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## A Conditioned Gaussian-Poisson Model for Default Phenomena

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# A CONDITIONED GAUSSIAN-POISSON MODEL FOR DEFAULT PHENOMENA

A Dissertation

Submitted to the Graduate Faculty of the  
Louisiana State University and  
Agricultural and Mechanical College  
in partial fulfillment of the  
requirements for the degree of  
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in

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by

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# Abstract

We introduce a new model to study the behavior of a portfolio of defaultable assets. We refer to this model as the Gaussian-Poisson model. It builds upon one-factor Gaussian copula models and Poisson models (specifically Cox processes). Our model utilizes a random variable  $Y$  along with probability measures  $\mathbb{P}^\bullet$  and  $\mathbb{P}^\dagger$ . The measures  $\mathbb{P}^\bullet$  and  $\mathbb{P}^\dagger$  will act as market pricing measures and are obtained via conditioning. The random variable  $Y$  will act as a default descriptor.

We provide the distribution of  $Y$  under both  $\mathbb{P}^\bullet$  and  $\mathbb{P}^\dagger$ . We use a conditional probability to examine expected portfolio and tranche losses, with applications including credit default swaps and collateralized debt obligations. The Gaussian-Poisson model requires a choice of an intensity model. We examine a portfolio loss' dependence upon parameters of the intensity model. Finally, we present three possible models of the intensity process.

# Chapter 1

## Introduction

Suppose we have a portfolio of  $m$  defaultable assets. It is of interest to study when these assets may default. Specifically, suppose we have an  $m$ -dimensional random variable

$$\tau = (\tau_1, \tau_2, \dots, \tau_m), \quad (1.1)$$

defined on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with the random variable

$$\tau_j : \Omega \rightarrow [0, T] \cup \{\infty\} \quad (1.2)$$

corresponding to name  $j$  in the portfolio, where  $0 < T < \infty$ . We define  $\tau$  to represent the default times of the  $m$  assets. We shall study the portfolio over the finite time interval  $[0, T]$ , and if  $\tau_j = \infty$ , we interpret that the asset does not default during our observation time. Rather than continuously observing the portfolio, we can observe it at a finite number of times,

$$0 = t_0 < t_1 < t_2 < \dots < t_n = T. \quad (1.3)$$

Hence, if  $\tau_j \in (t_{k-1}, t_k]$ , the default of name  $j$  during the  $k$ th interval is detected at time  $t_k$ .

A common framework from which to study the distribution of  $\tau$  is the one-factor Gaussian copula model. It models  $\tau_j$  using a standard Gaussian  $X_j$  with the distribution of  $\tau$  such that

$$X = (X_1, X_2, \dots, X_m) \quad (1.4)$$

is jointly distributed as an  $\mathbb{R}^m$ -valued Gaussian random variable. The model is determined by the correlations between the  $X_j$ . The correlation between  $X_i$  and

$X_j$  is defined as

$$\rho_{i,j} = \frac{\mathbb{E}[(X_i - E[X_i])(X_j - E[X_j])]}{\sqrt{\mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2} \sqrt{\mathbb{E}[X_j^2] - \mathbb{E}[X_j]^2}}, \quad (1.5)$$

i.e., the covariance of  $X_i$  and  $X_j$  divided by the product of their standard deviations. Since  $X_j$  is standard Gaussian for all  $j$ , we see this reduces to

$$\rho_{i,j} = \mathbb{E}[X_i X_j]. \quad (1.6)$$

A homogeneous portfolio is modeled by assuming there is a common correlation such that  $\rho_{i,j} = \rho$  for any two names.

Using the above model, we can obtain both the number of defaults during the  $k$ th time interval and the specific names which defaulted. Meanwhile, if one is simply concerned with the number of defaults, a Poisson process can be used as a model. A particular example would be in a homogeneous portfolio. Whereas the one-factor Gaussian copula relies upon correlation to determine the distribution of defaults, the Poisson model uses a default intensity. This intensity can be constant over time, a function of time, or a stochastic process. The richness of the model is determined by the complexity of the intensity. A Poisson process  $N(t)$  will observe the portfolio continuously and thereby detect defaults at times other than the previously mentioned observation times. However, we can focus on

$$\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_n, \quad (1.7)$$

where  $\tilde{N}_k$  is the Poisson random variable

$$N(t_k) - N(t_{k-1}). \quad (1.8)$$

Both of these models have deficiencies. The Poisson model can only be used when we either have a homogeneous portfolio or are simply interested in counting



defaults. However, even if we are only interested in counting defaults, a Poisson model has difficulty accounting for correlation between assets, which is especially important if the portfolio is inhomogeneous. On the other hand, many argue that Gaussian copula models rely too heavily on correlation between assets, since this value may change quickly. In fact, it received significant scrutiny following the financial crisis of 2008; as examples, see [33] and [41]. Models which study default behavior are especially of interest in relation to the study of two financial instruments, credit defaults swaps (CDSs) and collateralized debt obligations (CDOs). We give a brief introduction to these topics here.

A CDS is a contract between two parties, a seller and a buyer. It is a credit derivative, meaning that its value is based upon a certain entity's creditworthiness. The buyer agrees to make regular payments (premiums) to the seller, with the amount determined using the probability of an adverse credit event, and, in return, the seller will pay a lump sum to the buyer if a specified credit event occurs, such as a loan default or a credit rating drop.

A CDS can act quite similar to insurance. For instance, suppose that an investor holds a bond in a company but is worried about the bond losing value. Perhaps he may anticipate a credit rating drop or suspect the possibility of the company going bankrupt. Thus, in entering into a CDS contract as the buyer, he insures himself against this loss. Of course, if a credit event does not occur, then the buyer has lost money. However, he could have lost a significant amount more without protection, which is the essence of insurance. In this scenario, the purpose for which the seller purchased the CDS was to perform a hedge.

Another reason that an investor may take a position in a CDS contract is to speculate. For instance, suppose the credit event that a CDS contract is based upon is the default of a company's bond. The buyer of the contract is not even

required to hold this bond in order to participate in the swap. In this case, the buyer is not in the market for insurance purposes. Essentially, both parties are taking opposite position in a “bet” that the bond will or will not default.

CDOs are credit derivatives that comes in many different forms and can range from quite simple to incredibly complex. The underlying assets have a positive probability of default and include mortgages and CDS contracts. A CDO is divided into levels called *tranches*. The highest and lowest rated tranches are often called the *senior* and *equity* tranches, respectively. The purpose of these tranches are to organize payments. As an example, assume the underlying assets are mortgages, then the investors in the CDO will be receiving payments from the mortgage holders. Suppose that the equity tranche ranges from 0% to 3%. This range means that tranche is responsible for absorbing the first 3% of losses due to default. Thus, once the CDO has lost 3% of its principal, then the equity tranche will no longer receive any payments.

This dissertation introduces a new model to examine default behavior related to such financial instruments. We shall refer to our model as the Gaussian-Poisson model. It is designed to combine the Gaussian copula and Poisson models together so that they can each address the deficiencies of the other. The organization of the rest of this thesis is as follows.

**Chapter 2** consists entirely of background material. We discuss the two mathematical concepts, Poisson processes and Gaussian copulas, necessary to the rest of the paper.

In **Chapter 3**, we present the Gaussian-Poisson model. As its name suggests, it relies upon both a Gaussian and Poisson model. Hence, we first define the one-factor Gaussian copula model and Cox (doubly stochastic Poisson) process that will be used in our model. The intensity of the Cox process will be stochastic and

assumed to be dependent on the short rate (instantaneous interest rate). Both the Gaussian and Poisson models are defined on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . All random variables throughout this thesis are defined on the measurable space  $(\Omega, \mathcal{F})$  unless otherwise noted. We will proceed to define the key elements of the Gaussian-Poisson model, the probability measures  $\mathbb{P}^\bullet$  and  $\mathbb{P}^\dagger$  and random variable  $Y$ . Both measures will be defined via conditioning processes using the measure  $\mathbb{P}$ . Financially,  $\mathbb{P}^\bullet$  and  $\mathbb{P}^\dagger$  act as market pricing measures and  $Y$  is a default descriptor. We will provide the probability mass functions of  $Y$  under these two measures and prove that the conditional probabilities  $\mathbb{P}^\bullet[Y = y \mid \mathcal{G}]$  and  $\mathbb{P}^\dagger[Y = y \mid \mathcal{G}]$  are equal, where  $\mathcal{G}$  is the sigma-algebra that is generated by the short rate process.

**Chapter 4** focuses on using the Gaussian-Poisson model to provide results applicable to determining the fair market value of financial instruments. These calculations have an impact on CDS spreads and CDO tranche premiums. We dedicate a section to each of our two possible market pricing measures. In these sections, we calculate the present value of loss for a portfolio of defaultable assets and expected tranche losses for a CDO. We also examine the sensitivity of the model to intensity process parameters.

The Gaussian-Poisson model relies upon both a choice of a short rate model and an intensity model. Hence, the purpose of **Chapter 5** is to explore these models. After providing a simplistic example, we review popular short rate models, discussing their advantages and disadvantages. We conclude the chapter by proposing three intensity models and exploring their key properties.

A conclusion to the dissertation is offered in **Chapter 6**. We summarize the original results that are provided in this work while also looking ahead to future study.

The bibliography (**References**) lists works cited and consulted. Works that have been especially helpful in preparing this dissertation include: [21], [28], [35], and [44]. It also attempts to offer a broad overview of research in the field of credit derivatives and default behavior.

The purpose of the **Appendix** is to provide the necessary financial terminology required for this paper. It will act similar to a glossary. Credit defaults swaps and collateralized debt obligations were presented in this chapter and, hence, will not be discussed there.

# Chapter 2

## Poisson Processes and Gaussian Copula Models

In this chapter we describe the mathematical foundation of two key concepts that will be used in this work: Poisson processes and Gaussian copulas (specifically a one-factor model).

### 2.1 Poisson Processes

This section is motivated by our desire to study a type of stochastic processes called Cox processes which are also known as doubly stochastic Poisson processes. To begin, we recall the definition of a Poisson random variable.

**Definition 2.1.** *A discrete random variable  $N$  is called Poisson with parameter  $\lambda > 0$  if the probability mass function of  $N$  is*

$$\mathbb{P}[N = i] = e^{-\lambda} \frac{\lambda^i}{i!}, \quad (2.1)$$

for  $i = 0, 1, 2, \dots$

Poisson random variables are useful models for describing the number of events that occur in a finite time period when the events occur independently and at a known average rate. The parameter  $\lambda$  is that average rate,  $\mathbb{E}[N]$ , and is also the variance of  $N$ . Examples of events that follow a Poisson distribution can include

- Photons arriving at a telescope per second,
- Number of decays in a radioactive sample, and
- Mortgage defaults occurring per year in Baton Rouge, Louisiana.

The last example is from finance; applications in this area will be our focus. We will specifically focus on loan defaults and other credit-related events.

In this work, we wish to examine Poisson processes. However, the term “Poisson process” is a general one, and we will refer specifically to three types of Poisson processes: homogeneous, inhomogeneous, and doubly stochastic. The final type is the one that we are most interested in studying and is usually referred to as a Cox process; named after David Cox who first described doubly stochastic Poisson processes in 1955 [12]. The simplest type is a homogeneous Poisson process.

**Definition 2.2.** *A stochastic process  $N(t, \omega)$  is called a homogeneous Poisson process with intensity  $\lambda > 0$  if*

1.  $\mathbb{P}[\{\omega \mid N(0, \omega) = 0\}] = 1$
2.  $\mathbb{P}[\{\omega \mid N(\cdot, \omega) \text{ is right continuous with left-hand limits}\}] = 1$
3. *For any  $0 \leq s_1 < s_2 < \dots < s_k$ , the random variables*

$$N(s_1), N(s_2) - N(s_1), \dots, N(s_k) - N(s_{k-1}) \quad (2.2)$$

*are independent.*

4. *For any  $0 \leq s < t$ ,  $N(t) - N(s)$  is a Poisson random variable with parameter  $\Lambda_{s,t} = \lambda(t - s)$ ; i.e.,*

$$\mathbb{P}[N(t) - N(s) = i] = e^{-\Lambda_{s,t}} \frac{(\Lambda_{s,t})^i}{i!} \quad (2.3)$$

*for  $i = 0, 1, 2, \dots$ .*

Observe that the first two requirements, that  $N(t)$  begins at zero and is a càdlàg function almost surely, are logical constraints when we consider that  $N(t)$  is designed to count the number of arrivals by time  $t$ . At time  $t = 0$ , an arrival is not possible as time has not yet begun. Furthermore, suppose  $t$  is not an arrival time,

then there is a neighborhood around  $t$  for which there are no arrival times. Within this neighborhood the process is constant with value  $N(t)$ . Hence,

$$\lim_{x \rightarrow t} N(x) = N(t) \quad (2.4)$$

and the process is continuous at that time. If  $t$  is an arrival time, then for some time period before time  $t$ , we have the process is constant and equal to  $N(t) - 1$ . Also, there is a time period after time  $t$  for which the process is constant and equal to  $N(t)$ . Thus, we have

$$\lim_{x \rightarrow t^-} N(x) = N(t) - 1, \quad \lim_{x \rightarrow t^+} N(x) = N(t), \quad (2.5)$$

and can conclude that  $N(t)$  is a càdlàg function.

The parameter  $\lambda$  is referred to as the *intensity* of the Poisson process. A process satisfying the conditions of Definition 2.2 is called *homogeneous* since  $\lambda$  is a constant. In other words, the arrivals are expected to occur at the same rate, regardless of the time. By not requiring  $\lambda$  to be constant, we can define different types of Poisson processes. For instance, suppose we allow the intensity to be a function of time; the rate of arrivals is now inhomogeneous with respect to time. We define an inhomogeneous Poisson process as follows.

**Definition 2.3.** Let  $\lambda(t)$  be a non-negative deterministic function that, for all  $0 \leq s < t$ , satisfies

$$0 < \int_s^t \lambda(u) \, du < \infty, \quad (2.6)$$

and define

$$\Lambda_{s,t} = \int_s^t \lambda(u) \, du. \quad (2.7)$$

A stochastic process  $N(t, \omega)$  is called an *inhomogeneous Poisson process* with *instantaneous intensity function*  $\lambda(t)$  if:

1. Conditions 1, 2, and 3 from Definition 2.2 are satisfied,

2. For any  $0 \leq s < t$ ,  $N(t) - N(s)$  is a Poisson random variable with parameter

$\Lambda_{s,t}$ ; i.e.,

$$\mathbb{P}[N(t) - N(s) = i] = e^{-\Lambda_{s,t}} \frac{(\Lambda_{s,t})^i}{i!} \quad (2.8)$$

for  $i = 0, 1, 2, \dots$

A homogeneous Poisson process satisfies the above definition with constant intensity function  $\lambda(t) = \lambda$ . Also, observe that the Poisson parameter used is the average instantaneous intensity over the interval  $[s, t]$  times the length of the interval. Inhomogeneous Poisson processes can be quite useful as often phenomena that scientists wish to model have varying arrival rates over time. However, often an even more realistic assumption is to assume that not only is the intensity function non-constant but random as well. This assumption is especially useful for financial applications where there can be significant sources of uncertainty in the market. We now define a doubly stochastic Poisson process with a random intensity function; we shall refer to such a process as a *Cox process*.

**Definition 2.4.** Let  $\lambda(t, \omega)$  be a non-negative stochastic process that, for all  $0 \leq s < t$ , satisfies

$$0 < \int_s^t \lambda(u) du < \infty \quad (2.9)$$

almost surely. A stochastic process  $N(t, \omega)$  is called a Cox process (doubly stochastic poisson process) with intensity process  $\lambda(t, \omega)$  if

1. Conditions 1 and 2 from Definition 2.2 are satisfied

2. For any  $0 \leq s_1 < s_2 < \dots < s_k$ , given  $\lambda(u)$  for  $0 \leq u \leq s_k$ , the random variables

$$N(s_1), N(s_2) - N(s_1), \dots, N(s_k) - N(s_{k-1}) \quad (2.10)$$

are independent.



3. For any  $0 \leq s < t$ , given  $\lambda(u)$  for  $0 \leq u \leq t$ ,  $N(t) - N(s)$  is a Poisson random variable with parameter  $\Lambda_{s,t}$  as defined in Equation 2.7, i.e.,

$$\mathbb{P}[N(t) - N(s) = i \mid \lambda(u), 0 \leq u \leq t] = e^{-\Lambda_{s,t}} \frac{(\Lambda_{s,t})^i}{i!} \quad (2.11)$$

for  $i = 0, 1, 2, \dots$ .

Cox processes are often defined differently depending on the purpose for which they are being studied. For instance, often these processes are discussed using the language of random measures. However, that language requires additional background and is not required for this work. Alternatively, Cox processes often are defined as having the property

$$\mathbb{E}[e^{z(N(t)-N(s))} \mid \mathcal{F}_t^*] = \exp((e^z - 1)\Lambda_{s,t}), \quad (2.12)$$

for all  $0 \leq s < t$  and  $z \in \mathbb{C}$ , where

$$\mathcal{F}_t^* = \sigma(\{\lambda(u) \mid 0 \leq u \leq t\}), \quad (2.13)$$

and  $\sigma(\{\cdot \cdot \cdot\})$  is the sigma-algebra generated by  $\{\cdot \cdot \cdot\}$ . Equation 2.11 can then be deduced using this property by recalling that the characteristic function of a Poisson random variable with parameter  $\Lambda_{s,t}$  is equal to the right hand side of Equation 2.12. Another important property is that

$$\mathbb{E}[\theta^{N(t)-N(s)}] = \mathbb{E}[e^{-(1-\theta)(\Lambda(t)-\Lambda(s))}], \quad (2.14)$$

where the stochastic process  $\Lambda(t)$  is defined as

$$\Lambda(t)(\omega) = \left( \int_0^t \lambda(u) \, du \right)(\omega). \quad (2.15)$$

This property means that finding the probability generating function of  $N(t)$  is equivalent to finding the moment generating function of  $\Lambda(t)$ . Hence, if we know one distribution, we can obtain the other.

Finally, we wish to comment on the relationship between a Cox process and a Poisson Process with a deterministic intensity function. If  $N(t)$  is a Cox process on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , then it is not accurate to say that “ $N(t)$  has Poisson increments”, as  $N(t) - N(s)$  is only Poisson when conditioned to a specific sample path of the intensity process. In other words we must first be given  $\lambda(u)$ , for  $0 \leq s < t$ , so that  $\Lambda_{s,t}$  becomes deterministic. Then, after conditioning the Cox process  $N$  to  $\lambda$ , we have

$$N(t) - N(s) \sim \text{Poisson}(\Lambda_{s,t}), \quad (2.16)$$

thereby, obtaining an inhomogeneous Poisson process. This observation is important for simulations where one first simulates the intensity process  $\lambda(t)$ , obtains  $\Lambda(t)$  via integration, and proceeds to simulate  $N(t)$  as an inhomogeneous Poisson process with parameters  $\Lambda(t)$ .

To conclude this section, we calculate both the probability of  $i$  arrivals in a time interval and the expected value in that interval.

**Lemma 2.5.** *For a Cox process,  $N(t)$ , and for all  $0 \leq s < t$ , we have*

$$\mathbb{P}[N(t) - N(s) = i] = \mathbb{E} \left[ e^{-\Lambda_{s,t}} \frac{\Lambda_{s,t}^i}{i!} \right] \quad (2.17)$$

for  $i = 0, 1, 2, \dots$ , and

$$\mathbb{E}[N(t) - N(s)] = \mathbb{E}[\Lambda_{s,t}]. \quad (2.18)$$

*Proof.* Recall from Equation 2.13 that  $\mathcal{F}_t^*$  is the sigma-algebra generated by  $\lambda(u)$  for  $0 \leq u \leq t$ . An application of the law of total probability is that, for event  $A$ ,

$$\mathbb{P}[A] = \mathbb{E}[\mathbb{P}[A \mid \mathcal{F}_t^*]]. \quad (2.19)$$

Furthermore, for a random variable  $X$ , we have

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{F}_t^*]]. \quad (2.20)$$

Therefore, since

$$\mathbb{P}[N(t) - N(s) = i \mid \mathcal{F}_t^*] = e^{-\Lambda_{s,t}} \frac{\Lambda_{s,t}^i}{i!} \quad (2.21)$$

and

$$\mathbb{E}[N(t) - N(s) \mid \mathcal{F}_t^*] = \Lambda_{s,t}, \quad (2.22)$$

we have our desired result.  $\square$

## 2.2 Gaussian Copulas

We now proceed to discuss copulas, specifically the Gaussian copula model. Its use in finance became especially prominent following a 2000 paper by David Li [31]. Following the global economic crisis of 2008, it received significant scrutiny as one of the possible causes. However, despite its shortcomings, the model is practicably implementable and useful for our purposes. Our approach will not be to simply discard the copula model but rather attempt to improve upon it by adding another layer whereby additional market information can play a role.

We begin by defining a copula and then proceed with a discussion in order to show why that definition is relevant to our work.

**Definition 2.6.** *An  $m$ -dimensional copula is defined as a joint cumulative distribution function  $C : [0, 1]^m \rightarrow [0, 1]$  whose marginal distributions are uniform on  $[0, 1]$ .*

Suppose we have  $m$  real-valued random variables,

$$\tau_1, \tau_2, \dots, \tau_m$$

and that the  $j$ th cumulative distribution function,  $F_{\tau_j}$ , is strictly increasing and continuous, for all  $j$ . Define

$$U_j = F_{\tau_j}(\tau_j). \quad (2.23)$$

It is well known that  $U_j$  is uniformly distributed along the interval from 0 to 1.

We can see this fact easily by observing that

$$\begin{aligned}
F_{U_j}(u) &:= \mathbb{P}(U_j \leq u) \\
&= \mathbb{P}(F_{\tau_j}(\tau_j) \leq u) \\
&= \mathbb{P}(\tau_j \leq F_{\tau_j}^{-1}(u)) \\
&= F_{\tau_j}(F_{\tau_j}^{-1}(u)) \\
&= u,
\end{aligned} \tag{2.24}$$

for  $u \in [0, 1]$ . The equality

$$\mathbb{P}(F_{\tau_j}(\tau_j) \leq u) = \mathbb{P}(\tau_j \leq F_{\tau_j}^{-1}(u)) \tag{2.25}$$

is valid since the inverse is well defined due to assumption that  $F_{\tau_j}$  is strictly increasing and continuous.

Denote  $F_\tau$  as the joint cumulative distribution function for

$$\tau = (\tau_1, \tau_2, \dots, \tau_m) \tag{2.26}$$

and  $\Phi(t)$  as the standard Gaussian cumulative distribution function. We define

$$X_j = \Phi^{-1} \circ F_{\tau_j}(\tau_j) \tag{2.27}$$

implying that  $X_j$  is standard Gaussian since

$$\begin{aligned}
\mathbb{P}[X_j \leq t] &= \mathbb{P}[F_{\tau_j}(\tau_j) \leq \Phi(t)] \\
&= F_{U_j}(\Phi(t)) \\
&= \Phi(t).
\end{aligned} \tag{2.28}$$

We assume further that the joint distribution of  $\tau$  is such that

$$X = (X_1, X_2, \dots, X_m) \tag{2.29}$$

is an  $\mathbb{R}^m$ -valued Gaussian random variable. Since each  $X_j$  is standard Gaussian, the cumulative distribution function  $\Phi_R(x_1, x_2, \dots, x_m)$  of  $X$  is fully determined by the correlation matrix  $R$ , with

$$r_{ij} = \mathbb{E}[X_i X_j]. \quad (2.30)$$

In 1959 [45], Sklar showed that given a joint cumulative distribution function such as  $\Phi_R$ , there exists a unique copula function,  $C_X(u_1, u_2, \dots, u_m)$ , such that

$$C_X(\Phi(x_1), \Phi(x_2), \dots, \Phi(x_m)) = \Phi_R(x_1, x_2, \dots, x_m). \quad (2.31)$$

Since we assumed the joint distributions of  $\tau$  and  $X$  were equivalent, then  $F_\tau$  can be modeled using the Gaussian copula  $C_X$ . Thus, we can use the Gaussian framework to simulate the behavior of  $\tau$  using the equality of events,

$$[\tau_j \leq t] = [X_j \leq c_j(t)], \quad (2.32)$$

where

$$c_j(t) = \Phi^{-1} \circ F_{\tau_j}(t). \quad (2.33)$$

### 2.3 One-Factor Model

We are specifically interested in studying a one-factor Gaussian copula model. Let

$$Z, \epsilon_1, \epsilon_2, \dots, \epsilon_m$$

be independent standard Gaussian random variables, and assume that

$$\rho_j \in (0, 1) \text{ for } j \in \{1, 2, \dots, m\}.$$

Recall that we previously defined  $X_j$  as a standard Gaussian random variable via the assumption (2.27). We now further assume that

$$X_j = \sqrt{\rho_j} Z + \sqrt{1 - \rho_j} \epsilon_j. \quad (2.34)$$

Having  $X_j$  defined in this manner is consistent with the assumption that  $X_j$  is standard Gaussian as

$$\mathbb{E}[X_j] = \mathbb{E}[\sqrt{\rho_j}Z + \sqrt{1 - \rho_j}\epsilon_j] = \sqrt{\rho_j}\mathbb{E}[Z] + \sqrt{1 - \rho_j}\mathbb{E}[\epsilon_j] = 0, \text{ and} \quad (2.35)$$

$$\text{Var}[X_j] = \text{Var}[\sqrt{\rho_j}Z] + \text{Var}[\sqrt{1 - \rho_j}\epsilon_j] = \rho_j\text{Var}[Z] + (1 - \rho_j)\text{Var}[\epsilon_j] = 1. \quad (2.36)$$

Thus, the correlation matrix  $R$  fully determines the distribution of  $X$ . The correlation between two random variables,  $X_i$  and  $X_j$ , each with zero expectation and unit variance, is  $\mathbb{E}[X_iX_j]$ . Observe that, for  $i \neq j$ ,

$$\mathbb{E}[Z\epsilon_i] = \mathbb{E}[Z\epsilon_j] = \mathbb{E}[\epsilon_i\epsilon_j] = 0, \quad (2.37)$$

since  $Z, \epsilon_i, \epsilon_j$  are independent and standard Gaussian, implying that

$$\mathbb{E}[X_iX_j] = \sqrt{\rho_i\rho_j}\mathbb{E}[Z^2] = \sqrt{\rho_i\rho_j}. \quad (2.38)$$

For  $i = j$ , we have  $\mathbb{E}[X_iX_j] = \mathbb{E}[X_i^2] = 1$ . Therefore, the  $(i, j)$  entry of the correlation matrix  $R$  is

$$r_{ij} = \begin{cases} \sqrt{\rho_i\rho_j} & i \neq j \\ 1 & i = j \end{cases}. \quad (2.39)$$

If there exists  $\rho \in (0, 1)$  such that  $\rho_i = \rho$  for all  $i$ , then every entry of  $R$  equals  $|\rho| = \rho$  except on the diagonal. This model would represent a homogeneous portfolio.

# Chapter 3

## The Gaussian-Poisson Model

This chapter is designed to introduce the model that we wish to study. While it is motivated by financial ideas, we attempt to be as mathematically rigorous as possible with definitions made in mathematical terms intertwined with explanations of the model's relevance to finance. In the first two sections we discuss two different default models. The one-factor Gaussian copula model provides a framework to determine which names have defaulted within a particular time window. One can then determine how many assets have defaulted during each interval based upon that information. The Poisson model simply determines the number of assets that have defaulted. One cannot recover the names that have defaulted. The purpose of the remaining sections is to present a new framework which combines the previous two models together into one. We shall refer to this new model as the Gaussian-Poisson model. It shall be presented, at first, using only mathematical language. Once the model has been completely defined, we shall then proceed to describe its meaning relative to finance.

### 3.1 One-Factor Gaussian Copula Model

Suppose we have  $m$  assets in a portfolio and an  $m$ -dimensional random variable

$$\tau = (\tau_1, \tau_2, \dots, \tau_m). \quad (3.1)$$

The random variable,  $\tau_j$ , represents the time of default for the  $j$ th name and is assumed to be greater than zero. We will observe the portfolio a finite number of times,

$$0 = t_0 < t_1 < t_2 < \dots < t_n = T, \quad (3.2)$$

with  $t_0$  representing present time. Also, for notational convenience, we write

$$t_\infty = \infty, \quad (3.3)$$

and set  $\tau_j = t_\infty$  if name  $j$  does not default by time  $t_n$ .

**Definition 3.1.** *We define events,  $D_{j,k}$  and  $D_{j,\infty}$ , and random variables,  $\nu_k$  and  $\tilde{\nu}_k$ , in the following manner:*

$$\begin{aligned} D_{j,k} &= [t_{k-1} < \tau_j \leq t_k] \\ D_{j,\infty} &= [\tau_j = t_\infty] \\ \tilde{\nu}_k &= \sum_{j=1}^m \mathbb{1}_{D_{j,k}} \\ \nu_0 &= 0 \\ \nu_k &= \sum_{j=1}^m \mathbb{1}_{[\tau_j \leq t_k]}, \end{aligned} \quad (3.4)$$

where

$$j \in \{1, 2, \dots, m\} \text{ and } k \in \{1, 2, \dots, n\}.$$

These variables represent the following financial concepts. The event  $D_{j,k}$  occurs if the  $j$ th name defaults during the  $k$ th interval  $(t_{k-1}, t_k]$ . If name  $j$  does not default during our observations, i.e. by time  $T$ , then event  $D_{j,\infty}$  is said to have occurred. This event is equivalent to the event  $[\tau_j > t_n]$  as well. The following lemma will be used later in this chapter.

**Lemma 3.2.** *For fixed  $j \in \{1, \dots, m\}$ , the events*

$$D_{j,1}, D_{j,2}, \dots, D_{j,n}, D_{j,\infty}$$

*are disjoint and their union is  $\Omega$ .*

*Proof.* Let  $j$  be fixed and choose arbitrary  $k, k' \in \{1, \dots, n\}$  where  $k < k'$  without loss of generality. Since  $k < k'$ , we have  $t_k \leq t_{k'-1}$  implying the two intervals



$(t_{k-1}, t_k]$  and  $(t_{k'-1}, t'_k]$  are disjoint. Thus,  $\tau_j$  cannot be in both intervals, and we have

$$D_{j,k} \cap D_{j,k'} = \emptyset. \quad (3.5)$$

Also, since  $t_k \leq t_n < t_\infty$ ,

$$D_{j,k} \cap D_{j,\infty} = \emptyset \quad (3.6)$$

as well. Finally, either the event  $[\tau_j \in (0, t_n)]$  or the event  $[\tau_j = t_\infty]$  must occur implying that

$$\left( \bigcup_{k=1}^n D_{j,k} \right) \cup D_{j,\infty} = \Omega. \quad (3.7)$$

□

The lemma makes sense in the language of defaults. Recall that the event  $D_{j,\infty}$  occurs if a default does not occur during our observations. Hence the statement that the events are disjoint is equivalent to: a default cannot occur more than once, and if a default did not occur during our observation time period, we cannot state that a default occurred during the  $k$ th interval. Also, the statement that the union of the events equals the sample space is equivalent to saying that an asset must either default or not default. Furthermore, we observe that the lemma provides, for a fixed name  $j$ , that exactly one of the listed events must occur, i.e.,

$$\mathbb{1}_{D_{j,\infty}} + \sum_{k=1}^n \mathbb{1}_{D_{j,k}} = 1. \quad (3.8)$$

However, for fixed  $k$ , the events

$$D_{1,k}, D_{2,k}, \dots, D_{m,k}$$

are neither disjoint nor independent. In default language, two names can default during the same interval. Also, one asset's default is not assumed to be independent of another asset's default.

The variables  $\tilde{\nu}_k$  and  $\nu_k$  represent the number of assets whose defaults are detected at time  $t_k$  and by time  $t_k$ , respectively. No defaults have occurred at time  $t_0$  implying  $\nu_0 = 0$ . Note that, for  $k \neq 0$ ,

$$\nu_k = \sum_{j=1}^m \mathbb{1}_{[\tau_j \leq t_k]} = \sum_{j=1}^m \mathbb{1}_{[\cup_{i=1}^k D_{j,i}]} = \sum_{j=1}^m \sum_{i=1}^k \mathbb{1}_{[D_{j,i}]} = \sum_{i=1}^k \sum_{j=1}^m \mathbb{1}_{[D_{j,i}]} = \sum_{i=1}^k \tilde{\nu}_i. \quad (3.9)$$

Hence, if we know  $\tilde{\nu}_i$ , for all  $i \leq k$ , we can calculate  $\nu_k$ . Likewise, we can recover  $\tilde{\nu}_k$ , given  $\nu_{k-1}$  and  $\nu_k$ , via

$$\tilde{\nu}_k = \nu_k - \nu_{k-1}. \quad (3.10)$$

Also, observe that the total number of defaults,  $\nu_n$ , that occur by time  $t_n$  can be no greater than the number of assets,  $m$ .

In this chapter, we have yet to specify a model for the random variables,  $\tau_j$ . For the rest of this paper, we will assume they follow a one-factor Gaussian copula model; i.e., we assume the following definition.

**Definition 3.3.** *Let  $\tau = (\tau_1, \dots, \tau_m)$  be as above, and let*

$$X_j = \Phi^{-1} \circ F_{\tau_j}(\tau_j), \quad (3.11)$$

*for all  $j \in \{1, \dots, m\}$ . We assume that the distribution of  $\tau$  is such that the random variables  $X_j$  are jointly Gaussian. Specifically, we assume that there exist independent standard Gaussian random variables*

$$Z, \epsilon_1, \dots, \epsilon_m \quad (3.12)$$

*and constants*

$$\rho_1, \dots, \rho_m \in (0, 1) \quad (3.13)$$

*such that*

$$X_j = \sqrt{\rho_j} Z + \sqrt{1 - \rho_j} \epsilon_j \quad (3.14)$$

*for all  $j \in \{1, \dots, m\}$ .*

Since  $\rho_j \in (0, 1)$ ,  $X_j$  depends on both a global factor  $Z$  and an independent factor  $\epsilon_j$ . This model is referred to as a one-factor model due to there being only a single global factor. The usefulness of this model comes from the fact that we can model a default probability as the probability that the  $X_j$  random variable falls below a certain threshold; i.e.,

$$\mathbb{P}[\tau_j \leq t] = \mathbb{P}[X_j \leq c_j(t)], \quad (3.15)$$

where the function

$$c_j = \Phi^{-1} \circ F_{\tau_j} \quad (3.16)$$

is found using Equation 3.11. We shall write the threshold that is calculated using the  $j$ th distribution and  $k$ th observation as

$$c_{j,k} = \Phi^{-1} \circ F_{\tau_j}(t_k). \quad (3.17)$$

The default probabilities  $F_{\tau_j}(t_k)$  can be obtained using market data. Hence, the actual distribution of  $\tau_j$  is not required to provide the above thresholds.

At this point, we wish to stress that the definition of all events and random variables in Definition 3.1 are dependent upon the  $\tau$  random variable. In other words, the number of defaults detected is determined by how many of the  $\tau_j$  fall in a certain interval. Thus, the model that is provided for the  $\tau_j$  variables completely determines the distribution for  $\nu_k$ . We chose to model  $\tau$  using the one-factor Gaussian copula framework. Hence, this framework also provides the model for the random variables,

$$\nu = (\nu_1, \dots, \nu_n) \quad (3.18)$$

and

$$\tilde{\nu} = (\tilde{\nu}_1, \dots, \tilde{\nu}_n). \quad (3.19)$$

Given our assumptions about the relationship between  $\tau$  and  $X$ , we can make the following useful observation, which transforms Definition 3.1 into the language of our specified model.

**Corollary 3.4.**

$$\begin{aligned}
D_{j,k} &= [c_{j,k-1} < X_j \leq c_{j,k}] \\
D_{j,\infty} &= [X_j > c_{j,n}] \\
\nu_k &= \sum_{j=1}^m \mathbb{1}_{[X_j \leq c_{j,k}]} \\
\tilde{\nu}_k &= \sum_{j=1}^m \mathbb{1}_{D_{j,k}}
\end{aligned} \tag{3.20}$$

*Proof.* These equalities follow from Equations 3.15 and 3.17. Furthermore, Equation 3.20 is acquired since the following events are equivalent:

$$[\tau_j = t_\infty] = [\tau_j > t_n] = [X_j > c_{j,n}]. \tag{3.21}$$

□

### 3.2 Poisson Model

Other researchers have used assumptions similar to what we have provided in the previous section, e.g. [35]. A model would be chosen for  $\tau$ , and then one could study how these models impacted which names defaulted and/or how many names defaulted. Models that specified whether an asset would default during a time period could be used to study the number of assets that defaulted during the time interval. A different method, if the actual names that defaulted were not of interest, would be for one to simply focus on how many names defaulted. This section will provide such a model.

As in the previous section, we assume that a portfolio holds  $m$  assets. The non-negative, integer-valued, increasing stochastic process  $N(t, \omega)$  shall be used to

represent the number of assets which have defaulted by time  $t$ . As in the previous section, the portfolio will be observed at the following times,

$$0 = t_0 < t_1 < t_2 < \dots < t_n = T. \quad (3.22)$$

We will use the notation

$$N_k = N(t_k), \quad (3.23)$$

for  $k \in \{0, 1, \dots, n\}$ , and define

$$\tilde{N}_k = N_k - N_{k-1}, \quad (3.24)$$

for  $k \neq 0$ , which represents the the number of names that defaulted in the interval  $(t_{k-1}, t_k]$ . Observe that  $N_k$  and  $\tilde{N}_k$  are defined analogously to  $\nu_k$  and  $\tilde{\nu}_k$ , respectively, in financial terminology. However, we currently make no assumption about a relationship between their distributions. One should view these variables as based upon completely separate models.

**Definition 3.5.** *Let*

$$\lambda(t, \omega) : [0, T] \times \Omega \rightarrow \mathbb{R} : (t, \omega) \mapsto \lambda(t, \omega)$$

*be a non-negative stochastic process that, for all  $0 \leq s < t$ , satisfies*

$$0 < \int_s^t \lambda(u) \, du < \infty \quad (3.25)$$

*almost surely. Define the stochastic process*

$$\Lambda(t, \omega) : [0, T] \times \Omega \rightarrow \mathbb{R} : (t, \omega) \mapsto \Lambda(t, \omega) \quad (3.26)$$

*by*

$$\Lambda(t, \omega) = \left( \int_0^t \lambda(u) \, du \right) (\omega). \quad (3.27)$$

The aggregate process  $\Lambda(t)$  is almost surely positive, for  $t > 0$ , and finite. We denote

$$\Lambda_{s,t} = \Lambda(t) - \Lambda(s) = \int_s^t \lambda(u) \, du. \quad (3.28)$$

Suppose further that we are modelling the interest rate using the short rate process  $t \mapsto r(t)$ . We shall assume that the process  $\lambda(t)$  is *not* independent of the sigma-algebra,

$$\mathcal{F}_t^r = \sigma(\{r(u) \mid 0 \leq u \leq t\}), \quad (3.29)$$

and that  $\lambda(t)$  is deterministic given  $r(t)$ .

We now explicitly define  $N(t)$  as a Cox process that is dependent on the intensity process  $\Lambda(t)$  and, hence, the short rate process  $r(t)$  as well.

**Definition 3.6.** *Define*

$$N(t, \omega) : [0, T] \times \Omega \rightarrow \mathbb{Z}_{\geq 0}$$

*as a stochastic process with intensity process  $\Lambda(t, \omega)$  such that:*

1.  $\mathbb{P}[\{\omega \mid N(0, \omega) = 0\}] = 1$
2.  $\mathbb{P}[\{\omega \mid N(\cdot, \omega) \text{ is right continuous with left-hand limits}\}] = 1$
3. *For any  $0 \leq s_1 < s_2 < \dots < s_k$ , the random variables*

$$N(s_1), N(s_2) - N(s_1), \dots, N(s_k) - N(s_{k-1}) \quad (3.30)$$

*are independent, conditional on  $\mathcal{F}_{s_k}^r$*

4. *For any  $0 \leq s < t$ , given  $r(u)$  for  $0 \leq u \leq t$ ,  $N(t) - N(s)$  is a Poisson random variable with parameter  $\Lambda_{s,t}$ , i.e.,*

$$\mathbb{P}[N(t) - N(s) = i \mid \mathcal{F}_t^r] = e^{-\Lambda_{s,t}} \frac{(\Lambda_{s,t})^i}{i!} \quad (3.31)$$

*for  $i = 0, 1, 2, \dots$*

For notational convenience we define  $\Lambda_k$  as

$$\Lambda_k = \Lambda_{t_{k-1}, t_k} = \Lambda(t_k) - \Lambda(t_{k-1}) = \int_{t_{k-1}}^{t_k} \lambda(u) du. \quad (3.32)$$

Equation 3.31 implies that

$$\mathbb{P}[\tilde{N}_k = i \mid \Lambda_k] = e^{-\Lambda_k} \frac{\Lambda_k^i}{i!}, \quad (3.33)$$

for all  $k \in \{1, 2, \dots, n\}$  and  $i = 0, 1, 2, \dots$ . Furthermore, Equation 3.30 implies the following random variables are independent, conditional on  $\mathcal{F}_T^r$ :

$$\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_n. \quad (3.34)$$

We assert that a doubly stochastic Poisson framework is a logical model for how many defaults occur during a time interval. Using a Poisson framework is already common in financial mathematics. However, allowing the intensity function to be random rather than deterministic seems to be more realistic as the rate of default is an estimation rather than a known quantity. We define the intensity process to be dependent upon the interest rate for two reasons. There are many available stochastic models of the short rate  $r(t)$ , and defaults are almost always positively correlated with higher interest rates. Thus, incorporating interest rate dependency into a default model certainly appears to be a logical choice.

### 3.3 A New Probability Measure $\mathbb{P}^\bullet$

In this chapter thus far, we have modeled the random variable  $\tau$  using a one-factor Gaussian copula framework. We also defined events,  $D_{j,k}$ , and random variables,  $\nu_k$  and  $\tilde{\nu}_k$ , in Definition 3.1 which were translated from  $\tau_j$  to  $X_j$  language in Corollary 3.4. Furthermore,  $N(t)$  was defined as a Cox Process with intensity function  $\Lambda(t)$  which was dependent upon the stochastic process  $r(t)$ . We now state an assumption which shall prove vital for the rest of the paper.

**Definition 3.7.** Let the Gaussian copula model  $X$  and the Cox process  $N(t)$  dependent on the short rate process  $r(t)$  be as defined previously in this chapter. Define the following sigma-algebras:

$$\begin{aligned}\sigma_X &= \sigma(X_1, X_2, \dots, X_m) \\ \sigma_N &= \sigma(\{N(u) \mid 0 \leq u \leq T\}) \\ \mathcal{G} &= \sigma(\{r(u) \mid 0 \leq u \leq T\}).\end{aligned}\tag{3.35}$$

We now add the further constraint that the Gaussian model is independent of the Poisson model under the probability measure  $\mathbb{P}$ . That is, we assume that  $\sigma_X$  is independent of the sigma-algebra generated by  $\sigma_N \cup \mathcal{G}$ .

Two events  $A$  and  $B$  are independent conditional on the sigma-algebra  $\mathcal{G}$  if

$$\mathbb{P}[A \cap B \mid \mathcal{G}] = \mathbb{P}[A \mid \mathcal{G}] \mathbb{P}[B \mid \mathcal{G}].\tag{3.36}$$

Given our model assumption, it makes sense that if  $A \in \sigma_X$  and  $B \in \sigma_N$ , then  $A$  and  $B$  should be independent conditional on  $\mathcal{G}$ . We prove this corollary.

**Corollary 3.8.** Suppose  $A \in \sigma_X$  and  $B \in \sigma_N$ . Then  $A$  and  $B$  are independent conditional on  $\mathcal{G}$  with

$$\mathbb{P}[A \cap B \mid \mathcal{G}] = \mathbb{P}[A] \mathbb{P}[B \mid \mathcal{G}].\tag{3.37}$$

*Proof.* The events  $A$  and  $B$  are independent if

$$\mathbb{P}[(A \cap B) \cap G] = \int_G \mathbb{P}[A \mid \mathcal{G}] \mathbb{P}[B \mid \mathcal{G}] d\mathbb{P}.\tag{3.38}$$

for all  $\mathcal{G}$ -measurable sets. Let  $G \in \mathcal{G}$ . Definition 3.7 provides the equalities

$$\begin{aligned}\mathbb{P}[(A \cap B) \cap G] &= \mathbb{P}[A] \mathbb{P}[B \cap G] \\ &= \mathbb{P}[A] \int_G \mathbb{P}[B \mid \mathcal{G}] d\mathbb{P} \\ &= \int_G \mathbb{P}[A] \mathbb{P}[B \mid \mathcal{G}] d\mathbb{P} \\ &= \int_G \mathbb{P}[A \mid \mathcal{G}] \mathbb{P}[B \mid \mathcal{G}] d\mathbb{P},\end{aligned}\tag{3.39}$$



where the first and last equalities follow from the definition of independent sigma-algebras. The second and third equalities follow from the definition of conditional probability and linearity of integration, respectively.  $\square$

Recall that all random variables thus far have been defined on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . We wish to construct a new probability measure  $\mathbb{P}^\bullet$  on  $(\Omega, \mathcal{F})$ , and we shall denote its associated expectation as  $\mathbb{E}^\bullet$ . We begin by making a few observations. We will use the following notation henceforth,

$$[\nu = N] = [\nu_1 = N_1, \nu_2 = N_2, \dots, \nu_n = N_n] \text{ and} \quad (3.40)$$

$$[\tilde{\nu} = \tilde{N}] = [\tilde{\nu}_1 = \tilde{N}_1, \tilde{\nu}_2 = \tilde{N}_2, \dots, \tilde{\nu}_n = \tilde{N}_n]. \quad (3.41)$$

**Lemma 3.9.** *We have the following equivalency of events,*

$$[\nu = N] = [\tilde{\nu} = \tilde{N}] \quad (3.42)$$

*Proof.* ( $\subseteq$ ) Assume that the event  $[\nu = N]$  occurs and let  $k \in \{1, 2, \dots, n\}$ . We also know  $\nu_0 = N_0$  since both are equal to zero by definition. Thus, we have

$$\tilde{\nu}_k = \nu_k - \nu_{k-1} = N_k - N_{k-1} = \tilde{N}_k, \quad (3.43)$$

using Equations 3.10 and 3.24. Hence, the event  $[\tilde{\nu} = \tilde{N}]$  occurs as well.

( $\supseteq$ ) Let  $k \in \{1, 2, \dots, n\}$  and assume the occurrence of the event  $[\tilde{\nu} = \tilde{N}]$ . The assumption provides the equality,

$$\sum_{i=1}^k \tilde{\nu}_i = \sum_{i=1}^k \tilde{N}_i. \quad (3.44)$$

Using Equation 3.9 with the observation,

$$\begin{aligned} N_k &= N_k - N_0 \\ &= N_k - (N_{k-1} - N_{k-1}) - \dots - (N_1 - N_1) - N_0 \\ &= (N_k - N_{k-1}) + \dots + (N_1 - N_0) \\ &= \sum_{i=1}^k \tilde{N}_i, \end{aligned} \quad (3.45)$$

we can conclude that  $\nu_k = N_k$ . Therefore, the event  $[\nu = N]$  occurs.  $\square$

Define the set

$$\mathcal{I} = \left\{ (i_1, i_2, \dots, i_n) \in \{0, 1, \dots, m\}^n \left| \sum_{k=1}^n i_k \leq m \right. \right\}, \quad (3.46)$$

where we denote an element of this set as  $i \in \mathcal{I}$ . The event  $[\tilde{\nu} = i]$  is to be understood as implying  $\tilde{\nu}_k = i_k$  for all  $k$ . Observe that the event  $[\tilde{\nu} = \tilde{N}]$  can be written as the disjoint union over  $I$  of events  $E_i$ , where

$$E_i = [\tilde{\nu} = i] \cap [\tilde{N} = i]. \quad (3.47)$$

Recall we assumed, in Definition 3.7, that the Gaussian and Poisson models were independent; hence, the two events in Equation 3.47 are independent implying that

$$\mathbb{P}[E_i] = \mathbb{P}[\tilde{\nu} = i] \mathbb{P}[\tilde{N} = i]. \quad (3.48)$$

With the above information we can prove the following lemma.

**Lemma 3.10.** *The probability that the random variables  $\nu_k$  and  $N_k$  agree for each  $k \in \{1, \dots, n\}$  is positive and can be calculated as*

$$\mathbb{P}[\nu = N] = \sum_{i \in \mathcal{I}} \mathbb{P}[\tilde{\nu} = i] \mathbb{E} \left[ \prod_{k=1}^n e^{-\Lambda_k} \frac{\Lambda_k^{i_k}}{i_k!} \right], \quad (3.49)$$

*Proof.* Using Lemma 3.9 and Equation 3.47, we see that

$$[\nu = N] = \bigcup_{i \in \mathcal{I}} E_i. \quad (3.50)$$

Since the events  $E_i$  are disjoint, we have

$$\begin{aligned} \mathbb{P}[\nu = N] &= \sum_{i \in \mathcal{I}} \mathbb{P}[E_i] \\ &= \sum_{i \in \mathcal{I}} \mathbb{P}[\tilde{\nu} = i] \mathbb{P}[\tilde{N} = i]. \end{aligned} \quad (3.51)$$

by applying Equation 3.48. Both probabilities are non-zero implying each term and hence the sum itself is positive. The Cox process  $N(t)$  conditioned to the sigma-algebra  $\mathcal{G}$  has independent time increments through time  $t_n = T$ . Thus,

$$\begin{aligned}\mathbb{P}[\tilde{N} = i] &= \mathbb{E}\left[\mathbb{P}[\tilde{N} = i \mid \mathcal{G}]\right] \\ &= \mathbb{E}\left[\prod_{k=1}^n \mathbb{P}[\tilde{N}_k = i_k \mid \mathcal{G}]\right] \\ &= \mathbb{E}\left[\prod_{k=1}^n e^{-\Lambda_k} \frac{\Lambda_k^{i_k}}{i_k!}\right],\end{aligned}\tag{3.52}$$

which combines with Equation 3.51 to complete the proof.  $\square$

We can now define the new probability measure  $\mathbb{P}^\bullet$  as a conditional probability in the following way.

**Definition 3.11.** *Given an event  $A \in \mathcal{F}$ , we define the probability that  $A$  occurs according to the measure  $\mathbb{P}^\bullet$  as*

$$\mathbb{P}^\bullet[A] = \mathbb{P}[A \mid \nu = N]\tag{3.53}$$

We shall refer to the probability that an event occurs according to this new measure as the “ $\bullet$ -probability.” The expectation  $\mathbb{E}^\bullet$  with respect to  $\mathbb{P}^\bullet$  shall be referred to as the “ $\bullet$ -expectation.” The measure  $\mathbb{P}^\bullet$  is truly a probability measure since we proved in Lemma 3.10 that

$$\mathbb{P}[\nu = N] > 0.\tag{3.54}$$

### 3.4 A New Random Variable $Y$

In this section, we wish to define an  $\mathcal{S}$ -valued random variable

$$Y = (Y_1, Y_2, \dots, Y_m)$$

on the measurable space  $(\Omega, \mathcal{F})$ , where

$$\mathcal{S} = \{1, 2, \dots, n, \infty\}^m.\tag{3.55}$$

We write a point  $y \in \mathcal{S}$  as

$$y = (y_1, y_2, \dots, y_m).$$

Recall that Lemma 3.2 says that exactly one of the events,

$$D_{j,1}, \dots, D_{j,n}, D_{j,\infty}, \tag{3.56}$$

occurs.

**Definition 3.12.** *Let  $\omega \in \Omega$ . For each  $j \in \{1, \dots, m\}$ , set  $Y_j(\omega)$  equal to the unique  $k \in \{1, \dots, n, \infty\}$  such that*

$$\omega \in D_{j,k}.$$

*We define the random variable  $Y : \Omega \rightarrow \mathcal{S}$  as*

$$Y(\omega) = (Y_1(\omega), \dots, Y_m(\omega)). \tag{3.57}$$

We are also interested in counting how many times the value  $k$  appears in  $y$ , and we define a function  $\tilde{\nu}^*$  to serve this purpose.

**Definition 3.13.** *Let  $y \in \mathcal{S}$ . Define the function*

$$\tilde{\nu}^* : \mathcal{S} \rightarrow \mathcal{I} \tag{3.58}$$

*as*

$$y \mapsto (\tilde{\nu}_1^*(y), \dots, \tilde{\nu}_n^*(y)) \tag{3.59}$$

*where*

$$\tilde{\nu}_k^*(y) = |\{j \mid y_j = k\}|. \tag{3.60}$$

Finally, we make several useful observations in the following lemma including a proof that  $\tilde{\nu}^*$  does, in fact, map into  $\mathcal{I}$ .

**Lemma 3.14.** *Using Definitions 3.12 and 3.13, we have:*

1. The equivalency of events,

$$[Y = y] = \bigcap_{j=1}^m D_{j,y_j} \quad (3.61)$$

2. The function  $\tilde{\nu}^* : \mathcal{S} \rightarrow \mathcal{I}$  is surjective

3. The pointwise equality of random variables,

$$\tilde{\nu} = \tilde{\nu}^*(Y). \quad (3.62)$$

*Proof.* Let  $\omega \in \Omega$ .

1. Observe that

$$Y_j(\omega) = y_j \iff \omega \in D_{j,y_j} \quad (3.63)$$

implying that

$$[Y_j = y_j] = D_{j,y_j}. \quad (3.64)$$

Since the event  $[Y = y]$  is the intersection of the events  $[Y_j = y_j]$  for all  $j \in \{1, \dots, m\}$ , we obtain the desired result.

2. We wish to prove that  $\tilde{\nu}^*(\mathcal{S}) = \mathcal{I}$ . Let  $y \in \mathcal{S}$ . Notice that the inequalities

$$0 \leq \tilde{\nu}_k^*(y) \leq m \text{ and } \sum_{k=1}^n \tilde{\nu}_k^*(y) \leq m \quad (3.65)$$

follow from the definition of  $\tilde{\nu}_k^*(y)$ . Thus,  $\tilde{\nu}^*(y) \in \mathcal{I}$ . Now, let  $i \in \mathcal{I}$ . We will construct a  $\zeta \in Y$  such that  $\zeta \mapsto i$ . If  $i_1$  equals zero, then do nothing, but if  $i_1 > 0$ , then set

$$\zeta_1, \dots, \zeta_{i_1} = 1.$$

If  $i_2 = 0$ , then do nothing. If  $i_1 = 0$  and  $i_2 > 0$ , then set

$$\zeta_1, \dots, \zeta_{i_1} = 2,$$

but if  $i_1, i_2 > 0$ , then set

$$\zeta_{i_1+1}, \dots, \zeta_{i_1+i_2} = 2.$$

Proceed in this manner through  $i_n$ . Set the remaining  $\zeta_j = \infty$  through  $j = m$ .

Thus,  $\zeta \in \mathcal{S}$  and  $\tilde{\nu}^*(\zeta) = i$ . Therefore,  $\tilde{\nu}^*$  maps  $Y$  onto  $\mathcal{I}$ .

3. Suppose  $Y(\omega) = y$ , then we observe that

$$\begin{aligned} \tilde{\nu}_k^*(y) &= |\{j \mid y_j = k\}| \\ &= \sum_{j=1}^m \mathbb{1}_{D_{j,k}}(\omega) \\ &= \tilde{\nu}_k, \end{aligned} \tag{3.66}$$

which provides the desired result.

□

### 3.5 The Distribution of $Y$ Under $\mathbb{P}^\bullet$

We begin this section by providing a few notations.

**Definition 3.15.** *We shall use the following notations for the rest of the paper*

$$\begin{aligned} \phi(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \\ \Phi(x) &= \int_{-\infty}^x \phi(s) \, ds \\ P_{j,k}(x) &= \mathbb{P}[D_{j,k} \mid Z = x] \end{aligned} \tag{3.67}$$

where  $Z$  is the global factor in the Gaussian copula model.

We note that  $Y$  is a discrete random variable that has  $(n+1)m$  possible values. We wish to calculate the distribution of the random variable  $Y$  under the probability measure  $\mathbb{P}^\bullet$ . However, it shall first be necessary to find the probability mass function  $\mathbb{P}[Y = y]$ .

**Lemma 3.16.** *The distribution of the random variable  $Y$  under the probability measure  $\mathbb{P}$  is*

$$\mathbb{P}[Y = y] = \int_{\mathbb{R}} \left( \prod_{j=1}^m P_{j,y_j}(x) \right) \phi(x) dx, \quad (3.68)$$

where

$$P_{j,k}(x) = \Phi \left( \frac{c_{j,k} - \sqrt{\rho_j}x}{\sqrt{1 - \rho_j}} \right) - \Phi \left( \frac{c_{j,k-1} - \sqrt{\rho_j}x}{\sqrt{1 - \rho_j}} \right), \quad (3.69)$$

for  $k \in \{1, \dots, n\}$ , and

$$P_{j,\infty}(x) = 1 - \Phi \left( \frac{c_{j,n} - \sqrt{\rho_j}x}{\sqrt{1 - \rho_j}} \right) \quad (3.70)$$

*Proof.* Using Lemma 3.14, we can calculate the probability mass function as

$$\begin{aligned} \mathbb{P}[Y = y] &= \mathbb{P} \left[ \bigcap_{j=1}^m D_{j,y_j} \right] \\ &= \mathbb{E} \left[ \mathbb{P} \left[ \bigcap_{j=1}^m D_{j,y_j} \mid Z \right] \right] \\ &= \mathbb{E} \left[ \prod_{j=1}^m \mathbb{P}[D_{j,y_j} \mid Z] \right] \\ &= \int_{\mathbb{R}} \left( \prod_{j=1}^m P_{j,y_j}(x) \right) \phi(x) dx, \end{aligned} \quad (3.71)$$

where we used the fact that, given  $Z$ , events  $D_{j,k}$  and  $D_{j,k'}$  are independent under the probability measure  $\mathbb{P}$ , provided that  $k \neq k'$ . Using Corollary 3.4, we have  $P_{j,k}(x)$  can be written as

$$\mathbb{P}[D_{j,k} \mid Z = x] = \mathbb{P}[X_j \leq c_{j,k} \mid Z = x] - \mathbb{P}[X_j \leq c_{j,k-1} \mid Z = x]. \quad (3.72)$$

Furthermore,

$$\begin{aligned} [X_j \leq c_{j,k}] &= \left[ \sqrt{\rho}Z + \sqrt{1 - \rho}\epsilon_j \leq c_{j,k} \right] \\ &= \left[ \epsilon_j \leq \frac{c_{j,k} - \sqrt{\rho_j}Z}{\sqrt{1 - \rho_j}} \right], \end{aligned} \quad (3.73)$$

implying that

$$\begin{aligned}\mathbb{P}[X_j \leq c_{j,k} \mid Z = x] &= \mathbb{P}\left[\epsilon_j \leq \frac{c_{j,k} - \sqrt{\rho_j}Z}{\sqrt{1 - \rho_j}} \mid Z = x\right] \\ &= \Phi\left(\frac{c_{j,k} - \sqrt{\rho_j}x}{\sqrt{1 - \rho_j}}\right).\end{aligned}\tag{3.74}$$

Therefore, we have proven Equation 3.69, and Equation 3.70 follows by observing that

$$\mathbb{P}[D_{j,\infty}] = 1 - \mathbb{P}[X_j \leq c_{j,n}].\tag{3.75}$$

□

We now proceed to provide the  $\bullet$ -probability of the event  $[Y = y]$ . We will find the following notation useful:

$$\Delta_i = \{\zeta \in \mathcal{S} \mid \tilde{\nu}^*(\zeta) = i\}.\tag{3.76}$$

**Theorem 3.17.** *The distribution of the random variable  $Y$  under the probability measure  $\mathbb{P}^\bullet$  is*

$$\mathbb{P}^\bullet[Y = y] = \frac{\mathbb{P}[Y = y] \mathbb{E}\left[\prod_{k=1}^n e^{-\Lambda_k} \frac{\Lambda_k^{\tilde{\nu}_k^*(y)}}{\tilde{\nu}_k^*(y)!}\right]}{\sum_{\zeta \in \mathcal{S}} \mathbb{P}[Y = \zeta] \mathbb{E}\left[\prod_{k=1}^n e^{-\Lambda_k} \frac{\Lambda_k^{\tilde{\nu}_k^*(\zeta)}}{\tilde{\nu}_k^*(\zeta)!}\right]},\tag{3.77}$$

for  $y \in \mathcal{S}$ .

*Proof.* By definition,

$$\begin{aligned}\mathbb{P}^\bullet[Y = y] &= \mathbb{P}[Y = y \mid \nu = N] \\ &= \frac{\mathbb{P}([Y = y] \cap [\nu = N])}{\mathbb{P}[\nu = N]}.\end{aligned}\tag{3.78}$$

We begin by examining the event in the numerator. Using Lemmas 3.9 and 3.14, we have

$$[\nu = N] = [\tilde{\nu} = \tilde{N}] = [\tilde{\nu}^*(Y) = \tilde{N}].\tag{3.79}$$



Hence,

$$\begin{aligned} [Y = y] \cap [\nu = N] &= [Y = y] \cap [\tilde{\nu}^*(Y) = \tilde{\nu}^*(y)] \cap [\tilde{N} = \tilde{\nu}^*(y)] \\ &= [Y = y] \cap [\tilde{N} = \tilde{\nu}^*(y)] \end{aligned} \quad (3.80)$$

since

$$[Y = y] \subseteq [\tilde{\nu}^*(Y) = \tilde{\nu}^*(y)]. \quad (3.81)$$

Recall that we assumed the Gaussian and Poisson models were independent which implies the independence of the events  $D_{j,k}$  and  $[\tilde{N} = \tilde{\nu}^*(y)]$ . Since  $[Y = y]$  is the intersection of events of the form  $D_{j,k}$ , we see that

$$\mathbb{P}([Y = y] \cap [\tilde{N} = \tilde{\nu}^*(y)]) = \mathbb{P}[Y = y] \mathbb{P}[\tilde{N} = \tilde{\nu}^*(y)] \quad (3.82)$$

Thus, upon applying Equation 3.52, we obtain

$$\mathbb{P}([Y = y] \cap [\nu = N]) = \mathbb{P}[Y = y] \mathbb{E} \left[ \prod_{k=1}^n e^{-\Lambda_k} \frac{\Lambda_k^{\tilde{\nu}_k^*(y)}}{\tilde{\nu}_k^*(y)!} \right] \quad (3.83)$$

Next, consider the denominator. We have previously shown in Lemma 3.10 that

$$\mathbb{P}[\nu = N] = \sum_{i \in \mathcal{I}} \mathbb{P}[\tilde{\nu} = i] \mathbb{E} \left[ \prod_{k=1}^n e^{-\Lambda_k} \frac{\Lambda_k^{i_k}}{i_k!} \right] \quad (3.84)$$

Using the surjectivity of  $\tilde{\nu}^*$ , we obtain, for each  $i \in \mathcal{I}$ , an element  $y^i \in \mathcal{S}$  such that

$$\tilde{\nu}^*(y^i) = i. \quad (3.85)$$

By substituting, we have

$$\mathbb{P}[\nu = N] = \sum_{i \in \mathcal{I}} \mathbb{P}[\tilde{\nu}^*(Y) = \tilde{\nu}^*(y^i)] \mathbb{E} \left[ \prod_{k=1}^n e^{-\Lambda_k} \frac{\Lambda_k^{i_k}}{i_k!} \right]. \quad (3.86)$$

Also, notice that

$$[\tilde{\nu}^*(Y) = \tilde{\nu}^*(y^i)] = \bigcup_{\zeta \in \Delta_i} [Y = \zeta], \quad (3.87)$$

where the union is clearly disjoint. Since we also have the disjoint union

$$\mathcal{S} = \bigcup_{i \in \mathcal{I}} \Delta_i, \quad (3.88)$$

we can write the below double sum as a single sum which provides

$$\begin{aligned}
\mathbb{P}[\nu = N] &= \sum_{i \in \mathcal{I}} \left( \left( \sum_{\zeta \in \Delta_i} \mathbb{P}[Y = \zeta] \right) \mathbb{E} \left[ \prod_{k=1}^n e^{-\Lambda_k} \frac{\Lambda_k^{i_k}}{i_k!} \right] \right) \\
&= \sum_{i \in \mathcal{I}} \sum_{\zeta \in \Delta_i} \mathbb{P}[Y = \zeta] \mathbb{E} \left[ \prod_{k=1}^n e^{-\Lambda_k} \frac{\Lambda_k^{\tilde{\nu}_k^*(\zeta)}}{\tilde{\nu}_k^*(\zeta)!} \right] \\
&= \sum_{\zeta \in \mathcal{S}} \mathbb{P}[Y = \zeta] \mathbb{E} \left[ \prod_{k=1}^n e^{-\Lambda_k} \frac{\Lambda_k^{\tilde{\nu}_k^*(\zeta)}}{\tilde{\nu}_k^*(\zeta)!} \right].
\end{aligned} \tag{3.89}$$

Therefore, we have provided both the numerator and the denominator of  $\mathbb{P}^\bullet[Y = y]$ , concluding the proof.  $\square$

It will also be important for us to find the  $\bullet$ -probability that  $Y = y$  conditional on the sigma-algebra  $\mathcal{G}$ , which was defined in Equation 3.35. For notational convenience, define the random variable

$$\Psi_\zeta = \mathbb{P}[Y = \zeta] \prod_{k=1}^n e^{-\Lambda_k} \frac{\Lambda_k^{\tilde{\nu}_k^*(\zeta)}}{\tilde{\nu}_k^*(\zeta)!}, \tag{3.90}$$

for each  $\zeta \in \mathcal{S}$ , where the variable is non-deterministic since  $\Lambda_k$  is random.

**Theorem 3.18.** *We have the following conditional probability,*

$$\mathbb{P}^\bullet[Y = y \mid \mathcal{G}] = \frac{\mathbb{P}[Y = y] \prod_{k=1}^n \frac{\Lambda_k^{\tilde{\nu}_k^*(y)}}{\tilde{\nu}_k^*(y)!}}{\sum_{\zeta \in \mathcal{S}} \mathbb{P}[Y = \zeta] \prod_{k=1}^n \frac{\Lambda_k^{\tilde{\nu}_k^*(\zeta)}}{\tilde{\nu}_k^*(\zeta)!}}, \tag{3.91}$$

for  $y \in \mathcal{S}$ .

*Proof.* The conditional probability  $\mathbb{P}^\bullet[Y = y \mid \mathcal{G}]$  is the  $\mathcal{G}$ -measurable random variable such that

$$\mathbb{P}^\bullet[[Y = y] \cap G] = \int_G \mathbb{P}^\bullet[Y = y \mid \mathcal{G}] d\mathbb{P}^\bullet \tag{3.92}$$

for each  $G \in \mathcal{G}$ . Let us now fix  $G \in \mathcal{G}$ . Using  $\mathbb{P}^\bullet[\nu = N]$  as calculated in the proof of Theorem 3.17, we have

$$\begin{aligned} \mathbb{P}^\bullet[[Y = y] \cap G] &= \mathbb{P}([Y = y] \cap G \mid \nu = N) \\ &= \frac{\mathbb{P}([Y = y] \cap G \cap [\nu = N])}{\sum_{\zeta \in \mathcal{S}} \mathbb{P}[Y = \zeta] \mathbb{E} \left[ \prod_{k=1}^n e^{-\Lambda_k} \frac{\Lambda_k^{\tilde{\nu}_k^*(\zeta)}}{\tilde{\nu}_k^*(\zeta)!} \right]}, \end{aligned} \quad (3.93)$$

where the denominator is equivalent to

$$\mathbb{E} \left[ \sum_{\zeta \in \mathcal{S}} \Psi_\zeta \right]. \quad (3.94)$$

Since  $[Y = y] \in \sigma_X$  and  $[\tilde{N} = \tilde{\nu}^*(y)] \cap G \in \sigma(\sigma_N \cup G)$ , we can write the numerator as

$$\mathbb{P}[Y = y] \mathbb{P}([\tilde{N} = \tilde{\nu}^*(y)] \cap G) \quad (3.95)$$

using Definition 3.7. Furthermore,

$$\begin{aligned} \mathbb{P}([\tilde{N} = \tilde{\nu}^*(y)] \cap G) &= \mathbb{E} \left[ \mathbb{1}_G \mathbb{P}[\tilde{N} = \tilde{\nu}^*(y) \mid \mathcal{G}] \right] \\ &= \mathbb{E} \left[ \mathbb{1}_G \prod_{k=1}^n \mathbb{P}[\tilde{N}_k = \tilde{\nu}_k^*(y) \mid \mathcal{G}] \right] \\ &= \mathbb{E} \left[ \mathbb{1}_G \prod_{k=1}^n e^{-\Lambda_k} \frac{\Lambda_k^{\tilde{\nu}_k^*(y)}}{\tilde{\nu}_k^*(y)!} \right]. \end{aligned} \quad (3.96)$$

Thus, we have

$$\begin{aligned} \mathbb{P}^\bullet[[Y = y] \cap G] &= \frac{\mathbb{P}[Y = y] \mathbb{E} \left[ \mathbb{1}_G \prod_{k=1}^n e^{-\Lambda_k} \frac{\Lambda_k^{\tilde{\nu}_k^*(y)}}{\tilde{\nu}_k^*(y)!} \right]}{\mathbb{E} \left[ \sum_{\zeta \in \mathcal{S}} \Psi_\zeta \right]} \\ &= \mathbb{E} \left[ \frac{\Psi_y}{\left[ \sum_{\zeta \in \mathcal{S}} \Psi_\zeta \right]} \mathbb{1}_G \right]. \end{aligned} \quad (3.97)$$

We claim that

$$\frac{\Psi_y}{\sum_{\zeta \in \mathcal{S}} \Psi_\zeta} \quad (3.98)$$

is  $\mathcal{G}$ -measurable and

$$\mathbb{E} \left[ \frac{\Psi_y}{\mathbb{E} \left[ \sum_{\zeta \in \mathcal{S}} \Psi_\zeta \right]} \mathbb{1}_G \right] = \mathbb{E}^\bullet \left[ \frac{\Psi_y}{\sum_{\zeta \in \mathcal{S}} \Psi_\zeta} \mathbb{1}_G \right], \quad (3.99)$$

outsourcing the proof to Lemma 3.19 below. Therefore, we can conclude that

$$\begin{aligned} \mathbb{P}^\bullet[Y = y] \cap G &= \mathbb{E}^\bullet \left[ \frac{\Psi_y}{\sum_{\zeta \in \mathcal{S}} \Psi_\zeta} \mathbb{1}_G \right] \\ &= \int_G \frac{\Psi_y}{\sum_{\zeta \in \mathcal{S}} \Psi_\zeta} d\mathbb{P}^\bullet \end{aligned} \quad (3.100)$$

which implies that

$$\begin{aligned} \mathbb{P}^\bullet[Y = y \mid \mathcal{G}] &= \frac{\Psi_y}{\sum_{\zeta \in \mathcal{S}} \Psi_\zeta} \\ &= \frac{\prod_{k=1}^n e^{-\Lambda_k} \mathbb{P}[Y = y] \prod_{k=1}^n \frac{\Lambda_k^{\tilde{\nu}_k^*(y)}}{\tilde{\nu}_k^*(y)!}}{\prod_{k=1}^n e^{-\Lambda_k} \sum_{\zeta \in \mathcal{S}} \mathbb{P}[Y = \zeta] \prod_{k=1}^n \frac{\Lambda_k^{\tilde{\nu}_k^*(\zeta)}}{\tilde{\nu}_k^*(\zeta)!}} \\ &= \frac{\mathbb{P}[Y = y] \prod_{k=1}^n \frac{\Lambda_k^{\tilde{\nu}_k^*(y)}}{\tilde{\nu}_k^*(y)!}}{\sum_{\zeta \in \mathcal{S}} \mathbb{P}[Y = \zeta] \prod_{k=1}^n \frac{\Lambda_k^{\tilde{\nu}_k^*(\zeta)}}{\tilde{\nu}_k^*(\zeta)!}}. \end{aligned} \quad (3.101)$$

□

The below lemma provides the relationship between the expectations  $\mathbb{E}^\bullet$  and  $\mathbb{E}$  which we utilized above.

**Lemma 3.19.** *For all non-negative  $\mathcal{G}$ -measurable functions  $f$  on  $\Omega$ , we can calculate the  $\bullet$ -expectation of  $f$  using  $\mathbb{E}$  as*

$$\mathbb{E}^\bullet[f] = \mathbb{E} \left[ \frac{\sum_{\zeta \in \mathcal{S}} \Psi_\zeta}{\mathbb{E} \left[ \sum_{\zeta \in \mathcal{S}} \Psi_\zeta \right]} f \right]. \quad (3.102)$$

*Specifically, we have*

$$\mathbb{E}^\bullet \left[ \frac{\Psi_y}{\sum_{\zeta \in \mathcal{S}} \Psi_\zeta} \mathbb{1}_G \right] = \mathbb{E} \left[ \frac{\Psi_y}{\mathbb{E} \left[ \sum_{\zeta \in \mathcal{S}} \Psi_\zeta \right]} \mathbb{1}_G \right]. \quad (3.103)$$

*for each  $y \in \mathcal{S}$  and  $G \in \mathcal{G}$ .*

*Proof.* Let  $G \in \mathcal{G}$ , then we have the equalities

$$\begin{aligned}\mathbb{E}^\bullet[\mathbb{1}_G] &= \mathbb{P}^\bullet[G] \\ &= \sum_{\zeta \in \mathcal{S}} \mathbb{P}^\bullet[[Y = \zeta] \cap G].\end{aligned}\tag{3.104}$$

Using our previous calculation for  $\mathbb{P}^\bullet[[Y = \zeta] \cap G]$  in Equation 3.97, we can write

$$\begin{aligned}\mathbb{E}^\bullet[\mathbb{1}_G] &= \sum_{\zeta \in \mathcal{S}} \mathbb{E} \left[ \frac{\Psi_\zeta}{\mathbb{E} \left[ \sum_{\zeta \in \mathcal{S}} \Psi_\zeta \right]} \mathbb{1}_G \right] \\ &= \mathbb{E} \left[ \frac{\sum_{\zeta \in \mathcal{S}} \Psi_\zeta}{\mathbb{E} \left[ \sum_{\zeta \in \mathcal{S}} \Psi_\zeta \right]} \mathbb{1}_G \right].\end{aligned}\tag{3.105}$$

By taking linear combinations over different choices of  $G$ , we obtain, using the linearity of expectation,

$$\mathbb{E}^\bullet[f] = \mathbb{E} \left[ \frac{\sum_{\zeta \in \mathcal{S}} \Psi_\zeta}{\mathbb{E} \left[ \sum_{\zeta \in \mathcal{S}} \Psi_\zeta \right]} f \right]\tag{3.106}$$

for all simple  $\mathcal{G}$ -measurable functions  $f$ . Thus, Equation 3.106 follows for all non-negative  $\mathcal{G}$ -measurable functions  $f$  by application of the standard monotone convergence argument.

By definition,  $\Lambda_k$  is positive and  $\mathcal{G}$ -measurable implying

$$\prod_{k=1}^n e^{-\Lambda_k} \frac{\Lambda_k^{\tilde{\nu}_k^*(\zeta)}}{\tilde{\nu}_k^*(\zeta)!}\tag{3.107}$$

is as well. Hence,  $\Psi_y$  and  $\sum_{\zeta \in \mathcal{S}} \Psi_\zeta$  are positive  $\mathcal{G}$ -measurable functions on  $\Omega$  since  $\mathbb{P}(Y = \zeta) > 0$ . Therefore,

$$\frac{\Psi_y}{\sum_{\zeta \in \mathcal{S}} \Psi_\zeta} \mathbb{1}_G\tag{3.108}$$

is non-negative and  $\mathcal{G}$ -measurable, and Equation 3.103 follows immediately.  $\square$

### 3.6 An Alternative Probability Measure $\mathbb{P}^\dagger$

We note the following corollary to Lemma 3.10.

**Corollary 3.20.** *The random variable  $\mathbb{P}[\nu = N \mid \mathcal{G}]$  is positive with*

$$\mathbb{P}[\nu = N \mid \mathcal{G}] = \sum_{i \in \mathcal{I}} \mathbb{P}[\tilde{\nu} = i] \prod_{k=1}^n e^{-\Lambda_k} \frac{\Lambda_k^{i_k}}{i_k!} \quad (3.109)$$

*Proof.* Recall that the event  $[\nu = N]$  can be partitioned using the subsets  $E_i$ , for  $i \in \mathcal{I}$ , where  $E_i$  was defined in Equation 3.47. Thus, we can partition  $[\nu = N \cap G]$  using the subsets  $E_i \cap G$ , for any  $G \in \mathcal{G}$ . Hence,

$$\mathbb{P}[\nu = N \mid \mathcal{G}] = \sum_{i \in \mathcal{I}} \mathbb{P}[E_i \mid \mathcal{G}]. \quad (3.110)$$

Using Corollary 3.8, we have that

$$\begin{aligned} \mathbb{P}[E_i \mid \mathcal{G}] &= \mathbb{P}[\tilde{\nu} = i \cap [\tilde{N} = i] \mid \mathcal{G}] \\ &= \mathbb{P}[\tilde{\nu} = i] \mathbb{P}[\tilde{N} = i \mid \mathcal{G}]. \end{aligned} \quad (3.111)$$

Thus, we conclude that

$$\begin{aligned} \mathbb{P}[\nu = N \mid \mathcal{G}] &= \sum_{i \in \mathcal{I}} \mathbb{P}[\tilde{\nu} = i \mid \mathcal{G}] \mathbb{P}[\tilde{N} = i \mid \mathcal{G}] \\ &= \sum_{i \in \mathcal{I}} \mathbb{P}[\tilde{\nu} = i] \prod_{k=1}^n e^{-\Lambda_k} \frac{\Lambda_k^{i_k}}{i_k!}, \end{aligned} \quad (3.112)$$

which provides the desired result since each term is positive.  $\square$

Recall, the probability measure  $\mathbb{P}^\bullet$  is defined on the measurable space  $(\Omega, \mathcal{F})$  as

$$\begin{aligned} \mathbb{P}^\bullet[A] &:= \mathbb{P}[A \mid \nu = N] \\ &= \frac{\mathbb{P}[A \cap [\nu = N]]}{\mathbb{P}[\nu = N]} \\ &= \frac{\mathbb{E}[\mathbb{P}[A \cap [\nu = N] \mid \mathcal{G}]]}{\mathbb{E}[\mathbb{P}[\nu = N \mid \mathcal{G}]]} \end{aligned} \quad (3.113)$$

for  $A \in \mathcal{F}$ . We observe that the equality

$$\frac{\mathbb{E}[\mathbb{P}[A \cap [\nu = N] \mid \mathcal{G}]]}{\mathbb{E}[\mathbb{P}[\nu = N \mid \mathcal{G}]]} = \mathbb{E}\left[\frac{\mathbb{P}[A \cap [\nu = N] \mid \mathcal{G}]}{\mathbb{P}[\nu = N \mid \mathcal{G}]}\right] \quad (3.114)$$

is true only if the random variables

$$\mathbb{P}[\nu = N \mid \mathcal{G}] \quad \text{and} \quad \frac{\mathbb{P}[A \cap [\nu = N] \mid \mathcal{G}]}{\mathbb{P}[\nu = N \mid \mathcal{G}]} \quad (3.115)$$

are uncorrelated, where the second random variable exists since the denominator is positive. The left side of Equation 3.114 is the  $\bullet$ -probability of the event  $A$ . Hence, we can define a probability measure  $\mathbb{P}^\dagger$ , using the right side of Equation 3.114, that is similar to  $\mathbb{P}^\bullet$  in idea but produces different values.

**Definition 3.21.** *Define the  $\dagger$ -probability of the event  $A$  as*

$$\mathbb{P}^\dagger[A] = \mathbb{E} \left[ \frac{\mathbb{P}[A \cap [\nu = N] \mid \mathcal{G}]}{\mathbb{P}[\nu = N \mid \mathcal{G}]} \right] \quad (3.116)$$

for  $A \in \mathcal{F}$ .

One significant advantage to this measure is that

$$\mathbb{P}^\dagger[G] = \mathbb{P}[G] \quad (3.117)$$

for all  $G \in \mathcal{G}$ . Thus, the conditioning does not impact the behavior of the interest rate process. Furthermore, it provides

$$\mathbb{E}^\dagger[f] = \mathbb{E}[f] \quad (3.118)$$

for any non-negative  $\mathcal{G}$ -measurable function  $f$ . This property will be extremely useful in the next chapter. We prove these statements in the following lemma.

**Lemma 3.22.** *We have the equalities*

$$\mathbb{P}^\dagger[G] = \mathbb{P}[G] \quad (3.119)$$

$$\mathbb{E}^\dagger[f] = \mathbb{E}[f]$$

for any  $G \in \mathcal{G}$  and non-negative  $\mathcal{G}$ -measurable function  $f$ .

*Proof.* Let  $G \in \mathcal{G}$ . For any  $A \in \mathcal{F}$ , we have that

$$\begin{aligned} \mathbb{P}[G \cap A \mid \mathcal{G}] &= \mathbb{E}[\mathbb{1}_G \mathbb{1}_A \mid \mathcal{G}] \\ &= \mathbb{1}_G \mathbb{E}[\mathbb{1}_A \mid \mathcal{G}] \\ &= \mathbb{1}_G \mathbb{P}[A \mid \mathcal{G}]. \end{aligned} \quad (3.120)$$

Using the previous equation, with  $A = [\nu = N]$ , we can see that

$$\begin{aligned}\mathbb{P}^\dagger[G] &:= \mathbb{E}\left[\frac{\mathbb{P}[G \cap [\nu = N] \mid \mathcal{G}]}{\mathbb{P}[\nu = N \mid \mathcal{G}]}\right] \\ &= \mathbb{E}\left[\mathbb{1}_G \frac{\mathbb{P}[\nu = N \mid \mathcal{G}]}{\mathbb{P}[\nu = N \mid \mathcal{G}]}\right] \\ &= \mathbb{P}[G],\end{aligned}\tag{3.121}$$

and, hence,  $\mathbb{E}^\dagger[\mathbb{1}_G] = \mathbb{E}[\mathbb{1}_G]$ . Using the linearity of expectation, we obtain the desired result for simple  $\mathcal{G}$ -measurable functions. Therefore, using the standard monotone convergence argument, we have that

$$\mathbb{E}^\dagger[f] = \mathbb{E}[f]\tag{3.122}$$

for any non-negative  $\mathcal{G}$ -measurable function  $f$ .  $\square$

We now calculate the  $\dagger$ -probability of the event  $[Y = y]$  and prove that  $\mathbb{P}^\bullet$  and  $\mathbb{P}^\dagger$  provide the same conditional probability that  $[Y = y]$  occurs, when conditioned to  $\mathcal{G}$ . We will utilize the random variables  $\Psi_\zeta$ , as defined in Equation 3.90, and  $\psi_\zeta$ , defined as

$$\psi_\zeta = \mathbb{P}[Y = \zeta] \prod_{k=1}^n \frac{\Lambda_k^{\tilde{\nu}_k^*(\zeta)}}{\tilde{\nu}_k^*(\zeta)!}.\tag{3.123}$$

**Theorem 3.23.** *The distribution of the random variable  $Y$  under the probability measure  $\mathbb{P}^\dagger$  is*

$$\mathbb{P}^\dagger[Y = y] = \mathbb{E}\left[\frac{\mathbb{P}[Y = y] \prod_{k=1}^n \frac{\Lambda_k^{\tilde{\nu}_k^*(y)}}{\tilde{\nu}_k^*(y)!}}{\sum_{\zeta \in \mathcal{S}} \mathbb{P}[Y = \zeta] \prod_{k=1}^n \frac{\Lambda_k^{\tilde{\nu}_k^*(\zeta)}}{\tilde{\nu}_k^*(\zeta)!}}\right],\tag{3.124}$$

for  $y \in \mathcal{S}$ . Furthermore, we have the conditional probability

$$\mathbb{P}^\dagger[Y = y \mid \mathcal{G}] = \frac{\mathbb{P}[Y = y] \prod_{k=1}^n \frac{\Lambda_k^{\tilde{\nu}_k^*(y)}}{\tilde{\nu}_k^*(y)!}}{\sum_{\zeta \in \mathcal{S}} \mathbb{P}[Y = \zeta] \prod_{k=1}^n \frac{\Lambda_k^{\tilde{\nu}_k^*(\zeta)}}{\tilde{\nu}_k^*(\zeta)!}},\tag{3.125}$$

for  $y \in \mathcal{S}$ .



*Proof.* Let  $y \in \mathcal{S}$ . In Theorem 3.18, we proved that

$$\begin{aligned}\mathbb{P}[[Y = y] \cap [\nu = N] \cap G] &= \mathbb{P}[Y = y] \mathbb{E} \left[ \mathbb{1}_G \prod_{k=1}^n e^{-\Lambda_k} \frac{\Lambda_k^{\tilde{\nu}_k^*(y)}}{\tilde{\nu}_k^*(y)!} \right] \\ &= \mathbb{E} \left[ \mathbb{1}_G \mathbb{P}[Y = y] \prod_{k=1}^n e^{-\Lambda_k} \frac{\Lambda_k^{\tilde{\nu}_k^*(y)}}{\tilde{\nu}_k^*(y)!} \right],\end{aligned}\tag{3.126}$$

for any  $G \in \mathcal{G}$ , implying that

$$\mathbb{P}[[Y = y] \cap [\nu = N] \mid \mathcal{G}] = \Psi_y.\tag{3.127}$$

Since  $\Omega$  is equal to the union over  $\mathcal{S}$  of the disjoint events  $[Y = \zeta]$ , we have that

$$\begin{aligned}\mathbb{P}[\nu = N \mid \mathcal{G}] &= \sum_{\zeta \in \mathcal{S}} \mathbb{P}[[Y = \zeta] \cap [\nu = N] \mid \mathcal{G}] \\ &= \sum_{\zeta \in \mathcal{S}} \Psi_\zeta.\end{aligned}\tag{3.128}$$

Equation 3.101 provides the equality

$$\frac{\Psi_y}{\sum_{\zeta \in \mathcal{S}} \Psi_\zeta} = \frac{\psi_y}{\sum_{\zeta \in \mathcal{S}} \psi_\zeta}.\tag{3.129}$$

Thus, we have

$$\begin{aligned}\frac{\mathbb{P}[[Y = y] \cap [\nu = N] \mid \mathcal{G}]}{\mathbb{P}[\nu = N \mid \mathcal{G}]} &= \frac{\Psi_y}{\sum_{\zeta \in \mathcal{S}} \Psi_\zeta} \\ &= \frac{\psi_y}{\sum_{\zeta \in \mathcal{S}} \psi_\zeta},\end{aligned}\tag{3.130}$$

implying that

$$\mathbb{P}^\dagger[Y = y] = \mathbb{E} \left[ \frac{\psi_y}{\sum_{\zeta \in \mathcal{S}} \psi_\zeta} \right],\tag{3.131}$$

which proves Equation 3.124.

Using Equation 3.120, we have

$$\mathbb{P}[[Y = y] \cap [\nu = N] \cap G \mid \mathcal{G}] = \mathbb{1}_G \mathbb{P}[[Y = y] \cap [\nu = N] \mid \mathcal{G}]\tag{3.132}$$

for  $G \in \mathcal{G}$ . Therefore,

$$\begin{aligned}
\mathbb{P}^\dagger[(Y = y) \cap G] &= \mathbb{E} \left[ \frac{\mathbb{P}[(Y = y) \cap [\nu = N] \cap G \mid \mathcal{G}]}{\mathbb{P}[\nu = N \mid \mathcal{G}]} \right] \\
&= \mathbb{E} \left[ \mathbb{1}_G \frac{\mathbb{P}[(Y = y) \cap [\nu = N] \mid \mathcal{G}]}{\mathbb{P}[\nu = N \mid \mathcal{G}]} \right] \\
&= \mathbb{E} \left[ \mathbb{1}_G \frac{\psi_y}{\sum_{\zeta \in \mathcal{S}} \psi_\zeta} \right] \\
&= \mathbb{E}^\dagger \left[ \mathbb{1}_G \frac{\psi_y}{\sum_{\zeta \in \mathcal{S}} \psi_\zeta} \right].
\end{aligned} \tag{3.133}$$

The last equality is valid by Lemma 3.22, since

$$\mathbb{1}_G \frac{\psi_y}{\sum_{\zeta \in \mathcal{S}} \psi_\zeta} \tag{3.134}$$

is  $\mathcal{G}$ -measurable and non-negative; Equation 3.125 immediately follows.  $\square$

### 3.7 The Gaussian-Poisson Framework as a Model of Default Behavior

In this brief section, we will use the mathematical framework established in the previous sections to discuss the Gaussian-Poisson model for default phenomena. We will later use this model to study default probabilities and their affect on the price of various financial instruments.

Definition 3.11 provided the probability measure  $\mathbb{P}^\bullet$ , which was defined, using the probability measure  $\mathbb{P}$ , conditional on  $[\nu = N]$ . In financial terms, we shall consider  $\mathbb{P}^\bullet$  as a market pricing measure. Requiring  $[\nu = N]$  is equivalent to saying that the number of defaults detected at each observation time must be the same for both the Gaussian copula model and the Poisson model. We showed that requiring an equal number of defaults to have occurred before time  $t_k$  for all  $k$ ,  $[\nu = N]$ , was equivalent to requiring an equal number of defaults to have occurred during the  $k$ th time interval for all  $k$ ,  $[\tilde{\nu} = \tilde{N}]$ .

Another market pricing measure,  $\mathbb{P}^\dagger$ , was presented in Definition 3.21. The sigma-algebra  $\mathcal{G}$ , which was used to create this measure, is generated by the interest

rate process. We will frequently condition using  $\mathcal{G}$  in the following chapter. Hence, the measure  $\mathbb{P}^\dagger$  has a distinct advantage in that it does not affect the distribution of the interest rate process.

The  $m$ -dimensional random variable  $Y$  is designed to work as a default descriptor. If name  $j$  does not default by time  $t_n$ , then  $Y_j = \infty$ . Otherwise,  $Y_j = k$  for some  $k \in \{1, \dots, n\}$  and name  $j$  defaults during the  $k$ th time interval. Thus,  $Y$  gives a complete picture of which asset is observed to default at which time. Furthermore, the random variable  $\tilde{\nu}(Y)$  provides the number of defaults that occur during each time interval; i.e., for  $i \in \mathcal{I}$ , the equality  $\tilde{\nu}(Y) = i$  means that value  $i_k$  is the number of defaults detected at the observation point  $t_k$ .

We summarize the Gaussian-Poisson model as follows.

- Its purpose is to model default phenomena by utilizing both a one-factor Gaussian copula model and a Poisson model.
- The key elements to the model are the default descriptor  $Y$  and the market pricing measures  $\mathbb{P}^\bullet$  and  $\mathbb{P}^\dagger$ .
- The variable  $Y$  provides the time at which an asset's default is recorded. Also,  $\tilde{\nu}(Y)$  encapsulates the number of defaults which occur during each time interval.
- Each random variable  $Y_j$  is dependent upon the random variables  $D_{j,k}$ , for  $k \in \{1, \dots, n, \infty\}$ , where these variables model times of default,  $\tau_j$ , using the Gaussian copula model via

$$[\tau_j \leq t_k] = [X_j \leq c_{j,k}] \quad (3.135)$$

- The probability measure  $\mathbb{P}$  would be used as the market pricing measure in the Gaussian copula model. However, the market pricing measure used in the Gaussian-Poisson model can be either  $\mathbb{P}^\bullet$  or  $\mathbb{P}^\dagger$ . These probability measures assume that  $[\nu = N]$  thereby incorporating the Cox process into the model.

Analysis conducted using the probability mass functions  $\mathbb{P}^\bullet[Y = y]$  and  $\mathbb{P}^\dagger[Y = y]$  can provide insight into the default behavior of the random variable  $\tau$ . Both original models by themselves have deficiencies. Some researchers argue that the Gaussian copula model relies too heavily on correlations between the assets to determine default probabilities. However, while a Cox process can prove beneficial by incorporating interest rate dependence, we can only determine the number of assets which default not specific names. The Gaussian-Poisson model keeps the advantages of the two models while addressing their weaknesses.

# Chapter 4

## Expected Losses and Parameter Sensitivities

The previous chapter presented the Gaussian-Poisson model. In this chapter, we will prove several results using the mathematical framework in a financial context. Our calculations are relevant to the pricing of various financial instruments. We also examine parameter sensitivity. Throughout the chapter, we assume that we are using the Gaussian-Poisson model with default descriptor  $Y$ . We shall consider both market pricing measures  $\mathbb{P}^\bullet$  and  $\mathbb{P}^\dagger$ .

### 4.1 The Market Pricing Measure $\mathbb{P}^\bullet$

The results in this section include calculating expected losses using the market pricing measure  $\mathbb{P}^\bullet$ . We will examine both portfolio and tranche losses. We will also calculate partial derivatives with respect to a parameter  $\alpha$ . These calculations will help to illustrate a disadvantage to using  $\mathbb{P}^\bullet$ .

#### 4.1.1 The Present Value of a Portfolio's Loss

Suppose we have a portfolio comprised of  $m$  assets with a principal of  $P_j$  invested in asset  $j$ . We assume the probabilities of default for each asset are given by the Gaussian-Poisson model with  $\mathbb{P}^\bullet$  as the market pricing measure. Also, suppose that the asset has a recovery rate  $R_j$ . If asset  $j$  defaults, the loss to the portfolio will be

$$\ell_j = (1 - R_j)P_j. \quad (4.1)$$

**Definition 4.1.** *Define the set  $\mathcal{S}_{j,k} \subseteq \mathcal{S}$  as*

$$\mathcal{S}_{j,k} = \{y \in \mathcal{S} \mid y_j = k\}. \quad (4.2)$$

Define the random variables  $L, L^* : \Omega \rightarrow [0, \infty)$  as

$$\begin{aligned} L &= \sum_{k=1}^n \sum_{j=1}^m e^{-\int_{t_0}^{t_k} r(t) dt} \ell_j \mathbf{1}_{D_{j,k}} \\ L^* &= \sum_{k=1}^n \sum_{j=1}^m \ell_j e^{-\int_{t_0}^{t_k} r(t) dt} \frac{\sum_{y \in \mathcal{S}_{j,k}} \psi_y}{\sum_{\zeta \in \mathcal{S}} \psi_\zeta}. \end{aligned} \quad (4.3)$$

The real number  $L(\omega)$  represents the portfolio's loss, given in time- $t_0$  money, over the time interval  $[0, T]$ . We refer to

$$e^{-\int_{t_0}^{t_k} r(t) dt} \quad (4.4)$$

as a discount factor. See Appendix A for a further discussion of these market concepts. We have that

$$\begin{aligned} \frac{\sum_{y \in \mathcal{S}_{j,k}} \psi_y}{\sum_{\zeta \in \mathcal{S}} \psi_\zeta} &= \frac{\sum_{y \in \mathcal{S}_{j,k}} \mathbb{P}[Y = y] \prod_{k=1}^n \frac{\Lambda_k^*(y)}{\tilde{\nu}_k^*(y)!}}{\sum_{\zeta \in \mathcal{S}} \mathbb{P}[Y = \zeta] \prod_{k=1}^n \frac{\Lambda_k^*(\zeta)}{\tilde{\nu}_k^*(\zeta)!}} \\ &= \sum_{y \in \mathcal{S}_{j,k}} \mathbb{P}^\bullet[Y = y \mid \mathcal{G}] \\ &= \mathbb{P}^\bullet[D_{j,k} \mid \mathcal{G}]. \end{aligned} \quad (4.5)$$

implying that

$$L^* = \sum_{k=1}^n \sum_{j=1}^m \ell_j e^{-\int_{t_0}^{t_k} r(t) dt} \mathbb{P}^\bullet[D_{j,k} \mid \mathcal{G}]. \quad (4.6)$$

Since  $\mathbb{P}^\bullet$  is our market pricing measure, we refer to  $\mathbb{E}^\bullet[L]$  as the present value of the portfolio's loss. We will provide a formula, shortly, to calculate  $\mathbb{E}^\bullet[L]$ .

Suppose the process  $\Lambda(t)$  is dependent upon a parameter  $\alpha$ . Suppose further that  $r(t)$  is not dependent upon  $\alpha$ . That is, we assume for the rest of the chapter that  $\partial_\alpha \Lambda(t) \neq 0$ , for some  $\alpha$ , and  $\partial_\alpha r(t) = 0$  for all  $\alpha$ , where we denote, for notational convenience,  $\partial_\alpha f$  as the partial derivative of  $f$  with respect to  $\alpha$ . We are interested in calculating the dependence of  $\mathbb{E}^\bullet[L]$  on the parameter  $\alpha$ , where  $L$  and  $L^*$  are both viewed as functions of  $\alpha$  and  $\omega \in \Omega$ . We now proceed to show that  $\mathbb{E}^\bullet[L]$  can be calculated as  $\mathbb{E}^\bullet[L^*]$ . We also find  $\mathbb{E}^\bullet[\partial_\alpha L^*]$ .

**Theorem 4.2.** *The present value of the loss is given by*

$$\mathbb{E}^\bullet[L] = \mathbb{E}^\bullet[L^*] = \sum_{k=1}^n \sum_{j=1}^m \ell_j \mathbb{E}^\bullet \left[ e^{-\int_{t_0}^{t_k} r(t) dt} \mathbb{P}^\bullet[D_{j,k} \mid \mathcal{G}] \right]. \quad (4.7)$$

Furthermore, let  $\alpha^*$  be a real number such that  $\partial_\alpha \Lambda_z$  exists at  $\alpha^*$ , for each  $z \in \{1, \dots, n\}$ . Then the partial derivative  $\partial_\alpha L^*$  exists, and we can calculate  $\mathbb{E}^\bullet[\partial_\alpha L^*]$  at  $\alpha^*$  as

$$\mathbb{E}^\bullet[\partial_\alpha L^*](\alpha^*) = \sum_{k=1}^n \sum_{j=1}^m \ell_j \mathbb{E}^\bullet \left[ e^{-\int_{t_0}^{t_k} r(t) dt} \partial_\alpha \mathbb{P}^\bullet[D_{j,k} \mid \mathcal{G}](\alpha^*, \cdot) \right], \quad (4.8)$$

where the derivative  $\partial_\alpha \mathbb{P}^\bullet[D_{j,k} \mid \mathcal{G}]$  is given by

$$\frac{\sum_{y \in \mathcal{S}_{j,k}} \sum_{\zeta \in \mathcal{S}} \psi_y \psi_\zeta \sum_{z=1}^n \frac{\partial_\alpha \Lambda_z}{\Lambda_z} (\tilde{\nu}_z^*(y) - \tilde{\nu}_z^*(\zeta))}{\left( \sum_{\zeta \in \mathcal{S}} \psi_\zeta \right)^2}. \quad (4.9)$$

*Proof.* We begin by observing that

$$\begin{aligned} \mathbb{E}^\bullet[L] &= \mathbb{E}^\bullet[\mathbb{E}^\bullet[L \mid \mathcal{G}]] \\ &= \sum_{k=1}^n \sum_{j=1}^m \ell_j \mathbb{E}^\bullet \left[ \mathbb{E}^\bullet \left[ e^{-\int_{t_0}^{t_k} r(t) dt} \mathbb{1}_{D_{j,k}} \mid \mathcal{G} \right] \right] \\ &= \sum_{k=1}^n \sum_{j=1}^m \ell_j \mathbb{E}^\bullet \left[ e^{-\int_{t_0}^{t_k} r(t) dt} \mathbb{E}^\bullet[\mathbb{1}_{D_{j,k}} \mid \mathcal{G}] \right] \\ &= \sum_{k=1}^n \sum_{j=1}^m \ell_j \mathbb{E}^\bullet \left[ e^{-\int_{t_0}^{t_k} r(t) dt} \mathbb{P}^\bullet[D_{j,k} \mid \mathcal{G}] \right] \\ &= \mathbb{E}^\bullet[L^*], \end{aligned} \quad (4.10)$$

since  $r(t)$  is, by definition,  $\mathcal{G}$ -measurable.

Using Equation 4.10, we have that

$$\begin{aligned} \mathbb{E}^\bullet[\partial_\alpha L^*](\alpha^*) &= \mathbb{E}^\bullet \left[ \partial_\alpha \left( \sum_{k=1}^n \sum_{j=1}^m \ell_j e^{-\int_{t_0}^{t_k} r(t) dt} \partial_\alpha \mathbb{P}^\bullet[D_{j,k} \mid \mathcal{G}] \right) (\alpha^*, \cdot) \right] \\ &= \sum_{k=1}^n \sum_{j=1}^m \ell_j \mathbb{E}^\bullet \left[ e^{-\int_{t_0}^{t_k} r(t) dt} \partial_\alpha \mathbb{P}^\bullet[D_{j,k} \mid \mathcal{G}](\alpha^*, \cdot) \right] \end{aligned} \quad (4.11)$$

since  $\partial_\alpha r(t) = 0$ . Applying the quotient rule, we obtain

$$\partial_\alpha \mathbb{P}^\bullet[D_{j,k} \mid \mathcal{G}] = \frac{\sum_{y \in S_{j,k}} \sum_{\zeta \in \mathcal{S}} \psi_\zeta \partial_\alpha \psi_y - \psi_y \partial_\alpha \psi_\zeta}{\left(\sum_{\zeta \in \mathcal{S}} \psi_\zeta\right)^2} \quad (4.12)$$

Using the product and power rules, we see that

$$\begin{aligned} \partial_\alpha \psi_\zeta &= \mathbb{P}[Y = \zeta] \partial_\alpha \left( \prod_{q=1}^n \frac{\Lambda_q^{\tilde{\nu}_q^*(\zeta)}}{(\tilde{\nu}_q^*(\zeta))!} \right) \\ &= \mathbb{P}[Y = \zeta] \left( \prod_{q=1}^n \frac{\Lambda_q^{\tilde{\nu}_q^*(\zeta)}}{(\tilde{\nu}_q^*(\zeta))!} \right) \left( \sum_{z=1}^n \frac{\tilde{\nu}_z^*(\zeta)}{\Lambda_z} \partial_\alpha \Lambda_z \right) \\ &= \psi_\zeta \left( \sum_{z=1}^n \frac{\partial_\alpha \Lambda_z}{\Lambda_z} \tilde{\nu}_z^*(\zeta) \right). \end{aligned} \quad (4.13)$$

Therefore,

$$\psi_\zeta \partial_\alpha \psi_y - \psi_y \partial_\alpha \psi_\zeta = \psi_y \psi_\zeta \sum_{z=1}^n \frac{\partial_\alpha \Lambda_z}{\Lambda_z} (\tilde{\nu}_z^*(y) - \tilde{\nu}_z^*(\zeta)), \quad (4.14)$$

and Equation 4.9 is proven.  $\square$

Notice that we have calculated  $\mathbb{E}^\bullet[\partial_\alpha L^*]$  rather than  $\partial_\alpha \mathbb{E}^\bullet[L^*]$ , with the latter being equal to  $\partial_\alpha \mathbb{E}^\bullet[L]$ . These values correspond to the expected value of the change in loss and the change in expected loss. The latter is more useful since it provides us with the change in value of the portfolio's loss. Of course, it is possible that these two values are actually equal. In order to prove equality, we would need to interchange the order of integration and differentiation. However,  $\mathbb{E}^\bullet$  is dependent on the value of  $\alpha$  as well, presenting a significant obstacle to calculating  $\partial_\alpha \mathbb{E}^\bullet[L]$ .

Suppose we have entered into a credit default swap (CDS) contract with our portfolio of  $m$  assets as its underlying. In order to price this CDS, one must have a model, such as ours, that calculates  $\mathbb{E}^\bullet[L]$ . Thus, it is one of the motivations of the Gaussian-Poisson model. A significant difference between our model's assumption and the current standard assumption is that most models assume the default event  $D_{j,k}$  and the short rate  $r(t)$  are independent under their market pricing measure



$P$ . This assumption is made because it allows one to write

$$E\left[e^{-\int_{t_0}^{t_k} r(t) dt} \mathbb{1}_{D_{j,k}}\right] = E\left[e^{-\int_{t_0}^{t_k} r(t) dt}\right] E[\mathbb{1}_{D_{j,k}}]. \quad (4.15)$$

While this simplification can make computations easier, it can be argued that the assumption is unrealistic. Interest rates can, in fact, affect an entity's ability to make debt payments. Thus, in our model,  $\mathbb{P}^\bullet[D_{j,k}]$  is dependent upon the Cox process,  $N(t)$ , whose intensity process is defined as dependent upon the short rate. This factor was one of the primary motivations for our model to be defined in this manner.

#### 4.1.2 Portfolio Loss Under a Model Simplification

It can also be useful to know the sign of a derivative. For instance, we might have that if the parameter  $\alpha$  increases, then the present value of the loss always decreases. We will make several calculations in the rest of this chapter to examine the sign of certain derivatives. These signs will depend on us making the assumption that we are only examining one time interval; i.e., there is a single observation point and  $n = 1$ . For the rest of this section we will make that assumption, although we will restate it in any definition or theorem to which the assumption is relevant. We will also simplify three notations henceforth.

$$\begin{aligned} \mathcal{S}_j &= \mathcal{S}_{j,1} \\ \tilde{\nu}^*(\cdot) &= \tilde{\nu}_1^*(\cdot) \\ \Lambda &= \Lambda_1. \end{aligned} \quad (4.16)$$

For a fixed  $\alpha^*$ , suppose that the partial derivative  $\partial_\alpha \Lambda$  exists and  $\text{sgn}(\partial_\alpha \Lambda(\alpha^*, \cdot))$  is constant almost surely. Then we can refer to the value  $\text{sgn}(\partial_\alpha \Lambda(\alpha^*))$ . Our next significant result is that the sign of  $\mathbb{E}^\bullet[\partial_\alpha L^*]$  at  $\alpha^*$  is determined by and equal to the sign of  $\partial_\alpha \Lambda$  at  $\alpha^*$ . In order to prove this result, we need to first provide a definition and two lemmas.

**Definition 4.3.** Assume that  $n = 1$  and define the following sets and quantities:

$$\begin{aligned}
\Delta_j(i) &= \{y \in \mathcal{S}_j \mid \tilde{\nu}^*(y) = i\} \\
\theta_j(i) &= |\Delta_j(i)| \\
\Delta(i) &= \{\zeta \in \mathcal{S} \mid \tilde{\nu}^*(\zeta) = i\} \\
\theta(i) &= |\Delta(i)|,
\end{aligned} \tag{4.17}$$

for  $j \in \{1, \dots, m\}$  and  $i \in \{0, 1, \dots, m\}$ .

The set  $\Delta(i)$  contains all possible market outcomes  $\zeta$  such that the total number of defaults during our observation time is  $i$ , with  $\theta(i) < |\mathcal{S}| < \infty$  the number of outcomes which meet this criteria. The set  $\Delta_j(i)$  is a proper subset of  $\Delta(i)$ , hence  $\theta_j(i) < \theta(i)$ , with the added constraint that the outcome  $y$  must be in the set  $\mathcal{S}_j$ . That is, this outcome must have encoded that name  $j$  was one of the  $i$  names which defaulted. Notice that  $\Delta_j(0)$  must, therefore, be empty.

**Lemma 4.4.** Assume that  $n = 1$ . We can calculate the cardinalities,

$$\begin{aligned}
\theta_j(0) &= 0, \\
\theta_j(a) &= \binom{m-1}{a-1}, \text{ and} \\
\theta(b) &= \binom{m}{b},
\end{aligned} \tag{4.18}$$

where  $a \in \{1, \dots, m\}$ ,  $b \in \{0, 1, \dots, m\}$ , and  $j \in \{1, \dots, m\}$ .

*Proof.* Set  $j \in \{1, \dots, m\}$ . Since,  $y_j = 1$  for each  $y \in \mathcal{S}_j$ , then  $\tilde{\nu}^*(y) > 0$  and  $\Delta_j(0) = \emptyset$  implying that

$$\theta_j(0) = 0. \tag{4.19}$$

Set  $a \in \{1, \dots, m\}$  and  $b \in \{0, 1, \dots, m\}$ . We can use a combinatorial argument to find  $\theta_j(a)$  and  $\theta(b)$ . Let us refer to  $y_1, y_2$ , etc., as slots and observe that each  $y$  has  $m$  slots. Since  $n = 1$ , each slot has the value 1 or  $\infty$ . Thus, an element  $y$  is

determined by choosing which slots contain a 1. If  $y \in \Delta_j(a)$ , then  $y$  has  $a$  slots with the value 1. However, one of these slots must be slot  $j$ . Thus, we have  $m - 1$  slots from which we must choose  $a - 1$  of them to give the value 1 implying that

$$\theta_j(a) = \binom{m-1}{a-1}. \quad (4.20)$$

If  $\zeta \in \Delta(b)$ , then,  $\zeta$  has  $b$  slots with the value 1. Therefore, we have  $m$  slots from which we must choose  $b$  of them to give the value 1 implying that

$$\theta(b) = \binom{m}{b}. \quad (4.21)$$

□

In order to find  $\text{sgn}(\mathbb{E}^\bullet[\partial_\alpha L^*])$  we must make an additional simplification to the model. We will assume that the  $\bullet$ -probability of default given the value of the global factor  $Z$  is the same for any two names  $j_1$  and  $j_2$ , i.e.,

$$P_{j_1,k}(x) = P_{j_2,k}(x), \quad (4.22)$$

for all  $x \in \mathbb{R}$ . This can be accomplished by assuming that the model has a common correlation  $\rho$  between any two names and that there exists a  $c \in \mathbb{R}$  such that

$$F_{\tau_j}(t_1) = \Phi(c) \quad (4.23)$$

for all  $j \in \{1, \dots, m\}$ . This assumption allows us to make the following useful observation.

**Lemma 4.5.** *Assume that  $n = 1$  and that there are real numbers  $\rho \in (0, 1)$  and  $c$  such that  $\rho_j = \rho$  and  $c_{j,1} = c$  for all  $j \in \{1, \dots, m\}$ . Then, for each  $i \in \{1, \dots, m\}$ ,*

$$\mathbb{P}[Y = y] = \mathbb{P}[Y = \zeta], \quad (4.24)$$

for all  $y, \zeta \in \Delta(i)$ .

*Proof.* Set  $i \in \{1, \dots, m\}$  and  $y, \zeta \in \Delta(i)$ . Since  $n = 1$ ,  $y_j$  and  $\zeta_j$  have either the value 1 or  $\infty$  for each  $j$ . Using Lemma 3.16 with  $c_{j,1} = c$  and  $\rho_j = \rho$ , we have the equalities

$$\begin{aligned}\mathbb{P}[Y = y] &= \int_{\mathbb{R}} \left( \prod_{j=1}^m P_{j,y_j}(x) \right) \phi(x) \, dx \quad \text{and} \\ \mathbb{P}[Y = \zeta] &= \int_{\mathbb{R}} \left( \prod_{j=1}^m P_{j,\zeta_j}(x) \right) \phi(x) \, dx,\end{aligned}\tag{4.25}$$

where

$$\begin{aligned}P_{j,1}(x) &= \Phi\left(\frac{c - \sqrt{\rho}x}{\sqrt{1 - \rho}}\right) \quad \text{and} \\ P_{j,\infty}(x) &= 1 - \Phi\left(\frac{c - \sqrt{\rho}x}{\sqrt{1 - \rho}}\right).\end{aligned}\tag{4.26}$$

Observe that both  $P_{j,1}$  and  $P_{j,\infty}$  have no  $j$  dependence. Therefore,

$$\mathbb{P}[Y = y] = \int_{\mathbb{R}} (P_{1,1}(x))^i (P_{1,\infty}(x))^{m-i} \phi(x) \, dx = \mathbb{P}[Y = \zeta].\tag{4.27}$$

□

Using the previous work in this section, we can now proceed to prove the following theorem.

**Theorem 4.6.** *Assume that  $n = 1$  and that there are real numbers  $\rho \in (0, 1)$  and  $c$  such that  $\rho_j = \rho$  and  $c_{j,1} = c$  for all  $j \in \{1, \dots, m\}$ . Let  $\alpha^*$  be a real number such that  $\partial_\alpha \Lambda$  exists at  $\alpha^*$ . If  $\text{sgn}(\partial_\alpha \Lambda(\alpha^*, \cdot))$  is constant almost surely, then we have the equality,*

$$\text{sgn}(\mathbb{E}^\bullet[\partial_\alpha L^*](\alpha^*)) = \text{sgn}(\partial_\alpha \Lambda(\alpha^*)).\tag{4.28}$$

*Proof.* Using Theorem 4.2 we have that

$$\mathbb{E}^\bullet[\partial_\alpha L^*](\alpha^*) = \sum_{j=1}^m \ell_j \, \mathbb{E}^\bullet \left[ e^{-\int_0^{t_1} r(t) \, dt} \partial_\alpha \mathbb{P}^\bullet[D_{j,1} \mid \mathcal{G}](\alpha^*, \cdot) \right]\tag{4.29}$$

where

$$\partial_\alpha \mathbb{P}^\bullet[D_{j,1} \mid \mathcal{G}] = \frac{\partial_\alpha \Lambda \sum_{y \in \mathcal{S}_j} \sum_{\zeta \in \mathcal{S}} \psi_y \psi_\zeta (\tilde{\nu}^*(y) - \tilde{\nu}^*(\zeta))}{\Lambda \left( \sum_{\zeta \in \mathcal{S}} \psi_\zeta \right)^2}. \quad (4.30)$$

Substituting, we observe that

$$\mathbb{E}^\bullet[\partial_\alpha L^*](\alpha^*) = \sum_{j=1}^m \mathbb{E}^\bullet \left[ \partial_\alpha \Lambda(\alpha^*, \cdot) \frac{\ell_j e^{-\int_{t_0}^{t_1} r(t) dt}}{\Lambda \left( \sum_{\zeta \in \mathcal{S}} \psi_\zeta \right)^2} \sum_{y \in \mathcal{S}_j} \sum_{\zeta \in \mathcal{S}} \psi_y \psi_\zeta (\tilde{\nu}^*(y) - \tilde{\nu}^*(\zeta)) \right]. \quad (4.31)$$

Hence, if the double sum is positive, we will have the desired result since

$$\frac{\ell_j e^{-\int_{t_0}^{t_1} r(t) dt}}{\Lambda \left( \sum_{\zeta \in \mathcal{S}} \psi_\zeta \right)^2} > 0. \quad (4.32)$$

For each  $a \in \{1, \dots, m\}$ , choose an element,  $y(a)$ , of the set  $\Delta_j(a)$ , and, for each  $b \in \{0, 1, \dots, m\}$ , choose an element,  $\zeta(b)$ , of the set  $\Delta(b)$ . Using Lemma 4.5, we have that

$$\psi_{y(a)} = \psi_y \text{ and } \psi_{\zeta(b)} = \psi_\zeta \quad (4.33)$$

for each  $y \in \Delta_j(a)$  and  $\zeta \in \Delta(b)$ . Furthermore, the sets  $\mathcal{S}_j$  and  $\mathcal{S}$  can be written as disjoint unions of subsets of the form  $\Delta_j(i)$  and  $\Delta(i)$ , respectively, where  $i$  runs from 0 to  $m$ .

Thus, we can rewrite the double sum as follows:

$$\begin{aligned} \sum_{y \in \mathcal{S}_j} \sum_{\zeta \in \mathcal{S}} \psi_y \psi_\zeta (\tilde{\nu}^*(y) - \tilde{\nu}^*(\zeta)) &= \sum_{a=1}^m \sum_{y \in \Delta_j(a)} \sum_{b=0}^m \sum_{\zeta \in \Delta(b)} \psi_y \psi_\zeta (a - b) \\ &= \sum_{a=1}^m \sum_{b=0}^m \sum_{y \in \Delta_j(a)} \sum_{\zeta \in \Delta(b)} \psi_{y(a)} \psi_{\zeta(b)} (a - b) \\ &= \sum_{a=1}^m \sum_{b=0}^m \theta_j(a) \theta(b) \psi_{y(a)} \psi_{\zeta(b)} (a - b) \\ &= \sum_{a=0}^m \sum_{b=0}^m \theta_j(a) \theta(b) \psi_{y(a)} \psi_{\zeta(b)} (a - b) \end{aligned} \quad (4.34)$$

The final equality is valid since  $\theta_j(0) = 0$  allowing us to ignore the fact that  $\psi_{y(0)}$  has no meaning; we can thus define it equal to 1 so that the above is well-defined.

Notice that

$$\Delta_j(a) \subseteq \Delta(a) \implies \psi_{y(a)} = \psi_{\zeta(a)}. \quad (4.35)$$

Denote

$$w_{a,b} := \psi_{y(a)}\psi_{y(b)}(a-b) = \psi_{y(a)}\psi_{\zeta(b)}(a-b) \quad (4.36)$$

and observe that

$$\begin{aligned} w_{b,a} &= \psi_{y(b)}\psi_{y(a)}(b-a) \\ &= -\psi_{y(a)}\psi_{y(b)}(a-b) \\ &= -w_{a,b}. \end{aligned} \quad (4.37)$$

We can write manipulate the double sum again to produce

$$\begin{aligned} \sum_{a=0}^m \sum_{b=0}^m \theta_j(a)\theta(b) w_{a,b} &= \sum_{a=0}^m \sum_{0 \leq b < a} \theta_j(a)\theta(b) w_{a,b} + \theta_j(b)\theta(a)w_{b,a} \\ &= \sum_{a=0}^m \sum_{0 \leq b < a} \theta_j(a)\theta(b) w_{a,b} - \theta_j(b)\theta(a)w_{a,b} \\ &= \sum_{a=0}^m \sum_{0 \leq b < a} (\theta_j(a)\theta(b) - \theta_j(b)\theta(a))w_{a,b}. \end{aligned} \quad (4.38)$$

When  $b < a$ , we can clearly see that  $w_{a,b} > 0$ . Finally, we claim that

$$\theta_j(a)\theta(b) - \theta_j(b)\theta(a) > 0 \quad (4.39)$$

as well, when  $b < a$ , with the proof given in Lemma 4.7 below. Therefore, we have

$$\sum_{y \in \mathcal{S}_j} \sum_{\zeta \in \mathcal{S}} \psi_y \psi_{\zeta} (\tilde{\nu}^*(y) - \tilde{\nu}^*(\zeta)) = \sum_{a=0}^m \sum_{0 \leq b < a} (\theta_j(a)\theta(b) - \theta_j(b)\theta(a))w_{a,b}, \quad (4.40)$$

with each term on the right side positive, yielding Equation 4.28.  $\square$

The result in the preceding theorem follows one's general intuition. Recall that  $\Lambda_1 = \mathbb{E}[N(t_1) \mid \mathcal{G}]$ ; it is the average intensity over the time interval  $[0, t_1]$ . Thus, we can expect  $N(t_1)$  and  $\Lambda_1$  to either increase or decrease together. Hence, it seems

logical to expect the number of defaults,  $\tilde{\nu}^*(Y)$ , and thereby the expected loss to change in the same manner.

Finally, the following claim was made in the previous theorem without proof. The proof is non-trivial so we provide it here.

**Lemma 4.7.** *Suppose  $a \in \{0, 1, \dots, m\}$  and  $b$  is such that  $0 \leq b < a$ . Then we have*

$$\theta_j(a)\theta(b) - \theta_j(b)\theta(a) > 0. \quad (4.41)$$

*Proof.* Let  $a \in \{0, 1, \dots, m\}$ . If  $b = 0$ , then  $\theta_j(b) = 0$  implying that

$$\theta_j(a)\theta(b) - \theta_j(b)\theta(a) = \theta_j(a)\theta(b) > 0. \quad (4.42)$$

Now, assume  $b$  is such that  $0 < b < a$  which implies that  $1/a < 1/b$ . Multiply both sides of the inequality by

$$\frac{(m-1)!m!}{(a-1)!(m-a)!(b-1)!(m-b)!}$$

obtaining the new inequality

$$\binom{m-1}{b-1} \binom{m}{a} < \binom{m-1}{a-1} \binom{m}{b} \quad (4.43)$$

with the left side equal to  $\theta_j(b)\theta(a)$  and the right side equal to  $\theta_j(a)\theta(b)$ . Therefore,

$$\theta_j(a)\theta(b) - \theta_j(b)\theta(a) > 0. \quad (4.44)$$

□

#### 4.1.3 Tranche Loss Sensitivity to an Intensity Parameter

As in the previous section, we assume that there is only one time interval of interest. However, we no longer assume that there are a common correlation  $\rho$  and common threshold  $c$ . We begin by defining random variables which will be the basis of our study in this new section.

**Definition 4.8.** Assume that  $n = 1$  and define the random variables

$$\begin{aligned}
l_e(\kappa) &= \min\{\tilde{\nu}^*(Y), \kappa\}, \\
l_s(\kappa) &= \tilde{\nu}^*(Y) - \min\{\tilde{\nu}^*(Y), \kappa\}, \\
l_e^*(\kappa) &= \kappa + \sum_{i=0}^{\kappa} (i - \kappa) \frac{\sum_{y \in \Delta(i)} \psi_y}{\sum_{\zeta \in \mathcal{S}} \psi_\zeta}, \quad \text{and} \\
l_s^*(\kappa) &= \sum_{i=\kappa+1}^m (i - \kappa) \frac{\sum_{y \in \Delta(i)} \psi_y}{\sum_{\zeta \in \mathcal{S}} \psi_\zeta},
\end{aligned} \tag{4.45}$$

for each  $\kappa \in \{0, 1, \dots, m\}$ .

The codomains of the four random variables above are

$$\{0, 1, 2, \dots, \kappa\}, \{0, 1, 2, \dots, m - \kappa\}, [0, \kappa], \text{ and } [0, m - \kappa], \tag{4.46}$$

respectively. We also note that

$$\begin{aligned}
\frac{\sum_{y \in \Delta(i)} \psi_y}{\sum_{\zeta \in \mathcal{S}} \psi_\zeta} &= \sum_{y \in \Delta(i)} \mathbb{P}^\bullet[Y = y \mid \mathcal{G}] \\
&= \mathbb{P}^\bullet[\tilde{\nu}^*(Y) = i \mid \mathcal{G}],
\end{aligned} \tag{4.47}$$

which provides

$$\begin{aligned}
l_e^*(\kappa) &= \kappa + \sum_{i=0}^{\kappa} (i - \kappa) \mathbb{P}^\bullet[\tilde{\nu}^*(Y) = i \mid \mathcal{G}] \quad \text{and} \\
l_s^*(\kappa) &= \sum_{i=\kappa+1}^m (i - \kappa) \mathbb{P}^\bullet[\tilde{\nu}^*(Y) = i \mid \mathcal{G}].
\end{aligned} \tag{4.48}$$

We now show that the  $\bullet$ -expectation of the random variables  $l_e(\kappa)$  and  $l_s(\kappa)$  can be calculated as the  $\bullet$ -expectation of  $l_e^*(\kappa)$  and  $l_s^*(\kappa)$ , respectively.

**Lemma 4.9.** Assume  $n = 1$ . We have the equalities

$$\begin{aligned}
\mathbb{E}^\bullet[l_e(\kappa)] &= \mathbb{E}^\bullet[l_e^*(\kappa)] = \mathbb{E}^\bullet\left[\kappa + \sum_{i=0}^{\kappa} (i - \kappa) \mathbb{P}^\bullet[\tilde{\nu}^*(Y) = i \mid \mathcal{G}]\right] \quad \text{and} \\
\mathbb{E}^\bullet[l_s(\kappa)] &= \mathbb{E}^\bullet[l_s^*(\kappa)] = \mathbb{E}^\bullet\left[\sum_{i=\kappa+1}^m (i - \kappa) \mathbb{P}^\bullet[\tilde{\nu}^*(Y) = i \mid \mathcal{G}]\right].
\end{aligned} \tag{4.49}$$



*Proof.* Observe that  $l_e(\kappa)$  and  $l_s(\kappa)$  can be written as

$$\begin{aligned}
l_e(\kappa) &= \min\{\tilde{\nu}^*(Y), \kappa\} \\
&= \sum_{i=0}^{\kappa} i \mathbb{1}_{[\tilde{\nu}^*(Y)=i]} + \sum_{i=\kappa+1}^m \kappa \mathbb{1}_{[\tilde{\nu}^*(Y)=i]} \\
&= \sum_{i=0}^{\kappa} (i - \kappa + \kappa) \mathbb{1}_{[\tilde{\nu}^*(Y)=i]} + \sum_{i=\kappa+1}^m \kappa \mathbb{1}_{[\tilde{\nu}^*(Y)=i]} \\
&= \kappa + \sum_{i=0}^{\kappa} (i - \kappa) \mathbb{1}_{[\tilde{\nu}^*(Y)=i]}
\end{aligned} \tag{4.50}$$

and

$$\begin{aligned}
l_s(\kappa) &= \tilde{\nu}^*(Y) - \min\{\tilde{\nu}^*(Y), \kappa\} \\
&= \max\{0, \tilde{\nu}^*(Y) - \kappa\} \\
&= \sum_{i=0}^{\kappa} 0 \mathbb{1}_{[\tilde{\nu}^*(Y)=i]} + \sum_{i=\kappa+1}^m (i - \kappa) \mathbb{1}_{[\tilde{\nu}^*(Y)=i]} \\
&= \sum_{i=\kappa+1}^m (i - \kappa) \mathbb{1}_{[\tilde{\nu}^*(Y)=i]}.
\end{aligned} \tag{4.51}$$

Hence, we have that

$$\begin{aligned}
\mathbb{E}^\bullet[l_e(\kappa)] &= \mathbb{E}^\bullet \left[ \kappa + \sum_{i=0}^{\kappa} (i - \kappa) \mathbb{1}_{[\tilde{\nu}^*(Y)=i]} \right] \\
&= \kappa + \sum_{i=0}^{\kappa} (i - \kappa) \mathbb{P}^\bullet[\tilde{\nu}^*(Y) = i] \\
&= \kappa + \sum_{i=0}^{\kappa} (i - \kappa) \mathbb{E}^\bullet[\mathbb{P}^\bullet[\tilde{\nu}^*(Y) = i \mid \mathcal{G}]] \\
&= \mathbb{E}^\bullet \left[ \kappa + \sum_{i=0}^{\kappa} (i - \kappa) \mathbb{P}^\bullet[\tilde{\nu}^*(Y) = i \mid \mathcal{G}] \right]
\end{aligned} \tag{4.52}$$

and

$$\begin{aligned}
\mathbb{E}^\bullet[l_s(\kappa)] &= \mathbb{E}^\bullet \left[ \sum_{i=\kappa+1}^m (i - \kappa) \mathbb{1}_{[\tilde{\nu}^*(Y)=i]} \right] \\
&= \sum_{i=\kappa+1}^m (i - \kappa) \mathbb{P}^\bullet[\tilde{\nu}^*(Y) = i] \\
&= \sum_{i=\kappa+1}^m (i - \kappa) \mathbb{E}^\bullet[\mathbb{P}^\bullet[\tilde{\nu}^*(Y) = i \mid \mathcal{G}]] \\
&= \mathbb{E}^\bullet \left[ \sum_{i=\kappa+1}^m (i - \kappa) \mathbb{P}^\bullet[\tilde{\nu}^*(Y) = i \mid \mathcal{G}] \right].
\end{aligned} \tag{4.53}$$

□

These random variables have financial significance, specifically to collateralized debt obligations (CDOs). CDOs are split into tranches as described in Chapter 1. Suppose that a CDO we wish to study consists of  $m$  CDS contracts and assume that, for each contract, the loss  $\ell_j$  upon default is the same for all  $m$  contracts. Then, in determining tranche attachment and detachments points, we can simply refer to how many units were lost. Both  $l_e(\kappa)$  and  $l_s(\kappa)$  represent tranche losses in this scenario. The equity tranche loss with detachment point  $\kappa$  is given by  $l_e(\kappa)$ , and the corresponding senior tranche loss with attachment point  $\kappa + 1$  is given by  $l_s(\kappa)$ .

The main result of this subsection is that the expected change in tranche loss, for both the equity and senior tranches, changes with  $\alpha$  in the same direction that the average intensity changes with  $\alpha$ , i.e.,

$$\text{sgn}(\mathbb{E}^\bullet[\partial_\alpha l_e^*(\kappa)]) = \text{sgn}(\partial_\alpha \Lambda) = \text{sgn}(\mathbb{E}^\bullet[\partial_\alpha l_s^*(\kappa)]). \tag{4.54}$$

We will use the following notation when proving the above statement.

$$p_i = \mathbb{P}[\tilde{\nu}^*(Y) = i] = \sum_{\mathbf{y} \in \Delta(i)} \mathbb{P}[Y = \mathbf{y}]. \tag{4.55}$$

**Theorem 4.10.** Assume that  $n = 1$ . Let  $\alpha^*$  be a real number such that  $\partial_\alpha \Lambda$  exists at  $\alpha^*$ . If  $\text{sgn}(\partial_\alpha \Lambda(\alpha^*, \cdot))$  is constant almost surely, then we have the equality,

$$\text{sgn}(\mathbb{E}^\bullet[\partial_\alpha l_e^*(\kappa)](\alpha^*)) = \text{sgn}(\partial_\alpha \Lambda(\alpha^*)). \quad (4.56)$$

*Proof.* First observe that

$$\begin{aligned} \mathbb{E}^\bullet[\partial_\alpha l_e^*(\kappa)](\alpha^*) &= \mathbb{E}^\bullet \left[ \partial_\alpha \left( \kappa + \sum_{i=0}^{\kappa} (i - \kappa) \mathbb{P}^\bullet[\tilde{\nu}^*(Y) = i \mid \mathcal{G}] \right) (\alpha^*, \cdot) \right] \\ &= \mathbb{E}^\bullet \left[ \sum_{i=0}^{\kappa} (i - \kappa) \partial_\alpha (\mathbb{P}^\bullet[\tilde{\nu}^*(Y) = i \mid \mathcal{G}])(\alpha^*, \cdot) \right]. \end{aligned} \quad (4.57)$$

We have that

$$\begin{aligned} \mathbb{P}^\bullet[\tilde{\nu}^*(Y) = i \mid \mathcal{G}] &= \frac{\sum_{y \in \Delta(i)} \mathbb{P}[Y = y] \frac{\Lambda^i}{i!}}{\sum_{\zeta \in \mathcal{S}} \mathbb{P}[Y = \zeta] \frac{\Lambda^{\tilde{\nu}^*(\zeta)}}{\tilde{\nu}^*(\zeta)!}} \\ &= \frac{\frac{\Lambda^i}{i!} \sum_{y \in \Delta(i)} \mathbb{P}[Y = y]}{\sum_{q=0}^m \frac{\Lambda^q}{q!} \sum_{\zeta \in \Delta(q)} \mathbb{P}[Y = \zeta]} \\ &= \frac{p_i \frac{\Lambda^i}{i!}}{\sum_{q=0}^m p_q \frac{\Lambda^q}{q!}}, \end{aligned} \quad (4.58)$$

providing the partial derivative

$$\begin{aligned} \partial_\alpha (\mathbb{P}^\bullet[\tilde{\nu}^*(Y) = i \mid \mathcal{G}]) &= \partial_\alpha \left( \frac{p_i \frac{\Lambda^i}{i!}}{\sum_{q=0}^m p_q \frac{\Lambda^q}{q!}} \right) \\ &= \frac{h_i}{\left( \sum_{q=0}^m p_q \frac{\Lambda^q}{q!} \right)^2}, \end{aligned} \quad (4.59)$$

where

$$\begin{aligned} h_i &= i p_i \frac{\Lambda^{i-1}}{i!} \partial_\alpha \Lambda \left( \sum_{q=0}^m p_q \frac{\Lambda^q}{q!} \right) - p_i \frac{\Lambda^i}{i!} \left( \sum_{q=0}^m q p_q \frac{\Lambda^{q-1}}{q!} \partial_\alpha \Lambda \right) \\ &= \partial_\alpha \Lambda \sum_{q=0}^m p_i p_q \frac{\Lambda^{i+q-1}}{i! q!} (i - q). \end{aligned} \quad (4.60)$$

Thus,

$$\mathbb{E}^\bullet[\partial_\alpha l_e^*(\kappa)](\alpha^*) = \mathbb{E}^\bullet \left[ \frac{\partial_\alpha \Lambda(\alpha^*, \cdot)}{\left( \sum_{q=0}^m p_q \frac{\Lambda^q}{q!} \right)^2} \sum_{i=0}^{\kappa} \sum_{q=0}^m p_i p_q \frac{\Lambda^{i+q-1}}{i! q!} (i - \kappa)(i - q) \right], \quad (4.61)$$

and it remains to show the double sum is positive in order to achieve our desired result. We can rewrite the above double sum as

$$\sum_{i=0}^{\kappa} \sum_{q=0}^{\kappa} w_{i,q}(i-\kappa)(i-q) + \sum_{i=0}^{\kappa} \sum_{q=\kappa+1}^m w_{i,q}(i-\kappa)(i-q), \quad (4.62)$$

where

$$w_{i,q} = w_{q,i} = p_i p_q \frac{\Lambda^{i+q-1}}{i! q!} > 0. \quad (4.63)$$

Observe that  $(i-\kappa)(i-q) > 0$  for  $i < \kappa < q$  implying that

$$\sum_{i=0}^{\kappa} \sum_{q=\kappa+1}^m w_{i,q}(i-\kappa)(i-q) > 0. \quad (4.64)$$

Furthermore,

$$\begin{aligned} \sum_{i=0}^{\kappa} \sum_{q=0}^{\kappa} w_{i,q}(i-\kappa)(i-q) &= \sum_{i=0}^{\kappa} \sum_{0 \leq q < i} w_{i,q}(i-\kappa)(i-q) + w_{q,i}(q-\kappa)(q-i) \\ &= \sum_{i=0}^{\kappa} \sum_{0 \leq q < i} w_{i,q}(i-\kappa)(i-q) - w_{i,q}(q-\kappa)(i-q) \\ &= \sum_{i=0}^{\kappa} \sum_{0 \leq q < i} w_{i,q}(i-q)^2 \\ &> 0. \end{aligned} \quad (4.65)$$

Therefore, the double sum in Equation 4.61 is positive as well.  $\square$

**Theorem 4.11.** *Assume that  $n = 1$ . Let  $\alpha^*$  be a real number such that  $\partial_{\alpha}\Lambda$  exists at  $\alpha^*$ . If  $\text{sgn}(\partial_{\alpha}\Lambda(\alpha^*, \cdot))$  is constant almost surely, then we have the equality,*

$$\text{sgn}(\mathbb{E}^{\bullet}[\partial_{\alpha}l_s^*(\kappa)](\alpha^*)) = \text{sgn}(\partial_{\alpha}\Lambda(\alpha^*)). \quad (4.66)$$

*Proof.* Using the work from the proof of Theorem 4.10, it should be clear that

$$\mathbb{E}^{\bullet}[\partial_{\alpha}l_s^*(\kappa)](\alpha^*) = \mathbb{E}^{\bullet} \left[ \frac{\partial_{\alpha}\Lambda(\alpha^*, \cdot)}{\left( \sum_{q=0}^m p_q \frac{\Lambda^q}{q!} \right)^2} \sum_{i=\kappa+1}^m \sum_{q=0}^m p_i p_q \frac{\Lambda^{i+q-1}}{i! q!} (i-\kappa)(i-q) \right]. \quad (4.67)$$

As in that proof, this proof is complete once we show that the double sum, which we decompose into two new double sums, is positive. Notice, that

$$\sum_{i=\kappa+1}^m \sum_{q=0}^{\kappa} w_{i,q}(i-\kappa)(i-q) > 0 \quad (4.68)$$

since  $i > \kappa \geq q$ . Also, similar to the previous proof we have

$$\sum_{i=\kappa+1}^m \sum_{q=\kappa+1}^m w_{i,q}(i-\kappa)(i-q) = \sum_{i=\kappa+1}^m \sum_{i < q \leq m} w_{i,q}(i-q)^2 > 0. \quad (4.69)$$

Thus, the two previous equations imply that the double sum in Equation 4.67 is, in fact, positive.  $\square$

## 4.2 The Market Pricing Measure $\mathbb{P}^\dagger$

As seen in the previous section, when we use  $\mathbb{P}^\bullet$  as the market pricing measure, we are unable to calculate  $\partial_\alpha \mathbb{E}^\bullet[\cdot \cdot \cdot]$  by interchanging the operators, since  $\mathbb{P}^\bullet$  has  $\alpha$ -dependence. This section will illustrate that using  $\mathbb{P}^\dagger$  as our market pricing measure allows us to avoid this disadvantageous result. One previously mentioned advantage to using  $\mathbb{P}^\dagger$  is that it agrees with  $\mathbb{P}$  on  $\mathcal{G}$ . While this observation is itself noteworthy, it will also provide us with the ability to find the  $\alpha$ -derivative of an expected loss.

Recall that the conditional probabilities  $\mathbb{P}^\bullet[Y = y \mid \mathcal{G}]$  and  $\mathbb{P}^\dagger[Y = y \mid \mathcal{G}]$  are both equal to the random variable

$$\frac{\psi_y}{\sum_{\zeta \in \mathcal{S}} \psi_\zeta}. \quad (4.70)$$

We will use this fact, along with work from the previous section that did not rely upon  $\mathbb{P}^\bullet$ , to quickly provide much of the work in this section. The equality of the conditional probabilities allows us to observe that

$$L^* = \sum_{k=1}^n \sum_{j=1}^m \ell_j e^{-\int_{t_0}^{t_k} r(t) dt} \mathbb{P}^\dagger[D_{j,k} \mid \mathcal{G}], \quad (4.71)$$

where  $L^*$  was defined in Definition 4.1. We will consider  $L^*$  as a function of both  $\alpha$  and  $\omega$  as we did before. We now provide the present value of a portfolio's loss

$L$  under the market pricing measure  $\mathbb{P}^\dagger$ , where  $L$  was defined in Definition 4.1 as well. We also show that  $\partial_\alpha \mathbb{E}^\dagger[L]$  can be calculated as  $\mathbb{E}[\partial_\alpha L^*]$ . Note that we assume that there are  $n$  observation times, where  $n$  is not restricted to be equal to 1.

**Theorem 4.12.** *The present value of the loss is given by*

$$\mathbb{E}^\dagger[L] = \mathbb{E}^\dagger[L^*] = \sum_{k=1}^n \sum_{j=1}^m \ell_j \mathbb{E}^\dagger \left[ e^{-\int_{t_0}^{t_k} r(t) dt} \mathbb{P}^\dagger[D_{j,k} \mid \mathcal{G}] \right]. \quad (4.72)$$

Furthermore, let  $\alpha^*$  be a real number such that  $\partial_\alpha \Lambda_z$  exists at  $\alpha^*$ , for each  $z \in \{1, \dots, n\}$ . Suppose there exists both a random variable  $g$  with finite expectation and open interval  $U = (\alpha^* - \delta, \alpha^* + \delta)$  with  $\delta > 0$  such that  $\partial_\alpha L^*$  exists and

$$|\partial_\alpha L^*(u, \omega)| \leq g(\omega) \quad (4.73)$$

for all  $u \in U$  and  $\omega \in \Omega$ . Then the partial derivative  $\partial_\alpha \mathbb{E}^\dagger[L]$  at  $\alpha^*$  can be calculated as

$$\partial_\alpha \mathbb{E}^\dagger[L](\alpha^*) = \mathbb{E}[\partial_\alpha L^*](\alpha^*) = \sum_{k=1}^n \sum_{j=1}^m \ell_j \mathbb{E} \left[ e^{-\int_{t_0}^{t_k} r(t) dt} \partial_\alpha \mathbb{P}^\dagger[D_{j,k} \mid \mathcal{G}](\alpha^*, \cdot) \right], \quad (4.74)$$

where the derivative  $\partial_\alpha \mathbb{P}^\dagger[D_{j,k} \mid \mathcal{G}]$  is given by

$$\frac{\sum_{y \in \mathcal{S}_{j,k}} \sum_{\zeta \in \mathcal{S}} \psi_y \psi_\zeta \sum_{z=1}^n \frac{\partial_\alpha \Lambda_z}{\Lambda_z} (\tilde{\nu}_z^*(y) - \tilde{\nu}_z^*(\zeta))}{\left( \sum_{\zeta \in \mathcal{S}} \psi_\zeta \right)^2}. \quad (4.75)$$

*Proof.* Combining the equality

$$\mathbb{P}^\bullet[Y = y \mid \mathcal{G}] = \mathbb{P}^\dagger[Y = y \mid \mathcal{G}] \quad (4.76)$$

with the proof of Theorem 4.2 proves Equations 4.72 and 4.75. We observe that  $L^*$  is non-negative and  $\mathcal{G}$ -measurable, which provides

$$\mathbb{E}^\dagger[L^*] = \mathbb{E}[L^*], \quad (4.77)$$

according to Lemma 3.22. The probability measure  $\mathbb{P}$  is not dependent on  $\alpha$ , which allows us to attempt to change the order of integration and differentiation.

For all  $h$  such that  $0 < |h| < \delta$ , we see that  $\alpha^* + h \in U$  and there exists, using the Mean Value Theorem, some  $u_{h,\omega}$  between  $\alpha^*$  and  $\alpha^* + h$  such that

$$\frac{L^*(\alpha^* + h, \omega) - L^*(\alpha^*, \omega)}{h} = \partial_\alpha L^*(u_{h,\omega}, \omega), \quad (4.78)$$

where  $u_{h,\omega}$  may be  $\omega$ -dependent as implied by the subscript. Hence, we have the bound

$$\left| \frac{L^*(\alpha^* + h, \omega) - L^*(\alpha^*, \omega)}{h} \right| = |\partial_\alpha L^*(u_{h,\omega}, \omega)| \leq g(\omega), \quad (4.79)$$

for all  $h \in (-\delta, \delta)$ . Since  $\mathbb{E}[g] < \infty$ , we can apply dominated convergence to conclude that

$$\lim_{h \rightarrow 0} \mathbb{E} \left[ \frac{L^*(\alpha^* + h, \cdot) - L^*(\alpha^*, \cdot)}{h} \right] = \mathbb{E} \left[ \lim_{h \rightarrow 0} \frac{L^*(\alpha^* + h, \cdot) - L^*(\alpha^*, \cdot)}{h} \right]. \quad (4.80)$$

Therefore, we have the desired result,

$$\begin{aligned} \partial_\alpha \mathbb{E}^\dagger[L](\alpha^*) &= \partial_\alpha \mathbb{E}^\dagger[L^*](\alpha^*) \\ &= \partial_\alpha \mathbb{E}[L^*](\alpha^*) \\ &= \mathbb{E}[\partial_\alpha L^*](\alpha^*) \\ &= \sum_{k=1}^n \sum_{j=1}^m \ell_j \mathbb{E} \left[ e^{-\int_{t_0}^{t_k} r(t) dt} \partial_\alpha \mathbb{P}^\dagger[D_{j,k} \mid \mathcal{G}](\alpha^*, \cdot) \right], \end{aligned} \quad (4.81)$$

since  $\partial_\alpha r(t) = 0$ , where the first, second, and third equalities follow from Equations 4.72, 4.77, and 4.80, respectively.  $\square$

We note that the random variable  $\partial_\alpha L^*(\alpha^*, \cdot)$  is  $\mathcal{G}$ -measurable since it is the pointwise limit of the  $\mathcal{G}$ -measurable functions

$$\frac{L^*(\alpha^* + h, \cdot) - L^*(\alpha^*, \cdot)}{h}. \quad (4.82)$$

This observation implies that Equation 4.74 is still valid if we replace  $\mathbb{E}$  with  $\mathbb{E}^\dagger$ . Furthermore, we can see that monotone and dominated convergence arguments can be used to justify interchanging the operators  $\mathbb{E}^\dagger$  and  $\partial_\alpha$  for any non-negative  $\mathcal{G}$ -measurable function despite the general dependence of  $\mathbb{P}^\dagger$  on  $\alpha$ . This statement is true since the  $\dagger$ -probability of a  $\mathcal{G}$ -measurable set and the  $\dagger$ -expectation of a  $\mathcal{G}$ -measurable function are given by  $\mathbb{P}$  and  $\mathbb{E}$ , respectively, which are not dependent on  $\alpha$ .

We assume for the rest of this chapter that  $n = 1$ . We show that, for a homogeneous portfolio, the sign of  $\partial_\alpha \mathbb{E}^\dagger[L]$  at  $\alpha^*$  is equal to the sign of  $\partial_\alpha \Lambda$  at  $\alpha^*$ , assuming that  $\text{sgn}(\partial_\alpha \Lambda(\alpha^*, \cdot))$  is constant almost surely.

**Theorem 4.13.** *Assume that  $n = 1$  and that there are real numbers  $\rho \in (0, 1)$  and  $c$  such that  $\rho_j = \rho$  and  $c_{j,1} = c$  for all  $j \in \{1, \dots, m\}$ . Furthermore, let  $\alpha^*$  be a real number such that  $\partial_\alpha \Lambda$  exists at  $\alpha^*$ . Suppose there exists both a random variable  $g$  with finite expectation and open interval  $U = (\alpha^* - \delta, \alpha^* + \delta)$  with  $\delta > 0$  such that  $\partial_\alpha L^*$  exists and*

$$|\partial_\alpha L^*(u, \omega)| \leq g(\omega) \tag{4.83}$$

for all  $u \in U$  and  $\omega \in \Omega$ .

If  $\text{sgn}(\partial_\alpha \Lambda(\alpha^*, \cdot))$  is constant almost surely, then we have the equality,

$$\text{sgn}(\partial_\alpha \mathbb{E}^\dagger[L](\alpha^*)) = \text{sgn}(\partial_\alpha \Lambda(\alpha^*)). \tag{4.84}$$



*Proof.* Theorem 4.12 implies that

$$\begin{aligned}
\partial_\alpha \mathbb{E}^\dagger[L](\alpha^*) &= \mathbb{E}[\partial_\alpha L^*](\alpha^*) \\
&= \sum_{j=1}^m \mathbb{E} \left[ \ell_j e^{-\int_{t_0}^{t_1} r(t) dt} \partial_\alpha \mathbb{P}^\dagger[D_{j,1} \mid \mathcal{G}](\alpha^*, \cdot) \right] \\
&= \sum_{j=1}^m \mathbb{E} \left[ \partial_\alpha \Lambda(\alpha^*, \cdot) \frac{\ell_j e^{-\int_{t_0}^{t_1} r(t) dt}}{\Lambda\left(\sum_{\zeta \in \mathcal{S}} \psi_\zeta\right)^2} \sum_{y \in \mathcal{S}_j} \sum_{\zeta \in \mathcal{S}} \psi_y \psi_\zeta (\tilde{\nu}^*(y) - \tilde{\nu}^*(\zeta)) \right].
\end{aligned} \tag{4.85}$$

The proof is concluded by observing that the double sum above is no different than the double sum in Theorem 4.6 and, hence, is positive, due to the work there and Lemma 4.7.  $\square$

We also wish to reconsider tranche losses under  $\mathbb{P}^\dagger$ . The random variables  $l_e(\kappa)$ ,  $l_s(\kappa)$ ,  $l_e^*(\kappa)$ , and  $l_s^*(\kappa)$  were defined in Definition 4.8. We observe that

$$\begin{aligned}
l_e^*(\kappa) &= \kappa + \sum_{i=0}^{\kappa} (i - \kappa) \mathbb{P}^\dagger[\tilde{\nu}^*(Y) = i \mid \mathcal{G}] \quad \text{and} \\
l_s^*(\kappa) &= \sum_{i=\kappa+1}^m (i - \kappa) \mathbb{P}^\dagger[\tilde{\nu}^*(Y) = i \mid \mathcal{G}],
\end{aligned} \tag{4.86}$$

for each  $\kappa \in \{0, 1, \dots, m\}$ . We next provide a corollary that follows immediately from the proof of Lemma 4.9. All remarks in that lemma are still true if  $\bullet$  is replaced by  $\dagger$ . In financial terms, this corollary states the expected number of losses for senior and equity tranches. We have that the expected losses incurred by the equity tranche with detachment point  $\kappa$  can be calculated using  $\mathbb{E}^\dagger[l_e^*(\kappa)]$ . Also,  $\mathbb{E}^\dagger[l_s^*(\kappa)]$  can be used to find the expected losses incurred by the senior tranche with attachment point  $\kappa + 1$ .

**Corollary 4.14.** *Assume  $n = 1$ . We have the equalities*

$$\begin{aligned}\mathbb{E}^\dagger[l_e(\kappa)] &= \mathbb{E}^\dagger[l_e^*(\kappa)] = \mathbb{E}^\dagger\left[\kappa + \sum_{i=0}^{\kappa} (i - \kappa) \mathbb{P}^\dagger[\tilde{\nu}^*(Y) = i \mid \mathcal{G}]\right] \quad \text{and} \\ \mathbb{E}^\dagger[l_s(\kappa)] &= \mathbb{E}^\dagger[l_s^*(\kappa)] = \mathbb{E}^\dagger\left[\sum_{i=\kappa+1}^m (i - \kappa) \mathbb{P}^\dagger[\tilde{\nu}^*(Y) = i \mid \mathcal{G}]\right].\end{aligned}\tag{4.87}$$

We conclude the section by proving that both senior and equity tranches are short intensity. That is, as the average intensity increases, the expected loss increases as well. A CDO with CDS contracts as the underlying assets is called a *synthetic* CDO. Thus, an investor in a senior or equity tranche of a synthetic CDO, with a locked-in spread, would receive lower spread payments on lower than expected outstanding notational and has to make higher than expected loss payments.

**Theorem 4.15.** *Assume that  $n = 1$ . Let  $\alpha^*$  be a real number such that  $\partial_\alpha \Lambda$  exists at  $\alpha^*$ . Suppose there exists both a random variable  $g$  with finite expectation and open interval  $U = (\alpha^* - \delta, \alpha^* + \delta)$  with  $\delta > 0$  such that  $\partial_\alpha l_e^*(\kappa)$  and  $\partial_\alpha l_s^*(\kappa)$  exist and*

$$\begin{aligned}|\partial_\alpha l_e^*(\kappa)(u, \omega)| &\leq g(\omega) \\ |\partial_\alpha l_s^*(\kappa)(u, \omega)| &\leq g(\omega)\end{aligned}\tag{4.88}$$

for all  $u \in U$  and  $\omega \in \Omega$ .

If  $\text{sgn}(\partial_\alpha \Lambda(\alpha^*, \cdot))$  is constant almost surely, then we have that

$$\text{sgn}(\partial_\alpha \mathbb{E}^\dagger[l_e(\kappa)](\alpha^*)) = \text{sgn}(\partial_\alpha \Lambda(\alpha^*)) = \text{sgn}(\partial_\alpha \mathbb{E}^\dagger[l_s(\kappa)](\alpha^*)).\tag{4.89}$$

*Proof.* Using Corollary 4.14, we obtain the equalities

$$\begin{aligned}\mathbb{E}^\dagger[l_e(\kappa)] &= \mathbb{E}[l_e^*(\kappa)] \quad \text{and} \\ \mathbb{E}^\dagger[l_s(\kappa)] &= \mathbb{E}[l_s^*(\kappa)],\end{aligned}\tag{4.90}$$

since  $\mathbb{P}^\dagger[\tilde{\nu}^*(Y) = i \mid \mathcal{G}]$  is by definition  $\mathcal{G}$ -measurable. A similar dominating convergence argument to the one we used in Theorem 4.12 provides that

$$\begin{aligned}\partial_\alpha \mathbb{E}^\dagger[l_e(\kappa)] &= \partial_\alpha \mathbb{E}[l_e^*(\kappa)] \\ &= \mathbb{E}[\partial_\alpha l_e^*(\kappa)]\end{aligned}\tag{4.91}$$

and

$$\begin{aligned}\partial_\alpha \mathbb{E}^\dagger[l_e(\kappa)] &= \partial_\alpha \mathbb{E}[l_e^*(\kappa)] \\ &= \mathbb{E}[\partial_\alpha l_e^*(\kappa)].\end{aligned}\tag{4.92}$$

Using the work from Theorem 4.10, we have that

$$\mathbb{E}[\partial_\alpha l_e^*(\kappa)](\alpha^*) = \mathbb{E}\left[\frac{\partial_\alpha \Lambda(\alpha^*, \cdot)}{\left(\sum_{q=0}^m p_q \frac{\Lambda^q}{q!}\right)^2} \sum_{i=0}^{\kappa} \sum_{q=0}^m p_i p_q \frac{\Lambda^{i+q-1}}{i! q!} (i - \kappa)(i - q)\right]\tag{4.93}$$

and

$$\mathbb{E}[\partial_\alpha l_s^*(\kappa)](\alpha^*) = \mathbb{E}\left[\frac{\partial_\alpha \Lambda(\alpha^*, \cdot)}{\left(\sum_{q=0}^m p_q \frac{\Lambda^q}{q!}\right)^2} \sum_{i=\kappa+1}^m \sum_{q=0}^m p_i p_q \frac{\Lambda^{i+q-1}}{i! q!} (i - \kappa)(i - q)\right].\tag{4.94}$$

We previously proved both of these double sum to be positive in Theorems 4.10 and 4.11, respectively. This observation provides the desired result.  $\square$

# Chapter 5

## Intensity and Interest Rate Models

In order to use the Gaussian-Poisson model to investigate default behavior, one must choose a model for its intensity process,  $\Lambda(t)$ . Thus far, we have specified only that  $\Lambda(t)$  is dependent on the short rate  $r(t)$ . This chapter is designed to discuss potential models for both  $\Lambda$  and  $r$ . The first section will present an intensity process which is designed to function primarily as a mathematical example. We will then proceed to discuss popular models of the short rate in the second section and present potential models for  $\Lambda(t)$  in the third section. Before we begin this chapter, we define  $B(t)$  as a Brownian motion.

**Definition 5.1.** *Define  $B(t, \omega)$  as a stochastic process such that:*

1.  $\mathbb{P}[\{\omega \mid B(0, \omega) = 0\}] = 1$ .
2. *For all  $0 \leq s \leq t$ , the random variable  $B(t) - B(s)$  is Gaussian with mean 0 and variance  $t - s$ .*
3. *For any  $0 \leq s_1 < s_2 < \dots < s_k$ , the random variables*

$$B(s_1), B(s_2) - B(s_1), \dots, B(s_k) - B(s_{k-1}) \tag{5.1}$$

*are independent.*

4.  $\mathbb{P}[\{\omega \mid B(\cdot, \omega) \text{ is continuous}\}] = 1$ .

### 5.1 An Example

Interest rate models ideally would not permit the short rate to be negative. We would like to assume, minimally, a positive interest rate at time  $t = 0$ . Thus, we

shall assume  $r(0) > 0$ . Define

$$r(t) = r(0) + B(t); \quad (5.2)$$

i.e., assume  $r(t) - r(0)$  is a Brownian motion. Furthermore, define

$$\lambda(t) = \alpha r(t) = \lambda(0) + \alpha B(t) \quad (5.3)$$

where  $\alpha$  is a positive constant, implying that  $\lambda(0) > 0$ . Thus, we are assuming that the instantaneous interest rate and instantaneous intensity are directly proportional. We can calculate the desired intensity process  $\Lambda(t)$  as

$$\begin{aligned} \Lambda(t) &= \int_0^t \lambda(u) \, du \\ &= \int_0^t \lambda(0) \, du + \alpha \int_0^t B(u) \, du \\ &= \lambda(0)t + \alpha \int_0^t B(u) \, du. \end{aligned} \quad (5.4)$$

The random variable

$$\left( \int_0^t B(u) \, du \right) (\omega) \quad (5.5)$$

is defined using Riemann-Stieltjes integration on each sample path. Thus, we can use integration by parts to obtain

$$\begin{aligned} \int_0^t B(u) \, du &= B(u)(u-t)|_0^t - \int_0^t (u-t)dB(u) \\ &= \int_0^t (t-u)dB(u) \end{aligned} \quad (5.6)$$

since  $(u-t)' \, du = du$ . The equality of the random variables is given with probability 1 since  $B(0) = 0$  almost surely. The integral on the right side is a Wiener integral since  $(t-u)$  is deterministic. The Wiener integral of a function  $f \in L^2[0, t]$  is a Gaussian random variable with mean 0 and variance

$$\|f\|^2 = \int_0^t f(u)^2 \, du. \quad (5.7)$$

Observe that  $t - u \in L^2[0, t]$  with

$$\begin{aligned} \int_0^t (t - u)^2 du &= \int_0^t w^2 dw \\ &= \frac{t^3}{3} \end{aligned} \tag{5.8}$$

where  $w = t - u$ . Thus, using Equation 5.4, we can see that, for each  $t > 0$ ,  $\Lambda(t)$  is a Gaussian random variable with

$$\Lambda(t) \sim \lambda(0)t + \alpha \mathcal{N}\left(0, \frac{t^3}{3}\right) \sim \mathcal{N}\left(\alpha r(0)t, \alpha^2 \frac{t^3}{3}\right). \tag{5.9}$$

Furthermore, recalling  $\Lambda_k := \Lambda(t_k) - \Lambda(t_{k-1})$  and  $\tilde{N}_k := N(t_k) - N(t_{k-1})$  where  $N(t)$  is our Cox process, we have that

$$\begin{aligned} \mathbb{E}[\tilde{N}_k] &= \mathbb{E}\left[\mathbb{E}[\tilde{N}_k \mid \mathcal{F}_{t_k}^*]\right] \\ &= \mathbb{E}[\Lambda_k] \\ &= \lambda(0)(t_k - t_{k-1}), \end{aligned} \tag{5.10}$$

where  $\mathcal{F}_{t_k}^*$  is the sigma-algebra generated by  $B(u)$  for  $u \in [0, t_k]$ .

Mathematically, this example is certainly intriguing. We assumed that  $r(t) - r(0)$  was a Brownian motion with a positive initial short rate and that the instantaneous intensity was simply a multiple of that instantaneous interest rate. These assumptions allowed us to make two mathematically nice conclusions. First, the aggregate intensity over the time interval  $[0, t]$  is distributed normally with expectation and variance dependent on the time  $t$ . Second, we were able to calculate the expected number of defaults in a time interval. Since  $N(t)$  is a Cox process, we know that the expectation of  $N(t) - N(s)$  conditional on  $\mathcal{F}_t^*$  is given using the stochastic process as  $\Lambda(t) - \Lambda(s)$ . Thus, we must know the expectation of  $\Lambda(t) - \Lambda(s)$  in order to find the unconditional expected number of defaults. Furthermore, in this example, we see that this expectation is simply the length of the time interval times the initial instantaneous intensity  $\lambda(0)$ .

This example has deficiencies as a potential financial model. First, the conclusion that the expected number of defaults in an interval is dependent only upon the length of the interval and the initial intensity is arguably too simplistic to be useful. Second, while we have defined the initial short rate as positive,  $r(t)$  may become negative later in time. Observe that  $r(t) \sim r(0) + \sqrt{t}Z$  where  $Z$  is the standard normal distribution. Hence, the probability that  $r(t)$  is negative is given by

$$\mathbb{P}\left[r(0) + \sqrt{t}Z < 0\right] = \Phi\left(-\frac{r(0)}{\sqrt{t}}\right). \quad (5.11)$$

This probability is always less than 0.5 but does approach that value as  $t \rightarrow \infty$ . The more significant problem is that a negative interest rate for too long a time causes the intensity  $\Lambda(t)$  to become negative as well. In fact, the probability of the event  $\Lambda(t) < 0$  occurring is not dependent on  $\alpha$  and is given as

$$\mathbb{P}\left[\alpha r(0)t + \alpha\sqrt{\frac{t^3}{3}}Z < 0\right] = \Phi\left(-\sqrt{3}\frac{r(0)}{\sqrt{t}}\right) \quad (5.12)$$

implying that, as  $t \rightarrow \infty$ , the probability approaches 0.5 as well. We could attempt to fix the problem of a negative intensity by redefining  $\alpha$  as a function  $\alpha(t)$  which equals a positive constant when  $r(t) \geq 0$  and 0 when  $r(t)$  is negative. Unfortunately,  $r(t)$  remains negative which is often an undesirable characteristic for a model of the short rate.

## 5.2 Short Rate Models

In the previous section, we decided to model  $r(t)$  as a Brownian motion beginning at  $r(0) > 0$ . For reasons, both discussed and not discussed in the previous section, this assumption is considered to be too simplistic. Two properties that an ideal model of the short rate would possess are mean reversion and  $r(t) > 0$  for all  $t$ . Modeling the interest rate as Brownian motion provides neither. This section will provide a few common models of the short rate that each possess at least one of these properties. These models are explored extensively in texts such as Hull's [21].

In order to discuss these short rate models, we need to both define an Itô process and provide the well-known Itô's lemma. We set  $\{\mathcal{F}_t : 0 \leq t \leq T\}$  as a filtration which meets two conditions:

1.  $B(t)$  is  $\mathcal{F}_t$ -measurable
2.  $B(t) - B(s)$  is independent of the sigma-algebra  $\mathcal{F}_s$ , for all  $s \leq t$ .

**Definition 5.2.** An  $\{\mathcal{F}_t\}$ -adapted process is a stochastic process  $X(t, \omega) : [0, T] \times \Omega \rightarrow \mathbb{R}$  such that  $X(t)$  is  $\mathcal{F}_t$ -measurable for each  $t \in [0, T]$ .

**Definition 5.3.** An Itô process is a stochastic process  $X(t, \omega) : [0, T] \times \Omega \rightarrow \mathbb{R}$  which can be written in the form

$$X(t) = X(0) + \int_0^t \mu(u) du + \int_0^t \sigma(u) dB(u), \quad (5.13)$$

for  $0 \leq t \leq T$ , where  $X(0)$  is  $\mathcal{F}_0$ -measurable and  $\mu$  and  $\sigma$  are  $\{\mathcal{F}_t\}$ -adapted processes such that

$$\int_0^T |\mu(u)| + \sigma(u)^2 du < \infty. \quad (5.14)$$

almost surely.

The integral with  $B(u)$  as the integrator is to be interpreted as an Itô integral since  $\sigma(u)$  is stochastic. Note that an Itô process  $X(t)$  is an  $\{\mathcal{F}_t\}$ -adapted process. We will use the shorthand notation,

$$dX(t) = \mu(t) dt + \sigma(t) dB(t), \quad (5.15)$$

to mean an Itô process  $X(t)$  such that Equation 5.13 is satisfied. An equation such as (5.15) above is called a *stochastic differential equation* (SDE). We shall present Itô's lemma using this notation. Recall that we use the notation  $\partial_x$  for  $\frac{\partial}{\partial x}$ , and we will also use  $\partial_x^2$  to denote  $\frac{\partial^2}{\partial x^2}$ .



**Lemma 5.4** (Itô's Lemma). *Let  $X(t)$  be an Itô process such that*

$$dX(t) = \mu(t) dt + \sigma(t) dB(t), \quad (5.16)$$

*Suppose  $h(t, x)$  is a continuous function with continuous partial derivatives  $\partial_t h$ ,  $\partial_x h$ ,  $\partial_x^2 h$  and define  $Y(t) = h(t, X(t))$ . Then,  $Y(t)$  is an Itô process and*

$$dY(t) = \left( \partial_t h + \mu(t) \partial_x h + \frac{\sigma(t)^2}{2} \partial_x^2 h \right) dt + \sigma(t) \partial_x h dB(t), \quad (5.17)$$

*where  $h$  is evaluated at  $(t, X(t))$  in each instance above, which we have omitted for conciseness.*

We shall refer to the “formula” in the lemma as Itô's formula. We will provide our short rate models below as Itô processes using the SDE notation. There are two types of short rate models, equilibrium and no-arbitrage. We shall focus primarily on the former with a couple brief mentions of the latter. As both types of models have advantages, this decision is primarily due to the fact that equilibrium models can be discussed using less financial terminology.

### 5.2.1 The Rendleman-Bartter Model

The Rendleman-Bartter model [40] assumes that  $r(t)$  is described by the SDE

$$dr(t) = \mu r(t) dt + \sigma r(t) dB(t), \quad (5.18)$$

where  $\mu$ ,  $\sigma$ , and the initial value  $r(0) > 0$  are constants. A process  $r$  of this form is said to follow geometric Brownian motion. We will provide a solution, which can be found in many texts such as Øksendal's [38], to the SDE. Uniqueness and existence results for SDEs are discussed in-depth in that text as well.

**Lemma 5.5.** *Let  $r(0)$  be a positive constant. The stochastic differential equation*

$$dr(t) = \mu r(t) dt + \sigma r(t) dB(t), \quad (5.19)$$

with initial value  $r(0)$ , has the solution

$$r(t) = r(0)e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B(t)}. \quad (5.20)$$

*Proof.* Define

$$x(t) = \log(r(0)) + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B(t), \quad (5.21)$$

for  $t \geq 0$ . Then  $x(0) = \log(r(0))$  almost surely, and we write  $x(t)$  in SDE form as

$$dx(t) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dB(t). \quad (5.22)$$

We observe that  $e^{x(t)}$  is the stochastic process  $r(t)$  defined in Equation 5.20. It remains to show that  $e^{x(t)}$  satisfies Equation 5.19. Let  $h(t, x) = e^x$  and note that  $h$  meets the criteria from Itô's lemma. Therefore, we can calculate

$$\begin{aligned} de^{x(t)} &= dh(t, x(t)) \\ &= \left( \left(\mu - \frac{\sigma^2}{2}\right)e^{x(t)} + \frac{\sigma^2}{2}e^{x(t)} \right) dt + \sigma dB(t) \\ &= \mu e^{x(t)} dt + \sigma e^{x(t)} dB(t), \end{aligned} \quad (5.23)$$

using Itô's formula, and thereby obtain the desired result.  $\square$

Using Equation 5.21, we can see that  $x(t)$  is normally distributed with parameters

$$\mathbb{E}[x(t)] = \log(r(0)) + \left(\mu - \frac{\sigma^2}{2}\right)t \quad \text{and} \quad \text{Var}[x(t)] = \sigma^2 t, \quad (5.24)$$

for each  $t$ . Hence  $r(t)$  follows a log-normal distribution with

$$\mathbb{E}[r(t)] = r(0)e^{\mu t} \quad \text{and} \quad \text{Var}[r(t)] = r(0)^2 e^{2\mu t} (e^{\sigma^2 t} - 1), \quad (5.25)$$

since  $r(0)$  is deterministic. Furthermore, unlike the example short rate in the first section,  $r(t)$  is positive for all  $t$ , since both  $r(0)$  and the exponential function are positive. Thus, geometric Brownian motion models appear to be an improvement over the Brownian motion example.

Stock prices are usually assumed to follow geometric Brownian motion. For instance, Black and Scholes used this assumption in their model to provide the famous Black-Scholes formula for stock options pricing [7]. However, a good model for stock prices is not necessarily a good model for interest rates. One must consider if this model has desired properties that interest rates have been shown to display.

A significant difference between the stock prices and interest rates is that interest rates usually exhibit mean reversion. Mean reversion is the tendency to be pulled back to some long-term average. An argument can be made that an interest model should have this characteristic. When interest rates are high, demand for loans is lower than normal causing a subsequent decrease in the interest rate to encourage borrowing. When interest rates are low, demand for loans is higher than normal causing a subsequent increase in the interest rate to take advantage of the high demand. The short rate  $r$  in this model does not incorporate mean reversion. In fact, as  $t \rightarrow \infty$ ,  $r(t)$  tends to either 0 or  $\infty$  depending on the sign of  $\mu - \frac{\sigma^2}{2}$ ; if  $\mu - \frac{\sigma^2}{2} = 0$  the behavior is erratic. While this model does have some advantages, such as the inability of  $r$  to become negative, its lack of mean reversion is considered to be a significant drawback.

### 5.2.2 The Vasicek Model

The Vasicek model [49] does incorporate mean reversion. It assumes that the short rate is an Ornstein-Uhlenbeck process [48]. That is,  $r(t)$  is an Itô process such that

$$dr(t) = a(\mu - r(t)) dt + \sigma dB(t), \quad (5.26)$$

where  $a, \mu, \sigma$ , and the initial value  $r(0)$  are positive constants. The constant  $\mu$  is the long term mean to which  $r$  is reverting. If  $\mu > r(t)$ , the drift is positive and  $r$  will move toward  $\mu$ , and if  $\mu < r(t)$ , the drift is negative and  $r$  will again move

toward  $\mu$ . The rate at which  $r$  reverts to  $\mu$  is  $a$ . As before, we can solve the SDE using Itô's formula. However, in order to find this solution, we will require an “integrating factor.”

**Lemma 5.6.** *Let  $r(0)$  be a positive constant. The stochastic differential equation*

$$dr(t) = a(\mu - r(t)) dt + \sigma dB(t), \quad (5.27)$$

*with initial value  $r(0)$ , has the solution*

$$r(t) = e^{-at}r(0) + \mu(1 - e^{-at}) + \int_0^t \sigma e^{a(u-t)} dB(u) \quad (5.28)$$

*Proof.* Define

$$x(t) = r(0) + \mu(e^{at} - 1) + \int_0^t \sigma e^{au} dB(u), \quad (5.29)$$

for  $t \geq 0$  and observe that

$$r(t) = e^{-at}x(t). \quad (5.30)$$

We see that  $x(0) = r(0)$  and

$$\mu(e^{at} - 1) = \int_0^t a\mu e^{au} du \quad (5.31)$$

Hence, we can write  $x(t)$  in SDE form as

$$dx(t) = a\mu e^{at} dt + \sigma e^{at} dB(t). \quad (5.32)$$

Let  $h(t, x) = e^{-at}x$ . We conclude the proof by using Itô's formula to show that  $e^{-at}x(t)$  does satisfy Equation 5.27:

$$\begin{aligned} d(e^{-at}x(t)) &= (-ae^{-at}x(t) + a\mu e^{at}e^{-at}) dt + \sigma e^{at}e^{-at} dB(t) \\ &= a(\mu - e^{-at}x(t)) dt + \sigma dB(t). \end{aligned} \quad (5.33)$$

□

The Wiener integral in the above lemma is Gaussian with mean 0 and variance

$$\int_0^t \sigma^2 e^{2a(u-t)} du = \frac{\sigma^2}{2a} (1 - e^{-2at}). \quad (5.34)$$

Hence, in the Vasicek model,  $r(t)$  is distributed as

$$\mathcal{N}\left(e^{-at}r(0) + \mu(1 - e^{-at}), \frac{\sigma^2}{2a}(1 - e^{-2at})\right), \quad (5.35)$$

with the mean moving from  $r(0)$  to  $\mu$  at the rate  $a$ . As  $t$  becomes infinitely large, the distribution of  $r(t)$  goes to

$$\mathcal{N}\left(\mu, \frac{\sigma^2}{2a}\right); \quad (5.36)$$

i.e.,  $\mu$  is the long term mean. Unfortunately, since,  $r(t)$  is Gaussian we have a positive probability that  $r$  can become negative. This probability is lower with a small volatility  $\sigma$  or a fast (large) reversion rate  $a$ . While the Vasicek model improves upon the previous model by incorporating mean reversion, the trade-off is the possibility of negative interest rates especially if volatility is high or reversion is slow. Hence, one might wish to use it to generate the Gaussian-Poisson model's intensity only in certain conditions. Still, one would have to incorporate into the intensity model a method to not allow  $\lambda(t)$  to become negative; as while negative interest is undesirable, a negative intensity has no meaning.

### 5.2.3 Additional Interest Rate Models

The Cox-Ingersoll-Ross (CIR) model [13] of the short rate satisfies the SDE

$$dr(t) = a(\mu - r(t)) dt + \sigma\sqrt{r(t)} dB(t) \quad (5.37)$$

where  $a, \mu, \sigma$ , and the initial value  $r(0)$  are positive constants. Just as in the Vasicek model,  $r$  reverts back to the long-term mean  $\mu$  at rate  $a$ . Unlike the Vasicek model,  $r(t)$  is, almost surely, non-negative for all  $t$  and strictly positive if  $2a\mu \geq \sigma^2$ ; see [5] for an in depth discussion of this property. Informally, if  $r$  moves toward 0, then

not only does the drift “pull”  $r$  away from zero toward  $\mu$ , but the diffusion term tends to zero. The CIR model is not Gaussian and, hence, is harder to analyze. In fact, it does not have a general closed-form solution. However, we can provide both the expectation and variance of  $r(t)$ .

**Lemma 5.7.** *In the CIR model, the short rate  $r(t)$  has parameters*

$$\begin{aligned}\mathbb{E}[r(t)] &= e^{-at}r(0) + \mu(1 - e^{-at}) \text{ and} \\ \text{Var}[r(t)] &= r(0)\frac{\sigma^2}{a}(e^{-at} - e^{-2at}) + \mu\frac{\sigma^2}{2a}(1 - e^{-at})^2\end{aligned}\tag{5.38}$$

for all  $t$ .

*Proof.* Similar to the Vasicek model, using Itô’s formula on  $e^{at}r(t)$  provides

$$r(t) = e^{-at}r(0) + \mu(1 - e^{-at}) + \int_0^t \sigma e^{a(u-t)} \sqrt{r(u)} dB(u).\tag{5.39}$$

Since the expectation of an Itô integral is 0, we have

$$\mathbb{E}[r(t)] = e^{-at}r(0) + \mu(1 - e^{-at}).\tag{5.40}$$

The variance of  $r(t)$  is equal to the variance of the Wiener integral, which is

$$\mathbb{E}\left[\left(\int_0^t \sigma e^{a(u-t)} \sqrt{r(u)} dB(u)\right)^2\right] = \mathbb{E}\left[\int_0^t \sigma^2 e^{2a(u-t)} r(u) du\right]\tag{5.41}$$

by Itô’s isometry. Furthermore,

$$\mathbb{E}\left[\int_0^t \sigma^2 e^{2a(u-t)} r(u) du\right] = \int_0^t \sigma^2 e^{2a(u-t)} \mathbb{E}[r(u)] du\tag{5.42}$$

since the inside of the integral is non-negative and the right side of the equation is finite as we will see below. Using the expectation we have already calculated, we have that

$$\begin{aligned}\text{Var}[r(t)] &= \sigma^2 e^{-2at} \int_0^t e^{2au} (e^{-au}r(0) + \mu(1 - e^{-au})) du \\ &= \sigma^2 e^{-2at} \int_0^t (r(0) - \mu)e^{au} + \mu e^{2au} du \\ &= \frac{\sigma^2(r(0) - \mu)}{a}(e^{-at} - e^{-2at}) + \frac{\sigma^2\mu}{2a}(1 - e^{-2at}) \\ &= r(0)\frac{\sigma^2}{a}(e^{-at} - e^{-2at}) + \mu\frac{\sigma^2}{2a}(1 - e^{-at})^2.\end{aligned}\tag{5.43}$$

□

All of the previous models in this section have been equilibrium models. The difference between these models and no-arbitrage models is that today's yield curve is an input for no-arbitrage models as opposed to an output for equilibrium models. Thus, equilibrium models will not necessarily fit current interest rates. The significance of this drawback is dependent on what one is trying to model. No-arbitrage models are able to specifically match today's yield curve by including a time-dependent function in the drift term of an Itô process.

Two examples of no-arbitrage models of the short rate are the one-factor Hull-White model and the Black-Karasinski model. The Hull-White model [22] is an extension of the Vasicek model. It is given by the SDE

$$dr(t) = a \left( \frac{\mu(t)}{a} - r(t) \right) dt + \sigma dB(t), \quad (5.44)$$

where  $\mu(t)$  is determined using the initial yield curve. The value to which the mean is reverting,  $\frac{\mu(t)}{a}$ , is time dependent. This model has the same drawback of the Vasicek model;  $r(t)$  can become negative.

The Black-Karasinski model [6] of the short rate requires  $r(t)$  to be positive and exhibits mean reversion. The SDE which defines this model describes the behavior of  $\log(r(t))$  as

$$d(\log(r(t))) = a(t) \left( \frac{\mu(t)}{a(t)} - \log(r(t)) \right) dt + \sigma(t) dB(t), \quad (5.45)$$

where often both  $a(t)$  and  $\sigma(t)$  are assumed to be constant rather than time dependent. Its disadvantage is a lack of analytical tractability. For instance, in all previous models we have discussed, bond prices can be provided in closed form using the model's parameters. However, the Black-Karasinski model does not have this property.

There is rarely a “perfect” model in any field; the study of interest rates is not an exception to that rule. As a model becomes more realistic, it necessarily becomes more complex. Complex models often are more difficult to analyze and simulate. One must decide how to properly balance complexity and realism by considering the desired application of the model.

### 5.3 The Intensity Process

This section will consider potential models of the intensity process. Our examination will utilize the two families of functions below and their properties.

**Definition 5.8.** *For any  $\alpha > 0$  and  $\beta > 1$ , define*

$$f_{\alpha,\beta}(x) = \frac{\beta}{\alpha} \left(\frac{x}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x}{\alpha}\right)^\beta} \quad (5.46)$$

and

$$F_{\alpha,\beta}(x) = 1 - e^{-\left(\frac{x}{\alpha}\right)^\beta}, \quad (5.47)$$

for  $x \in [0, \infty)$ , and set both functions equal to 0 when  $x \in (-\infty, 0)$ .

A random variable that follows the Weibull distribution with parameters  $\alpha$  (scale) and  $\beta$  (shape) has the above functions as its probability density function and cumulative distribution function, respectively. We have the relationship

$$F'_{\alpha,\beta}(x) = f_{\alpha,\beta}(x). \quad (5.48)$$

These functions have certain properties of which we will make use. For instance, although the parameter  $\beta$  is normally only required to be positive, we require it to be larger than 1 because of the shape it provides the two functions. Since  $F_{\alpha,\beta}$  is a cumulative distribution function with density  $f_{\alpha,\beta}$ , which is strictly positive for all  $x \in (0, \infty)$ , then  $0 < F_{\alpha,\beta}(x) < 1$  for all  $x \in (0, \infty)$ ,  $F_{\alpha,\beta}(0) = 0$ , and

$$\lim_{x \rightarrow \infty} F_{\alpha,\beta}(x) = 1.$$

We also prove the following useful lemma.



**Lemma 5.9.** *The point*

$$x_{\alpha,\beta} = \alpha \sqrt[\beta]{\frac{\beta-1}{\beta}} \quad (5.49)$$

*is the only inflection point of  $F_{\alpha,\beta}(x)$  and the only critical point of  $f_{\alpha,\beta}(x)$  in the interval  $(0, \infty)$ . The function  $F_{\alpha,\beta}$  is convex on the interval  $(0, x_{\alpha,\beta})$  and concave on the interval  $(x_{\alpha,\beta}, \infty)$ . The function  $f_{\alpha,\beta}$  attains its global maximum value at  $x_{\alpha,\beta}$ .*

*Proof.* Since  $F'_{\alpha,\beta} = f_{\alpha,\beta}$ , then  $F''_{\alpha,\beta} = f'_{\alpha,\beta}$ , and the inflection points of  $F_{\alpha,\beta}$  will be the critical points of  $f_{\alpha,\beta}$ . We have

$$\begin{aligned} f'_{\alpha,\beta}(x) &= \frac{\beta}{\alpha} \left( \frac{\beta-1}{\alpha} \left( \frac{x}{\alpha} \right)^{\beta-2} e^{-\left(\frac{x}{\alpha}\right)^\beta} - \frac{\beta}{\alpha} \left( \frac{x}{\alpha} \right)^{\beta-1} e^{-\left(\frac{x}{\alpha}\right)^\beta} \left( \frac{x}{\alpha} \right)^{\beta-1} \right) \\ &= \frac{\beta}{\alpha^2} \left( \frac{x}{\alpha} \right)^{\beta-2} e^{-\left(\frac{x}{\alpha}\right)^\beta} \left( \beta-1 - \beta \left( \frac{x}{\alpha} \right)^\beta \right) \end{aligned} \quad (5.50)$$

For  $x > 0$ , we have  $f'_{\alpha,\beta}(x) = 0$  when

$$\beta-1 - \beta \left( \frac{x}{\alpha} \right)^\beta = 0 \quad (5.51)$$

which only occurs at  $x = x_{\alpha,\beta}$ . Furthermore, we can see that  $f'_{\alpha,\beta}(x)$  is positive when  $x < x_{\alpha,\beta}$  and negative when  $x > x_{\alpha,\beta}$  since

$$\frac{\beta}{\alpha^2} \left( \frac{x}{\alpha} \right)^{\beta-2} e^{-\left(\frac{x}{\alpha}\right)^\beta} > 0. \quad (5.52)$$

Therefore, at  $x_{\alpha,\beta}$ ,  $F_{\alpha,\beta}$  changes from convex to concave and  $f_{\alpha,\beta}$  has a local maximum. The global maximum of  $f_{\alpha,\beta}$  occurs at  $x_{\alpha,\beta}$  since the function has no other local extrema and  $f_{\alpha,\beta}(0) = 0$ .  $\square$

Requiring  $\beta > 1$  rather than simply  $\beta > 0$  is what guarantees the inflection/critical point. This inflection/critical point will prove to be useful for our models and is why the requirement was made. Before presenting our first potential model for the intensity process, we need another definition.

**Definition 5.10.** Let  $r(t)$  be a stochastic process such that

$$\int_0^T |r(u)| du < \infty \quad (5.53)$$

almost surely. Define the stochastic process  $\bar{r}(t)$  as  $\bar{r}(0, \omega) = r(0)$  and

$$\bar{r}(t)(\omega) = \frac{1}{t} \left( \int_0^t r(u) du \right) (\omega) \quad (5.54)$$

for  $0 < t \leq T$ , for all  $\omega \in \Omega$ . Furthermore define the function  $R : [0, T] \rightarrow [0, \infty)$  as  $R(0) = \bar{r}(0) = r(0)$  and

$$R(t) = -\frac{1}{t} \log(\mathbb{E}[e^{-\bar{r}(t)t}]). \quad (5.55)$$

For a set sample path, we can calculate the average value of the rate  $r$  in the interval  $[0, t]$ ;  $\bar{r}(t)(\omega)$  is this average for a fixed  $\omega$ . This quantity should not be confused with the expected value  $\mathbb{E}[r(t)]$  which is calculated for a fixed  $t$  value. We observe that, given the definition of  $R(t)$ , we have the equality

$$e^{-R(t)t} = \mathbb{E}[e^{-\bar{r}(t)t}] \quad (5.56)$$

where  $\bar{r}(t)t$  is the average value of  $r$  over the time interval  $[0, t]$  times its length  $t$ . In financial terminology,  $R(t)$  is the continuously compounded interest rate per unit of time quoted for a term length of  $t$  beginning at time  $t_0$ .

### 5.3.1 A Deterministic Intensity Model

We now present our first potential model of the intensity. We shall refer to it as  $\Lambda^1(t)$ , where the superscript denotes this model as our first and is not to be interpreted as a power. This model will be the most simplistic of the three we wish to discuss and is dependent on which model of the short rate we use. We define the intensity function as follows:

$$\Lambda^1(t) = C \frac{t}{T} (F_{\alpha, \beta} \circ R)(t) \quad (5.57)$$

for each  $t \in [0, T]$ , where  $C > 0$  is a constant. We refer to  $\Lambda^1$  as an intensity function not a process since it is not random as  $R(t)$  is deterministic. Thus, a Poisson process defined using  $\Lambda^1$  would be an inhomogeneous Poisson process, not a Cox process. While we have mostly studied Cox processes, we still present this model for two reasons. First, it is a good example that allows us to illustrate desired qualities in an intensity model. Second, it could still be a useful model for someone who wished to use the Gaussian-Poisson Model but did not need the complexity that a random intensity process provides.

Recall, we have assumed earlier that  $\alpha > 0$  and  $\beta > 1$ . In order to specify an actual model, one needs to choose the constant  $C$  and the parameters  $\alpha$  and  $\beta$ . Using properties of  $F_{\alpha, \beta}$ , we see that  $\Lambda^1(t) < C \frac{t}{T}$  for each specific  $t$  and that  $\Lambda^1(t) < C$  for all  $t$ . Also, we have the following corollary to Lemma 5.9.

**Corollary 5.11.** *If we view the intensity  $\Lambda^1$  as a function of the rate  $R$ , then  $\Lambda^1$  has one inflection point,*

$$R_{\alpha, \beta} = \alpha \sqrt{\beta \frac{\beta - 1}{\beta}}, \quad (5.58)$$

*is convex on the interval  $(0, R_{\alpha, \beta})$ , and concave on the interval  $(R_{\alpha, \beta}, \infty)$ .*

Defining  $\Lambda^1$  in this manner has several advantageous properties. First, the change in the intensity  $\Lambda^1$  with respect to the change in rate  $R$  is positive for all rates; i.e., as interest rates increase, assets are more likely to default. Second, this derivative is increasing until it reaches the rate  $R_{\alpha, \beta}$  where it begins decreasing. We assert that this choice is logical. Rate changes at relatively low and high interest rates do not cause as large of a shock as rate changes in the middle. Third, we have also made the assumption that the intensity does not go to infinity as the rate does. We have capped the expected number of defaults in the entire time interval,

$$\mathbb{E}[N(T)] = \Lambda^1(T), \quad (5.59)$$

as being no larger than  $C$ . One could argue that  $C$  should be defined equal to the number of assets in the portfolio. That is, we should not “expect” more default than there are assets. Finally,  $R(t)$  can be calculated explicitly in closed form for any of our presented short rate models aside from the Black-Karasinski model. We could use simulations to find  $R(t)$  for the Black-Karasinski model.

One drawback of the model is that while the intensity is always positive for positive rates, it is 0 for rates  $R \leq 0$ . Recall that a non-positive intensity is forbidden. There are three potential ways to address this issue. First, one could only use models which only provide positive rates. Second, one could attempt to address the issue within the framework of the short rate model. Third, we could set a floor for the intensity at a minuscule value  $K$ . For instance, we could set

$$\Lambda^1(t) = \frac{t}{T}K \text{ if } (F_{\alpha,\beta} \circ R)(t) \leq \frac{K}{C}. \quad (5.60)$$

This final suggestion seems to be both the easiest fix and most logical. We could argue that, in the real world, there should always be at least some small positive probability of default.

An important question is how to determine which parameters  $\alpha$  and  $\beta$  to use. We suggest the following method. One should actually first determine the desired value of  $R_{\alpha,\beta}$  for the model by considering at what rate the instantaneous intensity should begin decreasing. This value could be estimated from market data. Then one should determine the shape parameter,  $\beta$ . With a large  $\beta$ , the intensity is close to either 0 or  $\frac{t}{T}C$  for most rates with a small region where  $\frac{\partial \Lambda^1}{\partial R}$  is quite large. When  $\beta$  is small, the change in intensity from 0 to  $\frac{t}{T}C$  occurs more gradually. Finally, the scale parameter,  $\alpha$ , is determined, once  $R_{\alpha,\beta}$  and  $\beta$  are chosen, via

$$\alpha = R_{\alpha,\beta} \sqrt[\beta]{\frac{\beta}{\beta - 1}}. \quad (5.61)$$

### 5.3.2 A Stochastic Intensity Model

Our second intensity model is essentially a stochastic version of the first. We define the intensity process  $\Lambda^2(t)$  as

$$\Lambda^2(t) = C \frac{t}{T} (F_{\alpha,\beta} \circ \bar{r})(t). \quad (5.62)$$

We presented all of the models of the short rate  $r(t)$  as Itô processes in SDE form. We can write this intensity model and the stochastic process  $\bar{r}(t)$  as SDEs, as well, using the Itô formula. We will use the following notations.

**Definition 5.12.** *Define the functions*

$$\begin{aligned} h(t, x) &= \frac{t}{T} C F_{\alpha,\beta}(x) \\ H(t, x) &= \frac{1}{t} x \end{aligned} \quad (5.63)$$

and the stochastic process

$$\hat{r}(t) = \int_0^t r(u) \, du. \quad (5.64)$$

**Theorem 5.13.** *Assume that the short rate model satisfies the condition*

$$\int_0^T |r(u)| \, du < \infty \quad (5.65)$$

almost surely. The stochastic process  $\Lambda^2(t)$  satisfies the stochastic differential equation

$$d\Lambda^2(t) = \frac{C}{T} \left( (F_{\alpha,\beta} \circ \bar{r})(t) + (r(t) - \bar{r}(t))(f_{\alpha,\beta} \circ \bar{r})(t) \right) dt. \quad (5.66)$$

*Proof.* The stochastic process  $\hat{r}(t)$  can be represented in SDE form as

$$d\hat{r}(t) = r(t) \, dt. \quad (5.67)$$

We observe that  $\bar{r}(t) = H(t, \hat{r}(t))$ , and  $H$  has the partial derivatives,

$$\partial_t H = -\frac{1}{t^2} x \quad \text{and} \quad \partial_x H = \frac{1}{t}. \quad (5.68)$$

It follows that  $\bar{r}(t)$  satisfies the SDE,

$$\begin{aligned}
d\bar{r}(t) &= dH(t, \hat{r}(t)) \\
&= \left( \partial_t H + r(t) \partial_x H + 0 \partial_x^2 H \right) dt + 0 \partial_x H dB(t) \\
&= \left( -\frac{1}{t^2} \hat{r}(t) + \frac{1}{t} r(t) \right) dt \\
&= \frac{r(t) - \bar{r}(t)}{t} dt,
\end{aligned} \tag{5.69}$$

where we omitted that  $H$  is evaluated at  $(t, \hat{r}(t))$ .

The intensity process can be written as

$$\Lambda^2(t) = h(t, \bar{r}(t)), \tag{5.70}$$

and we can calculate the partial derivatives of  $h$  as

$$\begin{aligned}
\partial_t h &= \frac{C}{T} F_{\alpha, \beta}(x) \quad \text{and} \\
\partial_x h &= \frac{t}{T} C f_{\alpha, \beta}(x).
\end{aligned} \tag{5.71}$$

Therefore,

$$\begin{aligned}
d\Lambda^2(t) &= dh(t, \bar{r}(t)) \\
&= \left( \partial_t h + \frac{r(t) - \bar{r}(t)}{t} \partial_x h \right) dt \\
&= \left( \frac{C}{T} F_{\alpha, \beta}(\bar{r}(t)) + \frac{r(t) - \bar{r}(t)}{t} \frac{t}{T} C f_{\alpha, \beta}(\bar{r}(t)) \right) dt \\
&= \frac{C}{T} \left( (F_{\alpha, \beta} \circ \bar{r})(t) + (r(t) - \bar{r}(t))(f_{\alpha, \beta} \circ \bar{r})(t) \right) dt.
\end{aligned} \tag{5.72}$$

□

The SDE in the theorem is a stochastic differential equation in the broadest sense of the term. Technically, since there is no  $dB(t)$  term, it is an ordinary differential equation with a stochastic function. The short rate model assumption in the theorem is necessary so that  $\hat{r}(t)$  is an Itô process which allows us to write, using the Itô formula,  $\bar{r}(t)$  and  $\Lambda^2(t)$  as Itô processes as well. Being able to write a

stochastic process as an Itô process is important especially for simulation studies. The theorem also produces the immediate corollary.

**Corollary 5.14.** *Under the short rate model assumption in Theorem 5.13, we have the following equality:*

$$t(F_{\alpha,\beta} \circ \bar{r})(t) = \int_0^t (F_{\alpha,\beta} \circ \bar{r})(u) \, du + \int_0^t (r(u) - \bar{r}(u))(f_{\alpha,\beta} \circ \bar{r})(u) \, du. \quad (5.73)$$

The second intensity model has similar advantages and disadvantages to our first. However,  $\Lambda^2(t)$  is a stochastic process as opposed to the deterministic function  $\Lambda^1(t)$ . This fact can either be a drawback or a selling point depending on the application. Furthermore, the intensity can become 0 again depending on our choice of the short rate model. If we were simulating  $\Lambda^2(t)$ , then we could fix this issue by setting

$$\Lambda^2(t) = \begin{cases} \frac{t}{T}K & \text{if } (F_{\alpha,\beta} \circ \bar{r})(t) \leq \frac{K}{C} \\ \frac{t}{T}C(F_{\alpha,\beta} \circ \bar{r})(t) & \text{if } (F_{\alpha,\beta} \circ \bar{r})(t) > \frac{K}{C} \end{cases} \quad (5.74)$$

where  $K$  is a sufficiently small positive value.

### 5.3.3 An Instantaneous Intensity Model

Our final model is potentially both the most complex and realistic. The previous two sections provided models of the aggregate intensity  $\Lambda(t)$ . The instantaneous intensity was never mentioned as the aggregate intensity process is all that is needed to define a Cox process. However, we did calculate  $\lambda^2(t)$  in Theorem 5.13. The aggregate and instantaneous intensity are defined such that

$$d\Lambda^2(t) = \lambda^2(t)dt, \quad (5.75)$$

implying that the coefficient of the  $dt$  term in Equation 5.66,

$$\frac{C}{T} \left( (F_{\alpha,\beta} \circ \bar{r})(t) + (r(t) - \bar{r}(t))(f_{\alpha,\beta} \circ \bar{r})(t) \right), \quad (5.76)$$

is the instantaneous intensity process  $\lambda^2(t)$ .

By examining  $\lambda^2(t)$ , we notice a potential problem with both of our first two models which we have not yet addressed; the instantaneous intensity can be negative at time  $t$  if the short rate at time  $t$ ,  $r(t)$ , is sufficiently smaller than the average short rate over the interval  $[0, t]$ ,  $\bar{r}(t)$ . We have previously addressed a non-positive aggregate intensity. However, we also need  $\Lambda(t) - \Lambda(s) > 0$ , for  $s < t$ , almost surely as well since this value is the parameter for the Poisson random variable  $N(t) - N(s)$ . A negative value for  $\lambda(t)$  violates this strictly increasing condition for  $\Lambda(t)$ .

There are ways to again adjust the model, especially when simulating the models over discrete time steps. Another option is to instead use an instantaneous intensity model  $\lambda^3(t)$  with appropriate properties: non-negativity and

$$0 < \int_s^t \lambda^3(u) du < \infty \quad (5.77)$$

almost surely, for all  $0 \leq s < t$ . In this case,  $\Lambda^3(t)$  is guaranteed to exist, be positive, and be strictly increasing. We now define the instantaneous intensity process as

$$\lambda^3(t) = \begin{cases} K & \text{if } (f_{\alpha,\beta} \circ r)(t) \leq \frac{K}{C} \\ C(f_{\alpha,\beta} \circ r)(t) & \text{if } (f_{\alpha,\beta} \circ r)(t) > \frac{K}{C} \end{cases}, \quad (5.78)$$

where  $C, K > 0$  are positive constants, implying that the instantaneous intensity is always positive. Its integrability will depend on the model for  $r(t)$ . If  $r(t)$  is an Itô process, then  $\lambda^3(t)$  is time-integrable, due to Itô's lemma [28], since  $f_{\alpha,\beta}$  is continuous with continuous partial derivatives. That is, the intensity process  $\Lambda^3$  exists and can be used to define a Cox process. However, while we know the process exists for appropriate short rate models, calculating

$$\Lambda^3(t, \omega) = \left( \int_0^t \lambda^3(u) du \right) (\omega) \quad (5.79)$$



is dependent on the particular model  $r(t)$ . Furthermore, obtaining a closed-form solution is not guaranteed. In fact, even integrating simple processes, such as geometric Brownian motion, with respect to time can be quite complicated, see [17]. As we previously mentioned, by creating a more realistic model, we necessarily increase its complexity. In order to appropriately study the viability of the model, future work would include an extensive simulation study.

If  $r(t)$  is positive almost surely, for all  $t \in [0, T]$ , then we can simplify the model by setting

$$\lambda^3(t) = C(f_{\alpha,\beta} \circ r)(t), \quad (5.80)$$

for all  $t$ . This model has convenient qualities. First, we have the following corollary to Lemma 5.9.

**Corollary 5.15.** *Define*

$$r_{\alpha,\beta} = \alpha \sqrt[\beta]{\frac{\beta - 1}{\beta}}. \quad (5.81)$$

*If we view  $\lambda^3$  as a function of the short rate  $r$ , then  $\lambda^3$  is strictly increasing on the interval  $[0, r_{\alpha,\beta})$ , obtains its global maximum when the short rate equals  $r_{\alpha,\beta}$ , and is strictly decreasing on the interval  $(r_{\alpha,\beta}, \infty)$ .*

This corollary shows that the instantaneous intensity is strongest at the rate  $r_{\alpha,\beta}$  and is smaller at rates farther away from this point: a property which we previously described as advantageous. Also, the parameter  $C$  allows us to control the maximum value  $\lambda^3(r_{\alpha,\beta})$ . Unlike our previous two models, we use the short rate at time  $t$  as opposed to the average short rate over the interval  $[0, t]$ . We argue that when attempting to determine the instantaneous intensity it is logical to use the instantaneous interest rate. Finally, we note the absence of  $\frac{t}{T}$  from this model. We inserted this value in the first two models since the aggregate intensity “ceiling” should increase with time. However, the instantaneous intensity at a particular

time should be determined by the rate at that time but not by the time itself. That is, if the short rate were constant over time, then the instantaneous intensity would be constant as well.

# Chapter 6

## Final Remarks and Future Work

The main goal of the Gaussian-Poisson model presented in this dissertation is to study default phenomena. We have presented a general model framework and provided results illustrating some of its properties. Specifically, we defined two probability measures  $\mathbb{P}^\bullet$  and  $\mathbb{P}^\dagger$  to act as market pricing measures and a random variable  $Y$  to act as a default descriptor. The two measures provide different probabilities for the event  $[Y = y]$ , but the conditional probabilities  $\mathbb{P}^\bullet[Y = y \mid \mathcal{G}]$  and  $\mathbb{P}^\dagger[Y = y \mid \mathcal{G}]$  were shown to be equal. Using these calculations, we were able to provide results in Chapter 4 involving expected losses. The measure  $\mathbb{P}^\bullet$  was defined in order to connect the Gaussian copula and Poisson models together by requiring that these two models provide an identical number of defaults over each time interval. However, the measure  $\mathbb{P}^\dagger$  was shown to have an important and useful property: the distribution of the interest rate process was not altered by using  $\mathbb{P}^\dagger$  rather than  $\mathbb{P}$  as the market pricing measure. Furthermore, this property allowed the  $\mathbb{E}^\dagger$ -expected losses in Chapter 4 to be given using the original expectation  $\mathbb{E}$ . Since the measure  $\mathbb{P}$  was unaffected by a change in an intensity parameter  $\alpha$ , we were able to use a dominated convergence argument to examine the sign of the  $\alpha$ -derivative of an expected loss under  $\mathbb{P}^\dagger$ . We only examined the model's sensitivity to intensity parameters. Further work should examine correlation sensitivity. Also, it would be interesting to examine default behavior if we continuously monitor the portfolio.

In order to explicitly use our model in practice, specific choices of models and parameters must be made. Methods of obtaining the  $\rho_j$  Gaussian copula param-

eters are well documented in the literature. Our model will require interest rate and intensity models to be provided. While short rate models have also been well studied, specific analysis on these models and their influence on the Gaussian-Poisson model is needed. Furthermore, a choice must be made on how to relate the intensity model to the short rate.

The success of the Gaussian-Poisson model is highly dependent on finding an effective intensity model. We specifically discussed utilizing the functions  $F_{\alpha,\beta}$  and  $f_{\alpha,\beta}$ . The purpose of Section 5.3 was not to propose one specific model. Rather we discussed three models that were developed with a few desired properties in mind. These models will need to be studied using real market data in order to determine their viability, or the data may suggest other intensity models. A truly in-depth analysis of the Gaussian-Poisson model to determine its viability will require significant future work. At this current stage of development, it is a mathematically intriguing idea. We believe that we have shown, using its mathematical properties, that the Gaussian-Poisson model merits further financial studies.

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# Appendix: Financial Terminology

If a definition refers to another term listed in this Appendix, then that other term will be underlined. The terms are listed alphabetically.

- **Derivative:** A *derivative* is a financial security that is dependent on the price of an underlying asset. Examples of underlying assets are stocks, bonds, commodities and interest rates.
- **Discount factor:** Assuming positive interest rates and a numeraire of dollars, a dollar today is not worth as much as a dollar tomorrow. In order to transition between dollars today and dollars in the future, we use a *discount factor*. Suppose we wish to know the value  $V$  today of receiving  $M$  dollars in 3 years. The discount factor  $D$  is the value such that  $V = DM$ . If  $R$  is the continuously compounded interest rate per year for a period starting today and ending in 3 years, then  $D = e^{-3R}$ .
- **Hedging:** *Hedging* is the act of attempting to reduce risk through trading.
- **Market Pricing Measure:** A *market pricing measure* is the probability measure that is used for pricing. Specifically, for event  $A$ , the probability of  $A$  is to be considered as the market price, relative to a given fixed numeraire, of an instrument  $I_A$ , where  $I_A$  has the payoff  $\mathbb{1}_A$ .
- **Numeraire:** A *numeraire* is the basic good by which other items are priced. For example, a diamond may be worth 5 units of numeraire if the numeraire is gold bars or 5000 units of numeraire if the numeraire is dollars. Numeraire also usually incorporates when the good would be delivered. For instance, a dollar today and a dollar in a year do have the same value due to interest rates. We will be using \$1 today, which will be referred to as time- $t_0$  money, as numeraire unless otherwise mentioned.
- **Portfolio:** A portfolio is a collection of assets.
- **Principal:** The principal is the initial investment.
- **Recovery Rate:** The *recovery rate* is the percentage of the principal that is not lost upon a default. E.g., a recovery rate of 0.4 with a principal of 100 units of numeraire implies a loss of 60 units of numeraire upon default.
- **Risk-Free Interest Rate:** The *risk-free interest rate* is the rate used in the risk-free world. It is used when pricing derivatives, such pricing requires the use of discount factors.
- **Risk-Neutral World:** The *risk-neutral world* assumes that investors do not require additional compensation for additional risk. Thus, all investments are expected to grow at the risk-free interest rate.

- **Short rate:** The *short rate*  $r(t)$  is the instantaneous risk-free interest rate at time  $t$ ; i.e.,  $r(t)$  is the interest rate at time  $t$  that applies to an infinitesimally small time. In a risk-neutral world, an investment of 1 unit of numeraire would grow to approximately  $e^{r(t)\Delta}$  units of numeraire over extremely small time increments  $\Delta$ .
- **Speculating** *Speculating* is the act of engaging in risky behavior to increase financial gains.

# Vita

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