Closed form solutions to the optimality equation of minimal norm actuation

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CLOSED FORM SOLUTIONS TO THE OPTIMALITY EQUATION OF MINIMAL NORM ACTUATION

A Thesis
Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Master of Science in The Department of Mechanical and Industrial Engineering

by
Jorge Antonio Guerra Contreras
B.S., Georgia Institute of Technology, 2012
December 2013
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# Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>a</td>
<td>arbitrary vector</td>
</tr>
<tr>
<td>A</td>
<td>state-space system parameter matrix</td>
</tr>
<tr>
<td>b</td>
<td>control input vector</td>
</tr>
<tr>
<td>B</td>
<td>state-space system parameter matrix</td>
</tr>
<tr>
<td>C</td>
<td>damping coefficient matrix</td>
</tr>
<tr>
<td>e_i</td>
<td>(i^{th}) unit vector</td>
</tr>
<tr>
<td>f</td>
<td>velocity gain vector</td>
</tr>
<tr>
<td>g</td>
<td>position gain vector</td>
</tr>
<tr>
<td>I</td>
<td>identity matrix</td>
</tr>
<tr>
<td>K</td>
<td>stiffness coefficient matrix</td>
</tr>
<tr>
<td>M</td>
<td>system mass matrix</td>
</tr>
<tr>
<td>m</td>
<td>number of eigenvalues to be assigned by partial pole placement</td>
</tr>
<tr>
<td>n</td>
<td>number of degrees-of-freedom</td>
</tr>
<tr>
<td>q</td>
<td>pole placement vector</td>
</tr>
<tr>
<td>s</td>
<td>complex poles</td>
</tr>
<tr>
<td>t</td>
<td>time</td>
</tr>
<tr>
<td>u</td>
<td>control input</td>
</tr>
<tr>
<td>V</td>
<td>matrix of open loop eigenvectors</td>
</tr>
<tr>
<td>v_i</td>
<td>(i^{th}) eigenvector</td>
</tr>
<tr>
<td>w</td>
<td>closed loop eigenvector</td>
</tr>
<tr>
<td>x</td>
<td>state-space position vector</td>
</tr>
<tr>
<td>y</td>
<td>controlled system output</td>
</tr>
</tbody>
</table>
\( z \)  
first-order state vector

\( \alpha_i \)  
optimal input vector coefficients

\( \beta \)  
control input vector controllability vector

\( \Gamma \)  
controllability matrix

\( \eta \)  
number of different optimal input vectors

\( \Lambda \)  
open loop pole matrix

\( \lambda_i \)  
open loop eigenvalue

\( \mu_i \)  
closed loop eigenvalue, closed loop poles

\( \vartheta, \xi, \tau_i \)  
pole-placement factors

\( \nu \)  
first order eigenvectors
Abstract

This research focused on the problem of minimal norm actuation in the context of partial natural frequency or pole assignment applied to undamped vibrating systems by state feedback control. The result of the research was the closed form solutions for the minimal norm control input and gain vectors. These closed form solutions should took open loop eigenpairs and the desired frequencies of the controlled system and outputted the optimal controller parameters. This optimization technique ensures that the system’s dynamics will be effectively controlled while keeping the controller effort minimal. The controller must then be able to shift only the desired the system poles anywhere in the complex s-plane in order to give the system certain desired characteristics with no spillover.

The open loop system dynamics were found by applying a discrete model of the studied vibrating system and then finding the eigenvalue problem associated with the second-order open loop system equations. A first order realization was then performed on the system in order to know its response to certain initial conditions. The system’s dynamics where to be modified via closed loop control.

Partial natural frequency assignment was chosen as the control technique so that certain system frequencies could be left untouched to ensure that the system will not respond in an unexpected manner. The control was to be optimized by minimizing the norm of the control input and gain vectors. A closed form solution for these vectors was found in so that these vectors could be simply calculated using an algorithm that takes the open loop eigenpairs and the desired eigenvalues of the system and outputs the two vectors. This closed form solution was successful implemented and the controller parameters found were applied to a vibrational system.
A simulation for the un-optimized and optimized cases was performed applying both controllers to the same system. The response and controller forces for both cases were plotted in MATLAB and compared. Both systems showed the desired system response meaning that they both had the same effect on the system. Inspecting both controller efforts showed that the optimal control case simulation showed less controller effort than the arbitrary case thus showing successful implementation of minimal norm actuation.
Chapter 1: Introduction and Literature Review

As defined by Ginsberg (2001) vibration is a natural phenomenon where physical systems experience oscillatory motion about a reference state, usually about a local or global equilibrium position. These motions are ubiquitous in the physical world; everything in nature experiences vibrational motion. This behavior is even observed in atoms where their vibration is a measure of the systems internal kinetic energy. Like in most other disciplines of system dynamics, there are applications where vibration is necessary and applications where these motions cause problems. Their desirability depends on the system’s characteristics, application and industrial purpose. Some examples of desired vibration are: the motion of a guitar string producing different musical notes depending on the vibrational frequency, the eardrums and vocal chords in humans that allow hearing and verbal communication and in the cone of a speaker that allow for the oscillations of pressure that make sound. In systems engineering undesired vibration can cause serious problems to the performance and integrity of mechanical systems. Structural vibrations in an airplane, a building’s motion due to an earthquake, an unbalanced shaft rotating and a car going over a rough patch of asphalt are all examples of undesired vibrations. These vibrations unwanted because they present forces that result in structural failure, wasted energy and unwanted noise. These problems are often small issues but over time, they can have expensive if not catastrophic results as shown in Loh and Chao (1996).

Mechanical vibrations can be further divided into two categories depending on whether there is an external force continuously acting on the system or whether the system’s motion is only dependent on an initial condition. Free vibrations deal with motions that are studied after the system has already been disturbed thus depending only on the systems initial
conditions. Forced vibrations deal with motions associated with external forces that are present during the time interval that the system is studied. Both types of vibrations are important in vibration control and analysis. These types of vibrations can be further categorized into random and deterministic vibrations. It is important to know the complete picture so that vibrations can be accurately analyzed and, if necessary, controlled.

As previously stated, undesirable vibrations can lead to a myriad of engineering problems. The obvious solution to solve these problems is vibration control so that the vibrations can be attenuated or ideally eliminated as shown in Ram and Elhay (1996). While vibration control seems like a solution for every case of undesired vibration, it is not always possible to effectively control every system. Some systems are uncontrollable in nature and other engineering solutions must be explored.

When the system’s vibration profile is known, it is often appropriate to use passive control, which is often called structural modification. This technique has been studied by Singh and Ram (2000) and involves adding specific inertia, energy dissipation and energy storage components to an existing structure. In practice, this is accomplished by adding masses, dampers or materials with different mechanical properties. These components are chosen so that the resulting frequencies of the system are moved away from the system’s exciting frequencies. This modification results in a more desirable vibration profile when compared to the original system. Suryawanshi, Shitole and Rahane (2012) present a famous example of this. The tuned mass-damper system installed in the Taipei 101 building in Taipei, Taiwan is an example of passive vibration control. Earthquakes are an important civil engineering consideration in Taiwan due to the island’s proximity to a fault line. The Taipei 101 features a tuned mass damper in its higher levels that attenuates the building’s motion due
to wind and land movement. Without this damping system it would have been impossible to build a skyscraper as tall as the Taipei 101 in Taiwan. Figure 1.1. Shows a schematic for a tuned mass damper system adapted from Suryawashi et al.

![Tuned Mass Damper General Schematic](image)

Figure 1.1. Tuned Mass Damper General Schematic

The other control technique is *active control*, which can be categorized as either open loop or closed loop control. In open loop controls, the system’s output has no effect on the controller’s action and the system expected motion must be well understood, estimated and monitored as stated in Ogata (2004). Closed loop control consists of taking the difference between the input signal and the output feedback signal from a sensor and feeding this error signal into the controller so that it may iteratively reduce the error between the desired output and the actual output. The controller consists of the error calculating function and the amplifier that mathematically manipulates the error signal. The feedback signal is acquired via sensors that monitor the system’s desired outputs such as position, and velocity for mechanical systems. In modern control engineering these values are converted from the mechanical domain to the electrical domain by the sensors. Once this signal is fed into the
controller it is then outputted as an electrical signal to the actuators. These actuators are attached to an of the system’s $n$ degrees of freedom (DOF) so that they can exert a controlling force resulting in a desired output. Figure 1.2 shows a closed loop system in block diagram form adapted from Ogata (2004).

Figure 1.2. Block Diagram of a Closed Loop Control System

Ogata (2010) gives three common control methods used to give a mechanical system desired dynamics. The control method selection depends on the amount of DOF that a system has and on the desired system output. A 1-DOF system can be controlled by single-input, single-output (SISO) control. A multiple DOF ($n- \text{DOF}, n \neq 1$) system must be controlled by either Multiple-input, Multiple-output (MIMO) control or by Single-input, Multiple output control (SIMO). In practice, a MIMO controller is more desirable than a SIMO controller because it allows for effective control of all the degrees of freedom while providing for more flexibility in the system control.

While active control is an effective control strategy when used on a controllable system, it is imperative that the controlled system dynamics will not further destabilize an already unstable system or even destabilize an already stable system. In a stable system, the system’s kinetic energy will become potential energy over time. In an unstable system the
system’s potential energy will become kinetic energy as time increases. It is a possibility that the systems dynamics will exaggerate a controller force or respond in an unexpected manner when exposed to a new external force. Stability is an important topic in the study of mechanical systems because it gives an insight on how the system will respond to an applied force. A controller, in essence, stabilizes a system around a new desired equilibrium. Stability theory is the foundation of control engineering.

The goal of this thesis is to derive a closed form solution that provides the actuator input vector that minimizes the norm of the input vector and the gain vector so that the system can be controlled effectively and efficiently via partial pole placement. Partial pole placement ensures that the controller will not affect all of the systems natural frequencies while changing only the frequencies that to be modified. By finding the optimal actuator input vector a system’s vibrations can be attenuated without having to use an excessive amount of actuator force. This allows for the use of smaller actuators and less energy in applications where space, weight and power are at a premium.

1.1 Control Via Pole Placement

The goal of pole-placement of full-state feedback is to improve the response of a system by shifting one or more desired poles further to the left hand plane if the system is controllable. This method employs an input vector that is multiplied by a gain vector. The new desired poles determine this gain vector as shown in Wonham (1967).

Many studies have been done on the feasibility of control via pole placement using a single input controller. These studies usually deal with control of flexible structures in order to improve the structure’s response to a stimulus. Schulz and Heimbold (1983) state that active control is achieved on flexible structures via dislocated Actuator/Sensor positions.
1.2 Spillover and Partial Pole Placement

It is important to have an accurate model in vibration control in order to maximize the effectively of the controller. Because these systems have an infinite number of degrees of freedom it is impossible to apply a controller to every single mode of the system. Controllers employ a lumped parameter model that has a much smaller number of degrees of freedom than the real system. This leads to there being leftover uncontrolled modes, which cause spillover as mentioned by Balas (2008). This spillover causes instability in the case that the force of the actuator excites these leftover modes. If this happens, the system may not be effectively controlled and even be made to vibrate even more than the uncontrolled system.

Datta, Elhay and Ram (1997) presented a method that uses a control system modeled by a second order differential equation to derive an explicit solution to the partial pole assignment problem. This solution allows for only a small part of the modal spectrum to be modified and it leaves the remaining modes unchanged. This results in the closed-loop system having the desired stability characteristics with much lower uncertainty than by other pole placement methods.

1.3 Optimization

An important topic in controls engineering is that of optimization. In applications where space, weight and energy are constraints, it is desirable to have an optimized control system. This system should have small enough actuators that use just enough power to effectively control motion. The method derived in this thesis is to find a formula that show the best actuator input vector in a vibrating system when multiple poles are to be modified via partial pole placement. The method derived here builds off the optimization method for single pole placement developed by Guzzardo, Pang and Ram (2013).
1.4 Practical Applications

There are many papers that show the versatility of pole placement controls. One important application is in aerospace engineering. Wang (2003) discusses this application where feedback control and piezoelectric actuators are used to attenuate the vibrations of flexible structures used in airplanes and rockets. Pearson, Goodall and Lyndon (1994) show that the structural vibrations in helicopters can be controlled using pole placement. Bishop, Paynter and Sunkel (1992) apply pole-placement to the control of a space station. While these papers show the application of pole-placement in aerospace engineering, this control theory can also be used in the control of mechanical systems and in the control of structures typically studied in civil engineering.

This paper focuses on the application of pole-placement in vibration control of structures. Structures that use vibration controllers are usually referred to as smart structures. These structures use actuators and computers to minimize vibrations that could lead to structural failure. As previously stated the placement of these actuators can be optimized so that the vibrations can be successfully controlled with minimal controller effort.

1.5 Thesis Outline

This thesis explores mechanical vibrations and their closed-loop control. Chapter 2 presents the derivation of a mathematical model used to determine the characteristics of open-loop vibrations of systems. This thesis will deal with a discrete model of vibrating systems due to discrete nature of pole placement control theory. Once the model has been derived, an example will be conducted to study the dynamics of the system.

Chapter 3 will integrate the partial pole placement technique developed by Datta, Elhay and Ram into the previously developed open loop model. This controller will be
applied to the previous chapter’s example and the, now, closed-loop system will be analyzed and its stability and response will be discussed in order to determine the effectively of this controller.

Chapter 4 will show the minimal norm control optimization technique and apply it to the partial-pole placement control described in chapter 3 by developing a closed form solution to the problem of calculating the optimal actuator input and gain vectors in a vibrating system where multiple poles are to be changed. This will build on the solution defined by Guzzardo, Pang and Ram (2013) for the optimal actuator placement when only one of the system’s poles is to be changed. This solution will allow for effective and efficient control of vibrating members. An example will then illustrate these claims.

Chapter 5 will present a simulation of a controlled vibrating system. The first simulation will be performed using an arbitrary input control demonstrating partial pole placement without minimal norm optimization. The second simulation will be performed on the same system but this time applying the optimized control derived in the preceding chapter. The control effort for both simulations will be shown.

Chapter 6 will be a conclusion of the preceding chapters where all the topics will be brought together and the applicability of the solution presented in chapter 4 will be discussed. This chapter will also feature a brief commentary on possible future work and experimental considerations. All of the code used in the examples and simulations will be featured in the appendix.
Chapter 2: Open Loop Vibration

2.1 System Equations of Motion

An equation of motion for mechanical systems can be directly derived from Newton’s second law of motion. If there are no external forces applied on the system, the second law of motion for a single mass is

\[ F = m\ddot{x} . \]  

(2.1)

When a system has multiple masses and multiple degrees-of-freedom (DOF) it is more convenient to write the system’s equations of motion in the matrix form

\[ M\dddot{x} + C\ddot{x} + Kx = 0 , \]  

(2.2)

where \( M \) is a \( n \times n \) positive definite matrix, \( C \) and \( K \) are \( n \times n \) semi-positive definite matrices consisting of the system’s damping and stiffness coefficients respectively. The vector \( x \) is an \( n \times 1 \)-position vector and each dot represents the time derivative. Thus the equations contain the position, velocity, and acceleration of the system.

The equations of motion may be solved using separation of variables. Take the general solution for a damped system

\[ x = ve^{\omega t} , \]  

(2.3)

which may be differentiated twice with respect to time and substituted into equation (2.2) to give

\[ s^2Mve^{\omega t} + sCve^{\omega t} + Kve^{\omega t} = 0 , \]  

(2.4)

Dividing the exponential terms out of (2.4) the equation becomes

\[ (s^2M + sC + K)v = 0 , \]  

(2.5)
It is important to note that while the solution $v = 0$ exists, it is trivial and of no use to the dynamic analysis of a system because this means that the system has no motion. In order to find the other solutions to the equation it is necessary to solve

$$\left| s^2M + sC + K \right| = 0 ,$$

(2.6)

This determinant, whose order is $n$ will yield a polynomial whose solutions are the eigenvalues, $s_i$, of the system. By applying each eigenvalue to (2.5), each corresponding eigenvector, $v_i$, is found. The eigenvectors presented in this paper will be scaled to have unit norm for the sake of definiteness, thus

$$v_i^T v_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \quad i, j = 1, 2, \ldots, n .$$

(2.7)

### 2.2 First Order Realization

The second order equations of motion shown in the previous section may be manipulated into a first order state-space system of equations so that the system’s response over time can be found. A state-space form is obtained from the original system equations by performing a first order realization of the system. This realization is given by

$$\begin{bmatrix} I & 0 \\ C & M \end{bmatrix} \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} - \begin{bmatrix} 0 & I \\ -K & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = 0 ,$$

(2.8)

where $I$ is an $n \times n$ identity matrix. For the sake of simplicity take (2.8) to be

$$A = \begin{bmatrix} I & 0 \\ C & M \end{bmatrix} , B = \begin{bmatrix} 0 & I \\ -K & 0 \end{bmatrix} \text{ and } z = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} ,$$

(2.9)

which may be written as

$$A \ddot{z} - Bz = 0 ,$$

(2.10)
It is possible to solve equation (2.10) by using the general solution for a damped system

\[ z = ve^{st}, \]  

(2.11)

where, \( \mathbf{v} \), is the vector

\[ \mathbf{v}_i = \begin{bmatrix} v_i \\ sv_i \end{bmatrix}, \quad i=1,2,…,2n \]  

(2.12)

Applying (2.12) to (2.10) reduces the problem to

\[ (\mathbf{A} - s\mathbf{B})\mathbf{v} = 0, \]  

(2.13)

which is the generalized eigenvalue problem for the first order state space equation, it is important to note that this eigenvalue problem returns \( 2n \) eigenvalues instead to the \( n \) eigenvalues returned by equation (2.6). The eigenvalues returned by the first order system are the natural frequencies or poles of the system.

2.3 Eigenvalues, Eigenvectors and Stability

The motion of a mechanical system can be characterized using the system’s natural frequencies, or poles, and its mode shapes. The natural frequency is the frequency at which the system oscillates around its equilibrium position. The mode shapes are the direction in which the members of the system oscillate. These poles are found via the eigenvalue problems previously mentioned.

Eigenvalues depend on the makeup of the system. A second order system consisting only of springs will have only real eigenvalues, while a system made up of springs and dampers will have complex conjugate pairs. In this case the real part describes the rate of change in the magnitude of the motion while the imaginary part describes the frequency of the system’s response. This phenomenon is the key of stability analysis for linear systems.
The eigenvalues calculated from the second order system equation are related to the system’s poles by

\[ \lambda = -s^2, \quad i=1,2,...,n. \]  

(2.14)

The poles, \( s_i \), are the solutions to the eigenvalue problem applied to the first order realization of the equation of motion.

Eigenvectors are extremely important because they describe the overall motion of each degree of freedom in a mechanical system. Each second-order eigenvector consists of \( n \) rows in an \( n \)-DOF system. Each column element represents the motion of one of the DOF. Each eigenvector is dependent on an eigenvalue. These are called eigenpairs.

The system’s poles may be placed on a two-axis plane called the \( s \)-plane consisting of the imaginary axis (\( s \)-axis) and the real axis (Re-axis). When an eigenvalue is negative it is in the left-hand plane (LHP) and when it is real it is in the right-hand plane (RHP). This allows for the conclusion that any system with all its eigenvalues in the LHP is stable and any system with any of its eigenvalues in the RHP is unstable. Figure 2.1 from Franklin (1994) shows the behavior of poles located in different quadrants of the \( s \)-plane.

Figure 2.1. Pole Behavior on the \( s \)-plane
From this figure, it can be seen that the complex part of the poles are what describes a system’s oscillations while the real parts are what show if the system’s motion decreases to zero (stable) or increases to infinity (unstable). Systems with a real part of zero will simply oscillate indefinitely without decreasing or increasing amplitude and may be considered stable if system is a vibrational system.

### 2.4 Free Response of the System

The free response of the system describes how the system will react when an initial condition is applied at a time $t=0$ and when no additional external forces are present. The free response uses the first order realization of the system rather than it’s second order differential equation. The system’s output is given by

$$y(t) = \begin{bmatrix} x \\ \dot{x} \end{bmatrix},$$

(2.15)

where the position and velocities of each of the system’s degrees of freedom are contained if full system observability is assumed. The position vector, $x$, is given by

$$x = \sum_{i=1}^{2n} a_i s_i e^{s_i t},$$

(2.16)

which is a linear combination of the system’s $2n$ solutions. The velocity vector

$$\dot{x} = \sum_{i=1}^{2n} a_i s_i v_i e^{s_i t},$$

(2.17)

is simply found by differentiating (2.16). The coefficient, $a_i$, is determined by the system’s initial conditions. At $t=0$ (2.16) and (2.17) become

$$x(0) = \sum_{i=1}^{2n} a_i v_i,$$

(2.18)
By applying (2.12), these initial conditions can be written in matrix form as

$$Ua = \begin{pmatrix} x_0 \\ \dot{x}_0 \end{pmatrix},$$  

(2.18)

where

$$a = (a_1, a_2, \ldots, a_{2n})^T,$$  

(2.19)

and

$$U = [\nu_1, \nu_2, \ldots, \nu_{2n}].$$  

(2.20)

If $U$ is invertible, then the constants in $a$ are found using simple linear algebra. By calculating all relevant values, the position and velocity of each DOF of the system can be found at a point in time.

### 2.5 Example 1: System Response

The two degree-of-freedom system shown in figure 2.2 consists of two springs and two dampers. It is possible to show this system’s response applying the methods outlined in the previous section of this chapter.

![2-DOF Spring-Damper System](image)
In this example the system parameter matrices are

\[
M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.5 & -0.2 \\ -0.2 & 0.2 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 8 & -4 \\ -4 & 4 \end{bmatrix}.
\]

These matrices can be inputted directly into the first order realization (2.9) in order to put the system in state space form. The new state matrices are

\[
A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.5 & -0.2 & 1 & 0 \\ -0.2 & 0.2 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 8 & 4 & 0 & 0 \\ 4 & -4 & 0 & 0 \end{bmatrix},
\]

and the position and velocity vectors become

\[
z = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}.
\]

Applying (2.11) to arrive at the eigenvalue problem and solving for it gives the poles,

\[
S = \begin{bmatrix} -0.2980 + 3.2220i & 0 & 0 & 0 \\ 0 & -0.2980 - 3.2220i & 0 & 0 \\ 0 & 0 & -0.0520 + 1.2351i & 0 \\ 0 & 0 & 0 & -0.0520 + 1.2351i \end{bmatrix}.
\]

As expected there are 2\(n\) complex conjugate eigenvalues for the first order state space system. The negative nature of the real part of each pole places them on the left side of the Re-Im plane, which shows that the system is stable and that its oscillations will decay to zero after a discrete amount of time. The eigenvectors of the first order system are

\[
U = \begin{bmatrix} 0.0743 + 0.2224i & 0.0743 + 0.2224i & 0.2183 - 0.2788i & 0.2183 + 0.2788i \\ -0.0408 + 0.1386i & -0.0408 - 0.1386i & 0.3596 - 0.4464i & 0.3596 + 0.4464i \\ 0.6943 + 0.3057i & 0.6943 - 0.3057i & 0.3330 + 0.2842i & 0.3330 - 0.2842i \\ -0.4344 - 0.1728i & -0.4344 + 0.1728i & 0.5326 + 0.4674i & 0.5326 - 0.4674i \end{bmatrix}.
\]
The next step in finding the response of the system is finding the constants of \( a \) in (2.18).

First the initial conditions

\[
x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]

and

\[
\dot{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

These initial conditions give

\[
a = \begin{bmatrix}
0.2996 + 1.678i \\
0.2996 - 1.678i \\
0.0937 - 0.5183i \\
0.0937 + 0.5183i
\end{bmatrix}.
\]

By using these values in (2.16) and (2.17) it is possible to evaluate and plot the response of each degree-of-freedom of the system. Figure 2.3 shows the vibration response.

![Graphs showing vibration response](image_url)

Figure 2.3. Open Loop Response of Example 1
The response seen in Figure 2.3 is consistent with the expected response of the system. There is a clear decay in motion over time, which agrees with the stability analysis performed on the poles of the system. The response of the second mass is also as expected. The second mass has higher amplitude of motion because of the lower damping coefficient. The motion of this system may be controlled via pole placement where the system could be made to reach equilibrium sooner or oscillate at a different frequency. Optimizing this control so that it may be accomplished with smaller actuators and less energy is the goal of this thesis.
Chapter 3: Closed Loop Control

3.1 Closed Loop Equation of Motion

The response of physical systems can be modified via closed loop control if the system is observable and controllable. Feedback control may be used to either stabilize a system or change a systems response. Applying a controlling external force to the equation open loop equation of motion yields

$$\mathbf{M}\ddot{\mathbf{z}} + \mathbf{C}\dot{\mathbf{z}} + \mathbf{K}\mathbf{z} = \mathbf{b}\mathbf{u}(t),$$

(3.1)

where

$$\mathbf{u}(t) = \mathbf{f}^T\dot{\mathbf{z}} + \mathbf{g}^T\mathbf{z}, \quad \mathbf{f}, \mathbf{g} \in \mathbb{R}^n$$

(3.2)

is the control input and \(\mathbf{f}\) and \(\mathbf{g}\) are the velocity and position gains respectively. The \(n \times 1\) vector \(\mathbf{b}\) is the controller’s input selection vector. Like in the previous chapter, the differential equation (3.2) can be solved by setting

$$\mathbf{z} = \mathbf{w}e^{\mu},$$

(3.3)

Using (3.3) in (3.1) yields the eigenvalue problem for the controlled system

$$\left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)\mathbf{w} = \mathbf{b}\left(s\mathbf{f}^T + \mathbf{g}^T\right)\mathbf{w}, \quad \mu = s^2$$

(3.4)

which yields \(n\) eigenvalues \(\mu\) and eigenvectors.

3.2 Controllability

A system is completely controllable if the control input selector vector \(\mathbf{b}\) alters every output state in a finite time interval. Taking the linear first order controllability equation that takes into account both the system parameters and the control selection vector

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\mathbf{u},$$

(3.4)

where
and

\[ \beta = \begin{bmatrix} 0 \\ M^{-1}b \end{bmatrix}, \quad (3.6) \]

is a 2n x 1 vector containing the actuator selection vector and the mass matrix, \( M \). In order to verify the controllability of the system it is necessary to create a controllability matrix

\[ \Gamma = \begin{bmatrix} \beta & -A\beta & A^2\beta & \ldots & A^{2n-1}\beta \end{bmatrix}. \quad (3.7) \]

If rank(\( \Gamma \)) = 2n, then the system is fully controllable and an appropriate control technique may be applied to change the dynamics of the system.

### 3.3 Pole Placement Control

State feedback control or pole placement is an effective method of active (closed loop) vibration control. This method involves assigning pre-determined desired poles to a system. This allows the moving of the natural frequencies of an unstable system to the left hand side of the imaginary-real plane, moving the natural frequencies of an already stable system further to the left allow or simply moving the poles up and down the imaginary axis changing the vibrational frequency of the system.

If \( b^Tv_i \neq 0 \) the open loop eigenvalues \( \lambda_i \) may be modified into desired positive closed loop eigenvalues by applying the controller force, \( bg^Tv \). When \( b^Tv_i = 0 \) then the closed loop natural frequencies are the same as the open loop natural frequencies and the system is uncontrollable by this method.
3.4 Partial Pole Placement Control

Pole placement control involves the movement of all of the system’s eigenvalues. While this is an effective control technique, there is a risk of moving poles that were already at a desirable location in the real-imaginary plane to an undesirable location. Datta, Elhay and Ram (1997) provide a method for partial pole placement assignment. Using this technique, desired poles can be attained without affecting the location of poles that are to remain untouched. By pole placement the poles of (3.3) become the set
\[ \{ \mu_1, \mu_2, \ldots, s_{m+1}, \ldots, s_n \}. \] (3.8)

In order to apply these eigenvalues to the system, it is necessary to find the velocity and position gain vectors. The components of the velocity gain vector \( \mathbf{f} \) and the position gain vector \( \mathbf{g} \) are chosen as
\[ \mathbf{f} = M V \Lambda \mathbf{q}, \] (3.9)
and
\[ \mathbf{g} = -K V \mathbf{q}, \] (3.10)

Where the matrices
\[ \Lambda = \begin{bmatrix} s_1 \\ \vdots \\ s_m \end{bmatrix}, \quad \mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_n] \] (3.11)
contains the open loop poles and eigenvectors that are to be changed and
\[ \mathbf{q}_j = \frac{1}{b^T \mathbf{v}_j} \frac{\mu_j - s_j}{s_j} \prod_{i=1 \atop i \neq j}^{m} \frac{\mu_j - s_i}{s_i - s_j}, \quad j=1,2,\ldots,m \] (3.12)
is the pole placement vector. By applying these terms to system (closed loop system), and performing a first order realization, the dynamics of the controlled can be obtained. The modified first order realization for a controlled system is

\[
\begin{bmatrix}
I & 0 \\
(C - bf^T) & M
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\ddot{x}
\end{bmatrix} =
\begin{bmatrix}
0 & I \\
-(K - bg^T) & 0
\end{bmatrix}
\begin{bmatrix}
x \\
\dot{x}
\end{bmatrix}.
\]

This control technique has been explored and optimized by Guzzardo et al (2013). This paper discusses the optimization of the controller by minimizing the second norm, \(|bg^T|\) by deriving a closed form solution for the input selection vector, \(b\), and the position gain vector, \(g\), when \(M = I\), \(C = 0\) and multiple poles are to be changed. This is referred to as minimal norm actuation in context of natural frequency assignment. Provided in this paper is a closed form solution in the case that only one natural frequency of the system is to be assigned while keeping the rest of the natural frequencies unchanged by using partial pole placement. This paper also provides the equations for optimality for the control of several of a system’s natural frequencies. While these equations are stated, a closed form of their solution is not presented. This thesis builds upon these findings to present a closed form solution of these equations.

3.5 Example 2: Partial Pole Placement Control

The two degree-of-freedom system from the previous chapter is to be controlled via non-optimized partial pole placement. In this example controlling actuators are placed at both DOF. Figure 3.1 shows the placement of the actuators.
A controller may only be applied if the system is controllable. Applying the system parameters to (3.5) yields the state matrix

\[
A = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-8 & 4 & -0.5 & 0.2 \\
4 & -4 & 0.2 & -0.2
\end{bmatrix}.
\]

In this example an arbitrary control placement vector is chosen to apply pole-placement control to the system. This arbitrary vector is

\[
b = \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix},
\]

where the vector components are chosen to be

\[
\begin{bmatrix}
b_1 \\
b_2
\end{bmatrix} = \begin{bmatrix}
3 \\
1
\end{bmatrix}.
\]

This system has the same open loop dynamics as the example on the previous chapter of this thesis. The first-order poles of the systems may be applied to (3.12) in order to find the controller gain vectors. The control of this system is not optimized so it is appropriate to select an arbitrary control location vector \( b \). The desired system poles for this system are

\[
\{-0.2980 + 3.2220i, \ -0.2980 - 3.2220i, \ -0.5 + 1.0000i, \ -0.5 - 1.0000i\}.
\]
As seen on (3.19) only the last two system poles are to be changed and not the whole set of poles. This is the advantage of partial pole placement. Applying these new poles to (3.12) yields the pole placement vector

\[ q = \begin{bmatrix} -0.2682 + 0.1221i \\ -0.2682 - 0.1221i \end{bmatrix}. \]

This vector can be used to find both the position and velocity controller gains. Applying (3.20) to (3.9) and (3.10) yields

\[ f = \begin{bmatrix} -0.2481 \\ -0.3999 \end{bmatrix}, \]

which is the velocity control vector and the position gain vector

\[ g = \begin{bmatrix} 0.0567 \\ 0.1395 \end{bmatrix}. \]

Applying these vectors to the closed loop first order realization yields the new closed loop poles

\[ S = \begin{bmatrix} -0.2980 + 3.2220i & 0 & 0 & 0 \\ 0 & -0.2980 - 3.2220i & 0 & 0 \\ 0 & 0 & -0.5000 + 1.0000i & 0 \\ 0 & 0 & 0 & -0.5000 - 1.0000i \end{bmatrix}, \]

and the closed loop eigenvectors

\[ U = \begin{bmatrix} -0.0740 + 0.2226i & -0.0740 - 0.2226i & 0.3918 + 0.2898i & 0.3918 - 0.2898i \\ 0.0406 - 0.1387i & 0.0406 + 0.1387i & 0.5047 + 0.4859i & 0.5047 - 0.4859i \\ -0.6952 - 0.3048i & -0.6952 + 0.3048i & -0.4857 + 0.2469i & -0.4857 - 0.2469i \\ 0.4334 + 0.1723i & 0.4334 - 0.1723i & -0.7383 + 0.2617i & -0.7383 - 0.2617i \end{bmatrix}. \]

These results are consistent with the expected results of partial pole placement. Only the desired poles were changed and these new poles yield modified eigenvectors. Figure 3.2 shows the closed loop response of the controlled system.
The results shown in Figure 3.2 are as expected. The system now reaches equilibrium much faster than the response shown in example 1. The real part of the poles was moved further to the left side of the s-plane so that the function decays much faster. The imaginary part of the pole was moved closer to the Re axis so the system has a slower frequency. The following chapter deals with the optimization of this control when special structural characteristics are seen in the controllable system.
Chapter 4: Closed Form Solutions

4.1 Statement of Purpose

The problem considered in this thesis is that of minimal norm actuation in the context of natural frequency assignment by feedback control. Specifically the problem of deriving a closed form solution for finding the optimal actuator selection vector, \( \mathbf{b} \), so that pole placement control with minimal \( |\mathbf{b}\mathbf{g}^T| \) can be applied. This results in effective dynamic control while maintaining minimal controller effort. This is especially useful in applications where the controlling actuators are small or the application requires for minimal energy usage.

4.2 Modified Equations of Motion

The closed form solution for \( \mathbf{b} \) derived in this thesis requires that \( \mathbf{M} = \mathbf{I} \) and that \( \mathbf{C} = \mathbf{I} \). Applying these constraints, the controlled equation of motion (3.1) becomes

\[
\dot{\mathbf{z}} + \mathbf{Kz} = \mathbf{bg}^T \mathbf{z} .
\]  

(4.1)

Using separation of variables and applying the general solution for an undamped system

\[
\mathbf{z}(t) = \mathbf{w} \sin \gamma t ,
\]  

(4.2)

equation (4.1) becomes

\[
(\mathbf{K} - \mu \mathbf{I})\mathbf{w} = \mathbf{bg}^T \mathbf{w} , \quad \mu = \gamma^2 .
\]  

(4.3)

The control vectors used in (4.3) can be found analytically by applying the method derived in this thesis for optimal pole placement control.

4.3 Closed Form Solutions

The closed form solution for the position control vector \( \mathbf{g} \) to be used in pole placement is found analytically via Lemma 1. Note that this procedure uses the second order eigenvalues for the system but this method will also work with the first order eigenvalues as
used in Chapter 3. The first order eigenvalues can be easily obtainable by using the relationship in (4.3).

**Lemma 1**

Take the following expression

\[
\mathbf{g} = \sum_{k=1}^{m} \mu_k \mathbf{v}_k,
\]

where

\[
\mu_k = \frac{\lambda_k - \mu_k}{\mathbf{b}^T \mathbf{v}_k} \prod_{i=1, i \neq k}^{m} \frac{\lambda_k - \mu_i}{\lambda_k - \lambda_i},
\]

applies the new desired eigenvalues. The new eigenvalue set

\[
\{\mu_k\} = \{\mu_1, \ldots, \mu_m, \lambda_{m+1}, \ldots, \lambda_n\},
\]

Consists of \(m\) desired eigenvalues and \(n\) total eigenvalues. This Lemma is a direct adaptation of Theorem 3.2 shown in Datta, Elhay and Ram (1997).

Defining

\[
\xi_k = (\lambda_k - \mu_k) \prod_{i=1, i \neq k}^{m} \frac{\lambda_k - \mu_i}{\lambda_k - \lambda_i},
\]

then it follows from (4.5) and (4.7) that

\[
\xi_k = \mu_k \mathbf{b}^T \mathbf{v}_k.
\]

Also define,

\[
\tau_k = \mu_k \mathbf{v}_k.
\]

From (4.4), (4.7) and (4.10) the position control gain vector can be solved by the following closed form solution
To arrive at a closed form solution for the control input selection vector, \( \mathbf{b} \), it is necessary to recall that the objective of the optimization method put forward in this thesis is to minimize \( |\mathbf{b} \mathbf{g}^T| \). Noting the relationship between vector norms

\[
|\mathbf{b} \mathbf{g}^T| = ||\mathbf{g}||,
\]

and

\[
||\mathbf{a}||^2 = \mathbf{a}^T \mathbf{a},
\]

for any real vector \( \mathbf{a} \) give a basis for relating both vectors. It is therefore sufficient to minimize \( \mathbf{g}^T \mathbf{g} \) subject to the constraint

\[
\mathbf{b}^T \mathbf{b} = 1,
\]

in order to achieve the desired optimization objective. Applying equation (4.10) to the above relationships gives

\[
\mathbf{g}^T \mathbf{g} = \frac{\tau_1^T \tau_1}{(\mathbf{b}^T \mathbf{v}_1)^2} + \frac{\tau_2^T \tau_2}{(\mathbf{b}^T \mathbf{v}_2)^2} + \ldots + \frac{\tau_m^T \tau_m}{(\mathbf{b}^T \mathbf{v}_m)^2}.
\]

Noting that \( \tau_i^T \tau_j = 0 \) for \( i \neq j \) and applying this to the Lagrangian term put forward by Guzzardo et al. (2013), the input vector associated with the problem is found via

\[
L(b) = \frac{\tau_1^T \tau_1}{(\mathbf{b}^T \mathbf{v}_1)^2} + \frac{\tau_2^T \tau_1}{(\mathbf{b}^T \mathbf{v}_1)^2} + \ldots + \frac{\tau_m^T \tau_1}{(\mathbf{b}^T \mathbf{v}_m)^2} + \theta \mathbf{b}^T \mathbf{b},
\]

where \( \theta \) is a Lagrange multiplier. By differentiating (4.15) with respect to \( \mathbf{b} \) and setting it equal to zero, operation yields
where $\mathbf{e}_i$ is the $i^{th}$ unit vector. The equations in matrix (4.16) may be written, for simplicity, in the vector summation form,

$$\mathbf{b} = \mathbf{a}_1 \mathbf{v}_1 + \mathbf{a}_2 \mathbf{v}_2 + \ldots + \mathbf{a}_n \mathbf{v}_n,$$

(4.17)

where

$$\mathbf{a}_k = \frac{\tau_k^T \tau_k}{\theta (\mathbf{b}^T \mathbf{v}_k)^3}.$$

(4.18)

Substituting (4.16) in (4.17) and using the orthonormal relations in (2.7) gives

$$\mathbf{a}_k = \frac{\tau_k^T \tau_k}{\theta \mathbf{a}_k^3},$$

(4.19)

which may also be written as

$$\mathbf{a}_k = \frac{\sqrt{|\tau_k|}}{\sqrt[4]{\theta}}.$$

(4.20)

Substituting (4.18) and (4.21) in (4.14) gives

$$\mathbf{b}^T \mathbf{b} = \sum_{k=1}^{m} \alpha_k^2 = \frac{1}{\sqrt{\theta}} \sum_{k=1}^{m} |\tau_k| = 1,$$

(4.21)

by virtue of (2.7) the lagrangian is found by

$$\theta = \left( \sum_{k=1}^{m} |\tau_k| \right)^2.$$

(4.22)

The corresponding control gain vector is
which may be written directly as a function of the eigenvalue modification term (4.8) and the eigenvectors

\[ g = \frac{1}{b^r v_1^T} \tau_1 + \frac{1}{b^r v_2^T} \tau_2 + \ldots + \frac{1}{b^r v_m^T} \tau_m, \tag{4.23} \]

Lemma 2

\[ |bg^r| = \sqrt{\theta} \tag{4.25} \]

Proof

It follows from (4.18) that

\[ bv_i = \alpha_i \quad i = 1, 2, \ldots, m \tag{4.26} \]

by virtue of the orthonormal conditions (2.7), and hence (4.26) gives

\[ g = \frac{1}{\alpha_1} \tau_1 + \frac{1}{\alpha_2} \tau_2 + \ldots + \frac{1}{\alpha_m} \tau_m, \tag{4.27} \]

so that

\[ g^r g = \frac{\tau_1^T \tau_1}{\alpha_1^2} + \frac{\tau_2^T \tau_2}{\alpha_2^2} + \ldots + \frac{\tau_m^T \tau_m}{\alpha_m^2}. \tag{4.28} \]

From (4.20) we have

\[ \alpha_k^2 = \frac{\tau_k^T \tau_k}{\sqrt{\theta}}, \tag{4.29} \]

and (4.29) may be written in the form

\[ g^r g = \sqrt{\theta} \left( \sqrt{\frac{\tau_1^T \tau_1}{\alpha_1^2}} + \sqrt{\frac{\tau_2^T \tau_2}{\alpha_2^2}} + \ldots + \sqrt{\frac{\tau_m^T \tau_m}{\alpha_m^2}} \right), \tag{4.30} \]

which gives

\[ g^r g = \theta, \tag{4.31} \]

\[ \theta \]
by virtue of (4.23). Noting that

\[ |bg^T| = |g^Tg|, \tag{4.32} \]

which proves Lemma 2.

**Corollary 1**

The problem of minimal norm actuation has in general \(2^{m-1}\) different solutions up to a sign change.

**Proof**

Each unit norm eigenvector \(v_i\) is unique, up to a sign factor. By using all possible combinations of the vectors in the set \(\pm v_1, \pm v_2, \ldots, \pm v_m\), Equation (4.17) may generally yield \(\eta \equiv 2^m\) different optimal input vectors \(b_i, b_1, b_2, \ldots, b_\eta\). Clearly, if \(b\) is one optimal input vector with corresponding control gain vector \(g\) then \(-b\) is also an optimal input vector with corresponding gain vector \(-g\). The consequence is that there are generally \(2^{m-1}\) essentially different solutions to the minimal norm actuation problem. Note that by (4.11) and (4.23), \(\theta\) is independent of the optimal \(b_i\) chosen.

It is useful to device an algorithm for calculating due to the number of equations needed to find the optimal pole placement input vector. Algorithm 1 spells out the procedure for finding \(b\). This algorithm shows which system parameters are needed and what equations or definitions are to be used in conjunction with these parameters.
**Algorithm 1**

*Input:*

1. Mass matrix $\textbf{M} = \textbf{I}$ and $n \times n$ symmetric positive definite stiffness matrix $\textbf{K}$. The normalized eigenpairs of the matrix pencil $\textbf{K} - \lambda \textbf{M}$ are

$$\{\lambda_i, \textbf{v}_i\}, \quad \textbf{v}_i^T \textbf{v}_i = 1, \quad i = 1, 2, \ldots, n$$

2. Closed-loop eigenvalues

$$\mu_i = \begin{cases} 
\mu_i, & i = 1, 2, \ldots, m \\
\lambda_i, & i = m + 1, m + 2, \ldots, n 
\end{cases}$$

*Algorithm:*

3. Calculate $\xi_i, \quad i = 1, 2, \ldots, m$ from (4.7)

4. Calculate $\tau_i, \quad i = 1, 2, \ldots, m$ from (4.9)

5. Calculate $\theta$, from (4.22)

6. Calculate $\alpha_i, \quad i = 1, 2, \ldots, m$ from (4.21)

7. Calculate $\textbf{b}$, from (4.17)

8. Calculate $\textbf{g}$ from (4.24)

*Output:*

Optimal actuating input vector $\textbf{b}$, and its corresponding control gain vector $\textbf{g}$, which assign the eigenvalues of the closed loop system as desired in 2 above, and have the property that $|\textbf{bg}^T| = \sqrt{\theta}$ is the minimal norm over all possible vectors $\textbf{b}$.

It is noted that by permuting the sign of the open-loop normalized eigenvectors $\textbf{v}_i$, $i = 1, 2, \ldots, m$, other optimal input location vectors $\textbf{b}$ may be obtained. It follows therefore
that in general there are $2^m$ solutions to the problem. All solutions yield the same minimal norm $|bg^\top| = \sqrt{\theta}$. The following example demonstrates the above observations.

4.4 Example 3

Taking $M = I$, and the stiffness matrix

$$K = \begin{bmatrix} 5 & -3 \\ -3 & 13 \end{bmatrix},$$

and solving for the stiffness matrix eigenvalues (this is adequate since $M = I$ and $C = 0$). The second order eigenpairs of the system are

$$\begin{cases} \lambda_1 = 4 & v_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \\ \lambda_2 = 14 & v_2 = \frac{1}{\sqrt{10}} \begin{pmatrix} -1 \\ 3 \end{pmatrix}. \end{cases}$$

These eigenvalues are to be modified to

$$\mu_1 = 5, \quad \mu_2 = 20,$$

so that the vibrating frequency of the system may be changed via pole placement control with minimal norm actuation. Using equation (4.7)

$$\xi_1 = -1.6, \quad \xi_2 = -5.4,$$

and applying these values to equation (4.10) yields

$$\tau_1 = \begin{pmatrix} -1.5179 \\ -0.5060 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1.7076 \\ -5.1229 \end{pmatrix},$$

So that (4.22) gives the lagrangian multiplier $\theta = 49$. It follows from (4.21) that the optimal constants are

$$\alpha_1 = \frac{8}{\sqrt{35}}, \quad \alpha_2 = \frac{27}{\sqrt{35}}.$$

These constants along with the system’s open loop eigenvectors make up the optimal control input vector $b$. This optimal vector is parallel to the system’s eigenvalues.
Using these constants along with equations (4.17) and (4.24) yield the control’s input and gain vectors

\[ \mathbf{b} = \begin{pmatrix} 0.1758 \\ 0.9844 \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} -1.2307 \\ -6.8910 \end{pmatrix}. \]

So that

\[ \mathbf{bg}^T = \begin{bmatrix} -0.2164 & -1.2115 \\ -1.2115 & -6.7836 \end{bmatrix}, \]

and

\[ |\mathbf{bg}^T| = \sqrt{\theta} = 7. \]

While the calculated values make sense it is necessary to perform a check in order to verify that the control is indeed optimized. In order to do this \(|\mathbf{bg}^T|\) was calculated for various iterations of \(\mathbf{b}\). These results are shown in Figure 4.1.

![Figure 4.1: The norm of \(\mathbf{bg}^T\) as a function of \(-1 \leq b_1 \leq 1\), \(b_2 = \sqrt{1-b_1^2}\). To check the results the parameter \(b_1\) was changed in the interval \(-1 \leq b_1 \leq 1\) where \(b_2\) was chosen to satisfy \(b_2 = \sqrt{1-b_1^2}\). Figure 4.1 shows that the minimal actuator input norm, \(\min |\mathbf{bg}^T| = 7\), is found at two points, \(A\) and \(B\), which are marked by triangles in the figure.](image)

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Point $B$ corresponds to the optimal input vector

$$
b = \begin{pmatrix} 0.1758 \\ 0.9844 \end{pmatrix}
$$

where the unit norm eigenvectors

$$
\begin{align*}
v_1 &= \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\
v_2 &= \frac{1}{\sqrt{10}} \begin{pmatrix} -1 \\ 3 \end{pmatrix}
\end{align*}
$$

were used in the calculation. Point $A$ corresponds to the optimal input placement vector

$$
b = \begin{pmatrix} -0.7313 \\ 0.6821 \end{pmatrix}
$$

which is obtained by equation (4.23). The unit norm eigenvectors $v_1 = -v_1$ and $v_2 = v_2$ were used in the calculations.
Chapter 5: Control Simulation

While the previous chapters illustrate the implementation and optimization of pole placement control, these methods do not show a clear picture of the result of the optimization. In order to demonstrate the successful implementation of this optimization, a vibrating system will be simulated and controlled via non-optimized and optimized pole placement. The controller effort of these two systems are then shown and compared.

5.1 3-Degree-of-Freedom Vibrational System Open Loop Response

The 3-DOF system shown below in Figure 5.1 is to be controlled via partial pole-placement.

![Figure 5.1 Controlled 3-DOF System](image)

Before any work in the control of the system can be accomplished it is necessary to perform open loop analysis so that its dynamics are known. This particular system is characterized by its stiffness matrix because $\mathbf{M} = \mathbf{C}$ and $\mathbf{I} = \mathbf{0}$. The stiffness matrix for this system is given as

$$
\mathbf{K} = \begin{bmatrix}
10 & -6 & 0 \\
-6 & 10 & -4 \\
0 & -4 & 4 \\
\end{bmatrix}.
$$

The eigenvalues of the system are

$$
\mathbf{S} = \begin{bmatrix}
0.8944 & 0 & 0 \\
0 & 6.4408 & 0 \\
0 & 0 & 16.6648 \\
\end{bmatrix}.
$$
and the corresponding eigenvectors are

\[
\mathbf{v}_1 = \begin{bmatrix} -0.3747 \\ -0.5686 \\ -0.7323 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0.6598 \\ 0.3914 \\ -0.6414 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} -0.6514 \\ 0.7235 \\ -0.2285 \end{bmatrix}.
\]

Using the first order realization of the system equation of motion as done in Chapter 2 and using the initial conditions

\[
\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},
\]

and

\[
\dot{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},
\]

yields the system response. The position response of this system is shown in Figure 5.2.

![Figure 5.2 Open Loop Response of 3-DOF System](image)

The oscillations shown in Figure 5.2 do not diminish over time due to the lack of a damping term in the system. This means that the second order eigenvalues of the system are all real.
The goal of the control in this simulation is to partially modify the frequencies at which the system oscillates using the arbitrary and the optimized partial pole-placement technique for an undamped system.

5.2 Control with Arbitrary Input Vector

The vibrating system is to be controlled using the methods put forward on Chapter 3. This method uses the same partial pole placement technique as the optimized version, but uses an arbitrary control input vector, $b$, rather than the one solved using the closed form solution put forward in this thesis. The new desired eigenvalues for the system are

$$
\mu = \begin{bmatrix}
2 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 16.6648
\end{bmatrix},
$$

which means that only two of the system’s eigenvalues are to be modified hence the need of partial pole placement. Using (3.12) to find the pole placement vector yields

$$
q = \begin{bmatrix}
0 \\
-0.1381 \\
0.4810
\end{bmatrix}.
$$

Placing this vector in (3.10) gives

$$
g = \begin{bmatrix}
1.2514 \\
0.1553 \\
-3.0752
\end{bmatrix},
$$

which is the position control vector. The arbitrary control input vector in this simulation is chosen to be

$$
b = \begin{bmatrix}
0.5 \\
0.5 \\
-1
\end{bmatrix}.
Using these vectors in the controlled first order realization yields the response shown in Figure 5.3.

![Figure 5.3](image)

**Figure 5.3. Arbitrary Vibration Control of 3-DOF System**

The control input force is shown in Figure 5.4.

![Figure 5.4](image)

**Figure 5.4. Controller Effort of Arbitrary Input Control**

### 5.3 Optimized Control

The optimal input and gain vectors are calculated applying Algorithm 1 outlined in Chapter 4. The same closed loop eigenvalues assigned in the previous simulation are to be
used for the optimized system so that the control efforts can be compared. Using equation (4.7) yields

$$\xi_1 = -0.4197, \text{ and } \xi_2 = 2.7549.$$ 

A $\xi_3$ term is not necessary because only the first two eigenvalues are to be changed. These values may be used in equation (4.9) to give

$$\tau_1 = \begin{bmatrix} 0.1573 \\ 0.2387 \\ 0.3074 \end{bmatrix}, \text{ and } \tau_2 = \begin{bmatrix} 1.8177 \\ 1.0783 \\ -1.7671 \end{bmatrix}.$$ 

These vectors are used in equation (4.22) to find the Lagrange multiplier

$$\theta = 10.0784,$$

which along with (4.20) yields the coefficients

$$\alpha_1 = 0.3636, \quad \alpha_2 = 0.9315, \quad \alpha_3 = 0.0034.$$ 

which, along with equations (4.27) and (4.17) yield

$$g = \begin{bmatrix} 2.3767 \\ 1.8217 \\ -1.0541 \end{bmatrix},$$

and

$$b = \begin{bmatrix} 0.4762 \\ 0.1603 \\ -0.8646 \end{bmatrix},$$

respectively. These values of $g$ and $b$ are optimal values for this system. These values can be put into (3.13) to find the response of the system. This response is the same frequencies and amplitudes as the response shown in Figure 5.3. From the two system responses shown it can be seen that the frequencies at which the system oscillates are the same as are the maximum motions.
Figure 5.5. Optimized Vibration Control of 3-DOF System

To confirm that the controller with the optimized input vector leads to smaller control effort than shown in Figure 5.5, the control effort for the optimized case is shown in Figure 5.6.

Figure 5.6. Controller Effort for Optimized Control

It is clear from both Figures that the effort for the optimized controller is indeed less than that for the unoptimized controller. This shows that the optimization has been successful. These
results don’t show a large difference in the overall controller effort. The arbitrary input control vector is close numerically to the optimal vector calculated. An arbitrary control vector that is further away from the optimal vector would show a more significant jump in controller efficiency.
Chapter 6: Conclusion

Optimized Partial Pole placement control is an effective control technique for mechanical vibrations. The dynamics of a controllable system may be changed to alter the vibrating frequency of the system. This control may be optimized via the method derived in this Thesis under the right conditions. In certain applications large powerful actuators may not be desirable because of size or energy constraints. Optimization can be useful for these applications so that a vibrational system can be successfully controlled without resorting to large actuators.

The pole placement technique presented may be used to either lower the vibration on a certain degree of freedom or simply move it to a different point in the s-plane. This would be useful in applications where vibration is desired at a specific frequency. Using partial pole placement instead of full pole placement allows for only specific frequencies to be modified rather than modifying the whole spectrum of a system’s natural frequencies. An advantage of this method is that spillover is avoided. This ensures that the systems internal dynamics do not interfere with the system’s control and an unwanted response is avoided.

The fourth chapter of this thesis provides a way to calculate the optimal controller input selection and gain vectors. The elegance of this optimization method is that a control scheme may be designed without any knowledge of the system beyond the open loop system dynamics and without knowledge of the system’s initial conditions which is a limitation of input \( u(t) \) based controls. A simple algorithm was proposed to design an optimal control scheme by simply plugging in values into closed form equations. While this technique was shown to work, it also has impractical limitations. The controllable system must have \( M=C \) and \( I=0 \) so that a closed form solution of the input and gain vectors may be derived. These
constraints make the derived solutions not applicable to real-world systems and limit this work in the theoretical domain.

The results in Chapter 5 shows that the optimization technique works well in minimizing the control effort needed to implement partial pole placement control in a mechanical system. The simulation used an arbitrary control input vector that was close to the optimal input vector meaning that the control effort for this system is close to the optimal control effort. Taking this into consideration the results still showed an improvement in the necessary control force.

Further research could apply the methods presented here to problems in the other energy domains. The differential equations present a generic way to describe many systems so pole placement is applicable many other energy domains seen in engineering. Also, the theoretical nature of this research limits its applicability to the real world. Further research could build upon the techniques shown here to optimize real systems and give palpable value to this optimization method.
References


Appendix

A. MATLAB Code for Example 1

clear all; % input system parameters
n=2; % two-dimensional system
m1=1; m2=1; % define mass of each dimension
k1=4; k2=4; % define spring constants
c1=0.3; c2=0.2; % damping constants
% First order realization
I=eye(n);
O=zeros(n,n);
M=[m1 0; 0 m2];
C=[c1+c2 -c2; -c2 c2];
K=[k1+k2 -k2; -k2 k2];
A=[0 I; -K 0];
B=[I 0; C M];
[U,S]=eig(A,B); % eigenvalues and eigenvectors
% Initial conditions
x0=[1 0]';
vel0=[0 1]';
a=U\[x0;vel0]; % calculate coefficients of solution

k=0;
for j=0:0.1:100 % define time range and step
  k=k+1;
t(k)=j; % time
  x1(k)=0; % initialize positions and velocities of masses
  x2(k)=0;
  vel1(k)=0;
  vel2(k)=0;
  for i=1:2*n % begin calculations for time step
    x1(k)=x1(k)+a(i)*U(1,i)*exp(S(i,i)*j); % position of mass 1
    x2(k)=x2(k)+a(i)*U(2,i)*exp(S(i,i)*j); % position of mass 2

    vel1(k)=vel1(k)+a(i)*S(i,i)*U(1,i)*exp(S(i,i)*j); % velocity
    % of mass 1
    vel2(k)=vel2(k)+a(i)*S(i,i)*U(2,i)*exp(S(i,i)*j); % velocity
    % of mass 2
  end
end
% remove discretization errors by rounding off any imaginary parts less
% than tolerance
tol=1e-10; % define tolerance setting
if imag(x1)<tol
  x1=real(x1);
end
if imag(x2)<tol
  x2=real(x2);
end
if imag(vel1)<tol
vel1=real(vel1);
end
if imag(vel2)<tol
vel2=real(vel2);
end
frame=601; % plot up to t=60
subplot(2,2,1) % top left box shows plot of mass 1 position
plot(t(1:frame),x1(1:frame),'-b','LineWidth',1)
axis([0 60 -1.5 1.5])
xlabel('Time, t')
ylabel('Position')
title('Position of Mass 1')
subplot(2,2,2) % top right box shows plot of mass 2 position
plot(t(1:frame),x2(1:frame),'-b','LineWidth',1)
axis([0 60 -1.5 1.5])
xlabel('Time, t')
ylabel('Position')
title('Position of Mass 2')
subplot(2,2,3) % bottom left box shows plot of mass 1 velocity
plot(t(1:frame),vel1(1:frame),'-b','LineWidth',1)
axis([0 60 -3 3])
xlabel('Time, t')
ylabel('Velocity')
title('Velocity of Mass 1')
subplot(2,2,4) % bottom right box shows plot of mass 2 velocity
plot(t(1:frame),vel2(1:frame),'-b','LineWidth',1)
axis([0 60 -3 3])
xlabel('Time, t')
ylabel('Velocity')
title('Velocity of Mass 2')

B. MATLAB Code for Example 2

clear all;

n=2; % two-dimensional system
m1=1; m2=1; % define mass of each dimension
k1=4; k2=4; % define spring constants
c1=0.3; c2=0.2; % define damping constants

I=eye(n); % identity matrix
O=zeros(n,n); % zero matrix
M=[m1 0; 0 m2]; % mass matrix
C=[c1+c2 -c2; -c2 c2]; % damping matrix
K=[k1+k2 -k2; -k2 k2]; % spring matrix
Ao=[O I; -K O];
Bo=[I O; C M];
[Uo,So]=eig(Ao,Bo);
b=[2 1]'; % arbitrary control selection vector
i=sqrt(-1);
for k=1:2*n
    vo(:,k)=Uo(1:n,k); \% Uo is given as normalized set
end;
so=[So(1,1); So(2,2); So(3,3); So(4,4)]; \% open loop eigenvectors
mu=[so(1); so(2); -0.5-1i; -0.5+1i]; \% define new eigenvalues to be assigned
Num1=((mu(3)-so(3))/so(3))*)((mu(4)-so(3))/(so(4)-so(3)));
Num2=((mu(4)-so(4))/(so(4))*)((mu(3)-so(4))/(so(3)-so(4)));
Den1=b'*vo(:,3);
Den2=b'*vo(:,4);
q(1,1)=Num1./Den1;
q(2,1)=Num2./Den2;
f=M*vo(:,3:4)*diag(so(3:4))*q;
g=-K*vo(:,3:4)*q; \% solve for new eigenvalues of the system
A=[0 I; \% first-order realization including control
   -(K-b*g') O];
B=[I 0; (C-b*f') M];
[U,S]=eig(A,B); \% eigenvalues and eigenvectors
\%
\% Initial conditions
x0=[0.5 0]'; \% define initial position
vel0=[0 1]'; \% define initial velocity
a=U*[x0;vel0]; \% calculate coefficients of solution
\%
\% Calculate solution for each time-step
k=0;
for j=0:0.1:100 \% define time range and step
    k=k+1;
    t(k)=j; \% time
    x1(k)=0; \% initialize positions and velocities of masses
    x2(k)=0;
    vel1(k)=0;
    vel2(k)=0;
    for i=1:2*n \% begin calculations for time step
        x1(k)=x1(k)+a(i)*U(1,i)*exp(S(i,i)*j); \% position of mass 1
        x2(k)=x2(k)+a(i)*U(2,i)*exp(S(i,i)*j); \% position of mass 2
        vel1(k)=vel1(k)+a(i)*S(i,i)*U(1,i)*exp(S(i,i)*j); \% velocity
            \% of mass 1
        vel2(k)=vel2(k)+a(i)*S(i,i)*U(2,i)*exp(S(i,i)*j); \% velocity
            \% of mass 2
    end
end
tol=1e-10; \% define tolerance setting
if imag(x1)<tol
    x1=real(x1);
end
if imag(x2)<tol
    x2=real(x2);
end
if imag(vel1)<tol
    vel1=real(vel1);
end
if imag(vel2)<tol
    vel2=real(vel2);
end
frame=601; \% plot to t=60
subplot(2,2,1) \% top left box shows plot of mass 1 position
Vita

Jorge was born in San Salvador, El Salvador in 1990. He then immigrated to the United States along with his family in 1999. An alumnus of Christopher Columbus High School in Miami, FL he then went on to university at Georgia Institute of Technology to major in Mechanical Engineering. Jorge Graduated from Georgia Tech with a Bachelor of Science in Mechanical Engineering.

Wanting to obtain a masters degree before working, Jorge started to pursue a graduate degree in mechanical engineering at the Louisiana State University in Baton Rouge, Louisiana. He will graduate in December 2013 and then pursue a career in industrial automation.