2008

Neglecting parameter changes in GARCH option pricing models and VAR

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NEGLECTING PARAMETER CHANGES IN GARCH OPTION PRICING MODELS AND VAR

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Economics

by
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August 2008

To My Family

Acknowledgements

I am truly grateful to my family for their endless support. They have always believed in me and it would have been impossible for me to obtain this degree without their constant support and love.

My advisor, Dr. Eric Hillebrand, has guided and supported me at all stages of my research. Without his encouragement and day-and-night help, I would not have made it. I also would like to thank my committee members Dr. Jimmy Hilliard, Dr. Carter Hill, and Dr. Dek Terrell for their valuable comments. I am also indebted to Dr. Faik Koray and Dr. Sudipta Sarangi for their support and guidance.

I have made a lot of good friends during my graduate study at LSU. Koray, Mohammed, Peren, Ayca, Serban, Beatrice, Steve, Kathy, Umut, Ayla, Tahsin, Jada, Alp, Sertac, Duyal and Deniz; we have shared many good memories and you have always been there for me. I truly appreciate your friendship and cannot imagine how difficult and boring my life in Baton Rouge would have been without you.

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Abstract

In GARCH models, neglecting parameter changes in the conditional volatility process results in biased estimation. The estimated sum of the autoregressive parameters of the conditional volatility converges to one. In Chapter 2, I analyze the effect of changes in the parameters of conditional volatility on European call option prices when these parameters are estimated ignoring the change-points. Simulation studies show that ignoring parameter changes in the conditional variance process of GARCH(1,1) models leads to biased estimates of option prices. The bias, measured in percentages, is most pronounced for out-of-the-money options, substantial for at-the-money options, and vanishes as options move deep-in-the-money.

The empirical study in Chapter 2 shows that the bias in option prices decreases when NGARCH model is used. NGARCH model captures the negative correlation between the stock price and volatility. To analyze this issue further, in Chapter 3, I analyze the effect of changes in the parameters of conditional volatility on European call option prices using Heston's and Nandi's (2000) closed-form GARCH option pricing model. Simulation studies show that option prices obtained by the closed-form expression are biased when parameter changes are ignored, but due to asymmetry effects the bias is less pronounced compared to the results in Chapter 2.

In Chapter 4, I analyze the effect of parameter changes in the conditional volatility process on Value-at-Risk (VaR) based on a GARCH model. Ignoring parameter changes results in biased VaR estimates. The bias is more pronounced when parameter changes imply a greater change in unconditional volatility. In addition, the sign of the bias is negatively related to the sign of the change in unconditional volatility.

Chapter 1

Introduction

Using volatility models has become an important part in empirical asset pricing and risk management. The ARCH model by Engle (1982) and its generalization, the GARCH model by Bollerslev (1986), allow for the variance of the underlying process to change in a discrete time framework. Bollerslev et al. (1992) and Bollerslev et al. (1994) provide an overview of ARCH-type models. Several studies in the existing literature on GARCH models show that estimations of various different specification of GARCH models indicate volatility clustering and high persistence in financial data. Engle and Bollerslev (1986) introduce the I-GARCH (Integrated-GARCH) model to capture the high persistence feature of asset returns. In this model, shocks to volatility do not decay over time. Also, several studies in the fractional integration literature find high persistence in stock returns. Ding et al. (1993), Ding and Granger (1996), and Baillie et al. (1996) are some of the important studies in the area.

Recently an increasing number of studies show that there may exist nonstationarities in the volatility of asset returns. Diebold and Inoue (2001) point out the possibility of confusing long memory and structural change. Some of the important studies in the long memory literature emphasizing the same phenomenon are Lobato and Savin (1998), Granger and Hyung (2004), Granger and Teräsvirta (2001), and Smith (2005). Perron and Qu (2004) show that estimation of the order of integration of a short memory process contaminated with structural changes is biased upwards and therefore it implies long memory. In GARCH models the issue was first brought up by Diebold (1986). Lamoureux and Lastrapes (1990) show in data and simulation experiments that the GARCH(1,1) model exhibits high persistence due to neglected changes in the constant term of the conditional variance process. Hillebrand (2005) shows that if there is a neglected parameter change in the conditional variance of a GARCH process, the sum of the maximum likelihood estimators of the autoregressive parameters of conditional volatility converges to one. Starica and Granger (2005) show that instead of assuming global stationarity in S&P 500 returns, assuming nonstationarity and approximating the nonstationary data-generating process by locally stationary models provides a better forecasting performance. The results from Markov-switching models of Hamilton and Susmel (1994) (for ARCH models) and Gray (1996) (for GARCH models) indicate that locally stationary models provide lower persistence estimates. In summary, the conclusion from these studies is that assuming global stationarity and a constant unconditional variance for a process contaminated with parameter changes results in high persistence estimates and poor forecasting performance.This dissertation analyzes the effect of ignored parameter changes in conditional variance process of the GARCH model on option prices and Value-at-Risk.

An option is a derivative security whose value depends on one or more underlying assets. A call (put) option gives its owner the right but not the obligation to buy (sell) its underlying asset at a specific price (strike price) and at a specific time. Options are widely used in financial markets for hedging risk. Mispriced options may lead to arbitrage opportunities and a substantial increase in the portfolio risk. An arbitrage opportunity exists if an option is overpriced or underpriced relative to its expected value. For example, if an option is underpriced, a financial institution can buy the option by issuing a bond at the same interest rate the option value is discounted from its expected value at the expiration and obtain risk-free profits when the option expires. In Chapters 2 and 3, I analyze the effect of a parameter change in conditional variance process of the GARCH model on option prices.

Value-at-Risk is an important risk measure that is widely used by financial institutions to report their market-risk exposures. It is used by regulatory agencies to control the risk exposures of financial institutions. The Basel Committee on Banking Supervision (1996) at the Bank for International Settlements requires banks to calculate and report VaR estimates daily. Based on these VaR estimates, financial institutions must hold a certain level of capital. If a bank overestimates VaR, the amount of capital that it is required to hold will be also overestimated, which may lead to substantial opportunity cost. If VaR is underestimated and in an adverse financial situation, banks will be exposed to bankruptcy risk. GARCH models are widely used to estimate VaR and I analyze the effect of a parameter change in conditional volatility process of GARCH models on VaR estimates.

Chapter 2

The Sensitivity of GARCH Option Pricing Models to Ignored Parameter Changes

2.1 Introduction

A voluminous literature has developed in the theory and practice of option pricing after Black and Scholes (1973) and Merton (1973). Volatility of the stock price process is by far the most important variable in these models. It is not directly observable and was assumed constant in the early studies. It is widely accepted that volatility and correlations in asset prices vary over time. Hull and White (1987) introduced stochastic volatility models in a continuous-time framework. In their model, there is no closed-form solution for option prices when the sources of randomness in volatility and stock price are correlated, and Monte Carlo simulation is used to obtain option prices. A closed-form solution is given by Heston (1993). The ARCH model by Engle (1982) and its generalization, the GARCH model by Bollerslev (1986), allow for the variance of the underlying process to change in a discrete time framework. An alternative approach to ARCH-type models in discrete time is the stochastic volatility (SV) model introduced by Taylor (1986). A recent survey of the SV literature is given by Broto and Ruiz (2004). For an extensive review of forecasting performance of various volatility models, see Poon and Granger (2003).

The gap between ARCH-type models and continuous time models is closed by Nelson (1990), Drost and Werker (1996), and Corradi (2000). Nelson (1990) shows that the GARCH(1,1) model, in its continuous time limit, converges to a continuous time stochastic volatility process. Drost and Werker (1996) show that the class of continuous GARCH models contains not only continuous time diffusion models but also jump-diffusion models. Corradi (2000) shows that, assuming α (the ARCH parameter in Equation (2.3)) vanishes in the continuous time limit, the limiting volatility process is deterministic. Duan (1997) proposes the augmented GARCH model, which encompasses many parametric GARCH models, and shows that the diffusion limit of the model also encompasses many diffusion processes commonly used in the literature.

In this chapter, we analyze the effect of ignored parameter changes in conditional variance process of the GARCH model on European option prices using Duan's (1995) GARCH option pricing model. Duan (1995) developed the theoretical foundation that allows the use of GARCH models for option valuation. Several improvements of the model and the methodology have been developed to incorporate various empirical facts about asset returns (e.g. fat tails, leverage effect). For an overview and comparison of GARCH option pricing models see Christoffersen and Jacobs (2004). Our purpose in this study is to understand the effect of ignored changes in the unconditional volatility of a GARCH model on European option prices. Ignoring structural breaks creates problems in any autoregressive model (see Hillebrand 2005, 2006). The model can be modified as desired to capture more features of the data. As long as it has autoregressive parameters, ignored parameter changes will result in high persistence estimates. For clarity of exposition, we choose the simplest GARCH specification.

The sensitivity of option prices to volatility is called Vega. Merton (1973) and Black and Scholes (1973), among many others, show that volatility and price of an option are positively related. Or in other words, Vega of an option is positive. Although in GARCH option pricing models conditional volatility is time-varying, unconditional volatility is assumed constant and is an important determinant of the option price. Therefore, if we assume a stationary process and constant unconditional volatility for the underlying asset of an option when this is in fact not the case, the estimated biased unconditional volatility will lead to a biased option price. For instance, if the process switches from high to low unconditional volatility in the mid-point of the sample and we ignore the parameter change, the estimated unconditional volatility will be somewhere in between the two regimes. Therefore, since Vega is positive, we expect that the estimated option price will be higher than the one that is obtained using data after the parameter change only. We show evidence for this intuition in Monte Carlo simulations and in an empirical study of S&P500 index prices.

Our simulation study shows that ignoring structural breaks in the unconditional volatility of the underlying security leads to biased estimates of European option prices. The bias is most pronounced for out-of-the-money options and increases as the out-of-the-moneyness gets deeper. The effect is smaller for in-the-money options and becomes negligible as the in-the moneyness gets deeper. We test S&P500 index returns for unknown change-points and apply the same methodology that we use in simulations to real data. Our change-point study on the S&P500 index supports our simulation results.

In the next section, we briefly review the Duan (1995) model. Section 2.3 describes the simulation methodology. The results follow in Section 2.4. In Section 2.5, we present a change-point study on the S&P500 index and apply the same pricing methodology to see if the empirical results are similar to simulation results. Section 2.6 summarizes the main results.

2.2 The Model

We use the GARCH(1,1) option pricing model of Duan (1995). Consider

$$\log(S_t/S_{t-1}) = r + \lambda\sqrt{h_t} - \frac{1}{2}h_t + \varepsilon_t, \tag{2.1}$$

$$\varepsilon_t|F_{t-1} \sim N\ (0, h_t)\text{, under measure } P, \tag{2.2}$$

$$h_t = \omega + \alpha\varepsilon_{t-1}^2 + \beta h_{t-1}, \tag{2.3}$$

where F_t is a σ-field of all information up to and including time t; ω constant, α and β are autoregressive GARCH(1,1) parameters; S_t is the underlying asset (in our case, stock) price at time t; r is the risk-free interest rate; λ is the unit risk premium, which represents preferences; ε_t is the normally distributed innovation with mean zero and variance h_t. Equation (2.1) is the standard asset pricing equation that models one-period returns at time t that depend on the constant risk-free interest rate, constant unit risk premium, time-varying variance, and a normally distributed random term with mean zero and variance h_t. Equation (2.2) shows how the error terms are distributed. Equation (3.2) is the GARCH(1,1) equation that specifies how the conditional variance h_t evolves over time.

The physical probability measure P models the dynamics of the stock price. It determines how likely it is that the stock price moves up or down. The valuation of contingent claims, however, necessitates a fair pricing mechanism. To achieve this, all we need to assume is that there does not exist any arbitrage opportunity (the so-called First Fundamental Theorem of Asset Pricing, see Harrison and Kreps (1979) and Delbaen and Schachermayer (1994)). This leads to a pricing mechanism that depends on the value of the stock price at maturity and the payoff function of the contingent claim, not on the probability to obtain the stock price at maturity. Therefore, the new arbitrage-free pricing mechanism results in a different probability measure Q than the physical probability measure P. For more details, see the discussion in Duan (1995, p 15-18).

DEFINITION 1. *Duan (1995) A pricing measure Q satisfies the locally risk-neutral valuation relationship if*

- *measure Q is absolutely continuous with respect to measure P, which means that the probability of an event in σ-field F_t under measure Q is zero if and only if the probability of the same event in σ-field F_t under measure P is zero.*

- $(S_t/S_{t-1})|F_{t-1}$ *follows a lognormal distribution (under measure Q) with* $\mathbb{E}^Q\left[(S_t/S_{t-1})|F_{t-1}\right] = e^r$.

The conditional variances under measures P and Q are required to be equal. We can therefore estimate the parameters of the conditional variance under measure P and use them to estimate the stock price under measure Q. Although this is not sufficient to eliminate the unit risk premium λ, which represents preferences, this property along with the conditional mean results in a well-specified model that does not depend on preferences locally.

Duan (1995) shows that the locally risk-neutral valuation relationship implies that under probability measure Q,

$$\log(S_t/S_{t-1}) = r - \frac{1}{2}h_t + \zeta_t, \tag{2.4}$$

where

$$\zeta_t|F_{t-1} \sim N\ (0, h_t), \tag{2.5}$$

and

$$h_t = \omega + \alpha\left(\zeta_{t-1} - \lambda\sqrt{h_{t-1}}\right)^2 + \beta h_{t-1}. \tag{2.6}$$

Then, the price of a European call option under probability measure Q is given as:

$$c = \exp\left[-r\left(T-t\right)\right]\mathbb{E}^Q\left[\max\left(S_T - X, 0\right)|F_t\right]. \tag{2.7}$$

The price of a European call option can be calculated by following the steps below:

Step 1 Obtain parameters from the model given in Equations (2.1) through (2.3), which is under the probability measure P.

Step 2 Using the parameters in Step 1, obtain Monte Carlo simulation prices $\hat{S}$ by using the model given by equations (2.4) through (2.6), which is under the locally risk-neutral probability measure Q.

Step 3 Assuming that the number of simulated sample paths is N and time to maturity is T, apply the Empirical Martingale Simulation Method (EMS) of Duan and Simonato (1998) as follows:

For $t = 1, 2, ...T$ and $i = 1, 2, ...N$,

$$S^*(t,i) = S_0 \frac{Z(t,i)}{\overline{Z}(t)}, \tag{2.8}$$

where $S^*(t,i)$ is the EMS corrected asset price for the i^{th} sample path at time t, and

$$Z(t,i) = S^*(t-1,i) \frac{\hat{S}(t,i)}{\hat{S}(t-1,i)}, \tag{2.9}$$

$$\overline{Z}(t) = \frac{1}{N} \exp(-rt) \sum_{i=1}^{N} Z(t,i). \tag{2.10}$$

Note that $\hat{S}(t,i)$ is the simulated asset price for the i^{th} sample path at time t in Step 2. Also, for all sample paths we set $\hat{S}(0,i)$ and $S^*(0,i)$ to the initial stock price S_0. We start with $t = 1$ and calculate Equation (2.9) for all sample paths (for all i's). Then, we obtain $\overline{Z}(1)$ from Equation (2.10) and calculate the EMS corrected asset prices for all sample paths at time 1 using Equation (2.8). We repeat this sequence for all t's until expiration.

Step 4 After we obtain EMS asset prices, the price of a European call option is estimated by:

$$\hat{c} = \frac{1}{N} \exp\left[-r(T-t)\right] \sum_{i=1}^{N} \max\left[S^*(T,i) - X, 0\right], \tag{2.11}$$

where X is the strike price, $S^*(T,i)$ is the i^{th} simulated price of the underlying asset at expiration (time T).

See Duan (1995), Duan and Simonato (1998), and Christoffersen and Jacobs (2004) for more details on the model and the estimation methodology.

2.3 Simulation Methodology

In order to show the effects of changes in the parameters of the model on option prices, we first simulate a series of stock prices under the physical probability measure P. The series has 4,000 observations with a parameter change at observation $2,001$. The stock price follows the process:

$$\log(S_t/S_{t-1}) = r + \lambda\sqrt{h_t} - \frac{1}{2}h_t + \varepsilon_t, \tag{2.12}$$

$$\varepsilon_t|F_{t-1} \sim N\ (0, h_t)\,,\ \text{under measure } P, \tag{2.13}$$

$$h_t = \omega_i + \alpha_i\varepsilon_{t-1}^2 + \beta_i h_{t-1},\ \text{for } i = 1,2. \tag{2.14}$$

Here, $i = 1$ denotes the first regime of $2,000$ observations and $i = 2$ denotes the second regime of $2,000$ observations. We set initial volatility equal to the unconditional mean.

After simulating the system above, we estimate the parameters (a) for the whole series and (b) for the last $2,000$ observations. We then simulate $10,000$ sample paths until the expiration date under the risk-neutral probability measure Q (Equations (2.4) through (2.6)) and apply the empirical martingale simulation method (described in Section 2.2). Then, the payoff of the call option for each sample path is calculated. Since expectation is taken under the risk-neutral probability measure Q, the European call price is calculated using Equation (2.11) for $N = 10,000$ sample paths. We repeat this process $5,000$ times for each of the several different scenarios presented in the tables and figures in the next section in order to get $5,000$ simulated call prices. Thus, each of the $5,000$ call prices is the mean of the call prices of $10,000$ simulations. Then, we analyze the distributional properties of these $5,000$ call price observations for each scenario.

2.4 Simulation Results

2.4.1 At-the-Money Options

Our objective is to show the effect of an ignored change in one of the parameters in the GARCH(1,1) conditional volatility process on at-the-money European call option prices. An option is said to be at-the-money if the price of underlying stock is equal to its strike price. It is said to be in-the-money (out-of-the-money) if the price of underlying stock is higher (lower) than its strike price. We run simulations with a single change point in one of the parameters of the conditional volatility h_t. Each simulated series has $4,000$ observations and the change in the parameter occurs at observation $2,001$.

Following Duan and Simonato (1998), we chose λ to be 0.01 and $r = 0$ in the simulations. Estimating the model without accounting for the changes in the parameters of the conditional volatility does not affect the estimation of λ since it is in the mean equation. Experimenting with different choices of λ did not affect the results. These results are available upon request.

The annualized volatility σ is equal to $\sqrt{250\omega/(1-\alpha-\beta)}$. Let σ_1 denote the annualized volatility of the first half of the series and it is set equal to 20%. Let σ_2 denote the annualized volatility of the second half of the series and it is set to different values. The changes in the parameters are set according to the considered change in σ_2. For example, when we study the effect of a change in the constant parameter ω, we initially set $\omega_1 = 3.2e-5$, $\alpha_1 = 0.20$, and $\beta_1 = 0.60$, so that the initial annualized volatility is 20%. To study the effect of the change in ω when the annualized volatility decreases to 15%, we change ω_2 accordingly to $1.80e-5$. The setup is analogous for changes in the other parameters.

We simulate $5,000$ series with $4,000$ observations each. After we simulate the series, we estimate two sets of parameters by maximum likelihood: one from the whole series without accounting for the parameter change that occurs at observation $2,001$ and one from the second segment of the series, after the parameter change occurs. For each of the $5,000$ series, we simulate $10,000$ sample paths and calculate call option prices for each sample path. Then, we take the mean of the $10,000$ option prices to calculate the Monte Carlo simulation price. This results in $5,000$ Monte Carlo simulation prices. Table 2.1 reports the means over these $5,000$ call prices. We calculate prices of call options with 5, 30, and 90 days to maturity. Initial stock price and strike price are set equal to \$100.

In Table 2.1, we study the effect of neglected changes in the constant ω. In all tables and figures, we included the zero parameter change as benchmark. Consistent with the results in Hille-

Table 2.1: The effect of a single neglected change-point in ω on the European at-the-money call option price. GARCH(1,1) Option Pricing Model Duan (1995) with $r = 0$, $\lambda = 0.01$ and $h_t = \omega_i 10\text{e-}6 + 0.20\varepsilon_{t-1}^2 + 0.60h_{t-1}$ for $i = 1,2$. $\hat{\theta} = \hat{\alpha} + \hat{\beta}$. Strike price = Initial stock price = \$100. Parameter estimates are the means of 5,000 simulations. $\hat{c}$ is the call price calculated using the second half of the sample and $\hat{c}_n$ is the call price calculated using the whole sample without taking the parameter change into account.The annualized volatility σ_1 of the first segment is always 0.20. Standard errors are given in parentheses.

Parameter Changes			Parameter Estimates (Whole Sample)					5-Day		30-Day		90-Day	
σ_2	ω_1	ω_2	$\hat{\omega}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\sigma}$	$\hat{c}$	$\hat{c}_n$	$\hat{c}$	$\hat{c}_n$	$\hat{c}$	$\hat{c}_n$
0.10	3.20e-5	0.80e-5	2.05e-6	0.16	0.83	0.99	0.196	0.49	0.98	1.34	2.54	2.37	4.34
			(1.2e-6)	(0.03)	(0.04)			(0.05)	(0.15)	(0.14)	(0.34)	(0.14)	(0.59)
0.15	3.20e-5	1.80e-5	1.72e-5	0.20	0.66	0.86	0.178	0.73	0.86	2.00	2.35	3.54	4.18
			(3.31e-6)	(0.02)	(0.04)			(0.03)	(0.03)	(0.07)	(0.07)	(0.13)	(0.12)
0.20	**3.20e-5**	**3.20e-5**	**3.24e-5**	**0.20**	**0.60**	**0.80**	**0.200**	**0.97**	**0.97**	**2.66**	**2.66**	**4.72**	**4.72**
			(4.62e-6)	**(0.02)**	**(0.04)**			**(0.03)**	**(0.03)**	**(0.09)**	**(0.07)**	**(0.17)**	**(0.13)**
0.25	3.20e-5	5.00e-5	3.31e-5	0.20	0.64	0.84	0.227	1.22	1.10	3.33	3.01	5.88	5.33
			(5.68e-6)	(0.02)	(0.04)			(0.04)	(0.03)	(0.11)	(0.08)	(0.22)	(0.15)
0.30	3.20e-5	7.20e-5	2.33e-5	0.19	0.72	0.91	0.260	1.46	1.26	3.99	3.40	7.06	6.04
			(5.94e-6)	(0.03)	(0.04)			(0.05)	(0.04)	(0.14)	(0.10)	(0.26)	(0.24)
0.35	3.20e-5	9.80e-5	1.33e-5	0.17	0.79	0.96	0.300	1.70	1.48	4.65	3.91	8.23	6.86
			(5.20e-6)	(0.03)	(0.04)			(0.06)	(0.06)	(0.17)	(0.16)	(0.30)	(0.27)
0.40	3.20e-5	12.8e-5	8.71e-5	0.16	0.83	0.99	0.354	1.94	1.72	5.31	4.48	9.41	7.73
			(2.80e-6)	(0.03)	(0.03)			(0.07)	(0.11)	(0.19)	(0.25)	(0.35)	(0.43)

brand (2005), we see that β is overestimated and that the greater the jump size in the annualized volatility, the greater the effect of a neglected change in ω on $\hat{\beta}$ and, thus, on $\hat{\theta} = \hat{\alpha} + \hat{\beta}$. We observe that $\hat{\theta}$ approaches 1 as the jump size increases. If the annualized volatility is lower for the second segment of the series, then the option price obtained from the whole sample is higher than the option price obtained using only data from the second segment, and vice versa. The reason is that the estimated annualized volatility from the whole sample is between the initial annualized volatility (which is set to 20%) and the annualized volatility of the second segment. For example, if the annualized volatility of the second segment of the series is 10%, then the estimated annualized volatility from the whole sample will be between 10% and 20% almost surely. Since higher annualized volatility results in a higher option value, the option price calculated from the whole series is above the option price obtained from the second segment. The opposite holds when the annualized volatility increases: If the annualized volatility for the second segment is higher, then the option price obtained from this segment will be above the option price obtained from the whole series.

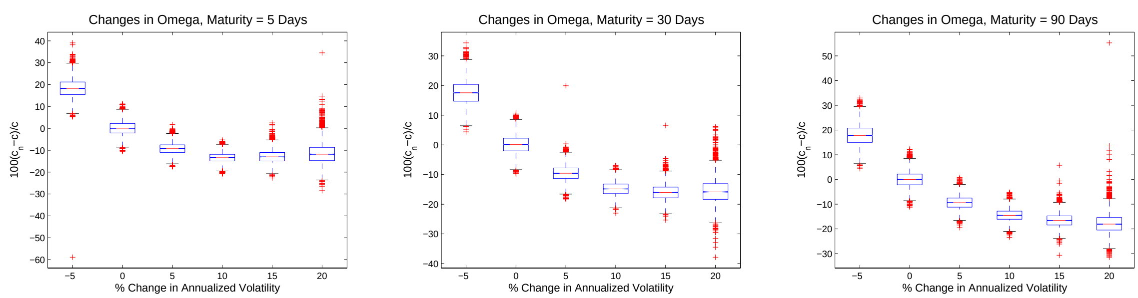

Figure 2.1: The effect of the change in parameter ω on European at-the-money call options in percentages. The vertical axis shows the percentage difference between the option price obtained from the whole sample ($\hat{c}_n$) without accounting for the parameter change, and the option price obtained from the second part of the sample ($\hat{c}$). The horizontal axis shows the percentage change in annualized volatility. For each value on the horizontal axis there are $5,000$ observations.

As can be seen in Table 2.1, if the annualized volatility is reduced to 10% after the parameter change, then the option price from the whole sample is roughly twice the option price obtained from the second segment. It is roughly 20% higher if the decrease in annualized volatility is 5%. This can also be seen in Figure 2.1, which plots the percentage difference in the option prices $\hat{c}_n$ obtained from the whole sample and option prices $\hat{c}$ obtained from only the second segment of the series (y-axis) for a given change in annualized volatility (x-axis). For each value of the change in annualized volatility, the figures provide a box and whisker plot. Lower quartile, median and upper quartile values are given by the lines in each box. The whiskers, which are the lines extending from each end of the boxes, give the values that correspond to 1.5 times the interquartile range away from the lower and upper quartiles. The values beyond the ends of the whiskers are the simulated date distribution tails. We observe that if annualized volatility decreases, the effect of a change in the constant parameter ω on option prices is greater than in the case where annualized volatility increases. If annualized volatility increases by 10% in the second half of the series the option price from the whole sample is around 10% less than the option price obtained from the second segment. If the annualized volatility decreases by 10%, the distortion of the option price ranges between 75% and 100%. The effect increases in magnitude for larger increases in annualized volatility but at a decreasing rate. For decreases in annualized volatility, the effect grows at an increasing rate.

The same conclusions can be drawn for changes in the parameters α and β. The results for these parameters are presented in Tables 2.2 and 2.3 and Figures 2.2 and 2.3. We see that the percentage differences are very close across the three parameters for the same change in annualized volatility. In the case of changes in α, the effect is slightly stronger. The reason for this is that a neglected change in the parameter α has the smallest effect on the estimation of the parameters of conditional volatility, consistent with the results in Hillebrand (2005). An increase in volatility after the ignored change-point results in lower estimates of θ for changes in α compared to changes in ω or β. This, in turn, results in lower estimated annualized volatilities for neglected changes in α compared to neglected changes in ω or β. Say that after the parameter change, annualized volatility in the second segment is increasing. Estimated annualized volatility falls in between the

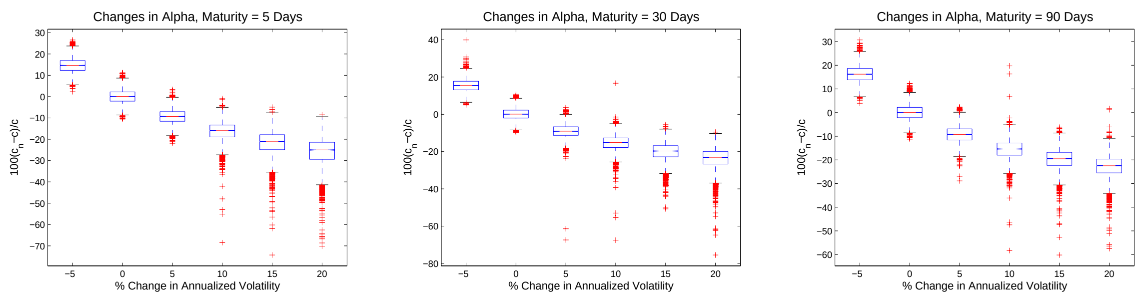

Figure 2.2: The effect of the change in parameter α on European at-the-money call options in percentages. The vertical axis shows the percentage difference between the option price obtained from the whole sample ($\hat{c}_n$) without accounting for the parameter change, and the option price obtained from the second part of the sample ($\hat{c}$). The horizontal axis shows the percentage change in annualized volatility. For each value on the horizontal axis there are $5,000$ observations. We did not include the -10% change in these experiments because of strict positivity constraint on the parameters.

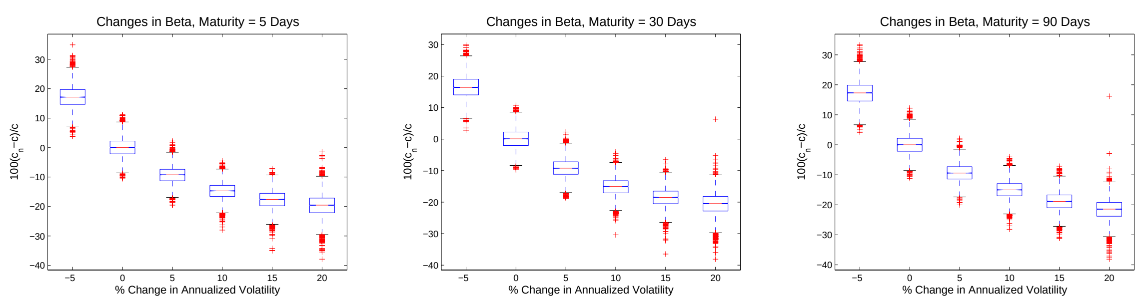

Figure 2.3: The effect of the change in parameter β on European at-the-money call options in percentages. The vertical axis shows the percentage difference between the option price obtained from the whole sample ($\hat{c}_n$) without accounting for the parameter change, and the option price obtained from the second part of the sample ($\hat{c}$). The horizontal axis shows the percentage change in annualized volatility. For each value on the horizontal axis there are $5,000$ observations. We did not include the -10% change in these experiments because of strict positivity constraint on the parameters.

Table 2.2: The effect of a single neglected change-point in α on the European at-the-money call option price. GARCH(1,1) Option Pricing Model Duan (1995) with $r = 0$, $\lambda = 0.01$ and $h_t = 3.20\text{e-}5 + \alpha_i \varepsilon_{t-1}^2 + 0.60 h_{t-1}$ for $i = 1, 2$. $\hat{\theta} = \hat{\alpha} + \hat{\beta}$. Strike price = Initial stock price = \$100. Parameter estimates are the means of 5,000 simulations. $\hat{c}$ is the call price calculated using the second half of the sample and $\hat{c}_n$ is the call price calculated using the whole sample without taking the parameter change into account.The annualized volatility σ_1 of the first segment is always 0.20. Standard errors are given in parentheses.

Parameter Changes			Parameter Estimates (Whole Sample)					5-Day		30-Day		90-Day	
σ_2	α_1	α_2	$\hat{\omega}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\sigma}$	$\hat{c}$	$\hat{c}_n$	$\hat{c}$	$\hat{c}_n$	$\hat{c}$	$\hat{c}_n$
0.15	0.20	0.044	2.11e-5	0.14	0.69	0.83	0.175	0.75	0.86	2.03	2.35	3.57	4.14
			(5.18e-6)	(0.02)	(0.06)			(0.02)	(0.02)	(0.05)	(0.06)	(0.08)	(0.10)
0.20	**0.20**	**0.200**	3.24e-5	**0.20**	**0.60**	**0.80**	**0.200**	**0.97**	**0.97**	**2.66**	**2.66**	**4.72**	**4.72**
			(4.62e-6)	**(0.02)**	**(0.04)**			**(0.03)**	**(0.03)**	**(0.09)**	**(0.07)**	**(0.17)**	**(0.13)**
0.25	0.20	0.272	3.02e-5	0.24	0.61	0.85	0.224	1.19	1.08	3.24	2.94	5.78	5.24
			(4.08e-6)	(0.02)	(0.03)			(0.07)	(0.04)	(0.02)	(0.10)	(0.32)	(0.18)
0.30	0.20	0.311	2.79e-5	0.26	0.62	0.88	0.244	1.40	1.17	3.76	3.17	6.72	5.67
			(3.88e-6)	(0.02)	(0.03)			(0.13)	(0.05)	(0.30)	(0.13)	(0.50)	(0.23)
0.35	0.20	0.335	2.62e-5	0.28	0.63	0.91	0.262	1.61	1.25	4.23	3.36	7.52	6.01
			(3.42e-6)	(0.02)	(0.03)			(0.20)	(0.07)	(0.41)	(0.16)	(0.68)	(0.29)
0.40	0.20	0.350	2.51e-5	0.29	0.63	0.92	0.277	1.80	1.32	4.64	3.52	8.18	6.29
			(4.16e-6)	(0.02)	(0.03)			(0.27)	(0.08)	(0.57)	(0.19)	(0.81)	(0.35)

Table 2.3: The effect of a single neglected change-point in β on the European at-the-money call option price. GARCH(1,1) Option Pricing Model Duan (1995) with $r = 0$, $\lambda = 0.01$ and $h_t = 3.20\text{e-}5 + 0.20 \varepsilon_{t-1}^2 + \beta_i h_{t-1}$ for $i = 1, 2$. $\hat{\theta} = \hat{\alpha} + \hat{\beta}$. Strike price = Initial stock price = \$100. Parameter estimates are the means of 5,000 simulations. $\hat{c}$ is the call price calculated using the second half of the sample and $\hat{c}_n$ is the call price calculated using the whole sample without taking the parameter change into account.The annualized volatility σ_1 of the first segment is always 0.20. Standard errors are given in parentheses.

Parameter Changes			Parameter Estimates (Whole Sample)					5-Day		30-Day		90-Day	
σ_2	β_1	β_2	$\hat{\omega}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\sigma}$	$\hat{c}$	$\hat{c}_n$	$\hat{c}$	$\hat{c}_n$	$\hat{c}$	$\hat{c}_n$
0.15	0.60	0.444	2.24e-5	0.20	0.62	0.82	0.177	0.73	0.86	2.02	2.35	3.55	4.16
			(4.33e-6)	(0.02)	(0.05)			(0.02)	(0.02)	(0.06)	(0.06)	(0.18)	(0.11)
0.20	**0.60**	**0.600**	3.24e-5	**0.20**	**0.60**	**0.80**	**0.200**	**0.97**	**0.97**	**2.66**	**2.66**	**4.72**	**4.72**
			(4.62e-6)	**(0.02)**	**(0.04)**			**(0.03)**	**(0.03)**	**(0.09)**	**(0.07)**	**(0.17)**	**(0.13)**
0.25	0.60	0.672	2.75e-5	0.20	0.67	0.87	0.223	1.21	1.10	3.28	2.98	5.85	5.30
			(4.33e-6)	(0.02)	(0.04)			(0.05)	(0.04)	(0.14)	(0.09)	(0.26)	(0.17)
0.30	0.60	0.711	2.05e-5	0.20	0.72	0.92	0.253	1.45	1.23	3.90	3.30	6.92	5.88
			(3.77e-6)	(0.02)	(0.03)			(0.08)	(0.05)	(0.21)	(0.12)	(0.37)	(0.22)
0.35	0.60	0.735	1.57e-5	0.20	0.76	0.96	0.283	1.67	1.38	4.48	3.64	7.95	6.44
			(3.04e-6)	(0.02)	(0.03)			(0.12)	(0.08)	(0.29)	(0.18)	(0.51)	(0.29)
0.40	0.60	0.750	1.29e-5	0.19	0.78	0.97	0.311	1.90	1.52	5.03	3.98	8.90	6.96
			(4.76e-6)	(0.02)	(0.03)			(0.16)	(0.10)	(0.40)	(0.25)	(0.66)	(0.38)

initial value and the value of the second segment. Therefore, the higher the estimate of annualized volatility, the closer the option price from the whole series is to the option price from the second segment. Hence, the low estimates of θ that we obtain if we neglect a change-point in α relative to neglecting a change-point in ω or β result in relatively lower estimates of annualized volatility on the whole sample.

Table 2.4: True option prices for change in ω. European at-the-money call option prices if parameter change points and values are known. GARCH(1,1) Option Pricing Model Duan (1995) with $r = 0$, $\lambda = 0.01$ and $h_t = \omega_i + 0.20\varepsilon^2_{t-1} + 0.60h_{t-1}$ for $i = 1, 2$. Strike price = Initial Stock Price = \$100. Call prices are the means of $5,000$ Monte Carlo simulation prices. For each Monte Carlo simulation, we use $10,000$ sample paths.

Parameter Changes			Call Prices c		
σ_2	ω_1	ω_2	5-day	30-day	90-day
0.10	3.20e-5	8.00e-6	0.49	1.33	2.36
0.15	3.20e-5	1.80e-5	0.73	2.00	3.54
0.20	3.20e-5	3.20e-5	0.97	2.66	4.72
0.25	3.20e-5	5.00e-5	1.22	3.33	5.89
0.30	3.20e-5	7.20e-5	1.46	4.00	7.07
0.35	3.20e-5	9.80e-5	1.70	4.66	8.25
0.40	3.20e-5	1.28e-4	1.94	5.32	9.42

Table 2.5: True option prices for change in α. European at-the-money call option prices if parameter change points and values are known. GARCH(1,1) Option Pricing Model Duan (1995) with $r = 0$, $\lambda = 0.01$ and $h_t = 3.2\text{e-}5 + \alpha_i\varepsilon^2_{t-1} + 0.60h_{t-1}$ for $i = 1, 2$. Strike price = Initial stock price = \$100. Call prices are the means of $5,000$ Monte Carlo simulation prices. For each Monte Carlo simulation, we use $10,000$ sample paths.

Parameter Changes			Call Prices c		
σ_2	α_1	α_2	5-day	30-day	90-day
0.15	0.20	0.044	0.75	2.04	3.57
0.20	0.20	0.200	0.97	2.66	4.72
0.25	0.20	0.272	1.19	3.24	5.80
0.30	0.20	0.311	1.41	3.78	6.77
0.35	0.20	0.335	1.63	4.28	7.65
0.40	0.20	0.350	1.85	4.75	8.41

Consistent with the results in Hillebrand (2005), we observe that the effect of a neglected parameter change in β on the *parameter estimates* of conditional volatility is smaller than the

Table 2.6: True option prices for change in β. European at-the-money call option prices if parameter change points and values are known. GARCH(1,1) Option Pricing Model Duan (1995) with $r = 0$, $\lambda = 0.01$ and $h_t = 3.2\text{e-}5 + 0.20\varepsilon_{t-1}^2 + \beta_i h_{t-1}$ for $i = 1, 2$. Strike price = Initial stock price = \$100. Call prices are the means of $5,000$ Monte Carlo simulation prices. For each Monte Carlo simulation, we use $10,000$ sample paths.

Parameter Changes			Call Prices c		
σ_2	β_1	β_2	5-day	30-day	90-day
0.15	0.60	0.444	0.73	2.02	3.55
0.20	0.60	0.600	0.97	2.66	4.72
0.25	0.60	0.672	1.21	3.29	5.86
0.30	0.60	0.711	1.45	3.91	6.96
0.35	0.60	0.735	1.69	4.53	8.04
0.40	0.60	0.750	1.93	5.12	9.06

effect of a change in ω. Since the estimated annualized volatilities from the whole sample in both cases are close to each other, however, the effect on *option prices* is similar to the effect of a change in ω. We also observe that as the magnitude of the parameter change increases, the variance of the observations in Figure 2.1 increases. This also holds for changes in the parameters α and β (see Figures 2.2 and 2.3, respectively).

For comparison, we report the true option prices assuming knowledge of the data-generating parameter values from the second segment in Tables 2.4, 2.5 and 2.6. To gauge bias and estimator variance, Table 2.7 reports the root mean-square errors for at-the-money options.

Table 2.7: Root Mean Square Errors for at-the-money European call option prices. σ_2 denotes annualized volatility of the segment after the change.

Root Mean Square Errors for At-the-Money Options

	Omega			Alpha			Beta		
σ_2	5-day	30-day	90-day	5-day	30-day	90-day	5-day	30-day	90-day
0.10	0.51	1.24	2.05	–	–	–	–	–	–
0.15	0.14	0.36	0.64	0.11	0.32	0.59	0.13	0.34	0.62
0.20	**0.03**	**0.07**	**0.13**	**0.03**	**0.07**	**0.13**	**0.03**	**0.07**	**0.13**
0.25	0.12	0.33	0.57	0.12	0.31	0.57	0.12	0.32	0.58
0.30	0.20	0.60	1.05	0.24	0.60	1.08	0.22	0.61	1.07
0.35	0.23	0.76	1.39	0.37	0.88	1.53	0.31	0.85	1.54
0.40	0.25	0.87	1.73	0.49	1.14	1.93	0.39	1.07	1.97

2.4.2 In-the-Money and Out-of-the-Money Options

In percentage terms, the effect of ignored changes in the parameters of conditional volatility on out-of-the-money option prices is substantially greater than it is on at-the-money option prices. The effect increases as the out-of-the-moneyness gets deeper. The price of a deep-out-of-the-money option is usually very close to zero and a change in the unconditional volatility that affects the probability of the option finishing in the money at expiration date has a large percentage impact on the price. Analogously, the price of a deep-in-the-money option is high and a change in the unconditional volatility that affects the probability of the option finishing in the money at expiration date has a small percentage impact on the price. Therefore, the price of a deep-in-the-money option is much less affected by an ignored change in one of the parameters of the conditional volatility process than the price of a deep-out-of-the-money option. This effect can also be seen in Table 2.9, where we report root mean square errors and the bias is measured in dollar terms. Results for in-the-money and out-of-the-money options in the case of a change in ω are given in Table 2.8. Results are similar for changes in α and β and are available upon request. Also, results for deep-in-the-money and deep-out-of-the-money options are available upon request.

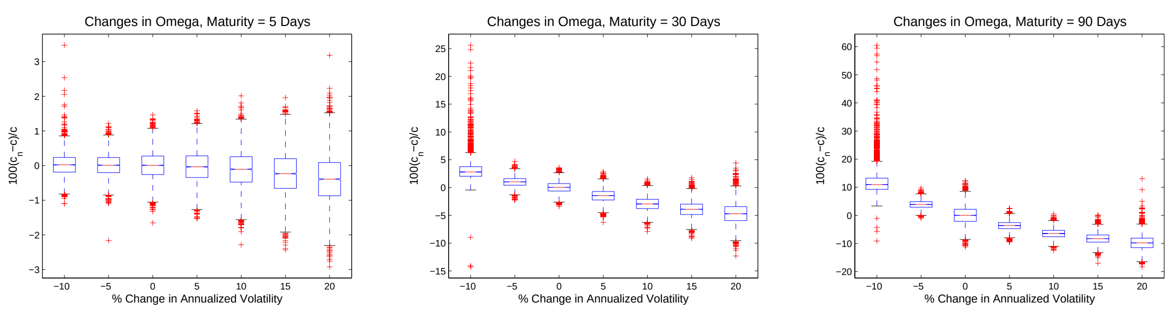

Figure 2.4: The effect of the change in parameter ω on European in-the-money call options in percentages. The vertical axis shows the percentage difference between the option price obtained from the whole sample ($\hat{c}_n$) without accounting for the parameter change, and the option price obtained from the second part of the sample ($\hat{c}$). The horizontal axis shows the percentage change in annualized volatility. For each value on the horizontal axis there are $5,000$ observations.

Table 2.8: The effect of a single neglected change-point in ω on the European in-the-money and out-of-the-money call option prices. GARCH(1,1) Option Pricing Model Duan (1995) with $r=0$, $\lambda=0.01$ and $h_t=\omega_i 10\text{e-}6+0.20\varepsilon_{t-1}^2+0.60h_{t-1}$ for $i=1,2$. $\hat\theta=\hat\alpha+\hat\beta$. Strike price is equal to (initial stock price/1.10) for in-the-money options and (initial stock price/0.90) for out-of-the-money options. Parameter estimates are the means of $5,000$ simulations. $\hat c$ is the call price calculated using the second half of the sample and $\hat c_n$ is the call price calculated using the whole sample without taking the parameter change into account.The annualized volatility σ_1 of the first segment is always 0.20. Standard errors are given in parentheses.

			Out-of-the-Money						In-the-Money					
Parameter Changes			5-Day		30-Day		90-Day		5-Day		30-Day		90-Day	
σ_2	ω_1	ω_2	$\hat c$	$\hat c_n$	$\hat c$	$\hat c_n$	$\hat c$	$\hat c_n$	$\hat c$	$\hat c_n$	$\hat c$	$\hat c_n$	$\hat c$	$\hat c_n$
0.10	3.20e-5	8.00e-6	0.00 (0.01)	0.00 (0.008)	0.01 (0.09)	0.26 (0.16)	0.11 (0.06)	1.21 (0.46)	9.09 (0.02)	9.09 (0.03)	9.10 (0.09)	9.38 (0.18)	9.24 (0.08)	10.34 (0.45)
0.15	3.20e-5	1.80e-5	0.00 (0.002)	0.00 (0.0004)	0.05 (0.014)	0.13 (0.021)	0.56 (0.07)	0.94 (0.08)	9.09 (0.02)	9.09 (0.02)	9.17 (0.05)	9.26 (0.06)	9.73 (0.10)	10.10 (0.11)
0.20	**3.20e-5**	**3.20e-5**	**0.00 (0.001)**	**0.00 (0.001)**	**0.21 (0.04)**	**0.21 (0.03)**	**1.30 (0.12)**	**1.30 (0.09)**	**9.09 (0.03)**	**9.09 (0.03)**	**9.34 (0.07)**	**9.34 (0.07)**	**10.45 (0.15)**	**10.45 (0.13)**
0.25	3.20e-5	5.00e-5	0.00 (0.002)	0.00 (0.001)	0.48 (0.06)	0.35 (0.04)	2.19 (0.17)	1.76 (0.11)	9.10 (0.03)	9.10 (0.03)	9.63 (0.10)	9.49 (0.08)	11.31 (0.19)	10.89 (0.15)
0.30	3.20e-5	7.20e-5	0.01 (0.01)	0.01 (0.006)	0.84 (0.09)	0.55 (0.06)	3.19 (0.22)	2.34 (0.16)	9.11 (0.04)	9.10 (0.03)	10.00 (0.12)	9.70 (0.09)	12.25 (0.24)	11.45 (0.18)
0.35	3.20e-5	9.80e-5	0.03 (0.009)	0.02 (0.006)	1.27 (0.12)	0.87 (0.11)	4.25 (0.27)	3.08 (0.23)	9.14 (0.04)	9.11 (0.04)	10.42 (0.15)	10.02 (0.13)	13.23 (0.28)	12.13 (0.25)
0.40	3.20e-5	1.28e-4	0.06 (0.02)	0.03) (0.01)	1.76 (0.14)	1.25 (0.19)	5.36 (0.32)	3.89 (0.39)	9.17 (0.05)	9.14 (0.05)	10.89 (0.17)	10.38 (0.20)	14.26 (0.33)	12.87 (0.38)

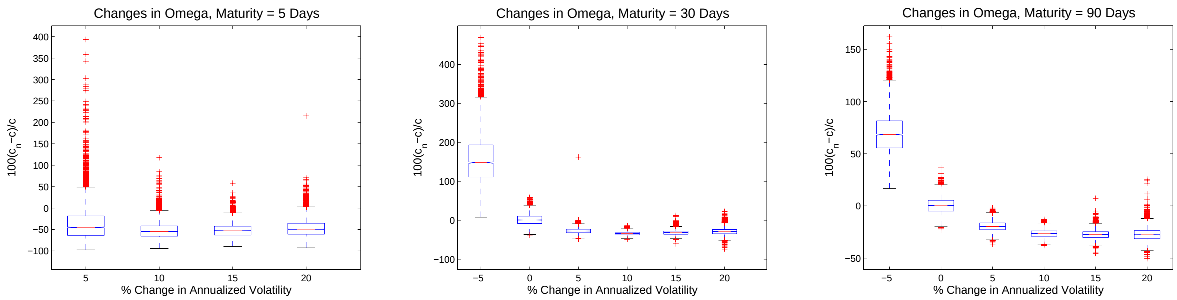

Figure 2.5: The effect of the change in parameter ω on European out-of-the-money call options in percentages. The vertical axis shows the percentage difference between the option price obtained from the whole sample ($\hat{c}_n$) without accounting for the parameter change, and the option price obtained from the second part of the sample ($\hat{c}$). The horizontal axis shows the percentage change in annualized volatility. For each value on the horizontal axis there are $5,000$ observations. For 5 days to maturity, a large proportion of the simulated option prices are close to zero in the case of negative changes and no change in annualized volatility. We therefore excluded these cases.

Table 2.9: Root mean square errors for European in-the-money and out-of-the-money call options when ω changes. Results for ω from Table 2.7 are added for easy comparison. σ_2 denotes annualized volatility of the segment after the change.

Root Mean Square Errors for changes in ω.

	At-the-Money			In-the-Money			Out-of-the-Money		
σ_2	5-day	30-day	90-day	5-day	30-day	90-day	5-day	30-day	90-day
0.10	0.51	1.24	2.05	0.03	0.33	1.19	0.01	0.30	1.19
0.15	0.14	0.36	0.64	0.02	0.11	0.39	0.00	0.08	0.39
0.20	**0.03**	**0.07**	**0.13**	**0.03**	**0.07**	**0.13**	**0.00**	**0.03**	**0.09**
0.25	0.12	0.33	0.57	0.03	0.16	0.44	0.00	0.14	0.45
0.30	0.20	0.60	1.05	0.03	0.31	0.81	0.01	0.29	0.86
0.35	0.23	0.76	1.39	0.04	0.43	1.12	0.02	0.42	1.19
0.40	0.25	0.87	1.73	0.06	0.55	1.44	0.03	0.54	1.51

2.5 Empirical Results

To study the effects of possible change-points in real data, we consider S&P500 index returns. The sample ranges from February 1, 1997 to August 20, 2007. First, we test for an unknown change-point using the statistic proposed by Kokoszka and Leipus (1999, 2000). We find that there is a single structural break on April 28, 2003 at 1% significance level. Looking at Figure 2.6, we see that until this date the index returns exhibit relatively higher volatility compared to the post-2003

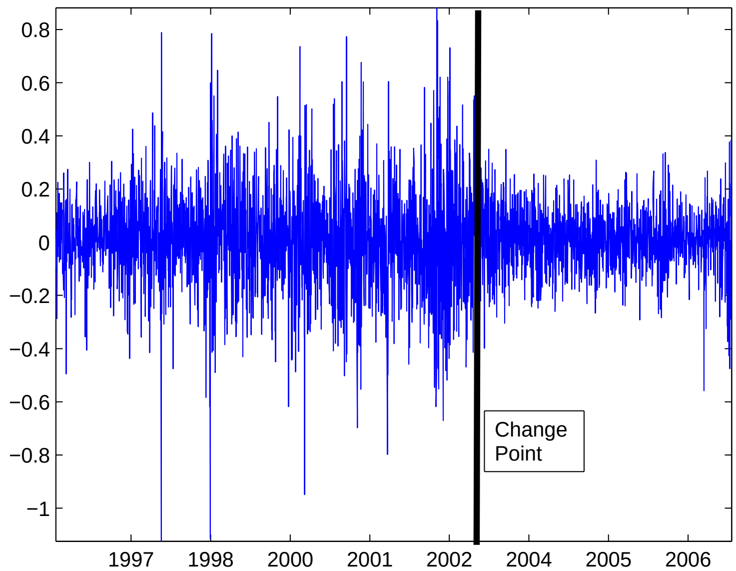

Figure 2.6: S&P500 returns between February 1, 1997 and August 20, 2007. Change-point at April 28, 2003 according to Kokoszka and Leipus (1999, 2000).

period.

If the GARCH(1,1) model (Equations (2.1) to (2.3)) is estimated without accounting for the detected change-point, we get the following results (standard errors in parentheses):

$$h_t = \underset{(5.69\text{e-}7)}{1.336\text{e-}6} + \underset{(0.013)}{0.072}\varepsilon_{t-1}^2 + \underset{(0.013)}{0.917}h_{t-1}.$$

We assume a zero interest rate for an easier comparison of option prices with different strikes and maturities. The estimated annualized unconditional volatility $\hat{\sigma}$ is equal to 0.18 and $\hat{\lambda}$ is equal to 0.07 ($s.e. = 0.02$).

If the model is estimated by segmenting the sample according to the estimated parameter change, we obtain the following results:

1-Feb-1997 through 28-Apr-2003:

$$h_t = \underset{(2.15\text{e-}6)}{9.332\text{e-}6} + \underset{(0.024)}{0.099}\varepsilon_{t-1}^2 + \underset{(0.031)}{0.849}h_{t-1},$$

$$\hat{\lambda}_1 = \underset{(0.024)}{0.057} \text{ and } \hat{\sigma}_1 = 0.212.$$

29-Apr-2003 through 20-Aug-2007:

$$h_t = \underset{(6.72\text{e-}7)}{2.038\text{e-}6} + \underset{(0.015)}{0.052}\varepsilon_{t-1}^2 + \underset{(0.021)}{0.911}h_{t-1},$$

$$\hat{\lambda}_2 = \underset{(0.022)}{0.08} \text{ and } \hat{\sigma}_2 = 0.116.$$

Table 2.10: European Call Prices - GARCH(1,1) Model. K is the strike price and, $S = \$1445$, is the initial stock price. $\hat{c}$ is the call price calculated by using the second half of the sample and $\hat{c}_n$ is the call price calculated by using the whole sample without taking the parameter change into account.

	5-Day		30-Day		90-Day	
S/K	$\hat{c}$	$\hat{c}_n$	$\hat{c}$	$\hat{c}_n$	$\hat{c}$	$\hat{c}_n$
0.90	0.00	0.00	0.11	1.88	3.00	13.67
1.00	8.40	12.96	22.61	34.61	39.81	60.34
1.10	131.37	131.35	131.59	134.02	135.38	146.92
1.15	188.48	188.47	188.52	189.11	189.44	195.96

The GARCH(1,1) estimation results show that there is a substantial shift in annualized unconditional volatility of the S&P500 return series. The estimated persistence parameter $\hat{\theta}$ equals approximately 0.99 if we ignore the change-point. European call option prices are given in Table 2.10 for different levels of moneyness S/K, where S is always equal to \$1445, the index price on Aug 20, 2007. The results strongly support our simulation experiments. If the option is at-the-money ($S/K = 1$), we observe roughly a 150% distortion from ignoring the change-point. If the option is out-of-the-money ($S/K < 1$), the effect is bigger. If the option is in-the-money ($S/K > 1$), the effect is smaller. The effect increases with time to maturity.

One of the widely accepted features of financial data is that it exhibits asymmetry between returns and volatility. It is well documented that asset returns are negatively correlated with volatility, which means that a negative shock to returns increases volatility more than a positive shock. This is the so-called leverage effect. To capture this feature of asset returns, Engle and Ng (1993) developed the Non-linear Asymmetric GARCH (NGARCH) model. The conditional variance process of the NGARCH(1,1) model is the following:

$$h_t = \omega + \alpha(\varepsilon_{t-1} - \gamma\sqrt{h_{t-1}})^2 + \beta h_{t-1}, \tag{2.15}$$

where γ is called the leverage parameter and all other variables are defined as before.

Under the locally risk-neutral measure Q, along with Equations (2.4) and (2.5), the conditional variance process follows:

$$h_t = \omega + \alpha\left(\zeta_{t-1} - (\lambda+\gamma)\sqrt{h_{t-1}}\right)^2 + \beta h_{t-1}, \tag{2.16}$$

where ζ_{t-1} is a different random variable than ε_{t-1} due to the measure change.

If the NGARCH(1,1) model (Equations (2.1), (2.2) and (2.15)) is estimated under measure P without accounting for the structural break, we get the following results (standard errors in parentheses):

$$h_t = \underset{(5.14\text{e-}7)}{2.284\text{e-}6} + \underset{(0.012)}{0.0695}(\varepsilon_{t-1} - \underset{(0.113)}{1.056}\sqrt{h_{t-1}})^2 + \underset{(0.027)}{0.841}h_{t-1}.$$

$$\hat{\lambda}_1 = \underset{(0.02)}{0.02} \text{ and } \hat{\sigma} = 0.214.$$

If the model is estimated by segmenting the sample according to the estimated parameter change, we get the following: 1-Feb-1997 through 28-Apr-2003:

$$h_t = \underset{(1.297\text{e-}6)}{7.135\text{e-}6} + \underset{(0.01)}{0.0496}(\varepsilon_{t-1} - \underset{(0.212)}{1.889}\sqrt{h_{t-1}})^2 + \underset{(0.053)}{0.739}h_{t-1},$$

$$\hat{\lambda}_1 = \underset{(0.017)}{0.005} \text{ and } \hat{\sigma}_1 = 0.227.$$

29-Apr-2003 through 20-Aug-2007:

$$h_t = \underset{(1.33\text{e-}6)}{4.59\text{e-}6} + \underset{(0.019)}{0.062}(\varepsilon_{t-1} - \underset{(0.282)}{1.549}\sqrt{h_{t-1}})^2 + \underset{(0.08)}{0.712}h_{t-1},$$

$$\hat{\lambda}_2 = \underset{(0.024)}{0.05} \text{ and } \hat{\sigma}_2 = 0.121.$$

Table 2.11: European Call Prices - NGARCH(1,1) Model. K is the strike price and, $S = \$1445$, is the initial stock price. $\hat{c}$ is the call price calculated by using the second half of the sample and $\hat{c}_n$ is the call price calculated by using the whole sample without taking the parameter change into account.

	5-Day		30-Day		90-Day	
S/K	$\hat{c}$	$\hat{c}_n$	$\hat{c}$	$\hat{c}_n$	$\hat{c}$	$\hat{c}_n$
0.90	0.00	0.00	0.00	0.00	1.50	3.08
1.00	5.98	6.26	20.70	21.75	40.55	48.97
1.10	131.35	131.36	131.97	132.26	138.22	144.79
1.15	188.47	188.48	188.56	188.71	191.35	195.96

The parameter estimates of the NGARCH(1,1) model exhibit similar results obtained from GARCH(1,1). The persistence parameter $\alpha(1+\gamma^2)+\beta$ is estimated at approximately 0.99. The unconditional volatility changes substantially in the second segment of the sample. European call option prices are reported in Table 2.11. The results exhibit similar characteristics. The effect of the neglected parameter change decreased somewhat for at-the-money and out-of-the-money options.

2.6 Summary

We analyzed the effect of ignored parameter changes in the parameters of the conditional variance of a GARCH(1,1) model on option prices. Ignoring such parameter changes and assuming a constant unconditional volatility result in biased parameter estimates. Unconditional volatility is an important determinant of option prices and biased estimation leads to biased option prices. Our Monte Carlo simulation experiments provide evidence for this intuition. For at-the-money options, we observe substantial price distortions as a result of the ignored change-point. The bias is more pronounced for out-of-the-money options and increases as options move deeper out-of-the-money. The bias decreases for in-the-money options and becomes negligible as options move deeper in-the-money. In our simulation experiments, we observe that negative changes in unconditional volatility affect option prices relatively more than positive changes in unconditional volatility. An empirical analysis of S&P500 index returns supports our simulation experiments and provides evidence that if parameter changes in the conditional variance are ignored, option prices are biased. Using the NGARCH model, which accounts for the leverage effect, decreases the bias, but does not eliminate it.

Chapter 3

Closed-form GARCH Option Pricing Model and Ignored Parameter Changes

3.1 Introduction

One of the main disadvantages of GARCH option pricing models is the fact that there is no closed-form solution and Monte Carlo simulation methods have to be used to price European options. Heston and Nandi (2000) propose a GARCH option pricing model that has a closed-form solution for the option price. Their model captures both leverage effects and time-varying volatility, which have indisputably become two of the most important empirical facts about financial volatility time series.

As mentioned earlier in Chapter 1, the leverage effect refers to the negative correlation of volatility with stock returns. The GARCH models that capture the leverage effect generate negative skewness in the risk-neutral distribution of stock returns, which means that the probability mass in the negative tail of the return distribution is greater than in the positive tail of the distribution. Therefore, these models generate lower call option prices. In simpler words, a decrease in the volatility of stock returns leads to two effects: 1) a decrease in the value of a call option (due to positive Vega) and 2) higher stock returns, which in turn leads to an increase in the value of a call option. Our study simulation study shows that the first effect dominates the second one and the net effect is a decrease in the value of a call option.

In Section 2.5, we studied change-points in S&P500 index returns. Since the index return series switches from a high to low volatility regime on April 28, 2003 in our sample, the estimation of the unconditional volatility is overestimated when the regime-switch is ignored. This creates a positive bias in option prices. However, when we incorporate the leverage effect the bias decreases. Increases in volatility generate higher option prices, but because of the negative correlation between stock price and volatility, the stock price decreases and so does the value of a call option. In other words, the effect of an increase in volatility is partially offset by the decrease in the stock price.

In this chapter, I analyze this issue with a simulation study. I analyze the effect of ignored parameter changes in the parameters of the data-generating GARCH model on European call option prices using Heston's and Nandi's (2000) closed-form GARCH option valuation model. Heston's and Nandi's (2000) model is similar to Engle's and Ng's (1993) model with a slight change, which is essential in obtaining the closed-form solution. In a simulation study, I show that neglecting parameter changes in the Heston and Nandi (2000) closed-form model also creates biased option prices. The results are presented in Section 3.4.

I also present the disproof of the solution for the model when it has a higher order than $p = q = 1$. The closed-form solution of the model depends on the inversion theorem of characteristic functions. Specifically, if one can write the characteristic function of the stock price process, then by using numerical inversion methods, the probability density function (pdf) of the stock prices can be obtained. The pdf can then be used to evaluate option prices. The problem with the model for higher orders than $p = q = 1$ is in the derivation of the characteristic function. The details of the model and the disproof are presented in the next section. The closed-form solution of Heston and Nandi (2000) works well for $p = q = 1$ and can be used to estimate European option prices.

3.2 The Model

The closed-form solution presented in Heston and Nandi (2000) depends on the inversion of the characteristic function of the stock price process. I start with the brief description of the inversion method.

The characteristic function $\varphi_Y(\phi)$ of a random variable Y is defined as:

$$\varphi_Y(\phi) = \mathbb{E}[\exp(i\phi y)],$$

where $t \in \Re$ and i is the imaginary unit.

If the cumulative distribution function of Y is F_Y, then the Fourier Transform of the pdf f_Y of Y is:

$$\varphi_Y(\phi) = \int_{-\infty}^{\infty} \exp(i\phi y) f_Y(y) dy,$$

which can also be written as:

$$\varphi_Y(\phi) = \int_{-\infty}^{\infty} \exp(i\phi y) dF_Y(y),$$

where F_Y is the cdf of Y.

Then, by applying the inverse Fourier transform, the pdf of Y can be obtained from the characteristic function:

$$f_Y(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\phi y) \varphi_Y(\phi) d\phi.$$

Next, I will go through the derivation of the closed-form GARCH option pricing model. I follow the notation used in Chapter 2 and Heston and Nandi (2000). In the setup of the model in Heston and Nandi (2000), assuming time step length $\triangle = 1$, the logarithm of the stock price S_t follows a GARCH process given as:

$$\log(S_t) = \log(S_{t-1}) + r + \lambda h_t + \sqrt{h_t} z_t, \tag{3.1}$$

$$h_t = \omega + \sum_{i=1}^{p} \beta_i h_{t-i} + \sum_{i=1}^{q} \alpha_i (z_{t-i} - \gamma_i \sqrt{h_{t-i}})^2, \tag{3.2}$$

$$z_t | F_{t-1} \sim N(0,1), \tag{3.3}$$

where r is the risk-free interest rate; λ is the unit risk premium; ω constant, α_i and β_i are autoregressive parameters; γ_i is the leverage parameter; z_t is a normally distributed innovation with mean zero and variance one and F_t is a σ-field of all information up to and including time t. Given these specifications, the stock returns $\log(S_t/S_{t-1})$ follow a normal distribution with mean zero and time-varying variance h_t.

This specification of the model allows for a closed-form solution to exist. Feller (1971) and Kendall and Stuart (1977) show that probability density functions can be recovered by inverting characteristic functions, a brief description of which is given above. Heston and Nandi (2000) apply this method to the above model and obtain a closed-form solution. The details of the derivation below shows that the solution does not hold for orders of p and q higher than 1.

Let $x_t = \log(S_t)$ and let $f(t;T,\phi)$ be the conditional moment generating function of X_T,

$$f_t := f(t;T,\phi) = \mathbb{E}_t[\exp(\phi x_T)]. \tag{3.4}$$

For notational convenience, the function arguments are dropped hereafter. The functional form of the conditional moment generating function of x_T proposed by Heston and Nandi (2000) is given as follows:

$$f_t = \exp\left(\phi x_t + A_t + \sum_{i=1}^{p} B_{i,t} h_{t+2-i} + \sum_{i=1}^{q-1} C_{i,t} \left(z_{t+1-i} - \gamma_i \sqrt{h_{t+1-i}}\right)^2\right). \tag{3.5}$$

Next, by applying the law of iterated expectations to f_t we get

$$
\begin{aligned}
f_t &= \mathbb{E}_t\left[f_{t+1}\right] \\
&= \mathbb{E}_t\left[\exp\left\{\phi x_{t+1} + A_{t+1} + \sum_{i=1}^{p} B_{i,t+1} h_{t+3-i} + \sum_{i=1}^{q-1} C_{i,t+1}\left(z_{t+2-i} - \gamma_i\sqrt{h_{t+2-i}}\right)^2\right\}\right].
\end{aligned}
\tag{3.6}
$$

Substituting Equations (3.1) and (3.2) gives:

$$
f_t = \mathbb{E}_t\left[\exp\left\{
\begin{array}{l}
\phi\left(x_t + r + \lambda h_{t+1} + \sqrt{h_{t+1}} z_{t+1}\right) + A_{t+1} \\
+B_{1,t+1}\left(\beta_1 h_{t+1} + \alpha_1\left(z_{t+1} - \gamma_1\sqrt{h_{t+1}}\right)^2\right) \\
+B_{1,t+1}\left(\begin{array}{l} \omega + \sum_{i=1}^{p-1} \beta_{i+1} h_{t+1-i} \\ + \sum_{i=1}^{q-1} \alpha_{i+1}\left(z_{t+1-i} - \gamma_{i+1}\sqrt{h_{t+1-i}}\right)^2 \end{array}\right) \\
+\sum_{i=1}^{p-1} B_{i+1,t+1} h_{t+2-i} + C_{1,t+1}\left(z_{t+1} - \gamma_1\sqrt{h_{t+1}}\right)^2 \\
+\sum_{i=1}^{q-2} C_{i+1,t+1}\left(z_{t+1-i} - \gamma_{i+1}\sqrt{h_{t+1-i}}\right)^2
\end{array}
\right\}\right].
\tag{3.7}
$$

Rearranging Equation (3.7) gives:

$$
f_t = \mathbb{E}_t\left[\exp\left\{
\begin{array}{l}
\phi\left(x_t + r\right) + \phi\lambda h_{t+1} + \phi\sqrt{h_{t+1}} z_{t+1} + A_{t+1} \\
+B_{1,t+1}\beta_1 h_{t+1} + B_{1,t+1}\alpha_1\left(z_{t+1} - \gamma_1\sqrt{h_{t+1}}\right)^2 \\
+B_{1,t+1}\omega + B_{1,t+1}\sum_{i=1}^{p-1} \beta_{i+1} h_{t+1-i} \\
+B_{1,t+1}\sum_{i=1}^{q-1} \alpha_{i+1}\left(z_{t+1-i} - \gamma_{i+1}\sqrt{h_{t+1-i}}\right)^2 \\
+\sum_{i=1}^{p-1} B_{i+1,t+1} h_{t+2-i} + C_{1,t+1}\left(z_{t+1} - \gamma_1\sqrt{ht+1}\right)^2 \\
+\sum_{i=1}^{q-2} C_{i+1,t+1}\left(z_{t+1-i} - \gamma_{i+1}\sqrt{h_{t+1-i}}\right)^2
\end{array}
\right\}\right].
\tag{3.8}
$$

Now, define $\Omega = \alpha_1 B_{1,t+1} + C_{1,t+1}$, collect terms and rearrange:

$$f_t = \mathbb{E}_t \left[\exp \left\{ \begin{array}{l} \phi\left(x_t + r\right) + A_{t+1} + B_{1,t+1}\omega + \Omega\left(z_{t+1} - \gamma_1\sqrt{h_{t+1}}\right)^2 \\ + \left(\phi\lambda + B_{1,t+1}\beta_1\right) h_{t+1} + \phi\sqrt{h_{t+1}} z_{t+1} \\ + B_{1,t+1}\sum_{i=1}^{p-1}\beta_{i+1}h_{t+1-i} + \sum_{i=1}^{p-1} B_{i+1,t+1}h_{t+2-i} \\ + B_{1,t+1}\sum_{i=1}^{q-1}\alpha_{i+1}\left(z_{t+1-i} - \gamma_{i+1}\sqrt{h_{t+1-i}}\right)^2 \\ + \sum_{i=1}^{q-2} C_{i+1,t+1}\left(z_{t+1-i} - \gamma_{i+1}\sqrt{h_{t+1-i}}\right)^2 \end{array} \right\} \right]. \tag{3.9}$$

Complete the square by adding and subtracting $\gamma_1\phi h_{t+1}$ and $\frac{\phi^2}{4\Omega}h_{t+1}$. Rearranging the terms will give:

$$f_t = \mathbb{E}_t \left[\exp \left\{ \begin{array}{l} \phi\left(x_t + r\right) + A_{t+1} + \omega B_{1,t+1} + \Omega\left(z_{t+1} - \left(\gamma_1 - \frac{\phi}{2\Omega}\right)\sqrt{h_{t+1}}\right)^2 \\ + \left(\phi\lambda + B_{1,t+1}\beta_1 + \phi\gamma_1 - \frac{\phi^2}{4\Omega}\right) h_{t+1} + B_{1,t+1}\sum_{i=1}^{p-1}\beta_{i+1}h_{t+1-i} \\ + \sum_{i=1}^{p-1} B_{i+1,t+1}h_{t+2-i} + B_{1,t+1}\sum_{i=1}^{q-1}\alpha_{i+1}\left(z_{t+1-i} - \gamma_{i+1}\sqrt{h_{t+1-i}}\right)^2 \\ + \sum_{i=1}^{q-2} C_{i+1,t+1}\left(z_{t+1-i} - \gamma_{i+1}\sqrt{h_{t+1-i}}\right)^2 \end{array} \right\} \right]. \tag{3.10}$$

All the terms in Equation (3.10) are known at time t, except $\Omega\left(z_{t+1} - \left(\gamma_1 - \frac{\phi}{2\Omega}\right)\sqrt{h_{t+1}}\right)^2$. Apply the fact $\mathbb{E}\left[\exp(a(z+b)^2)\right] = \exp\left(-\frac{1}{2}\log(1-2a) + \frac{ab^2}{1-2a}\right)$, where z is a standard normal random variable, to $\Omega\left(z_{t+1} - \left(\gamma_1 - \frac{\phi}{2\Omega}\right)\sqrt{h_{t+1}}\right)^2$:

$$\mathbb{E}_t\left[\exp\left(\Omega\left(z_{t+1} - \left(\gamma_1 - \frac{\phi}{2\Omega}\right)\sqrt{h_{t+1}}\right)^2\right)\right] = \exp\left[-\frac{1}{2}\log(1-2\Omega) + \frac{\Omega\left(\frac{\phi}{2\Omega} - \gamma_1\right)^2 h_{t+1}}{1-2\Omega}\right].$$

Now, substituting this result in Equation (3.10) gives:

$$f_t = \exp\left\{\begin{array}{l} \phi\left(x_t + r\right) + A_{t+1} + \omega B_{1,t+1} - \dfrac{1}{2}\log(1-2\Omega) \\ + \left(\phi\lambda + B_{1,t+1}\beta_1 + \phi\gamma_1 - \dfrac{\phi^2}{4\Omega} + \dfrac{\Omega\left(\frac{\phi}{2\Omega} - \gamma_1\right)^2}{1-2\Omega}\right) h_{t+1} \\ +B_{1,t+1}\sum_{i=1}^{p-1}\beta_{i+1}h_{t+1-i} + \sum_{i=1}^{p-1} B_{i+1,t+1}h_{t+2-i} \\ +B_{1,t+1}\sum_{i=1}^{q-1}\alpha_{i+1}\left(z_{t+1-i} - \gamma_{i+1}\sqrt{h_{t+1-i}}nn\right)^2 \\ + \sum_{i=1}^{q-2} C_{i+1,t+1}\left(z_{t+1-i} - \gamma_{i+1}\sqrt{h_{t+1-i}}n\right)^2 \end{array}\right\}. \tag{3.11}$$

Complete the square by adding and subtracting $\frac{1}{2}\gamma_1^2 h_{t+1}$. Rearranging gives:

$$f_t = \exp\left\{\begin{array}{l} \phi x_t + \phi r + A_{t+1} + \omega B_{1,t+1} - \dfrac{1}{2}\log(1-2\Omega) \\ + \left(\phi(\lambda+\gamma_1) + B_{1,t+1}\beta_1 - \dfrac{1}{2}\gamma_1^2 + \dfrac{1/n2(\phi-\gamma_1)^2}{1-2\Omega}n\right) h_{t+1} \\ +B_{1,t+1}\sum_{i=1}^{p-1}\beta_{i+1}h_{t+1-i} + \sum_{i=1}^{p-1} B_{i+1,t+1}h_{t+2-i}n \\ +B_{1,t+1}\sum_{i=1}^{q-1}\alpha_{i+1}\left(z_{t+1-i} - \gamma_{i+1}\sqrt{h_{t+1-i}}\right)^2 \\ + \sum_{i=1}^{q-2} C_{i+1,t+1}\left(z_{t+1-i} - \gamma_{i+1}\sqrt{h_{t+1-i}}\right)^2 \end{array}\right\}. \tag{3.12}$$

Now, rearrange the guess function (equation (3.5)) to make it easily tractable when we solve for the coefficients $A(.), B_i(.)$ and $C_i(.)$.

$$f_t = \exp\left(\begin{array}{l} \phi x_t + A_t + B_{1,t}h_{t+1} + \sum_{i=1}^{p} B_{i+1,t}h_{t+1-i} \\ + \sum_{i=1}^{q-1} C_{i,t}\left(z_{t+1-i} - \gamma_i\sqrt{h_{t+1-i}}\right)^2 \end{array}\right). \tag{3.13}$$

Comparing Equations (3.12) and (3.13), we get the following results:

$$A_t = \phi r + A_{t+1} + \omega B_{1,t+1} - \frac{1}{2}\log(1-2\Omega), \tag{3.14}$$

$$B_{1,t} = \phi(\lambda+\gamma_1) + B_{1,t+1}\beta_1 - \frac{1}{2}\gamma_1^2 + \frac{1/2(\phi-\gamma_1)^2}{1-2\Omega} + B_{2,t+1}. \tag{3.15}$$

$$B_{i,t} = B_{1,t+1}\beta_i + B_{i+1,t+1}, \text{ for } 2 \leqslant i \leqslant p. \tag{3.16}$$

Equations (3.14)-(3.16) are the same as the ones given in Heston and Nandi (2000). However, the problem occurs in the $C_i(.)$ terms. The solution for the $C_i(.)$ terms in Heston and Nandi (2000) is given as follows:

$$C_{i,t} = \alpha_{i+1}B_{1,t+1} + C_{i+1,t+1}, \text{ for } 1 \leqslant i \leqslant q-1. \tag{3.17}$$

To see that Equation (3.17) does not give the correct result, let's take the $C_{1,t}$ terms from Equation (3.13) (For simplicity $i=1$ is chosen. The result holds for all i). Compare Equation (3.13)to Equation (3.17): In Equation (3.13), $C_{1,t}$ is the coefficient of $\left(z_t - \gamma_1\sqrt{h_t}\right)^2$. So, $C_{1,t}$ must be equal to the coefficients of $\left(z_t - \gamma_1\sqrt{h_t}\right)^2$ in Equation (3.12). For $i=1$, the last two terms in Equation (3.12) give:

$$\alpha_2 B_{1,t+1}\left(z_t - \gamma_2\sqrt{h_t}\right)^2 + C_{2,t+1}\left(z_t - \gamma_2\sqrt{h_t}\right)^2. \tag{3.18}$$

The solution for $C_{1,t}$ in Equation (3.17) is given as:

$$C_{1,t} = \alpha_2 B_{1,t+1} + C_{2,t+1}.$$

In order for this expression to satisfy Equation (3.13), $(\alpha_2 B_{1,t+1} + C_{2,t+1})$ from Equation (3.18) must be the coefficient of $\left(z_t - \gamma_1\sqrt{h_t}\right)^2$, whereas Equation (3.18) show that it is the coefficient of $\left(z_t - \gamma_2\sqrt{h_t}\right)^2$. Although it looks like a small typographical error at the end, it cannot be corrected in the way it is given. The guess function in Equation (3.5) has to be redefined and all the steps above must be repeated to see if the guess function and the iterated solution match. However, we have not been able to find a correct form of Equation (3.5) and to the best of our knowledge, there has been no correction provided in the literature yet.

The model works for the Heston and Nandi (2000) version of the GARCH(1,1) specification, which is given as:

$$\log(S_t) = \log(S_{t-1}) + r + \lambda h_t + \sqrt{h_t} z_t, \tag{3.19}$$

$$h_t = \omega + \beta h_{t-1} + \alpha \left(z_{t-1} - \gamma \sqrt{h_{t-1}} \right)^2, \tag{3.20}$$

$$z_t | F_{t-1} \sim N(0,1). \tag{3.21}$$

To calculate option prices we need the risk-neutral distribution of the stock price, which is shown by Heston and Nandi (2000) to be:

$$\log(S_t) = \log(S_{t-1}) + r - 0.5 h_t + \sqrt{h_t} z_t^*, \tag{3.22}$$

$$h_t = \omega + \beta h_{t-1} + \alpha \left(z_{t-1}^* - \gamma^* \sqrt{h_{t-1}} \right)^2, \tag{3.23}$$

where

$$z_t^* = z_t + (\lambda + 0.5)\sqrt{h_t},$$
$$\gamma^* = \gamma + \lambda + 0.5.$$

The moment generating function of the logarithm of the stock price is as follows:

$$f_t(\phi) = S_t^{\phi} \exp\left(A_t + B_t h_{t+1}\right), \tag{3.24}$$

where

$$A_t = A_{t+1} + \phi r + \omega B_{t+1} - 0.5 \log\left(1 - 2\alpha B_{t+1}\right), \tag{3.25}$$

$$B_t = \phi(\lambda + \gamma) - 0.5\gamma^2 + \beta_1 B_{t+1} + \frac{0.5(\phi - \gamma)^2}{1 - 2\alpha B_{t+1}}. \tag{3.26}$$

Equations (3.25) and (3.26) are solved recursively with the terminal conditions:

$$A_T = B_T = 0,$$

since x_T is known at time T.

Following Feller (1971) and Kendall and Stuart (1977), Heston and Nandi (2000) show that if the characteristic function of the logarithm of the stock price is $f(i\phi)$, then

$$E_t[Max(S_T - K, 0)] = \begin{bmatrix} f(1)\left(0.5 + \frac{1}{\pi}\int_0^\infty Re\left[\frac{K^{-i\phi}f(i\phi+1)}{i\phi f(1)}\right]d\phi\right) \\ -K\left(0.5 + \frac{1}{\pi}\int_0^\infty Re\left[\frac{K^{-i\phi}f(i\phi)}{i\phi}\right]d\phi\right) \end{bmatrix}, \tag{3.27}$$

where $Re[\ .\ ]$ is the real part of a complex number.

Given all the above information, the price of a European call option that expires at time T and with a strike price K is given as:

$$\begin{aligned} C &= e^{-r(T-t)}E_t^*[Max(S_T - K, 0)] \\ &= \begin{bmatrix} 0.5S_t + \frac{e^{-r(T-t)}}{\pi}\int_0^\infty Re\left[\frac{K^{-i\phi}f^*(i\phi+1)}{i\phi}\right]d\phi \\ -Ke^{-r(T-t)}\left(0.5 + \frac{1}{\pi}\int_0^\infty Re\left[\frac{K^{-i\phi}f^*(i\phi)}{i\phi}\right]d\phi\right) \end{bmatrix}, \end{aligned} \tag{3.28}$$

where $E^*[\ .\ ]$ is the expectation with respect to risk-neutral probability measure and $f^*()$ is the characteristic function of the risk-neutral model given in equations (3.22) and (3.23). We evaluate the integrals in Equation (3.28) numerically in C++ by using numerical integration routine qromo() provided in Press et al. (2002).

3.3 Simulation Methodology

First, a series of stock prices is simulated under the physical probability measure P. The series has $4,000$ observations with a parameter change at observation $2,001$. The model used is given in equations (3.17)-(3.19) with a parameter change at observation $2,001$:

$$\log(S_t) = \log(S_{t-1}) + r + \lambda h_t + \sqrt{h_t} z_t, \tag{3.29}$$

$$h_t = \omega_i + \beta_i h_{t-1} + \alpha_i \left(z_{t-1} - \gamma \sqrt{h_{t-1}} \right)^2, \text{ for } i = 1, 2. \tag{3.30}$$

$$z_t | F_{t-1} \sim N(0,1), \tag{3.31}$$

where $i = 1$ denotes the first regime of $2,000$ observations and $i = 2$ denotes the second regime of $2,000$ observations. We set initial volatility equal to the unconditional mean.

After simulating the system above, the parameters (a) for the whole series and (b) for the last $2,000$ observations are estimated by maximum likelihood. Estimated parameters are used to evaluate the characteristic function given in Equations (3.24)-(3.26) after replacing ϕ with $i\phi$ wherever it appears. Option prices are then estimated by using Equation (3.28). This process is repeated 500 times for each of the scenarios. The numbers in Tables 3.1 through 3.3 and Figures 3.1 through 3.3 are the means of these 500 call prices. For brevity, I analyze only the changes in ω. The changes in α and β are analogous.

3.4 Simulation Results

In the following, I analyze the effect of changes in ω of the conditional volatility process on at-the-money, out-of-the-money, and in-the-money European Call options. The model given above from Equation (3.29) to (3.31) is simulated for a single change in ω at observation 2001. The initial stock price S_t is assumed to be \$100. We keep the interest rate r at zero to make cross-maturity analysis easier. Following Heston and Nandi (2000), I choose $\gamma = 421$ and $\lambda = 0.2$.

The annualized volatility σ is equal to $\sqrt{250(\omega+\alpha)/(1-\gamma^2\alpha-\beta)}$. σ_1 denotes the annualized volatility of the first half of the series and is set equal to 20% initially. σ_2 denotes the annualized volatility of the second half of the series and is set to different values, which in turn determines the change in ω, as in the method followed in Chapter 2. I initially set $\omega_1 = 2.5\text{e-}5$, $\alpha_1 = 1.33\text{e-}6$, and $\beta_1 = 0.60$, so that the initial annualized volatility is 20%. For example, to study the effect of the change in ω when the annualized volatility increases to 35%, ω_2 is changed accordingly to 7.92e-5. The results are presented in Tables 3.1-3.3 and Figures 3.1-3.3 below.

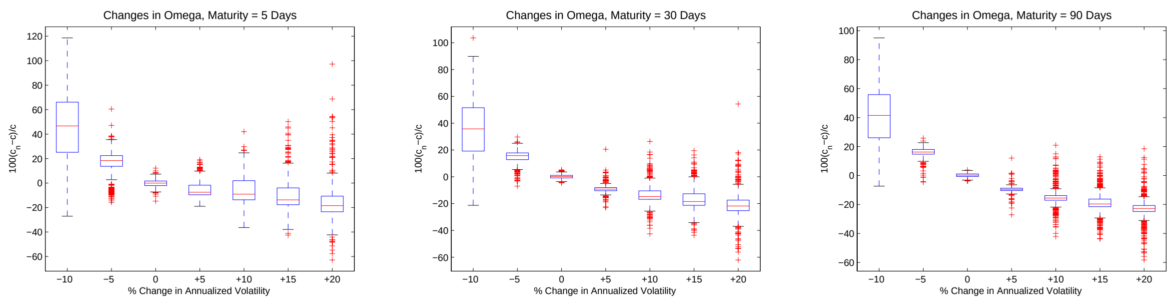

Figure 3.1: The effect of the change in parameter ω on European at-the-money call options in percentages. The vertical axis shows the percentage difference between the option price obtained from the whole sample ($\hat{c}_n$) without accounting for the parameter change, and the option price obtained from the second part of the sample ($\hat{c}$). The horizontal axis shows the percentage change in annualized volatility. For each value on the horizontal axis there are 500 observations.

When the parameter change is ignored, the estimated persistence of the model increases spuriously, as in the case of the Duan (1995) model analyzed in Chapter 2. This can be seen from the values of θ ($= \gamma^2\alpha + \beta$) in the tables. As the magnitude of the change in the annualized volatility σ increases, the overestimation of the persistence of the model increases. The persistence of the model implied by the initial values of the parameters is 0.84. Although the model's persistence does not change when ω changes from one regime to another, when the annualized volatility is decreased to 0.10 from 0.20 by decreasing ω, the estimated persistence increases to 0.99 if the parameter change is ignored. These findings are in line with the regime-switching literature reviewed

Table 3.1: The effect of a single neglected change-point in ω on the European at-the-money call option price. Heston and Nandi (2000) Closed-form GARCH(1,1) option pricing model with $r = 0$, $\lambda = 0.20$ and $h_t = \omega_i + 0.60h_{t-1} + 1.33\text{e-}6(z_{t-1} - 421\sqrt{h_{t-1}})^2$ for $i = 1, 2$. $\hat{\theta} = \hat{\gamma}^2\hat{\alpha} + \hat{\beta}$. Strike price = Initial stock price = \$100. Parameter estimates are the means of 500 simulations. $\hat{c}$ is the call price calculated using the second half of the sample and $\hat{c}_n$ is the call price calculated using the whole sample without taking the parameter change into account. The annualized volatility σ_1 of the first segment is always 0.20. Standard errors are given in parentheses.

Parameter Changes			Parameter Estimates (Whole Sample)						5-Day		30-Day		90-Day	
σ_2	ω_1	ω_2	$\hat{\omega}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\sigma}$	$\hat{c}$	$\hat{c}_n$	$\hat{c}$	$\hat{c}_n$	$\hat{c}$	$\hat{c}_n$
0.10	2.50e-5	5.28e-6	6.19e-6 (1.21e-6)	9.93e-7 (3.64e-7)	0.81 (0.07)	419.99 (3.59)	0.99	0.167	0.42 (0.04)	0.61 (0.13)	1.30 (0.04)	1.75 (0.31)	2.36 (0.05)	3.33 (0.45)
0.15	2.50e-5	1.35e-5	1.16e-5 (6.11e-6)	1.13e-6 (3.26e-7)	0.71 (0.10)	410.61 (43.31)	0.90	0.179	0.68 (0.08)	0.78 (0.06)	1.99 (0.07)	2.27 (0.10)	3.56 (0.07)	4.12 (0.13)
0.20	**2.50e-5**	**2.50e-5**	**2.51e-5 (7.16e-6)**	**1.34e-6 (2.30e-7)**	**0.59 (0.08)**	**423.80 (18.96)**	**0.84**	**0.200**	**0.88 (0.05)**	**0.88 (0.05)**	**2.62 (0.06)**	**2.62 (0.06)**	**4.73 (0.09)**	**4.73 (0.06)**
0.25	2.50e-5	3.97e-5	2.47e-5 (1.32e-5)	1.19e-6 (4.01e-7)	0.67 (0.12)	413.79 (36.08)	0.88	0.228	1.08 (0.09)	1.02 (0.07)	3.26 (0.10)	2.96 (0.10)	5.91 (0.12)	5.35 (0.13)
0.30	2.50e-5	5.78e-5	1.48e-5 (1.86e-5)	8.75e-7 (4.95e-7)	0.80 (0.13)	412.80 (23.68)	0.94	0.264	1.30 (0.14)	1.22 (0.15)	3.91 (0.15)	3.39 (0.34)	7.13 (0.16)	6.06 (0.54)
0.35	2.50e-5	7.92e-5	1.09e-5 (1.51e-5)	7.97e-7 (6.66e-7)	0.84 (0.10)	412.00 (37.50)	0.97	0.320	1.56 (0.16)	1.39 (0.18)	4.60 (0.19)	3.83 (0.43)	8.38 (0.19)	6.82 (0.71)
0.40	2.50e-5	1.04e-4	1.11e-5 (1.25e-5)	9.54e-7 (5.73e-7)	0.81 (0.08)	411.47 (33.06)	0.98	0.344	1.81 (0.20)	1.52 (0.22)	5.26 (0.27)	4.16 (0.54)	9.63 (0.29)	7.44 (0.87)

Table 3.2: The effect of a single neglected change-point in ω on the European in-the-money and out-of-the-money call option prices. Heston and Nandi (2000) closed-form GARCH(1,1) option pricing model with $r = 0$, $\lambda = 0.20$ and $h_t = \omega_i + 0.60h_{t-1} + 1.33\text{e-}6(z_{t-1} - 421\sqrt{h_{t-1}})^2$ for $i = 1, 2$. $\hat{\theta} = \hat{\gamma}^2\hat{\alpha} + \hat{\beta}$. Strike price is equal to (initial stock price/1.10) for in-the-money options and (initial stock price/0.90) for out-of-the-money options. Parameter estimates are the means of 500 simulations. $\hat{c}$ is the call price calculated using the second half of the sample and $\hat{c}_n$ is the call price calculated using the whole sample without taking the parameter change into account. The annualized volatility σ_1 of the first segment is always 0.20.Standard errors are given in parentheses.

			Out-of-the-Money						In-the-Money					
Parameter Changes			5-Day		30-Day		90-Day		5-Day		30-Day		90-Day	
σ_2	ω_1	ω_2	$\hat{c}$	$\hat{c}_n$	$\hat{c}$	$\hat{c}_n$	$\hat{c}$	$\hat{c}_n$	$\hat{c}$	$\hat{c}_n$	$\hat{c}$	$\hat{c}_n$	$\hat{c}$	$\hat{c}_n$
0.10	2.50e-5	5.28e-6	0.00 (0.001)	0.00 (0.003)	0.00 (0.002)	0.00 (0.01)	0.00 (0.006)	0.00 (0.13)	9.10 (0.001)	9.10 (0.003)	9.14 (0.004)	9.24 (0.07)	9.38 (0.02)	10.03 (0.28)
0.15	2.50e-5	1.35e-5	0.00 (0.002)	0.00 (0.002)	0.00 (0.003)	0.00 (0.002)	0.50 (0.03)	0.25 (0.09)	9.10 (0.002)	9.11 (0.002)	9.26 (0.02)	9.35 (0.03)	9.98 (0.08)	10.38 (0.05)
0.20	**2.50e-5**	**2.50e-5**	**0.00 (0.002)**	**0.00 (0.002)**	**0.00 (0.02)**	**0.00 (0.01)**	**0.89 (0.06)**	**0.89 (0.04)**	**9.11 (0.002)**	**9.11 (0.002)**	**9.48 (0.03)**	**9.48 (0.02)**	**10.80 (0.07)**	**10.80 (0.05)**
0.25	2.50e-5	3.97e-5	0.00 (0.004)	0.00 (0.003)	0.15 (0.04)	0.05 (0.03)	1.71 (0.09)	1.28 (0.10)	9.12 (0.004)	9.11 (0.005)	9.79 (0.06)	9.64 (0.05)	11.75 (0.10)	11.29 (0.11)
0.30	2.50e-5	5.78e-5	0.00 (0.007)	0.00 (0.007)	0.43 (0.08)	0.20 (0.15)	2.66 (0.13)	1.76 (0.42)	9.13 (0.01)	9.12 (0.01)	10.20 (0.10)	9.89 (0.20)	12.81 (0.14)	11.93 (0.46)
0.35	2.50e-5	7.92e-5	0.00 (0.007)	0.00 (0.008)	0.82 (0.11)	0.40 (0.22)	3.71 (0.17)	2.33 (0.59)	9.16 (0.02)	9.14 (0.02)	10.69 (0.13)	10.16 (0.28)	13.95 (0.18)	12.59 (0.62)
0.40	2.50e-5	1.04e-4	0.00 (0.007)	0.00 (0.008)	1.25 (0.17)	0.55 (0.30)	4.80 (0.26)	2.77 (0.77)	9.20 (0.03)	9.16 (0.03)	11.21 (0.20)	10.42 (0.37)	15.12 (0.27)	13.19 (0.76)

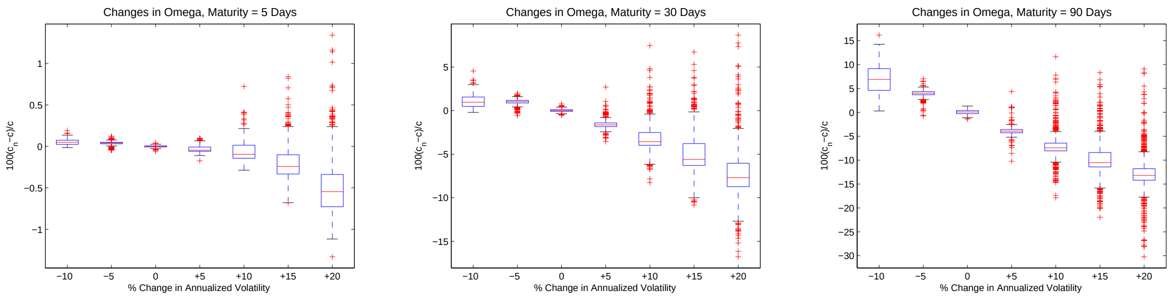

Figure 3.2: The effect of the change in parameter ω on European in-the-money call options in percentages. The vertical axis shows the percentage difference between the option price obtained from the whole sample ($\hat{c}_n$) without accounting for the parameter change, and the option price obtained from the second part of the sample ($\hat{c}$). The horizontal axis shows the percentage change in annualized volatility. For each value on the horizontal axis there are 500 observations.

in Chapter 2. We see that the direction of the change in unconditional volatility is negatively re-

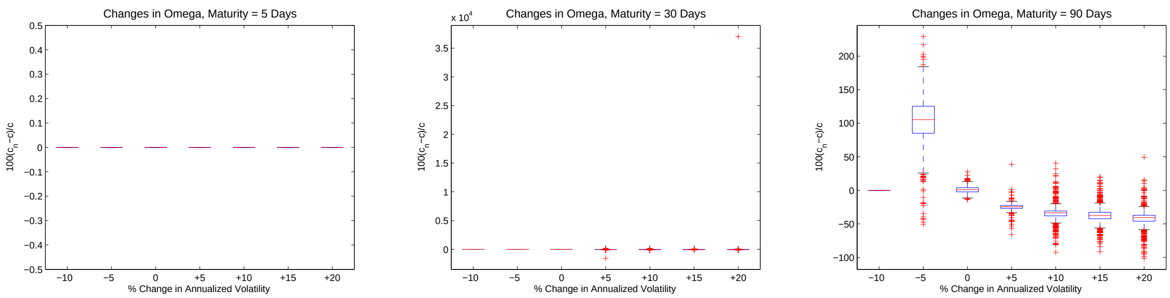

Figure 3.3: The effect of the change in parameter ω on European out-the-money call options in percentages. The vertical axis shows the percentage difference between the option price obtained from the whole sample ($\hat{c}_n$) without accounting for the parameter change, and the option price obtained from the second part of the sample ($\hat{c}$). The horizontal axis shows the percentage change in annualized volatility. For each value on the horizontal axis there are 500 observations.

lated to the direction of the bias in option prices. If unconditional volatility decreases for the second half of the series with the parameter change, then the bias in option prices will be positive. For example, from Table 1 we see that if unconditional volatility decreases from 0.20 to 0.10 or to 0.15 after the parameter change, the option price obtained by ignoring the parameter change is greater than the option price obtained from the second half of the series. Exactly the opposite happens if unconditional volatility increases. The main reason for this result is that if the unconditional volatility decreases after the parameter change, the estimated unconditional volatility will be greater than unconditional volatility of the second half of the series when the parameter change is ignored. Since the volatility and option prices are positively correlated, it results in positive bias in option prices in this case.

Table 3.3: Root Mean Square Errors for at-the-money European call option prices. σ_2 denotes annualized volatility of the segment after the change.

Root Mean Square Errors for changes in ω.

	At-the-Money			In-the-Money			Out-of-the-Money		
σ_2	5-day	30-day	90-day	5-day	30-day	90-day	5-day	30-day	90-day
0.10	0.23	0.54	1.06	0.01	0.12	0.70	0.00	0.00	0.10
0.15	0.12	0.31	0.58	0.00	0.10	0.40	0.00	0.00	0.26
0.20	**0.03**	**0.05**	**0.06**	**0.00**	**0.02**	**0.05**	**0.00**	**0.00**	**0.04**
0.25	0.09	0.31	0.59	0.00	0.16	0.47	0.00	0.10	0.44
0.30	0.17	0.62	1.20	0.01	0.38	0.99	0.00	0.27	0.85
0.35	0.24	0.88	1.72	0.03	0.60	1.49	0.00	0.47	1.05
0.40	0.37	1.22	2.36	0.05	0.87	2.07	0.00	0.77	1.87

When the parameter change is ignored, the bias in option prices as measured by percentage differences decreases as moneyness increases. In other words, ignoring parameter changes affects out-of-the-money options the most. The reason for this is the near-zero value of out-of-the-money options and the fact that the probability of the option finishing in the money is affected substantially when there is a change in unconditional volatility. The differences also increase with the time to maturity of the option. That is, as the time to maturity of the option increases, we see bigger percentage differences in option prices as well as more outliers when the parameter change is ignored.

If unconditional volatility decreases after the parameter change, the percentage differences due to ignoring parameter changes are smaller in the Heston and Nandi (2000) model than they are in the Duan (1995) model in Chapter 2. This result is seen especially when pricing at-the-money and out-of-the-money options. This difference stems from the fact that inclusion of the leverage parameter in the Heston and Nandi (2000) results in negative skewness in the risk-neutral distribution of returns. Negative skewness of the return distribution means that there is more probability mass in the negative tail of the distribution than in the positive tail. Therefore, models with negatively skewed return distributions generates lower prices for call options compared to models with symmetric return distributions. In other words, the leverage effect partially offsets the effect of a change in the volatility of stock returns on the value of options. For example, a decrease in volatility decreases the value of a call option since Vega is positive. This effect is partially offset by the increase in stock prices, which results from the negative correlation of stock returns and volatility.

3.5 Summary

In discrete-time modeling of financial returns, GARCH models play an important role since they capture the stochastic volatility and leptokurtic (fat tails) characteristics of financial return distributions. However, simulation methods have to be used to price options when they are used. Heston and Nandi (2000) provide a closed-form method of pricing European options by using a slightly altered GARCH model. This specification allows for asymmetric and leverage effects.

I analyze the effect of an ignored parameter change in the conditional variance process of the model on European call option prices. Models taking the leverage effect into account generate negatively skewed return distributions. Compared to the Black and Scholes (1973) or the Duan (1995) model, this results in lower prices for call options and a decrease in bias in option prices when parameter changes are ignored.

I also show that the solution of the Heston and Nandi (2000) model for GARCH models with orders of $p = q = 2$ or higher is not correct. Since majority of the GARCH option pricing models use GARCH(1,1) and there is little or no support for higher order GARCH models in option pricing, this result does not undermine the importance of the model.

Chapter 4

Sensitivity of VaR Models Using GARCH to Ignored Parameter Changes

4.1 Introduction

Value-at-Risk (VaR) has become a standard measure to numerically evaluate market risk exposures of a given portfolio of financial assets. For a given confidence level, VaR is defined as the quantile of that confidence level. An interpretation of VaR is therefore that with the probability specified at the confidence level, the loss of the portfolio is not going to exceed this quantile. Therefore, VaR is estimation of the left tail of the distribution of returns on the portfolio. VaR is usually calculated with 95% or 99% confidence level over 1 day, 10 days or 14 days.

The Basel Committee on Banking Supervision (1996) at the Bank for International Settlements requires banks and other financial institutions to calculate and report VaR estimates daily. The Bank for International Settlements requires a VaR estimation with a 99% confidence level over a 10-day period. Based on these VaR estimates, financial institutions must hold a certain level of capital. Financial institutions can report their risk exposures to the Securities and Exchange Commission by using 2-week VaR estimates with a level of 99% confidence.

The widespread use of VaR as a risk measure has increased the efforts to search for a good estimation method. In the literature, many different models and methods have been proposed, which are broadly categorized in three groups:

1. Historical simulation method,

2. Semi-parametric methods,

3. Parametric methods.

Historical simulation methods do not assume any parametric model. Returns are assumed to be independently and identically distributed (i.i.d.) and the estimation of VaR is based on a rolling window estimation scheme. However, it is a well-known fact that financial asset returns are not i.i.d. Although very easy to estimate, historical simulation methods are inaccurate due to the i.i.d. returns assumption.

Alternatively, one can use a semi-parametric method, such as filtered historical simulation (Barone-Adesi et al. (1998), Barone-Adesi et al. (1999), and Pritsker (2006)), extreme value theory (McNeil and Frey (2000)), or CAViaR (Engle and Manganelli (2004)).

The most frequently used model in the parametric approach is GARCH. Since estimating VaR means estimating a quantile of the distribution of returns, volatility modeling plays an important role in obtaining good VaR estimates. Various specifications of GARCH models have been proposed to model the volatility dynamics of financial return distributions (see Angelidis et al. (2004) and Duffie and Pan (1997) for an overview). In the literature, there has not been a consensus among researchers about which model is the best to capture the stylized facts of financial returns (e.g. fat-tails, leverage effect, volatility clustering, non-normality of distributions, etc.) and about how to generate better VaR results. The results vary depending on the assumptions about return distributions and volatility dynamics. The results also vary depending on the selection of portfolio and asset classes. The same method may provide results of different quality when applied to different portfolios.

Evidence of high persistence of volatility in financial returns has been observed by researchers in the literature. In the earlier chapters, I have cited several studies arguing that the high persistence in volatility dynamics may be a result of structural changes. I have shown that ignoring parameter changes in the conditional variance dynamics of GARCH models results in biased estimates of option prices. The purpose of this chapter is to analyze the effect of ignored parameter changes in the conditional variance dynamics of GARCH models on VaR. I follow the simulation methodology in earlier chapters and show that VaR estimates are biased when the parameter changes are ignored.

The rest of the chapter is organized as follows: The next section provides an overview of VaR estimation methodology. Section 4.3 explains the simulation methodology. Results are given in Section 4.4 and the last section concludes.

4.2 Evaluating Value-at-Risk

Let $r_t = \log(S_t) - \log(S_{t-1})$ be the daily return of a financial asset. A single asset portfolio is assumed to follow the GARCH(1,1) model given as:

$$r_t = \sqrt{h_t} z_t, \tag{4.1}$$

$$z_t | F_{t-1} \sim N\ (0,1), \tag{4.2}$$

$$h_t = \omega + \alpha \varepsilon_{t-1}^2 + \beta h_{t-1}, \tag{4.3}$$

where F_t is a σ-field of all information up to and including time t; z_t is a standard normal random variable; ω constant, α and β are autoregressive GARCH(1,1) parameters. Then the returns r_t are distributed normally with zero mean and variance h_t.

Formally, VaR at time t with a $1-\alpha$ confidence level is the solution to the following equation:

$$Pr(r_t \leq VaR_t^{\alpha}) = \int_{-\infty}^{VaR_t^{\alpha}} f(r_t) dr_t, \tag{4.4}$$

where $f(.)$ is the probability density function of a standard normal random variable. Then VaR for

1-day can be estimated as follows:

$$VaR_t^{\alpha} = \sqrt{\hat{h}_{t+1}} F^{-1}(\alpha), \tag{4.5}$$

where $F^{-1}(.)$ is the inverse cumulative distribution function of a standard normal random variable and $\hat{h}_{t+1}$ is the 1-day ahead forecast of the variance of r_t.

JPMorgan's RiskMetrics uses a special form the GARCH model given above, which is called exponentially weighted moving average model:

$$h_t = \alpha\varepsilon_{t-1}^2 + \beta h_{t-1}. \tag{4.6}$$

RiskMetrics does not estimate the parameter of the model and assumes that they are constant at $\beta = 0.94$ and $\alpha = 0.06$. The model is nonstationary since $\alpha + \beta$ is equal to 1. This means that shocks to the model are permanent and do not die out. This form of GARCH model is called Integrated-GARCH (IGARCH), which was proposed by Engle and Bollerslev (1986). The choice of these values is due to the high persistence in the volatility of financial returns reported by many studies. However, as mentioned in earlier chapters, high persistence in volatility may be due to ignored parameter changes and this is the main motivation of this chapter. RiskMetrics assumption about the persistence may create biased estimates of VaR values. In the next section, I analyze this issue in a simulation study.

4.3 Simulation Methodology and Results

Analogous to Chapters 2 and 3, I simulate the model given in Equations (4.1)-(4-3) with the parameter values given as:

$$r_t = \sqrt{h_t} z_t, \tag{4.7}$$

$$z_t|F_{t-1} \sim N\,(0,1)\,, \tag{4.8}$$

$$h_t = \omega_i + 0.20\varepsilon_{t-1}^2 + 0.60h_{t-1},\ \text{for } i = 1,2, \tag{4.9}$$

where $i = 1$ denotes the first $2,000$ observations and $i = 2$ denotes the second $2,000$ observations. The initial value of ω is set equal to 3.20e-5, so that the initial annualized unconditional volatility is 0.20. Volatility is changed for the second half of the series as specified in the tables.

The series has $4,000$ observations with a parameter change at observation $2,001$. I consider only changes in ω. After simulating the return series, the parameters (a) for the whole series and (b) for the last $2,000$ observations are estimated by maximum likelihood. 1-day, 5-day, and 10-day VaR values at 99% and 95% confidence levels are estimated by using the estimated parameters from (a) and (b). For 1-day VaR estimations, Equation (4.5) is used. One-day ahead forecasts of h_t are readily calculated since all the variables are known in Equation (4.3) at time t. Since $F^{-1}(0.01) = -2.3264$ and $F^{-1}(0.01) = -1.6449$, 1-day VaR estimates with 99% and 95% confidence levels are:

$$VaR_t^{\alpha} = -2.3264\sqrt{\hat{h}_{t+1}},$$

$$VaR_t^{\alpha} = -1.6449\sqrt{\hat{h}_{t+1}},\ \text{respectively.}$$

For longer horizon VaR estimations, Monte Carlo simulations with $10,000$ repetitions are used. For example, to calculate the 5-day VaR, we simulate $10,000$ return series given the model in Equations (4.1)-(4.3) until $t+5$ and calculate each of the $10,000$ VaR estimates with 99% and 95% confidence levels as:

$$VaR_t^{\alpha} = -2.3264\sqrt{5\hat{h}_{t+5}},$$

$$VaR_t^{\alpha} = -1.6449\sqrt{5\hat{h}_{t+5}},\ \text{respectively.}$$

Then, VaR is calculated by taking the mean over $10,000$ estimates. 10-day VaR is calculated analogously.

Table 4.1 reports the VaR estimates at the 99% confidence level and Table 4.2 reports the VaR estimates at the 95% confidence level.

The results show that the VaR estimates are biased when they are obtained ignoring the parameter changes. The direction of the bias depends on the direction of the change in unconditional volatility. For example, if annualized volatility of the second segment of the series is lower than that of the first segment, ignoring the parameter change overstates the VaR estimates. If annualized volatility of the second segment of the series is higher than that of the first segment and the parameter change is ignored, VaR is biased downward. This means that the risk of the portfolio is understated. The reason for this result is the spurious estimation of annualized volatilities when the parameter change is ignored. In the case of a decrease in annualized volatility for the second segment of the return series, ignoring the parameter change will result in overestimation of annualized volatility and therefore overestimation of the risk of a highly negative return.

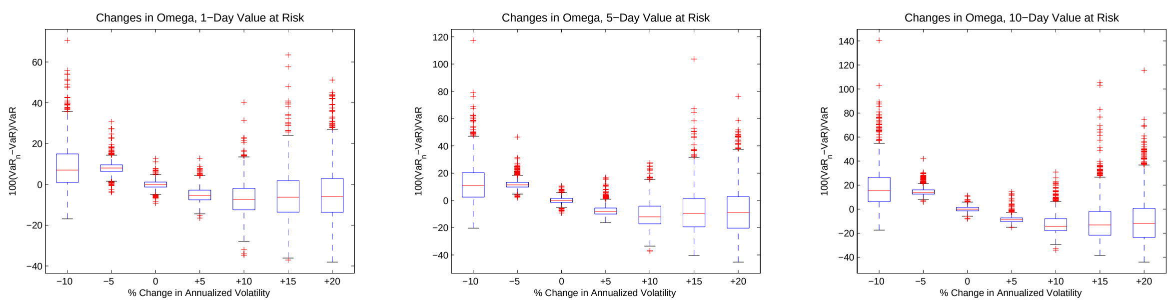

Figure 4.1: The effect of the change in parameter ω on VaR in percentages. The vertical axis shows the percentage difference between the VaR obtained from the whole sample (VaR_n) without accounting for the parameter change, and the VaR obtained from the second part of the sample (VaR). The horizontal axis shows the percentage change in annualized volatility. For each value on the horizontal axis there are 1000 observations.

Figure 4.1 shows that the bias increases with the magnitude of the change in annualized volatility. The intuition behind this result is that the greater the change in annualized volatility, the more pronounced the overestimation or underestimation of unconditional volatility. VaR at the 99% confidence level and VaR at the 95% confidence level are both quantile estimates of a standard

Table 4.1: The effect of a single neglected change-point in ω on 1-day, 5-day, and 10 VaR at the 99% confidence interval. $h_t = \omega_i + 0.20\varepsilon_{t-1}^2 + 0.60h_{t-1}$ for $i = 1,2$. $\hat{\theta} = \hat{\alpha} + \hat{\beta}$. Parameter estimates are the means of 1,000 simulations. *VaR* is the VaR calculated using the second half of the sample and VaR_n is the VaR calculated using the whole sample without taking the parameter change into account.The annualized volatility σ_1 of the first segment is always 0.20. Standard errors are given in parentheses.

Parameter Changes			Parameter Estimates (Whole Sample)					1-Day		5-Day		10-Day	
σ_2	ω_1	ω_2	$\hat{\omega}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\sigma}$	*VaR*	VaR_n	*VaR*	VaR_n	*VaR*	VaR_n
0.10	3.20e-5	0.80e-5	2.05e-6 (1.2e-6)	0.16 (0.03)	0.83 (0.04)	0.99	0.197	1.44 (0.30)	1.57 (0.39)	3.22 (0.29)	3.65 (0.77)	4.56 (0.20)	5.41 (0.96)
0.15	3.20e-5	1.80e-5	1.72e-5 (3.31e-6)	0.20 (0.02)	0.66 (0.04)	0.87	0.178	2.13 (0.44)	2.30 (0.45)	4.80 (0.43)	5.37 (0.55)	6.81 (0.30)	7.80 (0.41)
0.20	**3.20e-5**	**3.20e-5**	**3.24e-5 (4.62e-6)**	**0.20 (0.02)**	**0.60 (0.04)**	**0.80**	**0.200**	**2.89 (0.66)**	**2.89 (0.65)**	**6.44 (0.66)**	**6.44 (0.64)**	**9.09 (0.43)**	**9.09 (0.38)**
0.25	3.20e-5	5.00e-5	3.31e-5 (5.68e-6)	0.20 (0.02)	0.64 (0.04)	0.84	0.227	3.58 (0.80)	3.42 (0.89)	8.01 (0.79)	7.44 (1.06)	11.34 (0.52)	10.37 (0.72)
0.30	3.20e-5	7.20e-5	2.33e-5 (5.94e-6)	0.19 (0.03)	0.72 (0.04)	0.91	0.260	4.24 (0.93)	3.99 (1.13)	9.57 (0.97)	8.67 (1.79)	13.59 (0.69)	11.98 (1.68)
0.35	3.20e-5	9.80e-5	1.33e-5 (5.20e-6)	0.17 (0.03)	0.79 (0.04)	0.96	0.305	5.03 (1.10)	4.83 (1.49)	11.25 (1.10)	10.50 (2.87)	15.91 (0.74)	14.44 (3.37)
0.40	3.20e-5	12.8e-5	8.71e-5 (2.80e-6)	0.16 (0.03)	0.83 (0.03)	0.99	0.354	5.68 (1.22)	5.48 (1.66)	12.79 (1.21)	12.01 (3.41)	18.15 (0.82)	16.62 (4.34)

Table 4.2: The effect of a single neglected change-point in ω on 1-day, 5-day, and 10 VaR at the 95% confidence interval. $h_t = \omega_i + 0.20\varepsilon_{t-1}^2 + 0.60h_{t-1}$ for $i = 1,2$. $\hat{\theta} = \hat{\alpha} + \hat{\beta}$. Parameter estimates are the means of 1,000 simulations. *VaR* is the VaR calculated using the second half of the sample and VaR_n is the VaR calculated using the whole sample without taking the parameter change into account.The annualized volatility σ_1 of the first segment is always 0.20. Standard errors are given in parentheses.

Parameter Changes			1-Day		5-Day		10-Day	
σ_2	ω_1	ω_2	*VaR*	VaR_n	*VaR*	VaR_n	*VaR*	VaR_n
0.10	3.20e-5	8.00e-6	1.02 (0.21)	1.11 (0.27)	2.27 (0.21)	2.58 (0.54)	3.22 (0.14)	3.82 (0.68)
0.15	3.20e-5	1.80e-5	1.50 (0.31)	1.62 (0.32)	3.39 (0.30)	3.80 (0.39)	4.81 (0.21)	5.52 (0.29)
0.20	**3.20e-5**	**3.20e-5**	**2.04 (0.47)**	**2.04 (0.46)**	**4.56 (0.47)**	**4.56 (0.45)**	**6.43 (0.31)**	**6.43 (0.27)**
0.25	3.20e-5	5.00e-5	2.53 (0.57)	2.42 (0.63)	5.67 (0.56)	5.26 (0.75)	8.02 (0.37)	7.33 (0.51)
0.30	3.20e-5	7.20e-5	3.00 (0.66)	2.82 (0.80)	6.77 (0.69)	6.13 (1.27)	13.59 (0.49)	11.98 (1.19)
0.35	3.20e-5	9.80e-5	3.56 (0.78)	3.41 (1.06)	7.95 (0.78)	7.42 (2.03)	11.25 (0.52)	10.21 (2.39)
0.40	3.20e-5	1.28e-4	4.02 (0.87)	3.87 (1.17)	9.04 (0.85)	8.49 (2.41)	12.83 (0.58)	11.75 (3.07)

Table 4.3: Root Mean Square Errors

Omega

	99% Confidence			95% Confidence		
σ_2	1-day	5-day	10-day	1-day	5-day	10-day
0.10	0.65	0.89	1.28	0.46	0.63	0.91
0.15	0.48	0.80	1.08	0.34	0.57	0.76
0.20	0.35	0.37	0.38	0.29	0.32	0.33
0.25	0.90	1.20	1.21	0.64	0.85	0.85
0.30	1.16	2.01	2.33	0.82	1.42	1.65
0.35	1.51	2.96	3.68	1.07	2.09	2.60
0.40	1.67	3.49	4.60	1.18	2.47	3.25

normal variable. Therefore, the the percentage biases are the same for both cases. We only show the graph for VaR at 99% confidence level.

4.4 Summary

VaR as measure of risk exposure has become important since it is required by regulatory agencies such as the Securities and Exchange Commission and the Bank for International Settlements. Among parametric methods of estimating VaR, GARCH models are by far the most frequently used to capture the stylized facts of financial asset returns. Ignoring parameter changes in GARCH models generates spurious results. In a simulation experiment, I show that VaR estimations based on GARCH models are biased when parameter changes are ignored. Therefore, a change-point study is needed before estimating the parameters of GARCH models and VaR.

Chapter 5

Conclusions

Among stochastic volatility models, GARCH models are the most frequently used to forecast the volatility dynamics of financial asset returns. Several studies show that financial returns exhibit volatility clustering and fat-tails. GARCH models are not only capable of capturing these characteristics, but also can be specified in various ways to capture other stylized facts, such as negative correlation between asset returns and volatility and non-normality of asset returns.

High persistence of financial asset returns have been documented in many studies. In GARCH models, persistence of volatility is measured by the sum of estimated parameters. A value close to 1 is considered to indicate high persistence. In the literature, several studies show that when there are structural breaks in the data generating process and the sum of estimated autoregressive parameters approaches to 1, parameter estimates of GARCH models are spurious. An increasing number of studies show that the high persistence feature of financial asset returns may be due to structural breaks.

In this dissertation, I study the effect of this phenomenon on option prices and Value-at-Risk. Options are derivative securities, the values of which are derived from another underlying asset or assets. They are mostly used for hedging financial risk and are an important financial tool. Value-at-Risk is used to measure the downside risk of a portfolio and it has become a standard measure of market risk. The Bank for International Settlements require banks to report their VaR estimates daily. The Securities Exchange Commission allows financial institutions to use VaR as a measure of their risk exposures.

In Chapter 2, I analyze the effect of ignored parameter changes in the condition volatility of a GARCH model on European call option prices. Simulation studies show that the estimation of option prices is biased when parameter changes are ignored. The main reason of the bias is the spurious estimation of volatility. The value of an option depends positively on volatility of the underlying asset. The direction of the bias depends on whether the estimated unconditional volatility when parameter changes are ignored is higher or lower than unconditional volatility of the segment after parameter changes. In the case of a parameter change that results in a lower (higher) estimated unconditional volatility for the second segment of the series (the segment after parameter changes), the bias in the prices of options is positive (negative). In addition, it is observed that a negative change in unconditional volatility results in more pronounced bias than a positive change in unconditional volatility. In a change-point study on S&P500 returns, we find that there is a single structural break on April 28, 2003 at 1% significance level. The estimated unconditional volatility for the segment after the structural break is lower than the estimated unconditional volatility for the whole data without accounting for the structural break. Therefore, based on our simulation study, we expect that when the structural break is ignored, options should be overpriced. The results support this intuition.

The empirical study in Chapter 2 also shows that the bias in option prices decreases but does not disappear when GARCH models account for leverage effects. Leverage effects refer to the negative correlation between asset returns and volatility. When volatility is high, asset prices decrease and when volatility is low, asset prices increase. In Chapter 3, I analyze the effect of ignored parameter changes on European call options when leverage effects are accounted for. I use Heston's and Nandi's (2000) closed-form GARCH option pricing model to study this phenomenon. Results support the empirical finding of Chapter 2. The bias in option prices decreases when a leverage effect parameter is included in the model. The intuition behind the results is as follows: When volatility decreases, the price of a call option decreases. However, due to the leverage effects, the stock price increases. This, in turn, increases the price of a call option and as a result partially offsets the decrease due to the decrease in volatility. Therefore, taking leverage effects into account results in a less pronounced bias in call option prices. In Chapter 3, I also show that the closed-form solution for the GARCH option pricing model is incorrect for orders p and q greater than 1.

In Chapter 4, I analyze the effect of ignored parameter changes on VaR estimates that are based on GARCH models. I find that VaR estimates are biased when parameter changes are ignored. If a change in the parameters of the model increases (decreases) unconditional volatility, the bias in VaR estimates is negative (positive). The bias is positively correlated with the magnitude of change in unconditional volatility.

We see that in all the cases considered, ignoring parameter changes in conditional volatility results in biased estimates of option prices and VaR. Therefore, a change-point study is needed before estimating the parameters of GARCH models.

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