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A study of regret and rejoicing and a new MCDM method based on them

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A STUDY OF REGRET AND REJOICING AND A NEW MCDM METHOD BASED ON THEM

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ABSTRACT

Multi-criteria decision-making (MCDM) is one of the most widely used decision methodologies in the sciences, business, and engineering worlds. MCDM methods aim at improving the quality of decisions by making the process more explicit, rational, and efficient. One controversial problem is that some well-known MCDM methods, like the additive AHP methods and the ELECTRE II and III methods, may cause some types of rank reversal problems. Rank reversal means that the ranking between two alternatives might be reversed after some variation occurs to the decision problem, like adding a new alternative, dropping an old alternative or replacing a non-optimal alternative by a worse one etc. Usually such a rank reversal is undesirable for decision-making problems. If a method does allow it to happen, the validity of the method could be questioned. However, some recent studies indicate that rank reversals could also happen because of people's rational preference reversal which may be caused by their emotional feelings, like regret and rejoicing.

Since regret and rejoicing may play a pivotal role in evaluating alternatives in MCDM problems, sometimes the decision maker (DM) may want to anticipate these emotional feelings and consider them in the decision-making process. Most of the regret models in the literature use continuous functions to measure this emotional factor. This dissertation proposes to use an approach based on a linguistic scale and pairwise comparisons to measure a DM's anticipated regret and rejoicing feelings. The approach is shown to exhibit some key advantages over existing approaches. Next a multiplicative MCDM model is adopted to aggregate the alternatives' associated regret and rejoicing values with their performance values to get their final priorities and then rank them. A

simulated numerical example is used to illustrate the process of the proposed method. Some sensitivity analyses which aim at examining how changes of regret and rejoicing values might affect the ranking results of the decision problems are also developed. Then a fuzzy version of the new method is introduced and illustrated by a numerical example. Finally, some concluding remarks are made. Ranking intransitivity and some other issues about the proposed method are analyzed too.

CHAPTER 1. PRELIMINARY PROBLEM DESCRIPTION

Making all kinds of decisions is an indispensable part of our lives. From the ancient times to the modern age, people never stopped their efforts in seeking ways for making more reliable and scientifically sound decisions. For those daily life decision problems, such as which shirt should one wear to match a given suit and so on, one may quickly decide it just by using his/her personal preferences, experiences, and/or instincts. However, in many fields of engineering, business, government, and sciences, where decisions may be worth millions or billions of dollars, or decisions may have a significant impact on the welfare of the society, decision-making problems are usually too complex and anything but as simple as the above one.

For instance, many large companies and organizations face the problem of prioritizing a set of competing projects. Each one of these projects may have some short-term and long-term potential profits, costs and some negative or positive side effects. At the same time, there is a limited budget to be distributed among these projects. Some of the projects may not get funded at all. Besides these projects, the decision makers have also defined some criteria to be used to evaluate these projects. When faced with such decision-making problems, no single decision maker (DM) or group of decision makers can systematically consider all the available information simultaneously and reach the right decisions by just using their experiences or personal instincts. For such cases people need to use valid decision analysis approaches and tools in analyzing all the issues involved and eventually reaching the optimal decisions. They also need to do so in a way that can be easily and objectively explained to others and be defended to a wide audience of stakeholders. This is how and why the field of decision sciences has

emerged as an important scientific discipline in today's world.

In the past few decades, numerous decision-making methods and decision aid software packages have been proposed in the literature and are used in various areas. Among them, a class of methods known as multi-criteria decision-making (MCDM) is one of the most widely used decision-making methodologies in the sciences, business, and engineering worlds. MCDM methods aim at improving the quality of decisions by making the decision-making process more explicit, rational, and efficient. Some applications of MCDM include the use in civil and environmental engineering [Zavadskas, *et al*., 2004; Hobbs and Meier, 2000], in financial engineering [Zopounidis and Doumpos, 2000], in water resources planning [Raj, 1995], in waste water or solid waste management [Rogers and Bruen, 1999; Hokkanen and Salminen, 1997], and in credit risk assessment [Doumpos, *et al*., 2002].

Although MCDM has attracted the interest of researchers and practitioners for many years in a wide spectrum of areas, it is far from being mature and there are still a lot of unresolved issues. One intriguing problem is that oftentimes different methods may yield different answers when they are fed with exactly the same decision problem and data. Thus, the issue of evaluating the relative performance of different MCDM methods is naturally raised. This, in turn, raises the question of how one can evaluate them. Since it is practically impossible to know which one is the best alternative for a given decision problem, some kind of testing procedures need to be determined. One such procedure is to examine the validity of an MCDM method's mathematical process by checking the stability and validity of its proposed rankings.

The above subjects, along with some other related issues, have been studied by

many researchers in the MCDM area [Troutt, 1988; Buchanan, 1994]. In [Triantaphyllou, 2000] some test criteria for checking whether some kinds of ranking irregularities may happen with some MCDM methods were established to examine the relative performance of those methods. By using these test criteria, it was found that two well-known MCDM methods, the original AHP method and the revised AHP method both allow for some types of rank reversals to happen (the first case of rank reversal identified with the original AHP method was reported in [Belton and Gear, 1983]). Recently, two ELECTRE methods – ELECTRE II and III, were also found to suffer of similar rank reversal problems as the additive AHP methods as discussed in [Wang and Triantaphyllou, 2006] and [Wang and Triantaphyllou, 2008]. Rank reversal means that the ranking between two alternatives might be reversed after some variation occurs to the decision problem, like adding a new alternative, dropping an old alternative or replacing an old alternative by a worse one etc. For example, two alternatives A_1 and A_2 may be initially ranked as $A_1 \succ A_2$ (i.e., A_1 is more preferable than A_2). After a new alternative A_3 is introduced into the decision problem and the alternatives are ranked again by using the same method, the ranking between A_1 and A_2 may be reversed and become $A_2 \succ A_1$. Usually, such a rank reversal is undesirable. If a method does allow it to happen, the validity of the method could be questioned.

However, some studies have shown that it is not always unreasonable to have such rank reversals happening in MCDM problems. The critical question is to be able to distinguish why they happen. When a method exhibits rank reversals, is it because it accurately captured the way rational humans deal with decision-making and their preferences change or is it because the method has some kind of numerical

instabilities/mathematical defects? Let us put it more clearly through a metaphor: suppose a method is like a photo camera or X-ray image taking device. One takes a photo or takes an X-ray image of a subject and sees something strange in that image, like some very bright spots. Do these bright spots exist in reality or are purely the result of some kind of hardware defects?

For different MCDM methods and decision models, the answer to the above question could be very different. Some past research [Belton and Gear, 1983; Dyer, 1990a and 1990b; Triantaphyllou, 2000; Wang and Triantaphyllou, 2008] has shown that the rank reversal problems with the additive AHP and ELECTRE II and III methods are mainly due to these methods' own mathematical artifacts. However, rank reversals could also happen because people's rational preferences may change by their emotional feelings. Here is one such hypothetical example: suppose one is planning to buy a new car and a dealer offers two cars, say cars A and B. In this hypothetical scenario car A is cheaper than car B but car B is of better quality than car A. Then, one may decide to buy car A because it is cheaper. Next, suppose that besides the above two cars, the dealer introduces a third car C (let us call it a phantom alternative) which may not even be at stock at that dealership but it has been publicized by the media. This third car C is much more expensive than the previous two cars but it is of slightly better quality than car B. Knowing this situation about the third car, the perspective buyer may shift his/her preference and now choose car B instead of car A without actually changing anything regarding the two initial cars and the importance of the two evaluative criteria: cost and quality. When comparing car B with car C, the buyer feels very happy for getting a great deal by paying much less money to buy an almost equal quality car B. Thus for this

example, it is this anticipated *rejoicing* feeling that makes one unintentionally to reverse his/her preference between cars A and B.

Except rejoicing, another type of emotional feeling which can greatly influence people's preference in decision-making is *regret*. This type of emotional feeling comes from the fact that humans often base their choices on comparisons across the alternatives under consideration and relative to "what might have been" under another choice [Plous, 1993; Hastie and Dawes, 2001]. For example, suppose given are two alternatives A_1 and A_2 which have been evaluated in terms of three criteria. Assume that by using some MCDM method, the overall performance value of A_1 is better than that of A_2 but the individual performance value a_{1k} of alternative A_1 under criterion C_k is worse than that of alternative A_2 under the same criterion (denoted as a_{2k}). Then the decision maker who chooses A_1 and forgoes A_2 may experience a certain level of regret because the value a_{1k} is worse than a_{2k}. This regret feeling could be so strong that he/she may regret to have chosen A_1 instead of A_2. In order to avoid the above situation, sometimes the DM would want to anticipate the regret feeling and consider it in the decision-making process by making some tradeoffs for a more balanced alternative.

From the above examples, it can be seen that making a choice/decision, no matter what kind of, can be an intensive emotional experience. When making decisions, except those cognitive considerations about the decision problems themselves, sometimes people also need to consider some intense emotional factors, like regret and rejoicing. Psychologically speaking humans often behave based on a combination of reasons and emotions. It is natural that decisions should be made by the mind and also by the heart instead of by a complete rational mind which is dissociated from psychological feelings.

Studies on the notion of regret and rejoicing for decision-making under uncertainty have been carried out for over fifty years. However, it is just in recent years that these emotional factors began to be introduced in deterministic MCDM problems. Though there are some tentative works on this direction [Kujawski, 2005; Kaliszewski and Michalowski, 1998], more studies are needed to assess the impact that these emotional factors might bring to the MCDM problems and the role that they may play in evaluating alternatives. Meanwhile, an advanced model which can incorporate the notion of regret and rejoicing systematically in the MCDM modeling framework for conflicting decision criteria needs to be developed. These are the research subjects of this dissertation.

This dissertation is organized as follows. The next chapter presents a literature review on MCDM and some studies on rank reversals with the additive AHP methods and the ELECTRE II and III methods. The third chapter describes how regret and rejoicing are considered in the decision-making process and some regret models from the literature. The fourth chapter is the most intriguing one as it proposes to use a linguistic scale to measure regret and rejoicing and determine the alternatives' associated regret and rejoicing values by developing some regret/rejoicing matrices based on pairwise comparisons. In the fifth chapter, a multiplicative MCDM model is extended to combine the alternatives' associated regret and rejoicing values with their performance values in order to eventually determine their final priorities. The case of having intransitive rankings and some other issues about the new method are also discussed. In the sixth section, a numerical example is used to illustrate the process of the proposed new method. Then, some sensitivity analyses which aim at examining how changes of regret and rejoicing might affect the ranking results of the decision problems are developed. In the

seventh chapter, a fuzzy version of the new method is introduced and illustrated by a numerical example. In the last chapter, some concluding remarks are made on the main contributions in this dissertation and the meaning of those contributions. Finally, some possible future research directions are discussed.

CHAPTER 2. LITERATURE REVIEW ON MCDM

2.1 An Introduction to MCDM

A typical MCDM problem is concerned with the task of ranking a finite number of decision alternatives, each of which is explicitly described in terms of different characteristics (also often called attributes, decision criteria, or objectives) which have to be taken into account simultaneously (as in the previously mentioned project prioritization problem). Decision criteria may be quantitative (such as cost, age, weight, volume, etc) or qualitative (such as desirability, aesthetic appeal, style, etc). They can also be cost criteria (the lower the score is, the more preferable it is) or benefit criteria (the higher the score is, the more preferable it is). Different decision criteria may be associated with different units of measure. To combine them together, the criteria values may need to be normalized. Otherwise, combining them is equivalent to *"adding apples and oranges"*. Usually, the alternatives' performance values under the decision criteria and the criteria weights are viewed as the entries of a decision matrix defined as in Figure 1. The a_{ij} element of the decision matrix represents the performance value of the *i-th* alternative in terms of the *j-th* criterion. The parameter w_j represents the weight of the *j-th* criterion. Data for MCDM problems can be determined by direct observation (if they are easily quantifiable) or by indirect means if they are qualitative [Triantaphyllou *et al.*, 1994].

Another term that is also used frequently to mean the same type of decision models is multi-criteria decision analysis (MCDA). There is a subtle difference between these two terms. The term MCDM is often used to mean finding the best alternative in continuous decision spaces. However, in the setting of MCDA, the alternatives are not

known a priori but they can be determined by calculating the values of a number of discrete and/or continuous variables. Usually, an MCDA method aims at one of the following four goals, or "problematics" [Roy, 1985], [Jacquet-Lagreze and Siskos, 2001]:

Problematic 1:	Find the best alternative.
Problematic 2:	Group the alternatives into well-defined classes.
Problematic 3:	Rank the alternatives in order of total preference.
Problematic 4:	Describe how well each alternative meets all the criteria simultaneously.

Many interesting aspects of MCDA theory and practice are discussed in [Hobbs, 1986], [Hobbs, *et al.*, 1992], [Stewart, 1992], [Triantaphyllou, 2000], [Zanakis, *et al.*, 1995], and [Zanakis, *et al.*, 1998]. The terms MCDM and MCDA may also be used to denote the same class of models.

	Criteria			
	C_1	C_2	...	C_n
	$(w_1$	w_2	...	$w_n)$
Alternatives				
A_1	a_{11}	a_{12}	...	a_{1n}
A_2	a_{21}	a_{22}	...	a_{2n}
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
A_m	a_{m1}	a_{m2}	...	a_{mn}

Figure 1. Structure of a typical decision matrix.

From the early developments of the MCDM theories in the 1950s and 1960s, a

plethora of MCDM methods have been developed in the literature and new contributions are continuously coming forth in this area. There are also many ways to classify the existing MCDM methods. One of the ways is to classify MCDM methods according to the type of data they use. Thus, there are deterministic, stochastic, and fuzzy MCDM methods [Triantaphyllou, 2000]. Another way of classifying MCDM methods is according to the number of the decision makers involved in the decision process. Hence, there are single decision maker MCDM methods and group decision-making MCDM. For some representative articles in this area, see [George, *et al*., 1992], [Hackman and Kaplan, 1974], and [DeSanctis and Gallupe, 1987]. In this dissertation, the research concentrates on single decision maker deterministic MCDM problems which attempt to find the best alternative subject to a finite number of decision criteria.

2.2 Some Well-known MCDM Methods

Among the numerous MCDM methods, there are several prominent families that have enjoyed a wide acceptance in the academic area and many real-world applications. Each of these methods has its own characteristics and background logic. Next is a brief description of some of them.

2.2.1 The Analytic Hierarchy Process and Some of Its Variants

The Analytic Hierarchy Process (or AHP) method was developed by Professor Thomas Saaty [Saaty, 1980; Saaty, 1994; and Saaty and Vargas, 2000]. This decision-making method can help people set priorities and choose the best options by reducing complex decision problems to a system of hierarchies. Since its inception, it has evolved into several different variants and has been widely used to solve a broad range of multi-criteria decision problems [Vaidya and Kumar, 2006].

2.2.1.1 The Original Analytic Hierarchy Process

The AHP method uses the pairwise comparisons and eigenvector methods to determine the a_{ij} values and also the criteria weights w_j. The details about the pairwise comparisons and the eigenvector methods can be found in [Saaty, 1980; Saaty, 1994; and Saaty and Vargas, 2000]. In this method, a_{ij} represents the relative performance value of alternative A_i when it is considered in terms of criterion C_j. In the original AHP method, the a_{ij} values of the decision matrix need to be normalized vertically. That is, the elements of each column in the decision matrix add up to one. In this way, values with various units of measurement can be transformed into dimensionless ones. If all the criteria are benefit criteria, then according to the original AHP method, the best alternative is the one that satisfies the following expression:

$$P^{*}_{AHP} = \max_{i} P_i = \max_{i} \sum_{j=1}^{n} a_{ij} w_j \,, \qquad \text{for } i = 1, 2, 3, ..., m. \tag{2-1}$$

From the above formula, we can see that the original AHP method uses an additive expression to determine the final priorities of the. Next the revised AHP is introduced, which is also an additive variant of the original AHP method.

2.2.1.2 The Revised Analytic Hierarchy Process

The revised AHP model was proposed by Belton and Gear in [1983] after they first found a case of rank reversal that occurred when the original AHP method was used. In their case, the original AHP method was used to rank three alternatives in a simple test problem. Then a fourth alternative, identical to one of the three alternatives, was introduced in the original decision problem without changing any other data. The ranking of the original three alternatives was changed after the revised problem was ranked again

by the same method. The following is this rank reversal example from [Belton and Gear, 1983].

Suppose the decision matrix of a decision problem with three alternatives and three criteria is as follows:

	Criteria		
	C_1	C_2	C_3
	(1/3	1/3	1/3)
Alts.			
A_1	1	9	8
A_2	9	1	9
A_3	1	1	1

By using the original AHP method, the above decision matrix is normalized first by the column totals to get the relative data as follows:

	Criteria		
	C_1	C_2	C_3
	(1/3	1/3	1/3)
Alts.			
A_1	1/11	9/11	8/18
A_2	9/11	1/11	9/18
A_3	1/11	1/11	1/18

Then, it can be shown that the final AHP scores of the three alternatives are: (0.45, 0.47, 0.08). That is, $A_2 \succ A_1 \succ A_3$. Next, a new alternative A_4 which is identical to the existing alternative A_2 is added to the decision matrix. Now the normalized decision

matrix is as follows:

	Criteria		
	C_1	C_2	C_3
	(1/3	1/3	1/3)
Alts.			
A_1	1/20	9/12	8/27
A_2	9/20	1/12	9/27
A_3	1/20	1/12	1/27
A_4	9/20	1/12	9/27

By using the same AHP method, now the final AHP scores of these alternatives are: (0.37, 0.29, 0.06, 0.29). That is, the four alternatives are ranked as $A_1 \succ A_2 = A_4 \succ A_3$. This result contradicts the previous one in which $A_2 \succ A_1$.

According to Belton and Gear the root for this inconsistency is the fact that the relative values of the alternatives for each criterion sum up to one. So instead of having the relative values of the alternatives sum up to one, they proposed to divide each relative performance value by the maximum of the relative values. According to this variant, the a_{ij} values of the decision matrix need to be normalized by dividing the elements of each column in the decision matrix by the largest value in that column. As before, the best alternative is given again by the additive formula (2-1), but now the normalization is different.

$$P^*_{Revised-AHP} = \max_i P_i = \max_i \sum_{j=1}^{n} a_{ij} w_j \,, \qquad \text{for } i = 1, 2, 3, ..., m. \qquad (2\text{-}2)$$

The revised AHP was sharply criticized by Saaty in [1990]. After many debates and a heated discussion (e.g., [Dyer, 1990a; and 1990b], [Saaty, 1983; 1987; and 1990], and

[Harker and Vargas, 1990]), Saaty accepted this variant and now it is also called the ideal mode AHP [Saaty, 1994].

However, the revised AHP method was found to suffer of some other ranking problems even without the introduction of identical alternatives [Triantaphyllou and Mann, 1989; Triantaphyllou, 2000]. Most of the problematic situations of the additive AHP methods can be attributed to the required normalization (either by dividing by the sum of the elements or by the maximum value in a vector) and also the use of an additive formula on the data of the decision matrix for deriving the final preference values of the alternatives.

In the core step of one of the MCDM methods known as the Weighted Product Model (WPM) [Bridgeman, 1922; Miller and Starr, 1969], the use of an additive formula is avoided by using a multiplicative expression. This brought the development of a multiplicative version of the AHP method, known as the multiplicative AHP.

2.2.1.3 The Multiplicative Analytic Hierarchy Process

The use of multiplicative formulas in deriving the relative priorities in decision-making is not new [Lootsma, 1991]. A critical development appears to be the use of multiplicative formulations when one aggregates the performance values a_{ij} with the criteria weights w_j. In the WPM method, each alternative is compared with others in terms of the product of a number of ratios, one for each criterion. Each ratio is raised to the power of the relative weight of the corresponding criterion. In general, the following formula is used ([Bridgeman, 1922; Miller and Starr, 1969]) in order to compare two alternatives A_K and A_L:

$$R\left(\frac{A_K}{A_L}\right)=\prod_{j=1}^{n}\left(\frac{a_{Kj}}{a_{Lj}}\right)^{w_j} \tag{2-3}$$

If $R(A_K / A_L) > 1$, then A_K is more desirable than A_L (for the maximization case). Then the best alternative is the one that is better than or at least equal to all other alternatives.

Based on the WPM method, Barzilai and Lootsma in [1994] and Lootsma in [1999] proposed the multiplicative version of the AHP method. According to this method, the performance values a_{ij} and criteria weights w_j are not processed according to formula (2-1), but the WPM formula (2-3) is used instead. Furthermore, one can use a variant of formula (2-3) to compute preference values of the alternatives that in turn, can be used to rank them. The preference values can be computed as follows:

$$P_{i,multi-AHP}=\prod_{j=1}^{n}\left(a_{ij}\right)^{w_j} \tag{2-4}$$

Please note that if $P_i > P_j$, then $P_i / P_j > 1$, or equivalently, $P_i - P_j > 0$. That is, two alternatives A_i and A_j can be compared in terms of their preference values P_i and P_j by forming the ratios or, equivalently, the differences of their preference values.

By using the multiplicative formula, no matter how the decision matrix is normalized, the ratios of the alternatives' performance values are kept the same because the normalization factor is cancelled off in the multiplicative formula. Thus most of the ranking irregularities which occur to the additive AHP methods will not happen with the multiplicative AHP method. These properties of the multiplicative AHP method have been demonstrated theoretically in [Triantaphyllou, 2000].

2.2.2 The ELECTRE Methods

Another prominent role in MCDM methods is played by the ELECTRE approach

and its derivatives. The acronym ELECTRE stands for: *ELimination Et Choix Traduisant la REalité* (ELimination and Choice Expressing REality) [Roy, 1985]. This approach was first introduced in [Benayoun, *et al.*, 1966]. The main idea of this method is the proper utilization of what is called "outranking relations" to rank a set of alternatives. The ELECTRE approach uses the data of the decision problems along with some additional threshold values set by the decision makers to measure the degree to which each alternative outranks all others. Soon after the introduction of the first version known as ELECTRE I [Roy, 1968], this approach has evolved into a number of other variants. Among those variants, the ELECTRE II [Roy and Bertier, 1971, 1973] and the ELECTRE III [Roy, 1978] methods have been widely accepted in solving MCDM problems in the engineering world, like civil and environmental engineering [Hobbs and Meier, 2000].

For most ELECTRE methods, there are two main stages: the construction of the outranking relations and the exploitation of these relations to get the final ranking of the alternatives. Different ELECTRE methods may differ in how they define the outranking relations between the alternatives and how they apply these relations to get the final ranking of the alternatives. The construction of the outranking relations is based on the evaluation of two indices, the *concordance index* and the *discordance index*, defined for each pair of alternatives. The concordance index for a pair of alternatives *a* and *b* measures the strength of the hypothesis that alternative *a* is at least as good as alternative *b*. The discordance index measures the strength of evidence against this hypothesis [Belton and Stewart, 2001]. There are no unique measures of concordance and discordance indices. Since the ELECTRE approach is more complicated than the AHP

approach, the process of ELECTRE II is described next for a simple introduction of its logic.

In ELECTRE II, the concordance index *C(a, b)* for each pair of alternatives *(a, b)* is defined as follows:

$$C(a,b)=\frac{\sum_{i\in Q(a,b)} w_i}{\sum_{i=1}^{m} w_i}.$$

Where *Q (a, b)* is the set of criteria for which alternative *a* is equal or preferred to (i.e., at least as good as) alternative *b* and w_i is the weight of the *i-th* criterion. One can see that the concordance index is the proportion of the criteria weights allocated to those criteria for which *a* is equal or preferred to *b*. The discordance index *D (a, b)* for each pair *(a, b)* is defined as follows:

$$D(a,b)=\frac{\max_i [g_i(b)-g_i(a)]}{\delta}.$$

Where $g_i(a)$ and $g_i(b)$ represent the performance values of alternatives *a* and *b* in terms of criterion C_i and $\delta=\max_i |g_i(b)-g_i(a)|$ (i.e., the maximum difference on any criterion). This formula can only be used when the scores for different criteria are comparable. After computing the concordance and discordance indices for each pair of alternatives, two outranking relations are built between the alternatives by comparing the indices with two pairs of threshold values. They are referred to as the strong and weak outranking relations.

Next, two pairs of values (C^*, D^*) and (C^-, D^-) are defined as the concordance and discordance thresholds for the strong and weak outranking relations where $C^* > C^-$ and $D^* < D^-$. Then the outranking relations will be built based on the following rules:

(1) If $C(a, b) \geq C^*$, $D(a, b) \leq D^*$ and $C(a, b) \geq C(b, a)$, then alternative a is regarded as *"strongly outranking"* alternative b.

(2) If $C(a, b) \geq C^-$, $D(a, b) \leq D^-$ and $C(a, b) \geq C(b, a)$, then alternative a is regarded as *"weakly outranking"* alternative b.

The values of (C^*, D^*) and (C^-, D^-) are decided by the decision maker for a particular outranking relation. These threshold values may be varied to give more or less severe outranking relations; the higher the value of $C^*$and the lower the value of D^*, the more severe (i.e., stronger) the outranking relation is. That is, the more difficult it is for one alternative to outrank another one [Belton and Stewart, 2001]. After establishing the strong and weak outranking relations between the alternatives, the *descending* and *ascending distillation* processes are applied to the outranking relations to get two pre-orders of the alternatives. Next by combining the two pre-orders together, the overall ranking of the alternatives is determined. For a detailed description of the distillation processes, please refer to [Belton and Stewart, 2001] and [Rogers, *et al.*, 1999].

Compared with the simple process and precise data requirement of the AHP methods, ELECTRE methods apply some more complicated algorithms to deal with complex and imprecise information from the decision problems and rank the alternatives. The ELECTRE algorithms look reliable and in neat format. People believe that the process of this approach could lead to an explicit and logical ranking of the alternatives. However this is not always the case. In [Wang and Triantaphyllou, 2008], it was found that the ELECTRE II and III methods may cause some of the same ranking irregularity problems as the additive AHP methods because of its own mathematical artifacts.

2.2.3 Rank Reversals with the Additive AHP and the ELECTRE II and III Methods

As mentioned in Section 2.2.1.2, the revised AHP method was found to suffer of some other ranking problems even without the introduction of identical alternatives. Besides the rank reversal case found by Belton and Gear in [1983], some other types of ranking irregularities which happened with the additive AHP methods were reported in [Triantaphyllou and Mann, 1989; Triantaphyllou, 2000]. In one type of test, a decision problem is decomposed into a set of smaller problems, each defined on two alternatives at a time and the same number of criteria as in the original problem. The alternatives are ranked two at a time and also all of them simultaneously. Then, the ranking of the alternatives from the smaller problems may not follow the transitivity property or the combined ranking from the smaller problems may not be the same as the ranking deduced from the original un-decomposed problem. The reason is that the normalization factor might be different when alternatives are ranked two at a time or ranked all together. After the computations of the weighted sums, the overall performance values of the alternatives might also be different and that could alter their rankings. Another type of irregular ranking problem is that the indication of the optimal alternative may change when one of the non-optimal alternatives is replaced by a worse one (given that the other date of the decision problem remains unchanged). As discussed before, most of the problematic situations of the additive AHP methods can be attributed to their own mathematical artifacts.

Although research on the issue of rank reversals happened with the additive AHP methods has been carried out for more than thirty years, it is still a topic full of controversies. The AHP method has been widely used in many real-life decision

problems. Thousands of AHP applications have been reported in edited volumes and books (e.g., Golden, *et al.*, 1989, Saaty and Vargas, 2000) and on websites (e.g., www.expertchoice.com). However, the issue of ranking irregularities has not been fully known by regular users of these methods. *ExpertChoice* is popular decision support software which is based on the algorithm of the AHP method. Recently (i.e., in July of 2008), in an article from Blue Cross Blue Shield in Florida, the author said that Expert Choice helped them make decisions in an efficient way and avoid delays and manipulations by few DMs (http://extranet.expertchoice.com/public/Newsletter_July08.pdf). By using such kind of appealing software packages with friendly interface, the users usually are very confident that the software can lead them to the "right" decisions even though they may not be right scientifically. However, if the DMs know more about issues such as the problems related to irregular rankings which exist behind the used methods, they may have a more comprehensive and deeper understanding about the recommended ranking results from such software packages. Thus, it is imperative to bring to people's attention the analysis of the validity of MCDM/MCDA methods and the related issue of ranking irregularities.

For the same goal as above, in [Wang and Triantaphyllou, 2008], the ELECTRE II and III methods were studied in detail for the validity of their proposed rankings. It was found that these two methods might cause some of the same ranking irregularity problems as the additive AHP methods because of their own mathematical artifacts. One is that the indication of the optimal alternative may change when one of the non-optimal alternatives is replaced by a worse one. Another one is that the ranking of the alternatives may not follow the transitivity property when they are compared two at a time. The last

problem is that the ranking of the alternatives may be different when they are compared two at a time and also simultaneously. According to some computational experiments and real-life case studies in [Wang and Triantaphyllou, 2008], for the ELECTRE II and III methods, the rates of these types of ranking irregularities were rather significant (sometimes approaching 100%) in both the simulated decision problems and the studied real-life cases.

By analyzing the ranking processes of the ELECTRE II and III methods and some rank reversal cases which occurred when these methods were used, it was found that the main reason for the above rank reversals lies in the exploitation of the pairwise outranking relations [Wang and Triantaphyllou, 2008] which are the distillation processes of the ELECTRE II and III methods. The basic idea behind the distillation processes is to decide the rank of each alternative by the degree of how this alternative outranks all the other alternatives. Thus, the ranking of a specific alternative derived by these two methods depends on the performance of all the other alternatives currently under consideration and also the set of alternatives being compared. This causes the ranking of the alternatives to depend on each other and leads to the occurrence of the above mentioned ranking irregularities. For instance, when a non-optimal alternative is replaced by a worse one, the pairwise outranking relations related to it may be changed accordingly. Then the overall ranking of the entire alternative set, which depends on those pairwise outranking relations, may also be changed. The first change is reasonable when considering the fact that a non-optimal alternative has been replaced by a worse one. However, the second change may alter the indication of the best ranked alternatives, which is unreasonable and undesirable.

The publication of [Wang and Triantaphyllou, 2008], which is the paper titled as "Ranking Irregularities When Evaluating Alternatives by Using Some ELECTRE Methods", has stimulated some deeper discussions with others on the issue of rank reversals and how should researchers in this area evaluate the performance of different MCDM methods. Because of its significant potential to the decision-making problems related to civil and environmental engineering, the research in [Wang and Triantaphyllou, 2008] was funded by an Environmental Education 2003-2004 Award which was sponsored by the Office of Environmental Education, Office of the Governor, State of Louisiana.

2.2.4 Multi-Attribute Utility Analysis

Multi-attribute utility analysis (MAUA) is another type of systematic method for identifying and analyzing various alternatives and factors in order to arrive at a rational decision [Keeney and Raiffa, 1976; Kirkwood, 1997]. This approach transfers the performance value of an alternative under each decision criterion into a utility value according to some utility function for that criterion. The utility is a numerical value between 0 and 1 and it represents the preferability of the alternative under that decision criterion. Considering the weight of each criterion, the utility of each alternative under each criterion is multiplied by the weight of that criterion. The total utility of each alternative can be calculated by summing up the weighted utility values under all the decision criteria. Then the alternatives are ranked in terms of their total utilities.

One of the key assumptions behind the above utility model is that the DMs are "Rational Individuals" which are devoid of psychological influences or emotions [Luce, 1992]. Under this assumption, it is expected that DMs will always want to make choices

that can maximize the utilities of the chosen alternatives and the utilities of the alternatives are independent of each other. However, behavioral scientists have demonstrated that it is not always appropriate to relate decision rationality to utility maximization. Examples demonstrating systematic violations of the utility maximization principle can be found in [Allais, 1988; Ellsberg, 1961].

CHAPTER 3. STUDIES ON REGRET

Similar assumptions of a completely rational mind and utility maximization are also behind most of the MCDM methods which do not consider emotional feelings at all and always determine the alternatives with maximum overall performance values as the optimal solutions. In order to broaden the assumptions of classical utility theory, some alternative approaches have been proposed. In [Wierzbicki, 1980], an aspiration-based method which was developed according to Hebert Simon's bounded rationality principle [Simon, 1956] was proposed. Instead of identifying decisions with the maximum utility, this method helps a DM to identity prospective decisions which satisfy his/her preference expressed through setting scalarizing parameters for a so-called scalarizing function. In [Kahneman and Tversky, 1979], a new theory, called Prospect Theory, was developed. According to this theory, a DM must "edit" prospects (attributes of decisions) before selecting a decision in order to account for his/her risk attitude (risk seeking or risk averse).

Another direction of research is to incorporate behavioral issues into the analysis of decision-making problems, for example, strong emotional feelings like regret and rejoicing. These two emotional factors were first studied for decision-making under uncertainty. In [Sugden, 1985] regret was defined as "the painful sensation of recognizing that 'what is' compares unfavorably with 'what might have been'". The converse experience of a favorable comparison between the two is called "rejoicing". Some experimental studies confirm that for most individuals regret has the greater impact [Mellers, 2000]. In related research studies, regret is also the one that has received most of the attention.

3.1 Some Regret Models

One of the earliest regret models is known as the minimax regret model which was introduced by Savage [1951] and was first axiomatized by Milnor [1954]. This model defines regret as the difference between the actual performance value of each decision alternative and the best possible value among all alternatives for each state of nature. Suppose the utility value of an alternative A_i under a state of nature S_k is u_{ik}. Then, the decision maker who chooses A_i will experience a level of regret R_{ik} for the state of nature S_k where R_{ik} is defined as follows:

$$R_{ik} = \max_j (u_{jk}) - u_{ik}.$$

The DM would first determine the possible highest level of regret that could occur to each decision alternative, and then choose the alternative with the minimum of these maximum regret values [Zeelenberg, 1999]. Because this model decides the selection of alternatives totally by their regret values, it may lead to irrational choices. Such as a small disadvantage in a single decision criterion, no matter how large/small its importance is, may eliminate alternatives with more preferable performance values under more important criteria [Kujawski, 2005]. Given this undesirable property, the minimax regret model has not been used widely.

Later, Loomes and Sugden and also Bell proposed a regret theory (referred to as the RT-B/LS regret theory) simultaneously in 1982 for rational decision-making under uncertainty [Loomes and Sugden, 1982; Bell, 1982 and 1985]. In the RT-B/LS model, regret is defined as the psychological reaction that is caused by comparing an outcome under one state with the payoff one could have had by making a different choice under the same state. Except the notions of regret and its counterpart rejoicing, the RT-B/LS

model also considers disappointment and its counterpart elation. Disappointment and elation depend on the risk and opportunity of the selected action under a state of uncertainty [Browning and Hillson, 2004]. A rational individual feels some level of disappointment in decision-making under uncertainty when the outcome does not match up to expectations, and he/she experiences elation when the outcome exceeds expectation [Bell, 1985]. Anticipated disappointment and elation are not considerations or influences for deterministic choices. In contrast, a rational individual may experience regret and rejoicing when making decisions under certainty as well as uncertainty [Kujawski, 2005]. Since the research in this dissertation focuses on deterministic MCDM problems, disappointment and elation will not be considered.

The RT-B/LS model assumes that the levels of regret and rejoicing depend on the difference of the utilities between what is and what could have been. For example, the associated level of regret when comparing the utility value u_{ik} with the utility value u_{jk} is defined as follows:

$$R(u_{ik}, u_{jk}) = \begin{cases} R(u_{jk} - u_{ik}), & \textit{if} \;\; u_{ik} < u_{jk} \\ 0, & \textit{otherwise.} \end{cases}$$

Where u_{ik} is the classical utility of the *i-th* alternative in terms of the *k-th* criterion, and *R(.)* is a non-decreasing regret function which is further assumed to be convex [Kujawski, 2005].

In [Kujawski, 2005], a regret model called the Reference-Dependent Regret Model (RDRM) was proposed for deterministic decision-making. Kujawski argued that, in general, a person's level of regret when he/she chooses a multi-attribute alternative often depends explicitly on the absolute values of the utilities of the chosen and forgone alternatives (i.e., alternatives that were considered but not chosen) rather than simply

their differences. Thus, in his RDRM model, the anticipated regret when choosing u_{ik} and forgoing u_{jk} is defined as follows:

$$R(u_{ik}, u_{jk}) = \begin{cases} G(1-u_{ik}) - G(1-u_{jk}), & if \ \ u_{ik} < u_{jk} \\ 0, & otherwise. \end{cases}$$

Where *G(.)* is the regret-building function which measures the level of regret referenced to the maximum possible utility normalized to 1 and is defined as follows:

$$G(x) = \begin{cases} \dfrac{1}{1+(B/x)^{2\times S\times(B+x)}}, & if \ \ x > 0 \\ 0, & otherwise. \end{cases}$$

The two parameters *B* and *S* in the definition of *G(.)* are determined by querying the decision maker about the levels of regret that he/she experiences under each criterion [Kujawski, 2005]. The RDRM model defines the total level of regret for choosing A_i from a set *S* of *n* (where $n \geq 2$) alternatives with *m* criteria as follows:

$$R_i^S = (\frac{1}{n-1})\sum_{k=1}^{m} w_k \sum_{j=1}^{n} R(u_{ik}, u_{jk}). \tag{3-1}$$

The final utility of alternative A_i given the set *S* is defined as follows:

$$U_i^s = \sum_{k=1}^{m} w_k u_{ik} - R_i^S = \sum_{k=1}^{m} w_k u_{ik} - \sum_{k=1}^{m} w_k (\frac{1}{n-1})\sum_{l=1}^{n} R(u_{ik}, u_{lk}). \tag{3-2}$$

In the above formula, the first term is the classical utility of alternative A_i and the second term is the anticipated regret for choosing alternative A_i and forgoing all the other alternatives. Finally, the alternatives are ranked by their final utilities.

Among the previous regret models, the minimax and the RT-B/LS regret models were originally developed for decision-making under uncertainty. However, both of them can be tailored to be used in deterministic decision-making problems by identifying the states of nature with the criteria of a given MCDM problem. For instance, the notion of

regret in the RDRM model is defined by tailoring Bell's [1982] notion of anticipated regret for decision-making under uncertainty. In [Kaliszewski and Michalowski, 1998] it was also mentioned that the notion of regret becomes meaningful in deterministic multi-criteria decision problems if the notion of state is equated to the notion of attribute, and a state/attribute matrix conveys regret type of information (for example, the difference between ideal and actual values of the attributes).

It needs to be noted that the effect of the anticipated regret/rejoicing is different from the experienced emotions. In deterministic decision-making situations, decision makers do not have to experience the emotions in order to be influenced by them. Rather, they can predict the emotional consequences of different decision outcomes in advance, and opt for the choices that minimize the possibility of negative emotions [Zeelenberg *et al*., 2000]. As stated in [Kujawski, 2005], in the process of choosing a deterministic alternative, a rational individual may decide to trade off some benefits and forgo the alternative with the highest total value for a more balanced alternative in order to reduce his/her level of anticipated regret.

From the previous discussions it is clear that regret theory is based on two fundamental assumptions: (1) people experience the sensations of regret and rejoicing which can influence their current decision-making; and (2) when making decisions people try to anticipate and take into account feelings like regret and rejoicing [Loomes and Sugden, 1982; Kaliszewski and Michalowski, 1998]. Therefore, building an MCDM model that incorporates these emotional factors not only can provide a better description of human behavior in decision-making, but also offers the DMs the flexibility to trade off some economic benefits explicitly in order to gain a state of psychological satisfaction,

for prescriptive purposes [Bell, 1985].

3.2 An Alternative Way for Measuring Regret

In [Kujawski, 2005] it was asserted that the RDRM model satisfies three properties. The first property, referred to as the “independence of dominated alternatives” (IDA), seems to be an intuitive one. According to this property, given two alternatives A_i and A_j with $A_i \succ A_j$ the RDRM model preserves their ranking when a new alternative dominated by A_i is introduced or an old alternative dominated by A_j is dropped. However, as demonstrated in [Wang, Triantaphyllou, and Kujawski, 2008], the RDRM model may fail to satisfy this property. Next, a mathematical analysis why the RDRM model does not always follow the first property is described in detail.

3.2.1 Mathematical Analysis of the RDRM Model

Given a set S of n alternatives and m criteria, suppose that two alternatives, say alternatives A_i and A_j, are ranked as $A_i \succ A_j$. As described by formula (3-2), the RDRM utility for alternative A_i and A_j are calculated as follows:

$$U_i^s = \sum_{k=1}^{m} w_k u_{ik} - R_i^S = \sum_{k=1}^{m} w_k u_{ik} - \sum_{k=1}^{m} w_k (\frac{1}{n-1}) \sum_{l=1}^{n} R(u_{ik}, u_{lk})$$

$$U_j^s = \sum_{k=1}^{m} w_k u_{jk} - R_j^S = \sum_{k=1}^{m} w_k u_{jk} - \sum_{k=1}^{m} w_k (\frac{1}{n-1}) \sum_{l=1}^{n} R(u_{jk}, u_{lk})$$

For convenience of the discussion, let

$$R_i' = \sum_{k=1}^{m} w_k \sum_{l=1}^{n} R(u_{ik}, u_{lk}).$$

Then

$$U_i^s = \sum_{k=1}^{m} w_k u_{ik} - (\frac{1}{n-1}) R_i' .$$

Similarly, let

$$R_j' = \sum_{k=1}^{m} w_k \sum_{l=1}^{n} R(u_{jk}, u_{lk}) .$$

Then

$$U_j^s = \sum_{k=1}^{m} w_k u_{jk} - (\frac{1}{n-1}) R_j' .$$

The difference between U_i^s and U_j^s is

$$U_i^s - U_j^s = (\sum_{k=1}^{m} w_k u_{ik} - \sum_{k=1}^{m} w_k u_{jk}) - (\frac{1}{n-1})(R_i' - R_j') \tag{3-3}$$

Given that the two alternatives are ranked as $A_i \succ A_j$, we get

$$U_i^s - U_j^s > 0 \tag{3-4}$$

When introducing a new alternative A_k which is dominated by A_i, the value of R_i' remains unchanged while the value of R_j' may increase if A_k dominates A_j in terms of one or more criteria. Thus, in formula (3-3), the part $(R_i' - R_j')$ may become less than before. Meanwhile, the number of alternatives in the set S is increased by 1. Under the above possible changes, if the original value of $(R_i' - R_j')$ is positive, the term $(\frac{1}{n-1})(R_i' - R_j')$ in formula (3-3) will become smaller than before. Then the inequality relation in (3-4) still holds. However, if the original value of $(R_i' - R_j')$ is negative, the term $(\frac{1}{n-1})(R_i' - R_j')$ may become larger than before. Then the inequality relation in

(3-4) may be reversed and hence the ranking between A_i and A_j may be altered. This is how the RDRM model may fail to satisfy the property of independence of dominated alternatives.

The implication of the above problem is that when the concepts of regret and rejoicing are considered and defined in terms of all the available alternatives in accordance to formula (3-1) of the RDRM model, the anticipated regret and rejoicing associated with an alternative will be influenced by the number of the considered alternatives (i.e., the cardinality of the considered set of alternatives) along with their performance values. Then adding or deleting a dominated alternative (also called non-Pareto optimal alternative) might affect these values and subsequently the ranking of the alternatives.

3.2.2 An Alternative Way for Measuring Regret

As mentioned in [Wang, Triantaphyllou, and Kujawski, 2008], Quiggin in [1994] described a similar problem where manipulation of the set of the alternatives may yield irrational choices as the ranking of the alternatives might be "money pumped." That is, the ranking of the alternatives might be influenced by the introduction of dominated alternatives. In order to avoid being "money pumped", Quiggin [1994] proposed that the measure of regret should satisfy a property called the Irrelevance of Statewise Dominated Alternatives (ISDA). This property is similar to the IDA property. In order to satisfy the ISDA property, Quiggin [1994] proved that regret must be determined solely by the best attainable outcome in each state of the world, or equivalently, the best performance value of each decision criterion in MCDM problems.

This is in contrast with determining the regret associated with an alternative by

considering the entire set of alternatives, like averaging the regret contributions produced by comparing all available choice pairs. When Quiggin's idea is applied to model regret in MCDM problems, the regret associated with an alternative is determined only by comparing the chosen criteria values with the best criteria values. Then addition or deletion of dominated alternatives cannot affect the regret levels of the other alternatives because the best criteria values are kept the same.

However, the above idea may not make much sense as illustrated in the following hypothetical example. Suppose the scores of four students in some exam are according to the two scenarios as depicted in Tables 1 and 2:

Table 1. First scenario.

Student	Score
A1	30
A2	32
A3	31
A4	100

Table 2. Second scenario.

Student	Score
A1	30
A2	98
A3	97
A4	100

In the first scenario there is just one student who earned a very high score. Therefore, it is reasonable to expect that the student who earned only 30 points (i.e., the very bottom grade) feels some but limited regret for not having achieved a higher score because his/her performance is not as bad when it is compared to that of most of the other students. However, in the second scenario, the same student may feel much stronger regret for scoring only 30 points because he/she is the only student who has a very low score.

If one considers the previous two scenarios with a larger number of students (say 200 instead of 4), then the previous effects are much stronger. Therefore, intuitively in this example, it makes more sense to compute regret in terms of the entire set of alternatives. Thus, the concepts of regret and rejoicing may be more realistically expressed in terms of the criteria values of the entire set of alternatives than in terms of only the best criteria values. This is not the final suggestion. This point is further discussed in Section 4.3 where the influence of dominated alternatives is discussed.

Please note that a paper on the research problems discussed in Section 3.2 has been written and is in print for publication in the journal of *Systems Engineering*. It is a product of Xiaoting Wang's collaboration with Dr. Evangelos Triantaphyllou and Dr. Edouard Kujawski from the Naval Postgraduate School. For more detailed information about this paper, please refer to [Wang, Triantaphyllou, and Kujawski, 2008].

CHAPTER 4. A NEW WAY TO ASSESS THE ANTICIPATED REGRET AND REJOICING

From the descriptions in Section 2.1 it can be seen that a rather popular approach of measuring regret is to quantify regret by using some continuous functions. However, this approach may have some fundamental weaknesses.

4.1 Limitations of Measuring Regret by Using Continuous Functions

First of all, the definition of continuous regret functions may involve the determination of certain customizing parameters, such as the *B* and *S* parameters in the regret function *G(.)* of the RDRM model, as not all decision makers may behave in exactly the same way. Furthermore, it is not always clear how such parameters may be determined. It is also unclear whether such functions and their parameters should change from one criterion to another criterion within the same decision problem.

Another concern is raised from the fact that emotional feelings like regret and rejoicing vary more in a discrete manner than in a continuous manner. They may not always increase continuously with the increase of the difference between two compared performance values. For example, usually people feel a certain level of regret when the difference is beyond an echelon value or when one of the two compared performance values is below a cut-off point while the other one is above the cut-off point. Furthermore, the level of regret may not only depend on the difference but also on the context in which the difference occurs. For instance, consider three students taking an exam. Two of them scored 79 and 70 points while another one 69 points. If the cut-off point to get the passing grade C is 70, or else the grade will be F (fails the exam), then it is quite possible that the second student (who has earned 70 points) may not have a strong regret feeling when

comparing his/her score with the first one though their scores are 9 points apart. However, the third student may feel much stronger regret when comparing his/her score with the score of the second student though there is only 1 point difference.

As mentioned before, the RT-B/LS model assumes that the levels of regret and rejoicing depend on the difference of the two compared performance values. Thus Regret (69, 70) < Regret (70, 79) no matter what the background context is because Difference (69, 70) = 1 < Difference (70, 79) =9. However, this result may not always make sense as illustrated above. The RDRM model measures regret by considering the absolute values of the utilities of the chosen and forgone alternatives rather than simply their difference. The proposed approach is more reasonable than the RT-B/LS model. However, if a DM wants to describe a similar regret situation as that of the previous student scoring example, the two parameters *B* and *S* in the regret function *G(.)* need to be decided very carefully. Otherwise, it may produce the same result as that of the RT-B/LS model. Considering all the above issues, it can be seen that regret needs to be measured in a more realistic and flexible manner.

4.2 Measuring Regret and Rejoicing by Using Linguistic Terms

Please recall that decision criteria may be quantitative or qualitative. Regret and rejoicing are definitely qualitative aspects in decision problems. To deal with qualitative criteria, an approach proposed by Saaty [1980] as part of the AHP method has received widespread attention. One of the key steps of that approach is to ask a DM to select a linguistic statement (from a small set of linguistic statements) that best describes his/her assessment of the relative importance of two alternatives when they are considered in terms of a single criterion at a time. A total of 9 linguistic statements which include 4

intermediate values are used to choose from because some psychological studies [Miller, 1956] have shown that most individuals cannot simultaneously compare more than seven objects (plus or minus two). Each linguistic statement is also associated with a numerical value to reflect its natural importance.

The idea of pairwise comparisons along with the application of linguistic statements can also be used to measure a DM's anticipated regret and rejoicing values. (For simplicity, the following discussions are based on regret as rejoicing can be analyzed in an analogous manner.) According to the rule of 7 plus or minus two, a set of 9 linguistic choices which include 4 intermediate values can be developed and used to estimate a DM's anticipated regret value for choosing one alternative and forgoing another one under a specific criterion. Each linguistic term is attached to a numerical value as shown in Table 3.

Suppose we are considering two alternatives A_i and A_j and their performance values in terms of some benefit criterion C_k are a_{ik} and a_{jk}, respectively. If $a_{ik} \geq a_{jk}$, there is no regret for choosing a_{ik} over a_{jk}. Then the regret value is equal to the lowest level which is attached with a value 1. Otherwise, a linguistic statement should be selected from Table 3 and the corresponding numerical value will be attached to the associated regret value. Please note that the numerical value attached to the lowest linguistic term "no distinguishable regret" is 1. This is the case because in Section 4 some multiplicative formulas will be proposed to process the data and the identity under numerical multiplication is 1.

Table 3. Proposed scale for measuring pairwise regret values.

Linguistic Expression	Numerical Value
The feeling of regret when choosing alternative A_i over alternative A_j is not distinguishable.	1
The feeling of regret when choosing alternative A_i over alternative A_j is noticeable.	3
The feeling of regret when choosing alternative A_i over alternative A_j is strong.	5
The feeling of regret when choosing alternative A_i over alternative A_j is very strong.	7
The feeling of regret when choosing alternative A_i over alternative A_j is as strong as it can be.	9
The intermediate values of 2, 4, 6, and 8 are used when the DM feels that the best answer lies between two successive linguistic choices from the above list of choices.	2, 4, 6, 8

Following the above steps, the decision maker is able to fill in the entries of a pairwise comparison matrix for regret; one such matrix for each one of the decision criteria. These matrices are called here **pairwise regret matrices**. For simplicity, let us denote the entry of a typical pairwise regret matrix as r_{ij} (for $i, j = 1,2,3, \ldots, m$). Then r_{ij} = R(a_{ik}, a_{jk}), which is the anticipated regret for choosing alternative A_i and forgoing alternative A_j in terms of a specific decision criterion C_k. A complete pairwise regret matrix is shown in Figure 2.

$$
\begin{array}{cccccc}
 & & & \textbf{Alts.} & & \\
 & (A_1 & A_2 & \dots & A_m) & \\
\textbf{Alts.} & \text{-----} & \text{-----} & \text{-----} & \text{-----} & \\
A_1 & 1 & r_{12} & \dots & r_{1m} & \\
A_2 & r_{21} & 1 & \dots & r_{2m} & \\
. & . & . & . & . & \\
A_m & r_{m1} & r_{m2} & \dots & 1 &
\end{array}
$$

Figure 2. A typical pairwise regret matrix.

The entries of a pairwise regret matrix should satisfy the following two basic conditions:

(1) $r_{i\,i} = 1$, for any $i = 1, 2, 3, \ldots, m$;

(2) If $r_{i\,j} > 1$, then $r_{j\,i} = 1$, for any $i, j = 1, 2, 3, \ldots, m$.

The first rule means that there is no regret when an alternative is compared to itself. The second rule means that if there is a certain level of regret for choosing alternative A_i over alternative A_j, then there is no regret for choosing A_j over A_i.

The use of this set of linguistic terms to estimate a DM's anticipated regret feelings is in essence a mechanism for eliciting a hidden discrete regret function from the DM. This hidden discrete regret function might be different for different decision problems and/or decision criteria within the same problem. For example, a DM's perception of regret might be different stepwise functions for different decision criteria. By using the linguistic terms, the DM has the flexibility to decide the specific tendencies of his/her anticipated regret feelings according to the specific situations of his/her decision problems.

However, too much flexibility could sometime lead to arbitrary results. Thus

some consistency tests are needed to examine the general trend of a DM's assessments of regret and make sure that the assessed pairwise regret values do not violate some basic psychological principles. When examining the consistency of the pairwise regret values, a reference point needs to be decided. The reference point could be the chosen alternative's performance value or the forgone alternative's performance value. Generally speaking, under a given reference point, the bigger the difference between two compared performance values is, the more likely is that the DM may have a stronger regret feeling for choosing the worse performance value and forgoing the better performance value. That is, under a given reference point, the DM's perception of anticipated regret should be monotonically increasing with the increase of the difference between two compared performance values. Based on this principle, two tests are developed next to examine if there is any evident inconsistency within the DM' assessments of the pairwise regret values.

Without loss of generality, suppose that the performance values of m alternatives in terms of the *k-th* benefit criterion are sorted in ascending order such that $a_{1k} \leq a_{2k} \leq a_{3k} \leq a_{4k} \leq \ldots \leq a_{mk}$. By using the chosen performance value and the forgone performance value as reference points individually, the pairwise regret values $R(a_{ik}, a_{jk})$, for $i, j = 1, 2, 3, \ldots, m$ and $i \leq j$, should satisfy the following two conditions:

(1) $R(a_{ik}, a_{jk}) \leq R(a_{ik}, a_{(j+1)k}) \leq \ldots \leq R(a_{ik}, a_{mk})$.

For example: $R(a_{1k}, a_{2k}) \leq R(a_{1k}, a_{3k}) \leq R(a_{1k}, a_{4k}) \leq \ldots \leq R(a_{1k}, a_{mk})$.

(2) $R(a_{1k}, a_{jk}) \geq R(a_{2k}, a_{jk}) \geq \ldots \geq R(a_{(j-1)k}, a_{jk})$.

For example: $R(a_{1k}, a_{mk}) \geq R(a_{2k}, a_{mk}) \geq R(a_{3k}, a_{mk}) \geq \ldots \geq R(a_{(m-1)k}, a_{mk})$.

For each criterion, if the DM is consistent with his/her assessments, then his/her

anticipated regret values should satisfy the above two relations. Otherwise, the DM needs to re-assess the inconsistent parts of his/her assessments. The above tests are further illustrated in a numerical example in Section 6.

Please note that the proposed linguistic scale and definition of the pairwise comparisons are fundamentally different than those introduced by Saaty as part of the AHP method. In Saaty's scale, linguistic terms are used to assess the relative importance of two alternatives (that is, the ratio of their importance) in terms of each one of the decision criteria or the relative importance of two criteria at a time. Some examples of such linguistic expressions are "*A is more important than B*" or "*A is of the same importance as B*," or "A *is a little more important than B*," and so on [Saaty, 1980 and 1994]. According to Saaty's scale, the available numerical values for the pairwise comparisons are members of the set: {9, 8, 7, 6, 5, 4, 3, 2, 1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 1/9}. If in the evaluation of two alternatives, say A_i and A_j, the DM selects some entry from the scale with a value from the sub-interval [1, 9], then the reciprocal comparison of comparing alternative A_j with alternative A_i takes on the reciprocal of the previous value. That is, the value is in the interval [1/9, 1]. For instance, if a_{ij} = 7, then a_{ji} = 1/7. On the other hand, with the proposed linguistic scale, if r_{ij} takes a value between [1, 9], then r_{ji} will always have the value 1. This means that if there is a certain level of regret for choosing alternative A_i over alternative A_j in terms of criterion C_k, then there is no regret (i.e., the corresponding value is equal to 1) for choosing A_j over A_i under the same criterion. This follows directly from the definition of the concept of regret and the need to use multiplication in the related formulas.

Another major difference between the two linguistic scales is that the elicited

pairwise values are subject to different consistency tests. When using Saaty's scale to assess the relative importance of each pair of the alternatives (or criteria), if all the pairwise comparisons are perfectly consistent with each other, then the following relation should always be true for any three comparisons a_{ik} , a_{jk} , and a_{ij} [Saaty, 1980]:

$$a_{ik} \times a_{jk} = a_{ij} \text{ , for any } 1 \leq i, j, k \leq m.$$

The previous consistency test makes sense because of the very way pairwise comparisons are defined by Saaty; they are ratios of relative importance of two decision entities (alternatives or criteria). However, with the proposed linguistic scale, it is not required that a DM has to assess his/her level of regret as the ratio of two performance values. The DM has the flexibility to adapt the use of the new linguistic scale to the nature of a given decision problem. For example, a DM may decide his/her level of regret by comparing the chosen and forgone performance values individually with a specific threshold value as in the student score example. Thus their assessed pairwise regret values do not need to satisfy Saaty's consistency relation but instead they need to satisfy the two consistency tests developed previously.

CHAPTER 5. AN MCDM METHOD BASED ON REGRET AND REJOICING

At this point it is assumed that the DM has developed the regret and rejoicing pairwise comparison matrices for a given decision problem. In this section, a multiplicative MCDM approach is proposed to process the data in these matrixes and also to aggregate the regret and rejoicing values with the criterion values in order to derive the final priorities of the alternatives and then rank them.

5.1 A Multiplicative MCDM Model Based on Regret and Rejoicing

As mentioned in Chapter 2, some studies have reported that some types of rank reversals may occur with the original AHP method and the revised AHP method. Most of the problematic situations of the additive AHP methods can be attributed to the required normalization and also the use of an additive formula on the data of the decision matrix for deriving the overall performance values of the alternatives. However, in the WPM model and the multiplicative AHP method, the use of an additive formula is avoided by using a multiplicative expression. By using the multiplicative formula, no matter how the decision matrix is normalized, the ratios of the alternatives' performance values are kept the same because the normalization factor is cancelled off in the multiplicative formula. Thus most of the ranking irregularities which occurred with the additive AHP methods will not happen with the multiplicative AHP method.

Because of the above mentioned virtues, a similar multiplicatively formulated model is proposed to combine the alternatives' performance values with their associated regret and rejoicing values. First, a formula as the one in (2-4) is used to aggregate the alternatives' performance values under the different decision criteria. For an alternative A_i, its overall performance value is computed as follows:

$$P_i = \prod_{k=1}^{n} a_{ik}^{w_k} . \tag{5-4}$$

Next, the same formula is used to aggregate the alternatives' regret and rejoicing values under each one of the decision criteria. Then, the overall regret value of alternative A_i is as follows:

$$R_i = \prod_{k=1}^{n} r_{ik}^{w_k} . \tag{5-5}$$

Similarly, the overall rejoicing associated with alternative A_i is:

$$J_i = \prod_{k=1}^{n} j_{ik}^{w_k} . \tag{5-6}$$

In the above formulas, a_{ik} is the performance value of alternative A_i in terms of criterion C_k, and r_{ik} and j_{ik} are the anticipated regret and rejoicing values associated with A_i in terms of criterion C_k. To be consistent with the above multiplicative formulas, r_{ik} is defined as the geometric mean of the regret contributions generated when alternative A_i is compared with each of the other alternatives under the decision criterion C_k. That is,

$$r_{ik} = [\prod_{j=1}^{m} R(a_{ik}, a_{jk})]^{\frac{1}{m-1}} . \tag{5-7}$$

Similarly, j_{ik} is defined as follows:

$$j_{ik} = [\prod_{j=1}^{m} J(a_{ik}, a_{jk})]^{\frac{1}{m-1}} . \tag{5-8}$$

Next, a ratio formula is used to combine the alternatives' overall performance values, overall regret and rejoicing values together so that any potential normalization operation would not be able to affect the proportion of these three parts playing in the alternatives' final priority values. Assume that a DM wishes to consider his/her anticipated regret and rejoicing for a given MCDM problem which has m alternatives and

n benefit decision criteria. The formula for computing the final priority of each alternative is defined as follows:

$$P_i^* = \frac{P_i J_i}{R_i} = \frac{\prod_{k=1}^{n} a_{ik}^{w_k} \times \prod_{k=1}^{n} j_{ik}^{w_k}}{\prod_{k=1}^{n} r_{ik}^{w_k}} = \prod_{k=1}^{n} \left(\frac{a_{ik} j_{ik}}{r_{ik}}\right)^{w_k}, \quad \text{for } i = 1, 2, 3, ..., m. \tag{5-9}$$

Since regret is like a cost criterion (i.e., the smaller the value the better) it is placed in the denominator of the above formula. On the contrary, rejoicing is like a benefit criterion (i.e., the higher the value the better) thus it is placed in the numerator of the above formula.

A more general decision problem is assumed to have m alternatives and n decision criteria of which, without loss of generality, the first n_1 are benefit criteria and the remaining $(n-n_1)$ are cost criteria. When considering both the anticipated regret and rejoicing, the formula to compute the final priority of each alternative becomes:

$$P_i^* = \frac{P_i^B J_i^B J_i^C}{P_i^C R_i^B R_i^C} = \frac{\prod_{k=1}^{n_1} a_{ik}^{w_k} \times \prod_{k=1}^{n_1} j_{ik}^{w_k} \times \prod_{k=n_1+1}^{n} j_{ik}^{w_k}}{\prod_{k=n_1+1}^{n} a_{ik}^{w_k} \times \prod_{k=1}^{n_1} r_{ik}^{w_k} \times \prod_{k=n_1+1}^{n} r_{ik}^{w_k}} = \prod_{k=1}^{n_1} \left(\frac{a_{ik} j_{ik}}{r_{ik}}\right)^{w_k} \times \prod_{k=n_1+1}^{n} \left(\frac{j_{ik}}{a_{ik} r_{ik}}\right)^{w_k}. \tag{5-10}$$

Where R_i^B is the overall anticipated regret of alternative A_i under the benefit criteria;

J_i^B is the overall anticipated rejoicing of alternative A_i under the benefit criteria;

P_i^B is the overall performance value of alternative A_i under the benefit criteria;

R_i^C, J_i^C and P_i^C have the similar meaning as the above ones but in terms of the cost criteria.

As mentioned before, next either one of the following two rules could be used to rank two alternatives:

$$P_1^* - P_2^* > 0 \quad \Leftrightarrow \quad \frac{P_1^*}{P_2^*} > 1\,.$$

For example, to compare two alternatives A_i and A_j, the following ratio can be calculated (for simplicity, assume that all the criteria are benefit criteria):

$$R\left(\frac{A_i}{A_j}\right) = \frac{P_i^*}{P_j^*} = \frac{P_i}{P_j} \times \frac{J_i}{J_j} \times \frac{R_j}{R_i} = \prod_{k=1}^{n} \left(\frac{a_{ik}}{a_{jk}} \times \frac{j_{ik}}{j_{jk}} \times \frac{r_{jk}}{r_{ik}}\right)^{w_k} \tag{5-11}$$

In general, if $R(A_i / A_j) > 1$, it indicates that A_i is more preferable than A_j. For a stricter ranking, a threshold value could be used to decide if the difference between two priority values is significant enough to conclude with high confidence that one is more preferable than the other. For example, assume that we get $P_1^* > P_2^*$. Then in order to decide with high confidence that A_1 is more preferable than A_2, their final priorities may need to satisfy a more restrictive relation as follows:

$$\frac{P_1^* - P_2^*}{P_2^*} \geq some\ \ threshold\ \ value\,. \tag{5-12}$$

The above relation means that one priority value should be at least larger than the other one by a threshold percentage in order to conclude with high confidence that one is more preferable than the other.

The threshold value can be decided by the situation of a specific application. For a general purpose, it could be 10%. Please note that with the introduction of a threshold value, the rankings of the alternatives may become intransitive. For example, assume $P_1^* > P_2^* > P_3^*$. Under a certain threshold value, if both the difference between P_1^* and P_2^* and the difference between P_2^* and P_3^* are very small and could not satisfy inequality (5-12), A_1 will be ranked as equal to A_2 and A_2 will be ranked as equal to A_3.

From the transitivity point of view, one would expect that A_1 should also be ranked as equal to A_3. However, the difference between A_1 and A_3 may be large enough to satisfy inequality (5-12), and then A_1 will be ranked higher than A_3.

One needs to keep in mind the presence of a computability issue when using the proposed multiplicative formulas. This issue is associated with the scales that are used to measure the alternatives' performance values under the criteria. Scales for measurement can be nominal (for example, gender), ordinal (for example, degree of satisfaction), interval (for example, temperature) or ratio (for example, length). The difference between an interval scale and a ratio scale is that a ratio scale has a natural zero point but an interval scale does not. Because it has a natural zero, a ratio scale is unique under a positive multiplicative transformation. This means that any ratio scale can be multiplied by a positive constant and the result would still be a ratio scale of the same phenomenon, but just in different units [Drummond *et al.*, 2005]. This property is used, for example, to convert feet to yards, or meters to miles.

However, an interval scale has no natural zero. It is unique under a positive linear transformation [Drummond *et al.*, 2005]. This means that any interval scale x can be transformed to a scale y using a function $y = a + bx$, where a can be any constant and b can be any positive constant. The result will still be an interval scale of the same phenomenon, but in different units and with a different zero point. For instance, this property can be used to convert temperature from Fahrenheit (F) to Celsius (C) units.

As result of the above, ratios of differences between interval scores have meaning, but ratios of interval scores do not. For example, with temperature, it is correct to state that the difference between 80F and 40F is twice the difference between 60F and 40F, but

it is not correct to state that 80F is twice as hot as 40F. The first statement holds true whether the temperature is measured in F or C, while the second does not. For a ratio scale, both types of ratios have meaning. For instance, in length, it is both correct to state that the difference between 80 miles and 40 miles is twice as much as the difference between 60 miles to 40 miles, and 80 miles is twice as long as 40 miles. Both of the statements remain true no matter the lengths are measured in inches or miles.

Because of the above properties with ratio scale and interval scale, under the proposed multiplicative formulas, the ratio of two performance values measured by a ratio scale is the same no matter what unit is used. For example, the ratio of two monetary values expressed in Euros is the same as that of the two values expressed in US Dollars. But the ratio of two performance values measured by an interval scale might be different if they are transformed to other units. For instance, the ratio of two temperature values 40F and 60F is different when exactly the same temperature values are expressed in Celsius units. Please note that this problem lies in the use of the interval scale itself. Whether they are operated by additive or multiplicative formula, the proportions or the ratios of interval scores might be different if they are expressed in another unit. Users should be aware of this problem when they use interval scales. If it is unavoidable to use an interval scale to measure alternatives in terms of some criterion, it might be necessary to check how the ranking result might be changed when a different unit is used (such as the F and C units for temperature). If all criteria are measured by using ratio scales, there is no such problem.

5.2 Influence of Dominated Alternatives and the Intransitivity Problem

Though using the multiplicative formula can avoid the negative influence of

normalization operations, the way to measure regret/rejoicing introduces new interdependences into the ranking of the alternatives. As discussed in Section 3.2, when regret and rejoicing are measured by considering all available alternatives, the anticipated regret and rejoicing associated with one alternative will be influenced by the number of the considered alternatives (i.e., the cardinality of the set of the alternatives). By introducing or deleting a dominated alternative, the alternatives' associated regret and rejoicing values might be changed and then the ranking of them might also be altered or even completely reversed. It is not hard for someone to fabricate some nonexistent or arbitrary dominated alternatives and add them into the set of alternatives in order to boost his/her own preferred alternatives in an unfair way. Thus it is further suggested that dominated alternatives should be eliminated before using the proposed method to rank a set of alternatives and regret and rejoicing should be better measured by considering all available Pareto-optimal (i.e., nondominated) alternatives. In this manner the negative influence of dominated alternatives could be avoided and the idea of measuring regret and rejoicing by considering the existence of other alternatives instead of only the alternatives with the best criteria values is also considered to a certain degree.

Because of the same reason, the ranking of the alternatives by using the new method may not follow the transitivity property when they are ranked two at a time and regret and rejoicing are measured in terms of only the two alternatives. However, occurrence of this particular kind of intransitivity may not always be a negative aspect. Some studies [May, 1954; Tversky, 1969; Roberts, 1972] have shown that a rational DM may exhibit a certain level (although of limited size) of intransitivity in his/her comparison of alternatives in the decision-making process. Some degree of inconsistency

seems to be inherent in human decision-making. Thus it might be natural to allow certain intransitivity to exist in a decision-making process where emotional factors are involved.

However, the fact that intransitivity may be exhibited by rational decision makers does not mean that a very large number of intransitive cases are necessarily a benign aspect to have. As mentioned before, the ELECTRE II and III methods were shown in [Wang and Triantaphyllou, 2008] to exhibit very high frequencies of intransitivity within a large number of simulated problems and also on a random collection of real-life case studies. To get a feeling about the intransitivity rate of the proposed method, a similar test was carried out to the new method by using some simulated decision problems.

In the simulated decision problems, the number of the alternatives and the number of criteria were set to the following 7 values: 3, 4, 5, 6, 7, 8, 9. Thus, a total of 49 (that is, 7×7) different cases were examined with 5,000 randomly generated decision problems per case. For each simulated decision problem, the corresponding regret matrixes which satisfy the consistency tests described in Section 3.2 were also generated randomly. During the test, each simulated decision problem was decomposed into a set of smaller problems, each defined on two alternatives at a time and the same number of criteria as in the original problem. Then the alternatives in the smaller problems were ranked and the rankings were examined. Any occurred intransitivity among the paired rankings was recorded. Figure 3 shows the test results. In this figure, different curves correspond to cases with different numbers of alternatives; the horizontal axis stands for the number of criteria and the vertical axis is the rate of intransitivity that occurred in the 5,000 simulated decision problems.

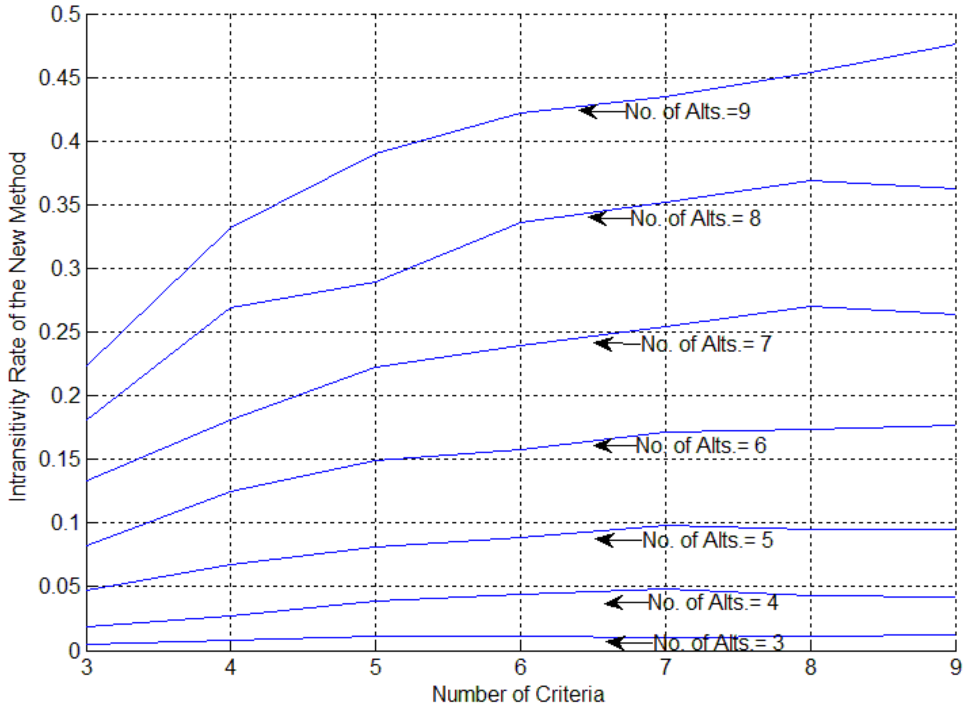

Figure 3. Intransitivity rate of the new method.

As reported in [Triantaphyllou, 2000] and [Wang and Triantaphyllou, 2008], the same kind of intransitivity could also happen with the original and the revised AHP methods, and the ELECTRE II and III methods. For a simple comparison, when the number of alternatives is 9 and the number of criteria is 7, according to the test results reported in those studies, the intransitivity rate of the original AHP method is about 10%; for the revised AHP method it is about 26%; for the ELECTRE II method, it is about 85%; while for the ELECTRE III method it is almost 100%. According to the results shown in Figure 3, for the new method the intransitivity rate is about 44%. However, the intransitivity cases will not happen with the new method when regret and rejoicing are not considered.

CHAPTER 6. A NUMERICAL EXAMPLE AND SOME SENSITIVITY ANALYSES

6.1 A Numerical Example

In this chapter a numerical example is used to illustrate the application of the proposed method. It is a simulated example and the data were generated randomly by a computer program. In this example, there are 4 alternatives and 3 criteria. The performance values of the alternatives under the three criteria are as follows:

	C_1	C_2	C_3
A_1	19	6	15
A_2	15	7	4
A_3	4	9	16
A_4	5	12	4

The weights of the criteria are: W = [0.35 0.42 0.23];

For simplicity, assume that the DM only wants to consider his/her anticipated regret. In terms of the three decision criteria, the corresponding regret matrixes are simulated as follows. The simulated pairwise regret matrix in terms of criterion C_1 is assumed to be as follows:

C_1	A_1	A_2	A_3	A_4
A_1	1	1	1	1
A_2	3	1	1	1
A_3	8	6	1	2
A_4	5	4	1	1

The simulated pairwise regret matrix in terms of criterion C_2 is assumed to be as follows:

C_2	A_1	A_2	A_3	A_4
A_1	1	8	8	8
A_2	1	1	3	6
A_3	1	1	1	4
A_4	1	1	1	1

The simulated pairwise regret matrix in terms of criterion C_3 is assumed to be as follows:

C_3	A_1	A_2	A_3	A_4
A_1	1	1	3	1
A_2	3	1	4	1
A_3	1	1	1	1
A_4	3	1	4	1

Now we need to examine if the above simulated regret values satisfy the two consistency relations described in Section 4.2. After some examination, it was found that they did satisfy the consistency tests. For instance, in terms of the second criterion, the four alternatives' performance values are 6, 7, 9, and12. They are in ascending order because of $a_{12} \leq a_{22} \leq a_{32} \leq a_{42}$. From the simulated regret matrix in terms of the second criterion, it can be seen that:

(1) $R(a_{12}, a_{22}) = 8 \leq R(a_{12}, a_{32}) = 8 \leq R(a_{12}, a_{42}) = 8$.

$R(a_{22}, a_{32}) = 3 \leq R(a_{22}, a_{42}) = 6$.

(2) $R(a_{12}, a_{42}) = 8 \geq R(a_{22}, a_{42}) = 6 \geq R(a_{32}, a_{42}) = 4$.

$\mathrm{R}(a_{12}, a_{32}) = 8 \geq \mathrm{R}(a_{22}, a_{32}) = 6.$

This shows that these regret values satisfy the two consistency relations. Similarly, the pairwise regret values under the other criteria can also be examined.

Next, by applying formula (5-7) to the above regret matrixes, we can get:

$$r_{11} = [\prod_{j=1}^{4} r(a_{11}, a_{j1})]^{\frac{1}{4-1}} = 1 \qquad r_{21} = [\prod_{j=1}^{4} r(a_{21}, a_{j1})]^{\frac{1}{4-1}} = \sqrt[3]{3} \qquad r_{31} = \sqrt[3]{96} \qquad r_{41} = \sqrt[3]{20}$$

$$r_{12} = \sqrt[3]{512} \qquad r_{22} = \sqrt[3]{18} \qquad r_{32} = \sqrt[3]{4} \qquad r_{42} = 1$$

$$r_{13} = \sqrt[3]{3} \qquad r_{23} = \sqrt[3]{12} \qquad r_{33} = 1 \qquad r_{43} = \sqrt[3]{12}$$

Where r_{ik} is the anticipated regret value associated with alternative A_i in terms of criterion C_k, for i = 1, 2, 3, 4 and k = 1, 2, 3. All of the above values can also be put in a table as follows (please note that the equivalent decimal expressions are used in this table):

r_{ik}	C_1	C_2	C_3
A_1	1.0000	8.0000	1.4422
A_2	1.4422	2.6207	2.2894
A_3	4.5789	1.5874	1.0000
A_4	2.7144	1.0000	2.2894

Then, by applying formula (5-9), we can get the final preference values of these alternatives.

$$P_1^* = \frac{P_1}{R_1} = \frac{\prod_{k=1}^{3} a_{1k}^{w_k}}{\prod_{k=1}^{3} r_{1k}^{w_k}} = \prod_{k=1}^{3} (\frac{a_{ik}}{r_{ik}})^{w_k} = (\frac{19}{1})^{w_1} \times (\frac{6}{8})^{w_2} \times (\frac{15}{\sqrt[3]{3}})^{w_3} = 4.256$$

$$P_2^* = 3.899 \qquad P_3^* = 3.74 \qquad P_4^* = 3.998$$

Without considering any threshold value for the final ranking of the alternatives, the above results indicate: $A_1 \succ A_4 \succ A_2 \succ A_3$.

If using the ratio formula (5-11), we can get:

$$R\left(\frac{A_1}{A_2}\right) = \frac{P_1^*}{P_2^*} = 1.1,\ R\left(\frac{A_1}{A_3}\right) = \frac{P_1^*}{P_3^*} = 1.14,\ R\left(\frac{A_1}{A_4}\right) = \frac{P_1^*}{P_4^*} = 1.06$$

$$R\left(\frac{A_2}{A_3}\right) = \frac{P_2^*}{P_3^*} = 1.04,\ R\left(\frac{A_2}{A_4}\right) = \frac{P_2^*}{P_4^*} = 0.98,\ R\left(\frac{A_3}{A_4}\right) = \frac{P_3^*}{P_4^*} = 0.94$$

The above ratios indicate the same ranking as before as it should be.

6.2 Some Sensitivity Analyses

Sometimes it is hard for a DM to precisely capture his/her perception of regret and rejoicing by using a specific linguist term. Thus, it is necessary to study how changes of regret and rejoicing values could affect the ranking results of the decision problems. There are many studies on sensitivity analysis for deterministic MCDM models [Masuda, 1990; Armacost and Hosseini, 1994]. Usually, a sensitivity analysis aims at examining how changes on the weights of the criteria or changes on the performance values of the alternatives could affect the ranking results of the decision problems. In [Triantaphyllou and Sanchez, 1997; Triantaphyllou, 2000], these two types of sensitivity problems were analyzed in detail and two corresponding sensitivity analysis approaches were proposed for three major MCDM methods which included the weighted sum model, the weighted product model, and the additive AHP methods. The first approach aims at determining what the smallest changes in the current weights of the criteria are which can alter the existing ranking of the alternatives. The second approach uses the same concept as the first one to determine how critical the various performance values of the alternatives (in

terms of a single decision criterion at a time) are in the ranking of the alternatives [Triantaphyllou, 2000]. Both of these two approaches can also be applied straightforwardly to the proposed new method for solving the same kinds of sensitivity analysis problems.

Following the same approach as in [Triantaphyllou, 2000], next some sensitivity analysis procedures are developed to determine what the minimum change is in a specific regret value such that the ranking between two alternatives will be altered.

6.2.1 Sensitivity Analysis in Terms of an Alternative's Aggregated Regret Value under a Given Criterion

Suppose the criteria of a decision problem are all benefit criteria and the DM is interested to see how change in a specific regret value might be able to alter the ranking between two alternatives A_i and A_j. Let r_{ik} represent the associated regret of alternative A_i under criterion C_k; let $T_{i\,k\,j}$ denote the change in the regret value r_{ik} (all the other regret values are kept the same) such that the ranking of alternatives A_i and A_j will be altered. First, let us assume, before the change of r_{ik}, the ranking between alternatives A_i and A_j is $A_i \succ A_j$. Then, by using the proposed method, the ratio $R(A_i\,/\,A_j)$ should be greater than 1 in order for alternative A_i to be more preferred than alternative A_j. That is:

$$R\left(\frac{A_i}{A_j}\right) = \frac{P_i}{P_j} \times \frac{J_i}{J_j} \times \frac{R_j}{R_i} = \prod_{k=1}^{n} \left(\frac{a_{ik}}{a_{jk}} \times \frac{j_{ik}}{j_{jk}} \times \frac{r_{jk}}{r_{ik}}\right)^{w_k} > 1 .$$

Let $R^{/}(A_i\,/\,A_j)$ denote the new ratio after the $T_{i\,k\,j}$ change has occurred on the regret value r_{ik}. The new ratio should be less than or equal to 1. Let $t_{i\,k\,j}$ denote the threshold value of $T_{i\,k\,j}$, which is the minimum change that has to occur in r_{ik} such that the original ranking between alternatives A_i and A_j will be altered. In this case, the new ratio will be as

follows:

$$R^{/}\left(\frac{A_i}{A_j}\right)=\frac{P_i^*}{P_j^*}=\frac{P_i}{P_j}\times\frac{J_i}{J_j}\times\frac{R_j}{R_i}=\prod_{k=1}^{n}(\frac{a_{ik}}{a_{jk}}\times\frac{j_{ik}}{j_{jk}})^{w_k}\times(\frac{r_{j1}}{r_{i1}})^{w_1}\times\ldots\times(\frac{r_{jk}}{r_{ik}^{/}})^{w_k}\times\ldots\times(\frac{r_{jn}}{r_{in}})^{w_n}\leq 1.$$

Let $r_{ik}^{/}=r_{ik}+\tau_{i,k,j}$, then

$$\begin{aligned}R^{/}\left(\frac{A_i}{A_j}\right)&=\prod_{k=1}^{n}(\frac{a_{ik}}{a_{jk}}\times\frac{j_{ik}}{j_{jk}})^{w_k}\times(\frac{r_{j1}}{r_{i1}})^{w_1}\times\ldots\times(\frac{r_{jk}}{r_{ik}+\tau_{i,k,j}})^{w_k}\times(\frac{r_{ik}}{r_{ik}})^{w_k}\times\ldots\times(\frac{r_{jn}}{r_{in}})^{w_n}\\&=(\frac{r_{ik}}{r_{ik}+\tau_{i,k,j}})^{w_k}\times\prod_{k=1}^{n}(\frac{a_{ik}}{a_{jk}}\times\frac{j_{ik}}{j_{jk}})^{w_k}\times(\frac{r_{j1}}{r_{i1}})^{w_1}\times\ldots\times(\frac{r_{jk}}{r_{ik}})^{w_k}\times\ldots\times(\frac{r_{jn}}{r_{in}})^{w_n}\\&=(\frac{r_{ik}}{r_{ik}+\tau_{i,k,j}})^{w_k}\times R\left(\frac{A_i}{A_j}\right)\leq 1.\end{aligned}$$

From the above relation and $R(A_i/A_j)>1$, we can get

$$\tau_{i,k,j}\geq r_{ik}\left(\sqrt[w_k]{R\left(\frac{A_i}{A_j}\right)}-1\right)>0. \tag{6-1}$$

Then,

$$r_{ik}^{/}=r_{ik}+\tau_{i,k,j}\geq r_{ik}+r_{ik}\left(\sqrt[w_k]{R\left(\frac{A_i}{A_j}\right)}-1\right)=r_{ik}\sqrt[w_k]{R\left(\frac{A_i}{A_j}\right)}.$$

Because here $R\left(\frac{A_i}{A_j}\right)>1$, then $\sqrt[w_k]{R\left(\frac{A_i}{A_j}\right)}>1$. Let $Q=\sqrt[w_k]{R\left(\frac{A_i}{A_j}\right)}-1$, then $Q>0$ and

$$r_{ik}^{/}\geq(Q+1)r_{ik}. \tag{6-2}$$

From inequality (6-2), it can be seen that if the regret value r_{ik} is increased by at least $Q\times 100\%$, the ranking between A_i and A_j will be altered.

Next, assume the ranking between alternatives A_i and A_j is $A_i \prec A_j$. Then $R(A_i/A_j)$ should be less than 1 and the new ratio $R^{/}(A_i/A_j)$ should be larger than or equal to 1. That is,

$$R^{/}\left(\frac{A_i}{A_j}\right)=\frac{P_i^*}{P_j^*}=\frac{P_i}{P_j}\times\frac{J_i}{J_j}\times\frac{R_j}{R_i}=\prod_{k=1}^{n}(\frac{a_{ik}}{a_{jk}}\times\frac{j_{ik}}{j_{jk}})^{w_k}\times(\frac{r_{j1}}{r_{i1}})^{w_1}\times\ldots\times(\frac{r_{jk}}{r_{ik}^{/}})^{w_k}\times\ldots\times(\frac{r_{jn}}{r_{in}})^{w_n}\geq 1.$$

Then

$$R^{/}\left(\frac{A_i}{A_j}\right)=\prod_{k=1}^{n}(\frac{a_{ik}}{a_{jk}}\times\frac{j_{ik}}{j_{jk}})^{w_k}\times(\frac{r_{j1}}{r_{i1}})^{w_1}\times\ldots\times(\frac{r_{jk}}{r_{ik}+\tau_{i,k,j}})^{w_k}\times(\frac{r_{ik}}{r_{ik}})^{w_k}\times\ldots\times(\frac{r_{jn}}{r_{in}})^{w_n}$$

$$=(\frac{r_{ik}}{r_{ik}+\tau_{i,k,j}})^{w_k}\times R\left(\frac{A_i}{A_j}\right)\geq 1.$$

From the above relation and $R(A_i / A_j) < 1$, we can get

$$\tau_{i,k,j}\leq r_{ik}\left(\sqrt[w_k]{R\left(\frac{A_i}{A_j}\right)}-1\right)<0. \tag{6-3}$$

Then,

$$r_{ik}^{/}=r_{ik}+\tau_{i,k,j}\leq r_{ik}+r_{ik}\left(\sqrt[w_k]{R\left(\frac{A_i}{A_j}\right)}-1\right)=r_{ik}\sqrt[w_k]{R\left(\frac{A_i}{A_j}\right)}.$$

This time $R\left(\frac{A_i}{A_j}\right)<1$, thus $\sqrt[w_k]{R\left(\frac{A_i}{A_j}\right)}<1$, $Q=\sqrt[w_k]{R\left(\frac{A_i}{A_j}\right)}-1<0$, and

$$r_{ik}^{/}\leq(1+Q)r_{ik}. \tag{6-4}$$

From inequality (6-4), it can be seen that if the regret value r_{ik} is decreased by at least $|Q|\times 100\%$, the ranking between A_i and A_j will be altered.

Furthermore, the following condition should also be satisfied for the changed regret value to be feasible:

$$r_{ik}^{/}=r_{ik}+\tau_{i,k,j}\geq 1,\ or\ \ \tau_{i,k,j}\geq 1-r_{ik}.$$

The above relation is true because 1 is the minimum regret value for the proposed multiplicative model and it cannot be decreased further. In summary, to alter the ranking

between two alternatives A_i and A_j, the value of $T_{i,k\,j}$; the change in the single regret value r_{ik} , should be within the following ranges:

$$\begin{cases} 0 < r_{ik}Q \le T_{i,k,j}, & \text{if originally } A_i \succ A_j, \\ 1 - r_{ik} \le T_{i,k,j} \le r_{ik}Q < 0, & \text{if originally } A_i \prec A_j. \end{cases}$$

Please note, if $T_{i,k\,j}$ is positive, it means that the regret value r_{ik} needs to be increased in order to alter the ranking between A_i and A_j. Otherwise, it needs to be decreased.

The same sensitivity analysis can also be carried out to determine what the change is in a specific rejoicing value which can alter the ranking between two alternatives A_i and A_j. Let $O_{i\,k,j}$ denote the change in the rejoicing value j_{ik} (all the other rejoicing values are kept the same) such that the ranking between A_i and A_j will be altered. Similar derivations indicated that $O_{i\,k,j}$ should be within the following ranges:

$$\begin{cases} 1 - j_{ik} \le O_{i,k,j} \le j_{ik}\left(\sqrt[w_k]{R\left(\dfrac{A_j}{A_i}\right)} - 1\right) < 0, & \text{if originally } A_i \succ A_j, \\ 0 < j_{ik}\left(\sqrt[w_k]{R\left(\dfrac{A_j}{A_i}\right)} - 1\right) \le O_{i,k,j}, & \text{if originally } A_i \prec A_j. \end{cases}$$

For instance, applying the above sensitivity analysis results to the example in Section 5.1, we can get a 3-D table as Table 4. In Table 4, the entry (i, k, j) is the Q value corresponding to $t_{i\ k\ j}$, the minimum change in r_{ik}. The value of Q is the minimum percentage that r_{ik} need to be changed such that the ranking between alternatives A_i and A_j will be altered. For instance, the entry (2, 1, 3) is 0.0146 which is the Q value corresponding to $t_{2\ 1\ 3}$. This value indicates that if the regret value of r_{21} is increased by at least 1.46% from the current value (i.e., 1.442) to $(1+0.0146)\times 1.442 = 1.463$, the ranking between A_2 and A_3 will be altered. Similarly, the entry (3, 2, 4) is -0.0276. It indicates that if the regret value of r_{32} is decreased by at least 2.76% from the current value (i.e., 1.5874)

to (1-0.0276)$\times$1.5874 = 1.5436, the ranking between A_3 and A_4 will be altered.

Table 4. Threshold values in relative terms for the example in Section 5.1.

Alt.(A_i)	Criterion C_k			Alt.(A_j)
	C_1	C_2	C_3	
A_1	*N/A*	0.0375	0.0204	A_2
A_1	*N/A*	0.0558	0.0302	A_3
A_1	*N/A*	0.0266	0.0145	A_4
A_2	-0.0302	-0.0362	-0.0200	A_1
A_2	**0.0146**	0.0176	0.0096	A_3
A_2	-0.0088	-0.0105	-0.0058	A_4
A_3	-0.0442	-0.0528	*N/A*	A_1
A_3	-0.0144	-0.0173	*N/A*	A_2
A_3	-0.0231	**-0.0276**	*N/A*	A_4
A_4	-0.0217	*N/A*	-0.0143	A_1
A_4	0.0089	*N/A*	0.0058	A_2
A_4	0.0236	*N/A*	0.0154	A_3

Please note, if originally r_{ik} = 1 (which means that there is no regret when alternative A_i is compared with all the others under the *k-th* criterion and the performance value of A_i is better than or equal to those of the others under the same criterion), it is infeasible to decrease or increase this regret value. Thus there is no corresponding feasible $T_{i,k\,j}$ and Q values. For instance, for the example in Section 5.1, the value of r_{11} is 1. Thus the Q values corresponding to $t_{1\ 1\ j}$, for j = 2, 3, and 4, are all infeasible

(represented by *N/A; not applicable*). For the same reason, there are no feasible Q values which correspond to $t_{3\,3j},$ for j = 1, 2, 4 and $t_{4\,2j},$ for j =1, 2, 3. Thus these entries are all represented by *N/A* in Table 4.

From formula (5-7), it can be seen that r_{ik} is the geometric mean of the $(m-1)$ regret contributions generated when alternative A_i is compared with the other $(m-1)$ alternatives under the decision criterion C_k. Thus, the change in r_{ik} is an aggregated effect of the possible changes in all these individual regret contributions. Sometimes, the DM may want to further find out how an individual regret contribution could affect the ranking results. By simple deduction, it can be seen that if all the other individual regret contributions are kept the same and only one of them need to be changed, then the change in this one should be at least as big as $(m-1)$ times of the previously derived value so that it could cause the ranking between two alternatives to be altered. This type of sensitivity analysis could be too cumbersome for a large-sized decision problem and result in too many sensitivity scenarios. However, in case it is needed for some small-sized decision problems, a formal mathematical derivation is also presented in the next section.

6.2.2 Sensitivity Analysis in Terms of an Alternative's Associated Pairwise Regret Values under a Given Criterion

Similarly, suppose that the interest is to alter the ranking between alternatives A_i and A_j and all the criteria are benefit criteria. Let $r(a_{ik}, a_{lk})$ represent the associated regret of alternative A_i when comparing its performance value a_{ik} with alternative A_l 's performance value a_{lk} under the criterion C_k, for l = 1, …, m; let $b_{i\,k\,l}$ denote the coefficient of the change in the regret value $r(a_{ik}, a_{lk})$ (all the other regret values are kept the same) such that the ranking between A_i and A_j will be altered. First, assume that

originally $A_i \succ A_j$, then $R(A_i / A_j) > 1$. Let $R^{/}(A_i / A_j)$ denote the new ratio after the change of $r(a_{ik}, a_{lk})$. The new ratio should be less than or equal to 1. It is given that

$$r_{ik} = [\prod_{j=1}^{m} r(a_{ik}, a_{jk})]^{\frac{1}{m-1}} = [r(a_{ik}, a_{1k}) r(a_{ik}, a_{2k}) ... r(a_{ik}, a_{lk}) ... r(a_{ik}, a_{mk})]^{\frac{1}{m-1}}.$$

Let $r^{/}(a_{ik}, a_{lk}) = \beta_{i,k,l} \times r(a_{ik}, a_{lk})$, then

$$\begin{aligned} r_{ik}^{/} &= [\prod_{j=1}^{m} r(a_{ik}, a_{jk})]^{\frac{1}{m-1}} \\ &= [r(a_{ik}, a_{1k}) r(a_{ik}, a_{2k}) ... r^{/}(a_{ik}, a_{lk}) ... r(a_{ik}, a_{mk})]^{\frac{1}{m-1}} \\ &= [r(a_{ik}, a_{1k}) r(a_{ik}, a_{2k}) ... \beta_{i,k,l} r(a_{ik}, a_{lk}) ... r(a_{ik}, a_{mk})]^{\frac{1}{m-1}} \\ &= (\beta_{i,k,l})^{\frac{1}{m-1}} r_{ik}. \end{aligned}$$

Let $(\beta_{i,k,l})^{\frac{1}{m-1}} = \beta_{i,k,l}^{/}$, then

$$\begin{aligned} R^{/}\left(\frac{A_i}{A_j}\right) &= \prod_{k=1}^{n} (\frac{a_{ik}}{a_{jk}} \times \frac{j_{ik}}{j_{jk}})^{w_k} \times (\frac{r_{j1}}{r_{i1}})^{w_1} \times ... \times (\frac{r_{jk}}{r_{ik}^{/}})^{w_k} \times (\frac{r_{ik}}{r_{ik}})^{w_k} \times ... \times (\frac{r_{jn}}{r_{in}})^{w_n} \\ &= \prod_{k=1}^{n} (\frac{a_{ik}}{a_{jk}} \times \frac{j_{ik}}{j_{jk}})^{w_k} \times (\frac{r_{j1}}{r_{i1}})^{w_1} \times ... \times (\frac{r_{jk}}{\beta_{i,k,l}^{/} r_{ik}})^{w_k} \times (\frac{r_{ik}}{r_{ik}})^{w_k} \times ... \times (\frac{r_{jn}}{r_{in}})^{w_n} \\ &= (\frac{1}{\beta_{i,k,l}^{/}})^{w_k} \times \prod_{k=1}^{n} (\frac{a_{ik}}{a_{jk}} \times \frac{j_{ik}}{j_{jk}})^{w_k} \times (\frac{r_{j1}}{r_{i1}})^{w_1} \times ... \times (\frac{r_{jk}}{r_{ik}})^{w_k} \times ... \times (\frac{r_{jn}}{r_{in}})^{w_n} \\ &= (\frac{1}{\beta_{i,k,l}^{/}})^{w_k} \times R\left(\frac{A_i}{A_j}\right) \leq 1. \end{aligned}$$

From the above relation and the relation $R(A_i / A_j) > 1$, we can get

$$\beta_{i,k,l}^{/} \geq \sqrt[w_k]{R\left(\frac{A_i}{A_j}\right)} > 1.$$

Then,

$$\beta_{i,k,l} \geq \left[R\left(\frac{A_i}{A_j} \right) \right]^{\frac{m-1}{w_k}} > 1. \tag{6-5}$$

Let $P = \left[R\left(\frac{A_i}{A_j} \right) \right]^{\frac{m-1}{w_k}} - 1$. It is now larger than 0. Then,

$$r^{/}(a_{ik}, a_{lk}) = \beta_{i,k,l} \times r(a_{ik}, a_{lk}) \geq \left[R\left(\frac{A_i}{A_j} \right) \right]^{\frac{m-1}{w_k}} \times r(a_{ik}, a_{lk}) = (1+P) \times r(a_{ik}, a_{lk}). \tag{6-6}$$

From the above inequality, it can be seen that if the regret value *r(a_{ik}, a_{lk})* is increased by at least $P \times 100\%$, the ranking between A_i and A_j will be altered. Similarly, if we assume $A_i \prec A_j$, by the same derivations as above, we can get:

$$\beta_{i,k,l} \leq \left[R\left(\frac{A_i}{A_j} \right) \right]^{\frac{m-1}{w_k}} < 1,$$

and

$$r^{/}(a_{ik}, a_{lk}) \leq (1+P) \times r(a_{ik}, a_{lk}).$$

Now $P = \left[R\left(\frac{A_i}{A_j} \right) \right]^{\frac{m-1}{w_k}} - 1 < 0$, which indicates that if the regret value *r(a_{ik}, a_{lk})* is decreased by at least $P \times 100\%$, the ranking between A_i and A_j will be altered.

Using the above results, DMs could examine how changes of regret or rejoicing values might affect the ranking results of the decision problems. It is believed that DMs can make more careful assessments about their regret and rejoicing feelings if they can see how sensitive the ranking results could be to the changes in these values. According to the results of sensitivity analysis, they may want to reassess some of their anticipated regret and rejoicing levels for better predication. Meanwhile, they can also obtain a more

comprehensive understanding about the ranking of the alternatives and finally choose the one that is more stable than the others.

CHAPTER 7. INTRODUCTION TO A FUZZY VERSION OF THE NEW METHOD

In Chapter 3 it is proposed to use crisp numbers to represent the natural importance of the linguistic terms for measuring regret and rejoicing. Because of the potential impreciseness within the linguistic terms, they can also be represented by fuzzy numbers. As pointed out in [Chen and Liao, 1996], fuzzy numbers employ a range of values instead of one crisp number, they are more in line with the uncertainty nature of many decision problems and the subjective nature of evaluations. The DM's assessments of anticipated regret and rejoicing feelings are subjective evaluations. The uncertainty and imprecision which is inherent in their assessments of regret and rejoicing can be accounted for by considering each of these emotional factors as fuzzy quantities, characterized by appropriate membership functions.

Except measuring regret and rejoicing by fuzzy numbers, for some decision problems, the performance values of the alternatives and the weights of the criteria may also need to be expressed by fuzzy numbers. For example, sometimes it is hard to assess precisely the performance values of the alternatives in terms of some qualitative criteria; it could also be hard to decide the weights of the criteria because of lack of complete information or some other potential vagueness in the decision problems. Under these situations, the data of the decision problems may need to be evaluated by using fuzzy numbers. With all these fuzzy data, a fuzzy version of the new method will be necessary. The fuzzy version of the proposed method shares the same core algorithm as before except that all the input data are fuzzy numbers and all the mathematical operations are fuzzy operations.

7.1 A Brief Introduction on Fuzzy Sets and Fuzzy Operations

Fuzzy set theory was developed for solving problems in which descriptions of activities and observations are imprecise, vague, and uncertain [Chen and Hwang, 1992]. Fuzzy sets are sets whose elements have different grades of membership in the interval [0, 1]. A membership function which assigns to each element a grade of membership is associated with each fuzzy set [Chen and Hwang, 1992]. Fuzzy sets were introduced by Lotfi A. Zadeh [1965] as an extension of the classical notion of sets. In classical set theory, the membership of elements in a set is assessed by binary values. If an element belongs to the set, its membership will be 1. Otherwise, it will be 0. Fuzzy sets generalize classical sets, since the indicator functions of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1 [Dubois and Prade, 1980].

A general definition of a fuzzy number is given by [Dubois and Prade, 1978; and 1980] as follows: any fuzzy subset M = $\{(x, u_M(x))\}$, where x takes its number on the real line R and $u_M(x) \in [0, 1]$. The membership function denotes the degree of truth that M takes a specific number x'. Two fuzzy numbers are equal if and only if they have the same membership functions. There are different types of fuzzy numbers. Two widely used are triangular type of fuzzy numbers and trapezoidal type of fuzzy numbers. Among these two, triangular fuzzy numbers are more often used because they are simpler compared to the more complex trapezoid fuzzy numbers. The triangular fuzzy numbers have lower, modal, and upper values. Let $\tilde{A}$ represent a triangular fuzzy number. Its membership function can be expressed as follows:

$$
f_{\tilde{A}}(x) = \begin{cases} f_{\tilde{A}}^{L}(x) = \dfrac{(x-l)}{(m-l)}, & if\ x \in [l, m] \\ f_{\tilde{A}}^{R}(x) = \dfrac{(x-u)}{(m-u)}, & if\ x \in [m, u] \\ 0, & otherwise. \end{cases}
$$

Where $f_{\tilde{A}}^{L}(x)$ and $f_{\tilde{A}}^{R}(x)$ are the left and the right spread membership functions. *l*, *m*, and *u* are real numbers and $l \leq m \leq u$, and they stand for the lower, the modal, and the upper values of fuzzy number $\tilde{A}$, respectively. As represented by the above membership function, fuzzy number $\tilde{A}$ can also be denoted as (*l*, *m*, *u*).

From the above description, it can be seen that fuzzy numbers are fuzzy sets which are characterized by different membership functions. For any application involving impreciseness and fuzziness, a vital step is the definition/generation of membership functions associated with fuzzy concepts. In general, there are two ways to generate membership functions. One is to define them subjectively. Interested readers may refer to [MacVicar-Whelan, 1978; Norwich and Turksen 1984; Turksen, 1991] where some membership function generation techniques that reflect subjective perception about vague or imprecise concepts were discussed. There are also some data-driven membership function generation techniques. A general overview of several methods for generating membership functions from domain data for fuzzy pattern recognition applications can be found in [Medasani, *et al.*, 1998]. Since then, more methods have been developed. An example is the fuzzy *c*-means variant for the generation of fuzzy term sets as developed by Liao et al. [2003]. After the membership function for each fuzzy variable is decided and the fuzzy data is collected, the next step is to apply the necessary operations on the fuzzy numbers.

Fuzzy number operations were first introduced by [Dubois and Prade, 1978; and 1980]. Let $\tilde{n}_1 = (n_{1l}, n_{1m}, n_{1u})$ and $\tilde{n}_2 = (n_{2l}, n_{2m}, n_{2u})$ represent two triangular fuzzy numbers. In [Laarhoven and Pedrycz, 1983], the basic operations of triangular fuzzy numbers are defined as follows:

(1) Addition: $\tilde{n}_1 \oplus \tilde{n}_2 = (n_{1l} + n_{2l}, n_{1m} + n_{2m}, n_{1u} + n_{2u})$

(2) Negation: $\ominus\, \tilde{n}_1 = (-n_{1u}, -n_{1m}, -n_{1l})$

(3) Multiplication: $\tilde{n}_1 \otimes \tilde{n}_2 = (n_{1l} \times n_{2l}, n_{1m} \times n_{2m}, n_{1u} \times n_{2u})$

(4) Division: $1/\tilde{n}_1 \cong (1/n_{1u}, 1/n_{1m}, 1/n_{1l})$

For the special case of raising a triangular fuzzy number to the power of another triangular fuzzy number, the approximation $\tilde{n}_1^{\tilde{n}_2} \cong ({n_{1l}}^{n_{2l}}, {n_{1m}}^{n_{2m}}, {n_{1u}}^{n_{2u}})$ can be used.

After the fuzzy data are processed by the proposed new method, the final priorities of the alternatives will also be fuzzy numbers. Since a fuzzy number represents many possible real numbers that have different membership values, it is not easy to compare the final ratings to determine which alternatives are preferred [Chen and Hwang, 1992]. Many fuzzy ranking methods have been developed to compare fuzzy numbers. For some review of these methods, interested readers may refer to [Bortolan and Degani, 1985; Chen and Hwang, 1992; Chang and Lee, 1994; Dubois and Prade, 1999; Lee-Kwang and Lee, 1999]. Under a given situation, usually people decide which method should be used by considering the complexity of the algorithm, its flexibility, accuracy, ease of interpretation and the shape of the fuzzy numbers which are used [Triantaphyllou, 2000]. The selection is also closely related to the application of the MCDM methods.

7.2 A Numerical Example on the Fuzzy Version of the New Method

In this section, a similar numerical example as the one in Section 5.1 is used to demonstrate how the fuzzy version of the proposed method can be implemented. For simplicity, triangular fuzzy numbers are used to represent the fuzzy data in this example. The fuzzy performance values and the fuzzy weights of the criteria come from a simple fuzzification of the corresponding crisp data in the original example. The original crisp data become the modals of the corresponding fuzzy data. The lower and upper parts of the fuzzy data are constructed by choosing a certain value as the spreads of the triangular fuzzy numbers.

In the original version of the proposed method, 9 crisp numbers are used to represent the natural importance of the linguistic terms. For the fuzzy version of the method, the fuzzy triangular numbers attached to the fuzzy linguistic terms will be constructed based on them. The 9 crisp numbers are used as the modals of the corresponding 9 triangular fuzzy numbers. For simplicity, value 1 is chosen as both the left and the right spreads of these fuzzy numbers except for the two end values. For the lowest linguistic term, the lower value of its associated fuzzy number should not be smaller than the minimum value 1 of the original scale. Thus its left spread is 0 and its lower and modal values are the same. For the highest linguistic term, the upper value of its associated fuzzy number should not be higher than the original maximum value 9. Then, its right spread is set to be 0 and its modal and upper values are the same. The original linguistic terms are fuzzified to fit into the fuzzy situation. After all the above adjustments, the fuzzy linguistic terms and the triangular fuzzy numbers attached to them are as follows:

Table 5. A fuzzy scale for the fuzzy version of the new method.

Linguistic Expression	**Fuzzy value**
The feeling of regret when choosing alternative A_i over alternative A_j is barely distinguishable.	(1, 1, 2)
The feeling of regret when choosing alternative A_i over alternative A_j may be noticeable.	(2, 3, 4)
The feeling of regret when choosing alternative A_i over alternative A_j is nearly strong.	(4, 5, 6)
The feeling of regret when choosing alternative A_i over alternative A_j is almost very strong.	(6, 7, 8)
The feeling of regret when choosing alternative A_i over alternative A_j is almost as strong as it can be.	(8, 9, 9)
The corresponding intermediate fuzzy numbers are used when the decision maker feels that the best answer lies between two successive fuzzy linguistic choices from the above list of choices.	(1, 2, 3), (3, 4, 5), (5, 6, 7), (7, 8, 9)

From now on, fuzzy alternatives and fuzzy criteria are denoted as $\hat{A}_i$ and $\hat{C}_k$ in order to distinguish them from their crisp version counterparts which are denoted as A_i and C_k. According to the previous description, the fuzzy performance values of the alternatives under the three fuzzy criteria are as follows:

	$\hat{C}_1$	$\hat{C}_2$	$\hat{C}_3$
$\hat{A}_1$	(18, 19, 20)	(5, 6, 7)	(14, 15, 16)
$\hat{A}_2$	(14, 15, 16)	(6, 7, 8)	(3, 4, 5)
$\hat{A}_3$	(3, 4, 5)	(8, 9, 10)	(15, 16, 17)
$\hat{A}_4$	(4, 5, 6)	(11, 12, 13)	(3, 4, 5)

The fuzzy weights of the fuzzy criteria are:

W = [(0.25, 0.35, 0.45) (0.32, 0.42, 0.52) (0.13, 0.23, 0.33)];

For crisp data, the sum of weights should be equal to 1. Now it is required that the sum of the modals of the fuzzy weights values should be equal to 1.

After replacing the original crisp regret values by the corresponding triangular fuzzy numbers, now the simulated fuzzy pairwise regret matrix in terms of the three criteria are as follows. The simulated fuzzy pairwise regret matrix in terms of criterion $\hat{C}_1$ is:

$\hat{C}_1$	$\hat{A}_1$	$\hat{A}_2$	$\hat{A}_3$	$\hat{A}_4$
$\hat{A}_1$	(1, 1, 2)	(1, 1, 2)	(1, 1, 2)	(1, 1, 2)
$\hat{A}_2$	(2, 3, 4)	(1, 1, 2)	(1, 1, 2)	(1, 1, 2)
$\hat{A}_3$	(7, 8, 9)	(5, 6, 7)	(1, 1, 2)	(1, 2, 3)
$\hat{A}_4$	(4, 5, 6)	(3, 4, 5)	(1, 1, 2)	(1, 1, 2)

The simulated fuzzy pairwise regret matrix in terms of criterion $\hat{C}_2$ is as follows:

$\hat{C}_2$	$\hat{A}_1$	$\hat{A}_2$	$\hat{A}_3$	$\hat{A}_4$
$\hat{A}_1$	(1, 1, 2)	(7, 8, 9)	(7, 8, 9)	(7, 8, 9)
$\hat{A}_2$	(1, 1, 2)	(1, 1, 2)	(2, 3, 4)	(5, 6, 7)
$\hat{A}_3$	(1, 1, 2)	(1, 1, 2)	(1, 1, 2)	(3, 4, 5)
$\hat{A}_4$	(1, 1, 2)	(1, 1, 2)	(1, 1, 2)	(1, 1, 2)

The simulated fuzzy pairwise regret matrix in terms of criterion $\hat{C}_3$ is as follows:

$\hat{C}_3$	$\hat{A}_1$	$\hat{A}_2$	$\hat{A}_3$	$\hat{A}_4$
$\hat{A}_1$	(1, 1, 2)	(1, 1, 2)	(2, 3, 4)	(1, 1, 2)
$\hat{A}_2$	(2, 3, 4)	(1, 1, 2)	(3, 4, 5)	(1, 1, 2)
$\hat{A}_3$	(1, 1, 2)	(1, 1, 2)	(1, 1, 2)	(1, 1, 2)
$\hat{A}_4$	(2, 3, 4)	(1, 1, 2)	(3, 4, 5)	(1, 1, 2)

For fuzzy data, the consistency tests are recommended to be applied on the modals of the fuzzy regret values. Since the modals of the above fuzzy regret values are the crisp regret values in the original example, they satisfy the consistency tests as examined before.

A fuzzy version of formula (4-7) is as follows:

$$\hat{r}_{ik} = [\prod_{j=1,\, i\neq j}^{m} \hat{r}(\hat{a}_{ik}, \hat{a}_{jk})]^{\frac{1}{m-1}}. \qquad (7\text{-}1)$$

Please note, in this formula, the fuzzy regret value produced when an alternative is compared with itself is not counted into the computation. It is set so in case that the upper values of the aggregated regret values are inflated improperly. By applying formula (7-1), the fuzzy regret values of the alternatives in terms of the three criteria are as follows:

$\hat{r}_{ik}$	$\hat{C}_1$	$\hat{C}_2$	$\hat{C}_3$
$\hat{A}_1$	(1.00, 1.00, 2.00)	(7.00, 8.00, 9.00)	(1.26, 1.44, 2.52)
$\hat{A}_2$	(1.26, 1.44, 2.52)	(2.15, 2.62, 3.83)	(1.82, 2.29, 3.42)
$\hat{A}_3$	(3.27, 4.58, 5.74)	(1.44, 1.59, 2.71)	(1.00, 1.00, 2.00)
$\hat{A}_4$	(2.29, 2.71, 3.91)	(1.00, 1.00, 2.00)	(1.82, 2.29, 3.42)

In the above table, the entries are $\hat{r}_{ik}$ which is the fuzzy anticipated regret value associated with fuzzy alternative $\hat{A}_i$ in terms of fuzzy criterion $\hat{C}_k$, for i = 1, 2, 3, 4 and k = 1, 2, 3. For example, $\hat{r}_{21} = (1.26 \ \ 1.44 \ \ 2.52)$.

The fuzzy version of formula (4-9) is as follows:

$$\hat{P}_i^* = \frac{\hat{P}_i \hat{J}_i}{\hat{R}_i} = \frac{\prod_{k=1}^{n} \hat{a}_{ik}^{\hat{w}_k} \times \prod_{k=1}^{n} \hat{j}_{ik}^{\hat{w}_k}}{\prod_{k=1}^{n} \hat{r}_{ik}^{\hat{w}_k}} = \prod_{k=1}^{n} \left(\frac{\hat{a}_{ik}\hat{j}_{ik}}{\hat{r}_{ik}}\right)^{\hat{w}_k}, \quad \text{for } i = 1, 2, 3, ..., m. \tag{7-2}$$

By applying formula (7-2), the fuzzy preference values of the fuzzy alternatives are:

$\hat{P}_1^* = (1.79,\ 4.26,\ 8.91)$, $\hat{P}_2^* = (1.74,\ 3.90,\ 8.67)$,

$\hat{P}_3^* = (1.56,\ 3.74,\ 8.44)$, $\hat{P}_4^* = (1.71,\ 4.00,\ 8.18)$.

Next, a method that ranks fuzzy numbers based on a distance measure is used to rank the above four fuzzy numbers. This method was introduced by Tran and Duckstein

in [2002] where they developed a new class of distance measures for interval numbers that takes into account all the points in both intervals and then used it to formulate the distance measure for fuzzy numbers. Their method for ranking fuzzy numbers is based on a comparison of the distance from fuzzy numbers to some predetermined targets: the crisp maximum (Max) and the crisp minimum (Min). The idea is that a fuzzy number is ranked first if its distance to the crisp maximum ($D_{\max}$) is the smallest but its distance to the crisp minimum ($D_{\min}$) is the greatest [Tran and Duckstein, 2002]. According to the results of some numerical examples in [Tran and Duckstein, 2002], their method overcomes several shortcomings such as the indiscriminative and counterintuitive behavior of several existing fuzzy ranking methods. Meanwhile, its computation process is simple and the concept is easy to be perceived by DMs.

In this method, the Max and Min are chosen as follows:

$$Max(I) \geq \sup\left(\bigcup_{i=1}^{I} s(A_i)\right), \qquad Min(I) \leq \inf\left(\bigcup_{i=1}^{I} s(A_i)\right).$$

In the above formulas, $s(A_i)$ is the support of fuzzy numbers A_i, $i = 1, \ldots, I$. In [Tran and Duckstein, 2002], formulas to compute $D_{\max}$ and $D_{\min}$ for some of the commonly used fuzzy numbers with two different weighting functions are also provided. Due to space limitation, they are not described here in detail. For more detailed information, interested readers can refer to their original paper. When the weighting function $f(\alpha)$ is set as $f(\alpha) = \alpha$ that means more weights are given to intervals at higher α levels, applying the $D_{\max}$ and $D_{\min}$ formulas for triangular fuzzy numbers to the previous four fuzzy preference values, the intermediate results are as follows:

$f(\alpha)=\alpha$	$\hat{P}_1^*$	$\hat{P}_2^*$	$\hat{P}_3^*$	$\hat{P}_4^*$
$D_{\max}$	5.7932	6.1988	6.4131	6.1496
$D_{\min}$	3.8748	3.4803	3.2671	3.4914

The above results indicate that the ranking of the four fuzzy alternatives is:

$$\hat{A}_1 \succ \hat{A}_4 \succ \hat{A}_2 \succ \hat{A}_3.$$

One can observe that this ranking is identical to that of the crisp case. To show that a ranking method could make a difference, another method that ranks fuzzy numbers with integral values as proposed by Liou and Wang in [1992] was also used to rank the fuzzy numbers. When the parameter α in this method is set equal to 0.5, the ranking of the fuzzy alternatives becomes:

$$\hat{A}_1 \succ \hat{A}_2 \succ \hat{A}_3 \succ \hat{A}_4.$$

Though fuzzy data offer the capability to deal with imprecise information, they may also complicate the analysis of decision problems. First, mathematical operations of fuzzy data are not easy. This may greatly increase the mathematical computations. Second, though there have been many (perhaps, too many) fuzzy ranking methods, it could still be hard to clearly distinguish which fuzzy numbers (priority values) are better or worse. As illustrated above, different ranking methods may lead to different ranking results. Moreover, within the same method, different setting of the same parameter could also lead to different results. Both situations could complicate the ranking of fuzzy priority values. Then, it may be difficult to decide which alternatives should be ranked higher and which lower. Thus, the DMs should carefully study these issues before they decide whether to use fuzzy data and the fuzzy version of a specific MCDM method.

Please note that the fuzzy version of the proposed MCDM method can be

accompanied with sensitivity analyses similar to ones developed for the crisp version of it. However, as this is rather straightforward to do, it has been omitted here for brevity.

CHAPTER 8. CONCLUDING REMARKS

In this chapter, the main contributions of the research in this dissertation and their significance are summarized. Then, some possible future research directions are proposed to expand the research in this dissertation.

8.1 Summary of the Research Contributions and Their Significance

In conclusion, the research in this dissertation has achieved the following main contributions. First, a new MCDM method is proposed. Besides the usual benefit and cost criteria, the new method is able to incorporate the effects of regret and rejoicing for decision makers who value these emotional factors in MCDM situations. Most of the current MCDM methods consider only the cognitive aspects of decision-making problems and assume that the DMs are complete rational humans which are dissociated from psychological feelings. The significance of this new model lies in that unlike those MCDM methods, it considers the notion of regret and rejoicing and provides a better description of human behavior in decision-making and offers the DMs the flexibility to trade off some economic benefits explicitly in order to gain a state of psychological satisfaction.

Second, within the new method, regret and rejoicing effects are determined by using linguistic terms. It is regarded as more reasonable and realistic to rational human behavior than using continuous functions. By using the linguistic terms, the DMs have the flexibility to decide their own regret/rejoicing levels and the specific tendencies of these feelings according to the specific situations of their decision problems. Furthermore, the proposed approach for eliciting regret and/or rejoicing by pairwise comparisons is flexible and adapts to the reactions of the individual DM and decision problem.

Third, by using the multiplicative formulas to compute the final priorities of the alternatives, the new method is immune to those rank reversal problems mentioned in the dissertation when regret and rejoicing are not considered. Then rank reversals may occur only as result of readjusting the effects of regret and/or rejoicing when the set of the alternatives is altered. It is a significant property of the new method. Because some well-known MCDM methods, like the additive AHP methods and the ELECTRE II and III methods, suffer from the rank reversals even without the consideration of regret and rejoicing. The effects of regret and rejoicing may be ignored if, for instance, their presence could be considered negligible when compared to the usual performance values of the alternatives under the benefit and cost criteria. Meanwhile, by using the multiplicative formulas, the new method is able to deal with qualitative and quantitative criteria expressed in different units of measurement.

Fourth, some sensitivity analysis procedures are developed for the proposed method. Sometimes it is hard for a DM to precisely capture his/her perception of regret and rejoicing by using a specific linguist term. Thus, it is significant and necessary to study how changes of regret and rejoicing values could affect the ranking results of the decision problems. It is believed that DMs can make more careful assessments about their regret and rejoicing feelings if they can see how sensitive the ranking results could be to the changes in these values. Another meaning of this contribution is that by using a sensitivity analysis, the DM can obtain a more comprehensive understanding about the ranking of the alternatives and choose the one that is more stable than the others.

Fifth, considering the potential impreciseness within the linguistic terms and the potential vagueness in the data of some decision problems, a fuzzy version of the new

method is also introduced. The uncertainty and imprecision which is inherent in a DM's assessments of regret and rejoicing can be accounted for by considering each of these emotional factors as fuzzy quantities, characterized by appropriate membership functions. Sometimes, the data of the decision problems may also need to be evaluated by using fuzzy numbers. With all these potential fuzzy data, a fuzzy version of the new method is necessary and significant. However, mathematical operations of fuzzy data may also complicate the analysis of decision problems. Thus, the DMs should think about it carefully before they decide whether to use fuzzy data and the fuzzy version of the new method.

Another significant contribution is that the introduction of emotional factors brings a new perspective to the issue of rank reversals. It was once thought that rank reversals resulted from a method's own mathematical artifacts are unacceptable. However, as illustrated by the car example, strong emotional feelings like regret and rejoicing could make a DM to change his/her preference about the alternatives unintentionally and then change the ranking of them. The reason of rank reversals also lies in the way that regret and rejoicing are measured. Since the feeling of regret and rejoicing comes from the comparison of one alternative with the others, it is unavoidable that the levels of these factors depend on the existence of other alternatives. Thus, for the new method which incorporates these emotional factors, the occurrence of some rank reversals might be natural and acceptable.

The main research in this dissertation has been summarized in the form of three journal articles. As mentioned previously, one of them has been published in a refereed journal. Another one is in print for publication also in a refereed journal. The latest one

which is on the study of regret and rejoicing with the collaboration with Dr. Edouard Kujawski, a professor at the Naval Postgraduate School, is going to be submitted for publication to the journal of Decision Sciences very soon. These articles have also been presented at several national and international conferences. For the details please see Part I of the reference list.

8.2 Future Research Directions

As mentioned before, the use of a set of linguistic terms to estimate a DM's anticipated regret feelings is in essence a mechanism for eliciting a hidden discrete regret function. It offers a DM the flexibility to decide the specific tendency of his/her regret feeling based on the specific situation of his/her decision problem. However, sometimes, some DMs may not be able to clearly capture the tendency of their perception of regret and rejoicing. Thus, in the future, some shapes of discrete functions may need to be developed to model some general situations of humans' perception of regret and rejoicing. These functions should be able to capture the realistic tendencies for most of the rational humans' perception of these emotional feelings. Except this possible direction, as with other aspects of decision making, other scales could be employed to quantify the linguistic terms and replace the original nine evenly distributed integer values. For example, exponential values [Lootsma, 1999] might be more applicable for situations where there is evidence that regret and rejoicing feelings vary by a certain geometric progression factor.

Undoubtedly, emotions and feelings are indispensable factors in humans' decision-making activities. More research is needed in this fascinating area. A long march

of my research in this area has lead to the completion of this dissertation. However, this is not the end but just the beginning.

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