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ORTHOGONAL GRASSMANNIANS AND HERMITIAN K-THEORY IN $\mathbb{A}^1$-HOMOTOPY THEORY OF SCHEMES

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
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Table of Contents

Acknowledgments ... **ii**

Abstract ... **v**

Chapter 1. Introduction ... **1**
- 1.1 Notations and Conventions ... 1
- 1.2 Introduction ... 1

Chapter 2. Notions from $\mathbb{A}^1$-Homotopy Theory ... **7**
- 2.1 Grothendieck Topologies on Sm/S ... 7
- 2.2 Colimits of Homotopy Equivalences ... 16

Chapter 3. $\mathbb{A}^1$-Representability of Hermitian K-Theory ... **22**
- 3.1 Grassmannians and Orthogonal Grassmannians ... 23
  - 3.1.1 Orthogonal Grassmannian and Stiefel Presheaf ... 33
  - 3.1.2 $Gr(n,\ H^\infty)$ and Relatives ... 36
  - 3.1.3 $BO(H^n)$ and $\mathbb{A}^1$-Contractibility of $St(H^n, H^\infty)$ ... 38
- 3.2 The Presheaves $\mathscr{F}h^{[0,\infty]}$ and $\mathscr{F}hO$ ... 43
  - 3.2.1 H-Space Structures on $\mathscr{F}h^{[0,\infty]}$ and $\mathscr{F}hO$ ... 45
  - 3.2.2 The Presheaf $\mathcal{K}^h$ and the Map $\mathscr{F}hO \xrightarrow{\hbar} \mathcal{K}^h$ ... 50
- 3.3 $\mathbb{A}^1$-Weak Equivalence of the Map $\hbar$ ... 57
  - 3.3.1 $\mathbb{A}^1$-Weak Equivalence of BO and $\mathcal{K}_0^h$ ... 57
  - 3.3.2 Grothendieck-Witt Groups and $\mathscr{F}hO$ in $\mathbb{A}^1$-Homotopy Category ... 61
  - 3.3.3 Some Properties of $\mathbb{A}^1$-Fibrations ... 71
  - 3.3.4 $\mathbb{A}^1$-Weak Equivalence of $\hbar$ ... 73

Chapter 4. A Realization Functor in $\mathbb{A}^1$-Homotopy Theory ... **76**
- 4.1 The Set of Real Points of a Smooth $\mathbb{R}$-Scheme ... 76
- 4.2 The Functor ρ_* and Some Properties ... 80
- 4.3 The Functor $\Delta^{op}PShv(Sm/\Bbbk) \xrightarrow{\rho^*} Top$ ... 89

Chapter 5. A Revised Proof of the $\mathbb{A}^1$-Representability Theorem . **94**
- 5.1 Introduction ... 94
- 5.2 Orthogonal Grassmannian GrO ... 95
- 5.3 Hermitian K-Theory Ppresheaf KO and the Map $GrO \xrightarrow{\hbar} KO$ ... 98
- 5.4 Grothendieck-Witt Group and $\pi_0 GrO\Delta^\bullet$ ... 101
- 5.5 The Classifying Spaces BO and $BO(H^n)$ of Orthogonal Group ... 111
- 5.6 An E_∞-Space Structure on Simplicial Presheaf $GrO\Delta^\bullet$ ... 119
- 5.7 $\mathbb{A}^1$-Weak Equivalence of the Map $\hbar$ ... 122

5.8 Concluding Remarks 126

References **127**

Vita **130**

Abstract

In this work we prove that the hermitian K-theory is geometrically representable in the $\mathbb{A}^1$-homotopy category of smooth schemes over a field. We also study in detail a realization functor from the $\mathbb{A}^1$-homotopy category of smooth schemes over the field $\mathbb{R}$ of real numbers to the category of topological spaces. This functor is determined by taking the real points of a smooth $\mathbb{R}$-scheme. There is another realization functor induced by taking the complex points with a similar description although we have not discussed this other functor in this thesis. Using these realization functors we have concluded in brief the relation of hermitian K-theory of a smooth scheme over the real numbers with the topological K-theory of the associated topological space of the real and the complex points of that scheme: The realization of hermitian K-theory induced taking the complex points is the topological K-theory of real vector bundles of the topological space of complex points, whereas the realization induced by taking the real points is a product of two copies of the topological K-theory of real vector bundles of the topological space of real points.

Chapter 1
Introduction

1.1 Notations and Conventions

Let me first mention some of the essential assumptions we would be making. In the entire work $\Bbbk$ will denote a perfect field of characteristic not equal to 2. The category of smooth schemes over the field $\Bbbk$ will be denoted by $Sm/\Bbbk$. For a given smooth scheme S we will denote the category of all smooth schemes over S by Sm/S.

A few remarks on notations. For a category $\mathcal{C}$, the category of simplicial presheaves of sets on $\mathcal{C}$ will be denoted by $\Delta^{op}PShv(\mathcal{C})$. If $\mathcal{C}$ is a site with respect to a Grothendieck topology τ then the category of simplicial sheaves on $\mathcal{C}$ will be denoted by $\Delta^{op}Shv_\tau(\mathcal{C})$. All the sheaves and presheaves of sets on a site can be considered simplicial of dimension 0 and they will be denoted by the same symbol even when considered as being simplicial. For a simplicial presheaf $\mathfrak{X} \in \Delta^{op}PShv(Sm/S)$ and an affine S-scheme $\operatorname{Spec} R$, we will denote the simplicial set $\mathfrak{X}(\operatorname{Spec} R)$ by $\mathfrak{X}(R)$ and ocassionally by $\mathfrak{X}_R$. For a map f of presheaves the map $f(\operatorname{Spec} R)$ will be denoted by $f(R)$ or by f_R.

1.2 Introduction

A framework for considering homotopical questions in algebraic geometry in generality was laid with the introduction of $\mathbb{A}^1$-homotopy theory by Morel and Voevodsky. They have defined realization functors from the $\mathbb{A}^1$-homotopy category of smooth schemes over complex numbers and over real numbers to the ordinary homotopy category of topological spaces. These functors were induced by taking the associated topological space of complex points of a variety over $\mathbb{C}$ and the topological space of real points of a variety over $\mathbb{R}$. One motivation for this work was to understand the image of hermitian K-theory under these and more general realization functors.

It was believed that the image of hermitan K-theory should be related to topological K-theory of real vector bundles. We considered a geometric representability of hermitian K-theory similar to the case of representability of algebraic K-theory by Grassmannians as one basic ingredient in a direct understanding this.

In this thesis, we have proved a geometric representability of hermitian K-theory similar to the representability of algebraic K-theory by Grassmannians (although the representability of hermitian K-theory was proved by Hornbostel in [H05], for our purposes, we needed a geometric representability). For a nondegenerate symmetric bilinear space (V, ϕ) over a field F, we have taken the open subscheme of nonvanishing sections determined by the universal bundle on the Grassmannian scheme $Gr(V)$. We have defined the orthogonal Grassmannian GrO in 5.2.3 as a colimit of these schemes after stabilizing with respect to the addition of hyperbolic planes. In theorem 5.7.1 we have proved that GrO represents the hermitian K-theory in the unstable $\mathbb{A}^1$-homotopy category of smooth scheme over a field of characterstic not equal to 2.

Although we haven't discussed stable version of this representability result, it can be extended to give the stable case as well. Also, another extension of the representability result discussed in this paper should be with respect to the algebraic analog of the real KR-theory of Atiyah.

It must be remarked that the organization of the proof is similar to the one for algebraic K-theory. There is one significant difference though, since the π_0 presheaf of hermitian K-theory is not constant in contrast to the case of algebraic K-theory, the identification of π_0 presheaves of GrO and hermitian K-theory involves some computation.

Next, I will present the detailed contents of the first four chapters of the thesis, for the contents of the fifth chapter see introduction of that chapter. In fact, the last chapter can be independently read without reading the first four chapters of the thesis. The reason for having that chapter seperate from the rest of the thesis is the fact that the proof of H-space structure on orthogonal Grassmannian using the machinery of Γ-spaces discussed in chapter 3 is not complete, and a complete proof for this would have been much more complicated. It has been proved in a different way using the theory of operads in chapter 5. While working the details of H-space structure, it so happened that the overall presentation of the representability theorem became much refined. But the theory of Γ-spaces in the wider context of this work is useful and can be explored further, that's why it has survived in this draft. Also, there are some computational details in the first four chapters that should help a reader in working through the fifth chapter.

In chapter 2, we have recalled some of the notions from $\mathbb{A}^1$-homotopy theory to set up notations. We choose a model structure as discussed in [J87] and [MV99] on the category $\Delta^{op}PShv(Sm/\Bbbk)$ of simplicial presheaves of sets on the category $Sm/\Bbbk$ of smooth schemes over a field $\Bbbk$. In a model category with an interval I, we provide a technical result in 2.2.4 which helps us in understanding the behaviour of I-homotopic maps under taking certain colimits. This result has been used later to prove the H-space structures on the presheaf $\mathscr{F}h^{[0,\infty]}$, and also in proving the $\mathbb{A}^1$-contractibility of Stiefel presheaves in 3.1.21.

The third chapter contains the most important result of this work, namely, the $\mathbb{A}^1$-representability theorem 3.3.18 (although a complete proof of this result appears in the last chapter in theorem 5.7.1). In the first section of this chapter we have recalled the definition of the usual Grassmannian scheme over a field $\Bbbk$ in 3.1.1, and then defined an open subscheme $Gr_{\Bbbk}(n, H^m)$ in 3.1. The $\mathbb{A}^1$-representability theorem states that hermitian K-theory is represented by a colimit, namely $\mathcal{G}rO$, of the schemes $Gr_{\Bbbk}(n, H^m)$ (see 3.2.2 and 3.3.18). To be able to do calculations, we have defined the presheaves $\mathscr{H}_n^m$ and $\mathscr{F}h_n^m$ in 3.1.10 and proved in 3.1.11 that the sheaf represented by $Gr_{\Bbbk}(n, H^m)$ is actually a Zariski sheafification of both of these presheaves. Later we have used the presheaf $\mathscr{F}h_n^m$ and it's derivatives almost all the times whenever we need to prove or define something with $Gr_{\Bbbk}(n, H^m)$.

We have also defined the presheaf $O(H^n)$ of isometries of hyperbolic space in 3.1.12. The Stiefel presheaves $St(H^n, H^m)$ have been defined in 3.1.14 and the presheaf $Gr(H^n, H^m)$ in 3.1.16. The name of Stiefel presheaves derives inspiration from the fact that they have a role similar to their topological counterparts: We have proved that $O(H^n)$ acts on $St(H^n, H^m)$ faithfully and transitively and the quotient space of this action is the presheaf $Gr(H^n, H^m)$ in 3.1.15. Later we have proved in 3.1.23 that $St(H^n, H^\infty)$ is $\mathbb{A}^1$-contractible. This allows us to identify the classifying space $BO(H^n)$ of $O(H^n)$ with the presheaf $Gr(H^n, H^\infty)$ in the $\mathbb{A}^1$-homotopy category (see 3.1.25).

In section 3.2 we have defined the presheaf $\mathscr{F}hO$ in 3.2.1 and the sheaf $\mathcal{G}rO$ in 3.2.2. We have proved that the presheaf $\mathscr{F}h^{[0,\infty]}$ is an H-space with the hope that this H-space structure extends to give us an H-space structure on $\mathscr{F}hO$ though we have not verified it yet. Next, in this section for sake of completeness we have recalled the definition of hermitian K-theory presheaf $\mathcal{K}^h$ in 3.2.13. Later we have defined the map $\hbar$ from the presheaf $\mathscr{F}hO$ to $\mathcal{K}^h$. This is a very important map and we prove in 3.3.18 that this is an $\mathbb{A}^1$-weak equivalence.

The last section collects necessary results in the proof of $\mathbb{A}^1$-weak equivalence of the map $\hbar$. First we define the presheaf BO in 3.3.1. We prove that there is an $\mathbb{A}^1$-

weak equivalence $\gamma : BO \to \mathcal{K}_0^h$, where $\mathcal{K}_0^h$ is the connected component of 0 of the hermitian K-theory (3.3.2, 3.3.3). Then we recall definition of the presheaf GW_0 of Grothendieck-Witt groups in 3.3.5 and prove that the Nisnevich sheafification of the presheaf $\pi_0^{\mathbb{A}^1}(\mathscr{F}hO)$ is isomorphic to the Nisnevich sheafification of GW_0 in 3.3.14. The last subsection 3.3.4 proves that $\hbar$ is an $\mathbb{A}^1$-weak equivalence (theorem 3.3.18).

In chapter 4, for the smooth schemes over $\mathbb{R}$, we want to study the relation of hermitian K-theory of a smooth $\mathbb{R}$-scheme with the topological K-theory of real and complex vector bundles over the topological space of it's real (and complex) points (4.1.1). This chapter is not written completely yet, more precisely the section 4.4 in which we propose to compute realization of presheaves $\mathcal{G}rO$ and $\mathscr{F}hO$ which represent the hermitian K-theory is not written yet. We have proved in the first section that the set of real points of a smooth real scheme can be given the structure of a smooth manifold in 4.1.3. In the next we have defined the functor ρ_* from the category of topological spaces to simplicial presheaves in 4.2.1, and proved that for a topological space S the simplicial presheaf $\rho_*(S)$ is homotopy invariant (4.2.3) and has the BG-property (4.2.5 and 4.2.7). It is also $\mathbb{A}^1$-local (4.2.11). The functor ρ_* sends weak equivalences of topological spaces to global weak equivalences of simplicial presheaves (4.2.12), and hence induces a map on the homotopy categories.

We have defined the functor ρ^* from the category of simplicial presheaves to the category of topological spaces in 4.3, and proved that the pair of functors (ρ_*, ρ^*) form an adjoint pair in 4.3.3. Then we have defined the left derived functor $L\rho^*$ of ρ^* in 4.3.7 using the topological space of real points of a smooth $\mathbb{R}$-scheme, this is one of the realization functors. The other realization functor can be defined using the topological space of complex points, but we have not considered this realization functor. The detailed computation of the image of hermitian K-theory under this realization functor has not been presented, although can be worked out with the results discussed in this chapter and the representability theorem.

Chapter 2
Basic Notions from $\mathbb{A}^1$-Homotopy Theory

In this chapter we recall the basic definitions and some of the results from $\mathbb{A}^1$-homotopy theory which we will need. The first section is a quick introduction of $\mathbb{A}^1$-homotopy theory with the purpose of setting up notations. In the second section we present a technical result from the general homotopy theory which helps us in understanding behavior of weak equivalences under taking colimits in some situations. A particular case which has been used many times from this section is the corollary 2.2.4.

2.1 Grothendieck Topologies on Sm/S

Definition 2.1.1. For a smooth S-scheme X, let $(f_\alpha : U_\alpha \to X)_\alpha$ be a finite family of étale morhpisms in Sm/S.

1. $(f_\alpha : U_\alpha \to X)_\alpha$ is called an étale cover of X, if X is union of the open sets $f_\alpha(U_\alpha)$.

2. $(f_\alpha : U_\alpha \to X)_\alpha$ is called a Nisnevich cover of X, if for every $x \in X$, there is an α and a $y \in U_\alpha$ which maps to x and $\kappa(x) \simeq \kappa(y)$.

3. $(f_\alpha : U_\alpha \to X)_\alpha$ is called a Zariski cover of X, if each f_α is open immersion and U_α cover X.

The collection of all the étale covers for schemes over S gives us a Grothendieck topology on the category Sm/S, see [Artin] and [M80]. Similarily we have the Nisnevich [N89], and the Zariski topologies on Sm/S. The category Sm/S together with these topologies is called an étale, a Nisnevich and a Zariski site according to the topology considered. All these topologies will be referred to as τ-topologies when a general situation valid for all the three topologies can to be considered simultaneously.

Definition 2.1.2. A category $\mathscr{I}$ is a left filtering category if the followig are true.

1. Given two objects $\alpha, \beta \in \mathscr{I}$, there is an object $\gamma \in \mathscr{I}$ and morphisms $\gamma \to \alpha$ and $\gamma \to \beta$, and

2. Given two morphisms $i_1, i_2 : \alpha \to \beta$ in $\mathscr{I}$, there is an object γ and a morphism $j : \gamma \to \alpha$ for which $i_1 j = i_2 j$.

Definition 2.1.3. A left filtering system in a category $\mathscr{C}$ is a functor $x : \mathscr{I} \to \mathscr{C}$ where $\mathscr{I}$ is a left filtering category.

Definition 2.1.4. Let $x : \mathscr{I} \to \mathscr{C}$ be a left filtering system in a category $\mathscr{C}$ and $X \in \Delta^{op}PShv(\mathscr{C})$ a simplicial presheaf of sets. We define a simplicial set X_x as the direct limit

$$X_x = \varinjlim_{U \in \mathscr{I}} X(x(U)) = \varinjlim_{\mathscr{I}^{op}} Xx.$$

This simplicial set is called the stalk of X at x and this construction gives us a functor $\Delta^{op}PShv(\mathscr{C}) \to \Delta^{op}Sets$. In particular, for a map $f : X \to Y$ of simplicial presheaves on a category $\mathscr{C}$ we have the induced map of simplicial sets $f_x : X_x \to Y_x$. This map is called the fiber of the map f at x.

Recall that a τ-point of a Grothenideck site T is a functor $Shv(T) \to Sets$ which commutes with finite limits and all colimits, [MV99, 2.1.2].

Definition 2.1.5. A family ζ of τ-points in $\mathscr{C}$ is called a conservative family of points if the following is true for all maps of simplicial presheaves $f : X \to Y$: The fiber of f at x is an isomorphism of simplicial sets for all $x \in \zeta$ if and only if the map of associated simplicial sheaves $aX \xrightarrow{\cong} aY$ is an isomorphism. A site $\mathscr{C}$ is said to have enough τ-points if there exists a conservative family of points in $\mathscr{C}$.

Theorem 2.1.6. *The site Sm/S has enough τ-points with respect to all the three topologies mentioned in the definition 2.1.1.*

Proof. We construct a conservative family ζ of τ-points for each of these sites. Let's first define $\bar{\tau}$-points which could be thought of as precursors of an actual τ-point for the étale, Nisnevich and the Zariski topologies on Sm/S.

1. For $\tau = \acute{e}t$, a $\overline{\acute{e}t}$-point in a scheme $X \in Sm/S$ is a morphism of schemes $x : \text{Spec } K \to X$, where K is a seperably closed field.

2. For $\tau = Nis$, a $\overline{Nis}$-point in a scheme $X \in Sm/S$ is a morphism $x : \text{Spec } K \to X$ such that the residue field of the image is K, K being any field.

Then it can be proved that all the categories defined below are left filtering categories:

1. For a point x of a smooth S-scheme X, let $\mathscr{I}_x^{zar}$ be the set of all open neighborhoods of the point x in the scheme X. $\mathscr{I}_x^{\text{zar}}$ becomes a category by taking a unique morphism between two objects U and V whenever $U \subset V$.

2. In both the cases of a $\bar{\tau}$-point x defined above, take $\mathscr{I}_x^{\tau}$ to be the set of pairs of the form $(f : U \to X,\ y : \text{Spec } K \to U)$ where f is an étale morphism of schemes, $U \in Sm$ and y is a $\bar{\tau}$-point of U with K being the residue field of theimage of y, and $fy = x$.

It can be seen that the construction of taking stalks relative to the three categories defined above give us τ-points in the three topologies $\tau = \acute{e}t,\ Nis$ and Zar. It is a matter of tedious verification using the construction of the sheafification functor to prove that the collection of all the τ-points obtained this way is a conservative family of points for the site Sm/S for $\tau = ét,\ Nis$ and Zar. $\square$

A comment on notations is in order: the τ-point determined by the $\bar{\tau}$-point x will also denoted by x.

Remark 2.1.7.

1. In view of the construction outlined in the proof of above theorem, we see that for a simplicial presheaf $\mathfrak{X} \in \Delta^{op}PShv(\nu)$ and for a τ-point x in the above conservative family, the stalk $\mathfrak{X}_x$ is the simplicial set $\text{colim}_{x \in U}\mathfrak{X}(U)$, where U is an open neighborhood of the point x in case of Zariski topology and U runs over all the S-schemes described above in the other two cases.

2. In the construction outlined above, in each case the category $\mathscr{I}_x^\tau$ has a cofinal subcategory $\mathscr{I}_{af,x}^\tau$ obtained by considering only the affine neighborhoods of x in case of Zariski topology, and only the affine schemes U in case of the other two topologies. This remark enables us to compute stalks of a presheaf using it's values on the affine schemes in Sm/S.

Definition 2.1.8. A map of simplicial presheaves in $\Delta^{op}PShv(\nu)$ is called a τ-simplicial weak equivalence (or just a simplicial weak equivalence when τ is understood from context) if all the fibers at τ-points in a conservative family of τ-points are weak equivalences of simplicial sets. With the choice of the simplicial weak equivalences as weak equivalences, the monomorphisms as cofibrations, and the appropriate class of fibrations defined by the right lifting property, the category $\Delta^{op}PShv(\nu)$ becomes a model category [J87]. We will denote the homotopy category of this model category by $Ho_\tau\Delta^{op}PShv(\nu)$.

Remark 2.1.9. The category $\Delta^{op}Shv(\nu_\tau)$ is also a model category with the choices of the class of τ-simplicial weak equivalences of sheaves as weak equivalences, the monomorphisms as cofibrations and fibrations determined by the right lifting property as proved in [MV99]. The homotopy category of this model category would be denoted by $Ho_\tau\Delta^{op}Shv(\nu_\tau)$.

The following lemma gives us a way to compute the Zariski stalks of some particular types of simplicial presheaves on $Sm/\mathbb{k}$. This computation will be useful later in proving Zariski weak equivalences of certain maps.

Lemma 2.1.10. *Let $\mathfrak{X}$ be a simplicial presheaf of $Sm/\mathbb{k}$ which can be extended to a simplicial presheaf $\widetilde{\mathfrak{X}}$ on the category of $\mathbb{k}$-schemes $Sch/\mathbb{k}$. Further, assume that*

$$\varinjlim_{\alpha} \mathfrak{X}(R_\alpha) \xrightarrow{\simeq} \widetilde{\mathfrak{X}}(\mathrm{Spec}\,(\varinjlim_{\alpha} R_\alpha)).$$

Then for a Zariski point $x \in X$ in $Sm/\mathbb{k}$, $\mathfrak{X}_x \simeq \widetilde{\mathfrak{X}}(\mathcal{O}_{X,x})$. In this situation, we will denote $\widetilde{\mathfrak{X}}$ simply by $\mathfrak{X}$.

Proof. This is just a restatement of the fact a Zariski point can be defined by the filtered system of affine open neighborhoods of $x \in X$. □

This lemma gives us a way of checking Zariski weak equivalences in some cases.

Lemma 2.1.11. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of simplicial presheaves on $Sm/\mathbb{k}$. Assume further that $\mathfrak{X}$ and $\mathfrak{Y}$ both extend to the category $Sch/\mathbb{k}$ of all $\mathbb{k}$-schemes. Then f is a Zariski weak equivalence if and only if for every regular local $\mathbb{k}$-algebra R which is a local ring of a smooth $\mathbb{k}$-scheme, the induced map $f_R : \mathfrak{X}(R) \to \mathfrak{Y}(R)$ (see 2.1.10) is a weak equivalence of simplicial sets .*

Definition 2.1.12. A map $f : \mathfrak{X} \to \mathfrak{Y}$ in the category $\Delta^{op}PShv(\nu)$ is a global weak equivalence if for every $X \in Sm/S$ the map of simplicial sets $\mathfrak{X}(X) \to$

$\mathfrak{Y}(X)$ is a weak equivalence of simplicial sets. Taking the global weak equivalences for weak equivalences, monomorphisms for cofibrations, and fibrations determined by the right lifting property we get a model structure on $\Delta^{op}PShv(\nu)$: See [J87, Thm 2.3]. We will denote the homotopy category of this model category by $Ho_{global}\Delta^{op}PShv(\nu)$.

Recall that the representable presheaf determined by a smooth S-scheme X in Sm/S, namely the presheaf $\mathrm{Hom}_{Sm/S}(\ ,X)$ is a sheaf with respect to all the three topologies $\acute{e}t$, Nis and Zar [M80]. We will denote this sheaf by the same letter X. For example, the smooth $\mathbb{k}$-scheme $\mathbb{A}^1 = \mathrm{Spec}\,\mathbb{k}[T]$ gives us the (pre-)sheaf of sets $\mathbb{A}^1$ on Sm/S. As mentioned in the beginning we can think of it as being simplicial (of dimension 0).

Definition 2.1.13. A simplicial presheaf $\mathfrak{Z}$ in the category $\Delta^{op}PShv(\nu_{\mathrm{nis}})$ is called $\mathbb{A}^1-$local if for every simplicial presheaf $\mathfrak{X}$ the canonical projection $\mathfrak{X}\times\mathbb{A}^1 \to \mathfrak{X}$ induces a bijection of Hom sets in the homotopy category

$$\mathrm{Hom}_{Ho_{Nis}(\Delta^{op}PShv(\nu))}(\mathfrak{X},\mathfrak{Z}) \to \mathrm{Hom}_{Ho_{Nis}(\Delta^{op}PShv(\nu))}(\mathfrak{X}\times\mathbb{A}^1,\mathfrak{Z}).$$

Definition 2.1.14. A morphism of simplicial presheaves $\mathfrak{X}\to\mathfrak{Y}$ is called an $\mathbb{A}^1$-weak equivalence if for every $\mathbb{A}^1$-local object $\mathfrak{Z}$ in $\Delta^{op}PShv(\nu)$ the induced map

$$\mathrm{Hom}_{Ho_{Nis}(\Delta^{op}PShv(\nu))}(\mathfrak{Y},\mathfrak{Z}) \to \mathrm{Hom}_{Ho_{Nis}(\Delta^{op}PShv(\nu))}(\mathfrak{X},\mathfrak{Z})$$

is a bijection.

Definition 2.1.15. The category $\Delta^{op}PShv(\nu)$ is a model catgeory with the choice of the class of $\mathbb{A}^1$-weak equivalences as weak equivalences, the class of monomorphisms as cofibrations and fibrations defined by the right lifting property, see [J87] and [MV99]. Homotopy category of this model category will be denoted by $\mathscr{H}(\mathbb{k})$, and it is called the *homotopy category of smooth* $\mathbb{k}$*-schemes*. If $\mathfrak{X}$ and $\mathfrak{Z}$ are simplicial presheaves on $Sm/\Bbbk$, the set $\mathrm{Hom}_{\mathscr{H}(\Bbbk)}(\mathfrak{X}, \mathfrak{Z})$ of morphisms in the $\mathbb{A}^1$-homotopy category will be denoted by $[\mathfrak{X}, \mathfrak{Z}]_{\mathbb{A}^1(\Bbbk)}$.

Observe that a global weak equivalence of simplicial presheaf is also a simplicial weak equivalence in all the three topologies, and a Nisnevich simplicial weak equivalence is also an $\mathbb{A}^1$-weak equivalence.

Remark 2.1.16. In their foundational work on the homotopy theory of schemes [MV99], Morel and Voevodsky have made the above three definitions for the category $\Delta^{op}Shv_{Nis}(\nu)$ of simplicial sheaves with respect to Nisnevich topology and proved the model structure. They use the notation $\mathscr{H}(\Bbbk)$ for the resulting homotopy category in their situation and call that the homotopy category of smooth $\Bbbk$-schemes. Since we would be working with presheaves, we have used this notation for presheaves. But homotopy theoretically there is no difference in working with either of the two model categories, since the resulting homotopy categories are naturally equivalent as described in the next paragraph.

Denoting the forgetful functor $\Delta^{op}Shv(\nu_{Nis}) \to \Delta^{op}PShv(\nu)$ by U, and the sheafification functor $\Delta^{op}PShv(\nu) \to \Delta^{op}Shv(\nu_{Nis})$ by a_{Nis}, we observe that the pair of functors $a_{Nis} : \Delta^{op}PShv(\nu) \rightleftarrows \Delta^{op}Shv(\nu_{Nis}) : U$ form an adjoint pair of functors (sheafification functor is left adjoint to the forgetful functor). We next observe two things: the first is that the unit of adjunction $1_{\Delta^{op}PShv(\nu)} \to U\ a_{Nis}$ is the map induced from a presheaf to the associated sheaf forgetting the sheaf structure, which becomes an isomporphism on passing to the associated homotopy categories (since we have enough Nis-points); and the second thing is that the counit of adjunction $a_{Nis}\ U \to 1_{\Delta^{op}Shv(\nu_{Nis})}$ is itself an isomprphism and hence it also produces an isomprphism on passing to the associated homotopy categories.

Therefore, the pair of functors induced by U and a_{Nis} provide an equivalence of the associated homotopy categories

$$a_{Nis} : Ho_{Nis}\Delta^{op}PShv(\nu) \rightleftarrows Ho_{Nis}\Delta^{op}Shv(\nu_{Nis}) : U.$$

verifying our assertion in the previous paragraph on using one of the two (slightly) different homotopy categories for dealing with schemes.

Now we are going to recall the definition of the naive homotopy of maps of simplicial presheaves.

Definition 2.1.17. If $X \in Sm/\mathbb{k}$. There are two maps of $\mathbb{k}$-algebras $\mathbb{k}[T] \to \Gamma(X, \mathcal{O}_X)$ given by $T \mapsto 0$ and $T \mapsto 1$. These two maps give us elements $i_0, , i_1 \in \mathrm{Hom}_{Sch}(X, \mathbb{A}^1)$ respectively. Let pt denote the unique final object in the category $\Delta^{op}PShv(\nu)$, the constant simplicial presheaf of singletons. The maps i_0 and i_1 give us two maps of presheaves (denoted by the same symbols) $i_0, i_1 : pt \to \mathbb{A}^1$. Let $f, g : \mathfrak{X} \to \mathfrak{Z}$ be two maps of simplicial presheaves on $Sm/\mathbb{k}$. We say that the maps f and g are naively $\mathbb{A}^1$-homotopic, if there is a map of simplicial presheaves $h : \mathfrak{X} \times \mathbb{A}^1 \to \mathfrak{Z}$ such that in the the diagram

$$\begin{tikzcd}
\mathfrak{X} \times pt \simeq \mathfrak{X} \arrow[r] \arrow[d, "1\times i_1"', "1\times i_0"] & \mathfrak{Z} \\
\mathfrak{X} \times \mathbb{A}^1 \arrow[ur, "h"'] &
\end{tikzcd}$$

we have $h \circ 1 \times i_0 = f$ and $h \circ 1 \times i_1 = g$. The set of naive homotopy classes of maps from $\mathfrak{X}$ to $\mathfrak{Z}$ will be denoted by $[\mathfrak{X}, \mathfrak{Z}]_{\mathbb{A}^1_{\mathrm{nv}}(\mathbb{k})}$.

Example 2.1.18. We describe a naive homotopy of maps from $\operatorname{Spec} A$ to $\operatorname{Spec} R$ of representable (pre)sheaves on $Sm/\mathbb{k}$. Any map of sheaves $\operatorname{Spec} A \xrightarrow{\alpha} \operatorname{Spec} R$ is given by a unique map of schemes, which in turn is determined by a unique map of $\mathbb{k}$-alegbras $R \xrightarrow{f_\alpha} A$. Two maps $\alpha, \beta : \operatorname{Spec} A \to \operatorname{Spec} R$ are naively $\mathbb{A}^1$-homotopic, if there is a map $h : R \to A[T]$ of $\mathbb{k}$-algebras such that in the diagram

$$\begin{tikzcd} R \arrow[r, "f_\alpha, f_\beta"] \arrow[d, "h"'] & A \\ A[T] \arrow[ur, "ev_0, ev_1"'] & \end{tikzcd}$$

$ev_0 \circ h = f_\alpha$ and $ev_1 \circ h = f_\beta$, where ev_0 and ev_1 are maps corresponding to evaluations at 0 and 1 respectively.

Lemma 2.1.19. *Two naively $\mathbb{A}^1$-homotopic maps induce equal maps in the homotopy category $\mathscr{H}(\mathbb{k})$ of smooth $\mathbb{k}$-schemes.*

Proof. If $p : \mathfrak{X} \times \mathbb{A}^1 \to \mathfrak{X}$ denotes the canonical projection, then for the maps i_0 and i_1 discussed in 2.1.17, the two compositions pi_0 and pi_1 are both equal to the identity $1_{\mathfrak{X}}$. Since p is isomorphism in the homotopy category $\mathscr{H}(\mathbb{k})$, the two maps i_0 and i_1 are equal in the $\mathscr{H}(\mathbb{k})$. Let $f,\ g : \mathfrak{X} \to \mathfrak{Z}$ be two naively homotopic maps of simplicial presheaves with a naive homotopy h. Then, in the homotopy catgeory $\mathscr{H}(\mathbb{k})$, we have $f = hi_0 = hi_1 = g$. □

Definition 2.1.20. Consider the affine scheme $\operatorname{Spec} \mathbb{k}[T_{i,j}] \quad (i, j = 1, ..., n)$. The $\mathbb{k}$-scheme $\mathcal{G}l_n(\mathbb{k})$, is the affine open subscheme of $\operatorname{Spec} \mathbb{k}[T_{i,j}] \quad (i, j = 1, ..., n)$, corresponding to localization of the polynomial algebra $\mathbb{k}[T_{i,j}]$ at the element

$$\det[T_{i,j}] = \Sigma_{\sigma \in \Sigma_n} \operatorname{sgn} \sigma\ T_{1,\sigma(1)} ... T_{n,\sigma(n)},$$

where Σ_n is the symmetric group on the set $\{1, ..., n\}$. Thus,

$$\mathcal{G}l_n(\mathbb{k}) = \operatorname{Spec} (\mathbb{k}[T_{i,j}]_{\det[T_{i,j}]}).$$

We will denote the representable presheaf $\operatorname{Hom}_{Sch}(\ \ , \mathcal{G}l_n(\mathbb{k}))$ also by $\mathcal{G}l_n(\mathbb{k})$. We know that if X is a smooth $\mathbb{k}$-scheme, the set $\mathcal{G}l_n(\mathbb{k})(X) = \operatorname{Hom}_{Sch}(X , \mathcal{G}l_n(\mathbb{k}))$, is the group $GL_n(\Gamma(X, \mathcal{O}_X))$ of units of the ring of global sections of $\mathcal{O}_X$. That is, the presheaf $\mathcal{G}l_n(\Bbbk)$ is in fact a presheaf of groups. An n-square invertible matrix M with entries in $\Bbbk$ determines a map $\mathcal{G}l_n(\Bbbk) \xrightarrow{M} \mathcal{G}l_n(\Bbbk)$ of presheaves by multiplication. It will be called the multiplication by the matrix M.

Remark 2.1.21. We have the notion of action of a presheaf of groups $\mathcal{G}$ on a preshead of sets $\mathfrak{X}$. It is a map of presheaves $m : \mathfrak{X} \times \mathcal{G} \to \mathfrak{X}$ satisfying the usual properties of a group action on a set, see [MV99]. In particular, we can consider the action of $\mathcal{G}l_n(\Bbbk)$ on a presheaf $\mathfrak{X}$.

2.2 Colimits of Homotopy Equivalences

This section is very technical and discusses some situations in which a system of homotopy equivalences induce weak-equivalence on colimits. Results discussed in this section have been used in later sections, particularily the corollary 2.2.4 has been used to prove the H-space structure on $\mathscr{F}h^{[0,\infty]}$ in section 3.2.1, and in proving the $\mathbb{A}^1$-contractibility of the Steifel presheaf in 3.1.23. Both these results have been used in the proof of Theorem 3.3.18. Although it must be made very clear in the beginning that the results of this section are not needed anymore because of the way the main theorem of this work, namely the $\mathbb{A}^1$-representability theorem in 3.3.18 has been proved in the last chapter of the thesis. The main objective of this section was to establish the necessary technical results needed in the following:

1. the $\mathbb{A}^1$-contractibility of Stiefel presheaves which has now been proved directly without any use of colimits, and

2. the H-space structures in section 3.2.1. Now this result has been replaced by a slightly different result using the machinery of E_∞-operads in section 5.6.

In a model category a homotopy between two maps is defined using cylinder objects. In the rest of this section, I will denote a functorially chosen cylinder object. We will discuss behaviour under taking colimits of a system of I-homotopic morphisms. The following observation is very useful.

Lemma 2.2.1. *In a model category given an I-homotopy commutative diagram*

$$\begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d, tail, "i"] & Y \\ X_2 \arrow[ur, "g"'] & \end{tikzcd}$$

where i is a cofibration and Y is fibrant, there exists a map $\tilde{g} : X_2 \to Y$ which is I-homotopic to g such that the diagram

$$\begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d, tail, "i"] & Y \\ X_2 \arrow[ur, "\exists\ \tilde{g}"'] & \end{tikzcd}$$

commutes.

Proof. In the pushout diagram

$$\begin{tikzcd} \{1\} \times X_1 \arrow[r, tail, "\sim"] \arrow[d, tail, "i"] & I \times X_1 \arrow[d, tail] \arrow[ddr, bend left, "I\times i"] & \\ X_2 \arrow[r, tail, "\sim"] \arrow[drr, bend right, "\sim", "\partial^0"'] & X_2 \coprod_{[1]} I \times X_1 \arrow[dr, tail, "\tilde{i}", "\sim"'] & \\ & & I \times X_2 \end{tikzcd}$$

$\tilde{i}$ is an acyclic cofibration. Since Y is fibrant, there is a map H in the following diagram, where $h = (f,\ gi)$ is the map defined by a simplicial homotopy between f and the composition gi and the map ∂^0 through a diagram similar to the one defining the map $\tilde{i}$:

$$\begin{tikzcd} X_2 \coprod_{[1]} I \times X_1 \arrow[r, "h=(f,\ gi)"] \arrow[d, "\tilde{i}"] & Y \\ I \times X_2 \arrow[ur, "\exists\ H"'] & \end{tikzcd}$$

Taking $\tilde{g} = H_0$, we get the map with the properties claimed in this lemma. $\square$

Lemma 2.2.2. *In a model category suppose there are morphisms*

$$A_1 \xrightarrow{i_1,j_1} A_2 \xrightarrow{i_2,j_2} A_3 \xrightarrow{i_3,j_3} \cdots \qquad A_n \xrightarrow{i_n,j_n} A_{n+1} \xrightarrow{i_{n+1},j_{n+1}} \cdots$$

with the property that for every positive integer k the morphisms i_k and j_k are I-homotopic. Then there exists a 'natural' zigzag

$$\mathrm{colim}_n(A_n, i_n) \xleftarrow{\simeq} C \xrightarrow{\simeq} B \xleftarrow{\simeq} \mathrm{colim}_n(A_n, j_n)$$

of weak equivalence.

Proof. We first consider the special case under the assumptions that all the i_k's are cofibrations and each of the A_k's are fibrant: In view of above lemma the following homotopy commutative diagram

$$\begin{tikzcd} A_1 \arrow[r, tail, "i_1"] \arrow[d, "\mathrm{Id}"] & A_2 \arrow[r, tail, "i_2"] \arrow[d, "\mathrm{Id}"] & A_3 \arrow[r, tail, "i_3"] \arrow[d, "\mathrm{Id}"] & \cdots \arrow[r, tail, "i_{n-1}"] & A_n \arrow[r, tail, "i_n"] \arrow[d, "\mathrm{Id}"] & A_{n+1} \arrow[r, tail, "i_{n+1}"] \arrow[d, "\mathrm{Id}"] & \cdots \\ A_1 \arrow[r, "j_1"] & A_2 \arrow[r, "j_2"] & A_3 \arrow[r, "j_3"] & \cdots \arrow[r, "j_{n-1}"] & A_n \arrow[r, "j_n"] & A_{n+1} \arrow[r, "j_{n+1}"] & \cdots \end{tikzcd}$$

can be replaced by a commutative diagram:

$$\begin{tikzcd} A_1 \arrow[r, tail, "i_1"] \arrow[d, "\mathrm{Id}"] & A_2 \arrow[r, tail, "i_2"] \arrow[d, "g_2"] & A_3 \arrow[r, tail, "i_3"] \arrow[d, "g_3"] & \cdots \arrow[r, tail, "i_{n-1}"] & A_n \arrow[r, tail, "i_n"] \arrow[d, "g_n"] & A_{n+1} \arrow[r, tail, "i_{n+1}"] \arrow[d, "g_{n+1}"] & \cdots \\ A_1 \arrow[r, "j_1"] & A_2 \arrow[r, "j_2"] & A_3 \arrow[r, "j_3"] & \cdots \arrow[r, "j_{n-1}"] & A_n \arrow[r, "j_n"] & A_{n+1} \arrow[r, "j_{n+1}"] & \cdots \end{tikzcd}$$

in which each g_k is I-homotopic to the identity map and hence is a weak equivalence. Therefore, (g_k) induce a weak equivalence $\mathrm{colim}_n(A_n, i_n) \xrightarrow{\simeq} \mathrm{colim}_n(A_n, j_n)$ (not just a zigzag!).

In the general case, in the model category of $\mathbb{N}$-diagrams over the given model category, we can make a functorial acyclic cofibrant and acyclic fibrant replacement of the given homotopy commutative diagram to get the diagram

$$\begin{tikzcd}
X_1 \arrow[r, tail, "s_1"] \arrow[d, "c_1"', "\simeq"] & X_2 \arrow[r, tail, "s_2"] \arrow[d, "c_1"', "\simeq"] & X_3 \arrow[r, tail, "s_3"] \arrow[d, "c_1"', "\simeq"] & \cdots \arrow[r, tail, "s_{n-1}"] & X_n \arrow[r, tail, "s_n"] \arrow[d, "c_1"', "\simeq"] & X_{n+1} \arrow[r, tail, "s_{n+1}"] \arrow[d, "c_1"', "\simeq"] & \cdots \\
A_1 \arrow[r, "i_1"] \arrow[d, "\mathrm{Id}"'] & A_2 \arrow[r, "i_2"] \arrow[d, "\mathrm{Id}"'] & A_3 \arrow[r, "i_3"] \arrow[d, "\mathrm{Id}"'] & \cdots \arrow[r, "i_{n-1}"] & A_n \arrow[r, "i_n"] \arrow[d, "\mathrm{Id}"'] & A_{n+1} \arrow[r, "i_{n+1}"] \arrow[d, "\mathrm{Id}"'] & \cdots \\
A_1 \arrow[r, "j_1"] \arrow[d, "f_1"', "\simeq"] & A_2 \arrow[r, "j_2"] \arrow[d, "f_2"', "\simeq"] & A_3 \arrow[r, "j_3"] \arrow[d, "f_3"', "\simeq"] & \cdots \arrow[r, "j_{n-1}"] & A_n \arrow[r, "j_n"] \arrow[d, "f_n"', "\simeq"] & A_{n+1} \arrow[r, "j_{n+1}"] \arrow[d, "f_{n+1}"', "\simeq"] & \cdots \\
Y_1 \arrow[r, "t_1"] & Y_2 \arrow[r, "t_2"] & Y_3 \arrow[r, "t_3"] & \cdots \arrow[r, "t_{n-1}"] & Y_n \arrow[r, "t_n"] & Y_{n+1} \arrow[r, "t_{n+1}"] & \cdots
\end{tikzcd}$$

with the properties that:

1. each s_i is a cofibration and c_i a weak equivalence,
2. each Y_i is fibrant and f_i weak equivalence,
3. the upper and the lower horizontal squares of the diagram commute,
4. the outer diagram with respect to vertical compositions is, and homotopy commutative.

Then we have a weak equivalence $\mathrm{colim}_n(X_n, s_n) \xrightarrow{\simeq} \mathrm{colim}_n(Y_n, t_n)$ (from the special case of this lemma proved in the beginning); and, also the weak equivalences (since filtered colimits of weak equivalences are so) $\mathrm{colim}_n(X_n, s_n) \xrightarrow{\simeq} \mathrm{colim}_n(A_n, i_n)$ and $\mathrm{colim}_n(A_n, j_n) \xrightarrow{\simeq} \mathrm{colim}_n(Y_n, t_n)$. These equivalences give us the zigzag mentioned in this lemma. $\square$

Lemma 2.2.3. *Given a commutative diagram of the form*

$$\begin{tikzcd}
X_1 \arrow[r, "i_1^x"] \arrow[d, "g_1"'] & X_2 \arrow[r, "i_2^x"] \arrow[d, "g_2"'] & X_3 \arrow[r, "i_3^x"] \arrow[d, "g_3"'] & \cdots \arrow[r, "i_{n-1}^x"] & X_n \arrow[r, "i_n^x"] \arrow[d, "g_n"'] & X_{n+1} \arrow[r, "i_{n+1}^x"] \arrow[d, "g_{n+1}"'] & \cdots \\
Y_1 \arrow[r, "i_1^y"] & Y_2 \arrow[r, "i_2^y"] & Y_3 \arrow[r, "i_3^y"] & \cdots \arrow[r, "i_{n-1}^y"] & X_n \arrow[r, "i_n^y"] & X_{n+1} \arrow[r, "i_{n+1}^y"] & \cdots
\end{tikzcd}$$

such that there is another commutative diagram

$$\begin{tikzcd}
Y_1 \arrow[r, "i_1^y"] \arrow[d, "g_1"'] & Y_2 \arrow[r, "i_2^y"] \arrow[d, "g_2"'] & Y_3 \arrow[r, "i_3^y"] \arrow[d, "g_3"'] & \cdots & {} \arrow[r, "i_{n-1}^y"] & Y_n \arrow[r, "i_n^y"] \arrow[d, "g_n"'] & Y_{n+1} \arrow[r, "i_{n+1}^y"] \arrow[d, "g_{n+1}"'] & \cdots \\
X_2 \arrow[r, "i_2^x"] & X_3 \arrow[r, "i_3^x"] & X_4 \arrow[r, "i_4^x"] & \cdots & {} \arrow[r, "i_n^x"] & X_{n+1} \arrow[r, "i_{n+1}^x"] & X_{n+2} \arrow[r, "i_{n+2}^x"] & \cdots
\end{tikzcd}$$

with the property that in the category of morhpisms the two maps

$$\begin{pmatrix} X_k \\ \downarrow f_k \\ Y_k \end{pmatrix} \xrightarrow{i_k,\, j_k} \begin{pmatrix} X_{k+1} \\ \downarrow f_{k+1} \\ Y_{k+1} \end{pmatrix}$$

are I-homotopic for every k, where i_k is given by the pair of maps (i_k^x, i_k^y) and j_k is given by the pair $(g_k f_k,\, f_{k+1} g_k)$. Then there is a zigzag of weak equivalences $\operatorname{colim}\,(X_k, i_k^x) \leftarrow \ldots \rightarrow \operatorname{colim}\,(Y_k, i_k^y)$ *is a weak equivalence.*

Recall that an I-homotopy between the two maps

$$\begin{pmatrix} X_k \\ \downarrow f_k \\ Y_k \end{pmatrix} \xrightarrow{i_k,\, j_k} \begin{pmatrix} X_{k+1} \\ \downarrow f_{k+1} \\ Y_{k+1} \end{pmatrix}$$

is a given by two homotopies, $h_k^x : X_k \times I \to X_{k+1}$ between i_k^x and $g_k f_k$, and $h_k^y : Y_k \times I \to Y_{k+1}$ between i_k^y and $f_{k+1} g_k$, such that the diagram

$$\begin{tikzcd}
X_k \times I \arrow[r, "h_k^x"] \arrow[d, "f_k \times 1"'] & X_{k+1} \arrow[d, "f_{k+1}"'] \\
Y_k \times I \arrow[r, "h_k^y"] & Y_{k+1}
\end{tikzcd}$$

commutes.

Proof. Applying above lemma we get a commutative diagram of zigzags of the form

$$\begin{tikzcd}
\operatorname{colim}\,(X_k, i_k) \arrow[r, dashed, dash, "\exists\ zigzag", "of\ w.eq."'] \arrow[d, "f"'] & \operatorname{colim}\,(X_k, g_k f_k) \arrow[d, "f"', "isom"] \\
\operatorname{colim}\,(Y_k, j_k) \arrow[r, dashed, dash, "\exists\ zigzag", "of\ w.eq."'] & \operatorname{colim}\,(Y_k, f_{k+1} g_k)
\end{tikzcd}$$

in which the right vertical map induced by f_k is an isomorphism, it's inverse is given by the map induced from g_k. All the zigzags in horizontal rows together with this isomorphism give us the claimed weak equivalence. $\square$

Since in the category $\Delta^{op}PShv(Sm/\Bbbk)$ the sheaf $\mathbb{A}^1$ is a cylinder object and $\mathbb{A}$-homotopy is naive $\mathbb{A}^1$-homotopy, above lemma can be rewritten in the following form which has been used later.

Corollary 2.2.4. *Given a system of simplicial presheaves and morphisms* X_k, Y_k, i_k, j_k, f_k, g_k $(k \geq 1)$ *in* $\Delta^{op}PSh(Sm/\Bbbk)$ *similar to the one in the above lemma such that* $g_k f_k$ *is naively* $\mathbb{A}^1$*-homotopic to* i_k *and* $f_{k+1}\ g_k$ *is naively* $\mathbb{A}^1$*-homotopic to* j_k *for every* k*, we have a zigzag of* $\mathbb{A}^1$*-weak equivalence* colim $(X_k, i_k) \leftarrow \ldots \rightarrow$ colim (Y_k, j_k).

Chapter 3

$\mathbb{A}^1$-Representability of Hermitian K-Theory

In this chapter we prove our main result, the $\mathbb{A}^1$-representability theorem in 3.3.18. In the first section of this chapter, we recall the definition of Grassmannian scheme in 3.1.1 and, also consider it's functor of points in 3.1.4. Then we construct a smooth $\mathbb{k}$-scheme $Gr_{\mathbb{k}}(n, H^\infty)$ in 3.1.9 in a manner analogous to the construction of the Grassmannian scheme, and discuss it's functor of points in 3.1.10. We have referred to this scheme and some of it's derivatives considered later, for example in 3.2.1 and 3.1.19, collectively as the orthogonal Grassmannian. Next, we have defined, what we call the Stiefel presheaves in 3.1.1: These are analogues of the same kinds of objects considered in topology, and in fact we have shown in corollary 3.1.25 that at least in one aspect they behave exactly as in topology. We have proved that the Steifel presheaf $St(H^n, H^\infty)$ (3.1.21) is $\mathbb{A}^1$-contractible in 3.1.23. Using this result we relate the classifying space $BO(H^n)$ of orthogonal group presheaf $O(H^n)$ (3.1.12) with the presheaf $Gr(H^n, H^\infty)$ (3.1.16 and 3.1.19) via a zigzag of $\mathbb{A}^1$-weak equivalences in corollary 3.1.25.

In section 3.2 we have considered H-space structure on the presheaf $\mathscr{F}h^{[0,\infty]}$ in 3.2.6. As mentioned in the remark 3.2.7 we believe that this H-space structure extends to an H-space structure on the presheaf $\mathscr{F}hO$ (see 3.2.1) as well. Also, we have recalled a technical definition of the Hermitian K-theory presheaf $\mathcal{K}^h$ in 3.2.13 (and later the definition of the connected component $\mathcal{K}^h_0$ of 0 in 3.3.2). As we have cautioned the reader, this definition of hermitian K-theory is applicable only to the affine $\mathbb{k}$-schemes. We have also defined the map $\hbar$ in 3.2.19 from the presheaf $\mathscr{F}hO$ to the hermitian K-theory presheaf $\mathcal{K}^h$.

A part of our objective in this work has been to prove that orthogonal Grassmannian represents hermitian K-theory in the $\mathbb{A}^1$-homotopy theory: This has been done in the last section of this chapter, namely in the theorem 3.3.18. The third section collects results needed for this purpose. First, we identify the classifying space BO of orthogonal group (3.3.1) with the connected component of hermitian K-theory $\mathcal{K}_0^h$ (3.3.2) via the $\mathbb{A}^1$-weak equivalence γ in 3.3.1. Next we have recalled the definition of the presheaf of Grothendieck-Witt groups 3.3.5, and identified the Nisnevich sheafification of $\pi_0^{\mathbb{A}^1}\mathscr{F}hO$ with the Nisnevich sheafification of the presheaf of Grothendieck-Witt groups in 3.3.15. The remaining subsections patch all these informations together to prove $\mathbb{A}^1$-weak equivalence of the map $\hbar$ in 3.3.18.

3.1 Grassmannians and Orthogonal Grassmannians

On the category $Sm/\mathbb{k}$ we define some presheaves and consider their Zariski weak equivalence. One of these presheaves, which is actually a sheaf and is very important for algebraic K-theory, is the representable sheaf determined by a Grassmannian over $\mathbb{k}$. There are two other closely related presheaves on $Sm/\mathbb{k}$ which appear to be more understandable. We verify that there are Zariski weak equivalences between these. Then we consider the analogous situation in the case of hermitian K-theory, and define closely related presheaves and consider their Zariski weak equivalence. These constructions in the case of hermitian K-theory are foundational to our representability considerations. First we consider Grassmannians and their Zariski-relatives since many of the arguments involved in this case are needed in the case of hermitian K-theory. In this chapter we take two non-negative integers m and n, where $n \leq m$.

Definition 3.1.1. Let $I = \{i_1, ..., i_n\} \subset \{1, ..., m\}$ be a subset of cardinality n, and $A = \Bbbk[X^I_{i,j}],\ \ i = 1, ..., m;\ j = 1, ..., n$ be the polynomial ring in mn variables. Let ε_I be the ideal of A generated by $\{X^I_{i_\alpha,j} - \delta^j_\alpha\}$ $(\alpha, j = 1, ..., n)$, where δ denotes the Kronecker symbol. Let

$$A_I = \Bbbk[X^I_{i,j}]/\langle\{X^I_{i_\alpha,j} - \delta^j_\alpha\}\rangle$$

and $U_I = \operatorname{Spec} A_I$. For another subset J of cardinality n in $\{1, ..., m\}$, let M^I_J be the submatrix of $[X^I_{i,j}]$ corresponding to the columns in J. Let $U_{I,J}$ be the open subscheme of U_I obtained by localizing the ring A_I at the element $\det M^I_J \in A_I$. Note that $U_{I,I} = U_I$. The following map of rings written in the form of multiplication of matrices

$$\Bbbk[X^J_{i,j}] \to (\Bbbk[X^I_{i,j}]/\varepsilon_I)_{\det M^I_J}, \qquad\qquad [X^J_{i,j}] \mapsto [X^I_{i,j}](M^I_J)^{-1}.$$

induces a map of rings $(A_J)_{g_I} \xrightarrow{\det M^I_J} (A_I)_{g_J}$. Let $\phi_{I,J} : U_{I,J} \to U_{J,I}$ be the corresponding map of schemes. The map $\phi_{I,I}$ is the identity map. In view of lemma 3.1.2, the schemes U_I as I ranges over all the cardinality n subsets of $\{1, ..., m\}$, can be glued using the isomorphisms $\phi_{I,J}$ to give us a scheme over $\Bbbk$. By the construction this scheme is smooth over $\Bbbk$. We denote this scheme by $Gr_\Bbbk(n, m)$. This is one description of the Grassmannian scheme over the field $\Bbbk$ in [EH]. The representable sheaf $\operatorname{Hom}_{Sm/\Bbbk}(\ , Gr_\Bbbk(n, m))$ on $Sm/\Bbbk$ will be denoted by the same symbol $Gr_\Bbbk(n, m)$. The term Grassmannian will be used for both of these objects, the meaning should be clear from the context.

Lemma 3.1.2. *For any three subsets $I,\ J,\ K \subset \{1,...,m\}$, the diagram of rings*

$$\begin{tikzcd}
(A_J)_{g_I} \arrow[r, "\det M_J^I"] \arrow[d, "\mathrm{can}"] & (A_I)_{g_J} \arrow[d, "\mathrm{can}"] \\
((A_J)_{g_I})_{g_K} \arrow[r, "\det M_J^I"] & ((A_I)_{g_J})_{g_K} \\
(A_J)_{g_I g_K} \arrow[r, "\det M_J^I"] \arrow[u, "\simeq"] & (A_I)_{g_J g_K} \arrow[u, "\simeq"]
\end{tikzcd}$$

is commutative. Hence the map $\phi_{I,J}$ when restricted to the affine open subscheme $U_{I,J} \bigcap U_{I,K} \hookrightarrow U_I$ induces a map $\phi^K_{I,J} : U_{I,J} \bigcap U_{I,K} \to U_{J,I} \bigcap U_{J,K}$. These induced maps satisfy the relation $\phi^I_{J,K} \circ \phi^K_{I,J} = \phi^J_{I,K}$. We also have $\phi^I_{I,J} = \phi_{I,J}$ and $\phi^J_{I,I} = 1_{U_{I,J}}$. In particular, $\phi^I_{J,I} \circ \phi^J_{I,J} = \phi^J_{I,I} = 1_{U_{I,J}}$. Therefore, all the maps $\phi_{I,J}$ are isomorphisms, and we can glue the schemes $\{U_I\}$ with respect to the maps $\{\phi_{I,J}\}$.

Proof. The proof of this lemma is a diagram chase in commutative rings. □

Now we define the other two presheaves as mentioned earlier.

Definition 3.1.3. For a ring R, and a subset $I \subset \{1,...,m\}$ of cardinality n, let $\mathscr{M}_I(R)$ be the set of those $m \times n$ matrices with entries in R whose submatrix of rows corresponding to I is the identity matrix of size n. Consider the set $\amalg_I \mathscr{M}_I(R)$, where I runs over all the cardinality n subsets of $\{1,...,m\}$. Define an equivalence relation on $\amalg_I \mathscr{M}_I(R)$ by declaring two elements (M, I) and (N, J) to be equivalent if the submatrix M_J of M formed by rows corresponding the set J is invertible and $N = M(M_J)^{-1}$. Let us denote the set of the equivalence classes with respect to this equivalence relation by $\mathscr{M}^m_n(R)$. For a smooth $\Bbbk$-scheme X considering the set $\mathscr{M}^m_n(\Gamma(X, \mathcal{O}_X))$, we get a presheaf on $Sm/\Bbbk$. We denote this presheaf by $\mathscr{M}^m_n$.

For sake of completeness, we recall that a direct factor of the free R-module R^m is a subdmodule $P \subset R^m$ such that there is a map $R^m \to P$ whose restriction on

P is the identity map. Note that a submodule $P \subset R^m$ is a direct factor if and only if there is an R-submodule $Q \subset R^m$ such that $P \oplus Q = R^m$.

Definition 3.1.4. For a ring R, let $\mathscr{F}_n^m(R)$ be the set of rank n free direct factors of R^m. A map $R \to S$ of rings gives us a map $\mathscr{F}_n^m(R) \to \mathscr{F}_n^m(S)$ by sending P to the image of the composite map $P \otimes_R S \xrightarrow{i \otimes_R S} R^m \otimes_R S \xrightarrow{\simeq} S^m$, where i denotes the inclusion $P \subset R^m$. The assignment $X \mapsto \mathscr{F}_n^m(\Gamma(X, \mathcal{O}_X))$, where X is a smooth $\mathbb{k}$-scheme, defines a presheaf on $Sm/\mathbb{k}$ which we denote by $\mathscr{F}_n^m$.

Now we will define two maps of presheaves $Gr_{\mathbb{k}}(n,m) \xleftarrow{\lambda} \mathscr{M}_n^m \xrightarrow{\rho} \mathscr{F}_n^m$, which will be important in most computations. The notations set in the first part of the section 3.1 will be used throughout. First, the map λ. In case of an affine scheme $\operatorname{Spec} R \in Sm/\mathbb{k}$, let $\lambda_I(R) : \mathscr{M}_I(R) \to \operatorname{Hom}_{Sm/\mathbb{k}}(\operatorname{Spec} R,\ U_I)$ be the map defined by sending a matrix $(a_{i,j}) \in \mathscr{M}_I(R)$ to variables in the matrix $(X_{i,j}^I)$ in $\mathbb{k}$-algebra $(\mathbb{k}[X_{i,j}^I]/\varepsilon_I)$ in that order. The proof of lemma 3.1.5 is a diagram chase in commutative algebra and will be omitted.

Lemma 3.1.5. *Let $M \in \mathscr{M}_I(R)$ and $N \in \mathscr{M}_J(R)$ be two matrices such that $N = M(M_J)^{-1}$, that is, they determine the same element in $\mathscr{M}_n^m(R)$. Then the map $\lambda_I(R)(M)$ has image in $U_{I,J}$, and the map $\lambda_I(R)(N)$ has image in $U_{J,I}$. Furthermore, the diagram*

$$\begin{tikzcd}
 & & U_I \arrow[ddrr] & & \\
 & & U_{I,J} \arrow[u] \arrow[drr] \arrow[dd, "\phi_{I,J}"', "\simeq"] & & \\
\operatorname{Spec} R \arrow[uurr] \arrow[urr] \arrow[drr] \arrow[ddrr] & & & & \operatorname{Spec} A \\
 & & U_{J,I} \arrow[urr] \arrow[d] & & \\
 & & U_J \arrow[uurr] & &
\end{tikzcd}$$

commutes. In particular, the maps $\lambda_I(R)$ define a map $\mathscr{M}_n^m(R) \to \mathrm{Hom}_{Sm/\Bbbk}(\mathrm{Spec}\, R, Gr_{\Bbbk}(n,m))$.

As a consequence of above lemma, we have defined the map λ for affine schemes in $Sm/\Bbbk$. For a general scheme X, recall that we have defined $\mathscr{M}_n^m(X)$ to be $\mathscr{M}_n^m(\Gamma(X, \mathcal{O}_X))$. To define $\lambda(X)$, we use the canonical bijection $\mathrm{Hom}_{Sch}(X,\ \mathrm{Spec}\, R) \xrightarrow{\simeq} \mathrm{Hom}_{Rings}(R,\ \Gamma(X, \mathcal{O}_X))$. Taking $R = \Gamma(X, \mathcal{O}_X)$ to get a canonical map in $\mathrm{Hom}_{Sch}(X,\ \mathrm{Spec}\, \Gamma(X, \mathcal{O}_X))$ corresponding to the identity map of the ring $\Gamma(X, \mathcal{O}_X)$. This map then gives us the map $Gr_{\Bbbk}(n,m)(\Gamma(X, \mathcal{O}_X)) \to Gr_{\Bbbk}(n,m)(X)$. We define $\lambda(X)$ to be the composite map $\mathscr{M}_n^m(X) = \mathscr{M}_n^m(\Gamma(X, \mathcal{O}_X)) \to Gr_{\Bbbk}(n,m)(\Gamma(X, \mathcal{O})) \to Gr_{\Bbbk}(n,m)(X)$. This completes definition of the map λ.

Now we define the map $\rho : \mathscr{M}_n^m \to \mathscr{F}_n^m$. Let M be a matrix in $\mathscr{M}_I(R)$ corresponding to the set $I = \{i_1, ..., i_n\} \subset \{1, ..., m\}$ where $i_1 < ... < i_n$. The $n \times m$ matrix $N = (c_1, ..., c_m)$, where $c_{i_1} = e_1, ..., c_{i_n} = e_n$ and all other columns are 0 has the property that $NM = I_n$. Thus the map $R^n \xrightarrow{M} R^m$ has a section, so the map M is injective and $Im(M)$ is a free rank n submodule of R^m. Thus we get a direct factor $Im(M) \subset R^m$. If $(M, I) \in \mathscr{M}_I(R)$ and $(N, J) \in \mathscr{M}_J(R)$ determine the same element in $\mathscr{M}_n^m(R)$, then $N = M(M_J)^{-1}$. The following commutative triangle shows that the two submodules $Im(M)$ and $Im(N)$ of R^m are equal.

$$\begin{tikzcd} R^n \arrow[r, "M"] & R^m \\ R^n \arrow[u, "M_J^{-1}", "\simeq"'] \arrow[ur, "N"'] & \end{tikzcd}$$

This gives us a well defined map $\rho : \mathscr{M}_n^m \to \mathscr{F}_n^m$ sending the class of a matrix $M \in \mathscr{M}_I(R)$ to the rank n free submodule $Im(M)$ of the map $R^n \xrightarrow{M} R^m$.

Definition 3.1.6. The above discussion defines maps $Gr_{\Bbbk}(n,m) \xleftarrow{\lambda} \mathscr{M}_n^m \xrightarrow{\rho} \mathscr{F}_n^m$ of presheaves on $Sm/\Bbbk$.

Lemma 3.1.7. *The maps λ and ρ induce isomorphisms of the associated Zariski sheaves. In particular, they are Zariski, and hence, $\mathbb{A}^1$-weak equivalences.*

Proof. This will follow using lemma 2.1.10, if we show that for a local ring R the maps $\lambda(R)$ and $\rho(R)$ are isomorphisms. In rest of this proof R is assumed to be a local ring. We will construct the inverses of the maps $\lambda(R)$ and $\rho(R)$.

We construct the inverse map $\lambda(R)^{-1} : Gr_{\Bbbk}(n, m)(R) = \mathrm{Hom}_{Sm/\Bbbk}(\operatorname{Spec} R, Gr_{\Bbbk}(n, m)) \to \mathscr{M}_n^m(R)$. Since R is a local ring, any map $f : \operatorname{Spec} R \to X$ of schemes factors as in the diagram below, where U is any open neighborhood of the image of the maximal ideal of R.

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r, "f"] \arrow[d, "g"] & X \\ U \arrow[ur, "i"'] & \end{tikzcd}$$

In particular, a map $f : \operatorname{Spec} R \to Gr_{\Bbbk}(n, m)$ is actually a map of affine schemes $f_I : \operatorname{Spec} R \to U_I$, where $U_I = \operatorname{Spec} A_I$ is an open affine subscheme of $Gr_{\Bbbk}(n, m)$ defined in 3.1.1. This map corresponds to a map of rings

$$A_I = \Bbbk[X^I_{i,j}]/\langle\{X^I_{i_\alpha,j} - \delta^j_\alpha\}\rangle \to R$$

which gives us a matrix in $M_I \in \mathscr{M}_I(R)$, and defines a map $\zeta(R) : Gr_{\Bbbk}(n, m)(R) \to \mathscr{M}(R)$. We claim that $\zeta : Gr_{\Bbbk}(n, m) \to \mathscr{M}$ is an inverse of the map λ. We first prove that the composition $\zeta(R) \circ \lambda(R)$ is the identity of the set $\mathscr{M}(R)$. Let us consider the class M of $(M, I) \in \mathscr{M}(R)$. By defintion of the map λ, the map $\lambda(R)(M)$ actually maps $\operatorname{Spec} R$ into U_I, and is the map $\lambda_I(R)$. For another choice of index J to represent M, the maps λ_I and λ_J are related as described in the lemma 3.1.5. The argument in lemma 3.1.5 can be interpreted to conclude that the map $\zeta(R)$ takes the map $\lambda(R)(M)$ to M in $\mathscr{M}(R)$, proving that $\zeta(R) \circ \lambda(R) = I_{\mathscr{M}(R)}$. The proof for the other composition also follows from an argument similar to the one used in lemma 3.1.5.

To prove that the map $\rho(R)$ is bijective when R is a local ring, we define its inverse. Let $P \subset R^m$ be a free direct factor of rank n. Choose an isomorphism $R^n \xrightarrow[\alpha]{\simeq} P$. Let M_α be the matrix of the composition $R^n \xrightarrow{\subset \circ \alpha} R^m$. Since R is local, we can apply an argument similar to the row-reduction procedure for vector-spaces to prove that the matrix M_α has an invertible $n \times n$ submatrix. Let $M_{\alpha,I}$ be an ivertible submatrix formed by the rows corresponding to a subset $I = \{i_1, ..., i_n\} \subset \{1, ..., m\}$. Consider $M_\alpha (M_{\alpha,I})^{-1} \in \mathscr{M}_I(R)$. We claim that the class of $(M_\alpha(M_{\alpha,I})^{-1}, I)$ in $\mathscr{M}_n^m(R)$ does not depend on the choice of isomorphism α and the subset I. To see that choice of the isomorphism α does not matter, observe that for any other choice $\beta : R^n \to P$, we have a commutative diagram

$$\begin{tikzcd}
R^n \arrow[r, "\alpha"] \arrow[d, "\beta^{-1}\alpha"'] & P \subset R^m \arrow[d, "="] \\
R^n \arrow[r, "\beta"] & P \subset R^m
\end{tikzcd}$$

Using this diagram we see that the $M_\beta = M_\alpha T$, where T is the invertible matrix corresponding to the map $\beta^{-1}\alpha$. Thus, we have shown that two matrices for different choices of isomorphism of R^n and P are related by right mulitplication by an n-square invertible matrix. And, this also shows that if M_α and M_β are two matrices for different isomorphisms of R^n with P, then an n-square submatrix of M_β corresponding to a subset $I \subset \{1, ..., m\}$ is invertible if and only if the n-square submatrix of M_α corresponding to I is invertible. The computation

$$M_\beta M_{\beta,I}^{-1} = M_\alpha T (M_{\alpha,I}.T)^{-1} = M_\alpha T T^{-1} (M_{\alpha,I})^{-1} = M_\alpha M_{\alpha,I}^{-1} \tag{3.1.1}$$

shows that the choice of isomorphism does not matter. Thus we are left only to verify that for a given M_α, the two pairs $(M_\alpha(M_{\alpha,I})^{-1}, I) \in \mathscr{M}_I(R)$ and

$(M_\alpha(M_{\alpha,J})^{-1},\, J) \in \mathscr{M}_J(R)$ corresponding to different invertible submatrices $M_{\alpha,I}$ and $M_{\alpha,J}$ of M_α determine the same element in $\mathscr{M}_n^m(R)$. This follows from a computation similar to the one in (3.1) above. This defines a map $\xi(R) : \mathscr{F}_n^m(R) \to \mathscr{M}_n^m(R)$.The arguments that verify that definitions of ρ and $\xi(R)$ are independent of choices made can be repeated to see that $\xi(R)$ is inverse of the map $\rho(R)$. This completes proof of lemma 3.1.7. □

Remark 3.1.8. We can describe a Zariski sheafification of the presheaf $\mathscr{F}_n^m$ as follows. If $X \in Sm/\Bbbk$, let $\mathscr{D}f_n^m(X)$ be the set $\{F \subset \mathcal{O}_X^m | F$ is locally free subsheaf of rank n and the quotient $\mathcal{O}_X^m/F$ is locally free$\}$. It can be seen that $\mathscr{D}f_n^m$ is a Zariski sheaf on $Sm/\Bbbk$ and there is a natural map $\mathscr{F}_n^m \to \mathscr{D}f_n^m$ which induces an isomorphism of $\mathscr{D}f_n^m$ with the canonical Zariski sheafification of $\mathscr{F}_n^m$.

Now we consider the orthogonal counterparts of these presheaves. First, we are going to define an open subscheme of Grassmannian $Gr_\Bbbk(n, 2m)$. For notations used in the next definition see 3.1.1 in case of $Gr_\Bbbk(n, 2m)$: I and proving the following two results:J denote subsets of $\{1, ..., 2m\}$ of cardinality n. Let h_1 be the 2-square matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Also, for any two matrices a and b, let $a \perp b$ be the matrix $\begin{pmatrix} a & \\ & b \end{pmatrix}$, where a vacant place has entry 0. The notation H^m stands for the $2m$-dimensional hyperbolic space, that is, the free R-module R^{2m} together with the standard hyperbolic form. This form can be represented by the matrix $h_m = h_1 \perp ... \perp h_1$ (m copies).

We refer to [K90, Ch. I] for more details. The transpose of a matrix M will be denoted by M^t.

Definition 3.1.9 (Orthogonal Grassmannian)**.** Let $B_I = (A_I)_{g_I}$ be the localization of $A_I = \Bbbk[X^I_{i,j}]/\varepsilon_I$ at the element $g_I = \det([X^I_{i,j}]^t\, h_m\, [X^I_{i,j}]) \in A_I$. This defines an open subscheme $V_I = \operatorname{Spec} B_I \hookrightarrow U_I = \operatorname{Spec} A_I$. Let

$$V_{I,J} = \operatorname{Spec}\,(B_I)_{\det M^I_J} \hookrightarrow V_I$$

be the open subscheme obtained by localizing B_I at the element $\det M^I_J \in B_I$. The map

$$\Bbbk[X^J_{i,j}] \to (\Bbbk[X^I_{i,j}]/\varepsilon_I)_{\det M^I_J}, \qquad [X^J_{i,j}] \mapsto [X^I_{i,j}](M^I_J)^{-1}$$

induces a map of schemes $\phi_{I,J} : V_{I,J} \to V_{J,I}$. The result analogous to the one in lemma 3.1.2 is true in this case with exactly the same kind of proof. Therefore, we can glue the subschemes V_I of U_I to get a smooth $\Bbbk$-scheme. We will denote this scheme by $Gr_{\Bbbk}(n, H^m)$. It is an open subscheme of $Gr_{\Bbbk}(n, 2m)$. The representable sheaf on $Sm/\Bbbk$, $\operatorname{Hom}_{Sm/\Bbbk}(\ \ , Gr_{\Bbbk}(n, H^m))$ will also be denoted by $Gr_{\Bbbk}(n, H^m)$. We will call the scheme as well as the sheaf, $Gr_{\Bbbk}(n, H^m)$, the Orthogonal Grassmannian.

Definition 3.1.10. (The presheaves $\mathscr{H}^m_n$ and $\mathscr{F}h^m_n$).

1. For a ring R, consider the set $\amalg_I\ \mathscr{H}_I(R)$, where I runs over all the cardinality n subsets of $\{1, ..., 2m\}$ and, $\mathscr{H}_I(R)$ is the set of $2m \times n$ matrices M with entries in R whose submatrix of rows corresponding to the set I is the identity matrix of size n and which satisfy the property that $M^t\, h_m\, M$ is an invertible n-square matrix. Define an equivalence relation on $\amalg_I\ \mathscr{H}_I(R)$ by declaring two matrices $M \in \mathscr{H}_I(R)$ and $N \in \mathscr{H}_J(R)$ to be equivalent if the submatrix M_J of M formed by rows corresponding the set J is invertible and $N = M(M_J)^{-1}$. Let us denote the set of the equivalence classes with respect to this equivalence relation by $\mathscr{H}^m_n(R)$. For a smooth $\Bbbk$-scheme X considering

the set $\mathscr{H}_n^m(\Gamma(X, \mathcal{O}_X))$, we get a presheaf on $Sm/\Bbbk$. We denote this presheaf by $\mathscr{H}_n^m$.

2. For a ring R, let $\mathscr{F}h_n^m(R)$ be the set of free rank n direct factors of R^{2m} on which standard hyperbolic form on R^{2m} given by the matrix h_m is non-degenerate. A map $R \to S$ of rings gives us a map $\mathscr{F}_n^m(R) \to \mathscr{F}_n^m(S)$ by sending P to $P \otimes_R S$. The assignment $X \mapsto \mathscr{F}h_n^m(\Gamma(X, \mathcal{O}_X))$ where X is a smooth $\Bbbk$-scheme, defines a presheaf on $Sm/\Bbbk$, which we denote this presheaf by $\mathscr{F}h_n^m$.

We see that $Gr_{\Bbbk}(n, H^m) \subset Gr_{\Bbbk}(n, 2m)$, $\mathscr{H}_n^m \subset \mathscr{M}_n^{2m}$ and $\mathscr{F}h_n^m \subset \mathscr{F}_n^{2m}$ are subpresheaves.

Lemma 3.1.11. *The maps λ and ρ defined in* 3.1.6 *induce maps $Gr_{\Bbbk}(n, H^m) \xleftarrow{\bar{\lambda}} \mathscr{H}_n^m \xrightarrow{\bar{\rho}} \mathscr{F}h_n^m$ of presheaves such that the following diagram is commutative*

$$\begin{tikzcd}
Gr_{\Bbbk}(n, 2m) & \mathscr{M}_n^{2m} \arrow[l, "\lambda"'] \arrow[r, "\rho"] & \mathscr{F}_n^{2m} \\
Gr_{\Bbbk}(n, H^m) \arrow[u] & \mathscr{H}_n^m \arrow[l, "\bar{\lambda}"'] \arrow[r, "\bar{\rho}"] \arrow[u] & \mathscr{F}h_n^m \arrow[u]
\end{tikzcd}$$

and, $\bar{\lambda}$ and $\bar{\rho}$ are Zariski and hence $\mathbb{A}^1$-weak equivalences. (We will drop the bar signs on λ and ρ once we have sketched a proof of weak equivalence.)

Proof. Let us first verify that the map $\lambda : \mathscr{M}_n^{2m} \to Gr_{\Bbbk}(n, 2m)$ induces a map $\mathscr{H}_n^m \to Gr_{\Bbbk}(n, H^m)$. In case of an affine scheme $\operatorname{Spec} R \in Sm/\Bbbk$, we defined a map $\lambda_I(R) : \mathscr{M}_I(R) \to \operatorname{Hom}_{Sm/\Bbbk}(\operatorname{Spec} R,\ U_I)$ by sending a matrix $(a_{i,j}) \in \mathscr{M}_I(R)$ to variables in the matrix $(X_{i,j}^I)$ in $\Bbbk$-algebra $(\Bbbk[X_{i,j}^I]/\varepsilon_I)$. Observe that if we restrict $\lambda_I(R)$ to the subset $\mathscr{H}_I(R)$ of $\mathscr{M}_I(R)$, then we get a map into $\operatorname{Hom}_{Sm/\Bbbk}(\operatorname{Spec} R,\ V_I)$. The analogue of lemma 3.1.5 is also true. Thus, we get the induced map $\bar{\lambda} : \mathscr{H}_n^m \to Gr_{\Bbbk}(n,\ H^m)$, which makes the left half of above diagram commute.

To see that the map $\rho : \mathscr{M}_n^{2m} \to \mathscr{F}_n^{2m}$ induces a map $\bar{\rho} : \mathscr{H}_n^m \to \mathscr{F}h_n^m$, we need to check the following. Given a matrix $M \in \mathscr{H}_I(R)$, on the rank n free direct factor $Im(M) \subset R^{2m}$ (as discussed in 3.1.5), the hyperbolic form on R^{2m} restricts to a non-degenerate form. In view of the commutative diagram

$$\begin{tikzcd} Im(M) \arrow[d, "h_m|_{Im(M)}"] & \simeq & R^n \arrow[r, "M"] \arrow[d, "M^t\, h_m\, M"] & R^{2m} \arrow[d, "h_m"] \\ Im(M) & \simeq & R^n & R^{2m} \arrow[l, "M^t"] \end{tikzcd}$$

this follows from the assumption that the matrix $M^t\, h_m\, M$ is invertible. Thus we have verified the claim that we get the induced map $\bar{\rho}$.

The proof that the induced map $\bar{\lambda}$ and $\bar{\rho}$ are Zariski weak-equivalences follows by noting that the inverse maps ζ and ξ constructed in 3.1.7 induce inverses in this case as well. $\square$

3.1.1 Orthogonal Grassmannian and Stiefel Presheaf

Now we are going to define some more presheaves which will be helpful in relating $Gr_{\mathbb{k}}(n, H^m)$ to hermitian K-theory.

Definition 3.1.12. For a commutative ring R, let $O(H^n)(R)$ be the set of square matrices M of size $2n$ with entries in R which have the property that $M^t\, h_n\, M = h_n$. It can be seen that $O(H^n)(R)$ is a group under multiplication of matrices. The assignment sending a smooth $\mathbb{k}$-scheme X to $O(H^n)(\Gamma(X, \mathcal{O}_X))$ defines a presheaf of sets (actually a representable sheaf of groups) on $Sm/\mathbb{k}$. We will denote this presheaf by $O(H^n)$.

Definition 3.1.13. We define a presheaf $Gr_{\mathbb{k}}(H^n, H^m) \subset \mathscr{F}h_{2n}^m$ on $Sm/\mathbb{k}$ by taking only those free direct factors $P \subset H^m(R)$ of rank $2n$ on which the hyperbolic form on $H^m(R)$ restricts to a form isometric to the hyperbolic form on R^{2n}. This means that there exists an isomorphism $\alpha : R^{2n} \xrightarrow{\simeq} P$ such that the composite map $R^{2n} \xrightarrow{\alpha} P \xrightarrow{\subset} R^{2m}$ is given by a $2m \times 2n$-matrix M with the property that $M^t\, h_m\, M = h_n$.

Definition 3.1.14. (The Stiefel Presheaf $St(H^n,\, H^m)$). For a commutative ring R, let $St(H^n,\, H^m)(R)$ be the set of $2m \times 2n$ matrices M with entries in R and having the property that $M^t\, h_m\, M = h_n$. For a smooth scheme X in $Sm/\Bbbk$, the assignment $X \mapsto St(H^n,\, H^m)(\Gamma(X,\, \mathcal{O}_X))$, defines a presheaf denoted by $St(H^n,\, H^m)$. We will refer to the presheaf $St(H^n,\, H^m)$ as a Steifel presheaf.

The Stiefel presheaves are analogous to Stiefel varieties in topology, for the topological side of matter see [H66, Ch 8]. There is a right action of $O(H^n)$ on $St(H^n,\, H^m)$ by multiplication

$$St(H^n,\, H^m) \times O(H^n) \to St(H^n,\, H^m), \qquad (M,\, G) \mapsto MG.$$

This action is free, that is, for a matrix $M \in St(H^n,\, H^m)(R)$ and $G \in O(H^n)(R)$, if $MG = M$, then $G = I_{2n}$: Multiplying both sides on the left by the matrix $M^t\, h_m$, we get $M^t\, h_m\, MG = M^t\, h_m\, M$, or $h_n G = h_n$, and hence $G = I_{2n}$ since h_n is invertible. We will denote the quotient presheaf of this action by $St(H^n,\, H^m)/O(H^n)$.

If $M \in St(H^n,\, H^m)(R)$, then the map $R^{2n} \xrightarrow{M} R^{2m}$ has a retraction given by left multiplication with $h_n^{-1}\, M^t\, h_m$. Thus, $Im(M) \subset R^{2m}$ is a free direct factor of rank $2n$ of R^{2m}. Denoting the hyperbolic form on R^{2m} by ϕ_m, consider the commutative diagram

$$\begin{tikzcd}
& R^{2n} \arrow[r, "\simeq", "\alpha"'] \arrow[dl, "h_n"'] \arrow[d] \arrow[dl, phantom, "\circlearrowleft", shift left=3ex] & Im(M) \arrow[r, "\subset"] \arrow[d, "\phi_m|_{Im(M)}"] & R^{2m} \arrow[d, "\phi_m"] \arrow[dr, "h_m"] \arrow[dr, phantom, "\circlearrowleft"', shift right=3ex] & \\
R^{2n} \arrow[r, "\simeq"'] & (R^{2n})^* & Im(M)^* \arrow[l, "\simeq", "\alpha^*"'] & (R^{2m})^* \arrow[l, "\subset^*"'] & R^{2m} \arrow[l, "\simeq"']
\end{tikzcd}$$

in which α is an isomorphism of R^{2n} with $Im(M)$ such that the matrix of the upper horinzontal composition is M and, M^t is the matrix of the lower horizontal composition after the identification $(R^{2n})^* \simeq R^{2n}$: From the property $M^t\, h_m\, M =$

h_n, we see that the form induced on $Im(M)$ by the hyperbolic form on $H^m(R)$ is isometric to the hyperbolic form on R^n. Therefore, we get a map of presheaves $St(H^n, H^m) \to Gr(H^n, H^m)$ by sending a matrix $M \in St(H^n,\ H^m)(R)$ to the image $Im(M) \subset R^{2m}$. Also, for a matrix $G \in O(H^n(R))$ and $M \in St(H^n, H^m)(R)$, the image of the map $MG : R^{2n} \to R^{2m}$ is the same as the image of the map $M : R^{2n} \to R^{2m}$, since G is an automorphism of the hyperbolic form on R^n. Thus, we get an induced map

$$\gamma : St(H^n, H^m)/O(H^n) \to Gr(H^n, H^m).$$

Proposition 3.1.15. *The map* $\gamma : St(H^n, H^m)/O(H^n) \to Gr(H^n, H^m)$ *is an isomorphism and hence, an* $\mathbb{A}^1$*-weak equivalence.*

Proof. We prove that for every ring R, the map $St(H^n, H^m)/O(H^n)(R) \to Gr(H^n, H^m)(R)$, is a bijection. First we prove surjectivity of the map γ_R. Let $P \subset R^{2m}$ be a free direct factor of rank $2n$ on which the hyperbolic form restricts to a form isometric to the standard hyperbolic form on R^{2n}. This means, there exists an isomorphism $\alpha : R^{2n} \xrightarrow{\simeq} P$ such that the diagram used in defining the map γ commutes. But then the matrix M_α of the composite map $R^{2n} \xrightarrow{i \circ \alpha} R^{2m}$ is of rank $2n$ and represents embedding of the hyperbolic spaces $H^n(R) \hookrightarrow H^m(R)$ since $M_\alpha^t h_m M_\alpha = h_n$, and $\gamma_R(M_\alpha) = P$.

Next, we prove the injectivity of γ_R. Let M and N be two elements of $St(H^n, H^m)(R)$ such that $Im(M) = Im(N)$. Let $\alpha : R^{2n} \xrightarrow{\simeq} Im(M)$ and $\beta : R^{2n} \xrightarrow{\simeq} Im(N)$ be isometries. Then $G = \beta^{-1} \circ \alpha$ is an isometry of hyperbolic space on R^n and $M = NG$. This proves injectivity of the map γ_R. $\square$

Definition 3.1.16. (Canonical Map $Gr(H^n,\ H^m) \hookrightarrow Gr_{\Bbbk}(2n,\ H^m)$). We have a commutative diagram obtained by the canonical maps from Zariski sheafification

$$\begin{tikzcd}
Gr(H^n,\ H^m) \arrow[r, "\subset"] & \mathscr{F}h^m_{2n} \arrow[r, "\mathrm{can}"] & a_{Zar}\mathscr{F}h^m_{2n} \\
& \mathscr{H}^m_{2n} \arrow[u, "\rho"] \arrow[r, "\mathrm{can}"] \arrow[d, "\lambda"] & a_{Zar}\mathscr{F}h^m_{2n} \arrow[u, "\exists!\alpha"'] \arrow[dl, "\exists!\beta"] \\
& Gr_{\Bbbk}(2n,\ H^m) &
\end{tikzcd}$$

.

The maps α and β are both isomorphisms in view of lemma 3.1.11. Taking the composition $\beta\,\alpha^{-1}\,\mathrm{can}\ \subset$, we get a canonical inclusion $Gr(H^n,\ H^m) \hookrightarrow Gr_{\Bbbk}(2n,\ H^m)$ of presheaves. We will always identify $Gr(H^n,\ H^m)$ as a subpresheaf of $Gr_{\Bbbk}(2n,\ H^m)$ by means of this map.

3.1.2 $Gr(n,\ H^\infty)$ and Relatives

For every non-negative integer m, we have an orthogonal sum decomposition of the hyperbolic space H^{m+1} given by the matrix $\begin{pmatrix} h_m & 0 \\ 0 & h_1 \end{pmatrix}$ as $H^m \perp H$. This gives us the isometric embeddings $H^m \hookrightarrow H^{m+1}$, $(m \geq 0)$. These inclusions can also be expressed via the maps $R^{2m} \to R^{2m+2}$ given by the $(2m+2) \times 2m$ matrix $(h_m\ 0)^t$. This gives us the inclusions $\mathscr{H}^m_n \hookrightarrow \mathscr{H}^{m+1}_n$, $\mathscr{F}h^m_n \hookrightarrow \mathscr{F}h^{m+1}_n$ and $Gr_{\Bbbk}(H^n, H^m) \hookrightarrow Gr_{\Bbbk}(H^n, H^{m+1})$ of presheaves. We have the $O(H^n)$-equivariant inclusion $St(H^n, H^m) \hookrightarrow St(H^n, H^{m+1})$ as well given by the map $M \mapsto (M\ 0)^t$. In view of the following lemma 3.1.17, we also have a map of $\Bbbk$-schemes $Gr_{\Bbbk}(2n, H^m) \to Gr_{\Bbbk}(2n, H^{m+1})$ which induces an inclusion of functor of points of these schemes.

Lemma 3.1.17. *There is a map of $\Bbbk$-schemes $\jmath_m : Gr_{\Bbbk}(n, H^m) \hookrightarrow Gr_{\Bbbk}(n, H^{m+1})$ which corresponds to the inclusion of the functor of points* 3.1.4.

Proof. In 3.1.16, the composition

$$comp = \beta \circ \alpha^{-1} \circ \mathrm{can} : \mathscr{F}_n^{2m} \to Gr_{\Bbbk}(n,\, H^m)$$

is an $\mathbb{A}^1$-weak equivalence. That is, the orthogonal Grassmannian sheaf $Gr_{\Bbbk}(n,\, H^m)$ represents the presheaf $\mathscr{F}_n^{2m}$. There is the obvious natural inclusion $\mathscr{F}_n^{2m} \hookrightarrow \mathscr{F}_n^{2m+2}$. Therefore, by Yoneda there exists a unique morphism

$$\jmath_m : Gr_{\Bbbk}(n,\, H^m) \to Gr_{\Bbbk}(n,\, H^{m+1})$$

such that the following diagram commutes

$$\begin{tikzcd}
\mathscr{F}_n^{2m} \arrow[r, "comp"] \arrow[d] & Gr_{\Bbbk}(n,\, H^m) \arrow[d, "\jmath_m"] \\
\mathscr{F}_n^{2m+2} \arrow[r, "comp"] & Gr_{\Bbbk}(n,\, H^{m+1})
\end{tikzcd}$$

□

Remark 3.1.18. We hope that the maps $\jmath$ are all closed immersions.

Taking colimits of the systems $\mathscr{F}_n^{2m} \hookrightarrow \mathscr{F}_{n+1}^{2m}$, $Gr_{\Bbbk}(n,\, H^m) \xrightarrow{\jmath_m} Gr_{\Bbbk}(n,\, H^{m+1})$ and other presheaves defined before with respect to the natural inclusions (or inclusion-like) maps, we get the following presheaves.

Definition 3.1.19 (Orthogonal Grassmannians)**.** The colimits of these presheaves with respect to the above inclusions, are denoted by $Gr(n,\, H^\infty)$, $\mathscr{H}_n^\infty$, $\mathscr{F}h_n^\infty$, $Gr(H^n,\, H^\infty)$ and $St(H^n,\, H^\infty)$. The presheaves $\coprod_{n\geq 0} Gr(n,\, H^\infty)$, $\coprod_{n\geq 0} \mathscr{H}_n^\infty$, $\coprod_{n\geq 0} \mathscr{F}h_n^\infty$, $\coprod_{n\geq 0} Gr(H^n,\, H^\infty)$ and $\coprod_{n\geq 0} St(H^n,\, H^\infty)$ will be denoted by $Gr(\mathbb{N},\, H^\infty)$, $\mathscr{H}$, $\mathscr{F}h$, $Gr(H^{\mathbb{N}},\, H^\infty)$ and $St(H^{\mathbb{N}},\, H^\infty)$. The sheaves $Gr_{\Bbbk}(n,\, H^\infty)$ and $Gr(\mathbb{N},\, H^\infty)$ are representable. The presheaves $St(H^n,\, H^\infty)$ and $St(H^{\mathbb{N}},\, H^\infty)$ will collectively be referred to as Stiefel presheaves, and all others and their later derivatives as orthogonal grassmannians.

Lemma 3.1.20. *The diagram* (see 3.1.11)

$$\begin{tikzcd} Gr_{\Bbbk}(n,\, H^m) \arrow[d, "\jmath_m"] & \mathscr{H}_n^m \arrow[l, "\lambda"'] \arrow[r, "\rho"] \arrow[d] & \mathscr{F}h_n^m \arrow[d] \\ Gr_{\Bbbk}(n,\, H^{m+1}) & \mathscr{H}_n^{m+1} \arrow[l, "\lambda"'] \arrow[r, "\rho"] & \mathscr{F}h_n^{m+1} \end{tikzcd}$$

is commutative. In particular, there are maps $Gr_{\Bbbk}(n,\, H^{\infty}) \xleftarrow{\lambda} \mathscr{H}_n^{\infty} \xrightarrow{\rho} \mathscr{F}h_n^{\infty}$ *and* $Gr_{\Bbbk}(\mathbb{N},\, H^{\infty}) \xleftarrow{\lambda} \mathscr{H} \xrightarrow{\rho} \mathscr{F}h$.

Proof. Follows from chasing the definitions. □

Lemma 3.1.21. *The induced maps* $Gr_{\Bbbk}(n,\, H^{\infty}) \xleftarrow{\lambda} \mathscr{H}_n^{\infty} \xrightarrow{\rho} \mathscr{F}h_n^{\infty}$, $Gr_{\Bbbk}(\mathbb{N}, H^{\infty}) \xleftarrow{\lambda} \mathscr{H} \xrightarrow{\rho} \mathscr{F}h$, *and* $St(H^n,\; H^{\infty})/O(H^n) \to Gr(H^n,\; H^{\infty})$ *are Zariski, and hence,* $\mathbb{A}^1$*-weak equivalences*

Proof. The Zariski weak equivalence follows by noting that filtered colimits of weak equivalences of simplicial sets are weak equivalences. □

3.1.3 $BO(H^n)$ and $\mathbb{A}^1$-Contractibility of $St(H^n,\, H^{\infty})$

We prove that the Stiefel presheaves $St(H^n,\, H^{\infty})$ are $\mathbb{A}^1$-contractible, that is, the unique map $St(H^n,\, H^{\infty}) \to pt$ into the final object of the category $PShv(Sm/\Bbbk)$ is an $\mathbb{A}^1$-weak equivalence. We have already discussed in 3.1.1 and 3.1.19 the action of $O(H^n)$ on $St(H^n,\, H^m)$ and $St(H^n,\, H^{\infty})$, and proved the $\mathbb{A}^1$-weak equivalences $St(H^n,\, H^m)/O(H^n) \xrightarrow{\gamma} Gr(H^n,\, H^m)$ and $St(H^n,\, H^{\infty})/O(H^n) \to Gr(H^n,\, H^{\infty})$ in 3.1.14 and 3.1.21. Once we prove $\mathbb{A}^1$-contractibility of $St(H^n,\, H^{\infty})$, we would be able to identify the classifying space of $O(H^n)$ with the presheaf $Gr(H^n,\, H^{\infty})$ as in 3.1.25 below. We begin with a technical result.

Lemma 3.1.22. *Let $\mathcal{G}$ be a sheaf of groups and, $\mathcal{X}$ and $\mathcal{Y}$ be simplicial presheaves of sets on $Sm/\Bbbk$. Further suppose that $\mathcal{G}$ acts freely on $\mathcal{X}$ and $\mathcal{Y}$, and there is a*

$\mathcal{G}$ equivariant $\mathbb{A}^1$-weak equivalence $\mathcal{X} \to \mathcal{Y}$. Then the induced map $\mathcal{X}/\mathcal{G} \to \mathcal{Y}/\mathcal{G}$ is an $\mathbb{A}^1$-weak equivalence.

Proof. In the commutative diagram

$$\begin{tikzcd} \mathrm{hocolim}_{\mathcal{G}}\mathcal{X} \arrow[r, "\mathbb{A}^1"] \arrow[d, "\mathrm{can}"'] & \mathrm{hocolim}_{\mathcal{G}}\mathcal{Y} \arrow[d, "\mathrm{can}"] \\ \mathcal{X}/\mathcal{G} = \mathrm{colim}_{\mathcal{G}}\mathcal{X} \arrow[r] & \mathrm{colim}_{\mathcal{G}}\mathcal{Y} = \mathcal{Y}/\mathcal{G} \end{tikzcd}$$

the upper horizonatal map is an $\mathbb{A}^1$-weak equivalence by property of homotopy colimits. The proof will be complete if we show that the vertical map is a global weak equivalence: This follows from the fact that homotopy colimit is constructed pointwise, and for simplicial sets this is weak equivalence. □

A part of what follows has been discussed in [MV99, p. 128]. The classifying space of the presheaf $O(H^n)$ is the simplicial presheaf of sets defined in the following manner. For a ring R, the group $O(H^n)(R)$ can be thought of as a category with just one element and all the group elements as morphisms (and group law as the composition of morphisms). Denoting this category by $\widetilde{O}(H^n)(R)$, we get a presheaf $\widetilde{O}(H^n)$ of categories on $Sm/\mathbb{k}$ by defining

$$\widetilde{O}(H^n)(X) = \widetilde{O}(H^n)(\Gamma(X, \mathcal{O}_X)), \qquad X \in Sm/\mathbb{k}.$$

The classifying space of the presheaf $O(H^n)$ is defined to be the simplicial presheaf of sets

$$X \mapsto \mathscr{N}\widetilde{O}(H^n)(\Gamma(X, \mathcal{O}_X)), \qquad X \in Sm/\mathbb{k},$$

and will be denoted by $BO(H^n)$, here $\mathscr{N}$ denotes the nerve of a category.

For a presheaf $\mathcal{X}$ of sets on $Sm/\mathbb{k}$, let $E\mathcal{X}$ be the simplicial presheaf whose n^{th}-degree presheaf is $\mathcal{X}^{n+1}$, and face and degeneracies are defined by projections and diagonal maps. It has the characterstic property that for any any simplicial presheaf $\mathcal{Y}$, the natural map

$$\mathrm{Hom}_{\Delta^{op}PShv(Sm/\Bbbk)}(\mathcal{Y},\, E\mathcal{X}) \to \mathrm{Hom}_{PShv(Sm/\Bbbk)}(\mathcal{Y}_0,\, \mathcal{X})$$

is a bijection. We have natural isomorphism $E(\mathcal{X} \times \mathcal{Y}) \simeq E\mathcal{X} \times E\mathcal{Y}$

If G is a presheaf of groups, then EG is a simplicial presheaf of groups, whose group of 0^{th}-simplicies is G. Hence, G acts freely on EG on left and also on right. EG is $\mathbb{A}^1$-contractible. The morphism

$$EG \to BG$$

$$(g_0,\, g_1,\, ...,\, g_n) \mapsto (g_0 g_1^{-1},\, g_1 g_2^{-1},\, ...,\, g_{n-1} g_n^{-1},\, g_n)$$

induces an isomorphism $EG/G \simeq BG$. The map $EG \to BG$ is an universal G-torsor, [MV99, lemma 1.12, p. 128]. In particular, we have the universal $O(H^n)$-torsor $EO(H^n) \to BO(H^n)$.

A different version of the following proposition that has been helpful in the $\mathbb{A}^1$-representability theorem 5.7.1 has been completely proved in theorem 5.5.7 in the last chapter. The reason for survival of this proposition in this form and the attempted proof lies in the fact that some of the details used in the proof of theorem 5.5.7 have been more fully explained here.

Proposition 3.1.23. *The presheaf* $St(H^n,\, H^\infty)$ *is* $\mathbb{A}^1$*-contractible.*

To prove this proposition we will use a presheaf $\frac{O(H^N)}{O(H^{[n+1,\, N]})}$ defined as follows. For a ring R, if $T \in O(H^{N-n})(R)$, the $2N \times 2N$-matrix

$$\tilde{T} = \begin{pmatrix} I_{2n} & 0 \\ 0 & T \end{pmatrix}$$

is an isometry of $O(H^N)$. Let $M, N \in O(H^N)(R)$. Write $M \sim N$, if there is an isometry T of $O(H^{N-n})(R)$ such that $M\tilde{T} = N$. This defines an equivalence relation on the set $O(H^N)(R)$, let us denote the equivalence classes of this equivalence relation by $\frac{O(H^N)}{O(H^{[n+1,\,N]})}(R)$. We have the presheaf $\frac{O(H^N)}{O(H^{[n+1,\,N]})}$ of groups on $Sm/\Bbbk$ defined by

$$X \mapsto \frac{O(H^N)}{O(H^{[n+1,\,N]})}(\Gamma(X, \mathcal{O}_X)).$$

The map of presheaves $i : O(H^N) \to O(H^{N+1})$ defined by sending a matrix M to $\begin{pmatrix} M & 0 \\ 0 & I_2 \end{pmatrix}$ induces a map $i : \frac{O(H^N)}{O(H^{[n+1,\,N]})} \to \frac{O(H^{N+1})}{O(H^{[n+1,\,N+1]})}$. Let us denote the colimit of these presheaves by $\frac{O(H^\infty)}{O(H^{[n+1,\,\infty]})}$. For $M \in O(H^N)(R)$, the map

$$M \mapsto M.\begin{pmatrix} I_{2n} \\ 0 \end{pmatrix}$$

where 0 is the zero matrix of size $(2N - 2n) \times 2n$, induces a well defined map of presheaves

$$\varphi_N : \frac{O(H^N)}{O(H^{[n+1,\,N]})} \to St(H^n, H^N).$$

In fact this map sends the class of the matrix $M \in O(H^N)(R)$ to the $2N \times 2n$ submatrix of the first $2n$ columns of M. The diagram of presheaves

$$\begin{tikzcd}
\frac{O(H^N)}{O(H^{[n+1,\,N]})} \arrow[r, "\varphi_N"] \arrow[d, "i"] & St(H^n, H^N) \arrow[d, "i"] \\
\frac{O(H^{N+1})}{O(H^{[n+1,\,N+1]})} \arrow[r, "\varphi_{N+1}"] & St(H^n, H^{N+1})
\end{tikzcd}$$

commutes, where the right vertical map i is the canonical inclusion map. Therefore, we get a map

$$\varphi : \frac{O(H^\infty)}{O(H^{[n+1,\,\infty]})} \to St(H^n, H^\infty)$$

which is an $\mathbb{A}^1$-weak equivalence in view of the following lemma.

Lemma 3.1.24. *For a local ring R, the map $\varphi_{N,R}$ is an isomorphism, and hence the map φ_N is a Zariski and an $\mathbb{A}^1$-weak equivalence.*

Proof. We prove that for a local ring R, the group $O(H^N)(R)$ acts transitively on $St(H^n, H^N)(R)$ via left multiplication, the stabilizer at $\begin{pmatrix} I_{2n} \\ 0 \end{pmatrix}$ is $O(H^{n+1,N]})(R)$ and, the canonical identification of orbit space with the quotient of the group with respect to stabilzer in this case corresponds to the map $\varphi_{N,R}$. Let $M, N \in St(H^n, H^N)(R)$ be two elements. Then these determine two free direct factors of hyperbolic space $H^N(R)$ which would be isometric since on both these factors the form is induced from $H^N(R)$. Choosing an isometry α as in the diagram

$$\begin{tikzcd} Im(M) & \simeq & R^{2n} \arrow[r, "M"] \arrow[d, "\alpha"'] & R^{2N} \\ Im(N) & \simeq & R^{2n} \arrow[r, "N"] & R^{2N} \end{tikzcd}$$

we get an isometry of $Im(M) \xrightarrow{\bar{\alpha}} Im(N)$. By Witt's cancellation, the compliments of $Im(M)$ and $Im(N)$ in $H^N(R)$ are isometric via an isometry τ. Then $T = \bar{\alpha}\perp\tau$ is an isometry of $O(H^N)(R)$ and $TM = N$. This proves the transitivity of the action. A matrix calculation shows that stabilizer of $\begin{pmatrix} I_{2n} \\ 0 \end{pmatrix}$ is the subgroup $O(H^{n+1,N]})(R)$. Also, the claim on the identification of orbit space via the map $\varphi_{N,R}$ can also be checked. This completes the proof of this lemma. □

Proof of Proposition 3.1.23: See the proof of theorem 5.5.7.

Corollary 3.1.25. *There is a zig-zag*

$$Gr(H^n, H^\infty) \xleftarrow{\mathbb{A}^1-eq.} \bullet \xrightarrow{\mathbb{A}^1-eq.} BO(H^n)$$

of $\mathbb{A}^1$-weak equivalences, and hence there is an isomorphism in $\mathbb{A}^1$-homotopy category $BO(H^n) \simeq Gr(H^n, H^\infty)$.

Proof. Let us consider the diagonal $O(H^n)$-action on $St(H^n, H^\infty) \times EO(H^n)$. In the sequence

$$St(H^n, H^\infty) \xleftarrow{proj} St(H^n, H^\infty) \times EO(H^n) \xrightarrow{proj} EO(H^n)$$

of $O(H^n)$-equivariant maps of simplicial presheaves, both the projection maps are $\mathbb{A}^1$-weak equivalences, since $St(H^n, H^\infty)$ and $EO(H^n)$ are both $\mathbb{A}^1$-contractible. Therefore, in view of lemmas 3.1.22 and 3.1.21 , we have the induced $\mathbb{A}^1$-weak equivalences

$$\begin{tikzcd}
St(H^n, H^\infty)/O(H^n) \arrow[d, "\gamma"] & St(H^n, H^\infty) \times EO(H^n))/O(H^n) \arrow[l, "\mathbb{A}^1"', "proj"] \arrow[dl, dashed, "\mathbb{A}^1"] \arrow[d, "\mathbb{A}^1"', "proj"] \arrow[dl, phantom, "\circlearrowleft", shift right=8ex] \\
Gr(H^n, H^\infty) & EO(H^n)/O(H^n)
\end{tikzcd}$$

which give us the zig-zag of $\mathbb{A}^1$-equivalences

$$Gr(H^n, H^\infty) \xleftarrow{\mathbb{A}^1-eq.} \frac{St(H^n, H^\infty) \times EO(H^n))}{O(H^n)} \xrightarrow{\mathbb{A}^1-eq.} BO(H^n)$$

and the isomorphism $BO(H^n) \simeq Gr(H^n, H^\infty)$ in the $\mathbb{A}^1$-homotopy category. $\square$

3.2 The Presheaves $\mathscr{F}h^{[0,\infty]}$ and $\mathscr{F}hO$

In this section we recall the definition of the hermitian K-theory presheaf $\mathcal{K}^h$. We also define presheaves that help us understand the equivalence generated by the addition of a hyperbolic space to the presheaf $Gr_{\Bbbk}(n, H^\infty)$ and its relatives considered in the subsection 3.1.2. The hermitian K-theory presheaf receives a map from the orthogonal grassmannian presheaves in the $\mathbb{A}^1$-homotopy category.

In subsection 3.1.2, we considered isometric embeddings $H^m \hookrightarrow H^{m+1}$, and defined the presheaves $\mathscr{H}$, $\mathscr{F}h$ etc. Without explicitly defining we considered the infinite hyperbolic space H^∞ as the colimit of these embeddings $H^m \hookrightarrow H^{m+1}$, $m \geq 0$.

We should think of this colimit more formally as $\oplus_{\mathbb{N}} H$, where H is the standard hyperbolic plane R^2 with the form given by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let us write $\oplus_{\mathbb{N}} H$ as $H^{[0,\infty]}$ or simply H^{∞} as used earlier. For a non-negative integer r, define

$$H^{[-r,\infty]} = \oplus_{\{-1,...,-r\} \bigcup \mathbb{N}} H.$$

We should reinterpret the presheaves keeping this technical formality in mind. For instance, the presheaf $\mathscr{F}h_n^m$ is actually given by rank n direct factors of $\oplus_{\{0,...,m\}} H$. Let us write this as $\mathscr{F}h_n^{[0,m]}$. Similar remarks for other presheaves considered in subsection 3.1.2. For example, the presheaf $\mathscr{F}h_n^{\infty}$ is the colimit of

$$... \to \mathscr{F}h_n^{[0,m]} \to \mathscr{F}h_n^{[0,m+1]} \to ...$$

and we should write this as $\mathscr{F}h_n^{[0,\infty]}$. This kind of notational modification should be kept in mind for other presheaves as well if needed. The presheaves $\mathscr{F}h_n^{[-r,\infty]}$ and $\mathscr{F}h_n^{[0,\infty]}$ are isomorphic for every non-negative integer r: there is an isometry $H^{[-r,\infty]} \to H^{[0,\infty]}$ (we can define one by shifting the bases indexed over $\{-1,...,-r\} \bigcup \mathbb{N}$ and $\mathbb{N}$ respectively,) inducing an isomorphism. Now the presheaf $\mathscr{F}h$ defined earlier in 3.1.19 becomes $\mathscr{F}h^{[0,\infty]}$.

Definition 3.2.1 (The presheaf $\mathscr{F}hO$)**.** In 3.1.19, we have defined the presheaf $\mathscr{F}h^{[0,\infty]}$ as the disjoint union $\coprod_{n\geq 0} \mathscr{F}h_n^{[0,\infty]}$. Let R be a ring, and $P \xrightarrow{i} R^{2m}$ a direct factor giving an element in the set $\mathscr{H}_n^{[0,m]}(R)$. Then $H \perp P$, where summand H corresponds to the extra basis element of $H^{[-1,m]}$, determines a direct factor of rank $n+2$ in $\mathscr{F}h_{n+2}^{[-1,m]}(R)$. This association defines a map $H\perp : \mathscr{F}h^{[0,\infty]} \to \mathscr{F}h^{[-1,\infty]}$ of presheaves. Similarily we have maps of presheaves $\mathscr{F}h^{[-r,\infty]} \xrightarrow{H\perp} \mathscr{F}h^{[-r-1,\infty]}$, for every non-negative integer r. The presheaf $\mathscr{F}hO$ is the colimit

$$\varinjlim_{r\geq 0} (\mathscr{F}h^{[0,\infty]} \xrightarrow{H\perp} \mathscr{F}h^{[-1,\infty]} \xrightarrow{H\perp} ... \xrightarrow{H\perp} \mathscr{F}h^{[-r,\infty]} \xrightarrow{H\perp} \mathscr{F}h^{[-r-1,\infty]} \xrightarrow{H\perp} ...).$$

Definition 3.2.2. Keeping the notational adjustment described in this subsection in mind, we now have smooth $\mathbb{k}$-schemes $Gr_{\mathbb{k}}(n,\, H^{[-r,\, m]})$ $(r,\, m \geq 0)$ and the sheaves represented by these. We can also consider the colimit as $m \to \infty$ to get $Gr_{\mathbb{k}}(n,\, H^{[-r,\, \infty]})$ and then the colimit as $r \to \infty$ to get a representable sheaf. We will denote this representable sheaf by $\mathcal{G}rO$. We have the $\mathbb{A}^1$-weak equivalences $\mathscr{F}h_n^{[-r,\, m]}(R) \to Gr_{\mathbb{k}}(n,\, H^{[-r,\, m]})$ $(r,\, m \geq 0)$ as defined in 3.1.16. These induce a Zariski weak equivalence

$$\chi : \mathscr{F}hO \to \mathcal{G}rO$$

which is Zariski sheafification of $\mathscr{F}hO$.

3.2.1 H-Space Structures on $\mathscr{F}h^{[0,\, \infty]}$ and $\mathscr{F}hO$

The only purpose of this subsection was to prove the H-space structures on $\mathscr{F}hO$ which has been used in an essential way in the proof of the thoerem 3.3.18. Since the complete argument using Γ-spaces was becoming more and more complicated, we resolved the issue of H-space structure using operads in section 5.6. This section survives only because of the discussion it offers on Γ-spaces. Now that we know with the H-space structure through the work in section 5.6, the Γ-space point of view can be used in studying the ring structure on hermitian K-theory.

We make some preliminary observations regarding Γ-objects in a category [S74]. Recall that the category Γ has as its objects all the finite sets; and, for any two given finite sets S and T, a morphism $S \to T$ is a map $\theta : S \to \mathscr{P}(T)$ from the set S to the set $\mathscr{P}(T)$ of all the subsets of T with the property that for any elements $i,\ j(\neq i)$ in S, the subsets $\theta(i)$ and $\theta(j)$ are disjoint (composition defined in the cited reference). A Γ-object in a category $\mathscr{C}$ is defined to be a functor $\Gamma^{\mathrm{op}} \to \mathscr{C}$ satisfying the properties listed in [S74].

The opposite category Γ^{op} is *equivalent* to the category of finite pointed sets $Sets^{\mathrm{f}}_{+}$: given a finite set S, write S_+ for set $S\bigcup\{+\}$ (which is the set S equipped with a base point). For a morphism $\alpha : S \to T$, define a map $\overline{\alpha} : T_+ \to S_+$ by sending $j \in T_+$ to i if $j \in \alpha(i)$, otherwise to the base point of S_+. This assignement provides us a functor from Γ^{op} to $Sets^{\mathrm{f}}_{+}$. To define the inverse functor, for a pointed set $S\bigcup\{+\}$ consider the set S, and for a given map of pointed sets $\beta : T\bigcup\{+\} \to S\bigcup\{+\}$, consider the map $\underline{\beta} : S \to \mathscr{P}(T)$ defined by sending $s \in S$ to the set $\beta^{-1}(s) \subset T$. Further the category $Sets^{\mathrm{f}}_{+}$ is equivalent to its full subcategory $\mathscr{N}^{\mathrm{f}}_{+}$ consisting of sets of the form $[n]_+ = \{+, 1, ..., n\}, n \geq 0$: To see this equivalence one can choose an isomorphism (and it's inverse) of a finite set pointed set with some pointed $[n]_+$ to define a functor. Therefore, the categry Γ^{op} is equivalent to the category $\mathscr{N}^{\mathrm{f}}_{+}$. As a result of this equivalence, to define a Γ-object in a category $\mathscr{C}$, we need to define the objects in $\mathscr{C}$ corresponding only to the the finite sets $[n]$ in Γ.

We also have a functor $\Delta \to \Gamma$ from the category of finite ordered ordinals Δ into Γ defined by sending the naturally ordered set $< n >= \{0, 1, ..., n\}$ to the set $[n] = \{1, ..., n\}$, and a nondecreasing map $\alpha :< m >\to< n >$ to the map (denoted by the same letter) $\alpha : [m] \to \mathscr{P}([n])$ defined by $j \mapsto \{i \in [n] : \alpha(j-1) < i \leq \alpha(i)\} \subset [n]$. We use this functor to recognize a Γ-object in a category as a simplicial object in that category.

Now we will define an $\mathbb{A}^1$-special Γ-presheaf of sets X on the category $Sm/\Bbbk$ of smooth $\Bbbk$-schemes, for which X_1 would be the presheaf $\mathscr{F}h^{[0,\infty]}$. Using the equivalence of the categories Γ^{op} and $\mathscr{N}^{\mathrm{f}}_{+}$ we define X as a functor $X : \mathscr{N}^{\mathrm{f}}_{+} \to PShv(Sm/\Bbbk)$.

Definition 3.2.3 (The Γ-presheaf X on $Sm/\mathbb{k}$)**.** For any two non-negative integers N and n, let us first define a presheaf of sets X_n^N on $Sm/\mathbb{k}$. For an affine smooth $\mathbb{k}$-scheme $Z = \text{Spec } R$, we define $X_n^N(Z)$ (which we also write as $X_n^N(R)$) as the set of n-tuples $(P_1, ..., P_n)$ where each P_i is a free direct summand of the hyperbolic space $H(R)^{[0,N]}$ such that the hyperbolic form restricted to P_i is non-degenerate and for all $i,\ j(\neq i)$, $P_i,\ P_j$ are mutually orthogonal. For a $\mathbb{k}$-scheme V, we define $X_n^N(V) = X_n^N(\Gamma(V, O_V))$. And for a map of schemes the corresponding map for X_n^N is defined by the maps induced between hyperbolic spaces by the map of the rings of global sections. For integers N and a commutative ring R, we have the isometric embeddings of the hyperbolic spaces $H(R)^{[0,N]} \hookrightarrow H(R)^{[0,N+1]}$ which induce compatible maps of presheaves $X_n^N \to X_n^{N+1}$.

For each non-negative integer N, using the presheaves X_n^N we define a Γ-presheaf X^N on $Sm/\mathbb{k}$ as a functor $X^N : \mathscr{N}_+^{\mathrm{f}} \to PShv(Sm/\mathbb{k})$. We take $X^N([n]_+)$ to be X_n^N and, for a map $\gamma : [n]_+ \to [m]_+$ in the category $\mathscr{N}_+^{\mathrm{f}}$, we take the map of presheaves $X_n^N \to X_m^N$ induced by taking the orthogonal direct sum of the appropriate factors: An n-tuple $(P_1, ..., P_n)$ is mapped to the m-tuple $(Q_1, ..., Q_m)$ where $Q_i = \oplus_{j\in\gamma^{-1}(i)} P_j = \Sigma_{j\in\gamma^{-1}(i)} P_j$. These maps are compatible with the maps $X_n^N \to X_n^{N+1}$ described above. We have a directed system of Γ-presheaves $X^N,\ N \geq 0$. The Γ-presheaf X is defined by taking

$$X_n = X([n]_+) = \varinjlim_{N\geq 0} X_n^N$$

and the induced maps. We will consider X as a Γ-simplicial presheaf (simplicially constant) as well, if needed.

Remark 3.2.4. Going back to notations used earlier in 3.1.10 and 3.1.19, we see that X_1^N is the presheaf $\mathscr{F}h^{[0,N]}$ and X_1 is the presheaf $\mathscr{F}h^{[0,\infty]}$. Thus, in view of 3.1.11 that the presheaves X_1^N and X_1 are representable.

We want to prove that the Γ-simplicial presheaf X is $\mathbb{A}^1$-special (precisley stated later).

Proposition 3.2.5. *The Γ-presheaf X is $\mathbb{A}^1$-special: that is, the map $p^n = \prod X(p_i) : X_n \to X_1 \times ... \times X_1$ induced by the maps $p_i : [n]_+ \to [1]_+$ $(i = 1, ..., n)$ which send $i \in [n]_+$ to $1 \in [1]_+$ and other elements of $[n]_+$ to the base point of $[1]_+$ is an $\mathbb{A}^1$-weak equivalence.*

Proof. We prove the $\mathbb{A}^1$-weak equivalence only in the case $n = 2$, since the arguments involved can be extended to other values of n: So we prove that the map $p_2 : X_2 \to X_1 \times X_1$ is an $\mathbb{A}^1$-weak equivalence. Let's first consider the maps $p_2^N : X_2^N \to X_1^N \times X_1^N$, and $\lambda_2^N : X_1^N \times X_1^N \to X_2^{2N}$: Here p_2^N is the map used in defining the Γ-presheaf; and on an S-scheme V, λ_2^N is defined by sending a pair (P_1, P_2) in $X_1^N(\Gamma(V, \mathcal{O}_V)) \times X_1^N(\Gamma(V, \mathcal{O}_V))$ the pair $(B_1^N A_0^N P_1, B_2^N A_0^N P_2)$ in $X_1^{2N}(\Gamma(V, \mathcal{O}_V))$, where $B_1^N = (e_1, e_3, ...e_{2N-1}, e_2, e_4, ..., e_{2N})$ and $B_2^N = (e_2, e_4, ..., e_{2N}, e_1, e_3, ..., e_{2N-1})$ are $4N$-square matrices (written using the standard hyperbolic basis vectors) and $A_0^N = \begin{pmatrix} I_{H^N} \\ O \end{pmatrix}$ is a $4N \times 2N$ matrix. Recall that the simplicial presheaves X_1 and X_2 are defined as the respective colimits of the systems $X_1^N \to X_1^{N+1}$ and $X_2^N \to X_2^{N+1}$ where maps are the ones mentioned in the defintion of the Γ-simplicial presheaf X. Fixing an $N \geq 2$, for every $k \geq 0$ let us consider the following objects and morphisms in the category $\Delta^{op}PShv(Sm/\mathbb{k})$: $\mathfrak{X}_k = X_2^{2^{k-1}N}$, $\mathfrak{Y}_k = X_1^{2^{k-1}N} \times X_1^{2^{k-1}N}$, $f_k = p_2^{2^{k-1}N}$, $g_k = \lambda_2^{2^{k-1}N}$ and, take the morphisms $i_k^2 : \mathfrak{X}_k \to \mathfrak{X}_{k+1}$ and $i_k^1 : \mathfrak{Y}_k \to \mathfrak{Y}_{k+1}$ to be the ones used in the definition of the Γ-simplicial presheaf X. Then colim $(\mathfrak{X}_k, i_k^2) = X_2$, colim $(\mathfrak{Y}_k, i_k^1) = X_1 \times X_1$ and the maps induced on these colimits by f_k and g_k are the maps p_2 and λ_2.

Using the criterion given in corollary 2.2.4, proof would be complete once we show that for all values of k the pairs of maps $(g_k \circ f_k,\ i_k^2)$ and $(f_{k+1} \circ g_k,\ i_k^1)$ are naively $\mathbb{A}^1$-homotopic in the category of morphisms. Both the maps i_k^1 and i_k^2 send a pair (P_1, P_2) to the pair $(A_0^{2^{k-1}N}P_1, A_0^{2^{k-1}N}P_2)$, and both the compositions $g_k \circ f_k$ and $f_{k+1} \circ g_k$ send (P_1, P_2) to $(B_1^{2^{k-1}N}A_0^{2^{k-1}N}P_1, B_2^{2^{k-1}N}A_0^{2^{k-1}N}P_2)$. Therefore to get a naive $\mathbb{A}^1$-homotopy, it suffices to prove that the maps given by the matrices $B_1^{2^{k-1}N}$ and $B_2^{2^{k-1}N}$ are naively $\mathbb{A}^1$-homotopic to the identity map. It should be kept in mind that homotopies to be considered should respect orthogonality of the pairs of spaces. We provide this homotopy in two stages: A map of the form

$$(B_1^N A_0^N, B_2^N A_0^N) \rightsquigarrow (TB_2^N A_0^N + (1-T)B_1^N A_0^N, B_2^N A_0^N) \rightsquigarrow (B_2^N A_0^N, B_2^N A_0^N)$$

defines a homotopy of $(B_1^N A_0^N, B_2^N A_0^N)$ with $(B_2^N A_0^N, B_2^N A_0^N)$; and then, to get the homotopy of $(B_2^N A_0^N, B_2^N A_0^N)$ with identity, we observe that $B_1^{2^{k-1}N}$ and $B_2^{2^{k-1}N}$ are both even permutation matrices, and even permutation matrices are naively $\mathbb{A}^1$-homotopic to the identity. It should be remarked as an aid to the reader willing to produce these homotopies in cases $n \geq 3$, we can use homotopies of above form in three stages: first, change (B_1^N, B_2^N, B_3^N) to (B_1^N, B_3^N, B_3^N), and then to (B_3^N, B_3^N, B_3^N). $\square$

Corollary 3.2.6. *The presheaf $\mathscr{F}hO^{[0,\infty]}$ is an H-space in the $\mathbb{A}^1$-homotopy category.*

Proof. We have already noticed that the presheaf $\mathscr{F}hO^{[0,\infty]}$ is just the presheaf X_1 in the Γ-presheaf X. Since X is $\mathbb{A}^1$-special, it follows from [S74] that X_1 is an H-space. $\square$

Remark 3.2.7. We are very sure that the presheaf $\mathscr{F}hO$ is also an H-space in the $\mathbb{A}^1$-homotopy category. A strong indication for this fact lies in our observation that all the known constructions of taking colimits of an H-space with respect to ‘addition’ of an element result in an H-spaces. See the construction of T_A in [S74], for example.

3.2.2 The Presheaf $\mathcal{K}^h$ and the Map $\mathscr{F}hO \xrightarrow{\hbar} \mathcal{K}^h$

We will recall informally some of the general notions from category theory relevant to us in the definition of the hermitian K-theory presheaf $\mathcal{K}^h$. A comprehensive reference for the symmetric monoidal categories is the book [Mc71]. A large part of material relevant to us in this subsection can be found in more details in the lecture notes [G76].

Definition 3.2.8. Let $\mathcal{C}$ be a monoidal category acting on a category $\mathcal{D}$, and $+ : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$ be the action. A category $< \mathcal{C}, \mathcal{D} >$ has been defined in [G76]. It has the same objects as the category $\mathcal{D}$, and a morphism $X \to Y$ in $< \mathcal{C}, \mathcal{D} >$ is an equivalence class of tuples $(X, Y, A, A + X \to Y)$, where A is an object in $\mathcal{C}$, X and Y are objects in $\mathcal{D}$, and $A + X \to Y$ is a morphism in $\mathcal{D}$; and, the tuples $(X, Y, A, A + X \to Y)$ and $(X, Y, A', A' + X \to Y)$ are equivalent if there is an isomorphism $A \xrightarrow{\alpha} A'$ such that diagram

$$\begin{tikzcd} A + X \arrow[r, "\alpha+1_X", "\simeq"'] \arrow[d] & A' + X \arrow[dl] \\ Y & \end{tikzcd}$$

commutes.

Definition 3.2.9. If a symmetric monoidal category $\mathcal{C}$ acts on category $\mathcal{D}$, then $\mathcal{C}$ acts on the product $\mathcal{C} \times \mathcal{D}$ via the diagonal action: $A + (B, X) = (A + B, A + X)$, where A, B are objects of $\mathcal{C}$ and X is an object of $\mathcal{D}$. The category $\mathcal{C}^{-1}\mathcal{D}$ is the category $< \mathcal{C}, \mathcal{C} \times \mathcal{D} >$. We have an invertible action of $\mathcal{C}$ on $\mathcal{C}^{-1}\mathcal{D}$ given by $A + (B, X) = (B, A + X)$.

Remark 3.2.10. In particular, if $\mathcal{C}$ is a symmetric monoidal category, we have the category $\mathcal{C}^{-1}\mathcal{C}$ on which $\mathcal{C}$ acts invertibly. Let every morphism in $\mathcal{C}$ be a isomorphism. The functor $+ : \mathcal{C}^{-1}\mathcal{C} \times \mathcal{C}^{-1}\mathcal{C} \to \mathcal{C}^{-1}\mathcal{C}$ defined by

$$(M, N) + (M', N') = (M + M', N + N')$$

makes $\mathcal{C}^{-1}\mathcal{C}$ a symmetric monoidal category. In fact, with respect to this functor $\mathcal{C}^{-1}\mathcal{C}$ is an H-group. The assignment $A \mapsto (0, A)$, defines a functor

$$\mathrm{can} : \mathcal{C} \to \mathcal{C}^{-1}\mathcal{C}.$$

The functor $\mathcal{C} \xrightarrow{\mathrm{can}} \mathcal{C}^{-1}\mathcal{C}$ is a group completion of $\mathcal{C}$: that is, the induced map

$$(\pi_0\mathcal{C})^{-1}H_p(\mathcal{C}) \to H_p(\mathcal{C}^{-1}\mathcal{C})$$

is an isomorphism for every $p \geq 0$ ([G76], p. 221); and, $\mathcal{C}^{-1}\mathcal{C}$ is group complete (i.e., it is an H-space whose π_0 is a group).

Remark 3.2.11. Let $\mathcal{P}(R)$ denote the category of finitely generated projective R-modules over a commutative ring R. We know that given a map $R_1 \to R_2$ of commutative rings, $M \mapsto M \otimes_{R_1} R_2$ defines an exact functor $\mathcal{P}(R_1) \to \mathcal{P}(R_2)$. For this functor there is a canonical isomorphism $\mathrm{Hom}_{R_1-\mathrm{Mod}}(M,\ R_1) \otimes_{R_1} R_2 \simeq \mathrm{Hom}_{R_2-\mathrm{Mod}}(M \otimes_{R_1} R_2,\ R_2)$, since M is finitely generated and projective (check this part..).

Definition 3.2.12. For a commutative ring R and a nonnegative integer r, let us define a category $\mathcal{S}_{R,r}$. In what follows for a given R-module M, the R-module $\mathrm{Hom}_{R-\mathrm{Mod}}(M, R)$ will be denoted by M^*. The objects in $\mathcal{S}_{R,r}$ are the pairs (M, ϕ), where M is a finitely generated projective R-submodule considered as a direct factor $M \subset H^{[-r,\infty]}(R)$ (see 3.2 for notations), $\phi = h|_M$ (h being the standard hyperbolic form on $H^{[-r,\infty]}(R)$) and $\phi : M \xrightarrow{\simeq} M^*$ is an isomorphism. A morphism $(M, \phi) \to (N, \psi)$ is given by an isomorphism $\alpha : M \to N$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "\simeq", "\phi"'] \arrow[d, "\alpha"', "\simeq"] & M* \\ N \arrow[r, "\simeq", "\psi"'] & N^* \arrow[u, "\simeq", "\alpha^*"'] \end{tikzcd}$$

commutes. The category $\mathcal{S}_{R,r}$ is a small category.

We also have a category $\widetilde{\mathcal{S}}_R$ whose objects are the pairs (M, ϕ) where M is a finitely generated projective R-module and $M \xrightarrow{\phi} M^*$ is an isomorphism; and, the morphisms are given by isomorphisms of R-modules as in the category $\mathcal{S}_{R,r}$. The canonical forgetful embedding of $\mathcal{S}_{R,r}$ into the category $\widetilde{\mathcal{S}}_R$ is an equivalence of categories via a choice of an isomorphism of an object in $\widetilde{\mathcal{S}}_R$ with an object in $\mathcal{S}_R$. Let (M, ϕ) and (N, ψ) be two objects of $\widetilde{\mathcal{S}}_R$. Let $\phi \oplus \psi$ be the map $M \oplus N \to M^* \oplus N^*$ given by the matrix $\begin{pmatrix} \phi & \\ & \psi \end{pmatrix}$, where direct sum is formed in the category $\widetilde{\mathcal{S}}_R$. The sum $(M, \phi) + (N, \psi) = (M \oplus N, \phi \perp \psi)$, where $\phi \perp \psi$ is the composition $M \oplus N \xrightarrow{\phi \oplus \psi} M^* \oplus N^* \xrightarrow{\text{can } \simeq} (M \oplus N)^*$, makes $\widetilde{\mathcal{S}}_R$ a symmetric monoidal category. This defines a symmetric monoidal structure on $\mathcal{S}_{R,r}$ as well via the aforementioned equivalence, though it involves a choice of direct sum. If $X \in Sm/\mathbb{k}$ is an smooth $\mathbb{k}$-scheme, we will denote the category $\widetilde{\mathcal{S}}_{\Gamma(X, \mathcal{O}_X)}$ simply by $\widetilde{\mathcal{S}}_X$ and $\mathcal{S}_{\Gamma(X, \mathcal{O}_X), r}$ by $\mathcal{S}_{X,r}$.

Given a map $R_1 \xrightarrow{f} R_2$ of commutative rings and an object (M, ϕ) in $\mathcal{S}_{R_1,r}$, the finitely generated projective R_2-module $M \otimes_{R_1} R_2$ can be thought of as a direct factor of $H^{[-r,\infty]}(R_2)$ via the composition $M \otimes_{R_1} R_2 \subset H^{[-r,\infty]}(R_1) \otimes_{R_1} R_2 \xrightarrow{\text{can } \simeq} H^{[-r,\infty]}(R_2)$. We have the induced isomorphism $M \otimes_{R_1} R_2 \xrightarrow{\phi \otimes 1} M^* \otimes_{R_1} R_2$. In view of the canonical isomorphism $M^* \otimes_{R_1} R_2 \simeq (M \otimes_{R_1} R_2)^*$ mentioned in 3.2.11, the composition $M \otimes_{R_1} R_2 \xrightarrow{\phi \otimes 1} M^* \otimes_{R_1} R_2 \simeq (M \otimes_{R_1} R_2)^*$ is an isomorphism. Let us denote this composition by $\phi \otimes R_2$. By defining $(M, \phi) \mapsto (M \otimes_{R_1} R_2,\ \phi \otimes R_2)$, we get a functor $\mathcal{S}_{R_1,r} \to \mathcal{S}_{R_2,r}$. The assignment $R \mapsto \mathcal{S}_{R,r}$ gives us a functor from the category of commutative rings to the category of all small categories. As mentioned above images of this functor actually are symmetric monoidal categories, though functors are not monoidal. Though it should be noted that using the universal property of direct sums, these functors can be made into symmetric monoidal functors. We also have the induced functor $\mathcal{S}_{R_1,r}^{-1}\mathcal{S}_{R_1,r} \to \mathcal{S}_{R_2,r}^{-1}\mathcal{S}_{R_2,r}$ between the categories defined in 3.2.10. Note that the category $\mathcal{S}_{R_1,r}^{-1}\mathcal{S}_{R_1,r}$ does not depend on the choice of direct sums. We will denote the category $\mathcal{S}_{R,r}^{-1}\mathcal{S}_{R,r}$ by $\mathcal{P}_{\hbar,r}(R)$. If X is a scheme, we will denote the category $\mathcal{P}_{\hbar,r}(\Gamma(X, \mathcal{O}_X))$ simply by $\mathcal{P}_{\hbar,r}(X)$. When $r = 0$, we will in general omit the subscript standing for r in all the three categories defined above: Thus, for example, the category $\mathcal{S}_{R,0}$ will be denoted simply by $\mathcal{S}_R$.

Definition 3.2.13 (The presheaf $\mathcal{K}^h$ and the hermitian K-groups)**.** For a smooth $\mathbb{k}$-scheme X, we have the category $\mathcal{P}_{\hbar,0}(X)$ defined in 3.2.12. The functor $X \mapsto \mathcal{P}_{\hbar,0}(X)$ is a presheaf of small categories on $Sm/\mathbb{k}$. Let $\mathscr{N}\mathcal{C}$ denote the nerve of a small category $\mathcal{C}$. The assignment $X \mapsto \mathscr{N}\mathcal{P}_{\hbar,0}(X)$ defines a presheaf of simplicial sets on $Sm/\mathbb{k}$. We will denote this presheaf by $\mathcal{K}^h$ and call it the hermitian K-theory presheaf. The hermitian K-groups of a smooth affine $\mathbb{k}$-scheme X are defined as the homotopy groups of the simplicial set $\mathscr{N}\mathcal{P}_{\hbar,0}(X)$ at 0. These are denoted by $K_n^h(X),\ n \geq 0$. Thus

$$K_n^h(X) = \pi_n(\mathscr{N}\mathcal{P}_{\hbar,0}(X), 0), \qquad n \geq 0,\ X \text{ affine}.$$

Remark 3.2.14. It can be seen that the hermitian K-theory presheaf $\mathcal{K}^h$ defined here is the same as the one defined in [H05] in 1.3(1), 1.5 and 1.7 for affine $\mathbb{k}$-schemes in $Sm/\mathbb{k}$ since the corresponding categories are equivalent when interpreted reasonably: The smallness of the category $P(A)$ and the related issues in the definition of the hermitian K-theory space have not been addressed in [H05], cf. the functoriality remark in [H05]. Another remark is also in order that we have not used the most general definition which is applicable to other additive categories with duality, since we will not need those. It should be noted for later use that for affine schemes in $Sm/\Bbbk$, the presheaf $\mathcal{K}^h$ is homotopy invariant and it has the Nisnevich-Mayer-Vietoris property: See [H05] Corollaries 1.12 and 1.14.

Definition 3.2.15. If X is a smooth $\Bbbk$-scheme, $X \mapsto \mathscr{N}\mathcal{S}_{X,r}$ and $X \mapsto \mathscr{N}\mathcal{P}_{\hbar,r}(X)$ define simplicial presheaves of sets on $Sm/\Bbbk$. Let us denote these presheaves by $\mathscr{N}\mathcal{S}_{-,r}$ and $\mathscr{N}\mathcal{P}_{\hbar,r}$ respectively. The canonical group completion functor $\mathcal{S}_{X,r} \xrightarrow{\text{can}} \mathcal{P}_{\hbar,r}(X)$ recalled in 3.2.10 defines a map (again denoted by 'can')

$$\text{can} : \mathscr{N}\mathcal{S}_{-,r} \to \mathscr{N}\mathcal{P}_{\hbar,r}$$

of the simplicial presheaves. In particular, when $r = 0$ we get the map

$$\text{can} : \mathscr{N}\mathcal{S}_{-,0} \to \mathcal{K}^h$$

of $\mathscr{N}\mathcal{S}_{-,0}$ into the hermitian K-theory presheaf $\mathcal{K}^h$.

Definition 3.2.16. We have already mentioned the group completion functor $\mathcal{S}_{R,r} \xrightarrow{\text{can}} \mathcal{S}_{R,r}^{-1}\mathcal{S}_{R,r} = \mathcal{P}_{\hbar,r}(R)$, see 3.2.10, 3.2.13 and 3.2.15. The addition of a hyperbolic plane $\mathcal{S}_{R,r} \xrightarrow{H\perp} \mathcal{S}_{R,r+1}$ induces a functor $H\perp : \mathcal{P}_{\hbar,r}(R) \to \mathcal{P}_{\hbar,r+1}(R)$: Just to avoid ambiguity, this induced functor $H\perp$ sends an element $((M,\phi),\ (N,\psi))$ in $H\perp : \mathcal{P}_{\hbar,r}(R)$ to the pair $((M,\phi),\ (N\perp H, \psi\perp h))$ in $\mathcal{P}_{\hbar,r+1}(R)$ (and morphisms are defined in the obvious way). The diagram of presheaves of simplicial sets

$$\begin{tikzcd}
\mathscr{N}\mathcal{S}_{-,r} \arrow[r, "\text{can}"] \arrow[d, "H\perp"] & \mathscr{N}\mathcal{P}_{\hbar,r} \arrow[d, "H\perp"] \\
\mathscr{N}\mathcal{S}_{-,r+1} \arrow[r, "\text{can}"] & \mathscr{N}\mathcal{P}_{\hbar,r+1}
\end{tikzcd}$$

is commutative, (the functor 'can' has been defined in 3.2.16). It should be noted that the map on the right is a homotopy equivalence for every integer r, since the functor $H\perp : \mathcal{P}_{\hbar,r}(R) \to \mathcal{P}_{\hbar,r+1}(R)$ induces a homotopy equivalence of categories, see [Q73]. Also, the presheaves of categories $\mathcal{P}_{\hbar,r}$ and $\mathcal{P}_{\hbar,0}$ are isomorphic, giving us an isomorphism of simplicial presheaves $\alpha_r : \mathscr{N}\mathcal{P}_{\hbar,r} \to \mathcal{K}^h$. Therefore, we get the commutative diagram

$$\begin{tikzcd}
\mathscr{N}\mathcal{S}_{-,0} \arrow[r, "\text{can}"] \arrow[d, "H\perp"] & \mathscr{N}\mathcal{P}_{\hbar,0} \arrow[r, "="] \arrow[d, "H\perp"] & \mathcal{K}^h \arrow[d, "\alpha_1\circ H\perp", "\simeq"'] \\
\mathscr{N}\mathcal{S}_{-,1} \arrow[r, "\text{can}"] \arrow[d, "H\perp"] & \mathscr{N}\mathcal{P}_{\hbar,1} \arrow[r, "\alpha_1"] \arrow[d, "H\perp"] & \mathcal{K}^h \arrow[d, "\alpha_2\circ H\perp\circ\alpha_1^{-1}", "\simeq"'] \\
\mathscr{N}\mathcal{S}_{-,2} \arrow[r, "\text{can}"] \arrow[d, "H\perp"] & \mathscr{N}\mathcal{P}_{\hbar,2} \arrow[r, "\alpha_2"] \arrow[d, "H\perp"] & \mathcal{K}^h \arrow[d, "\alpha_3\circ H\perp\circ\alpha_2^{-1}", "\simeq"'] \\
\cdot & \cdot & \cdot
\end{tikzcd}$$

in which the right vertical maps are homotopy equivalences. We will denote all the homotopy equivalences $\alpha_{r+1} \circ H\perp \circ \alpha_r^{-1}$ by $H\perp$ and all the compositions $\mathscr{N}\mathcal{S}_r \xrightarrow{\text{can}} \mathscr{N}\mathcal{P}_{\hbar r} \xrightarrow{\alpha_r} \mathcal{K}^h$ by 'can'.

In 3.2.1 we have defined the presheaf $\mathscr{F}hO$. Now we are going to define a map $\mathscr{F}hO \xrightarrow{\hbar} \mathcal{K}^h$. The goal of this work is to prove that $\hbar$ is an $\mathbb{A}^1$-weak equivalence.

Definition 3.2.17. For a commutative ring R and a nonnegative integer r, we denote by $\mathscr{F}\hbar^{[-r,\infty]}(R)$, the full subcategory of $\mathcal{S}_{R,r}$ defined as follows. The set of objects of $\mathscr{F}\hbar^{[-r,\infty]}(R)$ is the set $\mathscr{F}h^{[-r,\infty]}(R)$ (see 3.1.19 and 3.2.1). As morphisms we take isometries: if M, N are two objects in $\mathscr{F}\hbar^{[-r,\infty]}(R)$ determined by non-degenerate direct factors $M \subset H^{[-r,m]}(R)$ and $N \subset H^{[-r,n]}(R)$, then a morphism $M \to N$ is given by an isometry $M \xrightarrow{\simeq} N$. If $X \in Sm/\Bbbk$, we will denote the category $\mathscr{F}\hbar^{[-r,\infty]}(\Gamma(X, \mathcal{O}_X))$ simply by $\mathscr{F}\hbar^{[-r,\infty]}(X)$. We have the presheaves $\mathscr{F}\hbar^{[-r,\infty]}$ of categories on $Sm/\Bbbk$ defined by sending X to the category $\mathscr{F}\hbar^{[-r,\infty]}(X)$. There is a natural map of presheaves $H\perp : \mathscr{F}\hbar^{[-r,\infty]} \to \mathscr{F}\hbar^{[-r-1,\infty]}$ induced by adding a hyperbolic plane for the extra basis element.

Definition 3.2.18. Let M be a non-degenerate symmetric bilinear form which is a direct factor of some hyperbolic space $H^{[-r,\infty]}(R)$. If $h_M : M \to M^*$ denote the restriction of the adjoint isomorphism of the hyperbolic space $H^{[-r,m]}(R)$, then h_M is an isomorphism (see 3.2 and the paragraph just before the definition 3.1.9 for explanation of $H^{[-r,m]}(R)$). Thus, an object M in $\mathscr{F}\hbar^{[-r,\infty]}(R)$ gives us an object (M, h_M) in the category $\mathcal{S}_{R,r}$ (see 3.2.12). For a commutative ring R, this defines a functor $h^r : \mathscr{F}\hbar^{[-r,\infty]}(R) \to \mathcal{S}_{R,r}$, and a map (again denoted by h^r)

$$h^r : \mathscr{N}\mathscr{F}\hbar^{[-r,\infty]} \to \mathscr{N}\mathcal{S}_{-,r}$$

of simplicial presheaves of sets on $Sm/\Bbbk$. The diagram

$$\begin{tikzcd}
\mathscr{N}\mathscr{F}\hbar^{[-r,\infty]} \arrow[r, "h^r"] \arrow[d, "H\perp"] & \mathscr{N}\mathcal{S}_{R,r} \arrow[d, "H\perp"] \\
\mathscr{N}\mathscr{F}\hbar^{[-r-1,\infty]} \arrow[r, "h^{r+1}"] & \mathscr{N}\mathcal{S}_{R,r+1}
\end{tikzcd}$$

commutes.

Definition 3.2.19. The diagram of simplicial presheaves of sets on $Sm/\Bbbk$, in which the vertical maps on the right are homotopy equivalences as discussed in 3.2.16, $r \geq 0$

$$\begin{tikzcd}
\mathscr{N}\mathscr{F}\hbar^{[-r,\infty]} \arrow[r, "h^r"] \arrow[d, "H\perp"] & \mathscr{N}\mathcal{S}_{-,r} \arrow[r, "\mathrm{can}"] \arrow[d, "H\perp"] & \mathcal{K}^h \arrow[d, "H\perp", "\simeq"'] \\
\mathscr{N}\mathscr{F}\hbar^{[-r-1,\infty]} \arrow[r, "h^{r+1}"] \arrow[d, "H\perp"] & \mathscr{N}\mathcal{S}_{-,r+1} \arrow[r, "\mathrm{can}"] \arrow[d, "H\perp"] & \mathcal{K}^h \arrow[d, "H\perp", "\simeq"'] \\
\cdot & \cdot & \cdot
\end{tikzcd}$$

is commutative. We also have

$$\varinjlim_{H\perp} \mathscr{N}\mathscr{F}\hbar^{[-r,\infty]} = \mathscr{N}(\varinjlim_{H\perp} \mathscr{F}\hbar^{[-r,\infty]}).$$

Thus, taking the vertical colimits in the above diagram, we get a map of simplicial presheaves of sets on $Sm/\Bbbk$

$$\mathscr{N}(\varinjlim_{H\perp} \mathscr{F}\hbar^{[-r,\infty]}) \xrightarrow{\mathrm{can}\circ h} \mathcal{K}^h.$$

Since the presheaf of vertices of $\mathscr{N}(\varinjlim_{H\perp} \mathscr{F}\hbar)$ is the presheaf $\mathscr{F}hO$, we have defined a map

$$\hbar : \mathscr{F}hO \to \mathcal{K}^h$$

of the orthogonal Grassmannian (see 3.2.1) into the hermitian K-theory. Since $\mathcal{K}^h$ is a sheaf on affine smooth schemes and the natural map $\chi : \mathscr{F}hO \to \mathcal{G}rO$ defined in 3.2.2 is a Zariski sheafification, there is a unique map from $\mathcal{G}rO$ to $\mathcal{K}^h$ as well induced from $\hbar$. We will denote this map also by $\hbar$.

3.3 $\mathbb{A}^1$-Weak Equivalence of the Map $\hbar$

In this section we prove the $\mathbb{A}^1$-weak equivalence of the map $\hbar$ defined in 3.2.19 assuming a seemingly technical claim not yet verified, namely, the H-space structure on $\mathscr{F}hO$. In 3.3.1 we prove that the simplicial subpresheaf $\mathcal{K}_0^h$ (3.3.2) of connected component of 0 in $\mathcal{K}^h$, is $\mathbb{A}^1$-homotopic to the presheaf BO of classifying spaces of the presheaf O of isometries of the infinite hyperbolic space defined in 3.3.1.

3.3.1 $\mathbb{A}^1$-Weak Equivalence of BO and $\mathcal{K}_0^h$

Let us recall the definitions of simplicial presheaves BO and $\mathcal{K}_0^h$.

Definition 3.3.1. [The presheaf BO.] In 3.1.12, we have defined the presheaf $O(H^{[0,n]})$ of groups of isometries of the hyperbolic spaces $H^{[0,n]}$. For a commutative ring R, the standard embeddings $\mathrm{Aut}H^{[0,n]}(R) \hookrightarrow \mathrm{Aut}H^{[0,n+1]}(R)$ $(n \geq 0)$ of groups of automorphisms of hyperbolic spaces defined by $M \mapsto \begin{pmatrix} M & 0 \\ 0 & 1_{H(R)} \end{pmatrix}$, give us maps of presheaves of groups $O(H^{[0,n]}) \subset O(H^{[0,n+1]})$. The presheaf $O(H^{[0,\infty]})$ of groups on $Sm/\mathbb{k}$ is defined as the colimit of these presheaves. In this section we will denote the presheaf $O(H^{[0,\infty]})$ simply by O. Thus, for a smooth $\mathbb{k}$-scheme X, the group $O(X)$ is the colimit of the system

$$\cdots \to O(H^{[0,\,n]}(R)) \to O(H^{[0,\,n+1]}(R)) \cdots, \qquad\qquad M \mapsto \begin{pmatrix} M & 0 \\ 0 & 1_{H(R)} \end{pmatrix},$$

where $R = \Gamma(X\,\mathcal{O}_X)$. The groups $O(H^{[0,\,n]})(X)$ and $O(X)$ when thought of as categories with just one object and all the group elements as morphisms, would be written as $\widetilde{O}(H^{[0,\,n]})(X)$ and $\widetilde{O}(X)$ respectively. The simplicial presheaf BO is defined by taking nerve of this category, thus

$$BO(X) = \mathscr{N}\widetilde{O}(X)$$

for a smooth $\Bbbk$-scheme X.

Definition 3.3.2. [The presheaf $\mathcal{K}_0^h$ of connected component of 0.] In 3.2.13, we have defined the hermitian K-theory presheaf $\mathcal{K}^h$ as $X \mapsto \mathscr{N}\mathcal{P}_{\hbar,0}(X)$ where category $\mathcal{P}_{\hbar,0}(X)$ has been defined in 3.2.12. Let $\mathcal{P}_{\hbar}^{+}(X)$ denote the full subcategory of $\mathcal{P}_{\hbar,0}(X)$ of the connected component of $(0,\,0)$: An object $((M,\,\phi),\,(N,\,\psi))$ of $\mathcal{P}_{\hbar,0}(X)$ belongs to the category $\mathcal{P}_{\hbar}^{+}(X)$ if and only if $(M,\,\phi)$ and $(N,\,\psi)$ are stably isometric in the sense that they become isometric after addition of some hyperbolic planes. We have a presheaf $X \mapsto \mathcal{P}_{\hbar}^{+}(X)$ of categories on $Sm/\Bbbk$. The simplicial presheaf $\mathcal{K}_0^h$ is defined by taking nerve of this presheaf, thus

$$\mathcal{K}_0^h(X) = \mathscr{N}\mathcal{P}_{\hbar}^{+}(X).$$

This is called the connected component of 0 of the hermitian K-theory.

For every commutative ring R, there is a functor $\gamma_n : \widetilde{O}(H^{[0,\,n]})(R) \to \mathcal{P}_{\hbar}^{+}(R)$ defined by sending the unique object to $(H^{[0,n]}(R),\,H^{[0,n]}(R))$ and, to an automorphism u of $H^{[0,n]}(R)$ to the morphism

$$((0,\,0) + (H^{[0,n]}(R),\,H^{[0,n]}(R))) \xrightarrow{\{u_1,\,u_2\}} (H^{[0,n]}(R),\,H^{[0,n]}(R))$$

in $\mathcal{P}_\hbar^+(R)$, where u_1 is the canonical isometry $0 + H^{[0,n]}(R) \xrightarrow{\simeq} H^{[0,n]}(R)$, and u_2 is the composite $u \circ u_1$. The functors γ_n, $n \geq 0$ give us a map of presheaves of categories $\widetilde{O} \to \mathcal{P}_\hbar^+$, and hence a map of simplicial presheaves (denoted by γ)

$$\gamma : BO = \mathscr{N}\widetilde{O} \to \mathscr{N}\mathcal{P}_\hbar^+ = \mathcal{K}_0^h.$$

Lemma 3.3.3. *The map γ is an $\mathbb{A}^1$-weak equivalence of the simplicial presheaves.*

Proof. We have a commutative diagram of the form

$$\begin{tikzcd} BO \arrow[r, "\gamma"] \arrow[d, "\mathbb{A}^1"] & \mathcal{K}_0^h \arrow[d, "\mathbb{A}^1"] \\ {|BO\Delta^\bullet|} \arrow[r, "\gamma_\bullet"] & {|\mathcal{K}_0^h\Delta^\bullet|} \end{tikzcd}$$

in which the vertical maps are $\mathbb{A}^1$-weak equivalences from [MV99, cor 2.3.8, p. 53], where the bisimplicial presheaf $BO\Delta^\bullet$ is defined in [MV99, sect 2.3.2] and $|\ |$ denotes it's realization and $\gamma_\bullet$ the induced map. We prove that the lower horizontal map in this diagram is a Zariski and hence an $\mathbb{A}^1$-weak equivalence, which will complete the proof of this lemma. For this we prove that for every local ring R the simplicial ring $\Delta_R^\bullet$ gives us a weak equivalence $|BO\Delta_R^\bullet| \to |\mathcal{K}_0^h\Delta_R^\bullet|$ of simplicial sets.

Since the map $\gamma(X) : BO(X) \to \mathcal{K}_0^h(X)$ is a homology isomorphism for every smooth $\Bbbk$-scheme X, see arguments in the proof of theorem 7.4 in [S96], page 152, the map $\gamma_\bullet : |BO\Delta_R^\bullet| \to |\mathcal{K}_0^h\Delta_R^\bullet|$ is a homology isomorphism.

The homology weak equivalence $\gamma_\bullet$ would be a global weak equivalence, if we prove that $|BO\Delta_R^\bullet|$ is nilpotent, since $|\mathcal{K}_0^h\Delta_R^\bullet|$ is an H-space with respect to the H-space structure induced from $\mathcal{K}^h$. This is proved in the following lemma. $\square$

Lemma 3.3.4. *The simplicial group $|BO\Delta_R^\bullet|$ is nilpotent.*

Proof. We have a universal principal fibration of the form $O\Delta^{\bullet}_{R} \to |EO\Delta^{\bullet}_{R}| \to |BO\Delta^{\bullet}_{R}|$. Since $|EO\Delta^{\bullet}_{R}|$ is contractible, the fibre $O\Delta^{\bullet}_{R}$ is the loop space of $|BO\Delta^{\bullet}_{R}|$ and therefore the action of fundamental group of $|BO\Delta^{\bullet}_{R}|$ on its higher homotopy groups is given by the action of π_0 by conjugation on higher homotopy groups in case of the simplicial group $O\Delta^{\bullet}_{R}$. Hence to prove nilpotency of $|BO\Delta^{\bullet}_{R}|$, it suffices to show triviality of π_0-action by conjugation on higher homotopy groups for $O\Delta^{\bullet}_{R}$.

To prove the triviality of π_0-action on itself by conjugation, we prove that π_0 is commutative. For terminology and notations used in rest of this proof, see [Bak, Chapters 3 and 4]. This follows by writing down the Moore sequence of the simplicial group $O\Delta^{\bullet}_{R}$: In fact, we get that $\pi_0(O\Delta^{\bullet}_{R}) = KQ_1(R,0)/\sim$, a quotient of the abelian group $KQ_1(R,0) = \mathrm{GQ}(R,0)/\mathrm{EQ}(R,0)$, where $\mathrm{EQ}(R,0)$ is the group of infinite symmetric elementary matrices: After identifying the first two terms of the Moore sequence with the groups of infinite orthogonal matrices $\mathrm{GQ}(R, o)$ and $\mathrm{GQ}(R[x],0)$, for a given elementary symmetric matrix $e_{i,j}(\lambda)$ in $\mathrm{EQ}(R,0)$, consider the elementary matrix $e_{i,j}(x-\lambda x)$ in $\mathrm{EQ}(R[x],0)$. Finally to see that the π_0-action on other homotopy groups is trivial, we remark that the group $\pi_n(O\Delta^{\bullet}_{R}),\ (n \geq 1)$ is a subquotient of the group $\mathrm{GQ}(R[x_1,...,x_n],0)$, and π_0-action is via a conjugation through certain elements of $\mathrm{GQ}(R,0)/\mathrm{EQ}(R,0)$. Since all the matrices involved can be written using only finitely many non-identity blocks, such a conjugation can be arranged in the form

$$\begin{pmatrix} I_r & O & O \\ O & (\alpha_{i,j})_s & O \\ O & O & I \end{pmatrix} \cdot \begin{pmatrix} B_r & O & O \\ O & I_s & O \\ O & O & I \end{pmatrix} \cdot \begin{pmatrix} I_r & O & O \\ O & (\alpha_{i,j})_s^{-1} & O \\ O & O & I \end{pmatrix}$$

through multiplication with elementary symmetric matrices, where $(\alpha_{i,j})_s$ is an $s \times s$ matrix in $\mathrm{GQ}(R,0)$ and B_r is an $r \times r$ matrix in $\mathrm{GQ}(R[x_1,...,x_n],0)$, the proof of the lemma is complete. $\square$

3.3.2 Grothendieck-Witt Groups and $\mathscr{F}hO$ in $\mathbb{A}^1$-Homotopy Category

Now we move on to relate the presheaf $\mathscr{F}hO$ with Grothendieck-Witt groups in the $\mathbb{A}^1$-homotopy category of smooth $\Bbbk$-schemes. This will be one important component of the $\mathbb{A}^1$-representability result. First, we recall definition of the presheaf GW_0 (3.3.5) of Grothendieck-Witt groups, and it's relation with the hermitian K-theory presheaf $\mathcal{K}^h$ in 3.3.6. We establish the important fact in corollary 3.3.15 that the Nisnevich sheafifications of $\pi_0^{\mathbb{A}^1}(\mathscr{F}hO)$ and GW_0 are isomorphic via the map $\hbar$.

Definition 3.3.5 (Grothendieck-Witt group of a ring)**.** Let R be a commutative ring. Let $\widehat{W}(R)$ be the set of isometry classes of finitely generated non-degenerate symmetric bilinear spaces over R. With respect to orthogonal sum (see [S85, Chap 2, sec 1]) $\widehat{W}(R)$ is an abelian monoid. The Grothendieck-Witt group of R is the group completion of the abelian monoid $\widehat{W}(R)$, it is denoted by $GW_0(R)$. In particular, for a field $\mathbb{K}$ we have defined the Grothendieck-Witt group $GW_0(\mathbb{K})$ of $\mathbb{K}$.

Remark 3.3.6. The set of isomorphism classes of objects of the monoidal category $\mathcal{S}_{R,0}$ is the set $\widehat{W}(R)$, see 3.2.12. Also, the monoidal functor on $\mathcal{S}_{R,0}$ induces the monoid structure on $\widehat{W}(R)$. Since $\mathcal{S}_{R,0} \xrightarrow{\text{can}} \mathcal{P}_{\hbar,0}(R)$ is a group completion, from the definition of $\mathcal{K}^h$ in 3.2.13, we see that

$$GW_0(R) = \pi_0(\mathcal{P}_{\hbar,0}(R)) = K_0^h(R).$$

Thus, the 0^{th}-hermitian K-group of a ring is it's Grothendieck-Witt group.

We have the presheaf $\widehat{W}$ on $Sm/\Bbbk$ defined by $X \mapsto \widehat{W}(\Gamma(X, \mathcal{O}_X))$, where $\widehat{W}(R)$ for a commutative ring R has been introduced earlier in 3.3.6 and discussed in [K90] in details. There is a surjective map of presheaves $Ob\mathcal{S} \twoheadrightarrow \widehat{W}$.

We have the maps of preseaves of sets on $Sm/\Bbbk$

$$\begin{tikzcd} \mathrm{Ob}\,\mathcal{S} \arrow[r, "="] \arrow[d, two heads] & [\ ,Ob\mathcal{S}] \arrow[d, two heads] \\ \widehat{W} & [\ ,Ob\mathcal{S}]_{\mathbb{A}^1_{\mathrm{nv}(\Bbbk)}} \end{tikzcd}$$

in which we claim that there is a map of presheaves $\widehat{W} \xrightarrow{\exists} [\quad ,Ob\mathcal{S}]_{\mathbb{A}^1_{\mathrm{nv}(\Bbbk)}}$ making the above diagram commute. We start with some technical results. We have chosen the $2n$-square matrix

$$h_n = h_1 \perp \ldots \perp h_1 = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & \cdots & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix}$$

to represent the hyperbolic space $H^n(R) = H^{[0,\,n-1]}(R)$, see the paragraph preceding the definition 3.1.9. We are going to use some results on matrix computations for hyperbolic spaces and their isometries from the book [K90]. In that book they have used the matrix

$$\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

to represent the n-dimensional hyperbolic space. The two representations are related by a change of base. All the computations in rest of this subsection are based on the basis

$$\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

Lemma 3.3.7. *For every commutative ring R, the maps*

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} : \mathrm{Spec}\, R \to \mathcal{G}l_{2n}$$

of presheaves are naively $\mathbb{A}^1$*-homotopic.*

Proof. The element $\begin{pmatrix} I & 0 \\ -TI & I \end{pmatrix}$ is an element of the group of $\mathcal{G}l_{2n}(R[T])$. This element defines a naive $\mathbb{A}^1$-homotopy

$$h : \operatorname{Spec} R \times \mathbb{A}^1 \to \mathcal{G}l_{2n}$$

between the two maps $\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$ and $\begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}$. □

Remark 3.3.8.

1. Similar to the naive homotopy defined in the above lemma, we have naive homotopies of the maps determined by the elements $\begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$ and $\begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}$ with the map determined by the matrix $\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$.

2. For an endomorphism $R^n \to R^n \simeq \operatorname{Hom}_{R-Mod}(R^n, R)$ written as a matrix γ with the property that $\gamma = {}^t\gamma$ when we consider X_- (defined below), and the property $\gamma = -{}^t\gamma$ when we consider X_+, the matrices

$$X_-(\gamma) = \begin{pmatrix} I & 0 \\ \gamma & I \end{pmatrix} \text{ and } X_+(\gamma) = \begin{pmatrix} I & \gamma \\ 0 & I \end{pmatrix}$$

define isometries of the hyperbolic space $H^{[0,\,n]}(R)$ by the lemma 4.2.1 in [K90]. The matrices $\begin{pmatrix} I & 0 \\ T\gamma & I \end{pmatrix}$ and $\begin{pmatrix} I & T\gamma \\ 0 & I \end{pmatrix}$ define naive homotopies between $\begin{pmatrix} I & 0 \\ \gamma & I \end{pmatrix}$ and the matrices $X_-(\gamma)$ and $X_+(\gamma)$ respectively considered as maps of presheaves $\operatorname{Spec} R \to O(H^{[0,n-1]})$.

We have defined the representable sheaf $\mathcal{G}l_n$ in 2.1.20. For a matrix α defining an automorphism of the R^n, the matrix

$$H(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & ({}^t\alpha)^{-1} \end{pmatrix}$$

defines an isometry of the hyperbolic space $H^{[0,\,n-1]}(R)$. This isometry factors as in the following diagram

$$\begin{tikzcd} \operatorname{Spec} R \arrow[rr, "H(\alpha)"] \arrow[dr, "\alpha"'] & & O(H^{[0,\,n-1]}) \\ & \mathcal{G}l_n \arrow[ur, "H"'] & \end{tikzcd}$$

as a map of presheaves. If two maps $\alpha, \beta : \operatorname{Spec} R \to \mathcal{G}l_n$ of presheaves are naively $\mathbb{A}^1$-homotopic, then the maps $H(\alpha)$ and $H(\beta)$ would also be naively $\mathbb{A}^1$-homotopic.

Lemma 3.3.9. *The map*

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} : \operatorname{Spec} R \to \mathcal{G}l_{2n}$$

is naively $\mathbb{A}^1$-homotopic to the identity map $I_{2n} : \operatorname{Spec} R \to \mathcal{G}l_{2n}$. Consequently, the map

$$H\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} : \operatorname{Spec} R \to O(H^{[0,n-1]})$$

is $\mathbb{A}^1$-homotopic to the identity map $I_{4n} : \operatorname{Spec} R \to O(H^{[0,n-1]})$

Proof. This follows from the discussion above, the lemma 3.3.7 and part (1) of the remark 3.3.8 and the following matrix identity

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \cdot \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}.$$

$\square$

The map

$$\mu : O(H^{[0,\infty]}) \times Ob\mathcal{S} \to Ob\mathcal{S}$$

defined by $(\sigma, (M, \phi)) \mapsto (\sigma M, \phi')$, where ϕ' denotes the new induced form on σM, is an action of $O(H^{[0,\infty]})$ on $Ob\mathcal{S}$. For every non-negative integer n, the natural injection of presheaves $O(H^{[0,n]}) \xrightarrow{\text{can}} O(H^{[0,\infty]})$ induces an action of $O(H^{[0,n]})$ on $Ob\mathcal{S}$ via the composition

$$O(H^{[0,n]}) \times Ob\mathcal{S} \xrightarrow{\text{can}\times I} O(H^{[0,\infty]}) \times Ob\mathcal{S} \xrightarrow{\mu} \times Ob\mathcal{S}.$$

We will denote this composition also by μ.

Proposition 3.3.10. *Let (M, ϕ) and (N, ψ) be two isometric direct factors of a hyperbolic space $H^{[0,\infty]}(R)$. Then the two maps $\operatorname{Spec} R \to Ob\mathcal{S}$ determined by (M, ϕ) and (N, ψ) are naively $\mathbb{A}^1$-homotopic. Consequently, the assignment sending the isometry class of (M, ϕ) in $\widehat{W}(R)$ to the naive $\mathbb{A}^1$-homotopy class of the map determined by (M, ϕ) in $[\operatorname{Spec} R, Ob\mathcal{S}]_{\mathbb{A}^1_{\mathrm{nv}(\Bbbk)}}$ is well-defined and yields a map of presheaves $\zeta : \widehat{W} \to [\ , Ob\mathcal{S}]_{\mathbb{A}^1_{\mathrm{nv}(\Bbbk)}}$ such that the diagram of presheaves*

$$\begin{tikzcd} \mathrm{Ob}\,\mathcal{S} \arrow[r, "="] \arrow[d, two heads] & {[\ , Ob\mathcal{S}]} \arrow[d, two heads] \\ \widehat{W} \arrow[r, "\zeta"] & {[\ , Ob\mathcal{S}]_{\mathbb{A}^1_{\mathrm{nv}(\Bbbk)}}} \end{tikzcd}$$

commutes.

Proof. For a large enough positive number n, we have the isometries $M \perp \bar{M} \xrightarrow{\simeq} H^{[0,n']} \xleftarrow{\simeq} N \perp \bar{N}$, where $^{-}$ denotes the orthogonal complement. We also have the isometries

$$\bar{N} \perp H^{[0,n']} = \bar{N} \perp M \perp \bar{M} \simeq \bar{N} \perp N \perp \bar{M} \simeq H^{[0,n']} \perp \bar{M} = \bar{M} \perp H^{[0,n']}.$$

Thus, the given isometry $M \simeq N$ together with this, provides us an isometry σ of $H^{[0,2n'+1]}$ which takes M to N. Therefore, we can assume that there exists an isometry, say $\sigma : H^{[0,n]} \to H^{[0,\,n]}$, such that $\sigma(M) = N$. But then the isometry $\sigma\perp\sigma^{-1} = \begin{pmatrix} \sigma & \\ & \sigma^{-1} \end{pmatrix}$ of $H^{[0,\,2n+1]}$ under the natural inclusion $H^{[0,\,n]} \hookrightarrow H^{[0,\,2n+1]}$ maps the direct factor $(M,\,\phi)$ of $H^{[0,\,2n+1]}$ to the direct factor $(N,\,\psi)$ of $H^{[0,\,2n+1]}$. Thus, the two maps $\operatorname{Spec} R \to [\ \ , Ob\,\mathcal{S}]$ determined by $(M,\,\phi)$ and $(N,\,\psi)$ factor as the following maps

$$\begin{tikzcd}
 & O(H^{[0,2n+1]}) \times [\ \ , Ob\,\mathcal{S}] \arrow[dr, "\mu"] & \\
\operatorname{Spec} R \arrow[ur, "{(1, M)}"'] \arrow[dr, "{(\sigma\perp\sigma^{-1}, M)}"] & & [\ \ , Ob\,\mathcal{S}]. \\
 & O(H^{[0,2n+1]}) \times [\ \ , Ob\,\mathcal{S}] \arrow[ur, "\mu"'] &
\end{tikzcd}$$

The proof of this proposition is complete in view of the following lemma. $\square$

Lemma 3.3.11. *The map of presheaves* $\operatorname{Spec} R \xrightarrow{\sigma\perp\sigma^{-1}} O(H^{[0,2n+1]})$ *is naively* $\mathbb{A}^1$-*homotopic to identity.*

Proof. Let $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be the matrix representation of the isometry σ of $H^{[0,\,n]}(R)$. Consider the matrices

$$A = \begin{pmatrix} 0 & \gamma \\ -{}^t\gamma & {}^t\gamma\alpha \end{pmatrix}, B = \begin{pmatrix} 0 & \beta \\ -{}^t\beta & {}^t\delta\beta \end{pmatrix},$$

$$e_{12}(\alpha) = \begin{pmatrix} I & \alpha \\ 0 & I \end{pmatrix}, \text{and } e_{21}(-{}^t\delta) = \begin{pmatrix} I & 0 \\ -{}^t\delta & I \end{pmatrix}.$$

The maps $e_{12}(\alpha)$ and $e_{21}(-{}^t\delta) : \operatorname{Spec} R \to \mathcal{G}l_{2n}$ are naively $\mathbb{A}^1$-homotopic to the identity map (an explicit homotopy similar to the one in 3.3.9 can be defined). The isometry $\sigma\perp\sigma^{-1}$ of $H^{[0,\,2n+1]}$ has the following formula as described in [K90], Chap IV, sect 4, lemma 4.4.6, p 347:

$$\sigma \perp \sigma^{-1} = X_{-}(A).\, H(e_{12}(\alpha)).\, H(e_{21}(-{}^{t}\delta).\, X_{+}(B).\, X_{-}(A). H(e_{12}(\alpha)).\, H \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

In view of lemmas 3.3.7 and 3.3.9 and the remark 3.3.8, we see that the map $\sigma \perp \sigma^{-1}$ is naively $\mathbb{A}^1$-homotopic to identity. $\square$

We have defined a full inclusion of presheaves of categories $\mathscr{F}\hbar^{[0,\infty]} \subset \mathcal{S}$ in 3.2.17 (when $r = 0$). Taking objects, we get presheaves of sets $\mathscr{F}h^{[0,\infty]}$ and $Ob\mathcal{S}$ on $Sm/\mathbb{k}$, and a map of presheaves $\mathscr{F}h^{[0,\infty]} \to Ob\mathcal{S}$. As proved in the following lemma the presheaf $Ob\mathcal{S}$ is a sheaf on affine smooth $\mathbb{k}$-schemes. It can be seen that the inclusion $\mathscr{F}h^{[0,\infty]} \to Ob\mathcal{S}$ induces an isomorphism on Zariski-stalks. Therefore, $Ob\mathcal{S}$ is a Zariski-sheafification of the presheaf $\mathscr{F}h^{[0,\infty]}$ when restricted to the affine $\mathbb{k}$-schemes. We have already discussed in 3.1.16 that the map $\mathscr{F}h^{[0,\infty]} \to Gr_{\mathbb{k}}(H^{[0,\infty]})$ is a Zariski sheafification. Therefore, we get a unique map $Ob\mathcal{S} \to Gr_{\mathbb{k}}(H^{[0,\infty]})$ of presheaves on $Sm/\mathbb{k}$, which is a Zariski weak equivalence, and so an $\mathbb{A}^1$-weak equivalence.

In the commutative diagram

$$\begin{tikzcd} \mathscr{F}h^{[0,\infty]} \arrow[rr] \arrow[dr, "\text{can}"'] & & Ob\mathcal{S} \arrow[dl, dashed, "\exists!\,\kappa"] \\ & Gr_{\mathbb{k}}(H^{[0,\infty]}) & \end{tikzcd}$$

every map is an $\mathbb{A}^1$-weak equivalence. Therefore, in the $\mathbb{A}^1$-homotopy category, we get the following maps of presheaves of sets on $Sm/\Bbbk$

$$\begin{tikzcd}
 & Ob\mathcal{S} = [\ ,Ob\mathcal{S}] \arrow[r] \arrow[dl] \arrow[d, two heads] & {[\ ,Gr_{\Bbbk}(H^{[0,\infty]})]} \arrow[d, two heads] \\
\widehat{W} \arrow[r, two heads, "\zeta"] \arrow[drr, "\delta"'] \arrow[dd, "\mathrm{can}"] & {[\ ,Ob\mathcal{S}]_{\mathbb{A}^1_{\mathrm{nv}}(\Bbbk)}} \arrow[r, "\kappa"] \arrow[dr, phantom, "\circlearrowleft"] & {[\ ,Gr_{\Bbbk}(H^{[0,\infty]})]_{\mathbb{A}^1_{\mathrm{nv}}(\Bbbk)}} \arrow[d, "pr"] \\
 & & {[\ ,Gr_{\Bbbk}(H^{[0,\infty]})]_{\mathbb{A}^1(\Bbbk)}} \arrow[d, "\simeq"] \\
{[\ ,\mathcal{K}^h]_{\mathbb{A}^1(\Bbbk)}} & & {[\ ,\mathscr{F}h^{[0,\infty]})]_{\mathbb{A}^1(\Bbbk)}} \arrow[ll, "\hbar"]
\end{tikzcd}$$

where $[\ ,\ ]$ denotes the presheaf of set of maps in $PShv(Sm/\Bbbk)$ with every $\Bbbk$-scheme thought of as a representable presheaf, and $[\ ,\ ]_{\mathbb{A}^1_{\mathrm{nv}}(\Bbbk)}$ the presheaf of set of naive $\mathbb{A}^1$-homotopy classes of maps in $PShv(Sm/\Bbbk)$, pr the natural projection map, and δ the composite map $pr \circ \kappa \circ \zeta$ making that trinagle commute.

Lemma 3.3.12. *The presheaf $Ob\mathcal{S}$ is a sheaf on the Zariski site of affine smooth $\Bbbk$-schemes. In particular, the map κ is an isomorphism of sheaves on the Zariski site of affine smooth $\Bbbk$-schemes.*

Proof. It suffices to prove the Zariski sheaf condition for an open cover consisting of the distinguished open sets of an affine $\Bbbk$-scheme. That is, in a finite type $\Bbbk$-algebra R given elements f_1, ..., f_n which generate the ideal R, we have to prove that

$$Ob\mathcal{S}(R) \longrightarrow \textstyle\prod_{i=1}^n Ob\mathcal{S}(R_{f_i}) \Longrightarrow \prod_{i,j=1}^n Ob\mathcal{S}(R_{f_i f_j})$$

is an equalizer diagram. This follows from interpreting this as a formalism for patching projective modules with forms. $\square$

Proposition 3.3.13. *The map δ induces an isomorphism of the Nisnevich stalks.*

Proof. We need to prove that the map $\widehat{W}(R) \xrightarrow{\delta_R} [\ ,Gr_{\Bbbk}(H^{[0,\infty]})]_{\mathbb{A}^1(\Bbbk)}(R)$ is an isomorphism for a henselian local ring R. In the diagram of sets

$$\begin{tikzcd}
\widehat{W}(R) \arrow[r, two heads, "\zeta_R"] \arrow[rrd, "\delta_R"] \arrow[dd, "\text{can}"] & {[\operatorname{Spec} R,\, Ob\mathcal{S}]_{\mathbb{A}^1_{\text{nv}}(\Bbbk)}} \arrow[r, "\kappa_R"] \arrow[d, phantom, "\circlearrowleft"] & {[\operatorname{Spec} R,\, Gr_{\Bbbk}(H^{[0,\infty]})]_{\mathbb{A}^1_{\text{nv}}(\Bbbk)}} \arrow[d, "pr_R"] \\
 & \, & {[\operatorname{Spec} R,\, Gr_{\Bbbk}(H^{[0,\infty]})]_{\mathbb{A}^1(\Bbbk)}} \arrow[d, "\simeq"] \\
{[\operatorname{Spec} R,\, \mathcal{K}^h]_{\mathbb{A}^1(\Bbbk)}} & & {[\operatorname{Spec} R,\, \mathscr{F}h^{[0,\infty]})]_{\mathbb{A}^1(\Bbbk)}} \arrow[ll, "\hbar_R"']
\end{tikzcd}$$

the map κ_R is an isomorphism for all rings (not just local), whereas the map pr_R is surjective for henselian local rings in view of the following lemma 3.3.14. Also, $[\operatorname{Spec} R,\, \mathcal{K}^h]_{\mathbb{A}^1(\Bbbk)}$ coincides with the Grothendieck-Witt group of R, and is the group completion of the monoid $\widehat{W}(R)$ via the map 'can'. Since for a local ring $\widehat{W}(R)$ is a cancellative monoid in view of Witt's cancellation theorem [S85, thm 6.5, p.21], therefore, for local rings the map 'can' is injective. This map factors as

$$\begin{tikzcd}
\widehat{W}(R) \arrow[rr, two heads, "\delta_R"] \arrow[d, "\text{can}"] & & {[\operatorname{Spec} R,\, Gr_{\Bbbk}(H^{[0,\infty]})]_{\mathbb{A}^1(\Bbbk)}\ .} \arrow[d, "\simeq"] \arrow[lld] \arrow[ld, phantom, "\circlearrowleft"] \\
GW_0(R) = {[\operatorname{Spec} R,\, \mathcal{K}^h]_{\mathbb{A}^1(\Bbbk)}} & & {[\operatorname{Spec} R,\, \mathscr{F}h^{[0,\infty]})]_{\mathbb{A}^1(\Bbbk)}} \arrow[ll, "\hbar_R"]
\end{tikzcd}$$

And hence, δ_R is injective as well for henselian local rings. Therefore, δ_R is an isomorphism for henselian local rings. Taking the colimit $\ldots \xrightarrow{H\perp} \widehat{W}(R) \xrightarrow{H\perp} \widehat{W}(R) \xrightarrow{H\perp} \ldots$ with respect to adding the hyperbolic plane, we get $GW_0(R)$. Also, taking colimit of the presheaves of sets $[\ ,Gr_{\Bbbk}(H^{[0,\infty]})]_{\mathbb{A}^1(\Bbbk)}$ and $[\ ,\mathscr{F}h^{[0,\infty]})]_{\mathbb{A}^1(\Bbbk)}$ with respect to adding hyperbolic planes we get the presheaves $[\ ,\mathcal{G}rO]_{\mathbb{A}^1(\Bbbk)}$ and $[\ ,\mathscr{F}hO]_{\mathbb{A}^1(\Bbbk)}$ respectively: This can be checked using an explicit $\mathbb{A}^1$-fibrant model. Similar remarks for the presheaf $[\ ,\mathcal{K}^h]_{\mathbb{A}^1(\Bbbk)}$. These colimits give us the commutative diagram

$$\begin{tikzcd}
& GW_0(R) \arrow[rr, "\simeq"] \arrow[d, "\simeq"] & & {[\operatorname{Spec} R,\, \mathcal{G}rO]_{\mathbb{A}^1(\Bbbk)}} \arrow[dll] \arrow[d, "\simeq"] \arrow[dl, phantom, "\circlearrowleft"] . \\
GW_0(R) = {[\operatorname{Spec} R,\, \mathcal{K}^h]_{\mathbb{A}^1(\Bbbk)}} & & & {[\operatorname{Spec} R,\, \mathscr{F}hO]_{\mathbb{A}^1(\Bbbk)}} \arrow[lll, "\hbar_R"']
\end{tikzcd}$$

which proves that the map $\hbar_R : \mathscr{F}hO(R) \to \mathcal{K}^h(R)$ is an isomoprhism of the Nisnevich stalks. $\square$

Lemma 3.3.14. *For a presheaf $\mathcal{X}$, and an henselian local ring R, the canonical map of presheaves*

$$[\operatorname{Spec} R,\, \mathcal{X}]_{\mathbb{A}^1_{\mathrm{nv}}(\Bbbk)} \to [\operatorname{Spec} R,\, \mathcal{X}]_{\mathbb{A}^1(\Bbbk)}$$

is surjective. In particular, the map pr_R in the above proposition is surjective.

Proof. Let $\mathcal{X} \to \mathcal{X}_f$ be an $\mathbb{A}^1$-fibrant resolution of the presheaf $\mathcal{X}$. Then in view of [MV99, cor. 2.3.22, p 57] in the commutative diagram

$$\begin{tikzcd}
{[\operatorname{Spec} R,\, \mathcal{X}]_{\mathrm{nv}}} \arrow[r, two heads] \arrow[d, "pr_R"] & {[\operatorname{Spec} R,\, \mathcal{X}_f]_{\mathrm{nv}}} \arrow[d, "\simeq"] \\
{[\operatorname{Spec} R,\, \mathcal{X}]_{\mathbb{A}^1(\Bbbk)}} \arrow[r, "\simeq"] & {[\operatorname{Spec} R,\, \mathcal{X}_f]_{\mathbb{A}^1(\Bbbk)}}
\end{tikzcd}$$

the upper horizontal map is surjective (other maps are natural ones). Therefore, the map pr_R is surjective as well. $\square$

A consequence of this proposition is the following important result.

Corollary 3.3.15. *The Nisnevich sheafification of the presheaf $\pi_0^{\mathbb{A}^1}\mathscr{F}hO$ is isomorphic to the Nisnenich sheafification of the presheaf GW_0 via the map $\hbar$. Thus, we have the induced isomorphism*

$$\hbar : a_{Nis}(\pi_0^{\mathbb{A}^1}\mathscr{F}hO) \to a_{Nis}(GW_0).$$

3.3.3 Some Properties of $\mathbb{A}^1$-Fibrations

In this subsection we discuss a set of properties that help us to recognize $\mathbb{A}^1$-fibrations in some situations.

Lemma 3.3.16. *For a map $f : \mathcal{X} \to \mathcal{Y}$ of $\mathbb{A}^1$-fibrant presheaves, the following are equivalent:*

1. *f is an $\mathbb{A}^1$-fibration.*
2. *f is a Nisnevich fibration.*
3. *f is a global fibration.*

Proof. The implications (1) $\implies$ (2) $\implies$ (3) follow from definitions. We prove implication (3) $\implies$ (1). Let

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d, "\simeq"', "\alpha"] \arrow[dr, phantom, "(o)"] & X \arrow[d, "g"] \\ \mathcal{B} \arrow[r] & Y \end{tikzcd}$$

be a commutative diagram in which α is an $\mathbb{A}^1$-acyclic cofibration, and g has right lifting property with respect to the global weak equivalences. We prove a lifting $B \to X$ in this diagram. In the pushout of the pair of maps $A \rightarrowtail B,\ A \to X$

$$\begin{tikzcd} \mathcal{A} \arrow[rr] \arrow[dd, "\simeq"', "\alpha"] & & X \arrow[dl, "\alpha'"] \arrow[dd, "g"] \\ & P \arrow[dr, "\exists!"] \arrow[l, phantom, "\text{push}"] & \\ \mathcal{B} \arrow[ur] \arrow[rr] & & Y \end{tikzcd}$$

the induced map α' is an $\mathbb{A}^1$-acyclic cofibration (being push-out of such a map), and the existence of a lift in the diagram (o) reduces to existence of a map $f : P \to X$. Thus, in diagram (o) we can assume that A is an $\mathbb{A}^1$-fibrant (since X is assumed to be $\mathbb{A}^1$-fibrant). We factor the map $B \to Y$ as an $\mathbb{A}^1$-acyclic cofibration followed by an $\mathbb{A}^1$-fibration and consider the diagram

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, "p\circ\alpha", "\simeq"'] \arrow[dd, "\alpha", "\simeq"'] & & X \arrow[dd, "g"] \\
\text{push} & B' \arrow[dr, "q"] & \\
B \arrow[ur, "p"', "\simeq"] \arrow[rr] & & Y
\end{tikzcd}$$

In this diagram B' is $\mathbb{A}^1$-fibrant, since B is $\mathbb{A}^1$-fibrant and q is $\mathbb{A}^1$-fibration. Hence the map $p \circ \alpha : A \to B'$ is an $\mathbb{A}^1$-weak equivalence between two $\mathbb{A}^1$-fibrant objects, and therefore a global weak equivalence. Thus a lift $B' \to X$ can be constructed.

□

Lemma 3.3.17. *Let $\mathcal{X}$ be a simplicial presheaf and A, B be two homotopy invariant Nisnevich sheaves on Sm/S. Taking $\mathcal{X}_f$ as the canonical fibrant replacement of $\mathcal{X}$, if $B \to A$ and $\mathcal{X}_f \to A$ are any two maps, then the canonical map*

$$\mathcal{X} \times_A B \to \mathcal{X}_f \times_A B$$

is an $\mathbb{A}^1$-weak equivalence.

Proof. In the commutative diagram

$$\begin{tikzcd}
X \times_A B \arrow[r] \arrow[d] & X_f \times_A B \arrow[r] \arrow[d] & B \arrow[d] \\
X \arrow[r, "\mathbb{A}^1-w.\,eq."] & X_f \arrow[r] & A
\end{tikzcd}$$

the two squares are pullback squares. Lemma will follow if we show that the map $B \to A$ is an $\mathbb{A}^1$-fibration. For, the middle vertical map would be an $\mathbb{A}^1$-fibration since it is a pullback of an $\mathbb{A}^1$-fibration; and therefore, the upper-left horizontal map would be an $\mathbb{A}^1$-weak equivalence being a pullback of an $\mathbb{A}^1$-weak equivalence along an $\mathbb{A}^1$-fibration since the model category structure on $\Delta^{\mathrm{op}}PShv(Sm/\mathbb{k})$ is right proper, see [MV99, Thm 2.2.7].

Thus to prove this lemma we need to prove that any map of homotopy invariant Nisnevich sheaves is an $\mathbb{A}^1$-fibration. First we recall that a homotopy invariant Nisnevich sheaf is $\mathbb{A}^1$-fibrant, see lemmas 2.2.8 (page 34) and 2.2.28 (page 43) in [MV99]. Therefore, in view of the lemma 3.3.16 above, it suffices to prove that any map of presheaves of sets is a global fibration: To prove this, consider the commutative diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r] \arrow[d, "g"'] & S \arrow[d] \\
\mathcal{Y} \arrow[r] & T
\end{tikzcd}$$

in which the left vertical map is a global weak equivalence of simplicial presheaves, and S and T are two presheaves (i.e., simplicial of dimension 0). We need to prove that there is a lift $\mathcal{Y} \to S$ for this diagram. To construct a lift, we only need to consider the fact that above diagram factors as a diagram of the form

$$\begin{tikzcd}
\mathcal{X} \arrow[rr] \arrow[dr] \arrow[ddd, "g"'] & & S \arrow[ddd] \\
& \pi_0 S \arrow[ur] \arrow[d, "\tilde{g}"', "\simeq"] & \\
& \pi_0 T \arrow[dr] & \\
\mathcal{Y} \arrow[ur] \arrow[rr] & & T
\end{tikzcd}$$

in which $\tilde{g}$ is an isomorphism, and hence a lift $\pi_0 T \to S$ can be defined. □

3.3.4 $\mathbb{A}^1$-Weak Equivalence of $\hbar$

In this final subsection of the chapter we prove that the map $\hbar$ defined in 3.2.19 is an $\mathbb{A}^1$-weak equivalence.

Theorem 3.3.18 ($\mathbb{A}^1$-Representability Theorem)**.** *The map $\hbar : \mathscr{F}hO \to \mathcal{K}^h$ is an $\mathbb{A}^1$-weak equivalence.*

Proof. The commutative diagram of presheaves

$$\begin{tikzcd}
\coprod_{n\geq 0} Gr(H^n,\, H^{[0,\infty]}) \arrow[r] \arrow[d] \arrow[dr, phantom, "(1)"] & \coprod_{n\geq 0} \mathscr{F}h_n^{[0,\infty]} \arrow[d] \\
\coprod_{n\geq 0} pt \arrow[r, "H"] & \coprod_{n\geq 0} a_{Nis}(\pi_0^{\mathbb{A}^1}\mathscr{F}h_n^{[0,\infty]})
\end{tikzcd}$$

is a pullback diagram, where $a_{Nis}(\pi_0^{\mathbb{A}^1}\mathscr{F}h_n^{[0,\infty]})$ denotes the Nisnevich sheafification of the presheaf $\pi_0^{\mathbb{A}^1}\mathscr{F}h_n^{[0,\infty]}$, and $\coprod_{n\geq 0} pt$ is the constant Nisnevich sheaf. The constant sheaf $\coprod_{n\geq 0} pt$ is homotopy invariant; also, the sheaf $a_{Nis}(\pi_0^{\mathbb{A}^1}\mathscr{F}h_n^{[0,\infty]})$ which we have identified with the Nisnevich sheafification of the presheaf GW_0 of Grothendieck-Witt groups in 3.3.15, is homotopy invariant. Therefore, this pullback diagram is homotopy cartesian diagram in view of lemma 3.3.17.

We have proved in corollary 3.1.25 that there is a zigzag of $\mathbb{A}^1$-weak equivalences of presheaves $\coprod_{n\geq 0} BO(H^{n,[0,\infty]}) - - - \coprod_{n\geq 0} Gr(H^n,\, H^{[0,\infty]})$ (here the new notation instead of $BO(H^n)$ refers to our consideration of automorphisms of the hyperbolic spaces indexed on non-negative integers). Similar remarks for $\coprod_{n\geq 0} BO(H^{n,[-r,\infty]})$, where r is a positive integer. We have maps of presheaves $BO(H^{n,[-r,\infty]}) \to BO(H^{n+1,[-r-1,\infty]})$ induced by addition of hyperbolic space for the extra generator. For every integer r the colimit of the system

$$\ldots \xrightarrow{H\perp} BO(H^{n,[-r,\infty]}) \xrightarrow{H\perp} BO(H^{n+1,[-r-1,\infty]}) \xrightarrow{H\perp} \ldots$$

is isomorphic to BO, cf. 3.3.1. There is an $\mathbb{A}^1$-weak equivalence $BO \to \mathcal{K}_0^h$ in view of 3.3.1. Taking homotopy colimit of the diagram (1) with respect to addition of hyperbolic space $\perp H$ (as in the definition 3.2.1), we get the homotopy cartesian square

$$\begin{tikzcd}
\mathbb{Z}\times BO \arrow[r] \arrow[d] & \mathcal{G}rO \arrow[d] \\
\mathbb{Z} \arrow[r, "H"] & a_{Nis}(\pi_0^{\mathbb{A}^1}\mathcal{G}rO)
\end{tikzcd}$$

This square maps to the homotopy cartesian square

$$\begin{tikzcd}
\mathbb{Z} \times \mathcal{K}_0^h \arrow[r] \arrow[d] & \mathcal{K}^h \arrow[d] \\
\mathbb{Z} \arrow[r] & a_{Nis}GW_0
\end{tikzcd}$$

via the maps described in 3.2.19, 3.3.1 and 3.3.15. We have proved in 3.3.1 and 3.3.15 that all but possibly $\hbar$ in this map of squares are $\mathbb{A}^1$-weak equivalences. We claim that $\hbar$ is also an $\mathbb{A}^1$-weak equivalence. We have a map of fibrations

$$\begin{tikzcd}
BO \arrow[r, "\gamma"] \arrow[d] & \mathcal{K}_0^h \arrow[d] \\
\mathcal{G}rO \arrow[r, "\hbar"] \arrow[d] & \mathcal{K}^h \arrow[d] \\
a_{Nis}(\pi_0^{\mathbb{A}^1}\mathcal{G}rO) \arrow[r, "\zeta"] & a_{Nis}GW_0
\end{tikzcd}$$

where the maps γ, $\hbar$ and ζ have been defined in 3.3.1, 3.2.19 and 3.3.15, and we have proved that the maps γ and ζ are $\mathbb{A}^1$-weak equivalences. Since $\mathcal{G}rO$ and $\mathcal{K}^h$ are H-spaces, comparing the homotopy groups at Nisnevich stalks proves that the map $\hbar$ is also an $\mathbb{A}^1$-weak equivalence. $\square$

Chapter 4

A Realization Functor in $\mathbb{A}^1$-Homotopy Theory

In this chapter we will restrict our attention to the category of smooth schemes over the fields of real numbers. For an easy reference, we have recalled the definition of the set of real (and complex) points in 4.1.1 and proved that this can be given structure of a smooth manifold in 4.1.3. We have defined a functor $\rho_* : Top \to \Delta^{op}PShv(Sm/\mathbb{R})$ in 4.2.1 which sends a weak equivalence of topological spaces to a global and hence an $\mathbb{A}^1$-weak equivalence of presheaves, and hence, induces a functor from the ordinary homotopy category $HoTop$ of topological spaces to the $\mathbb{A}^1$-homotopy category of smooth $\mathbb{R}$-schemes, $\mathscr{H}(\mathbb{R})$. Thus, we can take the right derived functor $R\rho_*$ to be ρ_*. We have proved that for a topological space S, the simplicial presheaf $\rho_*(S)$ is homotopy invariant (4.2.3), it has BG-property (4.2.5 and 4.2.7), and it is $\mathbb{A}^1$-local (4.2.11). Then we have defined the functor ρ^* in 4.3.2 and proved that this is left adjoint of ρ_* in 4.3.3. In 4.3.7 we have defined it's left derived functor $L\rho^*$.

The results in this chapter are basic for understanding the realizations of the hermitian K-theory. In the next chapter we have commented on these realizations.

4.1 The Set of Real Points of a Smooth $\mathbb{R}$-Scheme

Definition 4.1.1 (Real and Complex points of a $\mathbb{R}$-scheme)**.** Let $X \in Sm/\mathbb{R}$ be a smooth scheme over $\mathbb{R}$. A point $x \in X$ is called a real point of X, if the function field $\kappa(x)$ of x is isomorphic to the field $\mathbb{R}$ of real numbers. The set of all real points of X is denoted by $X_{\mathbb{R}}$. Similarily we have the complex points, the set of complex points of X is denoted by $X_{\mathbb{C}}$.

We will now invesitgate the set of real points of a smooth scheme over $\mathbb{R}$, and describe a topology for it. First assume that $X = \mathrm{Spec}(\mathbb{R}[T_1, ..., T_n]/\ \langle f_1, ..., f_m\rangle)$ is affine $\mathbb{R}$-scheme (we are not assuming smoothness of X right now). Then we can identify the set of real points of X with the set of common zeros of the polynomials $f_1, ..., f_m$ in $\mathbb{R}^n$: Since a real point of X corresponds to a commutative triangle

$$\begin{tikzcd}
\mathrm{Spec}\ \mathbb{R} \arrow[r] \arrow[d, "\mathrm{Id}"'] & \mathrm{Spec}\ (\mathbb{R}[T_1, ..., T_n]/\langle f_1, ..., f_m\rangle) \arrow[dl] \\
\mathrm{Spec}\ \mathbb{R} &
\end{tikzcd}$$

whence all real points of X are in one to one correspondence with the set of all $\mathbb{R}$-algebra homomorphisms $\mathbb{R}[T_1, ..., T_n]/\langle f_1, ..., f_m\rangle \to \mathbb{R}$, which precisely is the zero set of polynomials $f_1, ..., f_m$ in $\mathbb{R}^n$. Therefore,

Lemma 4.1.2. *In case X is an affine $\mathbb{R}$-scheme, the set $X_\mathbb{R}$ of its real points is a closed subset of an Euclidean space $\mathbb{R}^n$ for some positive integer n.*

With this defintion of the space of real points of an affine $\mathbb{R}$-scheme X, we next claim that the topology of $X_\mathbb{R}$ does not depend on the ring used to represent the affine scheme: To see this, suppose an affine scheme X is considered as spectrum of two rings $\mathbb{R}[T_1, ..., T_n]/\langle f_1, ..., f_m\rangle$ and $\mathbb{R}[S_1, ..., S_r]/\langle g_1, ..., g_s\rangle$. Then these two rings would be isomorphic, and using the ring $\mathbb{R}[T_1, ..., T_n, S_1, ..., S_r]/\mathcal{I}$, where $\mathcal{I}$ is an ideal of $\mathbb{R}[T_1, ..., T_n, S_1, ..., S_r]$ and isomorphisms of the rings $\mathbb{R}[T_1, ..., T_n]/\langle f_1, ..., f_m\rangle$ and $\mathbb{R}[S_1, ..., S_r]/\langle g_1, ..., g_s\rangle$ into this new ring $\mathbb{R}[T_1, ..., T_n, S_1, ..., S_r]/\mathcal{I}$ obtained from the two rings by adding some more variables, we can assume that the two rings in different representations of an affine scheme vary only in number of vairables in the sense that one of starting with a ring, the other one is obtained by adding some more variables (and some relations).

So let us assume that $X = \text{spec } A = \text{spec } \mathbb{R}[T_1, ..., T_n]/\mathcal{I} = \text{spec } \mathbb{R}[T_1, ..., T_n, S_1, ..., S_m]/\mathcal{J}$. Then we have the following commutative diagram of rings:

$$\begin{tikzcd}
A & \mathbb{R}[T_1, ..., T_n] \arrow[l, "\mathrm{surj}"'] \\
& \mathbb{R}[T_1, ..., T_n, S_1, ..., S_m] \arrow[ul, "\mathrm{surj}", "\alpha"'] \arrow[u, "p"']
\end{tikzcd}$$

where the vertical homomorphism p is defined by $T_i \mapsto T_i,$ and $S_i \mapsto p_j(T_1, ..., T_n)$, p_j's being polynomials determined by the image of S_j under α. Note that we have many choices for the polynomials p_j's, but the point is that there is at least one such choice available, making above diagram commute. This diagram of commutative rings gives rise to the following diagram of real points of the corresponding affine schemes

$$\begin{tikzcd}
(\text{Spec } A)_{\mathbb{R}} \arrow[r] \arrow[d, "\alpha"] & \mathbb{R}^n \arrow[dl] \\
\mathbb{R}^{n+m} &
\end{tikzcd}$$

in which all the three maps are closed immersions: the vertical map is $(x_1, ..., x_n) \mapsto (x_1, ..., x_n, p_1(x_1, ..., x_n), ..., p_m(x_1, ..., x_n))$ (which can be checked to be a closed map by the limit point criterion for closed sets using the fact that polynomials are continuous maps). This means that the topology of $(\text{Spec } A)_{\mathbb{R}}$ as a closed subset of an Euclidean space is well defined.

Next, we claim that the construction of sending an affine real scheme to its space of real points is a functor from the category of affine real schemes to the the category of topological spaces, for which we need to prove the continuity of the induced map on the space of real points for a map of affine schemes. Let $\alpha : \text{Spec } B \to \text{Spec } A$ be a map of affine schemes. We can represent this situation in the diagram

$$\begin{tikzcd}
A \arrow[rr] & & B \\
\mathbb{R}[X_1, ..., X_n] \arrow[u] \arrow[r] & \mathbb{R}[X_1, ..., X_n, Y_1, ..., Y_m] \arrow[u] &
\end{tikzcd}$$

in which the lower horizontal map sends X_i to X_i. After taking the real points this diagram gives us the following maps:

$$\begin{tikzcd}
(\text{Spec } A)_{\mathbb{R}} \arrow[d] & (\text{Spec } B)_{\mathbb{R}} \arrow[l, "\alpha"'] \arrow[d] \arrow[dl] \\
\mathbb{R}^n & \mathbb{R}^{n+m} \arrow[l]
\end{tikzcd}$$

in which the diagonal map is continuous being composition of the right vertical closed immersion and the lower horizontal projection. This will prove continuity of the map α, since the diagonal map factors throught α and the left vertical closed immersion.

For a general $\mathbb{R}$-scheme X such a topology coming from Euclidean spaces on its set of real points can be given by using an open affine cover (X_α) of X, and using the topologies of $X_{\alpha_{\mathbb{R}}}$'s as described above to get a basis for the topology of $X_{\mathbb{R}}$: To be more explicit, a real point $x \in X$ lies in some open affine set X_α of X, and hence a basic open neighborhood of the point x in this topology would be an open neighborhood of the point x in the Euclidean topology of $X_{\alpha_{\mathbb{R}}}$.

We will use this topology on the set of real points of $\mathbb{R}$-schemes in rest of the work. For smooth $\mathbb{R}$-scheme X the topological space $X_{\mathbb{R}}$ is in fact a C^∞-manifold:

Lemma 4.1.3. *For a smooth scheme $X \in Sm/\mathbb{R}$, the topological space $X_{\mathbb{R}}$ of real points of X is a C^∞-manifold.*

Proof. For smooth $\mathbb{R}$-schemes, the space of real points is locally a zero set of a finite set of polynomials in some polynomial ring $\mathbb{R}[T_1, ..., T_n]$ whose jacobian has maximal rank. Therefore, it's C^∞-manifold by implicit function theorem. $\square$

4.2 The Functor ρ_* and Some Properties

Definition 4.2.1 (The functor ρ_*)**.** The functor $\rho_* : Top \to \Delta^{op}PShv(Sm/\mathbb{R})$ sends a topological space S to the presheaf of simplicial sets $\mathfrak{Map}$ ($\mathfrak{Sing}$ $-_{\mathbb{R}}, \mathfrak{Sing}$ S), and sends a continuos map to the one induced on presheaves defined above; where $\mathfrak{Sing} : Top \to \Delta^{op}Set$ is the singular simplicial set functor, and for a given simplicial set $\mathcal{T}$, $\mathfrak{Map}$ ($\mathfrak{Sing}$ $-_{\mathbb{R}}, \mathcal{T}$) is the functor that sends a smooth scheme X to the internal mapping simplicial set $\mathfrak{Map}$ ($\mathfrak{Sing}$ $X_{\mathbb{R}}, \mathcal{T}$) (usually we will drop 'internal' and call it just the mapping simplicial set, see reference in the *note* below).

Remark 4.2.2. For the sake of completeness let's recall the definition of the internal mapping simplicial set (this has been called the function complex in reference cited here) [GJ, sec. I.5]: For two given simplicial sets X and Y, its a simplicial set denoted by $\mathfrak{Map}$ (X, Y) (written as **Hom** (X, Y) in [GJ]) and having the set $Hom_{\Delta^{op}Set}(X \times \Delta^n, Y)$ as its set of n-simplices, and the boundary and face maps induced by those of the standard cosimplical simplical set defined using the simplicial sets $\Delta^n = Hom_{\mathrm{Cat}}(-, \underline{n})$ (Δ^n is called the simplicial n-simplex, $n \geq 0$).

Lemma 4.2.3. *For a smooth scheme X and a topological space S, the map*

$$\rho_*(S)(X) \to \rho_*(S)(X \times \mathbb{A}^1)$$

induced by natural projection is a weak equivalence of simplicial sets.

Proof. Lemma follows by noting that for any $\mathbb{R}$-scheme X, $(X \times \mathbb{A}^1)_{\mathbb{R}} = X_{\mathbb{R}} \times \mathbb{R}$; and, the map induced by the projection $X \times \mathbb{A}^1 \to X$ is the natural projection. $\square$

Definition 4.2.4. A cartesian square of $\mathbb{R}$-schemes (that is, pullback square in the category of schemes)

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "(*)"] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd}$$

is called a Nisnevich square or a distinguished square, if i is an open immersion and p is a étale morphism of schemes such that the induced map $(V - p^{-1}(U))_{red} \to (X - U)_{red}$ of reduced schemes is an isomorphism.

Definition 4.2.5. A simplicial presheaf $\mathfrak{X} \in \Delta^{op}PShv(\nu)$ is said to have Brown-Gersten (BG) propery, if for every Nisnevich square $(*)$ in the definition 4.2.4, the square $\mathfrak{X}(*)$ below is homotopy cartesian

$$\mathfrak{X}(*) \qquad \begin{array}{ccc} \mathfrak{X}(X) & \longrightarrow & \mathfrak{X}(V) \\ \downarrow & & \downarrow \\ \mathfrak{X}(U) & \longrightarrow & \mathfrak{X}(W) \end{array}$$

and, $\mathfrak{X}(\phi) = pt$.

Lemma 4.2.6. *Assume that a simplicial presheaf $\mathfrak{X}$ has BG-property and, for every $X \in Sm/\mathbb{k}$ the natural projection $X \times \mathbb{A}^1 \to X$ induces a weak equivalence of simplicial sets $\mathfrak{X}(X) \tilde{\to} \mathfrak{X}(X \times \mathbb{A}^1)$. Then $\mathfrak{X}$ is $\mathbb{A}^1$-local. Also, for every $\mathfrak{T} \in \Delta^{op}PShv(\nu)$,*

$$[\mathfrak{T}, \mathfrak{X}]_{\mathscr{H}(\mathbb{k})} = [\mathfrak{T}, \mathfrak{X}]_{\mathscr{H}(Nis)} = [\mathfrak{T}, \mathfrak{X}]_{\mathscr{H}(global)}.$$

Proof. Let $\sim$ denote the equivalence relation of Nisnevich simplicial homotopy. Then

$$[X, \mathfrak{X}]_{\mathscr{H}(Nis)} = \mathrm{Hom}_{\Delta^{op}PShv(Sm/\mathbb{k})}(X, \mathfrak{X}_{\mathrm{Nis},fib})/\sim$$

where $\mathfrak{X}_{\mathrm{Nis},fib}$ is a Nisnevich fibrant replacement of $\mathfrak{X}$. But

$$\mathrm{Hom}_{\Delta^{op}PShv(Sm/\mathbb{k})}(X, \mathfrak{X}_{\mathrm{Nis},fib})/\sim \; = \pi_0(\mathfrak{X}_{\mathrm{nis},fib}(X)),$$

which by definition equals $\pi_0(\mathfrak{X}(X))$. From the assumption on weak equivalence of simplicial sets $\mathfrak{X}(X)\tilde{\to}\mathfrak{X}(X \times \mathbb{A}^1)$, we also have that

$$\pi_0(\mathfrak{X}(X)) = \pi_0(\mathfrak{X}(X \times \mathbb{A}^1)),$$

which by a similar computation equals $= [X \times \mathbb{A}^1, \mathfrak{X}]_{\mathscr{H}(\Bbbk)}$. This proves that $\mathfrak{X}$ is $\mathbb{A}^1$-local. (It should be noted that we have not used the assumption that $\mathfrak{X}$ has BG-property).

The later half of the lemma follows from results in [MV99] once we know that it's true for representable ones. □

Lemma 4.2.7. *For any topological space S, $\rho_*(S)$ has BG-property.*

Proof. We need to prove that given any distinguished square $(*)$ as in the definition 4.2.4, the square $\rho_*(S)(*)$ below

$$\begin{array}{ccc}
\mathfrak{Map}\ (\mathfrak{Sing}\ X_{\mathbb{R}}, \mathfrak{Sing}\ S) & \longrightarrow & \mathfrak{Map}\ (\mathfrak{Sing}\ V_{\mathbb{R}}, \mathfrak{Sing}\ S) \\
\downarrow & & \downarrow \\
\mathfrak{Map}\ (\mathfrak{Sing}\ U_{\mathbb{R}}, \mathfrak{Sing}\ S) & \longrightarrow & \mathfrak{Map}\ (\mathfrak{Sing}\ W_{\mathbb{R}}, \mathfrak{Sing}\ S)
\end{array}$$

is homopty cartesian. This will follow from the fact that the square $(*_{\mathbb{R}})$:

$$(*_{\mathbb{R}}) \qquad \begin{array}{ccc}
W_{\mathbb{R}} & \xrightarrow{i^*_{\mathbb{R}}} & V_{\mathbb{R}} \\
\downarrow{\scriptstyle p^*_{\mathbb{R}}} & & \downarrow{\scriptstyle p_{\mathbb{R}}} \\
U_{\mathbb{R}} & \xrightarrow{i_{\mathbb{R}}} & X_{\mathbb{R}}
\end{array}$$

is homotopy cocartesian, which we prove in lemma 4.2.10, and from the facts listed below with some justifications and references for the same

1. The functor $\mathfrak{Sing}$ takes homotopy cocartesian squares of topological spaces to homotopy cartesian squares of simplicial sets, since its part of a Quillen equivalence, and

2. The functor $\mathfrak{Map}\,(-,\mathcal{T})$ treated as an endofunctor of the category of simplicial sets, takes a homotopy cocartesian squares to a homotopy cartesian squares when $\mathcal{T}$ is a Kan simplicial set: In our case the simlpicial set $\mathfrak{Sing}\ S$ is a Kan simplicial set.

□

Let's recall a result on weak equivalences of topological spaces from [DI04, corollary 2.3].

Lemma 4.2.8. *Let $f : X \to Y$ be a map of topological spaces, and let $\mathscr{U} = \{U_\alpha\}$ be an open cover of Y such that for all finite set of indices σ, $f^{-1}U_\sigma \to U_\sigma$ is a weak equivalence. Then $f : X \to Y$ is also a weak equivalence.*

Lemma 4.2.9. *For a Nisnevich square $(*)$, the square $(*_{\mathbb{R}})$ is bicartesian, that is, it is a*

1. *pushout (i.e., cocartesian) square, and hence there is a homeomorphism from the quotient space $U_{\mathbb{R}} \coprod_{W_{\mathbb{R}}} V_{\mathbb{R}}$ to the the topological space $X_{\mathbb{R}}$.*
2. *pullback (i.e., cartesian) square, and hence there is a homeomorphism of topological spaces $(U \times_X V)_{\mathbb{R}} \simeq U_{\mathbb{R}} \times_{X_{\mathbb{R}}} V_{\mathbb{R}}$*

Proof. (1) First observe that the map $U_{\mathbb{R}} \coprod V_{\mathbb{R}} \to X_{\mathbb{R}}$ is a local isomorphism and surjective, since it is the map of real points of an étale map. Hence, it is a topological quotient map. Also, the map $U_{\mathbb{R}} \coprod_{W_{\mathbb{R}}} V_{\mathbb{R}} \to X_{\mathbb{R}}$ is bijective and continuous. This gives us a sequence of compositions $U_{\mathbb{R}} \coprod V_{\mathbb{R}} \to U_{\mathbb{R}} \coprod_{W_{\mathbb{R}}} V_{\mathbb{R}} \to X_{\mathbb{R}}$ in which the first map is a quotient map and so is the composition. Informations in the last two sentences prove that in fact the map $U_{\mathbb{R}} \coprod_{W_{\mathbb{R}}} V_{\mathbb{R}} \to X_{\mathbb{R}}$ is also open, and hence a homeomorphism.

(2) It suffices to prove the lemma in case all the schemes U, V and X are affine, with

closed immersions $X_{\mathbb{R}} \hookrightarrow \mathbb{R}^n$, and $V_{\mathbb{R}} \hookrightarrow \mathbb{R}^m$, for some non-negative integers $n,\ m$, $U_{\mathbb{R}}$ being an open set of $X_{\mathbb{R}}$. There is a natural map $(U \times_X V)_{\mathbb{R}} \xrightarrow{\alpha,\ \text{say}} U_{\mathbb{R}} \times_{X_{\mathbb{R}}} V_{\mathbb{R}}$; this map is also a bijection (follows simply by recalling that the domain of this map is in bijection with all the possible factorizations of the identity map of Spec $\mathbb{R}$ as Spec $\mathbb{R} \to U \times_X V \to$ Spec $\mathbb{R}$. Appropriately modifying rings representing the schemes under consideration, we can fit the map α into a commutative triangle

$$\begin{tikzcd}
(U \times_X V)_{\mathbb{R}} \arrow[r, "\alpha"] \arrow[d] & U_{\mathbb{R}} \times_{X_{\mathbb{R}}} V_{\mathbb{R}} \arrow[dl, "j"] \\
\mathbb{R}^{n+m} &
\end{tikzcd}$$

where the vertical map is a closed immersion defining the topology of the space of real points of the scheme $U \times_X V$, and the map j is the obvious inclusion map. With these notations, note that j is also a closed immersion, since it is pullback of the closed immersion the diagonal map $\Delta : X_{\mathbb{R}} \to X_{\mathbb{R}} \times X_{\mathbb{R}}$ in the following pullback square

$$\begin{tikzcd}
(U \times_X V)_{\mathbb{R}} \arrow[r, "\alpha"] \arrow[d] & U_{\mathbb{R}} \times_{X_{\mathbb{R}}} V_{\mathbb{R}} \arrow[d] \\
X_{\mathbb{R}} \arrow[r, "\Delta"] & X_{\mathbb{R}} \times X_{\mathbb{R}}
\end{tikzcd}$$

proving the required homeomorphisms. □

Lemma 4.2.10. *For a distinguished square* $(*)$, *the square* $(*_{\mathbb{R}})$ *mentioned in the lemma 4.2.7 is homotopy cocartesian.*

Proof. This is proved by showing that the map α in the diagram below induced from the inner homotopy pushout square is a weak equivalence.

$$\begin{tikzcd}
W_{\mathbb{R}} \arrow[r] \arrow[d] & V_{\mathbb{R}} \arrow[d] \arrow[ddr, bend left] & \\
U_{\mathbb{R}} \arrow[r] \arrow[drr, bend right] & \text{htpy colim} \arrow[dr, "\alpha"] & \\
& & X_{\mathbb{R}}
\end{tikzcd}$$

which in turn will follow from the assertions below, lemma 4.2.8 cited above and induction

1. In the square $(*_{\mathbb{R}})$, if either $p_{\mathbb{R}}$ or $i_{\mathbb{R}}$ has a section, then α is a weak equivalence;
2. α is a weak equivalence if p (for any open immersion i) is an open immersion; and
3. there exists a finite open cover $\{X_\lambda\}$ of X such that in the following two squares, the squares $(*_\lambda)$ are Nisnevich squares; and for the the squares $(*_{\lambda_{\mathbb{R}}})$, at least one of the pevious two assertions holds good. Where $p|_{X_\lambda}$ and $i|_{X_\lambda}$ are written simply as p_λ and i_λ in the diagram on the right:

$$\begin{tikzcd}
W|_{X_\lambda} \arrow[r, "i^*_\lambda"] \arrow[d, "p^*_\lambda"] \arrow[dr, phantom, "(*_\lambda)"] & V|_{X_\lambda} \arrow[d, "p_\lambda"] \\
U \cap X_\lambda \arrow[r, "i_\lambda"] & X_\lambda
\end{tikzcd}
\qquad
\begin{tikzcd}
W_{\mathbb{R}}|_{U_\lambda} \arrow[r, "i^*_{\lambda_{\mathbb{R}}}"] \arrow[d, "p^*_{\lambda_{\mathbb{R}}}"] \arrow[dr, phantom, "(*_{\lambda_{\mathbb{R}}})"] & V_{\mathbb{R}}|_{U_\lambda} \arrow[d, "p_{\lambda_{\mathbb{R}}}"] \\
U_{\mathbb{R}} \cap U_{\lambda_{\mathbb{R}}} \arrow[r, "i_{\lambda_{\mathbb{R}}}"] & U_{\lambda_{\mathbb{R}}}
\end{tikzcd}$$

To complete the proof of this lemma we need to verify these three facts stated above. For the first fact, since the bicartesian square $(*_{\mathbb{R}})$ is homotopy cartesian when $V \to X$ has a section: Because, this section induces a section of $W \to U$ such that the diagram on left

$$\begin{tikzcd}
U_{\mathbb{R}} \arrow[r] \arrow[d] & W_{\mathbb{R}} \arrow[d] \\
X_{\mathbb{R}} \arrow[r] & V_{\mathbb{R}}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
W_{\mathbb{R}} \arrow[r] \arrow[d] & V_{\mathbb{R}} \arrow[d] \\
W_{\mathbb{R}} \arrow[r] & V_{\mathbb{R}}
\end{tikzcd}$$

commutes, and this diagram is retract of the diagram on the right. But right diagram is homotopy cocartesian, and so would be the retract diagram on the left. Then in the diagram

$$\begin{tikzcd} U_{\mathbb{R}} \arrow[r] \arrow[d] & W_{\mathbb{R}} \arrow[r] \arrow[d] & U_{\mathbb{R}} \arrow[d] \\ X_{\mathbb{R}} \arrow[r] & V_{\mathbb{R}} \arrow[r] & X_{\mathbb{R}} \end{tikzcd}$$

the outer square and the one on the left are both homotopy cartesian. Therefore the square on the right would also be homotopy cartesian. This proves part (1) of list of claims. For the second assertion when p is an open immersion (just a reminder that i is always an open immersion), we note that in this case the space $X_{\mathbb{R}}$ is the quotient of two of its open subsets $V_{\mathbb{R}}$ and $U_{\mathbb{R}}$, obtained by gluing them along the common open set $U_{\mathbb{R}} \cap V_{\mathbb{R}} = W_{\mathbb{R}}$. Also, recall that a homotopy colimit object of the diagram $(*_{\mathbb{R}})$ can be taken to be the topological space $U_{\mathbb{R}} \coprod_{p^*_{\mathbb{R}}} (I \times W_{\mathbb{R}}) \coprod_{i^*_{\mathbb{R}}} V_{\mathbb{R}}$, where I is the unit interval $[0,1]$, and $\coprod$ denotes the the mapping space topological space along the indicated map, and here we have used the two ends of the topological space $I \times W_{\mathbb{R}}$ to get and glue the two mapping spaces. To prove that the square $(*_{\mathbb{R}})$ is homotopy cocartesian, we prove that the map denoted by α and induced by the diagram $(*_{\mathbb{R}})$, namely, $\alpha : U_{\mathbb{R}} \coprod_{p^*_{\mathbb{R}}} (I \times W_{\mathbb{R}}) \coprod_{i^*_{\mathbb{R}}} V_{\mathbb{R}} \to X_{\mathbb{R}}$ is a weak equivalence. Since the space $U_{\mathbb{R}} \coprod_{p^*_{\mathbb{R}}} (I \times W_{\mathbb{R}}) \coprod_{i^*_{\mathbb{R}}} V_{\mathbb{R}} = [(U_{\mathbb{R}} \cup (W_{\mathbb{R}} \times I)) \coprod (V_{\mathbb{R}} \cup (W_{\mathbb{R}} \times I))]/\sim$, where in the space $(U_{\mathbb{R}} \cup (I \times W_{\mathbb{R}}))$, $W_{\mathbb{R}} \subset U_{\mathbb{R}}$ is identified with $\{0\} \times W_{\mathbb{R}}$; and in the space $(V_{\mathbb{R}} \cup (I \times W_{\mathbb{R}}))$, $W_{\mathbb{R}} \subset V_{\mathbb{R}}$ is identified with $\{1\} \times W_{\mathbb{R}}$; and, the quotient relation $\sim$ identifies the two copies of $I \times W_{\mathbb{R}}$ collapsing the unit interval I. Then α is the map which collapses the $I \times W_{\mathbb{R}}$ to $W_{\mathbb{R}}$, and therefore, it is a weak equivalence. The final (namely, the third) assertion follows from the fact that étale maps are local isomorphisms, so we get such an open cover of X. This completes proof of the lemma. $\square$

Lemma 4.2.11. *For every topological space S, the presheaf $\rho_*(S)$ is $\mathbb{A}^1$-local, and for every presheaf $\mathfrak{F}$ in $\Delta^{op}PShv(Sm/\mathbb{R})$*

$$[\mathfrak{F}, \rho_*(S)]_{Ho_{[\mathbb{A}^1]}(\Delta^{op}PShv(\nu))} = [\mathfrak{F}, \rho_*(S)]_{Ho_{[global]}(\Delta^{op}PShv(\nu))}.$$

Proof. Follows from lemmas 4.2.3, 4.2.6 and 4.2.7. □

Lemma 4.2.12. *For every weak equivalence $f : S \to T$ of topological spaces, the natural map of presheaves*

$$f_* : \mathfrak{Map}\ (\mathfrak{Sing}\ -_{\mathbb{R}},\ \mathfrak{Sing}\ S) \to \mathfrak{Map}\ (\mathfrak{Sing}\ -_{\mathbb{R}},\ \mathfrak{Sing}\ T)$$

is a global weak equivalence, and hence a simplicial and an $\mathbb{A}^1$-weak equivalence.

Proof. we need to prove that

$$f_* : \mathfrak{Map}\ (\mathfrak{Sing}\ X_{\mathbb{R}},\ \mathfrak{Sing}\ S) \to \mathfrak{Map}\ (\mathfrak{Sing}\ X_{\mathbb{R}},\ \mathfrak{Sing}\ T)$$

is a weak equivalence of simplicial sets for every smooth real scheme X. This follows by noting that

1. every weak equivalence $f : S \to T$ of topological spaces induces a simplicial weak equivalence of simplicial sets $\mathfrak{Sing}\ f : \mathfrak{Sing}\ S \to \mathfrak{Sing}\ T$; and
2. the internal mapping simplicial set functor, $\mathfrak{Map}(\mathcal{K},\ -)$, where $\mathcal{K}$ is a simplicial set, sends a simplicial weak equivalence of simplicial sets to a simplicial weak equialence of simplicial sets.

□

Rephrasing the contents of above lemma by saying that the functor $\rho_* : Top \to \Delta^{op}PShv(\nu)$ sends weak equivalences of topological spaces to global weak equivalences of simplicial presheaves, and since, the global weak quivalences of simplicial presheaves are also simplicial as well as $\mathbb{A}^1$-weak equivalences, we find that this functor ρ_* defines well defined functors on the associated homotopy categories (again denoted by ρ_*) in the string of compositions of functors induced by localizations as described below: $\rho_* : HoTop \to Ho_{[global]}\Delta^{op}PShv(\nu) \to Ho_{[Nis]}\Delta^{op}PShv(\nu) \to Ho_{[\mathbb{A}^1]}\Delta^{op}PShv(\nu)$.

Though we will define the left adjoint of the functor ρ_* in 4.3.2, let us prove the following corollary assuming that such a left adjoint has been defining. In this corollary the functor $L\rho^*$ denotes the left derived functor of ρ^* (which has been explicitly defined in the next section in 4.3.7).

Corollary 4.2.13. *The left adjoint $L\rho^*$ of the functor:*

$$\rho_* : HoTop \to Ho_{[global]}\Delta^{op}PShv(\nu)$$

sends the Nis-, and $\mathbb{A}^1$-weak equivalences to weak equivalences of topological spaces. In particular, it induces the left adjoint of ρ_* for all the three functors $\rho_* : HoTop \to Ho_{[global]}\Delta^{op}PShv(\nu)$, $\rho_* : HoTop \to Ho_{[Nis]}\Delta^{op}PShv(\nu)$ and $\rho_* : HoTop \to Ho_{[\mathbb{A}^1]}\Delta^{op}PShv(\nu)$.

Proof. This corollary will follow if we show that the left adjoint $L\rho^* : Ho_{[global]}\Delta^{op}PShv(\nu) \to HoTop$ sends $\mathbb{A}^1$-weak equivalences of simplicial presheaves to weak equivalences of topological spaces. Let a morphism $\alpha : \mathcal{X} \to \mathcal{Y}$ of simplicial presheaves be an $\mathbb{A}^1$-weak equivalence. We want to prove that the map $L\rho^*(\alpha) : L\rho^*(\alpha)(\mathcal{X}) \to L\rho^*(\alpha)(\mathcal{Y})$ is a weak equivalence of topological spaces. Since we are defining $L\rho^*$ as the left adjoint of the functor ρ_*, we have the following commutative diagram for every topological space T:

$$\begin{array}{ccc}
\mathrm{Hom}_{HoTop}(L\rho^*(\mathcal{X}),\ T) & \xrightarrow{\text{nat isom}} & \mathrm{Hom}_{Ho_{[global]}\Delta^{op}PShv(\nu)}(\mathcal{X},\ \rho_*(T)) \\
\Big\downarrow \overline{L\rho^*(\alpha)} & & \Big\downarrow \overline{\alpha} \\
\mathrm{Hom}_{HoTop}(L\rho^*(\mathcal{Y}),\ T) & \xrightarrow{\text{nat isom}} & \mathrm{Hom}_{Ho_{[global]}\Delta^{op}PShv(\nu)}(\mathcal{Y},\ \rho_*(T))
\end{array}$$

We have proved in 4.2.7 that $\rho_*(T)$ is $\mathbb{A}^1$-local and hence $\alpha_{\text{nis}} : \text{Hom}_{Ho_{[\text{nis}-simpl]}\Delta^{op}PShv(\nu)}(\mathcal{X},\ \rho_*(T)) \to \text{Hom}_{Ho_{[\text{nis}-simpl]}\Delta^{op}PShv(\nu)}(\mathcal{Y},\ \rho_*(T))$ is bijection by our assumption that α is an $\mathbb{A}^1$-weak equivalence. Therefore, using lemma 4.2.6 we see that $\overline{\alpha} = \alpha_{\text{nis}}$ is isomorphism, we see that $L\rho^*(\alpha) : \text{Hom}_{HoTop}(L\rho^*(\mathcal{X}),\ \ T) \to \text{Hom}_{HoTop}(L\rho^*(\mathcal{Y}),\ T)$ is also an isomorphism. This would mean that the induced map $L\rho^*(\alpha) : L\rho^*(\mathcal{X}) \to L\rho^*(\mathcal{Y})$ would be a simplicial weak equivalence and hence a weak equivalence of topological spaces as desired. $\square$

4.3 The Functor $\Delta^{op}PShv(Sm/\mathbb{k}) \xrightarrow{\rho^*} Top$

Having discussed the functor ρ_* in some details now we turn our attention to defining its left adjoint ρ^* and then it's left derived functor $L\rho^*$. Since any presheaf on ν is a colimit of representable presheaves, writing a presheaf $\mathcal{F} \in PShv(\nu)$ as a colimit $\mathcal{F} = \text{colim}_{X_i \to \mathcal{F}} X_i$, where X_i is a smooth real scheme in ν, we observe that since ρ^* is a left adjoint when defined, it should preserve colimits, and hence it suffices to define $\rho^* : \Delta^{op}PShv(\nu) \to Top$ for the representable presheaves, that is, we need to define ρ^*X only, for a smooth scheme $X \in \nu$. For a smooth scheme $X \in \nu$ and a topological space $S \in Top$, consider the following computations:

$$\begin{aligned}
\text{Hom}_{\Delta^{op}PShv(\nu)}(X,\ \rho_*S) &= \text{Hom}_{\Delta^{op}PShv(\nu)}(X,\ \mathfrak{Map}\ (\mathfrak{Sing}-_{\mathbb{R}}, \mathfrak{Sing}\ S)) \\
&= (\mathfrak{Map}\ (\mathfrak{Sing}-_{\mathbb{R}},\ \mathfrak{Sing}\ S)(X))_0 = \text{set of } 0-\text{simplices} \\
&\qquad (\ \text{[MV99], page 47, a property of } S(-,\ -)) \\
&= Hom_{\Delta^{op}Set}(\mathfrak{Sing}\ X_{\mathbb{R}},\ \mathfrak{Sing}\ S) \\
&= \text{Hom}_{Top}(|\mathfrak{Sing}\ X_{\mathbb{R}}|,\ S)
\end{aligned}$$

From this computation, we see how to define ρ^* for representable presheaves: If X is a smooth scheme in ν, we define $\rho^*X = |\mathfrak{Sing}X_{\mathbb{R}}|$. Since the topological realization functor $|\ |$ is a left adjoint (to the $\mathfrak{Sing}$ functor), it commutes with colimits, and we define

Definition 4.3.1. For a general presheaf $\mathcal{F}$ with $\mathcal{F} = \mathrm{colim}_{X_i \to \mathcal{F}} X_i$

$$\rho^* \mathcal{F} = \mathrm{colim}_{X_i \to \mathcal{F}} |\mathfrak{Sing}\ X_{i_{\mathbb{R}}}| = |\mathrm{colim}_{X_i \to \mathcal{F}} \mathfrak{Sing}\ X_{i_{\mathbb{R}}}|$$

To define the map induced by a map of presheaves, observe that given a map of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ with $\mathcal{F} = \mathrm{colim}_{X_i \to \mathcal{F}} X_i$ and $\mathcal{G} = \mathrm{colim}_{Y_j \to \mathcal{G}} Y_j$, we get compatible maps $X_i \to Y_j$ for each i and each j, by composidering the action of the composition $X_i \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}$ on the scheme Y_j, giving us the continuous map induced by ϕ in Top.

Given a simplicial presheaf $\mathcal{F} = (\mathcal{F}_n) \in \Delta^{op} PShv(\nu)$, note that the simplicial topological space $(\rho^* \mathcal{F}_n)_{n \geq 0}$ is 'good' [S74, Appendix]. In view of this note, we make the following defintion of the functor ρ^* for a general simplicial presheaf:

Definition 4.3.2. The functor $\rho^* : \Delta^{op} PShv(\nu) \to Top$ is defined by sending a simplicial presheaf $\mathcal{F} = (\mathcal{F}_n) \in \Delta^{op} PShv(\nu)$ to the topological space $|n \mapsto \rho^* \mathcal{F}_n|$, where for the simplicial topological space $(\rho^* \mathcal{F}_n)_{n \geq 0}$ we take its realization as defined in [S74, Appendix].

The functor ρ^* defined just now is in fact the left adjoint of the functor ρ_*, as proved below:

Lemma 4.3.3. *The pair of functors (ρ^*, ρ_*), $\rho^* : \Delta^{op} PShv(\nu) \rightleftarrows Top : \rho_*$, form an adjoint pair, ρ^* being the left adjoint and ρ_* the right adjoint.*

Proof. Consider the sequence of natural isomorphisms below for a simplicial presheaf $\mathcal{F} = (\mathcal{F}_n) \in \Delta^{op} PShv(\nu)$ and a topological space $S \in Top$:

$Hom_{Top}(\rho^* \mathcal{F},\ S) = Hom_{Top}(|n \mapsto \widetilde{\rho}^* \mathcal{F}_n|,\ S) = Hom_{\Delta^{op} Set}(n \mapsto \widetilde{\rho}^* \mathcal{F}_n,\ \mathfrak{Sing}\ S)$

$\square$

Having defined ρ^*, we now discuss its left derived functor $L\rho^* : Ho_{[global]} \Delta^{op} PShv$ $(\nu) \to HoTop$. First we define $L\rho^* X$ for a smooth real scheme $X \in Sm/\mathbb{R}$, considered as the constant simplicial presheaf. We must have $Hom_{HoTop}(L\rho^* X,\ S) = Hom_{Ho_{[global]}\Delta^{op}PShv(\nu)}(X,\ R\rho_* S)$. By means of following canonical isomorphisms

$$
\begin{aligned}
& Hom_{Ho_{[global]}\Delta^{op}PShv(\nu)}(X,\ R\rho_* S) \\
& = Hom_{Ho_{[global]}\Delta^{op}PShv(\nu)}(X,\ \mathfrak{Map}\ (\mathfrak{Sing}-_{\mathbb{R}},\ \mathfrak{Sing}\ S)) \\
& = \pi_0(\mathfrak{Map}\ (\mathfrak{Sing}X_{\mathbb{R}},\ \mathfrak{Sing}S)\ \text{(see remark 4.3.4)} \\
& = Hom_{\Delta^{op}Set}(\mathfrak{Sing}X_{\mathbb{R}},\ \mathfrak{Sing}S)/\sim \qquad \text{(where $\sim$ refers to simplicial homotopy} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{relation between two simplicial sets)} \\
& = Hom_{Top}(X_{\mathbb{R}},\ S)/\sim \qquad \text{(where $\sim$ refers to simplicial homotopy relation} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{between two topological spaces)} \\
& = Hom_{Top}(|\mathfrak{Sing}X_{\mathbb{R}}|,\ S)/\sim \\
& = Hom_{HoTop}(|\mathfrak{Sing}X_{\mathbb{R}}|,\ S)
\end{aligned}
$$

Remark 4.3.4. There exists a model category structure on the category $\Delta^{op}PShv(\nu)$, in which every Kan fibrant simplicial set determines a fibrant object, the representable objects are cofibrant, and weak equivalences are the global weak equivalences.

Remark 4.3.5. Before we formally make any definition based on above computations with hom-sets, we first extend this computation: For that purpose, let us remind ourselves of a way of viewing the representable simplicial presheaves: A representable simplicial presheaf is given by a simplicial 'scheme' $\mathfrak{X} = (X_n)$, where each X_n is an union of smooth real schemes in $Sm/\mathbb{R}$: As a caution we should keep in mind that such an union might sacrifice some finiteness properties of smooth $\mathbb{R}$-schemes. Thinking of a representable presheaf X_n as a constant simplicial presheaf, we have $\mathfrak{X} = \mathrm{colim}_{\Delta^{op}} X_n \tilde{\to} \mathrm{hocolim}_{\Delta^{op}} X_n$. Since $L\rho^*$ is intended to be a left adjoint, it must satisfy the relations of the kind

$$L\rho^*\mathfrak{X} = L\rho^*(\mathrm{hocolim}_{\Delta^{op}} X_n) = \mathrm{hocolim}_{\Delta^{op}} L\rho^* X_n = \mathrm{hocolim}_{\Delta^{op}} |\mathfrak{Sing} X_{n_{\mathbb{R}}}|.$$

Now the above computation extends as follows:

$$\begin{aligned}
& Hom_{Ho_{[global]}\Delta^{op}PShv(\nu)}(\mathfrak{X},\ R\rho_* S) \\
&= Hom_{Ho_{[global]}\Delta^{op}PShv(\nu)}(\mathfrak{X},\ \rho_* S) \\
&= Hom_{Ho_{[global]}\Delta^{op}PShv(\nu)}(\mathrm{colim}_{\Delta^{op}} X_n,\ \rho_* S) \\
&= Hom_{HoTop}(L\rho^*(\mathrm{colim}_{\Delta^{\mathrm{op}}} X_n),\ S) \\
&= Hom_{HoTop}(\mathrm{colim}_{\Delta^{op}} L\rho^* X_n,\ S) \\
&= Hom_{HoTop}(\mathrm{hocolim}_{\Delta^{op}} L\rho^* X_n,\ S) \qquad \text{since } L\rho^* X_n \text{ is `good' (see remark 4.3.6 below)} \\
&= Hom_{HoTop}(L\rho^*\mathfrak{X},\ S).
\end{aligned}$$

Remark 4.3.6. For a representable simplicial presheaf $\mathfrak{X} = (X_n)$, the simplicial topological space $(L\rho^* X_n)$ is good [[S74]AppendixA].

Above remark proves the canonical bijection of hom-sets through $L\rho^*$ and $\rho_* = R\rho_*$ as an adjoint pair for representable simplicial presheaves on ν. Therefore,

Definition 4.3.7. For a smooth real scheme $X \in Sm/\mathbb{R}$, considered as a constant simplicial presheaf, we define $L\rho^* X = |\mathfrak{Sing} X_{\mathbb{R}}|$. And, for a representable simplicial presheaf $\mathfrak{X} = (X_n)$ using lemma 4.3.6 we define

$$L\rho^*\mathfrak{X} = \mathrm{hocolim}_{\Delta^{op}} |\mathfrak{Sing} X_{n_{\mathbb{R}}}| \tilde{\to} \mathrm{colim}_{\Delta^{op}} L\rho^* X_n$$

For a general simplicial presheaf $\mathfrak{F} \in \Delta^{op}PShv(\nu)$, consider the weak equivalence $\Phi_{\mathfrak{R}}(\mathfrak{F}) \tilde{\to} \mathfrak{F}$ discussed in [MV99, remark 1.17 p. 53], here we have denoted by $\mathfrak{R}$ the class of all representable simplicial presheaves: This weak equivalence can be used to define $L\rho^*(\mathfrak{F})$ for a general simplicial presheaf $\mathfrak{F}$.

Theorem 4.3.8. *The pair of functors* $L\rho^* : Ho_{[global]}\Delta^{op}PShv(\nu) \rightleftarrows HoTop : R\rho_*$ *defined in this section form an adjoint pair of functors.*

Proof. Follows from [MV99, remark 1.17 page 53] and the computations in remark 4.3.5. $\square$

Chapter 5

A Revised Proof of the $\mathbb{A}^1$-Representability Theorem

5.1 Introduction

This chapter contains a complete proof of the representability theorem without any reference to the contents of the first four chapters as mentioned in the introduction.

In section 5.2, we have defined the orthogonal Grassmannian GrO as an open subscheme of the Grassmannian scheme. In section 5.3, we have recalled the defin-tion of hermitian K-theory presheaf KO and defined a map $\hbar : GrO \to KO$ from the orthogonal Grassmannian into the hermitian K-theory. In section 5.4, we have identified the π_0 presheaf of the singular simplicial presheaf $GrO\Delta^\bullet$ of GrO with the singular simplicial presheaf of Grothendieck-Witt groups in theorem 5.4.15.

In section 5.5, we have studied the classifying space of the orthogonal group $O(H^n)$. In proposition 5.5.2, we have idetinfied it with the connected component of 0 of the hermitian K-theory. We have defined the Stiefel presheaf in 5.5.4 and proved that it is contractibility in theorem 5.5.7. The Stiefel presheaves are analogs of the Stiefel manifolds in topology. Using this theorem, we have been able to identify the classifying $BO(H^n)$ of orthogonal group with the subpresheaf of GrO consisting of the hyperbolic spaces in proposition 5.5.10.

In section 5.6, we have provided the global sections of $GrO\Delta^\bullet$ with an operad structure which makes them H-spaces. As pointed out in remark 5.6.4, this op-erad structure gives us a choice of orthogonal sum of symmetric bilinear spaces in $GrO\Delta^\bullet$ and the map $\hbar : GrO \to KO$ becomes a H-space map with respect to the H-space structure of hermitian K-theory given by orthogonal sum.

In section 5.7, we have proved that the map $\hbar : GrO \to KO$ from GrO to the hermitian K-theory is an $\mathbb{A}^1$-weak equivalence. We have proved in theorem 5.7.1 that the map $\hbar$ induces a map of global fibrations $GrH\Delta^\bullet \to GrO\Delta^\bullet \to \pi_0\, GrO\Delta^\bullet$ and $|\,KO_{[0]}\,| \to |\,KO\Delta^\bullet| \to \pi_0 GW\Delta^\bullet$. In theorem 5.4.15, we have identified the π_0 presheaf of $GrO\Delta^\bullet$ with the π_0 presheaf of hermitian K-theory which is the presheaf of Grothendieck-Witt groups. And in proposition 5.5.10, we have identified the fibers of these two fibrations via the map induced from the map $\hbar$. The fact that $\hbar$ is a map of group complete H-spaces now finishes the proof of the representability theorem.

In the last section we have made some observations regarding some extensions and applications of this result.

5.2 Orthogonal Grassmannian GrO

First we will recall the definition of the Grassmannian scheme[1] Gr over a field F which represents the algebraic K-theory in the $\mathbb{A}^1$-homotopy category of smooth schemes [MV99, thm 4.3.13 p. 140, prop 4.3.7 p.138]. Then we will define an open subscheme of Gr denoted by GrO: We have called GrO the orthogonal Grassmannian.

Definition 5.2.1 (Grassmannian Scheme)**.** Let V be a F-vector space of dimension n, denote the corresponding rank n trivial bundle over $\operatorname{Spec} F$ also by V. The contravariant functor X_V from F-schemes to sets sending a F-scheme $T \xrightarrow{t} \operatorname{Spec} F$ to the set $X_V(T)$ of rank d subbundles of the pullback bundle $V_T = t^*V$ is represented by a scheme $Gr(d, V)$. We will denote the pullback of V to $Gr(d, V)$ by V_{Gr} unless it is necessary to specify d and V. There is a universal rank d subbundle $U_{Gr} \subset V_{Gr}$ over $Gr(d, V)$ with the property that for any F-scheme T there is a

[1]We may sometimes call scheme what is actually just an inductive colimit of schemes.

bijection between the of F-morphisms from T to $Gr(d, V)$ and the set of rank d subbundles of V_T given by

$$f \mapsto f^*U_{Gr} \subset f^*V_{Gr}$$

[DG80, chap 1, sect 1.3, 2.4] and [K69, sect 1]. Let $Gr(V)$ be the F-scheme $\coprod_{d\geq 0} Gr(d, V)$. An inclusion of vector spaces $V \subset W$ defines a natural transformation of functors $X_V \to X_W$ and closed immersions $Gr(d, V) \subset Gr(d, W)$ and $Gr(V) \subset Gr(W)$; if V is infinite dimensional we define $Gr(V)$ as the colimit over over all its finite dimensional subspaces. In particular, we have the scheme $Gr(F^\infty)$ for the infinite dimensional F-vector space F^∞; and, for $r \geq 0$, the scheme $Gr(F^r \oplus F^\infty)$ for the vector space $F^r \oplus F^\infty$. There are maps $Gr(F^r \oplus F^\infty) \to Gr(F^{r+1} \oplus F^\infty) \to \cdots$ corresponding to the direct sum of a copy of F for the new standard basis element $F^r \oplus F^\infty \to F^{r+1} \oplus F^\infty$. The colimit Gr of this system is the Grassmannian over F.

Definition 5.2.2. Let R be a F-algebra. Recall that the hyperbolic form on R^m can be described as follows: Let h_1 be the 2-square matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. For two matrices a and b let $a \perp b$ denote the matrix $\left(\begin{smallmatrix} a & \\ & b \end{smallmatrix}\right)$, where a vacant spot has the entry 0. The hyperbolic space $H^m(R)$ (which we will usually write as just H^m) is the free R-module R^{2m} together with the standard hyperbolic form given by the matrix $h_m = h_1 \perp \cdots \perp h_1$ (m copies) [K90, Ch. I]. A map of rings $R \to S$ induces a map hyperbolic spaces $H^n(R) \to H^n(R) \otimes S \simeq H^n(S)$. We have the presheaf H^n of sets on Sm/F given by $X \mapsto H^n(\Gamma(X, \mathcal{O}_X))$ and inclusions of presheaves $H^n \hookrightarrow H^{n+1}$ induced by the isometric embedding $H^n(R) \subset H^{n+1}(R) = H^n(R) \perp H(R)$. The presheaf H^∞ is the colimit of H^n over n.

Definition 5.2.3 (Orthogonal Grassmannian)**.** Let (V, ϕ) be a non-degenerate symmetric bilinear form space of dimension n over F (we will generally omit ϕ from the notation). On the bundle $V_{Gr(d,V)}$ we have the induced form $\phi : V_{Gr(d,V)} \to V^*_{Gr(d,V)}$ and the map $\phi|_{U_{Gr}} : U_{Gr} \to U^*_{Gr}$. The scheme $GrO(d, V)$ is the non-vanishing section of the rank 1 sheaf $\mathscr{H}om_{Gr(d,V)}(\wedge^d U_{Gr},\ \wedge^d U^*_{Gr})$ of the global section $\wedge^d(\phi|_{U_{Gr}})$; it is an open subscheme of $Gr(d, V)$. Denote the pullback of V to $GrO(d, V)$ by $V_{GrO(d,V)}$ and similarly the pullback of U_{Gr} by $U_{GrO(d,V)}$ (usually we will omit (d, V) from notation for simplicity). Analogous to the case of the scheme $Gr(d, V)$ for every F-scheme T there is a bijection between the set of F-morphisms $T \xrightarrow{f} GrO(d, V)$ and the set of those rank d subbundles of V_T on which the form ϕ restricts to a non-degenerate form given by

$$f \mapsto f^*(U_{GrO}) \subset f^* V_{GrO}$$

Let $GrO(V) = \coprod_{d \geq 0} GrO(d, V)$. An isometric embedding $(V, \phi) \hookrightarrow (W, \psi)$ defines a natural transformation of the functor of points of the schemes $Gr(d, V)$ and $Gr(d, W)$ and hence a map of schemes $Gr(d, V) \to Gr(d, W)$. We have an isomorphism $(W_{Gr(d,W)})_{Gr(d,V)} \simeq W_{Gr(d,V)}$ of the pullback bundles and under this isomorphism the pullback of the canonical restriction $\psi|_{U_{GrO(d,W)}} : U_{GrO(d,W)} \to U^*_{GrO(d,W)}$ is $\phi|_{U_{GrO(d,V)}} : U_{GrO(d,V)} \to U^*_{GrO(D,V)}$. Thus, we have the induced maps $GrO(d, V) \to GrO(d, W)$ and $GrO(V) \to GrO(W)$. If (V_α, ϕ_α) is an increasing system of non-degenerate symmetric spaces with colimit V, we define $GrO(V)$ as the colimit over $GrO(V_\alpha)$. In particular, denote by $GrO(H^\infty)$ the colimit of $GrO(H^n) \to GrO(H^{n+1}) (n \geq 0)$ induced by the standard isometric embedding $H^n \hookrightarrow H^{n+1}$; and, for every $r \geq 0$, by $GrO(H^r \perp H^\infty)$ the colimit of $GrO(H^r \perp H^n) \to GrO(H^r \perp H^{n+1})$ induced by $H^r \perp H^n \hookrightarrow H^r \perp H^{n+1}$. As in the case of Grassmannian, for all $r \geq 0$ there are maps $GrO(H^r \perp H^\infty) \xrightarrow{H\perp} GrO(H^{r+1} \perp H^\infty)$ corresponding to the orthogonal sum of a hyperbolic plane for the extra basis element: More precisely, on functor of points this map sends a direct factor $V \subset (H^r \perp H^\infty)(X)$ to $H \perp V \subset (H^{r+1} \perp H^\infty)(X)$. The orthogonal Grassmannian GrO is the colimit over r of the system

$$GrO(H^r \perp H^\infty) \xrightarrow{H\perp} GrO(H^{r+1} \perp H^\infty) \xrightarrow{H\perp} \cdots$$

5.3 Hermitian K-Theory Ppresheaf KO and the Map $GrO \xrightarrow{\hbar} KO$

In this section we precisely define the hermitian K-theory presheaf KO and then define a map $\hbar : GrO \to KO$. We will prove in theorem 5.7.1 that this map is an $\mathbb{A}^1$-weak equivalence.

Recall from [G76] that given a symmetric monoidal category $\mathcal{C}$ there is a category $\mathcal{C}^{-1}\mathcal{C}$, called group the completion of $\mathcal{C}$, whose objects are pairs of objects of $\mathcal{C}$ and whose morphisms are defined in [G76]. There is a functor $\mathcal{C} \to \mathcal{C}^{-1}\mathcal{C}$ called the group completion sending an object $X \in \mathcal{C}$ to the pair $(0, X)$.

Definition 5.3.1. Let $V = (V, \phi)$ be a non-degenerate symmetric bilinear form on an F-scheme X, that is, a vector bundle V on X together with a non-degenerate symmetric bilinear form ϕ on V. Let $\mathcal{S}_V$ be the category whose objects are finite rank subbundles of V on which the form ϕ restricts to a nondegenerate form. These subbundles will be direct factors of V. The morphisms in $\mathcal{S}_V$ are given by isometries. An isometric embedding $V \hookrightarrow W$ gives a full embedding $\mathcal{S}_V \to \mathcal{S}_W$ of categories where a subbundle of V is considered as a subbundle of W via the embedding $V \hookrightarrow W$. If V is a colimit of a direct system of non-degenerate symmetric bilinear forms (V_α, ϕ_α), we define $\mathcal{S}_V$ as the colimit of the categories $\mathcal{S}_{V_\alpha}$. For the standard hyperbolic space $H^\infty(X)$ on X [S10] we thus have the category $\mathcal{S}_{H^\infty(X)}$ which we will also denote by $\mathcal{S}_X$. The category $\mathcal{S}_X$ is equivalent to the category of inner product spaces over X that can be embedded in $H^\infty(X)$ and thus has a symmetric monoidal structure given by orthogonal sum, and $\mathcal{S}$ is a presheaf of categories on

Sm/F. In particular, for a ring R the category $\mathcal{S}_{\operatorname{Spec} R}$, which for simplicity will be written as $\mathcal{S}_R$, is equivalent to the category of inner product spaces over R (that is, finitely generated projective R-modules with a non-degenerate symmetric bilinear form) since every inner product space on R admits an isometric embedding into some hyperbolic space $H^n(R)$.

Remark 5.3.2. It follows from definition 5.2.3 that

$$GrO(V) = \operatorname{Ob}\mathcal{S}_V = \mathcal{N}_0\mathcal{S}_V \subset \mathcal{N}\mathcal{S}_V$$

where $\mathcal{N}$ denotes the nerve simplicial set of a category and $\mathcal{N}_0$ the set of vertices of nerve. Usually the hermitian K-theory is defined as the nerve simplicial presheaf $\mathcal{N}\mathcal{S}^{-1}\mathcal{S}$ of the group completion of $\mathcal{S}$, and $\hbar$ is just the map given by the composition

$$GrO(H^\infty) = \operatorname{Ob}\mathcal{S} = \mathcal{N}_0\mathcal{S} \subset \mathcal{N}\mathcal{S} \to \mathcal{N}\mathcal{S}^{-1}\mathcal{S}.$$

See 5.3.5 for a precise definition of $\hbar$.

To define the map $GrO \to \mathcal{KO}$ we introduce the presheaves of categories $\mathcal{S}_r$ for every positive integer r as follows. The hyperbolic space $H^r(X)\perp H^\infty(X)$ defines a symmetric monoidal category $\mathcal{S}_{H^r(X)\perp H^\infty(X)}$ which we will denote simply by $\mathcal{S}_{r,X}$; and, there is a functor $\mathcal{S}_{r,X} \xrightarrow{H\perp} \mathcal{S}_{r+1,X}$ mapping a direct factor $\mathcal{E} \subset H^r \perp H^\infty$ to $H\perp\mathcal{E} \subset H\perp H^r \perp H^\infty = H^{r+1}\perp H^\infty$ (it is not a monoidal functor). We have the presheaves of categories $\mathcal{S}_r$ which map $X \mapsto \mathcal{S}_{r,X}$, and the maps of presheaves $\mathcal{S}_r \xrightarrow{H\perp} \mathcal{S}_{r+1}$. The presheaf $\mathcal{S}_0$ is what we have denoted by $\mathcal{S}$ in the above definition.

Definition 5.3.3 (Hermitian K-theory KO and hermitian K-groups KO_i)**.** The group completion functor defines a map of presheaves of categories $\mathcal{S}_r \to \mathcal{S}_r^{-1}\mathcal{S}_r$

such that the diagram

$$\begin{tikzcd} \mathcal{S}_r \arrow[r] \arrow[d, "H\perp"] & \mathcal{S}_r^{-1}\mathcal{S}_r \arrow[d, "\simeq"', "(0,H)\perp"] \\ \mathcal{S}_{r+1} \arrow[r] & \mathcal{S}_{r+1}^{-1}\mathcal{S}_{r+1} \end{tikzcd} \tag{5.3.1}$$

commutes, and the right vertical map, which will also be denoted by $H\perp$, is a homotopy equivalence since all the categories $\mathcal{S}_r^{-1}\mathcal{S}_r$ are group complete and $H\perp$ is given by addition of an element. Denote the nerve simplicial presheaf $X \mapsto \mathscr{N}\mathcal{S}_r^{-1}\mathcal{S}_r(X)$ by $(KO)_r$. The hermitian K-theory simplicial presheaf KO is the colimit

$$KO = \mathrm{colim}_{r\geq 0}(\mathscr{N}\mathcal{S}_r^{-1}\mathcal{S}_r \xrightarrow{H\perp} \mathscr{N}\mathcal{S}_{r+1}^{-1}\mathcal{S}_{r+1} \xrightarrow{H\perp} \cdots)$$

As noted above in this colimit all the maps $(KO)_r \to (KO)_{r+1} \to KO$ are homotopy equivalences. In particular, $(KO)_r$ as well as KO are both models of hermitian K-theory. The hermitian K groups, which are also called the higher Grothendieck-Witt groups, of an affine scheme X are the homotopy groups of $KO(X)$ at the base point 0 and are denoted by $KO_i(X)$.

Remark 5.3.4. The hermitian K-theory presheaf KO defined here is homotopy equivalent to the one defined in [H05] in 1.3(1), 1.5 and 1.7 for affine F-schemes in Sm/F. The corollaries 1.12 and 1.14 in [H05] prove that for affine schemes in Sm/F, the presheaf KO is homotopy invariant and it has the Nisnevich-Mayer-Vietoris property.

Definition 5.3.5 (The Map $\hbar$ from GrO to KO)**.** The nerve simplicial presheaves in diagram 5.3.1 and the inclusion $GrO(H^r\perp H^\infty) = \mathscr{N}_0\mathcal{S}_r \subset \mathscr{N}\mathcal{S}_r$ as 0-simplicies give us the commutative diagram

$$\begin{tikzcd}
GrO(H^r \perp H^\infty) \arrow[r, "\subset"] \arrow[d, "H\perp"] & \mathcal{N}\mathcal{S}_r \arrow[r] \arrow[d, "H\perp"'] & \mathcal{N}\mathcal{S}_r^{-1}\mathcal{S}_r = (KO)_r \arrow[d, "H\perp", "\simeq"'] \\
GrO(H^{r+1} \perp H^\infty) \arrow[r, "\subset"] \arrow[d, "H\perp"] & \mathcal{N}\mathcal{S}_{r+1} \arrow[r] \arrow[d, "H\perp"'] & \mathcal{N}\mathcal{S}_{r+1}^{-1}\mathcal{S}_{r+1} = (KO)_{r+1} \arrow[d, "H\perp", "\simeq"'] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

of simplicial presheaves. Taking colimit over r (and composing horizontally) we get the map

$$\hbar : GrO \to KO$$

from orthogonal Grassmannian to the hermitian K-theory presheaf.

In theorem 5.7.1 we will prove that the map $\hbar$ is an $\mathbb{A}^1$-weak equivalence, and therefore, GrO represents the hermitian K-theory in the $\mathbb{A}^1$-homotopy category of smooth F-schemes.

5.4 Grothendieck-Witt Group and $\pi_0 GrO\Delta^\bullet$

In this section we will recall the definition of the singular simplicial presheaf $\mathcal{X}\Delta^\bullet$ of a presheaf $\mathcal{X}$. For the presheaf GrO, we will prove in theorem 5.4.15 that the π_0-presheaf of $GrO\Delta^\bullet$ gives an isomorphism via the map $\hbar$ to the presheaf $GW\Delta^\bullet$ of Grothendieck-Witt groups on affine sections $\operatorname{Spec} R$ for regular local rings R with $\frac{1}{2} \in R$.

Definition 5.4.1 (Grothendieck-Witt group of a ring)**.** Let $\widehat{W}(R)$ be the monoid of isometry classes of non-degenerate symmetric bilinear form spaces over R with respect to orthogonal sum [S85, Chap 2, sec 1]. The Grothendieck-Witt group $GW(R)$ of R is the group completion of the abelian monoid $\widehat{W}(R)$. The presheaf $\widehat{W}$ of Witt groups is defined by $X \mapsto \widehat{W}(\Gamma(X, \mathcal{O}_X))$, and the presheaf GW of Grothendieck-Witt groups is defined by $X \mapsto GW(\Gamma(X, \mathcal{O}_X))$.

Let us recall the definition of the singular simplicial presheaf $F\Delta^\bullet$ of a presheaf F, see [MV99, sect 2.3.2]. The construction of singular simplicial presheaf is useful because two naively $\mathbb{A}^1$-homotopic maps of presheaves define simplicially homotopic maps on the global sections, see the definition 5.4.3 and lemma 5.4.4.

Definition 5.4.2 (Singular simplicial presheaf $F\Delta^\bullet$)**.** Let $\Delta^\bullet$ be the standarad cosimplicial scheme $n \mapsto \Delta^n = \mathrm{Spec}\, \frac{F[T_0, T_1, \cdots, T_n]}{<T_0+T_1+\cdots+T_n=1>}$. Let F be a simplicial presheaf F. The singular simplicial presheaf $|\,F\Delta^\bullet\,|$ of F is the diagonal of the bisimplicial presheaf $n \mapsto F\Delta^n : X \mapsto F(\Delta^n \times X)$. If F is simplicially constant then this diagonal is the simplicial presheaf $F\Delta^\bullet$. The assignment $F \mapsto |\,F\Delta^\bullet\,|$ is a functor of simplicial presheaves such that the map $F \to |\,F\Delta^\bullet\,|$ is an $\mathbb{A}^1$-weak equivalence, see [MV99, cor 2.3.8, p. 53]. For a map $f : F \to G$ of simplicial presheaves, we will denote the corresponding map $|\,F\Delta^\bullet\,| \to |\,G\Delta^\bullet\,|$ by f.

Definition 5.4.3. Let $X \in Sm/F$ be a smooth F-scheme. The two maps of F-algebras $F[T] \to \Gamma(X, \mathcal{O}_X)$ given by $T \mapsto 0$ and $T \mapsto 1$ give us two maps of presheaves $i_0, i_1 : pt = \mathrm{Spec}\, F \to \mathbb{A}^1$ from the initial object pt of the category $PShv(Sm/F)$. Let $f, g : \mathcal{X} \to \mathcal{Z}$ be two maps of presheaves on Sm/F. We say that the maps f and g are naively A^1-homotopic, if there is a map $h : \mathcal{X} \times \mathbb{A}^1 \to \mathcal{Z}$ such that in the diagram

$$\begin{tikzcd} \mathcal{X} \times pt \simeq \mathcal{X} \arrow[r, "{f,g}"] \arrow[d, "1\times i_1"', "1\times i_0"] & \mathcal{Z} \\ \mathcal{X} \times \mathbb{A}^1 \arrow[ur, "h"'] & \end{tikzcd}$$

we have $h \circ 1 \times i_0 = f$ and $h \circ 1 \times i_1 = g$. The set of naive-homotopy classes of maps from $\mathcal{X}$ to $\mathcal{Y}$ is denoted by $[\mathcal{X}, \mathcal{Y}]_{\mathbb{A}^1_{\mathrm{nv}}}$

The following lemma has been used at a couple of ocassions in this article.

Lemma 5.4.4. *Let $\mathcal{X} \times \mathbb{A}^1 \xrightarrow{h} \mathcal{Y}$ be a naive $\mathbb{A}^1$-homotopy between two maps $h_0, h_1 : \mathcal{X} \to \mathcal{Y}$ of presheaves. Then for every F-algebra R, the induced maps*

$$h_0, h_1 : \mathcal{X}\Delta^\bullet_R \to \mathcal{Y}\Delta^\bullet_R$$

of the singular simplicial sets are simplicially homotopic.

Proof. This follows from [MV99, prop 2.3.4, p 53]. □

There is a surjective map of presheaves $GrO(H^\infty) \twoheadrightarrow \widehat{W}$ defined by sending a non-degenerate subbundle $\mathcal{E} \subset H^\infty$ to the class of $\mathcal{E}$ in $\widehat{W}$. Below we will define a surjective map $\widehat{W} \xrightarrow{\exists\,\zeta} \pi\, GrO(H^\infty)\Delta^\bullet$ making the diagram

$$\begin{tikzcd} & GrO(H^\infty) \arrow[dl, two heads] \arrow[dr, two heads] & \\ \widehat{W} \arrow[rr, dashed, two heads, "\zeta"] & & \pi_0\, GrO(H^\infty)\Delta^\bullet \end{tikzcd}$$

commute. The following discussion leads us to a definition of ζ in proposition 5.4.11. In what follows $\mathcal{G}l_n$ will denote the usual group scheme of the invertible n-square matrices of the ring of global sections. Let us define the presheaf of isometries of the hyperbolic space H^n. We will study this presheaf in detail later in next section.

Definition 5.4.5 (Orthogonal Group)**.** Let $V = (V, \phi)$ be a non-degenerate smmetric bilinear space over F. The presheaf $O(V)$ of isometries of V is defined by $X \mapsto O(V_X)$, where $V_X = (\mathcal{O}_X \otimes_F V, \phi)$ and $O(V_X)$ is the group of isometries of V_X. If $V \subset W$ is a non-degenerate subspace of W, there is a canonical inclusion of presheaves $O(V) \hookrightarrow O(W)$ defined by extending an isometry of V to W via the identity on the orthogonal complement of V in W. If $V = (V, \phi)$ is a symmetric bilinear space (possibly degenerate) over an F-vector space V (not necessarily finite dimensional), then we set

$$O(V) = \mathrm{colim}_{W \subset V} O(W)$$

where the indexing set runs over all the non-degenerate finite dimensional subspaces W of V.

Remark 5.4.6. If $V \xrightarrow{i} W$ is an inclusion of non-degenerate subspaces then we can describe the map $O(V) \to O(W)$ is given by a conjugation: Let $\bar{V}$ be the orthogonal complement of V in W. Denote the orthogonal sum decomposition of W by $V \perp \bar{V} \xrightarrow{(i,\bar{i})}$, by g the inverse of $(i, \bar{i})$. Then the map $O(V) \to O(W)$ is defined by $f \mapsto g(f \perp id_{\bar{V}})g^{-1}$. In particular, if $g : V \xrightarrow{\simeq} W$ is an isometry, the map $O(V) \to O(W)$ is given by conjugation with the isomtry g.

Remark 5.4.7. In particular, we have the presheaves $O(H^n)$ and $O(H^\infty)$ of groups of isometries of hyperbolic spaces. For an F-algebra R, the group $O(H^n)(R)$ can be described as the group of square matrices M of size $2n$ with entries in R which have the property that ${}^tM\, h_n\, M = h_n$. It is called the orthogonal group presheaf. In fact, $O(H^n)$ is a representable sheaf. The standard isometric embeddings $H^n(R) \hookrightarrow H^{n+1}(R) \hookrightarrow \cdots$ induce inclusion of presheaves $O(H^n) \hookrightarrow O(H^{n+1})$

$$M \mapsto \begin{pmatrix} M & 0 \\ 0 & I_H \end{pmatrix}.$$

The presheaf $O(H^\infty) = \mathrm{colim}_n O(H^n)$ is the colimit.

For a matrix α defining an automorphism of R^n, the matrix

$$H(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & ({}^t\alpha)^{-1} \end{pmatrix}$$

defines an isometry of the hyperbolic space $H^n(R)$. Varying α this gives a map of schemes $\mathcal{G}l_n \to O(H^n)$. By definition the diagram

$$\begin{tikzcd} \operatorname{Spec} R \arrow[rr, "H(\alpha)"] \arrow[dr, "\alpha"'] & & O(H^n) \\ & \mathcal{G}l_n \arrow[ur, "H"'] & \end{tikzcd}$$

commutes. If two maps α, $\beta : \operatorname{Spec} R \to \mathcal{G}l_n$ of presheaves are naively $\mathbb{A}^1$-homotopic, then the maps $H(\alpha)$ and $H(\beta)$ will also be naively $\mathbb{A}^1$-homotopic.

For every positive integer n the map

$$\mu : O(H^n) \times GrO(H^n) \to GrO(H^n)$$

defined by $(\sigma, (M, h|_M)) \mapsto (\sigma M, h|_{\sigma M})$, where $h|$ denotes the restriction of the form on hyperbolic spaces, is an action of $O(H^n)$ on $GrO(H^n)$.

Proposition 5.4.8. *Let $M, N \subset H^\infty(R)$ be two non-degenerate submodules of $H^\infty(R)$. Assume that $M = (M, h_{|M})$ and $N = (N, h_{|N})$ are isometric. Then the two maps* $\operatorname{Spec} R \to GrO(H^\infty)$ *determined by M and N are naively $\mathbb{A}^1$-homotopic.*

Proof. There is an m such that $M, N \subset H^m(R) \subset H^\infty(R)$. Let $\bar{M}$, $\bar{N}$ be the orthogonal complements of M, N in H^m, and denote by $i, \bar{i}, j, \bar{j}$ the inclusions $M, \bar{M}, N, \bar{N} \subset H^m(R)$. So, we have isometries $M \perp \bar{M} \xrightarrow{(i,\bar{i})} H^m \xleftarrow{(j,\bar{j})} N \perp \bar{N}$. By hypothesis, there is an isometry $\alpha : M \to N$. Consider the isometry σ of $H^m \perp H^m$ defined by the right vertical matrix in the commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\binom{i}{0}"] \arrow[d, "\alpha"] & H^m \perp H^m \arrow[d, "\sigma"] & M \perp \bar{M} \perp N \perp \bar{N} \arrow[l, "\begin{pmatrix} i\bar{i} & \\ & j\bar{j} \end{pmatrix}"'] \arrow[d, "\begin{pmatrix} \alpha & & & \\ & & & 1 \\ & & \alpha^{-1} & \\ & 1 & & \end{pmatrix}"] \\
N \arrow[r, "\binom{j}{0}"] & H^m \perp H^m & N \perp \bar{N} \perp M \perp \bar{M}. \arrow[l, "\begin{pmatrix} j\bar{j} & \\ & i\bar{i} \end{pmatrix}"']
\end{tikzcd}$$

In this diagram both horizontal compositions are the embeddings into the respective first factor. Therefore, the isometry $\sigma : H^{2m} \to H^{2m}$ satisfies $\sigma(M) = N$. Then the isometry $\sigma \perp \sigma^{-1} = \left(\begin{smallmatrix} \sigma & \\ & \sigma^{-1} \end{smallmatrix}\right)$ of H^{4m} under the natural inclusion $H^{2m} \hookrightarrow H^{4m}$

maps $M \subset H^{4m}$ to $N \subset H^{4m}$. Thus, in the diagram

$$\begin{tikzcd}[column sep=small]
& O(H^{4m}) \times GrO(H^{4m}) \arrow[dr, "\mu"] & \\
\operatorname{Spec} R \arrow[ur, "{(1, M)}"'] \arrow[dr, "{(\sigma\perp\sigma^{-1}, M)}"] & & GrO(H^{4m}) \\
& O(H^{4m}) \times GrO(H^{4m}) \arrow[ur, "\mu"'] &
\end{tikzcd} ,$$

the upper composition is given by M and the lower composition is given by N. The proof of the proposition is complete in view of the following lemma. $\square$

Lemma 5.4.9. *Given an isometry σ of $H(R^n) = (R^n \oplus R^n, h = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$ the map of presheaves $\operatorname{Spec} R \xrightarrow{\sigma\perp\sigma^{-1}} O(H^n \perp H^n)$ is naively $\mathbb{A}^1$-homotopic to identity.*

Proof. Let $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ be the matrix representation of the isometry σ. Thus, σ is invertible and ${}^t\sigma\, h\, \sigma = h$. From which we get that $\sigma^{-1} = \left(\begin{smallmatrix} {}^t\delta & {}^t\beta \\ {}^t\gamma & {}^t\alpha \end{smallmatrix}\right)$. Writing explicitly, we get the following relations among the blocks of σ

$$\alpha\, {}^t\delta + \beta\, {}^t\gamma = I, \quad \gamma\, {}^t\delta + \delta\, {}^t\gamma = \alpha\, {}^t\beta + \beta\, {}^t\alpha = 0 \ \text{ and}$$

$${}^t\alpha\, \delta +^t \gamma\beta = I, \quad {}^t\alpha\, \gamma +^t \gamma\, \alpha =^t \beta\, \delta +^t \delta\beta = 0.$$

Consider the matrices

$$A = \begin{pmatrix} 0 & \gamma \\ -{}^t\gamma & {}^t\gamma\alpha \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \beta \\ -{}^t\beta & {}^t\delta\beta \end{pmatrix},$$

$$e_{12}(\alpha) = \begin{pmatrix} I & \alpha \\ 0 & I \end{pmatrix}, \qquad \text{and} \qquad e_{21}(-{}^t\delta) = \begin{pmatrix} I & 0 \\ -{}^t\delta & I \end{pmatrix}.$$

Let $\tau = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}$ be the base-change isometry of hyperbolic spaces given by $H(R^n) \perp H(R^n)$ with the form $\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}$ and $H(R^n \oplus R^n)$ with the form $\begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix}$. Note that $\tau = \tau^{-1}$. We have the identity [K90, lemma 4.4.6, p 347]

$$\tau\,(\sigma\perp\sigma^{-1})\tau = X_-(A)\cdot H(e_{12}(\alpha))\cdot H(e_{21}(-{}^t\delta)\cdot X_+(B)\cdot X_-(A)\cdot H(e_{12}(\alpha))\cdot H \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$$

where for an endomorphism A of R^n having the property $A = -{}^tA$, the matrices

$$X_-(A) = \begin{pmatrix} I & 0 \\ A & I \end{pmatrix} \qquad \text{and} \qquad X_+(A) = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix}$$

define isometries of $H(R^n) \perp H(R^n)$. This is straightforward to check. The map $\operatorname{Spec} R \to O(H^n \perp H^n)$ defined by the matrix $X_-(A)$ is naively $\mathbb{A}^1$-homotopic to identity via the map $\left(\begin{smallmatrix} I & 0 \\ TA & I \end{smallmatrix}\right)$, similarly for $X_+(B)$. The maps defined by $H(e_{12}(\alpha))$ and $H(e_{21}(-{}^t\delta))$ are also naively $\mathbb{A}^1$-homotopic to identity via $H(e_{12}(T\alpha))$ and $H(e_{21}(-T\,{}^t\delta))$ respectively. In the matrix identity

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \cdot \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}$$

the three maps on the right defined by the given matrices are naively $\mathbb{A}^1$-homotopic to identity, and hence, so is the invertible matrix $\left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$ and the isometry $H\left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$. Thus, all the isometries involved in the product for the isometry $\tau(\sigma \perp \sigma^{-1})\tau$ are naively $\mathbb{A}^1$-homotopic to identity, and hence so is this isometry and the isometry $\sigma \perp \sigma^{-1}$. $\square$

Let Iso_d be the presheaf $X \mapsto Iso_d(X)$, where $Iso_d(X)$ denotes the set of isometry classes of rank d non-degenerate symmetric bilinear spaces over $\mathcal{O}_X$. It is a subpresheaf of $\widehat{W}$. The following two corollaries are direct consequences of proposition 5.4.8 in view of lemma 5.4.4.

Corollary 5.4.10. *Given a non-degenerate symmetric bilinear form (M, ϕ) of rank d over R, a choice of an isometric embedding $M \subset H^\infty(R)$ determines an element in the naive $\mathbb{A}^1$-homotopy class $[\operatorname{Spec} R, GrO(d, H^\infty)]_{\mathbb{A}^1_{\mathrm{nv}}}$. This element is independent of the choice of embedding and depends only on the isometry class of*

M, and hence $\exists!$ map $\zeta_d : Iso_d \to \pi_0 GrO(d, H^\infty)\Delta^\bullet$ such that the diagram

$$\begin{tikzcd} & GrO(d, H^\infty) \arrow[dl] \arrow[dr] & \\ Iso_d \arrow[rr, "\zeta_d"] & & \pi_0\, GrO(d, H^\infty)\Delta^\bullet \end{tikzcd}$$

commutes.

Since $\widehat{W} = \coprod_{d\geq 0} Iso_d$ above corollary gives the following map as well.

Corollary 5.4.11. *There is a unique surjective map $\zeta : \widehat{W} \to \pi_0\, GrO(H^\infty)\Delta^\bullet$ whose restriction to Iso_d is the map ζ_d defined in the previous corollary such that the diagram*

$$\begin{tikzcd} & GrO(H^\infty) \arrow[dl] \arrow[dr] & \\ \widehat{W} \arrow[rr, two heads, "\zeta"] & & \pi_0\, GrO(H^\infty)\Delta^\bullet \end{tikzcd}$$

commutes.

Remark 5.4.12. For a presheaf $\mathcal{X}$, it's $\mathbb{A}^1$-π_0 presheaf $\pi_0^{\mathbb{A}^1}\mathcal{X}$ is defined by $X \mapsto [X, \mathcal{X}]$, where $[\ ,\ ]$ denotes the set of maps in the homotopy category of F-smooth schemes. Let us recall how we identify the Grothendieck-Witt group of a ring R with $\pi_0^{\mathbb{A}^1} KO(R)$. Since KO has BG-property and is homotopy invariant for affines, a fibrant replacement $KO \to KO_f$ in the model category of Morel and Voevodsky is a global weak equivalence on affines, [H05]. Thus, in the following commutative diagram all the maps except possibly the left vertical one are isomorphisms

$$\begin{tikzcd} \pi_0^{\mathbb{A}^1} KO(R) \arrow[r, "\simeq"] & \pi_0^{\mathbb{A}^1} KO_f(R) \\ \pi_0 KO(R) \arrow[u] \arrow[r, "\simeq"] & \pi_0 KO_f(R). \arrow[u, "\simeq"'] \end{tikzcd}$$

And hence the left vertical map $\pi_0 KO(R) \to \pi_0^{\mathbb{A}^1} KO(R)$ is also an isomorphism.

For every positive integer r, the set of isomorphism classes of objects of the monoidal category $\mathcal{S}_{r,R} = \mathcal{S}_{r,\mathrm{Spec} R}$ defined in 5.3.1 is isomorphic to the set $\widehat{W}(R)$

via the map that sends $\mathcal{E} \subset H^r \perp H^\infty$ to the class of $\mathcal{E}$ in $\widehat{W}(R)$. Thus, the group completion $\mathcal{S}_{r,R} \to \mathcal{S}_{r,R}^{-1}\mathcal{S}_{r,R}$ used in defining KO in 5.3.3 gives us

$$KO_0(R) = \pi_0(\mathcal{S}_{r,R}^{-1}\mathcal{S}_{r,R}) \xrightarrow{\simeq} GW(R) \qquad (V, W) \mapsto V - W.$$

Thus, the 0^{th}-hermitian K-group of a ring is it's Grothendieck-Witt group.

We have a commutative diagram of presheaves of sets on Sm/F

$$\begin{tikzcd} & GrO(H^\infty) \arrow[dl] \arrow[dr, two heads] & \\ \widehat{W} \arrow[rr, two heads, "\zeta"] \arrow[d, "can"'] & & \pi_0\, GrO(H^\infty)\Delta^\bullet \arrow[d, "\hbar"] \\ \pi_0\,(KO)_0 \arrow[rr] & & \pi_0\,|\,(KO)_0\,\Delta^\bullet_R\,| \end{tikzcd}$$

where *can* is given by the group completion functor that was used in the definition of KO in 5.3.3, and the map $\pi_0\,(KO)_0 \to \pi_0\,|\,(KO)_0\,\Delta^\bullet\,|$ is induced by the map $(KO)_0 \to |\,(KO)_0\,\Delta^\bullet\,|$ in 5.4.2.

Proposition 5.4.13. *The map ζ induces an isomorphism on affine regular sections* $\operatorname{Spec} R$ *such that R is local and $\frac{1}{2} \in R$, and therefore it is a Zariski weak equivalence.*

Proof. We need to prove that the map $\widehat{W}(R) \xrightarrow{\zeta_R} \pi_0\, GrO(H^\infty)\Delta^\bullet_R$ is an isomorphism for a regular ring R with $\frac{1}{2} \in R$. In the diagram of sets

$$\begin{tikzcd} \widehat{W}(R) \arrow[r, two heads, "\zeta_R"] \arrow[d, "can"'] & \pi_0\, GrO(H^\infty)\Delta^\bullet_R \arrow[d, "\hbar"] \\ \pi_0(KO)_0(R) \arrow[r, "\simeq"] & \pi_0\,|\,(KO)_0\,\Delta^\bullet_R\,| \end{tikzcd} \tag{5.4.1}$$

the lower horizontal map $\pi_0(KO)_0(R) \to \pi_0\,|\,(KO)_0\,\Delta^\bullet_R\,|$ is an isomorphism because of homotopy invariance of hermitian K-theory [H05]. From remark 5.4.12, we see that the set $\pi_0\,(KO)_0(R)$ coincides with the Grothendieck-Witt group of R, and is the group completion of the monoid $\widehat{W}(R)$ via the group completion map *can*.

Since for a local ring $\widehat{W}(R)$ is a cancellative monoid in view of Witt's cancellation theorem [S85, thm 6.5, p.21], therefore, for local rings the map *can* is injective. And hence from commutativity of above diagram we see that the map ζ_R is injective as well for regular local ring R with $\frac{1}{2} \in R$. Therefore, ζ induces an isomorphism for such regular local rings. $\square$

Remark 5.4.14. The proposition 5.4.13 is true for the restriction map ζ_d from the presheaf Iso_d also, and hence, we have the isomorphism of the associated Zariski sheaves

$$a_{Zar}\, Iso_d \xrightarrow{\zeta_d} a_{Zar}\, \pi_0\, GrO(d,\, H^\infty)\Delta^\bullet.$$

The example of Parimala in [P76] shows that in general the map ζ_d is not an isomorphism before sheafification.

As a direct cosequence of this proposition we have the following important result.

Theorem 5.4.15. *For a regular local ring R such that $\frac{1}{2} \in R$, the set $\pi_0\, GrO\Delta^\bullet_R$ is isomorphic to the group $\pi_0\,|\,KO\,\Delta^\bullet_R\,| = \pi_0\, GW\Delta^\bullet_R = GW(R)$ via the map $\hbar$. Thus, the induced map*

$$\pi_0\, GrO\Delta^\bullet \xrightarrow{\hbar} \pi_0\,|\,KO\,\Delta^\bullet_R\,| = \pi_0 GW\Delta^\bullet$$

gives isomorphism on affine regular local sections $\operatorname{Spec} R$ *with* $\frac{1}{2} \in R$.

Proof. The colimit $\cdots \xrightarrow{H\perp} \widehat{W}(R) \xrightarrow{H\perp} \widehat{W}(R) \xrightarrow{H\perp} \cdots$ with respect to adding the hyperbolic plane is the Grothendieck-Witt group $GW(R)$. Thus, taking the colimit of the diagram 5.4.1 with respect to orthogonal sum of hyperbolic plane over $(KO)_r$ and $GrO(H^r \perp H^\infty)$ (see 5.2.3 and 5.3.3) we get the commutative diagram

$$\begin{tikzcd} GW(R) \arrow[r, "\zeta_R", "\simeq"'] \arrow[d, "\mathrm{Id}"] & \pi_0\, GrO\Delta^\bullet_R \arrow[dl, "\hbar_R"] \\ GW(R) = \pi_0|\,KO\Delta^\bullet_R\,| & \end{tikzcd}$$

in which the vertical map is identity (see remark 5.4.12) and ζ_R is an isomorphism by the previous proposition. Thus, the map $\hbar_R : \pi_0\, GrO\Delta^\bullet_R \to \pi_0\,|\,KO\,\Delta^\bullet_R\,|$ is also an isomorphism for a regular local ring R with $\frac{1}{2} \in R$. $\square$

5.5 The Classifying Spaces BO and $BO(H^n)$ of Orthogonal Group

We have defined the infinite orthogonal group $O = O(H^\infty)$ in 5.4.7 as the colimit $O(H^r) \xrightarrow{H\perp} O(H^{r+1}) \cdots$ given by addition of hyperbolic plane. In proposition 5.5.2 we will identify its classifying space BO with the connected component $KO_{[0]}$ of 0 of hermitian K-theory (to be defined in 5.5.1). In the second half of this section in the $\mathbb{A}^1$-homotopy category we will identify the classifying space $BO(H^n)$ of $O(H^n)$ (see 5.4.7) with the supresheaf $Gr(H^n,\, H^\infty)$ of $GrO(H^\infty)$ of those non-degenerate symmetric bilinear spaces which are isometric to the hyperbolic spaces H^n defined in 5.5.3. For this purpose we have defined the Stiefel presheaves $St(H^n,\, H^\infty)$ and proved in theorem 5.5.7 that they are contractible; this part of argument works just like the one in topology.

Definition 5.5.1 (Presheaf $KO_{[0]}$ of connected component of 0)**.** Let us denote the presheaf $\mathcal{S}_n^{-1}\mathcal{S}_n$ of categories defined in 5.3.1 by $h\mathcal{P}_n$. Let $h\mathcal{P}_n^0$ be the subpresheaf of connected component of $(0,\, H^n)$ in $h\mathcal{P}_n$: An object $((M,\, \phi),\, (N,\, \psi))$ of $h\mathcal{P}_n(X)$ belongs to the category $h\mathcal{P}_n^0(X)$ if and only if $(M,\, \phi)$ and $(N\perp H^n,\, \psi\perp h_n)$ are stably isometric in the sense that they become isometric after addition of an hyperbolic space. The maps $h\mathcal{P}_n \xrightarrow{(0,\,H)\perp} h\mathcal{P}_{n+1}$ given by addition of hyperbolic space induce homotopy equivalences of subpresheaves $h\mathcal{P}_n^0 \xrightarrow{(0,\,H)\perp} h\mathcal{P}_{n+1}^0$. The simplicial presheaf $KO_{[0]}$ is the colimit of nerve simplicial presheaves

$$KO_{[0]}(X) = \mathrm{colim}_{n\geq 0}(\cdots \xrightarrow{(0,\,H)\perp} \mathscr{N}(h\mathcal{P}_n^0) \xrightarrow{(0,\,H)\perp} \mathscr{N}(h\mathcal{P}_{n+1}^0) \xrightarrow{(0,\,H)\perp} \cdots)$$

in which all the maps $\mathscr{N}(h\mathcal{P}_n^0) \xrightarrow{(0,\,H)\perp} \mathscr{N}(h\mathcal{P}_{n+1}^0)$ are homotopy equivalences.

Let $\widetilde{O}(H^n)$ be the presheaf of categories having a unique object and all the automorphisms in $O(H^n)$ as the set of morphisms. For every $n \geq 0$, there is a functor $\gamma_n : \widetilde{O}(H^n)(R) \to h\mathcal{P}_0^n$ sending the unique object of the category $\widetilde{O}(H^n)(R)$ to $(0,\, H^n(R))$ and an automorphism u of $H^n(R)$ to the morphism

$$((0,\, 0) + (0,\, H^n(R))) \xrightarrow{\{0,u\}} (0,\, H^n(R))$$

in $h\mathcal{P}_n^0(R)$. The diagram

$$\begin{tikzcd}
\widetilde{O}(H^n) \arrow[r, "\gamma_n"] \arrow[d, "H\perp"] & h\mathcal{P}_n^o \arrow[d, "(0,H)\perp"] \\
\widetilde{O}(H^{n+1}) \arrow[r, "\gamma_{n+1}"] & h\mathcal{P}_0^{n+1}
\end{tikzcd}$$

commutes, and hence, taking the nerve simplicial presheaves and colimit we get a map

$$\gamma : BO = \mathscr{N}\widetilde{O} \to KO_{[0]}.$$

In the next proposition we will prove that the map γ is an $\mathbb{A}^1$-weak equivalence. For notations see the definition 5.4.2.

Proposition 5.5.2. *The map*

$$\gamma : |BO\Delta^\bullet| \to |KO_{[0]}\Delta^\bullet|$$

induced by γ is a global weak equivalence, and hence the map γ is an $\mathbb{A}^1$-weak equivalence.

Proof. In the commutative diagram

$$\begin{tikzcd}
BO \arrow[r, "\gamma"] \arrow[d, "\mathbb{A}^1"] & KO_{[0]} \arrow[d, "\mathbb{A}^1"] \\
|BO\Delta^\bullet| \arrow[r, "\gamma"] & |KO_{[0]}\Delta^\bullet|
\end{tikzcd}$$

the map γ is a global homology isomorphsim, that is, the map of simplicial sets $\gamma : |BO\Delta^\bullet| \to |KO_{[0]}\Delta^\bullet|$ is an homology isomorphism for every smooth F-scheme X as proved in [G76]. For a more accessible reference one can look at the proof of theorem 7.4 in [S96, p. 152]. Since $|BO\Delta^\bullet_R|$ and $|KO_{[0]}\Delta^\bullet_R|$ are both connected H-spaces with respect to the H-space structure induced from the operad structures discussed in section 5.6, the homology weak equivalence γ is a global weak equivalence. Therefore, the map γ is a global weak equivalence. $\square$

Definition 5.5.3 (Presheaves $GrO(V, V\perp H^m)$ and $GrO(V, V\perp H^\infty)$)**.** Let $V = (V, \phi)$ be an inner product space and $W = (W, \psi)$ be symmetric bilinear space (possibly degenerate and infinite dimensional) over F such that V is a subspace of W. For an F-algebra R, let $GrO(V, W)(R) \subset GrO(W)(R)$ be the set subbundles of $W_R = W \otimes_F R$ isometric to $V_R = V \otimes_F R$. The presheaf $GrO(V, W)$ is defined by $X \mapsto GrO(V, W)(\Gamma(X, \mathcal{O}_X)$. It is a subpresheaf of $GrO(W)$. In particular, we have the presheaves $GrO(V, V\perp H^m)$ and $GrO(V, V\perp H^\infty)$.

Definition 5.5.4 (The Stiefel presheaf $St(V, V\perp H^\infty)$)**.** Let $V = (V, \phi)$ be an inner product space and $W = (W, \psi)$ be symmetric bilinear space (possibly degenerate and infinite dimensional) over F such that V is a subspace of W. Let $St(V, W)(R)$ be the set of isometric embeddings of $V_R \hookrightarrow W_R$, where $V_R = V \otimes_F R$ and $W = W \otimes_F R$. We have the presheaf $St(V, W)$ on Sm/F defined by $X \mapsto St(V, W)(\Gamma(X, \mathcal{O}_X))$. In particular, we have the presheaves $St(V, V\perp H^m)$ and $St(V, V\perp H^\infty)$ which will be referred to as Stiefel presheaves.

Note that the set $St(H^n, H^n\perp H^m)(R)$ can also be described as the set of $(2m+2n)\times 2n$ matrices M with entries in R such that $M^t h_{m+n} M = h_n$. The natural map $St(H^n, H^n\perp H^m) \hookrightarrow St(H^n, H^n\perp H^{m+1})$ corresponding to the standard isometric embedding $H^n \perp H^m(R) \hookrightarrow H^n \perp H^{m+1}(R)$ is then given by $M \mapsto \binom{M}{0}$, where 0 is $2 \times 2n$ size zero matrix. We have the colimit presheaf $St(H^n, H^n \perp H^\infty)$.

Remark 5.5.5. We have a free right action of $O(V)$ on $St(V, W)$ by composition:

$$St(V, W) \times O(V) \to St(V, W), \qquad (i, \sigma) \mapsto i \circ \sigma.$$

The map $\eta : St(V, W) \to GrO(V, W)$ defined by $i : V \hookrightarrow W \mapsto Im(i)$, which is a non-degenerate subbundle of W isometric to V, induces an isomorphism

$$\eta : St(V, W)/O(V) \to GrO(V, W).$$

In theorem 5.5.7 we will prove that the Stiefel presheaf $St(V, V \perp H^\infty)$ is $\mathbb{A}^1$-contractible. Using this result we will identify the classifying space of $O(V)$ with the presheaf $GrO(V, V \perp H^\infty)$ in proposition 5.5.10. First, let us recall a general result.

Lemma 5.5.6. *Suppose that a group G acts freely on simplicial sets X and Y and there is a G-equivariant weak equivalence $X \to Y$. Then the induced map $X/G \to Y/G$ is a weak equivalence.*

Proof. We can assume that $X \to Y$ is a surjective trivial Kan fibration. We will prove that $X/G \to Y/G$ is a trivial Kan fibration, that is, a lift exists in the following diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & X/G \arrow[d] \\ \Delta^n \arrow[r] & Y/G. \end{tikzcd}$$

In fact we only need to prove that there is lift in the following diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & X/G \arrow[d] \\ \Delta^n \arrow[r, "id"] & \Delta^n. \end{tikzcd}$$

Since for a free G-space Y the principal G-bundle $Y \to Y/G = \Delta^n$ is trivial, we need to construct a lift in the left block of the diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & X/G \arrow[d] & X \arrow[l] \arrow[d] \\
\Delta^n \arrow[r, "id"] & \Delta^n & \Delta^n \times G. \arrow[l, "proj"']
\end{tikzcd}$$

This lift can be constructed by choosing the section of $proj$ given by $\sigma \mapsto (\sigma, 1)$ and constructing a lift for the outer diagram.

□

The map

$$O(V\perp H^\infty) \to St(V, V\perp H^\infty), \quad g \mapsto V \simeq Im(gV) \hookrightarrow V\perp H^\infty$$

induces an isomorphism

$$\frac{O(V\perp H^\infty)}{O(H^\infty)} \to St(V, V\perp H^\infty)$$

of the quotient presheaf with the Stiefel presheaf. In the following proposition we will prove that the Stiefel presheaf $St(V, V\perp H^\infty)$ is contractible by proving that the inclusion $O(H^\infty) \to O(V\perp H^\infty)$ is an $\mathbb{A}^1$-weak equivalence, and hence, the quotient presheaf $\frac{O(V\perp H^\infty)}{O(H^\infty)}$ is $\mathbb{A}^1$-contractible.

Theorem 5.5.7. *The inclusion $O(H^\infty) \hookrightarrow O(V\perp H^\infty)$ defined in* 5.4.7 *is an $\mathbb{A}^1$-weak equivalence, and hence, the Stiefel presheaf $St(V, V\perp H^\infty)$ is $\mathbb{A}^1$-contractible. More precisely, for every F-algebra R, the simplicial set $St(V, V\perp H^\infty)\Delta^\bullet_R$ is contractible.*

Proof. See the definition 5.4.2 and the comment following it. We prove that the inclusion $O(H^\infty)\Delta^\bullet_R \xrightarrow{\imath} O(V\perp H^\infty)\Delta^\bullet_R$ is a weak equivalence of simplicial sets. Since the simplicial groups $O(H^\infty)\Delta^\bullet$ and $O(V\perp H^\infty)\Delta^\bullet$ are H-groups, it suffices to prove that $\imath$ is a homology isomorphism for affine F-schemes. This is done in the following two lemmas. $\square$

Lemma 5.5.8. *The map* $H_*(\imath_R) : H_*(O(H^\infty)\Delta^\bullet_R) \to H_*(O(V\perp H^\infty)\Delta^\bullet_R)$ *is injective.*

Proof. Let $\xi \in H_p(O(H^\infty)\Delta^\bullet_R)$ such that $H_*(\imath)(\xi) = 0$ in $H_p(O(V\perp H^\infty)\Delta^\bullet_R)$. We can assume that there is a positive integer n such that ξ is given by the class of $\zeta \in H_p(O(H^n)\Delta^\bullet_R)$ and $H_p(\imath_R)(\zeta) = 0$ in $H_p(O(V\perp H^n)\Delta^\bullet_R)$. Let $\bar{V}$ be the orthogonal complement of V in $H^m(R)$ for some positive integer m, that is, $\bar{V}\perp V = H^m(R)$. Let $V\perp H^n \xrightarrow{g} \bar{V}\perp V\perp H^n$ be the inclusion, and hence we have a commutative diagram of isometric embeddings

$$\begin{tikzcd}
H^n(R) \arrow[r, "\imath"] \arrow[d, "can"] & V\perp H^n(R) \arrow[dl, "g"] \\
(H^m\perp H^n)(R) &
\end{tikzcd}$$

which induces the following commutative diagram

$$\begin{tikzcd}
O(H^n)\Delta^\bullet_R \arrow[rr, "\imath_R"] \arrow[dr, "can=H^m\perp"'] & & O(V\perp H^n)\Delta^\bullet_R \arrow[dl, "g"] \\
& O(H^m\perp H^n)\Delta^\bullet_R. &
\end{tikzcd}$$

Thus, we see that the image of ζ in $H_p(O(H^{m+n})\Delta^\bullet_R)$ is 0. Hence, $\xi = 0 \in H_p(O(H^\infty)\Delta^\bullet_R)$. $\square$

Lemma 5.5.9. *The map* $H_*(\imath_R) : H_*(O(H^\infty)\Delta^\bullet_R) \to H_*(O(V\perp H^\infty)\Delta^\bullet_R)$ *is surjective.*

Proof. Let $\xi \in H_p(O(V\perp H^\infty)\Delta^\bullet_R)$. We can choose a positive integer n such that ξ is the image of an element $\xi \in H_p(O(V\perp H^n)\Delta^\bullet_R)$.

We claim that there is an isometric embedding $(V\perp H^n)(R) \xrightarrow{k} H^m(R)$ and an isometry $\tau : (V\perp H^m)(R) \to (V\perp H^m)(R)$ such that the following diagram commutes and conjugation with τ is naively $\mathbb{A}^1$-homotopic to the identity

$$\begin{tikzcd}
O(V\perp H^n)(R) \arrow[r, "k"] \arrow[d, "can"] & O(H^m(R)) \arrow[d, "\imath_R"] \\
O(V\perp H^m)(R) \arrow[r, "conj\,\tau"] & O(V\perp H^m)(R)
\end{tikzcd} \tag{5.5.1}$$

where the map *can* is the canonical inclusion and $conj\,\tau$ is the conjugation with τ defined by $g \mapsto \tau \cdot g \cdot \tau^{-1}$.

To verify this, choose an isometric embedding $V \hookrightarrow H^r(R)$ and let $\bar{V}$ be the orthogonal complement of V in $H^r(R)$. Denote the orthogonal sum decomposition by $(j, \bar{j}) : V\perp\bar{V} \xrightarrow{\simeq} H^r(R)$ and let $\left(\begin{smallmatrix} p \\ \bar{p} \end{smallmatrix}\right) : H^r(R) \to V\perp\bar{V}$ be its inverse. The following diagram commutes

$$\begin{tikzcd}
(V\perp H^n)(R) \arrow[rr, "\left(\begin{smallmatrix} j & 0 \\ 0 & 1 \end{smallmatrix}\right)"] \arrow[d, "can", "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)"'] & & (H^r\perp H^n)(R) \arrow[d, "\imath_R"', "\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)"] \\
(V\perp H^r\perp H^n)(R) \arrow[rr, "\sigma_0=\left(\begin{smallmatrix} o & p & 0 \\ j & \bar{j}\bar{p} & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)"] & & (V\perp H^r\perp H^n)(R)
\end{tikzcd}$$

where σ_0 is the isometry given by the matrix $\left(\begin{smallmatrix} o & p & 0 \\ j & \bar{j}\bar{p} & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$.

Let $\sigma = \sigma_0\perp id_{\bar{V}}$ be the isometry of the hyperbolic space $(V\perp H^r\perp H^n\perp\bar{V})(R) \simeq H^{2r+n}(R)$. In view of lemma 5.4.9, the isometry $\tau_0 = \sigma\perp\sigma^{-1}$ of $H^{4r+2n}(R)$ is naively homotopic to the identity. Hence, the isometry $\tau = id_V\perp\tau_0$ of $V\perp H^{4r+2n} = V\perp H^m$, where $m = 4r + 2n$, is also naively homotopic to identity. The diagram below commutes

$$\begin{tikzcd}
(V\perp H^n)(R) \arrow[r, "k"] \arrow[d, "can"] & H^m(R) \arrow[d, "\imath"] \\
(V\perp H^m)(R) \arrow[r, "\tau"] & (V\perp H^m)(R)
\end{tikzcd}$$

and induces the commutative diagram as claimed in 5.5.1 in view of remark 5.4.6. Therefore, we have a commutative diagram of simplicial groups

$$\begin{tikzcd}
O(V\perp H^n)\Delta^\bullet_R \arrow[r, "k"] \arrow[d, "can"] & O(H^m)\Delta^\bullet_R \arrow[d, "\imath_R"] \\
O(V\perp H^m)\Delta^\bullet_R \arrow[r, "\tau"] & O(V\perp H^m)\Delta^\bullet_R
\end{tikzcd}$$

in which the map τ is simplicially homotopic to the identity, see lemma 5.4.4.

Let $\zeta = H_p(k)(\xi) \in H_p(O(H^m)\Delta^\bullet_R)$. Then $H_p(\imath_R)(H_p(k)(\xi)) = H_p(\imath_R)(\zeta)) = H_p(\tau)(H_p(can)(\xi)) = H_p(\tau)(\xi) = \xi \in H_p(O(V\perp H^m)\Delta^\bullet_R)$, since $H_p(\tau)$ is the identity. Thus, the map $\imath_R$ is surjective on homology. $\square$

Proposition 5.5.10. *For the $O(V)$-equivariant projection maps with diagonal action on $St(V, V\perp H^\infty) \times EO(V)$*

$$St(V, V\perp H^\infty) \xleftarrow{proj} St(V, V\perp H^\infty) \times EO(V) \xrightarrow{proj} EO(V)$$

in the induced zig-zag of simplicial presheaves

$$GrO(V, V\perp H^\infty)\Delta^\bullet \xleftarrow{global} \bullet \xrightarrow{global} |BO(V)\Delta^\bullet|$$

both the maps are global weak equivalences. Hence, in the $\mathbb{A}^1$-homotopy category we have an isomorphism $BO(V) \simeq GrO(V, V\perp H^\infty)$.

Proof. The $O(V)$-action on $St(V, V\perp H^\infty)$, $St(V, V\perp H^\infty) \times EO(V)$ and $EO(V)$ is free. Also, both the projection maps are global weak equivalences because of the global contractibility of $St(V, V\perp H^\infty)\Delta^\bullet$ (proved in 5.5.7) and $|EO(V)\Delta^\bullet|$. Therefore, in view of lemma 5.5.6 and remark 5.5.5, taking quotients we get the required zig-zag

$$GrO(V, V\perp H^\infty)\Delta^\bullet \simeq \frac{St(V, V\perp H^\infty)\Delta^\bullet}{O(V)\Delta^\bullet} \xleftarrow{global} \bullet \xrightarrow{global} \frac{|EO(V)\Delta^\bullet|}{O(V)} = |BO(V)\Delta^\bullet|$$

of global weak equivalences. Hence, this yields an isomorphism $BO(V) \simeq GrO(V, V\perp H^\infty)$ in the $\mathbb{A}^1$-homotopy category. $\square$

5.6 An E_∞-Space Structure on Simplicial Presheaf $GrO\Delta^\bullet$

In this section we will define an E_∞-operad $\mathcal{E}\Delta^\bullet$ in the category of simplicial presheaves on Sm/F, and show that the sections of $GrO\Delta^\bullet$ are H-spaces by giving them E_∞-space structures. We will also prove that the map $\hbar : GrO\Delta^\bullet \to |\,KO\Delta^\bullet\,|$ induced from the map defined in 5.3.5 is a H-space map.

Definition 5.6.1 (E_∞-operad $\mathcal{E}\Delta^\bullet$)**.** For a positive integer j and an F-algebra R, let $\mathcal{E}(j)(R)$ be the set of isometric embeddings of $(H^\infty\perp\cdots\perp H^\infty)(R)$ in $H^\infty(R)$. This set can be described as the following limit. Let $V=(V_1,\cdots,V_j)$ $(j\geq 1)$ be a j-tuple of non-degenerate subspaces of H^∞. Let $\mathcal{E}^V(j)(R)$ be the set of isometric embeddings of $(V_1\perp\cdots\perp V_j)(R)\hookrightarrow H^\infty(R)$. Note that an isometry in $\mathcal{E}^V(j)(R)$ factors as $(V_1\perp\cdots\perp V_j)(R)\hookrightarrow H^n(R)\to H^\infty(R)$ for some positive integer m. We have the presheaf $\mathcal{E}^V(j)$ of sets on Sm/F defined by $X\mapsto\mathcal{E}^V(j)(\Gamma(X,\mathcal{O}_X))$. For j-tuples V and V', define $V\subset V'$ if $V_i\subset V_1'\subset H^\infty$, $i=1,\cdots,j$. If $V\subset V'$, we have a map of presheaves $\mathcal{E}^{V'}(j)\to\mathcal{E}^V(j)$ given by restricting an isometric embedding $(V_1'\perp\cdots\perp V_j')(R)\to(H^\infty)(R)$ to $(V_1\perp\cdots\perp V_j)(R)$. The presheaf $\mathcal{E}(j)$ is the limit presheaf

$$\mathcal{E}(j)=\varprojlim_{V=(V_1,\cdots,V_j):V_i\subset H^\infty}\mathcal{E}^V(j).$$

In fact, we can restrict ourselves to j-tuples of the form $H^r=(H^r(R),\cdots,H^r(R))$ only. Define $\mathcal{E}(0)=pt$. Denote the presheaf $\mathcal{E}^{H^r}(j)$ by $\mathcal{E}^r(j)$. The symmetric group Σ_j acts on the presheaves $\mathcal{E}^r(j)$ on the right through permutation of blocks as follows: Let $\lambda:(H^r\perp\cdots\perp H^r)(R)\to H^\infty(R)$ be an isometric embedding in $\mathcal{E}^r(j)(R)$ and $\sigma\in\Sigma_j$ a permutation. Let i_σ be the isometry

$$(H^r\perp\cdots\perp H^r)(R)\xrightarrow{(e_{\sigma(1)},\cdots,e_{\sigma(j)})}(H^r\perp\cdots\perp H^r)(R)$$

determined by σ. Define $\lambda\sigma$ to be the composition $\lambda \circ i_\sigma$. This induces a Σ_j-action on $\mathcal{E}(j)$. This action is free: For, if $\alpha \circ \sigma = \alpha \circ id = \alpha$, then $\sigma = id$ since α is an isometry.

For positive integers $k; j_1 \cdots, j_k$, let $(\delta; \lambda_1, \cdots, \lambda_k) \in (\mathcal{E}(k) \times \mathcal{E}(j_1) \times \cdots \times \mathcal{E}(j_k)(R))$. Given a positive integer r, there exists a positive integer m such that each λ_i factors through an isometric embedding $(H^r \perp \cdots \perp H^r)(R) \xrightarrow{\lambda_i} H^m(R)$. The composite isometry $\delta \circ (\lambda_1 \perp \cdots \perp \lambda_k)$ is an element of the set $\mathcal{E}(j_1 + \cdots + j_k)(R)$. This defines a map

$$\mathcal{E}(k) \times \mathcal{E}(j_1) \times \cdots \times \mathcal{E}(j_k) \to \mathcal{E}(j_1 + \cdots + j_k)$$

$$(\delta; \lambda_1, \cdots, \lambda_k) \mapsto \delta \circ (\lambda_1 \perp \cdots \perp \lambda_k)$$

giving a Σ_j-free operad $\mathcal{E}$ in the category of presheaves on Sm/F. See [M72, def 1.1 p. 2] for the definition of an operad.

Proposition 5.6.2. *For every F-algebra R, $\mathcal{E}\Delta^\bullet_R = (\mathcal{E}(j)\Delta^\bullet_R)$ is an E_∞-operad.*

Proof. To prove this, we need to show that for all $j \geq 0$ the simplicial sets $\mathcal{E}(j)\Delta^\bullet_R$ are contractible. Note that each $\mathcal{E}(j)$ is isomorphic to $\mathcal{E}(1)$ (although not functorially). Thus, we need to prove the contractibility of $\mathcal{E}(1)\Delta^\bullet_R$ only. By definition of the Stiefel presheaves in 5.5.4, we see that $\mathcal{E}^r(1)$ is the presheaf $St(H^r, H^\infty)$ and

$$\mathcal{E}(1)\Delta^\bullet_R = \varprojlim_{r \geq 0} St(H^r, H^\infty)\Delta^\bullet_R. \tag{5.6.1}$$

Let $O(H^\infty) \to O(H^\infty)$ be the map of presheaves of groups defined by $\sigma \mapsto \begin{pmatrix} I_{2r} & \\ & \sigma \end{pmatrix}$. Denote the image of this morphism by $O(H^{r^\perp}) \subset O(H^\infty)$. We have inclusions of simplicial groups $O(H^{r+1^\perp})\Delta^\bullet_R \subset O(H^{r^\perp})\Delta^\bullet_R \subset O(H^\infty)\Delta^\bullet_R$. Since for an inclusion $H_0 \subset H_1 \subset G$ of simplicial groups the induced map of simplicial sets $\frac{G}{H_0} \to \frac{G}{H_1}$ is a Kan fibration of Kan complexes, hence, the map of quotient presheaves

$$\frac{O(H^{\infty})\Delta_R^{\bullet}}{O(H^{r+1^{\perp}})\Delta_R^{\bullet}} \to \frac{O(H^{\infty})\Delta_R^{\bullet}}{O(H^{r^{\perp}})\Delta_R^{\bullet}} \tag{5.6.2}$$

is a Kan fibration of Kan sets.

The map $O(H^{\infty}) \to St(H^r, H^{\infty})$ given by $\sigma \mapsto (\sigma(H^r) \hookrightarrow H^{\infty})$ induces an isomorphism $\frac{O(H^{\infty})}{O(H^{r^{\perp}})} \xrightarrow{\simeq} St(H^r, H^{\infty})$ such that the map $St(H^{r+1}, H^{\infty})\ \Delta_R^{\bullet} \to St(H^r, H^{\infty})\Delta_R^{\bullet}$ in the limit in 5.6.1 is the induced map in 5.6.2. In proposition 5.5.7 we have proved that the simplicial sets $St(H^r, H^{\infty})\Delta_R^{\bullet}$ are contractible. Thus, in the limit 5.6.1 each map is a Kan fibration of contractible Kan complexes, and hence, the limit presheaf is contractible. $\square$

Now we will define an action of the E_{∞} operad $\mathcal{E}\Delta^{\bullet}$ on $GrO\Delta^{\bullet}$, see the definition 5.2.3 and [M72, def. 1.2, p 3]. Note that the presheaf GrO can also be described as the colimit

$$GrO = \varinjlim_{U \subset H^{\infty}} GrO(U^{-} \perp U)$$

where U^{-} denotes the canonical subspace of $H^{-\infty}$ isometric to U, and, the colimit is taken over all the finite dimensional non-degenerate subspaces of H^{∞} as in the definition of orthogonal Grassmannian in 5.2.3 with respect to maps

$$GrO(U^{-} \perp U) \to GrO(V^{-} \perp V), \qquad W \mapsto U_V^{-} \perp W$$

corresponding to subspaces $U \subset V$ of H^{∞}, where U_V^{-} denotes the orthogonal complement of U^{-} in V^{-}.

We need to define compatible maps $\mathcal{E}(j) \times GrO \times \cdots \times GrO \to GrO$. Let $\delta \in \mathcal{E}(j)(R)$ and $(V_1, \cdots, V_j) \in (GrO \times \cdots \times GrO)(R)$. Choose non-degenerate subspaces $W_i \subset H^{\infty}$, $i = 1, \cdots, j$ such that $V_i \in GrO(W_i^{-} \perp W_i)\ (R)$. Let $W = (W_i, \cdots, W_j)$, then δ determines an isometric embedding $W_1 \perp \cdots \perp W_j \xrightarrow{\delta} H^{\infty}$. There is a subspace $U \subset H^{\infty}$ such that δ factors as $W_1 \perp \cdots \perp W_j \xrightarrow{\delta} U \subset H^{\infty}$. Let

$\delta^- : W_1^- \perp \cdots \perp W_j^- \to U^-$ be the isometric embedding induced by δ by symmetry. We have the isometric embedding $\left(\begin{smallmatrix}\delta^- & \\ & \delta\end{smallmatrix}\right) : W_1^- \perp \cdots \perp W_j^- \perp W_1 \perp \cdots \perp W_j \to U^- \perp U$. Denote the orthogonal complement of $W_1^- \perp \cdots \perp W_j^-$ in U^- by W_U^-. The element

$$\left(\begin{smallmatrix}\delta^- & \\ & \delta\end{smallmatrix}\right)(V_1 \perp \cdots \perp V_j) \perp W_U^-$$

of $GrO(U^- \perp U)$ determines a well-defined element $\delta \cdot (V_1, \cdots, V_j) \in GrO(R)$ and an action of the E_∞-operad $\mathcal{E}\Delta^\bullet$ on $GrO\Delta^\bullet$.

From this action of the operad $\mathcal{E}\Delta^\bullet$ on $GrO\Delta^\bullet$ and the general theory of E_∞-operads in the category of simplicial sets gives us the following.

Proposition 5.6.3. *For every F-algebra R, the simplicial set $GrO\Delta_R^\bullet$ is an H-space, and hence, in the $\mathbb{A}^1$-homotopy category the simplicial presheaf $GrO\Delta^\bullet$ is an H-space.*

Remark 5.6.4. Th H-space structure on GrO corresponds to a choice of a direct sum of two non-degenerate subspaces of $H^\infty \perp H^\infty$. The simplicial presheaf $|KO\Delta^\bullet|$ is also an H-space with respect to an orthogonal sum of forms and the map $\hbar$ is an H-space map.

5.7 $\mathbb{A}^1$-Weak Equivalence of the Map $\hbar$

In this section we will prove the main result of the paper:

Theorem 5.7.1 ($\mathbb{A}^1$-Representability Theorem)**.** *For every regular ring R with $\frac{1}{2} \in R$, the map $\hbar_R : GrO\Delta_R^\bullet \to |KO\Delta_R^\bullet|$ induced by the map $\hbar : GrO \to KO$ defined in 5.3.5 is a weak equivalence of simplicial sets. Therefore, the map $\hbar$ is an $\mathbb{A}^1$-weak equivalence.*

Proof. In proposition 5.5.10, we have obtained a zig-zag of $\mathbb{A}^1$-weak equivalences for every $n \geq 0$

$$GrO(H^n, H^n \perp H^\infty) \simeq \frac{St(H^n, H^n \perp H^\infty)}{O(H^n)} \xleftarrow{proj} \mathfrak{X}_n \xrightarrow{proj} \frac{EO(H^n)}{O(H^n)} = BO(H^n)$$

where $\mathfrak{X}_n = \frac{St(H^n, H^n \perp H^\infty) \times EO(H^n)}{O(H^n)}$ and the maps *proj* are the natural projections. As seen in 5.5.10, this zig-zag induces global weak equivalences of simplicial presheaves

$$GrO(H^n, H^n \perp H^\infty)\Delta^\bullet \xleftarrow{proj} |\mathfrak{X}_n \Delta^\bullet| \xrightarrow{proj} |BO(H^n)\Delta^\bullet|. \tag{5.7.1}$$

Consider the pullback diagram of simplicial sets

$$\begin{tikzcd} GrO(H^n, H^n \perp H^\infty)\Delta^\bullet \arrow[r] \arrow[d] & GrO(H^n \perp H^\infty)\Delta^\bullet \arrow[d] \\ pt \arrow[r, "H"] & \pi_0(GrO(H^n \perp H^\infty)\Delta^\bullet) \end{tikzcd} \tag{5.7.2}$$

where pt is the constant presheaf and $\pi_0(GrO(H^n \perp H^\infty)\Delta^\bullet)$ is the π_0 presheaf of the simplicial presheaf $GrO(H^n \perp H^\infty)\Delta^\bullet$.

The map $GrO(H^n \perp H^\infty) \to (KO)_n$ used to define the map $\hbar$ in 5.3.5 restricts to a map

$$GrO(H^n, H^n \perp H^\infty) \to \mathscr{N}(h\mathcal{P}_n^0) \tag{5.7.3}$$

See definitions 5.5.3 and 5.5.1. Let GrH be the colimit of $GrO(H^n, H^n \perp H^\infty) \xrightarrow{H\perp} GrO(H^{n+1}, H^{n+1} \perp H^\infty) \xrightarrow{H\perp} \cdots$ corresponding to the addition of hyperbolic planes. The colimit of 5.7.3 with respect to $H\perp$ gives us a map $GrH \to KO_{[0]}$ such that the diagram

$$\begin{tikzcd} GrH \arrow[r] \arrow[d] & KO_{[0]} \arrow[d] \\ GrO \arrow[r, "\hbar"] & KO \end{tikzcd} \tag{5.7.4}$$

commutes. Denote the map $GrH \to KO_{[0]}$ by $\hbar_0$.

Taking the colimit of diagram 5.7.2 with respect to $H\perp$, we get a pullback diagram

$$\begin{tikzcd} GrH\Delta^\bullet \arrow[r] \arrow[d] & GrO\Delta^\bullet \arrow[d] \\ pt \arrow[r, "H"] & \pi_0(GrO\Delta^\bullet). \end{tikzcd}$$

The colimit of 5.7.1 with respect to $H\perp$ gives us a zig-zag of global weak equivalences $GrH\Delta^\bullet \leftarrow \mathfrak{X} \rightarrow |\,BO\Delta^\bullet\,|$, where $\mathfrak{X}$ denotes the colimit of the middle term. In view of the next lemma 5.7.2, this pullback diagram maps into the pullback diagram

$$\begin{tikzcd} |\,KO_{[0]}\Delta^\bullet\,| \arrow[r] \arrow[d] & |\,KO\Delta^\bullet\,| \arrow[d] \\ pt \arrow[r] & \pi_0\,GW\Delta^\bullet \end{tikzcd}$$

via the maps $\hbar$ (5.3.5), $\hbar_0$ (5.7.2) and ζ (5.4.15). Thus, the maps of simplicial presheaves

$$\begin{tikzcd} GrH\Delta^\bullet \arrow[r, "\hbar_0"] \arrow[d] & |\,KO_{[0]}\Delta^\bullet\,| \arrow[d] \\ GrO\Delta^\bullet \arrow[r, "\hbar"] \arrow[d] & |\,KO\Delta^\bullet\,| \arrow[d] \\ \pi_0(GrO\Delta^\bullet) \arrow[r, "\zeta"] & \pi_0\,GW\Delta^\bullet \end{tikzcd}$$

induces a map fibrations of simplicial sets at regular affine sections. In the next lemma 5.7.2 we have proved that $\hbar_0$ is a global weak equivalence. We have also proved in theoren 5.4.15 that the induced map ζ_R is a weak equivalence of simplicial sets when R is regular and $\frac{1}{2} \in R$. Thus, the induced map

$$\hbar_R : GrO\Delta^\bullet_R \rightarrow KO\Delta^\bullet_R$$

is a weak equivalence of simplicial sets at the base point 0 for a regular ring R in which 2 is invertible. Since $\hbar_R$ is a map of group complete H-spaces as remarked in

5.6.4, it is a simplicial weak equivalence for such R. Thus, the map $\hbar$ is an $\mathbb{A}^1$-weak equivalence. $\square$

Lemma 5.7.2. *The following diagram of simplicial presheaves commutes upto a simplicial homotopy*

$$\begin{tikzcd} \frac{St(H^n, H^n\perp H^\infty)\times EO(H^n)}{O(H^n)} \arrow[r, "proj"] \arrow[d, "proj"] & BO(H^n) \arrow[d, "\gamma_n"] \\ \frac{St(H^n, H^n\perp H^\infty)}{O(H^n)} \arrow[r] & \mathscr{N}h\mathcal{P}_0^n \end{tikzcd} \tag{5.7.5}$$

where the maps proj are natural projection maps and the lower horizontal map is the composition $St(H^n, H^n\perp H^\infty) \xrightarrow{\alpha\mapsto Im\alpha} GrO(H^n, H^n\perp H^\infty) \xrightarrow{(5.7.3)} \mathscr{N}h\mathcal{P}_0^n$. *Furthermore, this induces the diagram*

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "proj"] \arrow[d, "proj"] & \lvert BO\Delta^\bullet \rvert \arrow[d, "\gamma"] \\ GrH\Delta^\bullet \arrow[r, "\hbar_0"] & \lvert KO_{[0]}\Delta^\bullet \rvert \end{tikzcd} \tag{5.7.6}$$

which is commutative upto a simplicial homotopy, and hence the map $\hbar_0 : GrH\Delta^\bullet \to \lvert KO\Delta^\bullet \rvert$ *is a global weak equivalence.*

Proof. The presheaf $\frac{St(H^n, H^n\perp H^\infty)}{O(H^n)}$ is nerve of the discrete category having the sections in $\frac{St(H^n, H^n\perp H^\infty)}{O(H^n)}$ as its object.

Also, the simplicial presheaf $\frac{St(H^n, H^n\perp H^\infty)\times EO(H^n)}{O(H^n)}$ is nerve of the category $\mathcal{C}$ whose objects are the objects of the discrete category $\frac{St(H^n, H^n\perp H^\infty)}{O(H^n)}$, and a morphism from $H^n \xrightarrow{i} H^n\perp H^\infty$ to $H^n \xrightarrow{j} H^n\perp H^\infty$ is give by an isometry $\sigma : H^n \to H^n$ such that $j \circ \sigma = i$. Let us denote such a morphism by (i, σ, j). In diagram 5.7.5, the composition

$$\frac{St(H^n, H^n\perp H^\infty) \times EO(H^n)}{O(H^n)} \xrightarrow{proj} BO(H^n) \to \mathscr{N}h\mathcal{P}_0^n$$

is induced by the functor $F : \mathcal{C} \to h\mathcal{P}_0^n$ which sends the object $(H^n \xrightarrow{i} H^n \perp H^\infty) \in \mathcal{C}$ to $(0, H^n) \in h\mathcal{P}_0^n$, and sends the morphism (i, σ, j) to the morphism given by the isometry σ in the category $h\mathcal{P}_0^n$. The other composition in this diagram is induced by the functor $G : \mathcal{C} \to h\mathcal{P}_0^n$ defined by $(H^n \xrightarrow{i} H^n \perp H^\infty) \mapsto (0, Imi) \in h\mathcal{P}_0^n$. Any choice of an isometry of $Imi \to H^n$ defines a natural transformation $G \to F$ upto an element of the group $O(H^n)$, thus the diagram 5.7.5 commutes upto a simplicial homotopy.

The remaining part of this lemma is a formal consequence of the first half of the proof. $\square$

5.8 Concluding Remarks

In this final section we will comment on a few aspects of the representability theorem that we did not address in this paper.

This representability theorem should give us a better way to study the multiplicative structure on hermitian K-theory. It should also allow us to understand the analog of Atiyah's real KR-theory. This unstable $\mathbb{A}^1$-representability result can be extended to a stable geometric representability result for the hermitian K-theory as given by the spectrum $\mathbb{K}O$ defined in [H05].

With the $\mathbb{A}^1$-representability theorem, it can be proved that the realization of hermitian K-theory by taking the complex points is the topological K-theory of real vector bundles, whereas the realization by taking the real points is a product of two copies of topological K-theory of real vector bundles.

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