since p is polynomial, $\{\lambda \mid p(\lambda) = 0\}$ has measure zero. Thus when n is even $\hat{N}_0$ is conull and when n is odd $\hat{N}_1$ is conull. $\square$

Note that both $\hat{N}_0$ and $\hat{N}_1$ are G-invariant since they are G orbits. The orbit $\hat{N}_0$ can be identified with the space of symplectic forms on $\mathrm{V} \times \mathrm{V}$ and the orbit $\hat{N}_1$ can be identified with the space of alternating forms on $\mathrm{V} \times \mathrm{V}$ with nullity 1.

3.5 The Plancherel Theorem

Let n be even and λ be in $\hat{N}_0$. Choose a B_λ-symplectic basis $\{x_1, ..., x_m, y_1, ..., y_m\}$ $(n = 2m)$ and write $\mathrm{V} = X+Y$ where $X = \mathrm{span}\{x_1, ..., x_m\}$ and $Y = \mathrm{span}\{y_1, ..., y_m\}$. Let π_λ be defined by

$$[\pi_\lambda(x, y, w)h](x') = e^{i\lambda(w)} e^{\frac{i}{2}B_\lambda(x+2x', y)} h(x' + x)$$

for $h \in \mathbf{L}^2(X, \mathbb{C})$. By Theorem 3.24, π_λ is an irreducible representation of G and any irreducible representation of G whose restriction to V is a $\overline{\sigma}_\lambda$-multiplier representation must be equivalent to π_λ. Furthermore, by Corollary 1.17, the set $\{[\pi_\lambda] : \lambda \in \hat{N}_0\}$ is almost all of $\hat{G}$. Let $T_{f,\lambda}$ be defined for $f \in L^1(G) \bigcap L^2(G)$ by

$$[T_{f,\lambda}h](x') = \int\int\int f(x, y, w)[\pi_\lambda(x, y, w)h](x')dx\,dy\,dw$$

where dx and dy are the normalized Lebesgue measures on X and Y, respectively, as determined by the above basis and dw is normalized Lebesgue measure on $N \cong \mathbb{R}^k$. If F_3 is the Fourier transform in w and $K_{f,\lambda}(x', x) = \int [F_3 f](x - x', y, \lambda) e^{iB_\lambda(\frac{x+x'}{2}, y)} dy$, then

$$\begin{aligned}
[T_{f,\lambda}(f)h](x') &= \int\int\int f(x, y, w)[\pi_\lambda(x, y, w)h](x')dx\,dy\,dw \\
&= \int\int\int f(x, y, w)e^{i\lambda(w)} e^{\frac{i}{2}B_\lambda(x+2x', y)} h(x' + x)dx\,dy\,dw \\
&= \int\int [F_3 f](x, y, \lambda) e^{\frac{i}{2}B_\lambda(x+2x', y)} h(x' + x)dx\,dy \\
&= \int (\int [F_3 f](x - x', y, \lambda) e^{iB_\lambda(\frac{x+x'}{2}, y)} dy) h(x)dx \\
&= \int K_{f,\lambda}(x', x)h(x)dx.
\end{aligned}$$

Let $u = x - x'$, $v = \frac{x+x'}{2}$, and $F_2 : L^2(Y) \to L^2(X)$ be the Fourier transform discussed at the end of section 3.2. Then $dx\,dx' = du\,dv$ and

$$\begin{aligned}
\|K_{f,\lambda}\|_2^2 &= \int\int |K_{f,\lambda}(x',x)|^2 dx\,dx' \\
&= \int\int |K_{f,\lambda}(u,v)|^2 du\,dv \\
&= \int\int |\int [F_3 f](u,y,\lambda) e^{-iB_\lambda(y,v)} dy|^2 du\,dv \\
&= \int\int |[F_2F_3 f](u,v,\lambda)|^2 du\,dv \\
&= \int\int |[F_3 f](u,y,\lambda)|^2 du\,dy.
\end{aligned}$$

Now fix a basis $\{e_1, ..., e_n\}$ of V and let m_0 be the Lebesgue measure on V determined by this basis. Define $dm_\lambda(x,y) = dx\,dy$. Then the measures dm_λ and dm_0 are related by $dm_\lambda = \delta(\lambda)dm_0$ where $\delta(\lambda) = \sqrt{\det[B_\lambda(e_i,e_j)]}$ (see section 3.2). Then

$$\begin{aligned}
[T_{f,\lambda}h](x') &= \int\int\int f(x,y,w)[\pi_\lambda(x,y,w)h](x') dm_\lambda(x,y) dw \\
&= \int\int\int f(x,y,w)[\pi_\lambda(x,y,w)h](x')\delta(\lambda) dm_0(x,y) dw \\
&= \delta(\lambda)[\pi_\lambda(f)h](x')
\end{aligned}$$

and so $[\pi_\lambda(f)h](x') = \delta(\lambda)^{-1}[T_{f,\lambda}h](x')$. This implies that

$$\begin{aligned}
\mathrm{Tr}(\pi_\lambda(f)\pi_\lambda(f)^*) &= \mathrm{Tr}(\delta(\lambda)^{-1}T_{f,\lambda}\delta(\lambda)^{-1}T_{f,\lambda}^*) \\
&= \|\delta(\lambda)^{-1}K_{f,\lambda}\|_2^2 \\
&= \int\int |[F_3 f](x,y,\lambda)|^2 \delta(\lambda)^{-2} dm_\lambda(x,y) \\
&= \int\int |[F_3 f](x,y,\lambda)|^2 \delta(\lambda)^{-1} dm_0(x,y).
\end{aligned}$$

Thus, if $dm_0 dw$ is chosen to be Haar measure and $d\lambda$ is the dual measure to dw,

$$\begin{aligned}\int \mathrm{Tr}(\pi_\lambda(f)\pi_\lambda(f)^*)\delta(\lambda)d\lambda &= \int\int\int |[F_3 f](x,y,\lambda)|^2 dm_0(x,y)d\lambda \\ &= \int\int\int |f(x,y,w)|^2 dm_0(x,y)dw \\ &= \|f\|_2^2.\end{aligned}$$

That is, Plancherel measure is given by $d\mu(\pi_\lambda) = \sqrt{\det[B_\lambda(e_i,e_j)]}d\lambda$.

Let $A \in \mathrm{GL}(V)$. Since $[B_{A\cdot\lambda}] = (A^{-1})^t[B_\lambda]A^{-1}$, $\det[B_{A\cdot\lambda}] = |A|^{-2}\det[B_\lambda]$. Since $d\lambda$ is dual to dw and $d(A\cdot w) = d(\tilde{A}w) = |\tilde{A}|dw = |A|^{n-1}dw$, $d(A\cdot\lambda) = |A|^{1-n}d\lambda$. Then $d\mu(A\cdot\pi_\lambda) = d\mu(\pi_{A\cdot\lambda}) = \sqrt{\det[B_{A\cdot\lambda}(e_i,e_j)]}d(A\cdot\lambda) = |A|^{-1}\sqrt{\det[B_\lambda(e_i,e_j)]}|A|^{1-n}d\lambda = |A|^{-n}d\mu(\pi_\lambda)$. Thus we see that μ has the expected relative invariance factor $\Delta(A) = |A|^{-n}$.

Now let n be odd, λ be in $\hat{N}_1$, and χ be in the linear dual of $\mathrm{rad}(B_\lambda)$. Choose a basis $\{x_1,...,x_m,y_1,...,y_m,z\}$ $(n = 2m+1)$ as in Proposition 3.21 and write $\mathrm{V} = X + Y + Z$ where $X = \mathrm{span}\{x_1,...,x_m\}$, $Y = \mathrm{span}\{y_1,...,y_m\}$, and $Z = \mathrm{span}\{z\}$. Let dx, dy, and dz be the normalized Lebesgue measures on X, Y, and Z, respectively. Let $\pi_{\lambda,\chi}$ be defined by

$$[\pi_{\lambda,\chi}(x,y,z,w)h](x') = e^{i\chi(z)}e^{i\lambda(w)}e^{\frac{i}{2}B_\lambda(x+2x',y)}h(x'+x)$$

for $h \in L^2(X,\mathbb{C})$. Then $\pi_{\lambda,\chi}$ is an irreducible representation of G and any irreducible representation of G whose restriction to V is a $\overline{\sigma}_\lambda$-multiplier representation must be equivalent to $\pi_{\lambda,\chi}$ for some χ. Furthermore, the set $\{[\pi_{\lambda,\chi}] :\ \lambda \in \hat{N}_0,\ \chi \in \mathrm{rad}(B_\lambda)^*\}$ is almost all of $\hat{G}$. Let $T_{f,\lambda,\chi}$ be defined for $f \in L^1(G)\bigcap L^2(G)$ by

$$[T_{f,\lambda,\chi}h](x') = \int\int\int f(x,y,z,w)[\pi_{\lambda,\chi}(x,y,z,w)h](x')dm_\lambda(x,y,z)dw$$

where $dm_\lambda(x,y,z) := dx\,dy\,dz$ and dw is normalized Lebesgue measure on N. Let F_3 and F_4 denote the Fourier transforms in z and w, respectively, and $K_{f,\lambda,\chi}(x',x) =$

$\int[F_3F_4f](x-x',y,\chi,\lambda)e^{iB_\lambda(\frac{x+x'}{2},y)}dy$. Then

$$\begin{aligned}
[T_{f,\lambda,\chi}(f)h](x') &= \int\int\int f(x,y,z,w)[\pi_{\lambda,\chi}(x,y,z,w)h](x')dx\,dy\,dz\,dw \\
&= \int\int\int f(x,y,z,w)e^{i\chi(z)}e^{i\lambda(w)}e^{\frac{i}{2}B_\lambda(x+2x',y)}h(x'+x)dx\,dy\,dz\,dw \\
&= \int\int[F_3F_4f](x,y,\chi,\lambda)e^{\frac{i}{2}B_\lambda(x+2x',y)}h(x'+x)dx\,dy \\
&= \int(\int[F_3F_4f](x-x',y,\chi,\lambda)e^{iB_\lambda(\frac{x+x'}{2},y)}dy)h(x)dx \\
&= \int K_{f,\lambda,\chi}(x',x)h(x)dx.
\end{aligned}$$

Let $u=x-x'$, $v=\frac{x+x'}{2}$, and $F_2: L^2(Y)\to L^2(X)$ be the Fourier transform as discussed in section 3.2. Then $dx\,dx' = du\,dv$ and

$$\begin{aligned}
\|K_{f,\lambda,\chi}\|_2^2 &= \int\int|K_{f,\lambda,\chi}(x',x)|^2dx\,dx' \\
&= \int\int|K_{f,\lambda,\chi}(u,v)|^2du\,dv \\
&= \int\int|\int[F_3F_4f](u,y,\chi,\lambda)e^{-iB_\lambda(y,v)}dy|^2du\,dv \\
&= \int\int|[F_2F_3F_4f](u,v,\chi,\lambda)|^2du\,dv \\
&= \int\int|[F_3F_4f](u,y,\chi,\lambda)|^2du\,dy.
\end{aligned}$$

Now fix a basis $\{e_1,...,e_n\}$ of V and let m_0 be the Lebesgue measure determined by this basis. Choose dm_0dw to be Haar measure on G. The measures dm_λ and dm_0 are related by $dm_\lambda = \delta(\lambda)dm_0$ where $\delta(\lambda)=\sqrt{\det[B_\lambda(e_i,e_j)]_{i,j\neq n}}$ (see section 3.2). Thus

$$\begin{aligned}
[T_{f,\lambda,\chi}h](x') &= \int\int\int f(x,y,z,w)[\pi_{\lambda,\chi}(x,y,z,w)h](x')dm_\lambda(x,y,z)dw \\
&= \int\int\int f(x,y,z,w)[\pi_{\lambda,\chi}(x,y,z,w)h](x')\delta(\lambda)dm_0(x,y,z)dw \\
&= \delta(\lambda)[\pi_{\lambda,\chi}(f)h](x').
\end{aligned}$$

So $[\pi_{\lambda,\chi}(f)h](x') = \delta(\lambda)^{-1}[T_{f,\lambda,\chi}h](x')$. This implies that

$$\begin{aligned}
\mathrm{Tr}(\pi_{\lambda,\chi}(f)\pi_{\lambda,\chi}(f)^*) &= \mathrm{Tr}(\delta(\lambda)^{-1}T_{f,\lambda,\chi}\delta(\lambda)^{-1}T^*_{f,\lambda,\chi}) \\
&= \|\delta(\lambda)^{-1}K_{f,\lambda,\chi}\|_2^2 \\
&= \int\int |[F_3F_4f](x,y,\chi,\lambda)|^2\delta(\lambda)^{-2}dm_\lambda(x,y) \\
&= \int\int |[F_3F_4f](x,y,\chi,\lambda)|^2\delta(\lambda)^{-1}dm_0(x,y).
\end{aligned}$$

So, if $d\lambda$ is dual to dw and $d\chi$ is dual to dz,

$$\begin{aligned}
\int\int \mathrm{Tr}(\pi_{\lambda,\chi}(f)\pi_{\lambda,\chi}(f)^*)\delta(\lambda)d\chi\, d\lambda &= \int\int\int\int |[F_3F_4f](x,y,\chi,\lambda)|^2 dm_0(x,y)d\chi\, d\lambda \\
&= \int\int\int\int |f(x,y,w)|^2 dm_0(x,y)dz\, dw \\
&= \|f\|_2^2
\end{aligned}$$

since F_3 and F_4 are unitary. That is, Plancherel measure is given by

$$d\mu(\pi_{\lambda,\chi}) = \sqrt{\det[B_\lambda(e_i,e_j)]_{i,j\neq n}}d\chi\, d\lambda.$$

We now demonstrate that μ has the expected relative invariance factor $\Delta(A) = |A|^{-n}$. Since Δ is a homomorphism it suffices to demonstrate this when A is diagonal. Let A_n be the matrix obtained by deleting the n-th row and column. Then $[B_{A\cdot\lambda}(e_i,e_j)]_{i,j\neq n} = (A_n^{-1})^t[B_\lambda(e_i,e_j)]_{i,j\neq n}A_n^{-1}$. Thus $\det[B_{A\cdot\lambda}(e_i,e_j)]_{i,j\neq n} = |A_n|^{-2}\det[B_\lambda(e_i,e_j)]_{i,j\neq n}$. Also $A\cdot\chi(z) = \chi(a_n^{-1}z) = a_n^{-1}\chi(z)$. Thus

$$\begin{aligned}
d\mu(A\cdot\pi_{\lambda,\chi}) &= d\mu(\pi_{A\cdot\lambda,A\cdot\chi}) \\
&= \sqrt{\det[B_{A\cdot\lambda}(e_i,e_j)]_{i,j\neq n}}d(A\cdot\chi)\, d(A\cdot\lambda) \\
&= |A_n|^{-1}\sqrt{\det[B_\lambda(e_i,e_j)]_{i,j\neq n}}a_n^{-1}d\chi\, |A|^{1-n}d\lambda \\
&= |A|^{-n}d\mu(\pi_{\lambda,\chi})
\end{aligned}$$

as expected.

Free Three Step Nilpotent Lie Groups

4.1 Definitions and Notation

Let V be an n dimensional real vector space. The free 3-step nilpotent Lie group G is the connected Lie group with Lie algebra $\mathfrak{g} = \mathfrak{f}_n, 3$. Multiplication in G expressed in terms of exponential coordinates is

$$(v, w, \sigma)\cdot(v', w', \sigma') = (v+v', w+w'+\frac{1}{2}[v, v'], \sigma+\sigma'+\frac{1}{2}([v, w']-[v', w])+\frac{1}{12}[v-v', [v, v']].$$

Let $N = \exp(\mathrm{V}^{(2)} \bigoplus \mathrm{V}^{(3)})$. N is then a normal, abelian subgroup of G of dimension $k = \frac{n^2-n}{2} + \frac{n^3-n}{3}$. Since N is abelian, $\hat{N} \simeq (\mathrm{V}^{(2)} \bigoplus \mathrm{V}^{(3)})^* \simeq (\mathrm{V}^{(2)})^* \bigoplus (\mathrm{V}^{(3)})^* \simeq \mathbb{R}^k$. Let $(\lambda, \phi) \in \hat{N}$, $g = (v, w, \sigma)$, and $n = (0, w', \sigma')$. Since

$$\begin{aligned}
gng^{-1} &= (v, w, \sigma)(0, w', \sigma')(-v-, w, -\sigma) \\
&= (v, w' + w, \sigma' + \sigma + 1/2[v, w'])(-v, -w, -\sigma) \\
&= (0, w', \sigma' + [v, w']),
\end{aligned}$$

the action of G on $\hat{N}$ is given by $g \cdot (\lambda, \phi)(n) = (\lambda, \phi)(gng^{-1}) = (\lambda, \phi)(n + [v, w])$. For any $\phi \in (\mathrm{V}^{(3)})^*$ and $v \in \mathrm{V}$, define $c_v\phi \in (\mathrm{V}^{(2)})^*$ by $c_v\phi(w) = \phi([v, w])$. Then the G-orbit of (λ, ϕ) is $G \cdot (\lambda, \phi) = \{(\lambda + c_v\phi, \phi) \mid v \in \mathrm{V}\}$. Thus the stabilizer of (λ, ϕ) is $G^{(\lambda,\phi)} = \{(v, w, \sigma) \mid c_v\phi = 0\}$. The quotient $G^{(\lambda,\phi)}/N \cong \{v \in \mathrm{V} \mid c_v\phi = 0\}$ will be denoted by V_ϕ.

We consider the action of GL(V) as a subgroup of Aut(G) on G. We use the notation $A\cdot(v, w, \sigma) = (Av, \tilde{A}w, \tilde{\tilde{A}}\sigma)$ where $\tilde{A}$ is defined as in section 3.1 and $\tilde{\tilde{A}}$ is defined by $\tilde{\tilde{A}}[v, v' \wedge v''] = [Av, Av' \wedge Av'']$ and linearity. Then $dm(g) = dv\, dw\, d\sigma$ where dv, dw, and $d\sigma$ are Lebesgue measures on V, $\mathrm{V}^{(2)}$, and $\mathrm{V}^{(3)}$, respectively, is relatively invariant with respect to the action of GL(V) with factor $\Delta(A) = |A||\tilde{A}||\tilde{\tilde{A}}|$. As before, we may assume that A is diagonal. Then $|\tilde{A}| = |A|^{n-1}$ and $\tilde{\tilde{A}}[e_i, e_j \wedge e_k] =$

$a_ia_ja_k[e_i, e_j \wedge e_k]$. So $\tilde{\tilde{A}}$ is diagonal and $|\tilde{\tilde{A}}| = \prod_{i\neq j\neq k\neq i} a_i^2a_j^2a_k^2 \ \prod_{i\neq j} a_i^2 a_j = |A|^d$ where $d = 3(n-1) + \frac{n(n-1)(n-2)}{3}$. Thus $\Delta(A) = |A|^{n+d}$.

4.2 The Dual

The dual of G is determined using Mackey induction in the following manner. Each (λ, ϕ) in $\hat{N}$ can be extended to a multiplier representation $\pi_{(\lambda,\phi)}$ of $G^{(\lambda,\phi)}$ defined by $\pi_{(\lambda,\phi)}(v, w, \sigma) = e^{i\lambda(w)}e^{i\phi(\sigma)}I$. Since

$$\begin{aligned}\pi_{(\lambda,\phi)}((v, w, \sigma)\cdot(v', w', \sigma')) &= \pi_{(\lambda,\phi)}(v + v', w + w' + \frac{1}{2}v\wedge v', \sigma+\sigma' \\ &\qquad + \frac{1}{2}([v, w'] - [v', w]) + \frac{1}{12}[v - v', v\wedge v']) \\ &= e^{i\lambda(w+w'+\frac{1}{2}v\wedge v')}e^{i\phi(\sigma+\sigma')}I \\ &= e^{\frac{i}{2}\lambda(v\wedge v')}e^{i\lambda(w)}e^{i\lambda(w')}e^{i\phi(\sigma)}e^{i\phi(\sigma')}I \\ &= e^{\frac{i}{2}\lambda(v\wedge v')}\pi_{(\lambda,\phi)}(v, w, \sigma)\pi_{(\lambda,\phi)}(v', w', \sigma'),\end{aligned}$$

the multiplier for $\pi_{(\lambda,\phi)}(v, w, \sigma)$ is $m_\lambda(v, v') = e^{\frac{-i}{2}\lambda(v\wedge v')}$. Next all $\overline{m}_\lambda$-multiplier representations of V_ϕ must be determined. As before, $B_\lambda(v, v') = e^{i\lambda(v\wedge v')}$ defines a skew-symmetric bilinear form on $\mathrm{V}_\phi \times \mathrm{V}_\phi$. By Proposition 3.23, the irreducible $\overline{m}_\lambda$-multiplier representations of V_ϕ are in one-to-one correspondence with $\mathrm{rad}(B_\lambda)^*$. So for each $\chi \in \mathrm{rad}(B_\lambda)^*$ there exists a unique equivalence class $\overline{\chi}$ of irreducible $\overline{m}_\lambda$-multiplier representations of V_ϕ. Thus, by Theorem 1.10, the dual of G is equivalent to the set $\{\mathrm{ind}_{G^{(\lambda,\phi)}}^{G}(\pi_{(\lambda,\phi)} \times \overline{\chi}) \mid (\lambda, \phi) \in \hat{N} \text{ and } \chi \in \mathrm{rad}(B_\lambda)^*\}$.

4.3 A Conull Set

Set $\hat{N}_0 = \{(\lambda, \phi) \in \hat{N} \mid G\cdot(\lambda, \phi) \text{ is maximal}\}$. Since $\hat{N}_0$ is a set of G orbits it is G-invariant. Thus, by Corollary 1.17, the set of representations of G induced by the representations in $\hat{N}_0$ is almost all of $\hat{G}$. When $n = 2$ or 3, $\dim(\mathrm{V}) \geq \dim((\mathrm{V}^{(2)})^*)$. Thus maximal orbits are of the form

$$G\cdot(\lambda, \phi) = (\mathrm{V}^{(2)})^* \times \{\phi\}. \tag{4.8}$$

For $n = 2$, equation (4.8) holds if there is any $v \in \mathrm{V}$ such that $c_v\phi \neq 0$, i.e., if $\phi \neq 0$. Thus $\hat{N}_0 = \{(\lambda, \phi) \in \hat{N} \mid \phi \neq 0\}$, which is obviously conull in $\hat{N}$.

For $n = 3$, equation (4.8) holds if and only if the map $v \mapsto c_v\phi$ is injective. Thus $\hat{N}_0 = \{(\lambda, \phi) \in \hat{N} \mid v \mapsto c_v\phi \text{ is injective}\}$. This set is conull in $\hat{N}$ if and only if the set $\Phi = \{\phi \,|\, v \mapsto c_v\phi \text{ is injective}\}$ is conull in $(\mathrm{V}^{(3)})^*$.

To see that Φ is conull, fix a basis $\{e_1, e_2, e_3\}$ of V. Let $w_1 = e_2 \wedge e_3$, $w_2 = e_3 \wedge e_1$, and $w_3 = e_1 \wedge e_2$. Then $\{w_j\}$ is a basis of $\mathrm{V}^{(2)}$. Associate to each $\phi \in (\mathrm{V}^{(3)})^*$ the 3×3 matrix $[\phi] = (\phi_{i,j})$ where $\phi_{i,j} = \phi[e_i, w_j]$. The Jacobi identity, $[e_1, w_1] + [e_2, w_2] + [e_3, w_3] = 0$, and the linearity of ϕ imply that $\mathrm{Tr}[\phi] = 0$, i.e., $[\phi] \in \mathfrak{sl}(3)$. The map $\phi \mapsto [\phi]$ is well-defined and bijective. So $(\mathrm{V}^{(3)})^* \cong \mathfrak{sl}(3)$ and Φ corresponds to the set $\Phi' = \{[\phi] \,|\, \det[\phi] \neq 0\}$. Since $\det[\phi]$ is polynomial in the components of ϕ, Φ' is conull in $\mathfrak{sl}(3)$. Thus Φ is conull in $(\mathrm{V}^{(3)})^*$.

When $n \geq 4$, $\dim\,(V) < \dim\,(V^{(2)})^*$. Thus $G \cdot (\lambda, \phi) = \{(\lambda + c_v\phi, \phi) \,|\, v \in V\}$ is maximal when the map $v \mapsto c_v\phi$ is injective. As in the $n = 3$ case, $\hat{N}_0$ is conull if and only if $\Phi = \{\phi \,|\, v \mapsto c_v\phi \text{ is injective}\}$ is conull in $(\mathrm{V}^{(3)})^*$. To see that Φ is conull proceed as in the $n = 3$ example. Fix a basis $\{e_i\}_{i=1,\ldots,n}$ of V and from it obtain a basis $\{e_i \wedge e_j\}_{i<j}$ of $\mathrm{V}^{(2)}$. Then associate to each ϕ the $n \times k$ matrix $[\phi] = (\phi[e_i, e_p \wedge e_q])$. Then $v \mapsto \phi$ is injective if and only if $[\phi]$ has maximum rank. Let $p(\phi) := \sum |M|^2$ where the sum is over the $n \times n$ minors M of $[\phi]$ obtained by deleting $k - n$ columns. Then $p(\phi) \neq 0$ is equivalent to rank $[\phi] = n$. Thus $\Phi = \{\phi \,|\, p(\phi) \neq 0\}$. Since the measure of $\{\phi \,|\, p(\phi) = 0\}$ is zero, the set Φ is conull.

Now consider the conull set $\Phi = \{\phi \,|\, \phi \neq 0\}$ when $n = 2$.

Proposition 4.27. *The set* Φ *is a single* $\mathrm{SL(V)}$*-orbit.*

Proof. Let $\phi_0 = [e_1^*, e_1^* \wedge e_2^*]$, $\phi_1 = [e_2^*, e_1^* \wedge e_2^*]$, and $\phi = \alpha\phi_0 + \beta\phi_1$. We show that $\mathrm{SL(V)} \cdot \phi_0 = \Phi$. If $\phi \in \Phi$ then α or β is nonzero.

If $\alpha \neq 0$, let $A = \begin{bmatrix} \alpha & \beta \\ 1 & \frac{1+\beta}{\alpha} \end{bmatrix}$. Then $\det(A) = 1$ and

$$Ae_1 \wedge Ae_2 = (\alpha e_1 + e_2) \wedge (\beta e_1 + \frac{1+\beta}{\alpha} e_2) = (\alpha(\frac{1+\beta}{\alpha}) - \beta)\, e_1 \wedge e_2 = e_1 \wedge e_2.$$

So $\phi_0[Ae_1, Ae_1 \wedge Ae_2] = \phi_0[\alpha e_1 + e_2, e_1 \wedge e_2] = \alpha$ and $\phi_0[Ae_2, Ae_1 \wedge Ae_2] = \phi_0[\beta e_1 + \frac{1+\beta}{\alpha} e_2, e_1 \wedge e_2] = \beta$. Thus $A^{-1} \cdot \phi_0 = \phi$.

If $\beta \neq 0$, let $A = \begin{bmatrix} \alpha & \beta \\ \frac{\alpha-1}{\beta} & 1 \end{bmatrix}$. Then $\det(A) = 1$ and

$$Ae_1 \wedge Ae_2 = (\alpha e_1 + \frac{\alpha-1}{\beta} e_2) \wedge (\beta e_1 + e_2) = (\alpha - (\frac{\alpha-1}{\beta})\beta)\, e_1 \wedge e_2 = e_1 \wedge e_2.$$

So $\phi_0[Ae_1, Ae_1 \wedge Ae_2] = \phi_0[\alpha e_1 + \frac{\alpha-1}{\beta} e_2, e_1 \wedge e_2] = \alpha$ and $\phi_0[Ae_2, Ae_1 \wedge Ae_2] = \phi_0[\beta e_1 + e_2, e_1 \wedge e_2] = \beta$. Thus $A^{-1} \cdot \phi_0 = \phi$. □

The remainder of this section is devoted to obtaining a better understanding of the set $\Phi = \{\phi \,|\, v \mapsto c_v\phi \text{ is injective}\}$ when $n = 3$.

Proposition 4.28. *Suppose ϕ is in Φ and that in some basis* $[\phi] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & x \\ 0 & 1 & 0 \end{bmatrix}$ *for some x in $\mathbb{R}$. Then $\mathrm{GL(V)} \cdot \phi$ is seven dimensional and $\mathrm{GL(V)}_o^\phi$ is abelian.*

Proof. These facts will be established by calculating the Lie algebra of the stabilizer. If $A = e^{tX}$ stabilizes ϕ then $\phi[Ae_i, Ae_j \wedge Ae_k] = \phi[e_i, e_j \wedge e_k]$. Differentiating this equality yields

$$\phi[Xe_i, e_j \wedge e_k] + \phi[e_i, Xe_j \wedge e_k] + \phi[e_i, e_j \wedge Xe_k] = 0.$$

If $Xe_i = a_i^r e_r$, this is equivalent to

$$a_i^r \phi[e_r, e_j \wedge e_k] + a_j^r \phi[e_i, e_r \wedge e_k] + a_k^r \phi[e_i, e_j \wedge e_r] = 0.$$

Letting i, j, and k vary yields the following system:

$$a_1^2 - a_3^1 = 0 \tag{4.9}$$

$$a_1^3 - a_3^2 = 0 \tag{4.10}$$

$$xa_1^2 + 2a_1^1 + a_2^2 = 0 \tag{4.11}$$

$$2a_2^2 + a_3^3 - xa_3^1 = 0 \tag{4.12}$$

$$a_2^3 - xa_3^2 - a_1^2 = 0 \tag{4.13}$$

$$a_2^1 + 2xa_2^2 + xa_1^1 - a_1^3 = 0 \tag{4.14}$$

$$a_3^2 - a_2^1 = 0 \tag{4.15}$$

$$2a_3^3 + a_1^1 = 0 \tag{4.16}$$

$$a_3^1 + xa_3^2 - a_2^3 = 0. \tag{4.17}$$

Equations (4.10) and (4.15) imply $a_2^1 = a_1^3 = a_3^2$. Any two of equations (4.9), (4.13), and (4.17) imply $a_1^2 = a_3^1 = a_2^3 - xa_3^2$. Equations (4.11), (4.12), and (4.16) are equivalent to

$$a_2^2 = -2a_1^1 - xa_1^2$$
$$a_3^3 = -2a_2^2 + xa_3^1$$
$$a_1^1 = -2a_3^3.$$

Thus $a_1^1 = 4a_2^2 - 2xa_3^1 = -8a_1^1 - 4xa_1^2 - 2xa_3^1$. Since $a_1^2 = a_3^1$, $a_1^1 = \frac{-2}{3}xa_1^2$. Then $a_2^2 = \frac{4}{3}xa_1^2 - xa_1^2 = \frac{1}{3}xa_1^2$ and $a_3^3 = \frac{-2}{3}xa_1^2 + xa_3^1 = \frac{1}{3}xa_1^2$. Equation (4.14) is trivial when $x = 0$ and is equivalent to $a_1^1 = -2a_2^2$ when $x \neq 0$. This is consistent with the preceding. Thus a_3^2 and a_2^3 are free, the Lie algebra of the stabilizer is two dimensional, and the orbit is seven dimensional. If $\alpha = a_3^2$ and $\beta = a_2^3$, elements of the Lie algebra are of the form

$$X = \begin{bmatrix} \frac{2}{3}(\alpha x^2 - \beta x) & \alpha & \beta - \alpha x \\ \beta - \alpha x & \frac{-1}{3}(\alpha x^2 - \beta x) & \alpha \\ \alpha & \beta & \frac{-1}{3}(\alpha x^2 - \beta x) \end{bmatrix}.$$

Thus the Lie algebra has generators $\begin{bmatrix} \frac{2x^2}{3} & 1 & -x \\ -x & \frac{-x^2}{3} & 1 \\ 1 & 0 & \frac{-x^2}{3} \end{bmatrix}$ and $\begin{bmatrix} \frac{-2x}{3} & 0 & 1 \\ 1 & \frac{x}{3} & 0 \\ 0 & 1 & \frac{x}{3} \end{bmatrix}$. Since

$$\begin{bmatrix} \frac{2x^2}{3} & 1 & -x \\ -x & \frac{-x^2}{3} & 1 \\ 1 & 0 & \frac{-x^2}{3} \end{bmatrix} \begin{bmatrix} \frac{-2x}{3} & 0 & 1 \\ 1 & \frac{x}{3} & 0 \\ 0 & 1 & \frac{x}{3} \end{bmatrix} = \begin{bmatrix} \frac{-4x^3}{9} + 1 & \frac{-2x}{3} & \frac{x^2}{3} \\ \frac{x^2}{3} & 1 - \frac{x^3}{9} & \frac{-2x}{3} \\ \frac{-2x}{3} & \frac{-x^2}{3} & 1 - \frac{x^3}{9} \end{bmatrix}$$

and

$$\begin{bmatrix} \frac{-2x}{3} & 0 & 1 \\ 1 & \frac{x}{3} & 0 \\ 0 & 1 & \frac{x}{3} \end{bmatrix} \begin{bmatrix} \frac{2x^2}{3} & 1 & -x \\ -x & \frac{-x^2}{3} & 1 \\ 1 & 0 & \frac{-x^2}{3} \end{bmatrix} = \begin{bmatrix} \frac{-4x^3}{9} + 1 & \frac{-2x}{3} & \frac{x^2}{3} \\ \frac{x^2}{3} & 1 - \frac{x^3}{9} & \frac{-2x}{3} \\ \frac{-2x}{3} & \frac{-x^2}{3} & 1 - \frac{x^3}{9} \end{bmatrix}$$

all brackets in the Lie algebra are zero. Thus the connected component of the stabilizer is abelian. □

Every λ in $(\mathrm{V}^{(2)})^*$ induces a bilinear form $(v, v') \mapsto \lambda(v \wedge v')$ on $\mathrm{V} \times \mathrm{V}$. Let rad λ denote the radical of this bilinear form.

Lemma 4.29. *If ϕ is in Φ then there is some nonzero v in V such that v is not in rad $c_v\phi$.*

Proof. Suppose otherwise. Then $e_i \in$ rad $c_{e_i}\phi$ for $i = 1, 2,$ and 3. Thus

$$\phi[e_1, e_1 \wedge e_2] = 0 = \phi[e_1, e_3 \wedge e_1]$$

$$\phi[e_2, e_1 \wedge e_2] = 0 = \phi[e_2, e_2 \wedge e_3]$$

$$\phi[e_3, e_2 \wedge e_3] = 0 = \phi[e_3, e_3 \wedge e_1].$$

So $[\phi] = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ where a, b, and c are each nonzero and $a + b + c = 0$. Letting $e_1 \mapsto \frac{e_1}{a}$ and $e_2 \mapsto \frac{e_2}{b}$, $[\phi] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. Let $v = e_1 + e_3$. Then $c_v\phi(v \wedge e_2) = \phi[v, v \wedge e_2] = \phi[e_1 + e_3, (e_1 + e_3) \wedge e_2] = \phi[e_1, e_1 \wedge e_2] + \phi[e_1, e_3 \wedge e_2] + \phi[e_3, e_1 \wedge e_2] + \phi[e_3, e_3 \wedge e_2] = -3$. A contradiction. □

Lemma 4.30. *Let ϕ be in Φ. For all non-zero v in* V *the dimension of rad $c_v\phi$ is one.*

Proof. Any bilinear form on V must have either one dimensional or three dimensional radical. Only the zero form has a three dimensional radical. Since $c_v\phi \neq 0$, its radical must be one dimensional. □

Lemma 4.31. *If λ and λ' (both non-zero) are in* $(\mathrm{V}^{(2)})^*$ *and have identical radicals then they are linearly dependent.*

Proof. Since λ and λ' are both non-zero, they have one dimensional radicals. Let $v_1 \neq 0$ be in rad λ and rad λ'. Extend to a basis $\{v_1, v_2, v_3\}$. Then λ and λ' can be expressed as linear combinations of $v_2^* \wedge v_3^*$, $v_3^* \wedge v_1^*$, and $v_1^* \wedge v_2^*$. Since v_1 is in both radicals, both λ and λ' must be non-zero multiples of $v_2^* \wedge v_3^*$. □

Lemma 4.32. *Suppose the matrix of ϕ in some basis is* $[\phi] = \begin{bmatrix} 0 & 0 & a \\ b & 0 & c \\ 0 & d & 0 \end{bmatrix}$ *where $a, b,$ and d are non-zero and c is arbitrary. Then the basis may be scaled so that* $[\phi] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & x \\ 0 & 1 & 0 \end{bmatrix}$ *for some x.*

Proof. Let $\{e_1, e_2, e_3\}$ be a basis for which $[\phi] = \begin{bmatrix} 0 & 0 & a \\ b & 0 & c \\ 0 & d & 0 \end{bmatrix}$. Let $\alpha = (\frac{da^4}{b^2})^{\frac{1}{9}}$, $\beta = (\frac{bd^4}{a^2})^{\frac{1}{9}}$, and $\gamma = (\frac{ab^4}{d^2})^{\frac{1}{9}}$. Since $\alpha^2\gamma = (\frac{d^2a^8}{b^4}\frac{ab^4}{d^2})^{\frac{1}{9}} = a$, $\gamma^2\beta = (\frac{a^2b^8}{d^4}\frac{bd^4}{a^2})^{\frac{1}{9}} = b$, and $\beta^2\alpha = (\frac{b^2d^8}{a^4}\frac{da^4}{b^2})^{\frac{1}{9}} = d$, the matrix of ϕ with respect to the scaled basis $\{\frac{e_1}{\alpha}, \frac{e_2}{\beta}, \frac{e_3}{\gamma}\}$ is $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & x \\ 0 & 1 & 0 \end{bmatrix}$ where $x = \frac{c}{\beta^2\alpha}$. □

Proposition 4.33. *For almost every ϕ in Φ there is a change of basis A such that* $[A \cdot \phi] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & x \\ 0 & 1 & 0 \end{bmatrix}$ *for some x in $\mathbb{R}$.*

Proof. Using Lemma 4.29, pick v_1 such that v_1 is not in rad $c_{v_1}\phi$. By Lemma 4.30, rad $c_{v_1}\phi$ is one dimensional. Pick v_3 such that $\langle v_3 \rangle =$ rad $c_{v_1}\phi$. Since v_1 is not in $\langle v_3 \rangle$, v_1 and v_3 are linearly independent. Again by Lemma 4.30, rad $c_{v_3}\phi$ is one dimensional. Pick v_2 such that $\langle v_2 \rangle =$ rad $c_{v_3}\phi$.

Case 1:If v_1, v_2, and v_3 form a basis then the matrix of ϕ relative to this basis is $[\phi] = \begin{bmatrix} 0 & 0 & a \\ b & 0 & c \\ 0 & d & 0 \end{bmatrix}$ for some a, b, c, and d. By Lemma 4.32, the basis may be scaled so that $[\phi] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & x \\ 0 & 1 & 0 \end{bmatrix}$ for some x. Thus, in this case, the proposed change of basis exists.

Case 2: If v_1, v_2, and v_3 do not form a basis then $v_2 = av_1 + bv_3$ for some a and b (at least one nonzero). If $a = 0$ then $\langle v_2 \rangle = \langle v_3 \rangle$ and so rad $c_{v_1}\phi =$ rad $c_{v_3}\phi$. This would imply that $c_{v_1}\phi$ and $c_{v_3}\phi$ are linearly dependent which contradicts the requirement that $\det[\phi] \neq 0$. Thus $a \neq 0$ and without loss $v_2 = v_1 + bv_3$. Note that $c_v\phi(v_3 \wedge v_2) = c_v\phi(v_3 \wedge v_1)$ for all v.

Claim: There is a $v \in$ V such that $c_v\phi(v_3 \wedge v_2) = c_v\phi(v_3 \wedge v_1) = 1$.

If not then $c_v\phi(v_3 \wedge v_2) = 0$ for all v. Consider the Jacobi identity $\phi[v, v_3 \wedge v_2] + \phi[v_2, v \wedge v_3] + \phi[v_3, v_2 \wedge v] = 0$. Since $c_v\phi(v_2 \wedge v_3) = 0$ and $v_2 \in$ rad $c_{v_3}\phi$, $\phi[v_2, v \wedge v_3] = 0$. Since $v_3 \in$ rad $c_{v_1}\phi$ and $v_2 = v_1 + bv_3$, $0 = \phi[v_2, v \wedge v_3] = \phi[v_1 + bv_3, v \wedge v_3] = b\phi[v_3, v \wedge v_3]$ for all v. If $b \neq 0$ then $\phi[v_3, v \wedge v_3] = 0$, i.e., rad $c_{v_3}\phi = \langle v_3 \rangle =$ rad $c_{v_1}\phi$. This is a contradiction. Thus $b = 0$ and rad $c_{v_3}\phi =< v_1 >$. Choose e_2 so that $\{v_1, e_2, v_3\}$ is a basis and $\phi[v_1, v_1 \wedge e_2] = 1$. Then the Jacobi identity $\phi[v_1, e_2 \wedge v_3] + \phi[v_3, v_1 \wedge e_2] + \phi[e_2, v_3 \wedge v_1] = 0$ implies that $\phi[e_2, v_3 \wedge v_1] = 0$. Thus relative to this basis ϕ has the matrix $[\phi] = \begin{bmatrix} 0 & 0 & 1 \\ A & 0 & B \\ C & 0 & 0 \end{bmatrix}$ for some A, B, and nonzero C. But this is a contradiction since this matrix has determinant zero. Thus the claim holds.

Extend $\{v_1, v_3\}$ to a basis $\{v_1, v_2', v_3\}$ and let $v = \alpha v_1 + \beta v_2' + \gamma v_3$ be as in the claim. Then, since rad $c_{v_1}\phi = \langle v_3 \rangle$ and rad $c_{v_3}\phi = \langle v_1 + bv_3 \rangle$, $\phi[\beta v_2', v_3 \wedge v_1] = \phi[\beta v_2', v_3 \wedge (v_1 + bv_3)] = \phi[\alpha v_1 + \beta v_2' + \gamma v_3, v_3 \wedge (v_1 + bv_3)] = \phi[v, v_3 \wedge (v_1 + bv_3)] = 1$, since $v_3 \in$ rad $c_{v_1}\phi$ and $v_1 + bv_3 = v_2 \in$ rad $c_{v_3}\phi$. Scale v_2' so that $\phi[v_1, v_1 \wedge v_2'] = 1$ and scale v_3 so that $\phi[v_2', v_3 \wedge v_1] = 1$. Set $e_1 = v_1$, $e_3 = v_3$, and $e_2 = v_2' - \phi[v_2', v_1 \wedge v_2']v_1$. The set $\{e_1, e_2, e_3\}$ is a basis. Since

$$\phi[e_1, e_2 \wedge e_3] = 0 = \phi[e_1, e_3 \wedge e_1],$$

$$\phi[e_1, e_1 \wedge e_2] = \phi[v_1, v_1 \wedge v_2'] = 1,$$

$$\phi[e_2, e_3 \wedge e_1] = \phi[v_2', v_3 \wedge v_1] = 1,$$

$$\phi[e_3, e_1 \wedge e_2] = -\phi[e_2, e_3 \wedge e_1] - \phi[e_1, e_2 \wedge e_3] = -1,$$

$$\phi[e_3, e_3 \wedge e_1] = \phi[v_3, v_3 \wedge (v_1 + bv_3)] = 0,$$

and

$$\begin{aligned}
\phi[e_2, e_1 \wedge e_2] &= \phi[v_2' - \phi[v_2', v_1 \wedge v_2']v_1, v_1 \wedge (v_2' - \phi[v_2', v_1 \wedge v_2']v_1)] \\
&= \phi[v_2' - \phi[v_2', v_1 \wedge v_2']v_1, v_1 \wedge v_2'] \\
&= \phi[v_2', v_1 \wedge v_2'] - \phi[v_2', v_1 \wedge v_2']\phi[v_1, v_1 \wedge v_2'] \\
&= \phi[v_2', v_1 \wedge v_2'] - \phi[v_2', v_1 \wedge v_2']1 \\
&= 0,
\end{aligned}$$

the matrix of ϕ with respect to $\{e_1, e_2, e_3\}$ is $[\phi] = \begin{bmatrix} 0 & 0 & 1 \\ A & 1 & 0 \\ B & 0 & -1 \end{bmatrix}$ where $A = \phi[e_2, e_2 \wedge e_3]$ and $B = \phi[e_3, e_2 \wedge e_3] \neq 0$. Case 2 now splits into three separate subcases.

Subcase 2a ($A \neq 0$): The radical of $c_{e_2}\phi = A(e_2^* \wedge e_3^*) + (e_3^* \wedge e_1^*)$ is $\langle Ae_1 + e_2 \rangle$ since $[A(e_2^* \wedge e_3^*) + (e_3^* \wedge e_1^*)]((Ae_1 + e_2) \wedge \cdot) = [Ae_3^* - Ae_3^*](\cdot) = 0$. The radical of

$c_{Ae_1+e_2}\phi = A(e_1^* \wedge e_2^*) + A(e_2^* \wedge e_3^*) + (e_3^* \wedge e_1^*)$ is $\langle Ae_1 + e_2 + Ae_3\rangle$ since $[A(e_1^* \wedge e_2^*) + A(e_2^* \wedge e_3^*) + (e_3^* \wedge e_1^*)](Ae_1 + e_2 + Ae_3 \wedge \cdot) = [A^2 e_2^* - Ae_1^* + Ae_3^* - A^2 e_2^* - Ae_3^* + Ae_1^*](\cdot) = 0$. Set $e_1' = e_2$, $e_2' = Ae_1 + e_2 + Ae_3$, and $e_3' = Ae_1 + e_2$. Since $\begin{vmatrix} 0 & A & A \\ 1 & 1 & 1 \\ 0 & A & 0 \end{vmatrix} = A^2$, the set $\{e_1', e_2', e_3'\}$ forms a basis when $A \neq 0$ such that rad $c_{e_1'}\phi = \langle e_3'\rangle$ and rad $c_{e_3'}\phi = \langle e_2'\rangle$. Thus this case reduces to Case 1 and so the proposed change of basis exists.

Subcase 2b ($A = 0, B \neq 2$): If $A = 0$ then the matrix of ϕ with respect to $\{e_1, e_2, e_3\}$ is $[\phi] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ B & 0 & -1 \end{bmatrix}$. The radical of $c_{e_1+e_2}\phi = (e_1^* \wedge e_2^*) + (e_3^* \wedge e_1^*)$ is $< e_2 + e_3 >$ since $[(e_1^* \wedge e_2^*) + (e_3^* \wedge e_1^*)](e_2 + e_3 \wedge \cdot) = [-e_1^* + e_1^*](\cdot) = 0$. The radical of $c_{e_2+e_3}\phi = (e_3^* \wedge e_1^*) + B(e_2^* \wedge e_3^*) - (e_1^* \wedge e_2^*)$ is $< -Be_1 - e_2 + e_3 >$ since $[(e_3^* \wedge e_1^*) + B(e_2^* \wedge e_3^*) - (e_1^* \wedge e_2^*)](-Be_1 - e_2 + e_3 \wedge \cdot) = [Be_3^* + e_1^* - Be_3^* - Be_2^* + Be_2^* - e_1^*](\cdot) = 0$. Set $e_1' = e_1 + e_2$, $e_2' = -Be_1 - e_2 + e_3$, and $e_3' = e_2 + e_3$. Since $\begin{vmatrix} 1 & -B & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = B - 2$, the set $\{e_1', e_2', e_3'\}$ forms a basis when $B \neq 2$ such that rad $c_{e_1'}\phi =< e_3' >$ and rad $c_{e_3'}\phi =< e_2' >$. Thus this case reduces to Case 1 and so the proposed change of basis exists.

Subcase 2c ($A = 0, B = 2$): If $A = 0$ and $B = 2$ then the matrix of ϕ with respect to $\{e_1, e_2, e_3\}$ is $[\phi] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$. The proposed change of basis does not exist for this case. The $\phi \in \Phi$ for which this occurs is a set of measure zero. These two facts will be established by calculating the Lie algebra of the stabilizer.

If $X = (a_i^r)$ is in the Lie algebra of the stabilizer then (as was shown in the proof of Proposition 4.28) $a_i^r \phi[e_r, e_j \wedge e_k] + a_j^r \phi[e_i, e_r \wedge e_k] + a_k^r \phi[e_i, e_j \wedge e_r] = 0$. Letting i, j, and k vary yields the following system:

$$2a_1^3 - a_3^1 = 0 \tag{4.18}$$

$$a_1^2 - a_3^2 = 0 \tag{4.19}$$

$$2a_1^1 + a_2^2 - a_1^3 = 0 \tag{4.20}$$

$$2a_2^3 - a_2^1 = 0 \tag{4.21}$$

$$4a_3^3 + 2a_2^2 + a_3^1 = 0 \tag{4.22}$$

$$a_1^1 + a_2^2 + a_3^3 = 0 \tag{4.23}$$

$$2a_3^2 - 2a_1^2 = 0 \tag{4.24}$$

$$a_2^1 - 2a_2^3 = 0 \tag{4.25}$$

$$a_3^1 - 2a_1^3 - a_3^3 - a_1^1 - a_2^2 = 0. \tag{4.26}$$

Clearly (4.26) is dependent on (4.18) and (4.23), (4.24) is the same as (4.19), and (4.25) is the same as (4.21). Since (4.23) implies $a_3^3 = -a_1^1 - a_2^2$ and (4.18) implies $a_3^1 = 2a_1^3$, (4.22) implies $-4a_1^1 - 4a_2^2 + 2a_2^2 + 2a_1^3 = 0$ which is equivalent to $2a_1^1 + a_2^2 - a_1^3 = 0$, i.e., (4.20). Thus the system is equivalent to

$$a_3^1 = 2a_1^3$$

$$a_1^2 = a_3^2$$

$$a_3^1 = -4a_3^3 - 2a_2^2$$

$$a_2^1 = 2a_2^3$$

$$a_1^1 = -a_2^2 - a_3^3.$$

Choosing a_2^2, a_3^3, a_2^3, and a_3^2 determines all other a_j^i. Thus the stabilizer is four dimensional and the orbit is five dimensional. By Lemma 4.28, the change of basis does not exist. Also since the orbit is five dimensional in an eight dimensional space (namely, Φ), it has measure zero. $\square$

Proposition 4.34. *Suppose ϕ is in Φ and A is in* $\mathrm{GL(V)}$*. Then* $[A^{-1} \cdot \phi] = A^t[\phi](A^t)^{-1}|A|$.

Proof. Let $\tilde{A} : \mathrm{V}_2 \to \mathrm{V}_2$ be defined by $\tilde{A}(v \wedge v') = Av \wedge Av'$ and linearity. Let $\mathrm{cof}(A)$ denote the cofactor matrix of A, i.e., $\mathrm{cof}(A) = [(-1)^{i+j}|A_{i,j}|]$. It is well-known that $\mathrm{cof}(A) = (A^t)^{-1}|A|$. Recall that if e_1, e_2, and e_3 form a basis of V then $w_1 = e_2 \wedge e_3$, $w_2 = e_3 \wedge e_1$, and $w_3 = e_1 \wedge e_2$ form a basis of V_2. Write $A = (a^r_c)$, $\tilde{A} = (\tilde{a}^r_c)$, and $\mathrm{cof}(A) = (A^r_c)$ where r denotes the row and c denotes the column. Then $Ae_i = a^r_i e_r$ and $\tilde{A}w_j = \tilde{a}^r_j w_r$ where we sum over r. Also let $\psi = A^{-1} \cdot \phi$. Then $\psi_{i,j} = \psi[e_i, w_j] = \phi[Ae_i, \tilde{A}w_j] = \phi[a^r_i e_r, \tilde{a}^s_j w_s] = a^r_i \tilde{a}^s_j \phi_{r,s} = a^r_i \phi_{r,s} \tilde{a}^s_j$. Let $w_j = e_p \wedge e_q$. Since $\tilde{A}w_j = Ae_p \wedge Ae_q = a^r_p e_r \wedge a^s_q e_s = (a^1_p a^2_q (e_1 \wedge e_2) + a^1_p a^3_q (e_1 \wedge e_3) + a^2_p a^1_q (e_2 \wedge e_1) + a^2_p a^3_q (e_2 \wedge e_3) + a^3_p a^1_q (e_3 \wedge e_1) + a^3_p a^2_q (e_3 \wedge e_2) = (a^2_p a^3_q - a^3_p a^2_q) w_1 + (a^3_p a^1_q - a^1_p a^3_q) w_2 + (a^1_p a^2_q - a^2_p a^1_q) w_3 = |A_{1,j}| w_1 - |A_{2,j}| w_2 + |A_{3,j}| w_3 = A^i_j w_i$, the entries of $\tilde{A}$ are $\tilde{a}^i_j = A^i_j$. So $\psi_{i,j} = a^r_i \phi_{r,s} A^s_j$ which is the (i,j)-th entry of the product $A^t[\phi]\mathrm{cof}(A) = A^t[\phi](A^t)^{-1}|A|$. $\square$

For a fixed basis let ϕ_x denote the element of Φ that has matrix

$$[\phi_x] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & x \\ 0 & 1 & 0 \end{bmatrix}.$$

Proposition 4.35. *Each seven dimensional orbit contains a unique ϕ_x.*

Proof. Suppose $[\psi] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & x \\ 0 & 1 & 0 \end{bmatrix}$, $[\phi] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & y \\ 0 & 1 & 0 \end{bmatrix}$, and $\psi \in \mathrm{GL(V)} \cdot \phi$. Then $[\psi] = A[\phi]A^{-1}|A|$ for some $A \in \mathrm{GL(V)}$. Taking the determinant of this equation yields $|A|^3 = 1$ since $\det[\psi] = 1 = \det[\phi]$ and $|A^{-1}| = |A|^{-1}$. Thus $|A| = 1$ and $[\psi] = A[\phi]A^{-1}$. This implies that $[\psi]$ and $[\phi]$ have the same characteristic polynomials. Since these polynomials are $-\lambda^3+x\lambda+1$ and $-\lambda^3+y\lambda+1$, respectively, $x = y$. $\square$

Lemma 4.36. *Lebesgue measure on* $\mathfrak{sl}(3)$ *is invariant under the action of* $\mathrm{GL}(V)$ *defined by* $A \cdot X = AXA^{-1}$.

Proof. Let dX denote Lebesgue measure on $\mathfrak{sl}(3)$. Since

$$\begin{aligned}
\int f(X)d(AXA^{-1}) &= \int f(A^{-1}XA)dX \\
&= \int f(A^{-1}XA + A^{-1}YA)dX \\
&= \int f(A^{-1}(X+Y)A)dX \\
&= \int f(X+Y)d(AXA^{-1}),
\end{aligned}$$

the uniqueness of translation invariant measures implies that there is a positive constant $c(A)$ such that $d(A \cdot X) = c(A)dX$,i.e., dX is relatively invariant. Thus $A \mapsto c(A)$ is a homomorphism, whence $c(U) = 1$ for any unitary U. Since $A \in \mathrm{GL}(V)$ can be written as $A = U_1DU_2$ where U_1 and U_2 are unitary and D is diagonal, it may be assumed that A is diagonal. Since all matrices commute with diagonals, $A \cdot X = AXA^{-1} = XAA^{-1} = X$. This implies that $d(A \cdot X) = dX$. Therefore dX is invariant under the action of $\mathrm{GL}(V)$. $\square$

Proposition 4.37. *Let* $d\phi = d[\phi]$ *be the Lebesgue measure on* Φ_0 *inherited from Lebesgue measure on* $\mathfrak{sl}(3)$. *Then* $d[A \cdot \phi] = |A|^{-8}d[\phi]$ *for* $A \in \mathrm{GL}(V)$.

Proof. Since $[A \cdot \phi] = (A^{-1})^t[\phi]A^t|A^{-1}|$ and $X \mapsto AXA^{-1}$ is measure preserving, $d[A\cdot\phi] = d(|A^{-1}|[\phi])$. Since $\mathfrak{sl}(3)$ is eight dimensional, $d(|A^{-1}|[\phi]) = |A^{-1}|^8 d[\phi]$. $\square$

Thus $\bigcup_x \mathrm{GL(V)}\cdot\phi_x$ is almost all of Φ. Let $\Phi_0 = \{\phi : \mathrm{GL(V)}\cdot\phi \text{ is seven dimensional}\}$. Then for each $\phi \in \Phi_0$ there exist unique $x \in \mathbb{R}$ and $\overline{A} \in \mathrm{GL(V)}/\mathrm{GL(V)}^{\phi_x}$ such that $\phi = \overline{A}\cdot\phi_x$ (this is well-defined). This gives a one-to-one correspondence between Φ_0 and $\bigcup_x(\mathrm{GL(V)}/\mathrm{GL(V)}^{\phi_x}, x)$.

4.4 The Plancherel Theorem

Since $G\cdot(\lambda,\phi) = (\mathrm{V}^{(2)})^* \times \{\phi\}$ for all $(\lambda,\phi) \in \hat{N}_0$, (λ,ϕ) and $(0,\phi)$ induce to equivalent representations of G. Note that $\pi_{(0,\phi)}$ is a representation of $G^{(0,\phi)}$ since $m_0 \equiv 1$. Thus the essential dual of G is equivalent to the set

$$\{\gamma_{\phi,\chi} := \mathrm{ind}_{G^{(0,\phi)}}^{G}(\pi_{(0,\phi)} \times \chi)\,|\,(0,\phi) \in \hat{N}_0 \text{ and } \chi \in \hat{\mathrm{V}}_\phi\}.$$

The representation $\gamma_{\phi,\chi}$ of G acts on $\mathcal{H}_{\phi,\chi} = \{h : G \to \mathbb{C}\,|\,h(\xi g) = (\pi_{(0,\phi)} \times \chi)(\xi)h(g)\,\forall\,\xi \in G^{(0,\phi)} \text{ and } \int_{G^{(0,\phi)}\backslash G}\|h\|^2 d(G^{(0,\phi)}g) < \infty\}$ by $(\gamma_{\phi,\chi}(g)h)(g') = h(g'g)$.

We first consider the case $n = 2$. Then $\mathrm{V}_\phi \cong \mathbb{R}$. Let X_ϕ be the linear space orthogonal to V_ϕ. Then $\mathrm{V} = X_\phi \bigoplus \mathrm{V}_\phi$. Let x and y denote the X_ϕ and V_ϕ components of v. Since

$$\begin{aligned} g' &= (x'+y', w', \sigma') \\ &= (y', w' - \frac{1}{2}[y',x'], \sigma' + \frac{1}{2}[x',w'] - \frac{1}{12}[y',[y',x']] - \frac{1}{6}[x',[y',x']])(x',0,0), \end{aligned}$$

$\gamma_{\phi,\chi}(g)h \in \mathcal{H}_{\phi,\chi}$, and $c_{y'}\phi = 0$,

$$\begin{aligned} (\gamma_{\phi,\chi}(g)h)(g') &= (\pi_{(0,\phi)} \times \chi)(y', w' - \frac{1}{2}[y',x'], \sigma' + \frac{1}{2}[x',w'] \\ &\qquad - \frac{1}{12}[y',[y',x']] - \frac{1}{6}[x',[y',x']])(\gamma_{\phi,\chi}(g)h)(x',0,0) \\ &= \chi(y')\phi(\sigma' + \frac{1}{2}[x',w'] - \frac{1}{6}[x',[y',x']])(\gamma_{\phi,\chi}(g)h)(x',0,0). \end{aligned}$$

Since $||\chi(y')\phi(\sigma' + \frac{1}{2}[x',w'] - \frac{1}{6}[x',[y',x']])|| = 1$, it may be assumed without loss of generality that $\gamma_{\phi,\chi}(g)$ acts on $\mathcal{H}'_\phi = \{h : X_\phi \to \mathbb{C}\,|\,\int_{X_\phi}\|h(x)\|^2dx < \infty\} \cong \mathrm{L}^2(\mathbb{R})$.

Since

$$\begin{aligned}
(x',0,0)(x+y,w,\sigma) &= (x'+x+y, w+\frac{1}{2}[x',y], \sigma+\frac{1}{2}[x',w] \\
&\quad + \frac{1}{12}[x'-x-y,[x',y]]) \\
&= (y, w+\frac{1}{2}[x',y]-\frac{1}{2}[y,x'+x], \sigma+\frac{1}{2}[x',w] \\
&\quad + \frac{1}{12}[x'-x-y,[x',y]]-[w,x'+x] \\
&\quad + \frac{1}{4}[[y,x'+x],x'+x]-\frac{1}{4}[x'+x,[y,x']] \\
&\quad + \frac{1}{12}[x'+x-y,[y,x'+x]])(x'+x,0,0),
\end{aligned}$$

$h \in \mathcal{H}_{\phi,\chi}$, and $c_y\phi = 0$, the action of $\gamma_{\phi,\chi}(g)$ on $h \in \mathcal{H}'_\phi$ is defined by

$$\begin{aligned}
(\gamma_{\phi,\chi}(g)h)(x') &= (\gamma_{\phi,\chi}(g)h)(x',0,0) \\
&= h((x',0,0)(x+y,w,\sigma)) \\
&= (\pi_{(0,\phi)} \times \chi)(y, w+\frac{1}{2}[x',y]-\frac{1}{2}[y,x'+x], \sigma+\frac{1}{2}[x',w] \\
&\quad + \frac{1}{12}[x'-x-y,[x',y]]+\frac{1}{2}[x'+x, w+\frac{1}{2}[x',y]] \\
&\quad - \frac{1}{6}[x'+x,[y,x'+x]]+\frac{1}{12}[y,[y,x'+x]])h(x'+x,0,0) \\
&= \chi(y)\phi(\sigma+\frac{1}{2}[x',w]+\frac{1}{12}[x'-x,[x',y]]+\frac{1}{2}[x'+x,w] \\
&\quad - \frac{1}{6}[x'+x,[y,x'+x]]-\frac{1}{6}[x'+x,[y,x']])h(x'+x,0,0) \\
&= \chi(y)\phi(\sigma+\frac{1}{2}[x',w]+\frac{1}{12}[x'-x,[x',y]]+\frac{1}{2}[x'+x,w] \\
&\quad - \frac{1}{6}[x'+x,[y,x'+x]]-\frac{1}{6}[x'+x,[y,x']])h(x'+x).
\end{aligned}$$

For notational convenience, let

$$\xi(x',x,y) = \frac{1}{12}[x'-x,[x',y]]-\frac{1}{6}[x'+x,[y,x'+x]]-\frac{1}{6}[x'+x,[y,x']].$$

Let F_2,F_3, and F_4 be the Fourier transforms in y, w, and σ, respectively. Then

$$\begin{aligned} K_{\phi,\chi,f}(x',x) &:= \iiint \chi(y)\phi(\sigma + \frac{1}{2}[x+x',w] + \xi) f(x-x',y,w,\sigma)dy\,dw\,d\sigma \\ &= F_2F_3F_4 f(x-x',\chi + \phi(\xi(x',x-x',\cdot), c_{\frac{x+x'}{2}}\phi,\phi) \end{aligned}$$

and

$$\begin{aligned} \gamma_{\phi,\chi}(f)(h)(x') &= \iiiint f(x,y,w,\sigma)\gamma_{\phi,\chi}(x,y,w,\sigma)h(x')dx\,dy\,dw\,d\sigma \\ &= \iiiint f(x,y,w,\sigma)\chi(y)\phi(\sigma + \frac{1}{2}[2x'+x,w] \\ &\qquad + \xi(x',x,y))h(x'+x)dx\,dy\,dw\,d\sigma \\ &= \iiiint f(x-x',y,w,\sigma)\chi(y)\phi(\sigma + \frac{1}{2}[x'+x,w] \\ &\qquad + \xi(x',x-x',y))h(x)dx\,dy\,dw\,d\sigma \\ &= \int K_{\phi,\chi,f}(x',x)h(x)dx. \end{aligned}$$

Thus $\gamma_{\phi,\chi}(f)$ is a kernel operator and $\mathrm{Tr}(\gamma_{\phi,\chi}(f)\gamma_{\phi,\chi}(f)^*) = ||K_{\phi,\chi,f}||^2$. Let $u = \frac{x'+x}{2}$ and $u' = x - x'$. Then $du\,du' = dx\,dx'$ and

$$\begin{aligned} ||K_{\phi,\chi,f}||^2 &= \iint ||F_2F_3F_4 f(x-x',\chi + \phi(\xi(x',x-x',\cdot), c_{\frac{x+x'}{2}}\phi,\phi)||^2 dx\,dx' \\ &= \iint ||F_2F_3F_4 f(u',\chi + \phi(\xi(u,u',\cdot), c_u\phi,\phi)||^2 du\,du'. \end{aligned}$$

If $\phi = A\cdot\phi_0$ then $c_u\phi = c_u(A\cdot\phi_0) = A\cdot(c_{A^{-1}u}\phi_0) = c_{\alpha u}\phi_0$. Thus $||K_{\phi,\chi,f}||^2 = \iint ||F_2F_3F_4 f(u',\chi+\phi(\xi(u,u',\cdot), c_{\alpha u}\phi_0,\phi)||^2 du\,du'$. Let u go to $\frac{u}{\alpha}$. Then $||K_{\phi,\chi,f}||^2 = \iint ||F_2F_3F_4 f(u',\chi+\phi(\xi(u,u',\cdot), c_u\phi_0,\phi)||^2 \frac{du}{\alpha} du'$. Let $\lambda = c_u\phi_0$. Since λ varies over $(\mathrm{V}^{(2)})^*$ in a one-to-one way as u varies over X_{ϕ_0}, integration with respect to du is equivalent to integration with respect to $d\lambda$ (the dual measure to dw). Thus

$$\begin{aligned} \mathrm{Tr}(\gamma_{\phi,\chi}(f)\gamma_{\phi,\chi}(f)^*) &= ||K_{\phi,\chi,f}||^2 \\ &= \iint ||F_2F_3F_4 f(u',\chi+\phi(\xi(u,u',\cdot),\lambda,\phi)||^2\alpha^{-1}d\lambda\,du' \\ &= \iint ||F_2F_4 f(u',\chi+\phi(\xi(u,u',\cdot),w,\phi)||^2\alpha^{-1}dw\,du'. \end{aligned}$$

Integrating both sides with respect to $d\chi$ where $d\chi$ is dual to dy yields

$$\begin{aligned}
\int \mathrm{Tr}(\gamma_{\phi,\chi}(f)\gamma_{\phi,\chi}(f)^*)d\chi &= \iiint ||F_2F_4f(u',\chi+\phi\cdot\xi,w,\phi)||^2\alpha^{-1}dw\,du'\,d\chi \\
&= \iiint ||F_2F_4f(u',\chi+\phi\cdot\xi,w,\phi)||^2\alpha^{-1}d\chi\,dw\,du' \\
&= \iiint ||F_2F_4f(u',\chi,w,\phi)||^2\alpha^{-1}d(\chi-\phi\cdot\xi)\,dw\,du' \\
&= \iiint ||F_2F_4f(u',\chi,w,\phi)||^2\alpha^{-1}d\chi\,dw\,du'.
\end{aligned}$$

Recall that the set $\Phi = \{\alpha\phi_0 + \beta\phi_1 \,|\, \alpha,\beta \in \mathbb{R}\}$. Integrating both sides with respect to $\alpha\,d\phi$ where $d\phi = d\alpha\,d\beta$ is dual to $d\sigma$ yields

$$\begin{aligned}
\iint \mathrm{Tr}(\gamma_{\phi,\chi}(f)\gamma_{\phi,\chi}(f)^*)\alpha\,d\chi\,d\phi &= \iiiint ||F_2F_4f(u',\chi,w,\phi)||^2\alpha^{-1}dw\,du'\alpha\,d\chi\,d\phi \\
&= \iiiint ||F_2F_4f(u',\chi,w,\phi)||^2dw\,du'\,d\chi\,d\phi \\
&= \iint ||F_4f(u',y,w,\phi)||^2dy\,dw\,du'\,d\phi \\
&= \iint ||f(u',y,w,\sigma)||^2dy\,dw\,du'\,d\sigma \\
&= ||f||_2^2.
\end{aligned}$$

So Plancherel measure is $d\mu(\gamma_{\phi,\chi}) = \alpha d\chi\,d\phi = \alpha d\chi\,d\alpha\,d\beta$.

We show that μ has the correct relative invariance factor. Let $\phi = \alpha\phi_0 + \beta\phi_2$ and $B = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}$. Then, as was shown in section 2.2, $B \cdot \gamma_{\phi,\chi} = \gamma_{B\cdot\phi,B\cdot\chi}$. Since

$$\begin{aligned}
B\cdot\phi([e_1,[e_1,e_2]]) &= \phi([B^{-1}e_1,[B^{-1}e_1,B^{-1}e_2]]) \\
&= \phi([b_1^{-1}e_1,[b_1^{-1}e_1,b_2^{-1}e_2]]) \\
&= \frac{\alpha}{b_1|B|}
\end{aligned}$$

and

$$\begin{aligned}B\cdot\phi([e_2,[e_1,e_2]]) &= \phi([B^{-1}e_2,[B^{-1}e_1,B^{-1}e_2]])\\ &= \phi([b_2^{-1}e_2,[b_1^{-1}e_1,b_2^{-1}e_2]])\\ &= \frac{\beta}{b_2|B|},\end{aligned}$$

we see that $B\cdot\phi = \frac{\alpha}{b_1|B|}\phi_0 + \frac{\beta}{b_2|B|}\phi_1$. Also $B\cdot\chi(y) = \chi(B^{-1}y) = \chi(b_2^{-1}y) = b_2^{-1}\chi(y)$. Therefore

$$\begin{aligned}d\mu(B\cdot\gamma_{\phi,\chi}) &= d\mu(\gamma_{B\cdot\phi,B\cdot\chi})\\ &= \frac{\alpha}{b_1|B|}\frac{d\chi}{b_2}\frac{d\alpha}{b_1|B|}\frac{\beta}{b_2|B|}\\ &= \frac{\alpha d\chi\, d\alpha\, d\beta}{|B|^5}\\ &= |B|^{-5}d\mu(\gamma_{\phi,\chi}).\end{aligned}$$

Thus μ has the correct relative invariance factor.

When $n=3$, $\mathrm{V}_\phi = G^{(\lambda,\phi)}/N$ is trivial for all $\phi\in\Phi$ since $G^{(\lambda,\phi)} = N$. Thus $\chi\in\hat{\mathrm{V}}_\phi$ is trivial. So the induced representations $\gamma_{\phi,\chi}$ all act on $\mathcal{H}_\phi = \{h : G\to\mathbb{C}\,|\,h(\xi g) = \pi_{(0,\phi)}(\xi)h(g)\,\forall\,\xi\in N$ and $\int_{N\backslash G}\|h\|^2 d(Ng)<\infty\}$. The χ will now be dropped from the notation. Since $\gamma_\phi h\in\mathcal{H}_\phi$ and $g=(v,w,\sigma) = (0,w,\sigma+\frac{1}{2}[v,w])(v,0,0)$ for all $g\in G$,

$$(\gamma_\phi(g)h)(g') = \phi(\sigma'+\frac{1}{2}[v',w'])(\gamma_\phi(g)h)(v',0,0)$$

and

$$\begin{aligned}h(g'g) &= h(v'+v,w'+w,\sigma'+\sigma+\frac{1}{2}([v',w]-[v,w'])+\frac{1}{12}[v'-v,v'\wedge v])\\ &= \phi(\sigma'+\sigma+\frac{1}{2}([v',w]-[v,w'])+\frac{1}{12}([v'-v,v'\wedge v]\\ &\quad+\frac{1}{2}[v'+v,w'+w])h(v'+v,0,0)\\ &= \phi(\sigma'+\frac{1}{2}[v',w'])\phi(\sigma+[v',w]+\frac{1}{12}[v'-v,v'\wedge v]+\frac{1}{2}[v,w])h(v'+v,0,0).\end{aligned}$$

Thus $(\gamma_\phi(g)h)(v',0,0) = \phi(\sigma + [v',w] + \frac{1}{12}[v'-v, v'\wedge v] + \frac{1}{2}[v,w])h(v'+v,0,0)$. Since $N\backslash G \simeq \mathrm{V}$, the induced representation γ_ϕ can be modelled on $\mathrm{L}^2(\mathrm{V})$.

Let $K_{\phi,f}(v',v) = \phi(\frac{1}{12}[2v'-v, v'\wedge v])(F_2F_3f)(v-v', c_{\frac{v'+v}{2}}\phi, \phi)$. Then

$$\begin{aligned}
(\gamma_\phi(f)h)(v') &= \int f(g)(\gamma_\phi(g)h)(v')dg \\
&= \iiint f(v,w,\sigma)\phi(\sigma + [v',w] + \frac{1}{12}[v'-v, v'\wedge v] \\
&\quad + \frac{1}{2}[v,w])h(v'+v)dv\,dw\,d\sigma \\
&= \iiint f(v-v',w,\sigma)\phi(\sigma + [v',w] + \frac{1}{12}[2v'-v, v'\wedge v] \\
&\quad + \frac{1}{2}[v-v',w])h(v)dv\,dw\,d\sigma \\
&= \iiint f(v-v',w,\sigma)\phi(\sigma + \frac{1}{2}[v'+v,w] \\
&\quad + \frac{1}{12}[2v'-v, v'\wedge v])h(v)dv\,dw\,d\sigma \\
&= \int \phi(\frac{1}{12}[2v'-v, v'\wedge v])(F_2F_3f)(v-v', c_{\frac{v'+v}{2}}\phi, \phi)h(v)dv \\
&= \int K_{\phi,f}(v',v)h(v)dv.
\end{aligned}$$

Thus $\gamma_\phi(f)$ is a kernel operator and its norm is

$$\begin{aligned}
\mathrm{Tr}(\gamma_\phi(f)\gamma_\phi(f)^*) &= \|K_{\phi,f}\|^2 \\
&= \iint \|\phi(\frac{1}{12}[2v'-v, v'\wedge v])\|^2 \|(F_2F_3f)(v-v', c_{\frac{v'+v}{2}}\phi, \phi)\|^2 dv\,dv'.
\end{aligned}$$

By noting that $\|\phi(\frac{1}{12}[2v'-v, v'\wedge v])\|^2 = 1$ and changing to the variables $u = v - v'$ and $u' = \frac{v+v'}{2}$, we obtain

$$\mathrm{Tr}(\gamma_\phi(f)\gamma_\phi(f)^*) = \iint \|(F_2F_3f)(u, c_{u'}\phi, \phi)\|^2 du\,du'. \tag{4.27}$$

Suppose ϕ is in $\mathrm{GL(V)}\cdot\phi_x$. Then $\phi = A\cdot\phi_x$ for some A. Thus $c_{u'}\phi(v_1\wedge v_2) = c_{u'}A\cdot\phi_x(v_1\wedge v_2) = A\cdot\phi_x[u', v_1\wedge v_2] = \phi_x[A^{-1}u', A^{-1}v_1\wedge A^{-1}v_2] = c_{A^{-1}u'}\phi_x(A^{-1}v_1\wedge A^{-1}v_2) = \tilde{A}\cdot c_{A^{-1}u'}\phi_x(v_1\wedge v_2)$. This implies that $c_{u'}\phi = \tilde{A}\cdot c_{A^{-1}u'}\phi_x$. Thus (4.32) becomes

$$\mathrm{Tr}(\gamma_\phi(f)\gamma_\phi(f)^*) = \iint \|(F_2F_3f)(u, \tilde{A}\cdot c_{A^{-1}u'}\phi_x, \phi)\|^2 du\, du'. \tag{4.28}$$

Let $u' \mapsto Au'$. Then $du' \mapsto |A|du'$ and (4.28) becomes

$$\mathrm{Tr}(\gamma_\phi(f)\gamma_\phi(f)^*) = \iint \|(F_2F_3f)(u, \tilde{A}\cdot c_{u'}\phi_x, \phi)\|^2 |A| du\, du'. \tag{4.29}$$

Let $\lambda_x = c_{u'}\phi_x$. Since the set of contractions $\{c_{u'}\phi_x \,|\, u' \in \mathrm{V}\} = (\mathrm{V}^{(2)})^*$ in a one-to-one way, integrating u' over V is equivalent to integrating λ_x over $(\mathrm{V}^{(2)})^*$. Thus (4.29) becomes

$$\mathrm{Tr}(\gamma_\phi(f)\gamma_\phi(f)^*) = \iint \|(F_2F_3f)(u, \tilde{A}\cdot \lambda_x, \phi)\|^2 |A| du\, d\lambda_x. \tag{4.30}$$

Let $\lambda_x \mapsto \tilde{A}^{-1}\lambda_x$. Then $d\lambda_x \mapsto |\tilde{A}^{-1}|d\lambda_x$ and (4.30) becomes

$$\mathrm{Tr}(\gamma_\phi(f)\gamma_\phi(f)^*) = \iint \|(F_2F_3f)(u, \lambda_x, \phi)\|^2 |A||\tilde{A}^{-1}| du\, d\lambda_x.$$

Since $|\tilde{A}^{-1}| = |A|^2$,

$$\mathrm{Tr}(\gamma_\phi(f)\gamma_\phi(f)^*) = \iint \|(F_2F_3f)(u, \lambda_x, \phi)\|^2 |A|^3 du\, d\lambda_x. \tag{4.31}$$

Let $\lambda = \lambda_0$. If w_1^*, w_2^*, and w_3^* form a basis for $(\mathrm{V}^{(2)})^*$ then $\lambda_x = c_{u'}\phi_x = u_2'w_1^* + u_3'w_2^* + (u_1' + xu_2')w_3^*$. So the Jacobian of the transformation $\lambda \mapsto \lambda_x$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{bmatrix}$ which has determinate one. Thus $d\lambda_x = d\lambda$ for all x and (4.31) becomes

$$\mathrm{Tr}(\gamma_\phi(f)\gamma_\phi(f)^*) = \iint \|(F_2F_3f)(u, \lambda, \phi)\|^2 |A|^3 du\, d\lambda. \tag{4.32}$$

Since F_2 is unitary,

$$\mathrm{Tr}(\gamma_\phi(f)\gamma_\phi(f)^*) = \iint \|(F_3f)(u, w, \phi)\|^2 |A|^3 du\, dw. \tag{4.33}$$

Formal integration of both sides of (4.32) with respect to the measure $d\mu(\gamma_\phi) = |A|^{-3}d[\phi]$ (where $d[\phi]$ is normalized Lebesgue measure on $\mathfrak{sl}(3)$ and $\phi = A \cdot \phi_x$) yields

$$\begin{aligned}\int \mathrm{Tr}(\gamma_\phi(f)\gamma_\phi(f)^*)d\mu(\phi) &= \iiint \|(F_3 f)(u,w,\phi)\|^2 |A|^3 du\, dw\, |A|^{-3} d\phi \\ &= \iiint \|f(u,w,\sigma)\|^2 du\, dw\, d\sigma \\ &= \|f\|_2^2.\end{aligned}$$

Thus $d\mu(\gamma_\phi) = |A|^{-3}d[\phi]$ is the Plancherel measure. We will now show that μ decomposes over the orbit space.

Let $p : \Phi_0 \to \Phi_0/\mathrm{GL(V)} \cong \mathbb{R}$ map any ϕ to its orbit under GL(V). Let $E \subset \Phi_0/\mathrm{GL(V)}$, m be Lebesgue measure on $(\mathrm{V}^{(3)})^*$, and $O_x = \mathrm{GL(V)} \cdot \phi_x$. Consider the measure p_*m on the orbit space. Since $p^{-1}(E+y) = \cup_{x \in E+y} O_x = \cup_{x-y \in E} O_x = \cup_{w \in E} O_{w-y} = \cup_{w \in E} O_w - \phi_{y'}$, we have that $p_*m(E+y) = m(\cup_{w \in E} O_w - \phi_{y'}) = m(\cup_{w \in E} O_w) = p_*m(E)$, i.e., p_*m is translation invariant. If $\tilde{m}$ is a finite measure on $(\mathrm{V}^{(3)})^*$ equivalent to m, then $p_*\tilde{m} \sim p_*m$. So $p_*\tilde{m}(E) = 0 \iff p_*m(E) = 0 \iff p_*m(E+y) = 0 \iff p_*\tilde{m}(E+y) = 0$. Thus $p_*\tilde{m}$ is a σ-finite, quasi-invariant measure on $\mathbb{R}$ and, therefore, is equivalent to Lebesgue measure on $\mathbb{R}$. Thus p_*m is equivalent to Lebesgue measure. By Theorem 1.12, $m = \int m_x dx$ where m_x is supported on $p^{-1}(x) = \mathrm{GL(V)}/\mathrm{GL(V)}^{\phi_x}$. Furthermore, the a.e. uniqueness of the measures m_x shows that m_x inherits the relative invariance of m for a.e. x. Thus $d[\phi] = dm(\phi) = dm_x(\overline{A})dx$ where ϕ corresponds to $(\overline{A}, x)$ and, therefore, $d\mu(\gamma_\phi) = |\overline{A}|^{-3}dm_x(\overline{A})dx$. This is well defined as all elements of any stabilizer have determinant one. Thus Plancherel measure disintegrates over the fibers of p as $\mu = \int \mu_x dx$ where $d\mu_x(\overline{A}) = |\overline{A}|^{-3}d\mu_x(\overline{A})$.

If $B \in \mathrm{GL}(V)$ then $B \cdot \phi = BA \cdot \phi_x$. Thus $d\mu(B \cdot \gamma_\phi) = d\mu(\gamma_{B\cdot\phi}) = |BA|^{-3} d[B \cdot \phi]$. By Proposition 4.37, $d\mu(B \cdot \gamma_\phi) = |BA|^{-3}|B|^{-8} d[\phi] = |B|^{-11} d\mu(\gamma_\phi)$. Thus we see that μ has the expected relative invariance factor.

When $n \geq 4$, the dimension of $(\mathrm{V}^{(2)})^*$ is greater than n. Thus $c_{\mathrm{V}}\phi$ is an n dimensional subspace of $(\mathrm{V}^{(2)})^*$ for any ϕ in Φ. Since $(\mathrm{V}^{(2)})^*/c_{\mathrm{V}}\phi \cong \mathbb{R}^{\frac{n^2-3n}{2}}$, there is a invariant measure m_ϕ on $(\mathrm{V}^{(2)})^*/c_{\mathrm{V}}\phi$ such that $f \mapsto \iint f(\lambda + c_v\phi) dv\, dm_\phi(\overline{\lambda})$ is a Haar integral on $(\mathrm{V}^{(2)})^*$. Choose $s_\phi : (\mathrm{V}^{(2)})^*/c_{\mathrm{V}}\phi \to (\mathrm{V}^{(2)})^*$ so that $s_\phi(\overline{\lambda}) + c_{\mathrm{V}}\phi = \overline{\lambda}$. Then $\hat{G}_0 \cong \{(s_\phi(\overline{\lambda}), \phi) \,|\, \overline{\lambda} \in (\mathrm{V}^{(2)})^*/c_{\mathrm{V}}\phi, \phi \in \Phi\}$. The induced representation $\gamma_{\overline{\lambda},\phi} := \mathrm{ind}_N^G(s_\phi(\overline{\lambda}), \phi)$ acts on $\mathcal{H}_{\overline{\lambda},\phi} := \{h : G \to \mathbb{C} |\, h(ng) = (\overline{\lambda}, \phi)(n)h(g)\, \forall\, n \in N, g \in G$ and $\int_{\mathrm{V}} \|h\|^2 dv < \infty\}$ by $[\gamma_{\overline{\lambda},\phi}(g)h](g') = h(g'g)$. As before, the representation $\gamma_{\overline{\lambda},\phi}$ may be modelled on $\mathrm{L}^2(\mathrm{V})$. The action is then

$$[\gamma_{\overline{\lambda},\phi}(v,w,\sigma)h](v') = s_\phi(\overline{\lambda})(w)\phi(\sigma + [v', w] + \frac{1}{2}[v,w] + \frac{1}{12}[v' - v, v' \wedge v])h(v' + v).$$

Thus

$$\begin{aligned}
(\gamma_{\overline{\lambda},\phi}(f)h)(v') &= \int f(g)(\gamma_{\overline{\lambda},\phi}(g)h)(v')dg \\
&= \iiint f(v,w,\sigma)s_\phi(\overline{\lambda})(w)\phi(\sigma + [v',w] + \frac{1}{12}[v'-v, v' \wedge v] \\
&\qquad + \frac{1}{2}[v,w])h(v'+v)dv\, dw\, d\sigma \\
&= \iiint f(v - v', w, \sigma)s_\phi(\overline{\lambda})(w)\phi(\sigma + [v',w] + \frac{1}{12}[2v' - v, v' \wedge v] \\
&\qquad + \frac{1}{2}[v - v', w])h(v)dv\, dw\, d\sigma \\
&= \iiint f(v - v', w, \sigma)s_\phi(\overline{\lambda})(w)\phi(\sigma + \frac{1}{2}[v' + v, w] \\
&\qquad + \frac{1}{12}[2v' - v, v' \wedge v])h(v)dv\, dw\, d\sigma \\
&= \int \phi(\frac{1}{12}[2v' - v, v' \wedge v])(F_2F_3f)(v - v', s_\phi(\overline{\lambda}) + c_{\frac{v'+v}{2}}\phi, \phi)h(v)dv.
\end{aligned}$$

So $(\gamma_{\overline{\lambda},\phi}(f)h)(v') = \int_{\mathrm{V}} K_{\overline{\lambda},\phi,f}(v', v)h(v)dv$ where

$$K_{\overline{\lambda},\phi,f}(v', v) := \phi(\frac{1}{12}[2v' - v, v' \wedge v])(F_2F_3f)(v - v', s_\phi(\overline{\lambda}) + c_{\frac{v'+v}{2}}\phi, \phi).$$

Thus $\gamma_{\overline{\lambda},\phi}(f)$ is a kernel operator and its norm is

$$\begin{aligned}
\mathrm{Tr}(\gamma_{\overline{\lambda},\phi}(f)\gamma_{\overline{\lambda},\phi}(f)^*) &= \|K_{\overline{\lambda},\phi,f}\|^2 \\
&= \iint \|(F_2F_3f)(v-v', s_\phi(\overline{\lambda}) + c_{\frac{v'+v}{2}}\phi, \phi)\|^2 dv\, dv'.
\end{aligned}$$

Note we used that $\|\phi(\frac{1}{12}[2v'-v, v'\wedge v])\|^2 = 1$. Changing to the variables $u = v-v'$ and $u' = \frac{v+v'}{2}$, we obtain

$$\mathrm{Tr}(\gamma_{\overline{\lambda},\phi}(f)\gamma_{\overline{\lambda},\phi}(f)^*) = \iint \|(F_2F_3f)(u, s_\phi(\overline{\lambda}) + c_{u'}\phi, \phi)\|^2 du\, du'. \tag{4.34}$$

For fixed ϕ, integrating both sides of (4.33) with respect to m_ϕ yields

$$\begin{aligned}
\int \mathrm{Tr}(\gamma_{\overline{\lambda},\phi}(f)\gamma_{\overline{\lambda},\phi}(f)^*)dm_\phi(\overline{\lambda}) &= \iiint \|(F_2F_3f)(u, s_\phi(\overline{\lambda}) + c_{u'}\phi, \phi)\|^2 du\, du'\, dm_\phi(\overline{\lambda}) \\
&= \iint \|(F_2F_3f)(u, \lambda, \phi)\|^2 du\, d\lambda \\
&= \iint \|(F_3f)(u, w, \phi)\|^2 du\, dw.
\end{aligned}$$

Integration over Φ with respect to normalized Lebesgue measure then yields

$$\begin{aligned}
\iint \mathrm{Tr}(\gamma_{\overline{\lambda},\phi}(f)\gamma_{\overline{\lambda},\phi}(f)^*)dm_\phi(\overline{\lambda})\, d\phi &= \iiint \|(F_3f)(u, w, \phi)\|^2 du\, dw\, d\phi \\
&= \iiint \|f(u, w, \sigma)\|^2 du\, dw\, d\sigma \\
&= \|f\|^2.
\end{aligned}$$

Thus, at least formally, Placherel measure is $\mu = \int m_\phi d\phi$.

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Vita

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