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Graham's variety and perverse sheaves on the nilpotent cone

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GRAHAM'S VARIETY AND PERVERSE SHEAVES
ON THE NILPOTENT CONE

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
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by

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Table of Contents

Acknowledgments	ii
List of Tables	iv
Abstract	v
Chapter 1: Introduction	1
Chapter 2: Preliminaries	4
Chapter 3: Graham's Variety	16
Chapter 4: Graham's Fibers in Type A_n	22
Chapter 5: Perverse Sheaves in Type A_n	30
Chapter 6: Towards Result in Other Types	32
References	40
Vita	42

List of Tables

2.1	Fundamental Groups of Nilpotent Orbits in Classical Types	9
2.2	Weyl Groups for Classical Lie Algebras	13
6.1	T_{ad} -orbits and Fundamental Groups for E_6	37
6.2	T_{ad} -orbits and Fundamental Groups for E_7	38

Abstract

In recent work, Graham has defined a variety which maps to the nilpotent cone, and which shares many properties with the Springer resolution. However, Graham's map is not an isomorphism over the principal orbit, and for type A_n in particular, its fibers have a nice relationship with the fundamental groups of the nilpotent orbits. The goal of this dissertation is to determine which simple perverse sheaves appear when the Decomposition Theorem for perverse sheaves is applied in Graham's setting for type A_n , and to begin to answer this question in the other types as well.

In Chapter 1, we give some motivation and a brief description of this project. Then, Chapter 2 is a summary of several background topics. In Chapter 3, we review Graham's construction of his variety. In Chapter 4, we use results of Tymoczko to study the fibers of Graham's map in type A_n . Chapter 5 contains the conclusions in the perverse sheaf setting, and lastly, Chapter 6 contains results pertaining to Graham's fibers in the other types.

Chapter 1

Introduction

First, we will describe a well-known result in the area, one which helped to inspire the topic of this paper. The Springer Correspondence ([Spr76], [Spr78]) is a result dating back to the 1970s, but in the case of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, its statement was likely noted long before that. We will begin by describing this special case. One way to define $\mathfrak{sl}_n(\mathbb{C})$ is as the set of complex matrices with trace zero. Within this Lie algebra, the set of nilpotent matrices, matrices with eigenvalue zero, form what is called the nilpotent cone \mathcal{N} . From linear algebra, it is known that each conjugacy class of these matrices can be represented by a matrix in Jordan canonical form and that this matrix is unique up to re-arranging the order of the Jordan blocks. Thus, the sizes of the Jordan blocks, given by partitions of n , parametrize the conjugacy classes of matrices in \mathcal{N} . These conjugacy classes are also referred to as nilpotent orbits, since they are the orbits determined by the action of the Lie group $SL_n(\mathbb{C})$ on \mathcal{N} . The Springer Correspondence relates two features of a Lie algebra, and one of these is the nilpotent orbits. The other is the Weyl group, a finite reflection group associated to each Lie algebra. In the case of $\mathfrak{sl}_n(\mathbb{C})$, the Weyl group is the symmetric group S_n , and it is a fact from the representation theory of finite groups that the irreducible representations of S_n are parametrized by partitions of n . Thus, the Springer Correspondence for $\mathfrak{sl}_n(\mathbb{C})$ is a bijective relationship given by the partition classifications of nilpotent orbits and irreducible representations of the Weyl group. While this description lacks mention of geometry, Springer established the underlying geometric nature of this phenomenon by constructing

S_n representations in the cohomology groups of certain related varieties called Springer fibers.

Inspired by the example above, the Springer Correspondence applies to all reductive complex Lie algebras. However, it is not generally bijective between nilpotent orbits and irreducible Weyl group representations. In actuality, it associates to each Weyl group representation a particular local system on a nilpotent orbit. Local system here refers to a locally constant sheaf, and for each orbit, the local systems are in bijection with the representations of the fundamental group. One proof of the Springer Correspondence, provided by Borho and MacPherson, comes from an application of the Decomposition Theorem for perverse sheaves. (See [BBD82] and [BM83].) The relationship arises from the fact that each equivariant simple perverse sheaf on \mathcal{N} is associated to a particular local system on a nilpotent orbit. In order to apply the Decomposition Theorem, a map with particular properties is required, and in this case, it is a map to \mathcal{N} known as the Springer resolution. The main result of this paper is an application of the Decomposition Theorem in a situation related to the Springer resolution.

In [Gra], Graham defines a variety with a map to \mathcal{N} that is similar to the Springer resolution, and in fact, this map factors through the Springer resolution. He makes a useful connection between the root and weight lattices of the Lie algebra and toric varieties, and the construction of his variety is based on these toric varieties. A description of this variety and map can be found in Chapter 3, and much of this paper will be dedicated to better understanding the fibers of Graham's map. Chapter 4 focuses on the fibers in the case where \mathfrak{g} is of type A_n . Here, Tymoczko's results [Tym03] about affine pavings of Springer fibers are used to determine the Graham fibers over a particular Springer fiber component. We will show in Chapter 5 that when the Decomposition Theorem is applied in

this setting for a semisimple Lie algebra of type A_n , like $\mathfrak{sl}_n(\mathbb{C})$, the result is that every equivariant simple perverse sheaf appears. To contrast this result with the Springer resolution case, only the simple perverse sheaves associated to the trivial local system appear in the classical Springer Correspondence for this type. In Chapter 6, the results leading towards an understanding of Graham's fibers in the other types can be found.

Chapter 2

Preliminaries

In this section, we will briefly review the background topics necessary for understanding the results of the paper. Basic knowledge of Lie theory, representations of finite groups, homological algebra, and algebraic geometry are assumed. For the reader wishing to become more familiar with any of these topics, the recommended references are [Hum72], [Sen12], [FH91], [GM03], and [Har77].

Throughout, \mathfrak{g} will be used to denote a complex semisimple Lie algebra and G will denote the simply-connected algebraic group with Lie algebra \mathfrak{g} . See [War83] for the proof that such a group exists. All sheaves will be $\overline{\mathbb{Q}}_\ell$ -sheaves in the étale topology. For the reader unfamiliar with this setting, a good reference is [Mil80], but most statements involving $\overline{\mathbb{Q}}_\ell$ could also be understood as statements for \mathbb{C} . We will begin by defining the nilpotent cone of \mathfrak{g} . Included here are the partition classifications of the orbits and the fundamental groups of the orbits. Next, we will briefly introduce those definitions essential to the topic of perverse sheaves. Lastly, we will give a description of the Springer Correspondence in the setting of perverse sheaves, one of the key motivations for this paper.

The Nilpotent Cone and Nilpotent Orbits

The definitions and statements found here will follow the treatments given in [CM93] and Jantzen's *Nilpotent Orbits in Representation Theory* found in [AO03].

The Nilpotent Cone and the Action of G

Recall that \mathfrak{g} is a complex, semisimple Lie algebra, and let us define $\text{ad} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$ by $\text{ad}_x(y) = [x, y]$. Then, an element x in \mathfrak{g} is *nilpotent* if ad_x is a nilpotent endomorphism. Note that in the case where \mathfrak{g} is a subset of $GL_n(\mathbb{C})$, an element

x is nilpotent when it is a nilpotent matrix. The set of all nilpotent elements in \mathfrak{g} forms a normal variety \mathcal{N} that is called the *nilpotent cone*.

Here, we will follow the language of [FH91] and [CM93]. Let $\psi_g : G \rightarrow G$ be the automorphism of G defined by $\psi_g = ghg^{-1}$ and define $\psi : G \rightarrow \text{Aut}(G)$ to be the map sending g to ψ_g . The differential at the identity $d(\psi_g)_e$ is an automorphism of \mathfrak{g} for all $g \in G$. Define then the *adjoint representation* $\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g})$ to be the map sending g to $d(\psi_g)_e$. The image $\text{Ad}(G)$ is an algebraic group with Lie algebra \mathfrak{g} that we will call the *adjoint group* and denote G_{ad} . The kernel of Ad is the finite group Z which is the center of G , and thus, $G_{ad} \cong G/Z$. In general, since \mathfrak{g} is a complex semisimple Lie algebra, all of the complex algebraic groups with Lie algebra \mathfrak{g} can be viewed as G/Z_* where Z_* is some subgroup of the center Z . Thus, G_{ad} can be viewed as the smallest algebraic group for \mathfrak{g} .

The adjoint action of G partitions the nilpotent cone \mathcal{N} into finitely many *nilpotent orbits*. In general, we will use \mathcal{O}_x to denote the nilpotent orbit containing the element $x \in \mathcal{N}$. Note that $\mathcal{O}_x = G_{ad} \cdot x = \{\phi(x) | \phi \in G_{ad} \subset \text{Aut}(\mathfrak{g})\}$, and the nilpotent orbits are well-defined since, for any automorphism ϕ , $\text{ad}_{\phi(x)} = \phi^{-1} \text{ad}_x \phi$ is nilpotent exactly when x is nilpotent. Each of these orbits is a homogeneous complex manifold and can be given a symplectic structure through use of the Lie algebra dual.

The largest orbit with respect to dimension is called the *principal nilpotent orbit* and denoted \mathcal{O}_{prin} . It is open and dense in \mathcal{N} , and thus, has the property that $\dim \mathcal{O}_{prin} = \dim \mathcal{N} = \dim \mathfrak{g} - \text{rank } \mathfrak{g}$ where the rank is the size of a Cartan subalgebra, or equivalently, the number of simple roots. Overall, the nilpotent orbits are locally closed, so that the closure of an orbit is the union of that orbit and other orbits of smaller dimension. The trivial orbit \mathcal{O}_0 is the unique orbit of smallest dimension.

Partition Classification of Nilpotent Orbits

If \mathfrak{g} is a matrix Lie algebra, the adjoint action is determined by conjugation of the appropriate matrices, thus the nilpotent orbits correspond to conjugacy or similarity classes of matrices. For the classical Lie algebras, the nilpotent orbits can be described using partitions, which correspond to the sizes of the blocks in the Jordan canonical form of the nilpotent matrices. Following [CM93], we will denote a partition as $[p_r^{k_r} p_{r-1}^{k_{r-1}} \dots p_1^{k_1}]$ where p_i is a *part* and k_i is the *multiplicity* of the part p_i . If each part is distinct and decreasing with the indices, we will call this a *reduced* partition. For example, the partition $P = 4 + 4 + 2 + 1 + 1 + 1$ is denoted here in reduced form by $[4^2 2 1^3]$. For the partition classification of nilpotent orbits, we will need the definition of a *very even* partition. It is a partition having only even parts, each with even multiplicity. (See [CM93] for the proof of Theorem 2.1.)

Theorem 2.1. *The nilpotent orbits in the classical Lie algebras can be classified in the following manner.*

- For \mathfrak{sl}_n , there is a one-to-one correspondence between partitions of n and nilpotent orbits. The principal orbit \mathcal{O}_{prin} corresponds to the partition of the form $[n]$, and \mathcal{O}_0 corresponds to the partition $[1^n]$.
- For \mathfrak{so}_{2n+1} , the nilpotent orbits are in one-to-one correspondence with the set of partitions of $2n + 1$ where even parts occur with even multiplicity. The principal orbit \mathcal{O}_{prin} corresponds to the partition of the form $[2n + 1]$, and \mathcal{O}_0 corresponds to the partition $[1^{2n+1}]$.
- For \mathfrak{sp}_{2n} , the nilpotent orbits are in one-to-one correspondence with the partitions of $2n$ in which odd parts occur with even multiplicity. The principal orbit \mathcal{O}_{prin} corresponds to the partition of the form $[2n]$, and \mathcal{O}_0 corresponds to the partition $[1^{2n}]$.

- For \mathfrak{so}_{2n} , the nilpotent orbits are parametrized by partitions of $2n$ where even parts occur with even multiplicity, except very even partitions correspond to two distinct nilpotent orbits. The principal orbit \mathcal{O}_{prin} corresponds to the partition of the form $[2n - 1, 1]$, and \mathcal{O}_0 corresponds to the partition $[1^{2n+1}]$.

Bala-Carter Classification of Nilpotent Orbits

While the partition classification works in the classical types, another method must be used for the exceptional Lie algebras. In [BC76a] and [BC76b], Bala and Carter give a different method to parametrize the nilpotent orbits, which works for all types. A concise description of their method can be seen in [Som98], and we will combine this with the treatment from [AO03] to give a summary here.

First, a nilpotent element $x \in \mathfrak{g}$ is called a *distinguished nilpotent element* if each torus contained in the centralizer G^x is also contained in the center of G . A nilpotent orbit comprised of distinguished nilpotent elements is said to be a *distinguished nilpotent orbit*. In \mathfrak{sl}_n , the only distinguished nilpotent orbit is \mathcal{O}_{prin} , and in fact, \mathcal{O}_{prin} is distinguished in all types. The classification of all distinguished nilpotent orbits is the first ingredient in Bala and Carter's work.

Next, let \mathfrak{p} be a parabolic subalgebra of \mathfrak{g} . Then, $\mathfrak{p} = \mathfrak{u}_{\mathfrak{p}} + \mathfrak{l}$ where $\mathfrak{u}_{\mathfrak{p}}$ is the nilradical of \mathfrak{p} and the reductive Lie algebra \mathfrak{l} is called the Levi factor of \mathfrak{p} , or a *Levi subalgebra*. The second ingredient in the Bala-Carter classification is to associate to each nilpotent orbit \mathcal{O}_x , a Levi subalgebra \mathfrak{l} such that x is distinguished in \mathfrak{l} . Then, the orbit $\mathcal{O} = \mathfrak{l} \cap \mathcal{O}_x$ is a distinguished orbit in \mathfrak{l} . Up to conjugation, there is a unique Levi subalgebra \mathfrak{l} for each x . Thus, this gives a bijective relationship between nilpotent orbits in \mathcal{N} and conjugacy classes of pairs $(\mathfrak{l}, \mathcal{O})$, and listing all such pairs $(\mathfrak{l}, \mathcal{O})$ up to conjugacy gives a classification of the nilpotent orbits. Notationally, the orbits are labeled by the type of their Levi subalgebra, which can

be determined by the weighted Dynkin diagrams. This will be used in Chapter 6 when the exceptional Lie algebras E_6 and E_7 are discussed.

A topic related to the Bala-Carter classification is a *Richardson orbit*. Let \mathfrak{p} again be a parabolic subalgebra. The Richardson orbit is the unique nilpotent orbit whose intersection with the nilradical $\mathfrak{u}_{\mathfrak{p}}$ is open and dense in $\mathfrak{u}_{\mathfrak{p}}$. The Richardson orbit is always distinguished in \mathfrak{p} . Let B be a Borel subgroup with nilradical \mathfrak{n} , and let Φ^+ be the positive roots determined by B . For any $x \in \mathfrak{n}$, there is a decomposition $x = \sum_{\alpha \in \Phi^+} X_{\alpha}$ where X_{α} is a root vector in \mathfrak{g}_{α} . It can be shown that x is a Richardson element for B if and only if the X_{α} coefficient is nonzero for every simple root α . A proof of this can be found in [AO03]. Note that for any Levi subalgebra \mathfrak{l} , the Richardson orbit of the Borel (for \mathfrak{l}) gives the principal orbit of $\mathcal{M}_{\mathfrak{l}}$, which we have said is distinguished in all types. Thus, if $x \in \mathcal{N}$ is the sum of some subset of simple root vectors, it is contained in the principal orbit of the Levi subalgebra whose root system is determined by the subset of simple roots. The relevance of this fact will be seen in Chapters 4 and 6.

Fundamental Groups of Nilpotent Orbits

For any $x \in \mathfrak{g}$ and any complex algebraic group G_* with Lie algebra \mathfrak{g} , we can define the *centralizer of x in G_** as $G_*^x = \{g \in G_* \mid \text{Ad}(g)x = x\}$. Then, we can define the G_* -equivariant fundamental group for any nilpotent orbit \mathcal{O}_x to be $G_*^x / (G_*^x)^o$ where $(G_*^x)^o$ is the identity component of G_*^x . In the special case that G_* is the simply-connected group G , the G -equivariant fundamental group is also $\pi_1(\mathcal{O}_x)$, the fundamental group of the orbit \mathcal{O}_x .

Let P be a partition of the appropriate form from Theorem 2.1 for each classical Lie algebra, and let \mathcal{O}_P be the nilpotent orbit associated to P . In the following table from [CM93], we will give formulas for the fundamental group $\pi_1(\mathcal{O}_P)$ of each

orbit and the G_{ad} -equivariant fundamental group $A(\mathcal{O}_P)$. To simplify our formulas, let

a = the number of distinct odd parts in P ,

b = the number of (nonzero) distinct even parts in P , and

c = the greatest common divisor of all parts in P .

Also, a partition is called *rather odd* if all of its odd parts have multiplicity one.

Notice, a very even partition is trivially rather odd.

TABLE 2.1. Fundamental Groups of Nilpotent Orbits in Classical Types

Lie Algebra	$\pi_1(\mathcal{O}_P)$	$A(\mathcal{O}_P)$
\mathfrak{sl}_n	$\mathbb{Z}/c\mathbb{Z}$	1
\mathfrak{so}_{2n+1}	If P is rather odd, a central extension by $\mathbb{Z}/2\mathbb{Z}$ of $(\mathbb{Z}/2\mathbb{Z})^{a-1}$; otherwise, $(\mathbb{Z}/2\mathbb{Z})^{a-1}$	$(\mathbb{Z}/2\mathbb{Z})^{a-1}$
\mathfrak{sp}_{2n}	$(\mathbb{Z}/2\mathbb{Z})^b$	$(\mathbb{Z}/2\mathbb{Z})^b$ if all even parts have even multiplicity; otherwise, $(\mathbb{Z}/2\mathbb{Z})^{b-1}$
\mathfrak{so}_{2n}	If P is rather odd, a central extension by $\mathbb{Z}/2\mathbb{Z}$ of $(\mathbb{Z}/2\mathbb{Z})^{\max(0, a-1)}$; otherwise, $(\mathbb{Z}/2\mathbb{Z})^{\max(0, a-1)}$	$(\mathbb{Z}/2\mathbb{Z})^{\max(0, a-1)}$ if all odd parts have even multiplicity; otherwise, $(\mathbb{Z}/2\mathbb{Z})^{\max(0, a-2)}$

The fundamental groups for the exceptional types E_6 and E_7 are discussed in Tables 6.1 and 6.2 as needed.

Perverse Sheaf Review

Now, we will review the concept of perverse sheaves. References for this material are [BBD82], [Dim03], and [AO03]. Also, the notes [Ach07] give a nice construction in the case of ordinary sheaves. Recall that a “sheaf” here will be a $\overline{\mathbb{Q}}_\ell$ -sheaf, and the unfamiliar reader is advised to review this definition in [Mil80] or to see the abbreviated version in the appendix of [Car85].

Definitions for Conditions of Sheaves

Let X be a normal, connected complex variety. A locally constant, constructible $\overline{\mathbb{Q}}_\ell$ -sheaf for X is called a *local system*. The local systems on X are in bijection (up to isomorphism) with the representations of the fundamental group $\pi_1(X)$. For example, the constant sheaf $\overline{\mathbb{Q}}_{\ell, X}$ is the local system corresponding to the trivial representation.

Let \mathcal{S} be a finite collection of subspaces of X . Then \mathcal{S} is a *stratification* of X , and subset $S \in \mathcal{S}$ is a *stratum*, if

- X is the disjoint union of the strata $S \in \mathcal{S}$.
- Each stratum $S \in \mathcal{S}$ is a manifold.
- The closure of a stratum S , denoted by \overline{S} , is the union of strata.

In the case of \mathcal{N} , the nilpotent orbits \mathcal{O}_x form such a stratification.

The *support* of a sheaf \mathcal{F} , denoted $\text{supp } \mathcal{F}$, is the complement of all the open subsets $U \subset X$ such that $\mathcal{F}|_U = 0$. Note that if \mathcal{F} is constructible with respect to a stratification \mathcal{S} , then $\text{supp } \mathcal{F}$ will be the union of strata in \mathcal{S} . Let us use \mathcal{F}^\bullet to mean a complex of sheaves. Then we can define $\text{supp}^i \mathcal{F}^\bullet = \text{supp } H^i(\mathcal{F}^\bullet)$.

Let us denote by $D^b(X)$ the bounded derived category of sheaves on X . (See [GM03] for the construction of this category.) Let a be the constant map from X to some point. Then, for any complex of sheaves \mathcal{F}^\bullet , the *Verdier dual* is $\mathbb{D}\mathcal{F}^\bullet = R\mathcal{H}om(\mathcal{F}^\bullet, a^! \overline{\mathbb{Q}}_{\ell, X})$. See [[Dim03], Chapter 3] for a more-detailed discussion of this derived functor.

Perverse Sheaves

One approach to defining perverse sheaves involves constructing a t -structure in $D^b(X)$. Instead, here we will give the dimension support conditions for perverse sheaves and consider this as our definition. These conditions can be found in [BBD82], [Dim03] or [BM83].

Definition 2.2. Let X be a stratified topological space, and let \mathcal{F}^\bullet be a constructible complex in $D^b(X)$. Then, \mathcal{F}^\bullet is said to be a *perverse sheaf on X* if both of the following conditions hold:

- $\dim(\text{supp}^{-i} \mathcal{F}^\bullet) \leq i$
- $\dim(\text{supp}^{-i} \mathbb{D}\mathcal{F}^\bullet) \leq i$

Denote by $\mathcal{P}(X)$ the abelian category of perverse sheaves on X .

A definition can also be given using the notion of t -structures. See [BBD82] for this alternate approach.

The simple perverse sheaves are the interesection cohomology complexes, denoted $IC(\overline{S}, \mathcal{E})[\dim S]$ where S is a stratum for the stratified space X and \mathcal{E} is a local system on S . Here, $[1]$ denotes the shift functor in the derived category, and $[k] = [1]^k$ for some $k \in \mathbb{Z}$. These are objects in $\mathcal{P}(X)$ characterized by the following properties:

- $H^i(IC(\overline{S}, \mathcal{E})[\dim S]) = 0$ for all $i > \dim S$.
- $H^{-\dim S}(IC(\overline{S}, \mathcal{E})[\dim S])|_S = \mathcal{E}$.
- $\dim(\text{supp}^{-i} IC(\overline{S}, \mathcal{E})[\dim S]) < i$ for all $i < \dim S$.
- $\dim(\text{supp}^{-i} \mathbb{D}IC(\overline{S}, \mathcal{E})[\dim S]) < i$ for all $i < \dim S$.

Suppose \mathcal{E} is a local system on a nilpotent orbit \mathcal{O}_x corresponding to a representation of the G_* -equivariant fundamental group of \mathcal{O}_x . Then, $IC(\overline{\mathcal{O}_x}, \mathcal{E})[\dim \mathcal{O}_x]$ is G_* -equivariant in $\mathcal{P}(\mathcal{N})$.

A space X is called *rationaly smooth* if $H^{-\dim S}(IC(X, \overline{\mathbb{Q}}_{\ell_X})[\dim S]) = \overline{\mathbb{Q}}_{\ell_X}$ and $H^i(IC(X, \overline{\mathbb{Q}}_{\ell_X})[\dim S]) = 0$ for $i \neq -\dim S$. While a smooth space is necessarily rationaly smooth, a rationaly smooth space may have singularities. For example, the nilpotent cone is rationaly smooth, not smooth.

Decomposition Theorem

In this section, we will use the language of [BM83]. Let X and Y be irreducible complex algebraic varieties and let \mathcal{S} be a stratification of X . Let $f : Y \rightarrow X$ be a proper morphism of varieties for which the stratification \mathcal{S} makes it a *weakly stratified mapping*. That means, for any stratum S , the restriction to the preimage $f^{-1}(S)$ is a topological fibration over S with fibers $f^{-1}(s)$ for any $s \in S$. Let us define $d_s = \dim f^{-1}(s)$. The map f is called *semismall* if $2d_s \leq \dim X - \dim S$ for all strata S . Let us define f_* to be the push forward of f , thus it sends sheaves on Y to sheaves on X .

For any stratum S and any point $s \in S$, there is a basis for $H^{2d_x}(f^{-1}(s))$ indexed by the irreducible components of $f^{-1}(s)$, and the fundamental group $\pi_1(S)$ acts on $H^{2d_x}(f^{-1}(s))$ by permuting the irreducible components. Understanding this action gives us important information in the perverse sheaf setting.

The following theorem is from [BBD82], but the statement here will follow the one from [BM83].

Theorem 2.3 (Decomposition Theorem for Perverse Sheaves). *Let X , Y , and f be as above, and further assume that f is projective and semismall, Y is rationally smooth, and $\dim X = \dim Y$. Then*

$$Rf_* \overline{\mathbb{Q}}_{\ell_Y} = \bigoplus IC(S, \mathcal{E}) \otimes V_{(S, \mathcal{E})}$$

where the sum is over all strata S such that $2d_s = \dim X - \dim S$ and local systems \mathcal{E} that correspond to an irreducible representation $\psi_{\mathcal{E}}$ of $\pi_1(S)$ on $H^{2d_s}(f^{-1}s)$. The vector space $V_{(S, \mathcal{E})}$ has dimension equal to the multiplicity of $\psi_{\mathcal{E}}$ in the action of $\pi_1(S)$.

Springer Correspondence

The Springer Correspondence, as mentioned in Chapter 1, is a result due originally to Springer in [Spr76] and [Spr78] which relates the irreducible representations of the Weyl group of a Lie algebra to the local systems occurring on its nilpotent orbits. Here, we will review briefly the Weyl group, and then explain the perverse sheaf approach for proving the Springer Correspondence from [BM81].

Weyl Group

There is a finite group associated to every Lie algebra \mathfrak{g} called the Weyl group and denoted here as \mathscr{W} . This group permutes the roots in the root system and is generated by the reflections across the hyperplanes of the root system. It can also be defined as $N(T)/T$ for any maximal torus T in G where $N(T)$ is the normalizer of T in G . In the table below, \mathscr{W} is described for the classical Lie algebras.

TABLE 2.2. Weyl Groups for Classical Lie Algebras

Lie Algebra	\mathscr{W}
\mathfrak{sl}_n	S_n
\mathfrak{so}_{2n+1}	$S_n \times (\mathbb{Z}/2\mathbb{Z})^n$
\mathfrak{sp}_{2n}	$S_n \times (\mathbb{Z}/2\mathbb{Z})^n$
\mathfrak{so}_{2n}	$S_n \times (\mathbb{Z}/2\mathbb{Z})^{n-1}$

This group will play an important role in Chapter 4 due to its use in the Bruhat Decomposition for the flag variety G/B , the space of all Borel subgroups for G .

Springer Resolution

While \mathcal{N} is rationally smooth, it is not smooth, and there is a useful resolution of its singularities. Let $\widetilde{\mathcal{N}} = \{(x, B) | x \in \mathcal{N} \cap \text{Lie}(B) \text{ and } B \text{ is a Borel subgroup of } G\}$. Then, the map $\mu : \widetilde{\mathcal{N}} \rightarrow \mathcal{N}$ which forgets B is the Springer resolution, and the varieties \mathcal{B}_x called Springer fibers are the fibers of this map. These varieties have inspired extensive study over the past four decades. For example, they

are prominently featured in Spaltenstein's book [Spa82], and Springer himself has studied their purity in [Spr84]. De Concini, Lusztig, and Procesi have a paper [DLP88] focusing on their cohomology, and various other modern mathematicians are still exploring these varieties. Most relevant to the content of this paper, Borho and MacPherson [BM81] use the action of $\pi_1(\mathcal{O}_x)$ on the top degree cohomology of the Springer fibers to determine which simple perverse sheaves occur as summands when Theorem 2.3 is applied in the setting of the Springer resolution. This is essential to their proof of the Springer Correspondence.

Springer Correspondence

Let $\mathcal{P}(\mathcal{N})$ be the category of perverse sheaves on \mathcal{N} with respect to the stratification by G -orbits. Let $\overline{\mathbb{Q}}_{\ell, \widetilde{\mathcal{N}}}$ denote the constant sheaf on $\widetilde{\mathcal{N}}$. For any orbit \mathcal{O}_x , let \mathcal{L}_φ be the local system corresponding to the representation φ of $\pi_1(\mathcal{O}_x)$. Define $d_x := \dim \mathcal{B}_x$. All the results in the following fact can be seen in [BM83], but each statement was first proven elsewhere.

Fact 2.4. (a) *The Springer resolution is proper and semismall.*

(b) *$\widetilde{\mathcal{N}}$ is rationally smooth.*

(c) $\dim \widetilde{\mathcal{N}} = \dim \mathcal{N}$.

(d) $2d_x = \dim \mathcal{N} - \dim \mathcal{O}_x$ for any $x \in \mathcal{N}$.

The above statements tell us that the Springer resolution satisfies all the necessary conditions to apply the Decomposition Theorem for perverse sheaves, Theorem 2.3. Thus, we know that $R\mu_*\overline{\mathbb{Q}}_{\ell, \widetilde{\mathcal{N}}}$ is semisimple in $\mathcal{P}(\mathcal{N})$. Each simple perverse sheaf that occurs as a summand corresponds to a local system \mathcal{L}_φ on an orbit \mathcal{O}_x . Specifically, φ is an irreducible representation occurring as part of the action of $\pi_1(\mathcal{O}_x)$ on $H^{2d_x}(\mathcal{B}_x)$. In [BM81], Borho and MacPherson establish their

proof of the Springer Correspondence by showing that the local systems which appear are the same ones previously constructed by Springer, and furthermore, that the multiplicity with which they appear is the same as the dimension of the representation of the Weyl group to which they correspond. This means that in type A_n , the trivial local system, and only the trivial local system, appears for each orbit. We will see in Chapter 5 that more local systems occur for Graham's variety.

Chapter 3

Graham's Variety

Let \mathcal{O}_{prin} be the principal orbit in \mathcal{N} and \mathcal{O} be its universal cover. Since \mathcal{N} is a normal variety [Kos63], $\mathcal{N} = \text{Spec } \mathbb{R}(\mathcal{O}_{prin})$ where $\mathbb{R}(\mathcal{O}_{prin})$ is the ring of regular functions for the principal orbit. Let $\mathcal{M} = \text{Spec } \mathbb{R}(\mathcal{O})$. In [Gra], Graham defines a map $\varphi : \widetilde{\mathcal{M}} \rightarrow \mathcal{M}$ which is in some ways analogous to the Springer resolution for \mathcal{M} . He also gives a map $\tilde{\mu} : \widetilde{\mathcal{M}} \rightarrow \mathcal{N}$ which factors through both the Springer resolution and φ .

Here, we will describe the construction of $\widetilde{\mathcal{M}}$. This description will be slightly different than the one given in [Gra]. The reason for this will be seen in Section 3, where we will need the freedom to change our choice of Borel subgroup in order to implement Tymoczko's techniques. Let B be a Borel subgroup in G . Then $T := B/[B, B]$ is a torus which is canonically isomorphic to any maximal torus in B . We know that the center Z of G is contained in B , and $B_{ad} := B/Z$ is a Borel subgroup in G_{ad} . Thus, $T_{ad} := B_{ad}/[B_{ad}, B_{ad}]$ is a torus canonically isomorphic to any maximal torus in B_{ad} . Let $W_{ad} := \mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]$ where \mathfrak{u} is the nilradical of B_{ad} . We can imbue W_{ad} with the structure of an affine toric variety for T_{ad} by using the character group of T_{ad} as its lattice and by using the fundamental weights to generate its cone. By following the same construction as for W_{ad} but with the character group for T , we can construct a toric variety W for T such that $W/Z = W_{ad}$. For a more detailed description of the construction of a toric variety, see [Ful93] or [Gra]. The Borel B acts on W through projection onto T , so that the unipotent part of B acts trivially. Let $p_1 : \mathfrak{u} \rightarrow W_{ad}(= \mathfrak{u}/[\mathfrak{u}, \mathfrak{u}])$ and $p_2 : W \rightarrow$

$W_{ad}(= W/Z)$ be the B -equivariant projections. We can then define $\tilde{\mathbf{u}} = W \times_{W_{ad}} \mathbf{u} = \{(w, u) | p_1(u) = p_2(w)\}$.

Definition 3.1. *Graham's variety, $\tilde{\mathcal{M}}$, is $G \times^B \tilde{\mathbf{u}}$.*

Theorem 3.2 (Graham [Gra]). *Let $\tilde{\mathcal{M}}$ be as above. Then, the following diagram commutes.*

$$\begin{array}{ccc} \tilde{\mathcal{M}} & \xrightarrow{\gamma} & \tilde{\mathcal{N}} \\ \varphi \downarrow & & \downarrow \mu \\ \mathcal{M} & \longrightarrow & \mathcal{N} \end{array}$$

Here, φ is the composition of the normalization map for $\mathcal{M} \times_{\mathcal{N}} \tilde{\mathcal{N}}$ and the projection to \mathcal{M} , γ is the induced quotient map, and μ is the Springer resolution.

Note, the above diagram is not cartesian, and we will not need the fact that it commutes for this paper. The fact is included here to help describe Graham's work, but we will focus on the map $\tilde{\mu} := \mu \circ \gamma$. The fibers of γ are given by the fibers of the quotient map $\rho : W \rightarrow W_{ad}$, and Graham gives a method to determine the fibers of ρ in terms of the T_{ad} -orbits in W_{ad} .

In general, the orbits in an affine toric variety correspond to faces of the variety's cone. Let τ be a face in the cone for W , and denote the corresponding T -orbit in W by $\mathcal{O}(\tau)$. Then, $\mathcal{O}(\tau)$ is isomorphic to the torus $T(\tau) = \text{Hom}(\tau^\perp \cap \hat{T}, \mathbb{C}^*)$ where \hat{T} is the character group for T , i.e. the weight lattice. We can make analogous statements and definitions for the torus T_{ad} . Let us define $Z(\tau) = \ker(T(\tau) \rightarrow T_{ad}(\tau))$, which is a finite group. Then the character group for $Z(\tau)$ is $\hat{Z}(\tau) = \hat{T}(\tau)/\hat{T}_{ad}(\tau) = (\tau^\perp \cap \hat{T})/(\tau^\perp \cap \hat{T}_{ad})$. Graham proves in [Gra] that $\mathcal{O}(\tau) \rightarrow \mathcal{O}_{ad}(\tau)$ is a covering map with fibers $Z(\tau)$. Thus, understanding $Z(\tau)$, or equivalently $\hat{Z}(\tau)$, will give the fibers of the map ρ in terms of the torus orbits.

We can describe the faces of the cone in W_{ad} using subsets J of $\{1, 2, \dots, n\}$ since the cone of W_{ad} is generated by the fundamental weights $\{\omega_i\}_{i=1}^n$. Let τ_J be the face generated by the ω_j such that $j \in J$, and let $\mathcal{O}(\tau_J)$ denote the orbit corresponding to τ_J . To determine $\widehat{Z}(\tau_J)$, note that $(\tau^\perp \cap \widehat{T})/(\tau^\perp \cap \widehat{T}_{ad})$ maps injectively into $\widehat{T}/\widehat{T}_{ad} = Z$ which is the abstract fundamental group of the root system. To describe Z , we will follow the conventions found in Section 13.2 of [Hum72], where a table can be found that lists the dominant weights. The elements in this quotient group can be represented by particular dominant weights λ_i , where two dominant weights are in the same coset if their difference can be written with integer coefficients for all simple roots α_j . Thus, to describe the image of $\widehat{Z}(\tau_J)$ in Z , a coset $\lambda + \widehat{T}_{ad}$ is in the image if it has nonempty intersection with τ_J^\perp , which means the coefficients for all α_j in λ are integers if $j \in J$. This allows us to do the following calculations to determine $Z(\tau_J) = Z(J)$ for the T_{ad} -orbits in W_{ad} , and thus, the fibers of γ in terms of the T_{ad} -orbits.

Theorem 3.3. *Let J be a nonempty subset of $\{1, 2, \dots, n\}$. Then, the group $Z(J)$ is given as follows.*

$A_n :$	$Z(J) = \mathbb{Z}/c\mathbb{Z}$	where $c = \gcd(J \cup \{n + 1\})$
$B_n :$	$Z(J) = \mathbb{Z}/2\mathbb{Z}$	if all $j \in J$ are even
	$Z(J) = \{1\}$	otherwise
$C_n :$	$Z(J) = \mathbb{Z}/2\mathbb{Z}$	if $n \notin J$
	$Z(J) = \{1\}$	otherwise
$D_n :$	$Z(J) = Z$	if $n - 1, n \notin J$ and all $j \in J$ are even
	$Z(J) = \mathbb{Z}/2\mathbb{Z}$	if $n - 1, n \notin J$ and not all $j \in J$ are even
	$Z(J) = \mathbb{Z}/2\mathbb{Z}$	if exactly one of $n - 1$ and n is in J , all $j \in J$ such that $j < n - 1$ are even, and $n = 4k + 2$ for some $k \geq 1$
	$Z(J) = \{1\}$	otherwise

$$\begin{aligned}
E_6 : \quad Z(J) &= \mathbb{Z}/3\mathbb{Z} && \text{if none of } 1, 3, 5, 6 \text{ are in } J; \text{ otherwise } Z(J) = \{1\} \\
E_7 : \quad Z(J) &= \mathbb{Z}/2\mathbb{Z} && \text{if none of } 2, 5, 7 \text{ are in } J; \text{ otherwise } Z(J) = \{1\}
\end{aligned}$$

Proof. Assume first that \mathfrak{g} is of type A_n . Then from the table in Section 13.2 of [Hum72], we see that cosets in Z are given by 0 and

$$\begin{aligned}
\lambda_i = \frac{1}{n+1} [& ((n+1) - i)\alpha_1 + 2((n+1) - i)\alpha_2 + \dots + (i-1)((n+1) - i)\alpha_{i-1} \\
& + i((n+1) - i)\alpha_i + i((n+1) - (i+1))\alpha_{i+1} + \dots + i\alpha_n].
\end{aligned}$$

Let us denote by a_j the coefficient of α_j in the above formula. Then the question of calculating $Z(J)$ reduces to finding the conditions under which a_j is an integer. Define $d_j = \gcd(j, (n+1))$ and $c_j = \frac{n+1}{d_j}$. If $j \leq i$, we see that $a_j = \frac{1}{n+1}j((n+1) - i)$ is an integer when $(n+1)|ij$. Similarly, if $j > i$, $a_j = \frac{1}{n+1}i((n+1) - j)$ is an integer when $(n+1)|ij$. By our definitions of d_j and c_j , $(n+1)$ divides ij precisely when i is a multiple of c_j , and there are exactly $d_j - 1$ nonzero multiples of c_j less than $(n+1)$. Thus, there are d_j minimal dominant weights with an integer coefficient for α_j . To have integer coefficients for α_k as well when $k \neq j$, we would need i to simultaneously be a multiple of c_j and c_k . Thus, i must be a multiple of $\text{lcm}(c_j, c_k) = \frac{c_j c_k}{\gcd(c_j, c_k)}$. Since c_k and c_j both divide $(n+1)$, there is some integer b such that $(n+1) = b \frac{c_j c_k}{\gcd(c_j, c_k)}$. Combining this with the the fact that $n+1 = c_j d_j = c_k d_k$, we see that $b \frac{c_j}{\gcd(c_j, c_k)} = d_k$ and $b \frac{c_k}{\gcd(c_j, c_k)} = d_j$. Since $\frac{c_k}{\gcd(c_j, c_k)}$ and $\frac{c_j}{\gcd(c_j, c_k)}$ can have no common factors, $b = \gcd(d_j, d_k) = \gcd(j, k, (n+1))$, and there are exactly $b - 1$ nonzero i less than $(n+1)$ that are multiples of $\text{lcm}(c_j, c_k)$. The same reasoning will hold for any number of distinct roots, so we have the result that $Z(J) = \mathbb{Z}/c\mathbb{Z}$ where $c = \gcd(J \cup \{n+1\})$.

Assume now that \mathfrak{g} is of type B_n . Then we see that the cosets are represented by 0 and

$$\lambda_n = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \dots + n\alpha_n).$$

Thus, the coefficient on α_j is an integer exactly when j is even. Suppose \mathfrak{g} is of type C_n . Then the cosets are represented by 0 and

$$\lambda_i = \alpha_1 + 2\alpha_2 + \dots + (i-1)\alpha_{i-1} + i(\alpha_i + \dots + \alpha_{n-1} + \frac{1}{2}\alpha_n) \text{ for some odd } i \leq n.$$

Thus, the only coefficient that is not an integer is a_n , and $Z(J) = \mathbb{Z}/2\mathbb{Z}$ whenever $n \notin J$.

Let \mathfrak{g} be of type D_n . Then the cosets are represented by 0 and

$$\begin{aligned} \lambda_1 &= \alpha_1 + \alpha_2 + \dots + \alpha_{n-2} + \frac{1}{2}(\alpha_{n-1} + \alpha_n) \\ \lambda_{n-1} &= \frac{1}{2}[\alpha_1 + 2\alpha_2 + \dots + (n-2)\alpha_{n-2} + \frac{1}{2}\alpha_{n-1} + \frac{1}{2}(n-2)\alpha_n] \\ \lambda_n &= \frac{1}{2}[\alpha_1 + 2\alpha_2 + \dots + (n-2)\alpha_{n-2} + \frac{1}{2}(n-2)\alpha_{n-1} + \frac{1}{2}\alpha_n] \end{aligned}$$

The only way for λ_1 to be included is to have $n-1$ and n not be in J , since the coefficients in λ_1 on α_{n-1} and α_n are $\frac{1}{2}$. Thus, we should first consider the cases where $n-1$ and n are not in J . Then λ_{n-1} and λ_n have integer coefficients for all α_j with $j \in J$ exactly when all $j \in J$ are even. Next, let us consider when exactly one of $n-1$ and n is in J . Assume first that $n \in J$. It is still the case then that λ_1 and λ_n are not in $Z(J)$, but λ_{n-1} will be in $Z(J)$ if $n-2$ is divisible by 4 and all other $j \in J$ are even. Similarly, if $n-1 \in J$, λ_n will be in $Z(J)$ exactly when $n-2$ is divisible by 4 and all other $j \in J$ are even.

Let us now consider the exceptional types E_6 and E_7 . These are the only exceptional types for which Z is not the trivial group. For E_6 , the only nontrivial cosets

are represented by

$$\begin{aligned}\lambda_1 &= \frac{1}{3}(4\alpha_1 + 3\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6) \\ \lambda_3 &= \frac{1}{3}(5\alpha_1 + 6\alpha_2 + 10\alpha_3 + 12\alpha_4 + 8\alpha_5 + 4\alpha_6).\end{aligned}$$

Thus, the coefficients on α_2 and α_4 are integers in λ_1 and λ_3 , and $Z(J) = \mathbb{Z}/3\mathbb{Z}$ so long as J is a subset of $\{2, 4\}$. For E_7 , the only nontrivial coset is represented by

$$\lambda_2 = \frac{1}{2}(4\alpha_1 + 7\alpha_2 + 8\alpha_3 + 12\alpha_4 + 9\alpha_5 + 6\alpha_6 + 3\alpha_7).$$

Thus, the coefficients on α_1 , α_3 , α_4 , and α_6 are integers in λ_2 , and $Z(J) = \mathbb{Z}/2\mathbb{Z}$ exactly when J is a subset of $\{1, 3, 4, 6\}$. \square

Chapter 4

Graham's Fibers in Type A_n

G -orbits and Graham's Fibers

The Bala–Carter theorem associates to each orbit \mathcal{O}_x a pair $(\mathfrak{l}, \mathcal{O}_{x,\mathfrak{l}})$, where \mathfrak{l} is the smallest Levi subalgebra meeting \mathcal{O}_x and $\mathcal{O}_{x,\mathfrak{l}} = \mathcal{O}_x \cap \mathfrak{l}$. In these terms, the orbits \mathcal{O}_x that meet W_{ad} are those for which $\mathcal{O}_{x,\mathfrak{l}}$ is principal in \mathfrak{l} . In type A_n , all of the G -orbits are of this type, so that they all intersect W_{ad} . For the next two sections, we will only consider Lie algebras of type A_n .

Proposition 4.1. *Let \mathfrak{g} be a Lie algebra of type A_n . Let $J = \{d_1, d_2, \dots, d_r\}$ be a subset of $\{1, 2, \dots, n\}$ with the assumption that $d_i < d_j$ if $i < j$. Then τ_J is contained in the G -orbit given by the partition*

$$P(J) = [(n + 1 - d_r) (d_r - d_{r-1}) \dots (d_2 - d_1) d_1].$$

Proof. We take a representative X_J of a set J to be the sum of the root vectors X_{α_i} for all $i \in \{1, 2, \dots, n\} - J$ where α_i is a simple root following the notation of Humphreys [Hum72]. Since each T_{ad} -orbit is contained in some G -orbit, by calculating the Jordan canonical form for our representative, we are able to associate a single G -orbit to each J .

We will use the root vector conventions found in [CM93]. Let $X_J = \sum_{i \notin J} X_{\alpha_i}$. In type A_n , the root vector $X_{\alpha_i} = E_{i, i+1}$ where $E_{i,j}$ is a matrix with a one in the i th row and j th column and zeroes everywhere else. In this case, we see X_J is already in Jordan canonical form, and consecutive numbers not in J give us the sizes of the Jordan blocks. Thus, the formula for $P(J)$ is given by the distance between the consecutive elements in J , the distance between the largest element in J and $n + 1$, and the distance between the smallest element and zero. \square

Proposition 4.2. *Let \mathfrak{g} be a Lie algebra of type A_n , and let $x \in \tau_J$. Then, $Z(J) \cong \pi_1(\mathcal{O}_x)$.*

Proof. Let $J = \{d_1, d_2, \dots, d_r\}$. Then we know from 3.3 that $Z(J) = \mathbb{Z}/c\mathbb{Z}$ where $c = \gcd\{d_1, d_2, \dots, d_r, n + 1\}$. From the above result, we see that c is also the greatest common divisors of the parts in $P(J)$. From [CM93], Corollary 6.1.6, we see that $\pi_1(\mathcal{O}_x) = \mathbb{Z}/c\mathbb{Z}$ as well. \square

Cohomology of Graham's Fibers

For our perverse sheaf calculations, we will need to know more about the top degree cohomology of our Graham fibers $\widetilde{\mathcal{M}}_x$. This will be done by showing that each fiber contains $|\pi_1(\mathcal{O}_x)|$ affine spaces of dimension equal to the maximal possible dimension in the Springer fiber \mathcal{B}_x . In particular, we will use results of Tymoczko [Tym03] to find a particular affine space \mathcal{A}_x of maximal dimension inside \mathcal{B}_x and then show that over \mathcal{A}_x , Graham's map γ is a covering map with fibers $\pi_1(\mathcal{O}_x)$.

Fix $x \in \mathcal{N}$ and choose a basis $V' = \{v'_1, v'_2, \dots, v'_{n+1}\}$ so that x is in Jordan canonical form with the sizes of the blocks decreasing down the diagonal. Let B' be the Borel of upper triangular matrices on the basis V' . Let $P = [d_r \ d_{r-1} \ \dots \ d_1]$ with $d_i \geq d_{i-1}$ be the partition of $n + 1$ corresponding to x and denote by Y_P the Young diagram for P . Here, we follow the convention that Y_P is the left-justified array of boxes where the i th row from the bottom has d_i boxes. Next, we will describe two different labellings of this diagram. For the first, fill the blocks with $\{1, 2, \dots, n + 1\}$ in increasing order starting at the bottom left and moving up the columns, treating the columns left to right. We will call Y_P with this labelling Y_P^{Tym} . For the second, again label Y_P with $\{1, 2, \dots, n + 1\}$ in increasing order, but this time start at the top left and fill in the rows. We will call Y_P with this labelling Y_P^{Std} . Let σ be the permutation taking Y_P^{Std} to Y_P^{Tym} , i.e $\sigma(j)$ is the num-

ber in Y_P^{Tym} occupying the same box that j occupies in Y_P^{Std} .

Example 4.3. Let $x \in \mathcal{N}$ be an element in the orbit with partition $P = [2\ 2\ 1]$.

Then

$$x = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad Y_P^{Tym} = \begin{array}{|c|c|} \hline 3 & 5 \\ \hline 2 & 4 \\ \hline 1 & \\ \hline \end{array}, \quad Y_P^{Std} = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \hline \end{array},$$

and $\sigma = (1\ 3\ 2\ 5)$.

Let us call any two adjacent boxes in the same row of a labelled diagram a *pair* and use the notation $(i|j)$ to mean a pair where the label in the left box is i and the label in the right one is j . We can use the above labellings of Y_P to form nilpotent matrices by placing a 1 in the i th row and j th column for every pair $(i|j)$ in the labelled diagram and by filling the remaining entries with 0's. (In other words, this is the sum of all $E_{i,j}$ where $(i|j)$ is a pair.) Let M^{Std} be the matrix obtained this way from Y_P^{Std} , and note that M^{Std} is the matrix of x with respect to V' . The matrix M^{Tym} obtained from Y_P^{Tym} is not in Jordan canonical form, but there is a basis \widehat{V} for which M^{Tym} is the matrix of x with respect to \widehat{V} . In fact, $\widehat{V} = \{v'_{\sigma^{-1}(1)}, v'_{\sigma^{-1}(2)}, \dots, v'_{\sigma^{-1}(n+1)}\}$ where σ is the permutation described previously.

Let B be the Borel of upper triangular matrices on the basis \widehat{V} . Then, if we define the permutation matrix P_σ to be the matrix (with respect to V') with row

vectors $e_{\sigma(1)}$ through $e_{\sigma(n+1)}$, we see that $B = P_{\sigma^{-1}}B'P_{\sigma}$. Let $\{\alpha_i\}_1^n$ be the simple roots for B . Define $\alpha_{i,j} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-2} + \alpha_{j-1}$. Then x is the sum of all $\alpha_{i,j}$ such that $(i|j)$ is a pair in Y_P^{Tym} . Let $\beta_{j-1} = \alpha_{i,j}$ for each pair $(i|j)$ in Y_P^{Tym} . Note that these β 's are simple roots for B' which are positive for B .

Define $\mathcal{A}_x := B\sigma B \cap \mu^{-1}(x)$ where $B\sigma B$ is the Schubert cell associated to σ . For any $w \in \mathcal{W}$, we will denote by Φ_w the set of positive roots for B that become negative under the action of w . For any $x \in \mathcal{N}$, we will let Φ_x be the positive roots for B whose vectors appear as summands when x is decomposed as the sum of root vectors. We will denote by $\Phi_{w,x}$ the subset of Φ_w whose elements can be viewed as a sum of a root in Φ_w and a root in Φ_x . We have chosen B and Y_P^{Tym} such that Tymoczko's Theorem 22 and Theorem 24 in [Tym03] can be phrased as follows in the special case that the Hessenberg space is $\text{Lie } B$ and the corresponding Hessenberg variety is \mathcal{B}_x .

Theorem 4.4 (Tymoczko). *(a) Let $w \in \mathcal{W}$. The Schubert cells BwB intersect each Springer fiber in a paving by affines. The nonempty cells are BwB where $\text{Ad}(w^{-1})x \in \text{Lie } B$, and they have dimension*

$$|\Phi_w| - |\Phi_{w,x}|.$$

(b) The nonempty cells from (a) correspond to the permutations w such that $w^{-1}(Y_P^{Tym})$ has the property that $i < j$ for each pair $(i|j)$.

We can now use our labelled Young diagrams Y_P^{Tym} and $Y_P^{Std} = \sigma^{-1}(Y_P^{Tym})$ with the above theorem to find our affine space of maximal dimension in \mathcal{B}_x .

Lemma 4.5. *\mathcal{A}_x is nonempty, and it is an affine space of dimension $\frac{1}{2}(\dim \mathcal{N} - \dim \mathcal{O}_x) = \dim \mathcal{B}_x$.*

Proof. To see that \mathcal{A}_x is nonempty, we will need only to note that Y_P^{Std} is filled with labels that increase from left to right. Thus, the condition in Theorem 4.4 (b) is always satisfied. From the discussions in previous paragraphs, $\Phi_x = \{\alpha_{i,j} \mid (i|j) \text{ is a pair in } Y_P^{Tym}\}$. Given our description of the positive roots of B , we can restate our definition of Φ_σ as $\alpha_{i,j} \in \Phi_\sigma$ for $i < j$ if and only if $\sigma^{-1}(i) < \sigma^{-1}(j)$. Since we used our labellings of Y_P to define σ , Φ_σ can be seen from Y_P^{Tym} . Let us number the rows in Y_P^{Tym} in increasing order from top to bottom and number the columns in increasing order from left to right. For any label i in Y_P^{Tym} , let $\text{row}(i)$ be the row number of the row containing i and $\text{col}(i)$ be the column number of the column containing i . Using this notation, $\alpha_{i,j}$ is in Φ_σ if and only if $\text{row}(i) > \text{row}(j)$ and $\text{col}(i) \geq \text{col}(j)$. We can also see $\Phi_{\sigma,x}$ from Y_P^{Tym} . In particular, $\alpha_{i,j}$ is in $\Phi_{\sigma,x}$ if and only if $\text{row}(i) > \text{row}(j)$ and $\text{col}(i) > \text{col}(j)$. See Example 4.6.

Viewing Φ_σ and $\Phi_{\sigma,x}$ in this manner allows us to translate the formula from Theorem 4.4(a) into a statement about the number of blocks in the columns of Y_P . More specifically, if we let $\text{ht}(j)$ denote the number of blocks (or height) in the j th column of Y_P , then

$$|\Phi_\sigma| - |\Phi_{\sigma,x}| = \sum_{j=1}^{d_r} \sum_{i=1}^{\text{ht}(j)} (\text{ht}(j) - i)$$

where d_r is the largest part in the partition P and thus the number of columns in Y_P . From [CM93] Corollary 6.1.4, we see that the dimension of \mathcal{O}_x is $(n+1)^2 - \sum_{j=1}^{d_r} \text{ht}(j)^2$. Thus, we have

$$\begin{aligned} \dim \mathcal{N} - 2 \dim \mathcal{A}_x &= n(n+1) - 2 \sum_{j=1}^{d_r} \sum_{i=1}^{\text{ht}(j)} (\text{ht}(j) - i) \\ &= n^2 + n - 2 \sum_{j=1}^{d_r} \left(\text{ht}(j)^2 - \left(\frac{\text{ht}(j)^2 + \text{ht}(j)}{2} \right) \right) \end{aligned}$$

$$\begin{aligned}
&= n^2 + n - \sum_{j=1}^{d_r} \text{ht}(j)^2 + \sum_{j=1}^{d_r} \text{ht}(j) \\
&= (n+1)^2 - \sum_{j=1}^{d_r} \text{ht}(j)^2 \\
&= \dim \mathcal{O}_x.
\end{aligned}$$

Since, the dimension of \mathcal{A}_x is equal to the codimension of the orbit \mathcal{O}_x , we see that \mathcal{A}_x is of maximal possible dimension. \square

Example 4.6. Let $x \in \mathcal{N}$ be such that it corresponds to the partition $P = [3\ 3\ 1]$.

Then

$$Y_P^{Tym} = \begin{array}{|c|c|c|} \hline 3 & 5 & 7 \\ \hline 2 & 4 & 6 \\ \hline 1 & & \\ \hline \end{array},$$

$$\Phi_x = \{\alpha_{2,4}, \alpha_{3,5}, \alpha_{4,6}, \alpha_{5,7}\},$$

$$\Phi_\sigma = \{\alpha_{1,2}, \alpha_{1,3}, \alpha_{1,4}, \alpha_{1,5}, \alpha_{1,6}, \alpha_{1,7}, \alpha_{2,3}, \alpha_{2,5}, \alpha_{2,7}, \alpha_{4,5}, \alpha_{4,7}, \alpha_{6,7}\},$$

$$\text{and } \Phi_{\sigma,x} = \{\alpha_{1,4}, \alpha_{1,6}, \alpha_{1,5}, \alpha_{1,7}, \alpha_{2,5}, \alpha_{2,7}, \alpha_{4,7}\}.$$

Thus, $\dim \mathcal{A}_x = |\Phi_\sigma| - |\Phi_{\sigma,x}| = 5$. Notice that $\dim \mathcal{O}_x = 32$ and $\dim \mathcal{N} = 42$ so this agree with the above calculations.

Proposition 4.7. $\gamma^{-1}(\mathcal{A}_x)$ is the disjoint union of $|\pi_1(\mathcal{O}_x)|$ copies of \mathcal{A}_x .

Proof. We will show that if x is the sum of simple roots for any Borel in \mathcal{A}_x , then it must be the sum of the same simple roots for all other Borels in \mathcal{A}_x . Since how x decomposes as the sum of simple roots determines the fibers of Graham's map, this will be enough to tell us that the fiber over all of \mathcal{A}_x must be the same.

As defined earlier, we see that B' and B are in \mathcal{B}_x . Furthermore, $B' = \sigma B \in B\sigma B$. Suppose there is some Borel $B'' = bB'b^{-1}$ for some $b \in B$ such that $x \in \text{Lie } B''$. Then, B' and B'' are both in \mathcal{A}_x . Let T be a maximal torus contained in $B \cap B'$ and let U be the unipotent radical of B so that $B = TU$. Then we can assume $b \in U$. Let \mathfrak{u}' and \mathfrak{u}'' be the nilradicals of $\text{Lie } B'$ and $\text{Lie } B''$, respectively. Then $\mathfrak{u}'' = \text{Ad}(b)\mathfrak{u}'$ since $B'' = bB'b^{-1}$. Moreover, for any root vector X'_β for B' , $X''_\beta := \text{Ad}(b)X'_\beta$ is root vector for B'' . Thus, we can write $x = \sum c'_\beta X'_\beta$ in \mathfrak{u}' and $x = \sum c''_\beta X''_\beta = \sum c''_\beta \text{Ad}(b)X'_\beta$ in \mathfrak{u}'' . Then $\text{Ad}(b^{-1})x = \sum c''_\beta X'_\beta$. This means the fibers over B' and B'' are isomorphic if and only if for every simple root β for B' , $c'_\beta \neq 0$ is equivalent to $c''_\beta \neq 0$.

Since $T \subset B \cap B'$, we can assume that $\text{Lie } B'$ decomposes as

$$\text{Lie } B' = \text{Lie } T \oplus \left\{ \begin{array}{l} \text{some positive root} \\ \text{spaces for } B \end{array} \right\} \oplus \left\{ \begin{array}{l} \text{some negative root} \\ \text{spaces for } B \end{array} \right\}.$$

Thus, since $b \in U$, we know $\text{Ad}(b^{-1})X'_\beta = X'_\beta + \sum_{\alpha > \beta} a_\alpha X'_\alpha$ where α is a root and $\alpha > \beta$ means $\alpha - \beta$ is the sum of positive roots for B . We need to verify that none of the simple roots with nonzero coefficients in \mathfrak{u}' have zero coefficients in \mathfrak{u}'' , and hence, we need to show that the following *never* happens:

$$\left\{ \begin{array}{l} \text{a simple root } \beta \text{ for } B' \\ \text{such that } c''_\beta \neq 0 \end{array} \right\} + \left\{ \begin{array}{l} \text{a sum of positive} \\ \text{roots for } B \end{array} \right\} = \left\{ \begin{array}{l} \text{a simple root} \\ \text{for } B' \end{array} \right\}.$$

Luckily, we know the simple roots β above in terms of positive roots for B . They are the $\alpha_{i,j}$ in Φ_x and all other simple roots for B' are negative roots for B . Thus, since it can't happen that the sum of positive roots is negative, we need only consider the case where the difference of two roots in Φ_x is a sum of positive roots. For this, we will refer again to Tymoczko's work. In [Tym03] Definition 15, she defines a set of roots to be *non-overlapping* if no pair of roots α and β in the set

are such that $\alpha > \beta$. Thus, our previous statement reduces now to showing Φ_x is non-overlapping, but this fact is established in Tymoczko's proof of Theorem 22. We can also see this by examining Y_P^{Tym} and noting that there are no pairs $(i|j)$ and $(k|l)$ such that $i \leq k < l \leq j$. \square

Despite the title of this section, these propositions do not give a full understanding of Graham's fibers. However, they do give enough enlightenment that we can make statements in the perverse sheaf setting.

Chapter 5

Perverse Sheaves in Type A_n

First, let us establish that Theorem 2.3 applies in Graham's setting.

Proposition 5.1. *The map $\tilde{\mu}$ is proper and semismall. $\tilde{\mathcal{M}}$ is rationally smooth with $\dim \tilde{\mathcal{M}} = \dim \mathcal{N}$. Furthermore, $2d_x = \dim \mathcal{N} - \dim \mathcal{O}_x$ for any $x \in \mathcal{N}$.*

Proof. According to [Gra], π is finite and $\tilde{\mathcal{M}}$ is an orbifold. All of the properties except rational smoothness follow from the finiteness of π and Fact 2.4. For a proof that orbifolds are rationally smooth, see [Bri99]. \square

While still lacking the complete picture of the cohomology of Graham's fibers, the calculations in Section 3.2 are enough to make the following useful proposition:

Proposition 5.2. *For \mathfrak{g} in type A_n , $H^{2d_x}(\tilde{\mathcal{M}}_x)$ contains a copy of the regular representation of $\pi_1(\mathcal{O}_x)$.*

Proof. There is a basis of $H^{2d_x}(\tilde{\mathcal{M}}_x)$ indexed by irreducible components of maximal dimension. Therefore, determining how $\pi_1(\mathcal{O}_x)$ acts on the components is sufficient to understand how it acts on the whole space. Since $\dim \mathcal{A}_x$ is maximal in \mathcal{B}_x , it must be the case that the closure of \mathcal{A}_x is an irreducible component of maximal dimension in \mathcal{B}_x , and similarly, the closure of each copy of \mathcal{A}_x found in $\gamma^{-1}(\mathcal{A}_x)$ must be an irreducible component of maximal dimension in $\tilde{\mathcal{M}}_x$. From Proposition 4.2, we know that $\pi_1(\mathcal{O}_x)$ acts freely and transitively on the fibers of $\tilde{\mu}$, so also on these components in $\gamma^{-1}(\mathcal{A}_x)$. We see then that there must be a copy of the regular representation of $\pi_1(\mathcal{O}_x)$ contained in its action on $H^{2d_x}(\tilde{\mathcal{M}}_x)$. \square

With these propositions in place, we can now state our main result.

Theorem 5.3. *If \mathfrak{g} is a Lie algebra of type A_n , then every G -equivariant simple perverse sheaf occurs as a summand in the decomposition of $R\mu_*\overline{\mathbb{Q}}_{\ell, \widetilde{\mathcal{M}}}$.*

Proof. From Proposition 5.1, we know that the Decomposition Theorem applies. Then, since $\tilde{\mu}$ is semismall, the Decomposition Theorem tells us that $\mathrm{IC}(\mathcal{O}_x, \mathcal{L}_\varphi)$ occurs in $R\mu_*\overline{\mathbb{Q}}_{\ell, \widetilde{\mathcal{M}}}$ if and only if the representation φ occurs in the action of $\pi_1(\mathcal{O}_x)$ on $H^{2d_x}(\widetilde{\mathcal{M}}_x)$. From Proposition 5.2, we see that all the irreducible representations appear in this action. The result follows. \square

Chapter 6

Towards Result in Other Types

For the classical Lie algebras, the nilpotent orbits are parametrized with partition classifications. In the following theorem, we give the inclusion correspondence between T_{ad} -orbits and G -orbits in terms of these partitions. Note that multiple T_{ad} -orbits can be contained in the same G -orbit. The partition notation comes from [CM93], and *unreduced* refers to the possible need to combine like parts once the formula has been implemented.

Theorem 6.1. *Let G be of type B_n , C_n , or D_n . Each T_{ad} -orbit is contained in the G -orbit which has the partition classification given in the following way:*

Let $J = \{d_1, d_2, \dots, d_r\}$ be a subset of $\{1, 2, \dots, n\}$. Assume here that $d_i < d_j$ for $i < j$. We denote by $P(J)$ the unreduced partition associated to the set J .

$$B_n : P(J) = [(2(n - d_r) + 1) (d_r - d_{r-1})^2 \dots (d_2 - d_1)^2 d_1^2]$$

$$C_n : P(J) = [2(n - d_r) (d_r - d_{r-1})^2 \dots (d_2 - d_1)^2 d_1^2]$$

$$D_n : P(J) = [(2(n - d_r) - 1) (d_r - d_{r-1})^2 \dots (d_2 - d_1)^2 d_1^2 1] \text{ if } n - 1, n \notin J$$

$$P(J) = [(d_r - d_{r-1})^2 \dots (d_2 - d_1)^2 d_1^2] \text{ if } n - 1, n \in J$$

$$P(J) = [(n - d_{r-1})^2 (d_{r-1} - d_{r-2})^2 \dots (d_2 - d_1)^2 d_1^2] \text{ if } n - 1 \text{ or } n \in J$$

Proof. As before, we take a representative X_J of a set J to be the sum of the root vectors X_{α_i} for all $i \in \{1, 2, \dots, n\} - J$ where α_i is a simple root following the notation of Humphreys [Hum72]. Since each T_{ad} -orbit is contained in some G -orbit, by calculating the Jordan canonical form for our representative, we are able to associate a single G -orbit to each J . See the remark following this proof for the special case in D_n where Jordan canonical form does not determine the G -orbit.

We will again use the root vector conventions found in [CM93]. Following the previous notation, let $X_J = \sum_{i \notin J} X_{\alpha_i}$. Now, we must calculate the Jordan canonical form for our X_J in the classical types other than A_n .

Suppose we are in type B_n . Then, $X_{\alpha_i} = E_{i+1, i+2} - E_{n+i+2, n+i+1}$ if $1 \leq i \leq n-1$ and $X_{\alpha_n} = E_{1, 2n+1} - E_{n+1, 1}$. When putting X_J in Jordan canonical form, we see that $E_{i,j}$ and $E_{j,k}$ give rise to elements in the same block, and this determines the blocks, so long as each i occurs only once as the first indice and once as the second. From the definitions of X_{α} , we see that a maximal set of consecutive roots creates two blocks, each with size equal to the size of the set, except when one of the roots is α_n . The root vector X_{α_n} forms a single block of size three when considered alone. Consequently, if α_n is included in the set, the block formed has size one more than twice the size of the set. As before, we compute the sizes of the blocks by taking the distance between consecutive elements not in J . However, each distance now corresponds to two blocks instead of one, unless J does not contain n .

Now, suppose we are in type C_n . In this case, $X_{\alpha_i} = E_{i, i+1} - E_{n+i+1, n+i}$ if $1 \leq i \leq n-1$ and $X_{\alpha_n} = E_{n, 2n}$. As in type B_n , a maximal set of consecutive roots creates two blocks with size equal to the size of the set with the exception of when α_n is in the set. Any maximal set of k consecutive roots containing α_n forms a block of size $2k$. So again, we compute the sizes of the blocks by taking the distance between consecutive elements not in J with each distance corresponding to two blocks except when n is not contained in J .

Finally, suppose we are in type D_n . Then, $X_{\alpha_i} = E_{i, i+1} - E_{n+i+1, n+i}$ if $1 \leq i \leq n-1$ and $X_{\alpha_n} = E_{n-1, 2n} - E_{n, 2n-1}$. We now have three distinct cases to consider. To begin, suppose both $n-1$ and n are in J . Then, we can proceed as in type C_n . Next, suppose neither $n-1$ nor n is in J . This works as before except $X_{\alpha_{n-1}} + X_{\alpha_n}$ forms one block of size three and another of size one. Then, each

consecutive root for α_{n-1} adds two to the size of the larger block. This gives the block of size $2(n - d_r) - 1$ appearing in the formula. Lastly, suppose exactly one of n and $n - 1$ is in J . The block calculation follows the same in either case due to the definitions of $X_{\alpha_{n-1}}$ and X_{α_n} , so that only n appears in the formula. \square

Remark. A partition is called *very even* if it has only even parts, each with even multiplicity. In the case of type D_n , very even partitions give two distinct nilpotent orbits. In order to completely classify the correspondence between T_{ad} -orbits and G -orbits, the weighted Dynkin diagrams would need to be calculated if $P(J)$ is very even in type D_n . This can only happen if $P(J)$ is of the third type listed in Proposition 6.1, and the result should be that the orbit is determined by whether n or $n - 1$ is in J . Since no distinction between these two orbits is necessary for the purposes of this paper, the calculation is omitted. See [CM93] for the weighted Dynkin diagrams of the two orbits.

In Proposition 6.2 below, $Z(\mathcal{O}_x)$ is defined to be $Z(J)$ where $x \in \mathcal{O}(\tau_J)$. Although each nilpotent orbit \mathcal{O}_x contains multiple T_{ad} -orbits, the fiber $Z(J)$ is the same for all $\mathcal{O}(\tau_J)$ in any particular orbit. This can be seen from the calculations done for Proposition 3.3 and 6.1, and thus, $Z(\mathcal{O}_x)$ is well-defined.

Proposition 6.2. *Let x be in W_{ad} . Then, $Z(\mathcal{O}_x)$ is the kernel of the quotient map from $\pi_1(\mathcal{O}_x)$ to $A(\mathcal{O}_x)$.*

Proof. Let $\mathcal{O}(\tau)$ (respectively, $\mathcal{O}_{ad}(\tau)$) be the T -orbit in W (T_{ad} -orbit in W_{ad}) corresponding to the face τ of the cone of W (W_{ad}). We know the center Z of G is contained in T and acts on W . Let $y \in \mathcal{O}(\tau)$ and $Z^\tau := \text{Stab}_Z(y)$. We know $q : \mathcal{O}(\tau) \rightarrow \mathcal{O}_{ad}(\tau)$ is the quotient by the action of Z . Let $x \in \mathcal{O}_{ad}(\tau)$ be such that $q(y) = x$. By definition, $Z^\tau = Z \cap T^y$. Because of the way x and y are defined, we know $T^y \subseteq T^x$, and moreover, since the dimensions of the orbits for x and y

are the same, we know that T^y must be a collection of components for T^x . Thus, $(T^x)^\circ \subseteq T^y$ where T^y is connected since this is equivalent to its character group being torsion free. From the connectedness of T^y , we see $T^y = (T^x)^\circ$ and thus, $Z^\tau = Z \cap (T^x)^\circ$. Note that as above, we have $(G^x)^\circ \subseteq G^y$, but also $(T^x)^\circ \subseteq (G^x)^\circ$. Combining these statements, we have

$$Z^\tau = Z \cap (T^x)^\circ \subseteq Z \cap (G^x)^\circ \subseteq Z \cap G^y = Z^\tau.$$

Therefore, $Z^\tau = Z \cap (G^x)^\circ$ and $Z(\mathcal{O}_x) = Z/Z^\tau$ is a subgroup of $(G^x)/(G^x)^\circ = \pi_1(\mathcal{O}_x)$. Then, since $G_{ad} = G/Z$, we have $Z(\mathcal{O}_x) \subseteq \ker(\pi_1(\mathcal{O}_x) \rightarrow A(\mathcal{O}_x))$. Now that we have containment regardless of type, we will examine this kernel along with $Z(\mathcal{O}_x)$ in each type to see that $Z(\mathcal{O}_x)$ must, in fact, be the kernel.

Let P be the partition associated to x in types B_n , C_n , and D_n . To clarify some of the following argument, let us denote the multiplicity of a part d in a partition P to be $\text{mult}_P(d)$. Let us now suppose we are in type B_n . Then $P(J)$ can be of two forms. If $P(J)$ has exactly one odd part d with $\text{mult}_P(d) = 1$, then $Z(\mathcal{O}_P) = \mathbb{Z}/2\mathbb{Z}$ since all elements of J must be even if all parts with even multiplicity are even. If $P(J)$ has more than one odd part or an odd part with multiplicity greater than one, then $Z(\mathcal{O}_P) = \{1\}$ since this can only happen when not all elements in J are even.

Let us now consider type C_n . Again, $P(J)$ has two possible forms. If $P(J)$ is such that $\text{mult}_P(d)$ is even for all parts d , then we know $n \in J$ and $Z(\mathcal{O}_P) = \{1\}$. If $P(J)$ has exactly one part with odd multiplicity, then we know $n \notin J$ and $Z(\mathcal{O}_P) = \mathbb{Z}/2\mathbb{Z}$.

Let us finally suppose we are in type D_n . In this case, we have four possible forms for $P(J)$. If $P(J)$ is such that $\text{mult}_P(1) = 1$, $\text{mult}_P(d) = 1$ for some odd part d , and the rest of the parts are even numbers with even multiplicity, then

$Z(\mathcal{O}_P) = Z$ since this corresponds to J having all even elements and $n, n-1 \notin J$. If $P(J)$ is such that $\text{mult}_P(1) = 1$ and there is some odd part d with $\text{mult}_P(d) > 1$, then $Z(\mathcal{O}_P) = \mathbb{Z}/2\mathbb{Z}$ since this corresponds to not all elements of J being even and $n, n-1 \notin J$. If $P(J)$ has all even parts with even multiplicities, then this corresponds to the third condition for D_n in Theorem 3.3 and $Z(\mathcal{O}_P) = \mathbb{Z}/2\mathbb{Z}$. Lastly, if $P(J)$ has $\text{mult}_P(d)$ even for all parts d and at least one d is odd, then $Z(\mathcal{O}_P) = \{1\}$.

Since the form of our partition P tells us what $\pi_1(\mathcal{O}_P)$ and $A(\mathcal{O}_P)$ are in Table 2.1, we see from the above reasoning that $Z(\mathcal{O}_P)$ is isomorphic to the kernel of the quotient map from $\pi_1(\mathcal{O}_P)$ to $A(\mathcal{O}_P)$. For types E_6 and E_7 , we can examine Tables 6.1 and 6.2 below to see that the same is true in these types for any orbit \mathcal{O}_x with $x \in W_{ad}$. Thus, since $Z(\mathcal{O}_x) \subseteq \ker(\pi_1(\mathcal{O}_x) \twoheadrightarrow A(\mathcal{O}_x))$, we must have equality in all types. \square

Recall that the G -orbits which intersect W_{ad} are the ones whose Bala–Carter label corresponds to the principal nilpotent orbit of the Levi subalgebra. This connection with the Bala–Carter classification will be how we determine the containment of T_{ad} -orbits inside G -orbits for the exceptional Lie algebras E_6 and E_7 . In the case of E_7 , it was also necessary to calculate the weighted Dynkin diagrams when the same type of Levi subalgebra labels more than one orbit.

The following tables do not include the G -orbits which contain no T_{ad} -orbits. For any orbit \mathcal{O} , the G_{ad} -equivariant fundamental group $A(\mathcal{O})$ can be determined from the $\pi_1(\mathcal{O})$ column by taking the quotient by any $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$ factor present. The values for $\pi_1(\mathcal{O})$ and $A(\mathcal{O})$ come from [Ale05] and [Miz80]. They can also be found in [Car85] Section 13.1 and [CM93] Section 8.4, but with some errors in the results for E_7 .

TABLE 6.1. T_{ad} -orbits and Fundamental Groups for E_6

Bala–Carter	Subsets J corresponding to T_{ad} -orbits	$Z(\mathcal{O})$	$\pi_1(\mathcal{O})$
Triv.	$\{1, 2, 3, 4, 5, 6\}$	1	1
A_1	$\{1, 2, 3, 4, 5\}, \{1, 2, 3, 4, 6\}, \{1, 2, 3, 5, 6\},$ $\{1, 2, 4, 5, 6\}, \{1, 3, 4, 5, 6\}, \{2, 3, 4, 5, 6\}$	1	1
$2A_1$	$\{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 2, 4, 6\}, \{1, 2, 5, 6\},$ $\{1, 3, 4, 5\}, \{1, 3, 4, 6\}, \{1, 4, 5, 6\}, \{2, 3, 4, 5\},$ $\{2, 3, 4, 6\}, \{2, 3, 5, 6\}, \{3, 4, 5, 6\}$	1	1
$3A_1$	$\{1, 4, 5\}, \{1, 4, 6\}, \{2, 3, 5\}, \{3, 4, 5\}, \{3, 4, 6\}$	1	1
A_2	$\{1, 2, 3, 4\}, \{1, 2, 3, 6\}, \{1, 3, 5, 6\}, \{2, 4, 5, 6\}$	1	S_2
$A_2 + A_1$	$\{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}, \{2, 3, 4\},$ $\{2, 3, 6\}, \{2, 4, 5\}, \{2, 4, 6\}, \{3, 5, 6\}, \{4, 5, 6\}$	1	1
$2A_2$	$\{2, 4\}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$
$A_2 + 2A_1$	$\{1, 4\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{4, 6\}$	1	1
A_3	$\{1, 2, 3\}, \{1, 2, 6\}, \{1, 3, 6\}, \{1, 5, 6\}, \{2, 5, 6\}$	1	1
$2A_2 + A_1$	$\{4\}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$
$A_3 + A_1$	$\{1, 5\}, \{2, 3\}, \{2, 5\}, \{3, 6\}$	1	1
A_4	$\{1, 2\}, \{1, 3\}, \{2, 6\}, \{5, 6\}$	1	1
D_4	$\{1, 6\}$	1	1
$A_4 + A_1$	$\{3\}, \{5\}$	1	1
A_5	$\{2\}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$
D_5	$\{1\}, \{6\}$	1	1
E_6	\emptyset	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$

TABLE 6.2. T_{ad} -orbits and Fundamental Groups for E_7

Bala–Carter	Subsets J corresponding to T_{ad} -orbits	$Z(\mathcal{O})$	$\pi_1(\mathcal{O})$
Triv.	$\{1, 2, 3, 4, 5, 6, 7\}$	1	1
A_1	$\{1, 2, 3, 4, 5, 6\}, \{1, 2, 3, 4, 5, 7\}, \{1, 2, 3, 4, 6, 7\},$ $\{1, 2, 3, 5, 6, 7\}, \{1, 2, 4, 5, 6, 7\}, \{1, 3, 4, 5, 6, 7\},$ $\{2, 3, 4, 5, 6, 7\}$	1	1
$2A_1$	$\{1, 2, 3, 4, 6\}, \{1, 2, 3, 5, 6\}, \{1, 2, 3, 5, 7\},$ $\{1, 2, 4, 5, 6\}, \{1, 2, 4, 5, 7\}, \{1, 2, 4, 6, 7\},$ $\{1, 3, 4, 5, 6\}, \{1, 3, 4, 5, 7\}, \{1, 3, 4, 6, 7\},$ $\{1, 4, 5, 6, 7\}, \{2, 3, 4, 5, 6\}, \{2, 3, 4, 5, 7\},$ $\{2, 3, 4, 6, 7\}, \{2, 4, 5, 6, 7\}, \{3, 4, 5, 6, 7\}$	1	1
$(3A_1)''$	$\{1, 3, 4, 6\}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
$(3A_1)'$	$\{1, 2, 4, 6\}, \{1, 4, 5, 6\}, \{1, 4, 5, 7\}, \{1, 4, 6, 7\},$ $\{2, 3, 4, 6\}, \{2, 3, 5, 6\}, \{2, 3, 5, 7\}, \{3, 4, 5, 6\},$ $\{3, 4, 5, 7\}, \{3, 4, 6, 7\}$	1	1
A_2	$\{1, 2, 3, 4, 5\}, \{1, 2, 3, 4, 7\}, \{1, 2, 3, 6, 7\},$ $\{1, 2, 5, 6, 7\}, \{1, 3, 5, 6, 7\}$	1	S_2
$4A_1$	$\{1, 4, 6\}, \{3, 4, 6\}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}^1$
$A_2 + A_1$	$\{1, 2, 3, 5\}, \{1, 2, 3, 6\}, \{1, 2, 4, 5\}, \{1, 2, 4, 7\},$ $\{1, 2, 5, 6\}, \{1, 2, 5, 7\}, \{1, 3, 4, 5\}, \{1, 3, 4, 7\},$ $\{1, 3, 5, 6\}, \{1, 3, 5, 7\}, \{1, 5, 6, 7\}, \{2, 3, 4, 5\},$ $\{2, 3, 4, 7\}, \{2, 3, 6, 7\}, \{2, 4, 5, 6\}, \{2, 4, 5, 7\},$ $\{2, 4, 6, 7\}, \{3, 4, 6, 7\}, \{4, 5, 6, 7\}$	1	S_2
$A_2 + 2A_1$	$\{1, 4, 5\}, \{1, 4, 7\}, \{2, 3, 5\}, \{2, 3, 6\}, \{2, 4, 6\},$ $\{3, 4, 5\}, \{3, 4, 7\}, \{3, 5, 6\}, \{3, 5, 7\}, \{4, 5, 6\},$ $\{4, 5, 7\}, \{4, 6, 7\}$	1	1
$2A_2$	$\{2, 4, 5\}, \{1, 3, 5\}, \{2, 4, 7\}, \{1, 2, 5\}$	1	1
$A_2 + 3A_1$	$\{4, 6\}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

¹This was mistakenly stated to be trivial in [CM93] and [Car85].

TABLE 6.2. Continued

Bala–Carter	Subsets J corresponding to T_{ad} -orbits	$Z(\mathcal{O})$	$\pi_1(\mathcal{O})$
A_3	$\{1, 2, 3, 4\}, \{1, 2, 3, 7\}, \{1, 2, 6, 7\}, \{1, 3, 6, 7\}, \{2, 5, 6, 7\}$	1	1
$(A_3 + A_1)''$	$\{1, 3, 4\}, \{1, 3, 6\}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
$2A_2 + A_1$	$\{3, 5\}, \{4, 5\}, \{4, 7\}$	1	1
$(A_3 + A_1)'$	$\{1, 5, 6\}, \{1, 5, 7\}, \{2, 3, 4\}, \{2, 3, 7\}, \{2, 5, 6\}, \{2, 5, 7\}, \{3, 6, 7\}, \{1, 2, 4\}, \{1, 2, 6\}$	1	1
$A_3 + 2A_1$	$\{1, 4\}, \{3, 4\}, \{3, 6\}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
D_4	$\{1, 6, 7\}$	1	1
$A_3 + A_2$	$\{1, 5\}, \{2, 4\}, \{2, 5\}$	1	S_2
$A_3 + A_2 + A_1$	$\{4\}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
A_4	$\{1, 2, 7\}, \{1, 3, 7\}, \{2, 6, 7\}, \{5, 6, 7\}, \{1, 2, 3\}$	1	S_2
$(A_5)''$	$\{1, 3\}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}^2$
$D_4 + A_1$	$\{1, 6\}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
$A_4 + A_1$	$\{2, 3\}, \{2, 6\}, \{3, 7\}, \{5, 6\}, \{5, 7\}$	1	S_2
$A_4 + A_2$	$\{5\}$	1	1
$(A_5)'$	$\{1, 2\}, \{2, 7\}$	1	1
$A_5 + A_1$	$\{3\}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
D_5	$\{1, 7\}, \{6, 7\}$	1	1
A_6	$\{2\}$	1	1
$D_5 + A_1$	$\{6\}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
D_6	$\{1\}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
E_6	$\{7\}$	1	1
E_7	\emptyset	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

²This was mistakenly stated to be trivial in [CM93] and [Car85].

References

- [Ach07] P. Achar. Course notes from Applications of Homological Algebra: Introduction to Perverse Sheaves, 2007.
- [Ale05] A. Alexeevski. Component Groups of the Centralizers of Unipotent Elements in Semisimple Algebraic Groups. *Amer. Math. Soc. Transl. (2)*, 213:15 – 31, 2005.
- [AO03] J. Anker and B. Orsted. *Lie Theory: Lie algebras and representations*. Birkhäuser, Boston, 2003.
- [BBD82] A. Beilinson, J. Bernstein, and P. Deligne. Faisceaux pervers. *Astérisque*, 100:5–171, 1982.
- [BC76a] P. Bala and R. W. Carter. Classes of unipotent elements in simple algebraic groups I. *Math. Proc. Cambridge Philos. Soc*, 79(3):401–425, 1976.
- [BC76b] P. Bala and R. W. Carter. Classes of unipotent elements in simple algebraic groups II. *Math. Proc. Cambridge Philos. Soc*, 80:1–18, 1976.
- [BM81] W. Borho and R. MacPherson. Representations des groupes de Weyl et homologie d’intersection pour les varietes nilpotentes. *C. R. Acad. Sc. Paris, Ser. I Math*, 292:707–710, 1981.
- [BM83] W. Borho and R. MacPherson. Partial resolutions of nilpotent varieties. *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, *Astérisque*, 101-102:23–74, 1983.
- [Bri99] M. Brion. Rational smoothness and fixed points of torus actions. *Transformation Groups*, 4(2-3):127–156, June 1999.
- [Car85] Roger W. Carter. *Finite Groups of Lie Type: Conjugacy Classes and Complex Characters*. John Wiley and Sons, 1985.
- [CM93] D. H. Collingwood and W. M. McGovern. *Nilpotent Orbits in Semisimple Lie Algebras*. Van Nostrand Reinhold, New York, 1993.
- [Dim03] A. Dimca. *Sheaves in Topology*. Springer, Berlin, 2003.
- [DLP88] C. De Concini, G. Lusztig, and C. Procesi. Homology of the zero-set of a nilpotent vector field on a flag manifold. *Journal of the American Mathematical Society*, 1(1):15–34, 1988.
- [FH91] W. Fulton and J. Harris. *Representation Theory: A first course*. Springer-Verlag, New York, 1991.

- [Ful93] W. Fulton. *Introduction to Toric Varieties*. Princeton University Press, Princeton, New Jersey, 1993.
- [GM03] S. I. Gelfand and Y. I. Manin. *Methods of Homological Algebra*. Springer, New York, 2003.
- [Gra] W. Graham. Toric varieties and the principal nilpotent orbit.
- [Har77] R. Hartshorne. *Algebraic Geometry*. Springer, New York, 1977.
- [Hum72] J. E. Humphreys. *Introduction to Lie algebras and representation theory*. Springer Verlag, New York, 1972.
- [Kos63] B. Kostant. Lie Group Representations on Polynomial Rings. *Amer. J. of Math.*, 85(3):327 – 404, 1963.
- [Mil80] J.S. Milne. *Étale Cohomology*. Princeton University Press, New Jersey, 1980.
- [Miz80] K. Mizuno. The conjugate classes of unipotent elements of the Chevalley groups E_7 and E_8 . *Tokyo J. Math.*, 3(2):391 – 459, 1980.
- [Sen12] A. N. Sengupta. *Representing Finite Groups: A semisimple introduction*. Springer, New York, 2012.
- [Som98] E. Sommers. A generalization of the Bala-Carter theorem for nilpotent orbits. *International Mathematics Research Notices*, (11):539–562, 1998.
- [Spa82] N. Spaltenstein. *Classes unipotentes et sous-groupes de Borel*. Springer-Verlag, 1982.
- [Spr76] T.A. Springer. Trigonometric sums, Green functions of finite groups and representations of Weyl groups. *Inventiones Mathematicae*, 36(1):173–207, 1976.
- [Spr78] T.A. Springer. A construction of representations of Weyl groups. *Inventiones Mathematicae*, 44(3):279–293, October 1978.
- [Spr84] T.A. Springer. A purity result for fixed point varieties in flag manifolds. *J. Fac. Sci. Uni. Tokyo. Sect. 1 A*, 31(2):271 – 282, 1984.
- [Tym03] J.S. Tymoczko. *Decomposing Hessenberg varieties over classical groups*. PhD thesis, Princeton University, 2003.
- [War83] F.W. Warner. *Foundations of differentiable manifolds and Lie groups*. Springer-Verlag, New York - Berlin, 1983.

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