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Method of the Riemann-Hilbert Problem for the Solution of the Helmholtz Equation in a Semi-infinite Strip

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METHOD OF THE RIEMANN-HILBERT PROBLEM FOR THE SOLUTION OF THE HELMHOLTZ EQUATION IN A SEMI-INFINITE STRIP

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List of Abbreviations

1. RHP .... Riemann-Hilbert problem
2. PDE .... Partial differential equation
3. BVP .... Boundary value problem
4. FIT .... Finite integral transform
5. SL .... Sturm Liouville

Abstract

In this dissertation, a new method is developed to study BVPs of the modified Helmholtz and Helmholtz equations in a semi-infinite strip subject to the Poincare type, impedance and higher order boundary conditions. The main machinery used here is the theory of Riemann-Hilbert problems, the residue theory of complex variables and the theory of integral transforms. A special kind of interconnected Laplace transforms are introduced whose parameters are related through branch of a multi-valued function. In the chapter 1 a brief review of the unified transform method used to solve BVPs of linear and non-linear integrable PDEs in convex polygons is given. Then unified transform method is applied to the BVP of the modified Helmholtz equation in a semi-infinite strip subject to the Poincare type and impedance boundary conditions. In the case of BVP of the modified Helmholtz equation in a semi-infinite strip subject to the impedance boundary conditions, two scalar RHPs are derived, then the closed form solutions of the given BVP are derived. The difficulty in application of the unified transform method to BVP of the Helmholtz equation in a semi infinite strip is discussed later on. The chapter 2 contains application of the finite integral transform (FIT) method to study the BVP for the Helmholtz equation in a semi-infinite strip subject to the Poincare type and impedance boundary conditions. In the case of the impedance boundary conditions, a series representation of the solution of the BVP for the Helmholtz equation in a semi-infinite strip is derived. The Burniston-Siewert method to find integral representations of a certain transcendental equation is presented. The roots of this equation are required for both methods, the FIT method and the RHP based method. To implement the Burniston-Siewert method, we solve a scalar RHP on

several segments of the real axis.

In chapter 3, we have applied the new method to study the Poincare type and impedance BVPs for the Helmholtz equation in a semi-infinite strip. In the case of the Poincare type boundary conditions an order two vector RHP is derived. In general, it is not possible to find closed form solution of an order two vector RHP. In the case of the impedance boundary conditions two scalar RHPs are derived whose closed form solutions are found. Then the series representation for solution of the BVP of the Helmholtz equation in a semi-infinite strip subject to the impedance boundary conditions, is recovered using the inverse transform operator and the residue theory of complex variables. The numerical results are presented for various values of the parameters involved. It is observed that the FIT method and the new method generate exactly the same solution of the BVP of the Helmholtz equation in a semi-infinite strip subject to the impedance boundary conditions. In chapter 4, we have applied the new method to study the acoustic scattering from a semi-infinite strip subject to higher order boundary conditions. Two scalar RHPs are derived whose closed form solutions are found. A unique solution of the problem is obtained.

Chapter 1
Introduction

1.1 Historical Back Ground

D'Alembert and Euler discovered a general approach for solving a large class of two dimensional partial differential equations (PDEs). This approach includes separation of variables, and superimposing solutions of resulting ordinary differential equations. The method of separation of variables is actually the solution of a PDE by a transform pair. Examples of such pairs are Fourier transform and a variation of it are the Laplace transform, Mellin transform, sine transform, cosine transform and their discrete analogues. The transform method depends on the given PDE, domain, and the boundary conditions. Consider the general evolution equation $(\frac{\partial}{\partial t} + i\Sigma_{j=0}^{n}\alpha_j(-i\frac{\partial}{\partial x})^j)) \quad q(x,t) = 0,\ -\infty < x < \infty, \quad t > 0, \quad q(x,0) = q_0(x) \in S(\mathbb{R})$, where $\alpha_j \in \mathbb{R}, \quad S(\mathbb{R})$ is the space of Schwartz functions, $q(x,t)$ and its derivatives decay as $\mid x \mid \to \infty$, uniformly in t. This initial value problem can be solved by the Fourier transform:

$$q(x,t) = \frac{2}{\pi}\int_{-\infty}^{\infty} e^{ikx-\omega(k)t}\hat{q_0}(k)dk,$$
$$\omega(k) = \Sigma_{j=0}^{n}\alpha_j k^j, \quad \hat{q_0}(k) = \int_{-\infty}^{\infty} e^{-ikx}q_0(x)dx.$$

Consider the 2nd order initial boundary value problem:

$$i\frac{\partial q}{\partial t} + \frac{\partial^2 q}{\partial x^2} = 0, \quad 0 < x < \infty, \quad t > 0,$$
$$q(x,0) = q_0(x), \quad q(0,t) = f_0(t), \quad q(x) \in S(\mathbb{R}^+), \quad f_0(t) \in C^1,$$

$q_0(x), f_0(t)$ are compatible at $x = t = 0$. This initial boundary value problem can be solved by the sine transform:

$$
\begin{aligned}
q(x,t) &= \frac{2}{\pi}\int_0^\infty \sin(kx)[e^{-ik^2t}\hat{q_0}(t) + ik\int_0^t e^{-ik^2}(t-\tau)f_0(\tau)d\tau]dk, \\
\hat{q_0}(k) &= \int_0^\infty \sin(kx)q_0(x)dx.
\end{aligned}
$$

The transform method is used to solve a wide variety of initial boundary value problems, but for complicated problems, the classical transform method fails. For example, there does not exist classical transforms to solve even a 2nd order elliptic PDE in simple domains. The main difficulty with the classical transform method is the identification of a proper transform pair to be used. Some other available methods are the Wiener-Hopf factorization method and Sommerfield's integral representation method. The Wiener-Hopf technique is extensively used to solve many classical problems in acoustics, diffraction, electromagnetism, fluid mechanics etc. The unified transform method was introduced by A.S. Fokas, to solve boundary and initial value problems for two dimensional linear and non linear integrable PDEs [16]. This method was further developed in [15], [17], [18]. The unified transform method for boundary value problems (BVPS) for PDEs in convex polygons consists of three steps:

(a) Given a PDE, construct two compatible eigen value equations, which in accordance with the theory of non linear integrable PDEs, are called as Lax pair.

(b) Perform simultaneous spectral analysis of the Lax pair. This will generate an integral representation of $q(x_1, x_2)$ in terms of a function $\hat{q}(k)$, which is called as spectral function, and an integral representation of $\hat{q}(k)$. The integral representation of $\hat{q}(k)$ involves values of $q(x_1, x_2)$ and of its derivatives on the boundary of the domain. The implementation of this step for some simple

evolution equations, and for the Laplace equation in some simple domains, is explained in [16] and [15]. Implementation of this step for the Laplace equation in convex polygons is explored in [19].

(c) For the given appropriate boundary conditions, analyze the global relation satisfied by the boundary values of $q(x_1, x_2)$ and of its derivatives. This step is necessary because $\hat{q}(k)$ involves some unknown boundary data. This step is discussed in detail for some simple domains in [16] and [15]. This step for the Laplace equation in convex polygons is discussed in [19], and expressions for $\hat{q}(k)$ in terms of the given boundary data are given. To carry out this step, some specified domains are required, because for arbitrary domains this step becomes prohibitively complicated.

In the following section some terminology is defined.

1.2 Terminology

Definition 1.2.1. *To define Schwartz space of functions $S(\mathbb{R}^n)$, let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$ with $\alpha_j, \beta_j \geq 0$. Define*

$$\partial^\alpha \phi(x) = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \phi(x),$$
$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

Such multi-indices are denoted by $\alpha, \beta \geq 0$. Note that $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n$ denotes length of the multi-index α. A function $\phi(x) \in S(\mathbb{R}^n)$ if $\phi(x)$ is a smooth function on $\mathbb{R}^n$ and

$$\sup_{x \in \mathbb{R}^n} |x^\beta \partial^\alpha \phi(x)| < \infty,$$

holds for all multi-indices $\alpha, \beta \geq 0$. The space $S(\mathbb{R}^n)$ is a topological vector space.

For every pair of multi-indices $\alpha, \beta \geq 0$ and $\phi(x) \in S(\mathbb{R}^n)$, a norm on $S(\mathbb{R}^n)$ is defined as

$$\| \phi \|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} |x^\beta \partial^\alpha \phi(x)|.$$

A sequence of functions $\{\phi_n, n \in \mathbb{N}\}$, where $\phi_n(x) \in S(\mathbb{R}^n)$, converges to $\phi(x) \in S(\mathbb{R}^n)$ if

$$\| \phi_n(x) - \phi(x) \|_{\alpha,\beta} \quad \longrightarrow 0 \text{ as } n \quad \longrightarrow \infty.$$

Note 1.2.1. *Definition 1.2.1 reveals that the Schwartz space of functions $S(\mathbb{R}^n)$ consists of all smooth functions whose all the derivatives, and the functions themselves decay at infinity faster than reciprocal of any polynomial. $S(\mathbb{R}^n)$ is also referred as a space of rapidly decreasing functions.*

Example 1.2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by*

$$f(x) = e^{-|x|^2}.$$

Then $f(x) \in S(\mathbb{R}^n)$ because $f(x)$ is infinitely differentiable ($f(x) \in C^\infty(\mathbb{R}^n)$) and decays at infinity faster than reciprocal of any polynomial. If $p(x)$ is any polynomial then $q(x) = p(x)e^{-|x|^2}$ also belongs to $S(\mathbb{R}^n)$.

Example 1.2.2. *Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function with support in a compact set. Then $\sup_{x \in \mathbb{R}^n} |\partial^\alpha \phi(x)| < \infty$ for any multi-index $\alpha \geq 0$ because a continuous function on a compact set is bounded. The linear space of all such functions is denoted by $C_0^\infty(\mathbb{R}^n)$. If the support of $\phi \in B(0, r)$, then*

$$\sup_{x \in \mathbb{R}^n} |x^\beta \partial^\alpha \phi(x)| \leq r^{|\beta|} \sup_{x \in \mathbb{R}^n} |\partial^\alpha \phi(x)| < \infty,$$

holds for all multi-indices $\alpha, \beta \geq 0$. Hence $C_0^\infty(\mathbb{R}^n) \subset S(\mathbb{R}^n)$.

Example 1.2.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = e^{-x^2} \sin(e^{x^2}).$$

Then $f(x) \notin S(\mathbb{R})$ because $f'(x)$ is not decaying as $|x| \to \infty$.

Example 1.2.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{1}{(1+|x|^2)^n}$, where n is a non negative integer. The $f(x) \notin S(\mathbb{R})$ because $|x|^{2n} f(x)$ is not decaying as $|x| \to \infty$.*

1.2.1 Lax pair for linear PDEs

Proposition 1.2.1. *[17] Suppose $q(x, y)$ satisfies the PDE with constant coefficients*

$$L(\partial_x, \partial_y)q(x, y) = 0, \tag{1.1}$$

where $L(\partial_x, \partial_y)$ is a linear operator of ∂_x and ∂_y with constant coefficients. The PDE (1.1) possesses the Lax pair

$$\partial_x \mu(x, y, k) - ik\mu(x, y, k) = q(x, y), \quad k \in \mathbb{C}, \tag{1.2}$$

$$L(\partial_x, \partial_y)\mu(x, y) = 0, \tag{1.3}$$

where $\mu(x, y, k)$ is a scalar function. Note that if $q(x, y)$ satisfies $L(\partial_x, \partial_y)q(x, y) = 0$, then the equations (1.2) and (1.3) are compatible.

Proof. Apply the operator $L(\partial_x, \partial_y)$ on equation (1.2) to get

$$L(\partial_x, \partial_y)(\partial_x - ik)\mu(x, y, k)) = L(\partial_x, \partial_y)q(x, y), \quad k \in \mathbb{C}. \tag{1.4}$$

Operators $L(\partial_x, \partial_y)$ and $\partial_x - ik$ commute, so

$$(\partial_x - ik)L(\partial_x, \partial_y)\mu(x, y, k) = L(\partial_x, \partial_y)q(x, y). \tag{1.5}$$

Use the compatibility condition of equations (1.2) and (1.3) in equation (1.5) to get the given PDE $L(\partial_x, \partial_y)q(x, y) = 0$. □

Example 1.2.5. *Consider the linearized nonlinear Schrödinger equation*

$$i\partial_t q(x, t) + \partial_{xx} q(x, t) = 0, \quad 0 < x < \infty, \quad t > 0. \tag{1.6}$$

A Lax pair associated with the PDE (1.6) is

$$\mu_x(x,t,k) - ik\mu(x,t,k) = q(x,t), \quad k \in \mathbb{C},$$

$$\mu_t(x,t,k) + ik^2\mu(x,t,k) = i\partial_x q(x,t) - kq(x,t).$$

Proof. Using proposition 1.2.1, a Lax pair associated with PDE (1.6) is

$$\partial_x\mu(x,t,k) - ik\mu(x,t,k) = q(x,t), \quad k \in \mathbb{C}, \tag{1.7}$$

$$i\partial_t\mu(x,t,k) + \partial_{xx}\mu(x,t,k) = 0. \tag{1.8}$$

To eliminate $\partial_{xx}\mu(x,t,k)$ from equation (1.8), apply the operator ∂_x on equation (1.7), and simplify to get

$$\partial_{xx}\mu(x,t,k) = ik\partial_x\mu(x,t,k) + \partial_x q(x,t).$$

Use equation (1.7), to find the value of $\partial_x\mu(x,t,k)$ and insert that value in the above equation to get,

$$\begin{aligned}
\partial_{xx}\mu(x,t,k) &= ik[ik\mu(x,t,k) + q(x,t)] + \partial_x q(x,t),\\
\partial_{xx}\mu(x,t,k) &= (ik)^2\mu(x,t,k) + ikq(x,t) + \partial_x q(x,t).
\end{aligned} \tag{1.9}$$

Insert the value of $\partial_{xx}\mu(x,t,k)$ in equation (1.8) and simplify to get

$$\begin{aligned}
\partial_t\mu(x,t,k) &= i[(ik)^2\mu(x,t,k) + ikq(x,t) + \partial_x q(x,t)],\\
\partial_t\mu(x,t,k) + ik^2\mu(x,t,k) &= i\partial_x q(x,t) - kq(x,t).
\end{aligned} \tag{1.10}$$

□

Remark 1.2.1. *Another Lax pair associated with the linearized nonlinear Schrödinger equation (1.6) is*

$$\partial_t\mu(x,t,k) - ik\mu(x,t,k) = q(x,t), \quad k \in \mathbb{C}, \tag{1.11}$$

$$\partial_{xx}\mu(x,t,k) + k\mu(x,t,k) = -iq(x,t). \tag{1.12}$$

Example 1.2.6. *Consider the linearized Korteweg-de Vries equation*

$$\partial_t q(x,t) + \partial_{xxx} q(x,t) = 0. \tag{1.13}$$

A Lax pair of PDE (1.13) is

$$\mu_x(x,t,k) - ik\mu(x,t,k) = q(x,t) \quad k \in \mathbb{C}$$

$$\mu_t(x,t,k) - ik^3\mu(x,t,k) = -\partial_{xx}q(x,t) - ik\partial_x q(x,t) + k^2 q(x,t).$$

Proof. Using proposition 1.2.1, a Lax pair associated with PDE (1.13) is

$$\partial_x \mu(x,t,k) - ik\mu(x,t,k) = q(x,t), \quad k \in \mathbb{C}, \tag{1.14}$$

$$\partial_t \mu(x,t,k) + \partial_{xxx}\mu(x,t,k) = 0. \tag{1.15}$$

To eliminate $\partial_{xxx}\mu(x,t,k)$ from equation (1.15), apply the operator ∂_x on equation (1.14), and simplify to get

$$\partial_{xx}\mu(x,t,k) = ik\partial_x\mu(x,t,k) + \partial_x q(x,t).$$

Use equation (1.14) to the find value of $\partial_x\mu(x,t,k)$ and insert that value in the above equation to get,

$$\partial_{xx}\mu(x,t,k) = ik(ik\mu(x,t,k) + q(x,t)) + \partial_x q(x,t),$$

$$\partial_{xx}\mu(x,t,k) = (ik)^2\mu(x,t,k) + ikq(x,t) + \partial_x q(x,t). \tag{1.16}$$

Apply the operator ∂_x on the above equation and simplify to get

$$\partial_{xxx}\mu(x,t,k) = (ik)^3\mu(x,t,k) + (ik)^2 q(x,t) + ik\partial_x q(x,t) + \partial_{xx}q(x,t).$$

Insert the value of $\partial_{xxx}\mu(x,t,k)$ in equation (1.15) and simplify to get (1.17)

$$\partial_t\mu(x,t,k) = -[(ik)^3\mu(x,t,k) + (ik)^2 q(x,t) + ik\partial_x q(x,t) + \partial_{xx}q(x,t)],$$

$$\partial_t\mu(x,t,k) - ik^3\mu(x,t,k) = -\partial_{xx}q(x,t) - ik\partial_x q(x,t) + k^2 q(x,t).$$

□

Example 1.2.7. *Suppose $q(x,t)$ satisfies the evolution equation*

$$(\partial_t + \sum_{j=0}^{n_0} \alpha_j (-i\partial_x)^j) q(x,t) = 0, \; -\infty < x < \infty, \quad t > 0, \quad n_0 \in \mathbb{Z}^+. \tag{1.18}$$

where α_j, $0 \le j \le n_0$, are constants. A Lax pair associated with PDE (1.18) is

$$\begin{aligned} \mu_x(x,t,k) - ik\mu(x,t,k) &= q(x,t), \quad k \in \mathbb{C}, \\ \mu_t(x,t,k) + \sum_{j=0}^{n_0} \alpha_j k^j \mu(x,t,k) &= -q^*(x,t), \; \textit{where}, \\ q^*(x,t) &= \sum_{j=1}^{n_0} \alpha_j [(-i\partial_x)^{j-1} + k(-i\partial_x)^{j-2} + k^2(-i\partial_x)^{j-3} \\ &\quad + \cdots + k^{j-1}] q(x,t). \end{aligned}$$

Remark 1.2.2. *Another Lax pair associated with the linear PDE (1.18) is*

$$\partial_t \mu(x,t,k) - ik\mu(x,t,k) = q(x,t), \quad k \in \mathbb{C}, \tag{1.19}$$

$$L\mu(x,t,k) = 0, \textit{where}, \tag{1.20}$$

$$L = \partial_t + \sum_{j=0}^{n_0} \alpha_j (-i\partial_x)^j. \tag{1.21}$$

Example 1.2.8. *Consider the elliptic PDE*

$$(\partial_x^2 + \partial_y^2 + 4\alpha) q(x,y) = 0, \tag{1.22}$$

where α is a constant. For $\alpha = 0, -\beta^2, \beta^2$, $\beta \in \mathbb{R}$ equation (1.22) is the Laplace equation, the modified Helmholtz equation and the Helmholtz equation, respectively. A Lax pair for equation (1.22) is

$$\mu_x(x,y,k) - i(k + \frac{\alpha}{k})\mu(x,y,k) = \frac{1}{2}(q_x(x,y) - iq_y(x,y)) - \frac{i\alpha}{k} q(x,y) \quad k \in \mathbb{C}, \tag{1.23}$$

$$\mu_y(x,y,k) + (k - \frac{\alpha}{k})\mu(x,y,k) = \frac{1}{2}(iq_x(x,y) + q_y(x,y)) - \frac{\alpha}{k} q(x,y). \tag{1.24}$$

Let $z = x + iy$, $\quad \bar{z} = x - iy$, then we have the operators $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$. The elliptic PDE (1.22), and corresponding Lax pair defined by equations (1.23) and (1.24) become

$$q_{z\bar{z}}(z, \bar{z}) + \alpha q(z, \bar{z}) = 0, \tag{1.25}$$

$$\mu_z(z, \bar{z}, k) - ik\mu(z, \bar{z}, k) = q_z(z, \bar{z}), \quad k \in \mathbb{C}, \tag{1.26}$$

$$\mu_{\bar{z}}(z, \bar{z}, k) - i\frac{\alpha}{k}\mu(z, \bar{z}, k) = -\frac{i\alpha}{k}q(z, \bar{z}). \tag{1.27}$$

1.2.2 Simultaneous spectral analysis

Consider the generic case of a Lax pair i.e., it is assumed that while writing the Lax pair in a matrix form, a matrix is obtained that can be diagonalized. So, $Lq(x, y) = 0$ can be written as a compatibility condition of the two linear 1st order equations:

$$\mu_x(x, y, k) - if_1(k)\mu(x, y, k) = q_1(x, y, k), \tag{1.28}$$

$$\mu_y(x, y, k) - if_2(k)\mu(x, y, k) = q_2(x, y, k). \tag{1.29}$$

Note 1.2.2. *$f_1(k), f_2(k)$ are given analytic functions of $k \in \mathbb{C}$, and $q_1(x, y, k)$ and $q_2(x, y, k)$ are analytic functions of $k \in \mathbb{C}$ depending on $q(x, y)$ and its derivatives. To carry out the spectral analysis of equations (1.28) and (1.29), construction of a sectionally analytic function $\mu(x, y, k)$ in the complex k-plane i.e., $\mu(x, y, k) = \mu_j(x, y, k)$ for $k \in D_j$, $\quad \cup_{j=1}^{n} D_j = \mathbb{C}$ and each $\mu_j(x, y, k)$ is analytic in D_j, is required.*

Write equations (1.28) and (1.29) in the form

$$(e\mu(x, y, k))_x = eq_1, \quad (e\mu(x, y, k))_y = eq_2,$$

where $e = Exp[-if_1(k)x - if_2(k)y]$. A particular solution of equations (1.28) and (1.29) is

$$\mu_j(x, y, k) = \int_{\zeta_j}^{\zeta} e^{if_1(k)(x-x')+if_2(k)(y-y')}[q_1(x', y', k)dx' + q_2(x', y', k)dy'].$$

$\int_{\zeta_j}^{\zeta}$ denotes the line integral from the fixed point ζ_j to an arbitrary point $\zeta = x + iy$. The function $\mu_j(x, y, k)$ is a solution of the equations (1.28) and (1.29), even if the line integral is replaced by any smooth curve from ζ_j to ζ, and it is independent of choice of this curve. The compatibility of equations (1.28) and (1.29), and application of the Green's theorem imply that for any smooth closed curve

$$\oint eq_1(x, y, k)dx + eq_2(x, y, k)dy = \int\int [(eq_2(x, y, k))_x - (eq_1(x, y, k))_y]dxdy = 0.$$

It is shown in [18] that if $\zeta_j,\ j = 1, 2, 3, ..., n$ are the corners of a polygon then $\mu_j(x, y, k)$ is holomorphic in S_j, and $\cup_{j=1}^{n} S_j = \mathbb{C}$. Let L_{ij} be a curve in intersection of $S_i,$ and $S_j, \quad i \neq j$. Then

$$\mu_i(x, y, k) - \mu_j(x, y, k) = e^{if_1(k)x + if_2(k)y}\rho_{i,j}(k), \tag{1.30}$$

$$\rho_{i,j}(k) = \int_{\zeta_i}^{\zeta_j} e^{-if_1(k)x - if_2(k)y}[q_1(x, y, k)dx + q_2(x, y, k)dy]. \tag{1.31}$$

Using the Sokhotski-Plemelj formulae, it is possible to reconstruct the unique solution of this scalar Riemann Hilbert problem (RHP). Hence the required sectionally analytic function is

$$\mu(x, y, k) = \frac{1}{2\pi i}\sum_{i,j}\int_{L_{i,j}} e^{if_1(k')x + if_2(k')y}\frac{\rho_{i,j}(k')}{k' - k}dk'. \tag{1.32}$$

Equation (1.32) expresses $\mu(x, y, k)$ in terms of the spectral function $\hat{q}(k)$, where $\hat{q}(k) = \rho_{i,j}(k)$. The spectral function $\hat{q}(k)$ involves $q(x, y)$ and its derivatives along the boundary of the polygon. Let $L = \cup_{i,j=1}^{n} L_{i,j}$, then either equation (1.28) or (1.29) generates $q(x, y)$ in terms of the spectral function $\hat{q}(k)$ along the curve L. If the Lax pair is expressed in $(z, \bar{z})$ coordinates, then equations (1.28) and (1.29) become

$$\mu_z(z, \bar{z}, k) - if_1(k)\mu(z, \bar{z}, k) = q_1(z, \bar{z}, k), \tag{1.33}$$

$$\mu_{\bar{z}}(z, \bar{z}, k) - if_2(k)\mu(z, \bar{z}, k) = q_2(z, \bar{z}, k). \tag{1.34}$$

Now the particular solution

$$\mu_j(z,\overline{z},k)=\int_{z_j}^{z} e^{if_1(k)(z-z')+if_2(k)(\overline{z}-\overline{z}')}(q_1(z',\overline{z}')dz'+q_2(\overline{z},\overline{z}')d\overline{z}'), \tag{1.35}$$

is also well defined. This is obvious from the complex form of Green's theorem. Hence corresponding to equations (1.31) and (1.32)

$$\rho_{i,j}(k)=\int_{z_i}^{z_j} e^{-if_1(k)z-if_2(k)\overline{z}}(q_1(z,\overline{z},k)dz+q_2(z,\overline{z},k)d\overline{z}), \tag{1.36}$$

$$\mu(z,\overline{z},k)=\frac{1}{2\pi i}\sum_{i,j}\int_{L_{i,j}} e^{if_1(k')z+if_2(k')\overline{z}}\frac{\rho_{i,j}(k')}{k'-k}dk'. \tag{1.37}$$

1.2.3 Analysis of the global relation

The formulae representing the solution $q(x,y)$ of a given BVP depend on the given PDE and domain, these are valid for any boundary conditions, provided these boundary conditions generate a well posed BVP. A basic limitation of these formulae is that, these are derived under a priori assumption of existence of solutions. Also, for a given BVP, the spectral function $\hat{q}(k)$ contains some unknown boundary data. The part of $\hat{q}(k)$ involving unknown boundary data in terms of the given boundary conditions, can be expressed by the following three steps[18]:

(a) For convex closed polygons, the global relation is $\Sigma_{j=1}^{n}\rho_{j+1,j}(k)=0$, $k\in\mathbb{C}$, and for unbounded convex polygons the global relation is $\Sigma_{j=1}^{n-1}\rho_{j+1,j}(k)=0$, $k\in S_1\cap S_n$, where S_1 and S_n are sectors in complex k-plane. The definition of $\hat{q}(k)$ is used to express it in terms of the given boundary conditions and some unknown functions denoted by $\psi_j(k)$. Insert these expressions in the global relation to obtain an equation for the unknown functions $\psi_j(k)$.

(b) Use certain invariant transformations in the complex k-plane to construct a set of additional equations from the equation obtained in step (a).

(c) [6] shows that $\psi_j(k)$ can be obtained either through a system of algebraic equations or through the solution of a RHP which is obtained in step (b). For

implementation of step (c), it is observed that for general BVPs for elliptic equations, the functions $\psi_j(k)$ satisfy a RHP. This RHP can be obtained as follows:

1. Determine the domains in the complex k-plane, where each unknown function is bounded and analytic. These domains are separated by certain curves.
2. For each of these curves, use the equations obtained in step (b) to compute the jumps of these functions.

Theorem 1.2.1. *[17] Let Ω be a convex closed polygon in the complex z-plane, $z = x + iy$, with corners $z_1, z_2, \cdots, z_n$, a part of Ω is shown in figure 1.1. Let $q(x, y)$ be a real valued function satisfying the 2nd order PDE*

$$(\partial_x^2 + \partial_y^2 + 4\alpha)q(x, y) = 0, \quad (x, y) \in \Omega. \tag{1.38}$$

Suppose that appropriate boundary conditions are prescribed on the boundary of Ω such that there exists a solution $q(x, y)$ which is sufficiently smooth up to the boundary of Ω. Then $q(x, y)$ can be expressed as follows:

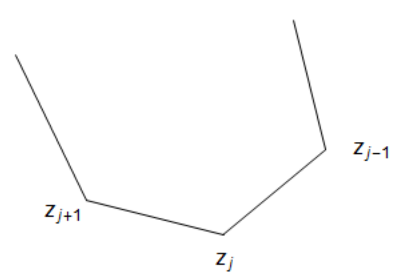

FIGURE 1.1. A part of a convex closed polygon Ω.

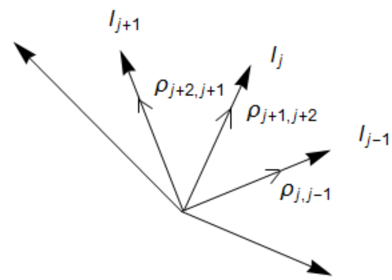

FIGURE 1.2. Contours and spectral functions for the Laplace equation in a convex closed polygon Ω.

1. *For the Laplace equation i.e. equation (1.38) with* $\alpha = 0$

$$\partial_z q(z,\bar{z}) = \frac{1}{2\pi}\sum_{j=1}^{n}\int_{l_j} e^{ikz}\rho_{j+1,j}(k)dk, \tag{1.39}$$

$$\rho_{j+1,j}(k) = \int_{z_{j+1}}^{z_j} e^{-ikz} q_z(z,\bar{z})dz,\ \ z_{n+1} = z_n. \tag{1.40}$$

l_j *are rays in complex* k*-plane oriented from zero to* ∞*, and defined by*

$$l_j = \{k \in \mathbb{C} : arg(k) = -arg(z_j - z_{j+1})\}, \quad j = 1,2,3,\cdots,n. \tag{1.41}$$

Contours and spectral functions for Laplace equation in a convex closed polygon are shown in figure 1.2.

2. *For the modified Helmholtz equation i.e. equation (1.38) with* $\alpha = -\beta^2$

$$q(z,\bar{z}) = \frac{1}{2\pi i}\sum_{j=1}^{n}\int_{l_j} e^{ikz-(\frac{i\beta^2}{k})\bar{z}}\rho_{j+1,j}(k)\frac{dk}{k}, \tag{1.42}$$

$$\rho_{j+1,j}(k) = \int_{z_{j+1}}^{z_j} e^{-ikz+(i\frac{\beta^2}{k})\bar{z}}(q_z(z,\bar{z})dz + i\frac{\beta^2}{k}q(z,\bar{z})d\bar{z}), z_{n+1} = z_n. \tag{1.43}$$

l_j *are the same as in case of Laplace equation, and are defined by equation (1.41), and improper integrals are assumed where needed. Contours and spectral functions for the modified Helmholtz equation in the convex closed polygon*

Ω *are the same as for the Laplace equation in the convex closed polygon* Ω *shown in figure 1.2.*

3. *For the Helmholtz equation i.e. equation (1.38) with* $\alpha = \beta^2$

$$q(z,\bar{z}) = \frac{1}{2\pi i}[\sum_{j=1}^{n} \int_{\tilde{l}_j} e^{ikz+(\frac{i\beta^2}{k})\bar{z}} \rho_{j+1,j}(k) \frac{dk}{k} + \sum_{j=1}^{2n} \int_{L_j} e^{ikz+(\frac{i\beta^2}{k})\bar{z}} \rho^{(j)}(k)] \frac{dk}{k}, \tag{1.44}$$

$$\rho_{j+1,j}(k) = \int_{z_{j+1}}^{z_j} e^{-ikz-(i\frac{\beta^2}{k})\bar{z}} (q_z(z,\bar{z})dz - i\frac{\beta^2}{k} q(z,\bar{z})d\bar{z}), z_{n+1} = z_n. \tag{1.45}$$

Define $\tilde{l}_j, L_j, \rho^{(j)}(k)$ *as:* $\tilde{l}_j$ *is union of two rays, originating from origin, given by*

$$\tilde{l}_j = \{k \in \mathbb{C} : [arg(k) = -arg(z_j - z_{j+1}), |k| > \beta] \cup$$
$$[arg(k) = \pi - arg(z_j - z_{j+1}), |k| < \beta]\}.$$

Note that L_j *are circular arcs formed by intersection of the ray* $\tilde{l}_j$ *with the*

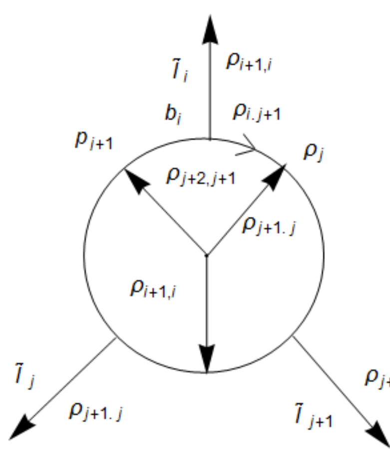

FIGURE 1.3. Contours and spectral functions for the Helmholtz equation in a convex closed polygon Ω.

circle $|k| = \beta$*; if* p_{j+1}, b_i, p_j *are points of intersection of the circle* $|k| = \beta$

with the rays $\{\tilde{l}_{j+1}, |k| < \beta\}$, $\{\tilde{l}_i, |k| > \beta\}$, $\{\tilde{l}_j, |k| < \beta\}$, *where* b_i *is between* p_{j+1} *and* p_j, *then* $\rho^{(j)}$ *on* $L_j = (b_i, \alpha_j)$ *is* $\rho_{i,j+1}$. *The spectral functions satisfy the global relation*

$$\sum_{j=1}^{n} \rho_{j+1,j}(k) = 0, \quad k \in \mathbb{C}. \tag{1.46}$$

If Ω *is open, the following modifications are made. The corners* z_1 *and* z_n *are moved to* ∞, *and assume that* $q(x, y)$ *has sufficient decay as* $z \to \infty$. *The spectral function* $\rho_{1,n}(k)$ *is zero, hence the summation in equations (1.39),(1.42) and (1.44) is only up to* $(n-1)$. *The spectral functions* $\rho_{2,1}(k)$ *and* $\rho_{n,n-1}(k)$ *are not defined for all* $k \in \mathbb{C}$ *but for* k *in* S_1 *and* S_n, *respectively.* S_1 *and* S_n *are defined as: for the Laplace and modified Helmholtz equations* S_1 *and* S_n *are the half planes defined by*

$$S_1 = \{k \in \mathbb{C}, arg(k) \in [-arg(z_2 - z_1), \pi - arg(z_2 - z_1)]\}, \tag{1.47}$$

$$S_n = \{k \in \mathbb{C}, arg(k) \in [-arg(z_{n-1} - z_n), \pi - arg(z_{n-1} - z_n)]\}. \tag{1.48}$$

For the Helmholtz equation, $\tilde{S}_1, \tilde{S}_n$ *are defined by equations (1.47) and (1.48) with* k *replaced by* λ. *Then* S_1 *and* S_n *are domains in complex* k*-plane obtained from the map* $\lambda = (1 - \frac{\beta^2}{|k|^2})k$ *of the sectors* $\tilde{S}_1$ *and* $\tilde{S}_n$ *respectively. The global relation is*

$$\sum_{j=1}^{n-1} \rho_{j+1,j}(k) = 0, \quad k \in S_1 \cap S_n. \tag{1.49}$$

1.3 Modified Helmholtz equation in a semi-infinite strip Ω

In this section we give a summary of the results obtained in [3] for a BVP of the modified Helmholtz equation in a semi-infinite strip subject to the Poincare type boundary conditions. Consider the modified Helmholtz equation

$$(\partial_x^2 + \partial_y^2 - 4\beta^2)q(x, y) = 0, \ \beta \in \mathbb{R}, \quad (x, y) \in \Omega, \tag{1.50}$$

where Ω is a semi-infinite strip with Poincare type boundary conditions shown in figure 1.4, with corners $z_1 = \infty$, $z_2 = 0$, $z_3 = ia$, $z_4 = \infty + ia$, $a > 0$.

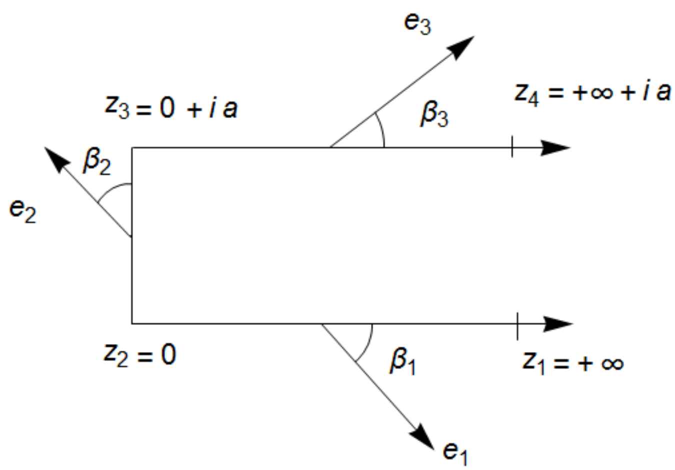

FIGURE 1.4. Semi-infinite strip with Poincare type boundary conditions.

The Poincare type boundary conditions are

$$\left.\frac{\partial q}{\partial \nu}\right|_{e_j} + \gamma_j q = g_j, \tag{1.51}$$

where $\left.\frac{\partial q}{\partial \nu}\right|_{e_j} = \bigtriangledown q \cdot e_j$ is the directional derivative in the direction e_j specified by constant β_j $(0 < \beta_j < \pi)$, γ_j is a real non negative constant, and g_j is a real valued function with appropriate smoothness and decay. The boundary conditions in equation (1.51) can be written as:

$$side1 : \cos\beta_1 q_x - \sin\beta_1 q_y + \gamma_1 q = g_1(x), \quad 0 < x < \infty, \quad y = 0, \tag{1.52}$$

$$side2 : \cos\beta_2 q_y - \sin\beta_2 q_x + \gamma_2 q = g_2(y), \quad x = 0, \quad 0 < y < a, \tag{1.53}$$

$$side3 : \cos\beta_3 q_x + \sin\beta_3 q_y + \gamma_3 q = g_3(x), \quad 0 < x < \infty, \quad y = a. \tag{1.54}$$

The functions $g_1(x)$, $g_3(x)$ vanish at the points $x = 0$ and $x = \infty$, $\sin\beta_j \neq 0$, $j = 1, 2, 3$. Let $z = x + iy$ and $\bar{z} = x - iy$, then $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$.

Equation (1.50) becomes

$$(\frac{\partial^2}{\partial z \partial \bar{z}} - \beta^2) q(z, \bar{z}) = 0. \tag{1.55}$$

In example 1.2.8, a Lax pair related to equation (1.55) is given by equations (1.26) and (1.27) with $\alpha = -\beta^2$. Simultaneous spectral analysis of the Lax pair in Ω yields a sectionally holomorphic function:

$$\mu(z, \bar{z}, k) = \frac{1}{2\pi i} \sum_{i,j=1}^{3} \int_{L_{i,j}} e^{i\acute{k}z + i\frac{\beta^2}{\acute{k}}\bar{z}} \frac{\rho_{i,j}(\acute{k})}{\acute{k} - k} d\acute{k}, \tag{1.56}$$

$$\rho_{i,j}(k) = \int_{z_i}^{z_j} e^{-ikz + i\frac{\beta^2}{k}\bar{z}} (q_z(z, \bar{z}) dz + i\frac{\beta^2}{k} q(z, \bar{z}) d\bar{z}), \;\; i = j + 1. \tag{1.57}$$

$L_{i,j}$ are curves formed by intersection of sectors $\tilde{S}_i$ and $\tilde{S}_j$ defined by equations

$$\begin{aligned} \tilde{S}_j &= \{\lambda \in \mathbb{C}, arg(\lambda) \in [-arg(z_{j-1} - z_j), \pi - arg(z_{j+1} - z_j)\}, \;\; 2 \leq j < n, \\ \tilde{S}_i &= \{\lambda \in \mathbb{C}, arg(\lambda) \in [-arg(z_{i-1} - z_i), \pi - arg(z_{i+1} - z_i)\} \;, 2 \leq j < n, \\ & \qquad i = j + 1. \end{aligned} \tag{1.58}$$

For the modified Helmholtz equation in semi-infinite strip Ω, $\tilde{S}_1$ and $\tilde{S}_n$ are the half planes, obtained by using equations (1.47) and (1.48) and replacing k by λ.

$$\begin{aligned} \tilde{S}_1 &= \{\lambda \in \mathbb{C}, arg(\lambda) \in [-arg(z_2 - z_1), \pi - arg(z_2 - z_1)]\} \\ &= \{\lambda \in \mathbb{C}, arg(\lambda) \in [-\pi, 0]\} \end{aligned}$$

$$\begin{aligned} \tilde{S}_n &= \{\lambda \in \mathbb{C}, arg(\lambda) \in [-arg(z_{n-1} - z_n), \pi - arg(z_{n-1} - z_n)]\} \\ \tilde{S}_n &= \{\lambda \in \mathbb{C}, arg(\lambda) \in [-arg(z_3 - z_4), \pi - arg(z_3 - z_4)]\} \\ &= \{\lambda \in \mathbb{C}, arg(\lambda) \in [-\pi, 0]\} \end{aligned}$$

Now S_j is found by the map $\lambda = (1 + \frac{\beta^2}{|k|^2})k$ from $\tilde{S}_j \to S_j$. It is observed that S_j coincide with $\tilde{S}_j$. Hence curves $L_{i,j}$ are just the curves l_j defined by equation (1.41). Now for the modified Helmholtz equation the sectionally holomorphic function in

complex k-plane becomes

$$\mu(z,\bar{z},k)=\frac{1}{2\pi i}\sum_{j=1}^{3}\int_{l_j}e^{i\acute{k}z+i\frac{\beta^2}{k'}\bar{z}}\frac{\rho_{i,j}(\acute{k})}{\acute{k}-k}d\acute{k}, \tag{1.59}$$

$$\rho_{j+1,j}(k)=\int_{z_{j+1}}^{z_j}e^{-ikz+i\frac{\beta^2}{k}\bar{z}}(q_z(z,\bar{z})dz+i\frac{\beta^2}{k}q(z,\bar{z})d\bar{z}),\quad j=1,2,3. \tag{1.60}$$

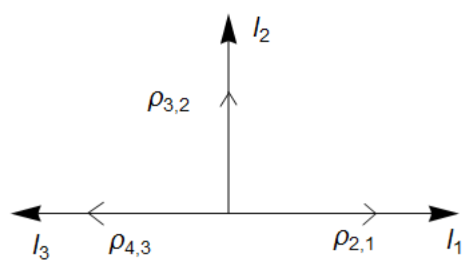

FIGURE 1.5. Contours and spectral functions for the modified Helmholtz equation in semi-infinite strip Ω.

In figure 1.5, contours and spectral functions for the modified Helmholtz equation in semi-infinite strip Ω, are obtained by using equation (1.41).

$$l_1=\{k\in\mathbb{C}:arg(k)=-arg(z_1-z_2)\}=\{k\in\mathbb{C}:arg(k)=0\}$$

$$l_2=\{k\in\mathbb{C}:arg(k)=-arg(z_2-z_3)\}=\{k\in\mathbb{C}:arg(k)=\frac{\pi}{2}\}$$

$$l_3=\{k\in\mathbb{C}:arg(k)=-arg(z_3-z_4)\}=\{k\in\mathbb{C}:arg(k)=-\pi\}$$

For the modified Helmholtz equation, $\tilde{S}_j$ coincides with S_j, so, S_1 and S_n are:

$$S_1=\{\lambda\in\mathbb{C},arg(k)\in[-\pi,0]\},$$

$$S_n=\{\lambda\in\mathbb{C},arg(k)\in[-\pi,0]\}.$$

Note that S_1 and S_n represent the lower half complex k-plane, thus their intersection is the lower half complex k-plane denoted by S. Using theorem 1.2.1 the global relation becomes

$$\sum_{j=1}^{n-1} \rho_{j+1,j}(k) = 0, \ j = 1, 2, 3, \quad \forall k \in S_1 \cap S_2 = S. \tag{1.61}$$

Now a relationship between the global relation and a closed 1-form $W(x, y, k), \ k \in \mathbb{C}$, is defined. A closed 1-form related to an arbitrary linear partial differential equation with constant coefficients is given in [20]. A closed 1-form for the modified Helmholtz equation (1.55) is

$$W(z, \bar{z}, k) = e^{-ikz+i\frac{\beta^2}{k}\bar{z}}(q_z(z, \bar{z})dz + i\frac{\beta^2}{k}q(z, \bar{z})d\bar{z}), \quad k \in \mathbb{C}.$$

Apply the differential operator

$$\begin{aligned} dW &= (e^{-ikz+i\frac{\beta^2}{k}\bar{z}}q_z(z, \bar{z}))_{\bar{z}}d\bar{z} \wedge dz + (i\frac{\beta^2}{k}e^{-ikz+i\frac{\beta^2}{k}\bar{z}}q(z, \bar{z}))_z dz \wedge d\bar{z} \\ &= e^{-ikz+i\frac{\beta^2}{k}\bar{z}}[(q_{z\bar{z}}(z, \bar{z}) + i\frac{\beta^2}{k}q_z(z, \bar{z}))d\bar{z} \wedge dz + (i\frac{\beta^2}{k}q_z(z, \bar{z}) + \beta^2 q(z, \bar{z})) \\ &\quad dz \wedge d\bar{z}], \end{aligned}$$

$dz \wedge d\bar{z} = -d\bar{z} \wedge dz$ use this relation to get,

$$dW = e^{-ikz+i\frac{\beta^2}{k}\bar{z}}(q_{z\bar{z}}(z, \bar{z}) - \beta^2 q(z, \bar{z}))d\bar{z} \wedge dz.$$

So, $W(z, \bar{z}, k), \ k \in \mathbb{C}$ is closed if and only if $q(z, \bar{z})$ satisfies equation (1.55). If the integrable PDE satisfied by $q(z, \bar{z})$ is valid in a closed simply connected domain D with boundary ∂D, then $dW = 0$ is equivalent to

$$\int_{\partial D} W(z, \bar{z}, k) = 0, \ k \in \mathbb{C}. \tag{1.62}$$

Hence equation (1.62) becomes global relation of the unified transform method for an integrable PDE in a closed polygon, the term coined in [16], [17] and [18]. In the given BVP of the modified Helmholtz equation in a semi-infinite strip Ω, equation

(1.62) with $Im(k) \le 0$ becomes the global relation defined by equation (1.61). Now, it is evident that for the BVP of the modified Helmholtz in a semi-infinite strip Ω, both the global relation and definition of spectral function $\{\rho_{j+1,j}(k)\}_{j=1}^{n-1}$ are a direct consequence of the closed 1-form $W(z,\bar{z},k)$.

1.3.1 Derivation of order two vector Riemann-Hilbert problem

It is shown in [18] that the generalized direct and inverse Fourier transform pair associated with modified Helmholtz equation (1.55) is

$$\rho_{i,j}(k) = \int_{z_i}^{z_j} e^{-(ikz+\frac{\beta^2}{ik}\bar{z})}(q_z(z,\bar{z})dz + i\frac{\beta^2}{k}q(z,\bar{z})d\bar{z}),$$
$$i = j+1,\ Im(k) \le 0,\ \text{for } j = 1,3 \text{ and } k \in \mathbb{C} \text{ for } j = 2, \tag{1.63}$$

$$q(z,\bar{z}) = \frac{1}{2\pi i}\sum_{j=1}^{3}\int_{l_j} e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\rho_{j+1,j}(k)\frac{dk}{k}. \tag{1.64}$$

Rays l_j are defined by equation (1.41). Use $\partial_z q = \frac{1}{2}(\partial_x q - i\partial_y q)$, $z = x$ along side(1), $z = iy$ along side(2) and $z = x + ia$ along side(3). Equation (1.63) yields

$$\rho_{2,1}(k) = \int_0^\infty e^{-(ik+\frac{\beta^2}{ik})x}(\frac{1}{2}\partial_x q - \frac{i}{2}\partial_y q - \frac{\beta^2}{ik}q)(x,0)dx,\ Im(k) \le 0, \tag{1.65}$$

$$\rho_{3,2}(k) = -i\int_0^a e^{(k+\frac{\beta^2}{k})y}(\frac{1}{2}\partial_x q - \frac{i}{2}\partial_y q - \frac{i\beta^2}{k}q)(0,y)dy,\ k \in \mathbb{C}, \tag{1.66}$$

$$\rho_{4,3}(k) = -e^{(k+\frac{\beta^2}{k})a}\int_0^\infty e^{-(ik+\frac{\beta^2}{ik})x}(\frac{1}{2}\partial_x q - \frac{i}{2}\partial_y q - \frac{\beta^2}{ik}q)(x,a)dx,\ Im(k) \le 0. \tag{1.67}$$

From equations (1.122), (1.123) and (1.124), find $q_y(x,0), q_x(0,y), q_y(x,a)$ as follows:

$$q_y(x,0) = \frac{1}{\sin\beta_1}[-g_1(x) + \cos\beta_1 q_x(x,0) + \gamma_1 q(x,0)], \tag{1.68}$$

$$q_x(0,y) = \frac{1}{\sin\beta_2}[-g_2(y) + \cos\beta_2 q_y(0,y) + \gamma_2 q(0,y)], \tag{1.69}$$

$$q_y(x,a) = \frac{1}{\sin\beta_3}[g_3(x) - \cos\beta_3 q_x(x,a) - \gamma_3 q(x,a)]. \tag{1.70}$$

From equation (1.68), use value of $q_y(x,0)$ in equation (1.65) and integrate by parts to get

$$\rho_{2,1}(k) = ih_1(k), \quad Im(k) \le 0, \tag{1.71}$$

$$\text{where} \quad h_1(k) = -J_1(ik)\Psi_1(-ik) + G_1(-ik) + \frac{e^{i\beta_1} d_0}{2\sin\beta_1}, \; arg(k) = 0, \tag{1.72}$$

$$J_1(k) = \frac{\gamma_1 + \frac{\beta^2}{k} e^{-i\beta_1} + k e^{i\beta_1}}{2\sin\beta_1}, \tag{1.73}$$

$$G_1(k) = \frac{1}{2\sin\beta_1} \int_0^\infty e^{(k+\frac{\beta^2}{k})x} g_1(x) dx, \; Re(k) \leq 0, \tag{1.74}$$

$$d_0 = q(0,0). \tag{1.75}$$

$\psi_1(k)$ is the unknown function defined by

$$\psi_1(k) = \int_0^\infty e^{(k+\frac{\beta^2}{k})x} q(x,0) dx, \quad Re(k) < 0. \tag{1.76}$$

From equation (1.69), use value of $q_x(0,y)$ in equation (1.66) and integrate by parts to get

$$\rho_{3,2}(k) = i[-J_2(k)\Psi_2(k) + G_2(k) - \frac{E(k)d_1 - d_0}{2e^{i\beta_2}\sin\beta_2}], \quad k \in \mathbb{C}, \tag{1.77}$$

$$J_2(k) = \frac{\gamma_2 - \frac{\beta^2}{k} e^{i\beta_2} - k e^{-i\beta_2}}{2\sin\beta_2}, \tag{1.78}$$

$$G_2(k) = \frac{1}{2\sin\beta_2} \int_0^a e^{(k+\frac{\beta^2}{k})y} g_2(y) dy, \quad k \in \mathbb{C}, \tag{1.79}$$

$$E(k) = e^{(k+\frac{\beta^2}{k})a}, \tag{1.80}$$

$$d_1 = q(0,a). \tag{1.81}$$

Note that $\psi_2(k)$ is the unknown function defined by

$$\psi_2(k) = \int_0^a e^{(k+\frac{\beta^2}{k})y} q(0,y) dy, \quad k \in \mathbb{C}. \tag{1.82}$$

From equation (1.70), use value of $q_y(x,a)$ in equation (1.67) and integrate by parts to get

$$\rho_{4,3}(k) = ih_3(k), \quad Im(k) \leq 0, \tag{1.83}$$

where $\quad h_3(k) = E(k)[-J_3(ik)\Psi_3(-ik) + G_3(-ik) + \dfrac{e^{-i\beta_3}d_1}{2\sin\beta_3}], \quad arg(k) = \pi,$

$$(1.84)$$

$$J_3(k) = \frac{\gamma_3 + \frac{\beta^2}{k}e^{i\beta_3} + ke^{-i\beta_3}}{2\sin\beta_3}, \tag{1.85}$$

$$G_3(k) = \frac{1}{2\sin\beta_3}\int_0^\infty e^{(k+\frac{\beta^2}{k})x} g_3(x)dx, \quad Re(k) \leq 0. \tag{1.86}$$

$\psi_3(k)$ is the unknown function defined by

$$\psi_3(k) = \int_0^\infty e^{(k+\frac{\beta^2}{k})x} q(x,a)dx, \quad Re(k) < 0. \tag{1.87}$$

Application of the abelian theorem to integrals defining $\psi_1(k)$ and $\psi_3(k)$ implies that $\psi_1(k)$ and $\psi_3(k)$ decay as $k \to 0$ or $k \to \infty$. Use values of $\rho_{2,1}(k)$, $\rho_{3,2}(k)$ and $\rho_{4,3}(k)$ from equations (1.71), (1.77) and (1.83) respectively, in the global relation defined by equation (1.61), and simplify to get

$$J_1(ik)\psi_1(-ik) + J_2(k)\psi_2(k) + E(k)J_3(ik)\psi_3(-ik) = G(k), \quad Im(k) \leq 0, \text{ where,} \tag{1.88}$$

$$\begin{aligned} G(k) = G_1(-ik) + G_2(k) + E(k)G_3(-ik) + \frac{d_0}{2}(\frac{e^{i\beta_1}}{\sin\beta_1} + \frac{e^{-i\beta_2}}{\sin\beta_2}) - \frac{d_1}{2}E(k) \\ \times (\frac{e^{-i\beta_2}}{\sin\beta_2} - \frac{e^{-i\beta_3}}{\sin\beta_3}). \end{aligned} \tag{1.89}$$

Take complex conjugate of equation (1.88) and replace k by $\bar{k}$ to get

$$\bar{J}_1(-ik)\psi_1(ik) + \bar{J}_2(k)\psi_2(k) + E(k)\bar{J}_3(-ik)\psi_3(ik) = \bar{G}(k), \quad Im(k) \geq 0. \tag{1.90}$$

From equation (1.90), find $\psi_2(k)$ in terms of $\psi_1(ik)$ and $\psi_3(ik)$. Using the resulting value of $\psi_2(k)$ in (1.88), and making use of equation (1.77), the result obtained is $\rho_{3,2}(k) = ih_2(k)$. Note that

$$\begin{aligned} h_2(k) = \frac{J_2(k)}{\bar{J}_2(k)}[\bar{J}_1(-ik)\Psi_1(ik) + E(k)\bar{J}_3(-ik)\Psi_3(ik) - \bar{G}(k)] + G_2(k) \\ - \frac{E(k)d_1 - d_0}{2e^{i\beta_2}\sin\beta_2}, \quad arg(k) = \frac{\pi}{2}. \end{aligned} \tag{1.91}$$

Both equations (1.88) and (1.90) are valid for $k \in \mathbb{R}$ (contour of RHP), $\psi_1(ik), \psi_3(ik)$ are holomorphic functions for $Im(k) > 0$, and $\psi_1(-ik)$, $\psi_3(-ik)$ are holomorphic functions for $Im(k) < 0$. Now $\psi_2(k)$ is eliminated from equations (1.88) and (1.90) to get

$$\frac{\bar{J}_1(-ik)}{\bar{J}_2(k)}\psi_1(ik) - \frac{J_1(ik)}{J_2(k)}\psi_1(-ik) + E(k)[\frac{\bar{J}_3(-ik)}{\bar{J}_2(k)}\psi_3(ik) - \frac{J_3(ik)}{J_2(k)}\psi_3(-ik)] = \frac{\bar{G}(k)}{\bar{J}_2(k)} - \frac{G(k)}{J_2(k)}, \quad k \in \mathbb{R}. \tag{1.92}$$

Replace k by $-k$ in equation (1.92) to get

$$\frac{\bar{J}_1(ik)}{\bar{J}_2(-k)}\psi_1(-ik) - \frac{J_1(-ik)}{J_2(-k)}\psi_1(ik) + E(-k) \times [\frac{\bar{J}_3(ik)}{\bar{J}_2(-k)}\psi_3(-ik) - \frac{J_3(-ik)}{J_2(-k)}\psi_3(ik)] = \frac{\bar{G}(-k)}{\bar{J}_2(-k)} - \frac{G(-k)}{J_2(-k)} \quad k \in \mathbb{R}. \tag{1.93}$$

Write equations (1.92) and (1.93) in matrix form

$$J(k)\begin{bmatrix} \psi_1(ik) \\ \psi_3(ik) \end{bmatrix} = \bar{J}(k)\begin{bmatrix} \psi_1(-ik) \\ \psi_3(-ik) \end{bmatrix} + \begin{bmatrix} f(k) \\ -f(-k) \end{bmatrix}, \quad k \in \mathbb{R}, \tag{1.94}$$

$$J(k) = \begin{bmatrix} \frac{\overline{J}_1(-ik)}{\overline{J}_2(k)} & E(k)\frac{\overline{J}_3(-ik)}{\overline{J}_2(k)} \\ \frac{J_1(-ik)}{J_2(-k)} & E(-k)\frac{J_3(-ik)}{J_2(-k)} \end{bmatrix}, \overline{J}(k) = \begin{bmatrix} \frac{J_1(ik)}{J_2(k)} & E(k)\frac{J_3(ik)}{J_2(k)} \\ \frac{\overline{J}_1(ik)}{\overline{J}_2(-k)} & E(-k)\frac{\overline{J}_3(ik)}{\overline{J}_2(-k)} \end{bmatrix}, \tag{1.95}$$

(1.96)

$$f(k) = \frac{\overline{G}(k)}{\overline{J}_2(k)} - \frac{G(k)}{J_2(k)}. \tag{1.97}$$

Equation (1.94) is the order two vector RHP which is equivalent to the BVP for the modified Helmholtz equation in a semi-infinite strip Ω subject to the Poincare type boundary conditions. Note that $\psi_1(ik), \psi_3(ik)$ are unknown holomorphic functions in the upper half complex k-plane, and $\psi_1(-ik)$, $\psi_3(-ik)$ are the unknown

holomorphic functions in the lower half complex k-plane. It is already observed that

$$\psi_1(k) = \circ(1), \quad \psi_3(k) = \circ(1) \text{ as } k \to 0 \text{ or } k \to \infty. \tag{1.98}$$

To obtain the standard form of the order two vector RHP, multiply equation (1.94) with $[J(k)]^{-1}$ to get

$$\begin{bmatrix} \psi_1(ik) \\ \\ \psi_3(ik) \end{bmatrix} = H(k) \begin{bmatrix} \psi_1(-ik) \\ \\ \psi_3(-ik) \end{bmatrix} + \begin{bmatrix} \mu_1(k) \\ \\ \mu_3(k) \end{bmatrix}, \quad k \in \mathbb{R}, \tag{1.99}$$

$$H(k) = \frac{1}{detJ(k)} \begin{bmatrix} H_{11}(k) & H_{12}(k) \\ \\ H_{21}(k) & H_{22}(k) \end{bmatrix}, \begin{bmatrix} \mu_1(k) \\ \\ \mu_3(k) \end{bmatrix} = [J(k)]^{-1} \begin{bmatrix} f(k) \\ \\ -f(-k) \end{bmatrix}. \tag{1.100}$$

Note that

$$H_{11}(k) = \frac{J_1(ik)J_3(-ik)}{J_2(k)J_2(-k)}E(-k) - \frac{\bar{J}_1(ik)\bar{J}_3(-ik)}{\bar{J}_2(k)\bar{J}_2(-k)}E(k), \tag{1.101}$$

$$H_{12}(k) = \frac{J_3(ik)J_3(-ik)}{J_2(k)J_2(-k)} - \frac{\bar{J}_3(ik)\bar{J}_3(-ik)}{\bar{J}_2(k)\bar{J}_2(-k)}, \tag{1.102}$$

$$H_{21}(k) = -\frac{J_1(ik)J_1(-ik)}{J_2(k)J_2(-k)} + \frac{\bar{J}_1(ik)\bar{J}_1(-ik)}{\bar{J}_2(k)\bar{J}_2(-k)}, \tag{1.103}$$

$$H_{22}(k) = -\frac{J_1(-ik)J_3(ik)}{J_2(k)J_2(-k)}E(k) + \frac{\bar{J}_1(-ik)\bar{J}_3(ik)}{\bar{J}_2(k)\bar{J}_2(-k)}E(-k). \tag{1.104}$$

Generally, it is not possible to find closed form solution of the order two vector RHP. The Closed form solution of the order two vector RHP is possible in some special cases like scalar and triangular [3].

Remark 1.3.1. *If $q(x,0) = O(x^{\delta_0})$, $\quad -1 < \delta_0 < 0$, i.e. $q(x,0)$ has a power singularity at $x = 0$ then integrals $\rho_{2,1}(k)$ and $\rho_{4,3}(k)$ defined by equations (1.65) and*

(1.66) respectively, are understood in regularized sense, and $d_0 = 0$. Correspondingly, if $q(x, a) = O(x^{\delta_1})$ as $x \to 0$ and $-1 < \delta_1 < 0$, then $d_1 = 0$, and $\rho_{3,2}(k)$ defined by equation (1.67) is understood in regularized sense[3].

1.3.2 Scalar cases

Let

$$J_{j2}(k) = \frac{J_j(ik)\bar{J}_2(k)}{\bar{J}_j(-ik)J_2(k)}, \quad j = 1, 3. \tag{1.105}$$

Using relation (1.105), rewrite relation (1.94) as follows:

$$\frac{\bar{J}_1(-ik)}{\bar{J}_2(k)}[\psi_1(ik) - J_{12}(k)\psi_1(-ik)] + \frac{E(k)\bar{J}_3(-ik)}{\bar{J}_2(k)}[\psi_3(ik) - J_{32}(k)\psi_3(-ik)] = \frac{\bar{G}(k)}{\bar{J}_2(k)} - \frac{G(k)}{J_2(k)}, \quad k \in \mathbb{R}, \tag{1.106}$$

$$\frac{\bar{J}_1(ik)}{\bar{J}_2(-k)}J_{12}(-k)[\psi_1(ik) - \frac{\psi_1(-ik)}{J_{12}(-k)}] + \frac{E(-k)\bar{J}_3(ik)}{\bar{J}_2(-k)}J_{32}(-k)[\psi_3(ik) - \frac{\psi_3(-ik)}{J_{32}(-k)} = \frac{G(-k)}{J_2(-k)} - \frac{\bar{G}(-k)}{\bar{J}_2(-k)}, \quad k \in \mathbb{R}. \tag{1.107}$$

Let

$$J_{j2}(k)J_{j2}(-k) = 1, \quad j = 1, 3. \tag{1.108}$$

Using this relation in equations (1.106) and (1.107), $\psi_1(ik) - J_{12}(k)\psi_1(-ik)$ and $\psi_1(ik) - J_{12}(k)\psi_1(-ik)$ are obtained as follows:

$$\psi_1(ik) = J_{12}(k)\psi_1(-ik) + \omega_1(k), \quad k \in \mathbb{R}, \tag{1.109}$$

$$\psi_3(ik) = J_{32}(k)\psi_3(-ik) + \omega_3(k), \quad k \in \mathbb{R}. \tag{1.110}$$

Equations (1.109) and (1.110) define two scalar RHPs, $\psi_1(ik)$, and $\psi_3(ik)$ are the holomorphic functions in upper half complex k-plane, where as $\psi_1(-ik)$ and

$\psi_3(-ik)$ are holomorphic functions in the lower half complex k-plane. Note that

$$\omega_1(k) = -\frac{E(k)\bar{J}_3(-ik)}{\bar{J}_1(-ik)}\omega_3(k) + \frac{1}{\bar{J}_1(-ik)}[\bar{G}(k) - \frac{\bar{J}_2(k)}{J_2(k)}G(k)],$$

$$\begin{aligned}\omega_3(k) = [&\frac{E(-k)J_3(-ik)}{J_2(-k)} - \frac{E(k)J_1(-ik)\bar{J}_3(-ik)}{\bar{J}_1(-ik)J_2(-k)}]^{-1}\times \\ &\{\frac{G(-k)}{J_2(-k)} - \frac{\bar{G}(-k)}{\bar{J}_2(-k)} - \frac{J_1(-ik)}{\bar{J}_1(-ik)J_2(-k)}[\bar{G}(k) - \frac{\bar{J}_2(k)}{J_2(k)}G(k)]\},\end{aligned} \tag{1.111}$$

$$\omega_1(k) = \circ(1) \text{ as } k \to \pm\infty, \text{ or } k \to 0 \text{ and } \omega_3(k) = \circ(1) \text{ as } k \to \pm\infty.$$

Conditions for the scalar RHPs are defined by equation (1.108). Simplification of these conditions in terms of β_j and γ_j results in the following relations:

$$j = 1 : e^{4i(\beta_1+\beta_2)} = 1, (2\beta^2 - \gamma_2^2)\sin 2\beta_1 - (2\beta^2 - \gamma_1^2)\sin 2\beta_2 = 0, \tag{1.112}$$

$$j = 3 : e^{4i(\beta_1\beta_3)} = 1, (2\beta^2 - \gamma_2^2)\sin 2\beta_3 - (2\beta^2 - \gamma_3^2)\sin 2\beta_3 = 0. \tag{1.113}$$

Since $\beta_j \in (0,\pi)$ and $\gamma_j > 0, (j = 1, 2, 3)$, the above relations generate

$$j = 1 : \beta_1 + \beta_2 = \frac{m\pi}{2}, 2\beta^2 - \gamma_2^2 + (-1)^m(2\beta^2 - \gamma_1^2) = 0, \quad m = 1, 2, 3, \tag{1.114}$$

$$j = 3 : \beta_2 - \beta_3 = \frac{m\pi}{2}, (-1)^m(2\beta^2 - \gamma_2^2) + 2\beta^2 - \gamma_3^2) = 0, \quad m = -1, 0, 1. \tag{1.115}$$

Hence conditions of the scalar RHPs given by equation (1.108) are simplified in terms of the parameters involved in the given BVP of the modified Helmholtz equation in a semi-infinite strip Ω i.e. $\beta_j \in (0,\pi)$ and $\gamma_j > 0, (j = 1, 2, 3)$, in the following cases:

1. $\gamma_1 = \gamma_3 = \sqrt{4\beta^2 - \gamma_2^2}, 0 < \gamma_j < 2|\beta|, \beta_1 = \pi - \beta_2, \beta_3 = \beta_2;$

2. $\gamma_1 = \sqrt{4\beta^2 - \gamma_2^2}, \gamma_3 = \gamma_2, 0 < \gamma_j < 2|\beta|, \beta_1 = \pi - \beta_2, \beta_3 = \beta_2 \pm \frac{\pi}{2};$

3. $\gamma_1 = \gamma_2, \gamma_3 = \sqrt{4\beta^2 - \gamma_2^2}, 0 < \gamma_j < 2|\beta|, \beta_1 = \pi - \beta_2 \pm \frac{\pi}{2}, \beta_3 = \beta_2;$

4. $\gamma_1 = \gamma_2 = \gamma_3, \beta_1 = \frac{\pi}{2} - \beta_2, \beta_3 = \frac{\pi}{2} + \beta_2, 0 < \beta_2 < \frac{\pi}{2}$;

5. $\gamma_1 = \gamma_2 = \gamma_3, \beta_1 = \frac{3\pi}{2} - \beta_2, \beta_3 = \beta_2 - \frac{\pi}{2}, \frac{\pi}{2} < \beta_2 < \pi$.

Boundary conditions (1.109) and (1.110) imply that the functions $\psi_1(-ik)$ and $\psi_3(-ik)$ can be analytically continued into $\mathbb{C}^+$ through the following relations:

$$\psi_1(-ik) = \frac{\psi_1(ik) - \omega_1(k)}{J_{12}(k)}, \quad k \in \mathbb{C}^+, \tag{1.116}$$

$$\psi_3(-ik) = \frac{\psi_3(ik) - \omega_3(k)}{J_{32}(k)}, \quad k \in \mathbb{C}^+. \tag{1.117}$$

Use values of $\psi_1(ik)$ and $\psi_3(ik)$ in equation (1.90), and simplify to get the function $\psi_2(k)$ as

$$\psi_2(k) = -\frac{\bar{J}_1(-ik)}{\bar{J}_2(k)}\psi_1(ik) - E(k)\frac{\bar{J}_3(-ik)}{\bar{J}_2(k)}\psi_3(ik) + \frac{\bar{G}(k)}{\bar{J}_2(k)}. \tag{1.118}$$

Now $\rho_{2,1}(k), \rho_{3,2}(k)$ and $\rho_{4,3}(k)$ given by equations (1.71), (1.77) and (1.83) respectively, are expressed in terms of $\psi_1(ik)$ and $\psi_3(ik)$, by using equations (1.116), (1.117), and (1.118). Then the inverse transformation equation (1.64) generates

$$q(z, \bar{z}) = I_0 + I_1 + I_2 + I_3. \tag{1.119}$$

Note that

$$\begin{aligned}
2\pi I_0 = &\int_0^{\infty} [\frac{\bar{J}_1(-ik)J_2(k)}{\bar{J}_2(k)}\omega_1(k) + G_1(-ik) + \frac{e^{i\beta_1}d_0}{2\sin\beta_1}]e^{ikz+\frac{\beta^2}{ik}\bar{z}}\frac{dk}{k} + \\
&\int_0^{i\infty} [-\frac{J_2(k)}{\bar{J}_2(k)}\bar{G}(k) + G_2(k) - \frac{E(k)d_1 - d_0}{2e^{i\beta_2}\sin\beta_2}]e^{ikz+\frac{\beta^2}{ik}\bar{z}}\frac{dk}{k} \\
&- \int_{-\infty}^{0} [G_3(-ik) + \frac{e^{-i\beta_3}d_1}{2\sin\beta_3}]E(k)e^{ikz+\frac{\beta^2}{ik}\bar{z}}\frac{dk}{k}, \\
I_1 = &-\frac{1}{2\pi}\int_{L^{++}} \frac{\bar{J}_1(-ik)J_2(k)}{\bar{J}_2(k)}\psi_1(ik)e^{ikz+\frac{\beta^2}{ik}\bar{z}}\frac{dk}{k}, \\
I_2 = &-\frac{1}{2\pi}\int_{-\infty}^{0} \frac{\bar{J}_3(-ik)J_2(k)}{\bar{J}_2(k)}\omega_3(k)E(k)e^{ikz+\frac{\beta^2}{ik}\bar{z}}\frac{dk}{k}, \\
I_3 = &\frac{1}{2\pi}\int_{L^{-+}} \frac{\bar{J}_3(-ik)J_2(k)}{\bar{J}_2(k)}\Psi_3(ik)E(k)e^{ikz+\frac{\beta^2}{ik}\bar{z}}\frac{dk}{k}.
\end{aligned} \tag{1.120}$$

Here $L^{++} = \{(i\infty, 0) \cup (0, \infty)\}$ and $L^{-+} = \{(-\infty, 0) \cup (0, i\infty)\}$ denote the boundaries of the first and second quadrant of the complex k-plane, drawn anti-clockwise. Note that the integrals I_0 and I_2 are expressed in terms of given boundary conditions, where as the integrals I_1 and I_3 involve the unknown functions $\psi_1(ik)$ and $\psi_3(ik)$ which are analytic in $\mathbb{C}^+$. The closed form solutions of the scalar RHPs (1.109) and (1.110) and hence the solution of given BVP of modified Helmholtz equation in a semi-infinite strip Ω subject to the Poincare type boundary conditions, are derived in [3].

1.3.3 Triangular cases

Let the relation defined by equation (1.108) is valid for $j = 1$, but is not valid for $j = 3$, i.e.

$$J_{12}(k)J_{12}(-k) = 1, \quad J_{32}(k)J_{32}(-k) \neq 1. \tag{1.121}$$

The first equation in relation (1.121) reveals that

(a) $\gamma_1 = \sqrt{4\beta^2 - \gamma_2^2}, \quad \beta_1 = \pi - \beta_2$ or

(b) $\gamma_1 = \gamma_2, \quad \beta_1 = \pi - \beta_2 \pm \frac{\pi}{2}$.

Using the above defined conditions (a) or (b) in the order two vector RHP (1.99), a triangular order two vector RHP is obtained [3]. The closed form solution of the triangular order two vector RHP, and hence the solution of given BVP of the modified Helmholtz equation in a semi-infinite strip Ω subject to the Poincare type boundary conditions, are derived in [3].

1.4 Impedance boundary conditions

This case is not explicitly discussed in [3]. Now we discuss this case here. Insert $\beta_1 = \beta_2 = \beta_3 = \frac{\pi}{2}$ in equations (1.122), (1.123) and (1.124). Then the impedance

boundary conditions are:

$$side1 : -q_y + \gamma_1 q = g_1(x), \quad 0 < x < \infty, \quad y = 0, \tag{1.122}$$

$$side2 : -q_x + \gamma_2 q = g_2(y), \quad x = 0, \quad 0 < y < a, \tag{1.123}$$

$$side3 : q_y + \gamma_3 q = g_3(x), \quad 0 < x < \infty, \quad y = a. \tag{1.124}$$

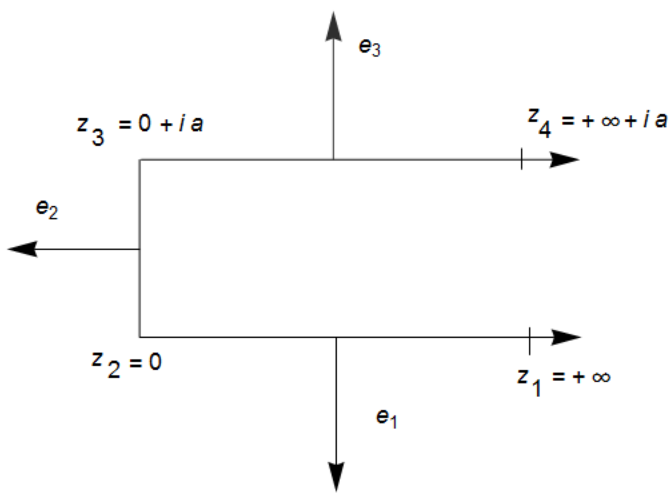

FIGURE 1.6. Impedance boundary condtions along the sides of Ω.

Figure 1.6 shows a semi-infinite strip Ω subject to the impedance boundary conditions. To find the corresponding scalar RHP in this case, consider the following results from equations (1.100), (1.101), (1.102), (1.103) and (1.104):

$$H(k) = \frac{1}{detJ(k)} \begin{bmatrix} H_{11}(k) & H_{12}(k) \\ & \\ H_{21}(k) & H_{22}(k) \end{bmatrix}, \tag{1.125}$$

$$detJ(k) = \frac{J_3(-ik)\overline{J}_1(-ik)E(-k) - J_1(-ik)\overline{J}_3(-ik)E(k)}{J_2(-k)\overline{J}_2(k)}, \tag{1.126}$$

$$H_{11}(k) = \frac{J_1(ik)J_3(-ik)}{J_2(k)J_2(-k)}E(-k) - \frac{\bar{J}_1(ik)\bar{J}_3(-ik)}{\bar{J}_2(k)\bar{J}_2(-k)}E(k), \tag{1.127}$$

$$H_{12}(k) = \frac{J_3(ik)J_3(-ik)}{J_2(k)J_2(-k)} - \frac{\bar{J}_3(ik)\bar{J}_3(-ik)}{\bar{J}_2(k)\bar{J}_2(-k)}, \tag{1.128}$$

$$H_{21}(k) = -\frac{J_1(ik)J_1(-ik)}{J_2(k)J_2(-k)} + \frac{\bar{J}_1(ik)\bar{J}_1(-ik)}{\bar{J}_2(k)\bar{J}_2(-k)}, \tag{1.129}$$

$$H_{22}(k) = -\frac{J_1(-ik)J_3(ik)}{J_2(k)J_2(-k)}E(k) + \frac{\bar{J}_1(-ik)\bar{J}_3(ik)}{\bar{J}_2(k)\bar{J}_2(-k)}E(-k), \tag{1.130}$$

$$\begin{bmatrix} \mu_1(k) \\ \\ \mu_2(k) \end{bmatrix} = [J(k)]^{-1} \begin{bmatrix} f(k) \\ \\ -f(-k) \end{bmatrix}. \tag{1.131}$$

From equation (1.95)

$$J[k]^{-1} = \frac{1}{detJ(k)} \begin{bmatrix} \frac{J_3(-ik)E(-k)}{J_2(-k)} & \frac{-\bar{J}_3(-ik)E(k)}{\bar{J}_2(k)} \\ \\ \frac{-J_1(-ik)}{J_2(-k)} & \frac{\bar{J}_1(-ik)}{\bar{J}_2(k)} \end{bmatrix}, \tag{1.132}$$

$$f(k) = \frac{\overline{G}(k)}{\bar{J}_2(k)} - \frac{G(k)}{J_2(k)}. \tag{1.133}$$

Note that $G(k)$ is given by substituting $\beta_1 = \beta_2 = \beta_3 = \frac{\pi}{2}$ in equation (1.89)

$$G(k) = G_1(-ik) + G_2(k) + G_3(-ik). \tag{1.134}$$

$G_1(-ik), G_2(k), E(k)$ and $G_3(-ik)$ are given below. To find $H_{11}(k), H_{12}(k), H_{21}(k)$ and $H_{22}(k)$ substitute $\beta_1 = \beta_2 = \beta_3 = \frac{\pi}{2}$ in equations (1.72), (1.84), (1.91),(1.73), (1.74), (1.78), (1.79), (1.85), (1.86) to get the following:

$$h_1(k) = -J_1(ik)\psi_1(-ik) + G_1(-ik) + \frac{id_0}{2},\ arg(k) = 0, \tag{1.135}$$

$$J_1(k) = \frac{ik^2 + \gamma_1 k - i\beta^2}{2k}, \tag{1.136}$$

$$G_1(k) = \frac{1}{2}\int_0^\infty e^{(k+\frac{\beta^2}{k})x} g_1(x)dx,\ Re(k) \leq 0, \tag{1.137}$$

$$d_0 = q(0,0), \tag{1.138}$$

$$
\begin{aligned}
h_2(k) = & \frac{J_2(k)}{\bar{J}_2(k)}[\bar{J}_1(-ik)\psi_1(ik) + E(k)\bar{J}_3(-ik)\psi_3(ik) - \bar{G}(k)] + G_2(k) \\
& i\frac{E(k)d_1 - d_0}{2}, \quad arg(k) = \frac{\pi}{2},
\end{aligned} \tag{1.139}
$$

$$
J_2(k) = \frac{ik^2 + \gamma_2 k - i\beta^2}{2k}, \tag{1.140}
$$

$$
G_2(k) = \frac{1}{2}\int_0^a e^{(k+\frac{\beta^2}{k})y} g_2(y)dy, \quad k \in \mathbb{C}, \tag{1.141}
$$

$$
E(k) = e^{(k+\frac{\beta^2}{k})a}, \tag{1.142}
$$

$$
d_1 = q(0, a), \tag{1.143}
$$

$$
h_3(k) = E(k)[-J_3(ik)\psi_3(-ik) + G_3(-ik) + \frac{-id_1}{2}], \quad arg(k) = \pi, \tag{1.144}
$$

$$
J_3(k) = \frac{-ik^2 + \gamma_3 k + i\beta^2}{2k}, \tag{1.145}
$$

$$
G_3(k) = \frac{1}{2}\int_0^\infty e^{(k+\frac{\beta^2}{k})x} g_3(x)dx, \quad Re(k) \leq 0. \tag{1.146}
$$

Since $J_2[-k] = \overline{J}_2(k)$, $J_2(k)J_2(-k) = \overline{J}_2(k)\overline{J}_2(-k)$, $J_3(ik)J_3(-ik) = \overline{J}_3(ik)\overline{J}_3(-ik)$, and $J_1(ik)J_1(-ik) = \overline{J}_1(ik)\overline{J}_1(-ik)$, equations (1.127), (1.128), (1.129), (1.130) and (1.132) become

$$
H_{11}(k) = \frac{J_1(ik)J_3(-ik)E(-k) - \overline{J}_1(ik)\overline{J}_3(-ik)E(k)}{J_2(k)J_2(-k)}, \tag{1.147}
$$

$$
H_{12}(k) = \frac{J_3(ik)J_3(-ik)\overline{J}_3(ik)\overline{J}_3(-ik)}{J_2(k)J_2(-k)} = 0, \tag{1.148}
$$

$$
H_{21}(k) = \frac{-J_1(ik)J_1(-ik) + \overline{J}_1(ik)\overline{J}_1(-ik)}{J_2(k)J_2(-k)} = 0, \tag{1.149}
$$

$$
H_{22}(k) = \frac{-J_1(-ik)J_3(ik)E(k) + \overline{J}_1(-ik)\overline{J}_3(ik)E(-k)}{J_2(k)J_2(-k)}, \tag{1.150}
$$

$$
J[k]^{-1} = \frac{1}{J_2(-k)detJ(k)} \begin{bmatrix} J_3(-ik)E(-k) & -\overline{J}_3(-ik)E(k) \\ -J_1(-ik) & \overline{J}_1(-ik) \end{bmatrix}. \tag{1.151}
$$

Note that $detJ(k)$ is given by equation (1.126). Use value of $J[k]^{-1}$ from equation (1.151) in equation (1.131) to get

$$\begin{bmatrix} \mu_1(k) \\ \\ \mu_3(k) \end{bmatrix} = \frac{1}{J_2(-k)detJ(k)} \begin{bmatrix} J_3(-ik)E(-k)f(k) + \overline{J}_3(-ik)E(k)f(-k) \\ -J_1(-ik)f(k) - \overline{J}_1(-ik)f(-k) \end{bmatrix}. \tag{1.152}$$

Use the value of $f(k)$ from equation (1.133) in equation (1.152) to get

$$\mu_1(k) = \frac{J_3(-ik)E(-k)\{\frac{\overline{G}(k)}{\overline{J}_2(k)} - \frac{G(k)}{J_2(k)}\} + \overline{J}_3(-ik)E(k)\{\frac{\overline{G}(-k)}{\overline{J}_2(-k)} - \frac{G(-k)}{J_2(-k)}\}}{J_2(-k)detJ(k)}, \tag{1.153}$$

$$\mu_3(k) = \frac{-J_1(-ik)\{\frac{\overline{G}(k)}{\overline{J}_2(k)} - \frac{G(k)}{J_2(k)}\} - \overline{J}_1(-ik)\{\frac{\overline{G}(-k)}{\overline{J}_2(-k)} - \frac{G(-k)}{J_2(-k)}\}}{J_2(-k)detJ(k)}. \tag{1.154}$$

Note that $E(k)$ and $G(k)$ are given by equations (1.142) and (1.134). From equations (1.136), (1.140), (1.145), (1.142) and (1.126) use the values of $J_1(k), J_2(k)$ and $J_3(k)$, $E(k)$ and $detJ(k)$ respectively, in equations (1.126), (1.147), (1.148), (1.149), (1.150), (1.153) and (1.154) to get

$$\frac{H_{11}(k)}{detJ(k)} = \frac{(k^2+\beta^2-k\gamma_1)(k^2-\beta^2+ik\gamma_2)}{(k^2+\beta^2+k\gamma_1)(k^2-\beta^2-ik\gamma_2)}, \tag{1.155}$$

$$\frac{H_{21}(k)}{detJ(k)} = 0, \tag{1.156}$$

$$\frac{H_{21}(k)}{detJ(k)} = 0, \tag{1.157}$$

$$\frac{H_{22}(k)}{detJ(k)} = \frac{(k^2-\beta^2+ik\gamma_2)(k^2+\beta^2+k\gamma_3)}{(k^2-\beta^2-ik\gamma_2)(k^2+\beta^2-k\gamma_3)}, \tag{1.158}$$

$$\begin{aligned} \mu_1(k) = {} & \frac{ie^{a(k+\frac{\beta^2}{k})}(k^2+\beta^2-k\gamma_3)}{Q(k)}[\frac{G(-k)}{k^2-\beta^2+ik\gamma_2} + \frac{\overline{G}(-k)}{k^2-\beta^2-ik\gamma_2}] - \\ & \frac{ie^{-a(k+\frac{\beta^2}{k})}(k^2+\beta^2-k\gamma_3)}{Q(k)}[\frac{G(k)}{k^2-\beta^2-ik\gamma_2} + \frac{\overline{G}(k)}{k^2-\beta^2+ik\gamma_2}], \end{aligned} \tag{1.159}$$

$$\begin{aligned} \mu_3(k) = {} & \frac{i(k^2+\beta^2+k\gamma_1)[(k^2-\beta^2-ik\gamma_2)G(-k) + (k^2-\beta^2+ik\gamma_2)\overline{G}(-k)]}{Q(k)[k^4+\beta^4+k^2(-2\beta^2+\gamma_2^2)]} + \\ & \frac{(k^2+\beta^2+k\gamma_1)[(-ik^2+i\beta^2+k\gamma_2)G(k) - i(k^2-\beta^2-ik\gamma_2)\overline{G}(k)]}{Q(k)[k^4+\beta^4+k^2(-2\beta^2+\gamma_2^2)]}. \end{aligned} \tag{1.160}$$

Note that

$$
\begin{aligned}
Q(k) &= J2(-k) det J(k) \\
&= \frac{i(k^2+\beta^2+k\gamma_1)(k^2+\beta^2-k\gamma_3)e^{-a(k+\frac{\beta^2}{k})}(-1+e^{2a(k+\frac{\beta^2}{k})})}{2k(k^2-\beta^2+ik\gamma_2)}.
\end{aligned}
\tag{1.161}
$$

From equations (1.155), (1.156), (1.157) and (1.158) use the values of $H_{11}(k), H_{12}(k)$, $H_{21}(k), H_{22}(k)$ in equation (1.125) to get

$$
H(k) = \begin{bmatrix} p_1(k) & 0 \\ 0 & p_3(k) \end{bmatrix}, \text{ where} \tag{1.162}
$$

$$
p_1(k) = \frac{(k^2+\beta^2-k\gamma_1)(k^2-\beta^2+ik\gamma_2)}{(k^2+\beta^2+k\gamma_1)(k^2-\beta^2-ik\gamma_2)}, \tag{1.163}
$$

$$
p_3(k) = \frac{(k^2-\beta^2+ik\gamma_2)(k^2+\beta^2+k\gamma_3)}{(k^2-\beta^2-ik\gamma_2)(k^2+\beta^2-k\gamma_3)}. \tag{1.164}
$$

From equation (1.162) use the value of $H(k)$ in equation (1.99), then the order two vector RHP becomes

$$
\begin{bmatrix} \psi_1(ik) \\ \psi_3(ik) \end{bmatrix} = \begin{bmatrix} p_1(k) & 0 \\ 0 & p_3(k) \end{bmatrix} \begin{bmatrix} \psi_1(-ik) \\ \psi_3(-ik) \end{bmatrix} + \begin{bmatrix} \mu_1(k) \\ \mu_3(k) \end{bmatrix}, \quad k \in \mathbb{R}. \tag{1.165}
$$

$\mu_1(k), \mu_3(k), p_1(k),$ and $p_3(k)$ are given by equations (1.159), (1.160), (1.163), (1.164). Equation (1.165) is equivalent to the following scalar RHPs:

$$
\psi_1(ik) = p_1(k)\psi_1(-ik) + \mu_1(k), \quad k \in \mathbb{R}, \tag{1.166}
$$

$$
\psi_3(ik) = p_3(k)\psi_3(-ik) + \mu_3(k), \quad k \in \mathbb{R}. \tag{1.167}
$$

Note that equations (1.166) and (1.167) indicate that the functions $\psi_1(ik)$ and $\psi_3(ik)$ can be analytically continued into $\mathbb{C}^+$ through the following relations.

$$
\psi_1(-ik) = \frac{1}{p_1(k)}\psi_1(ik) - \frac{\mu_1(k)}{p_1(k)}, \quad k \in \mathbb{C}^+, \tag{1.168}
$$

$$
\psi_3(-ik) = \frac{1}{p_3(k)}\psi_3(ik) - \frac{\mu_3(k)}{p_3(k)}, \quad k \in \mathbb{C}^+. \tag{1.169}
$$

Substitute $\rho_{j+1,j}(k) = ih_j(k), j = 1, 2, 3$ in equation (1.64) to get

$$q(z,\bar{z}) = \frac{1}{2\pi}\sum_{j=1}^{3}\int_{l_j} e^{ikz+(\frac{\beta^2}{ik})\bar{z}} h_j(k)\frac{dk}{k}. \tag{1.170}$$

Now, to find an expression for the solution $q(z, \bar{z})$, consider

$$\frac{1}{2\pi}\int_{l_1} h_1(k)e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k}$$

from equation (1.135) insert value of $h_1(k)$ in the above equation to get

$$= \frac{1}{2\pi}\int_{l_1}[-J_1(ik)\psi_1(-ik) + G_1(-ik) + \frac{id_0}{2}]e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k},\ arg(k) = 0,$$

use the value of $\psi_1(-ik)$ from equation (1.168) in the above equation

$$= \frac{1}{2\pi}\int_{l_1}[-J_1(ik)\{\frac{1}{p_1(k)}\psi_1(ik) - \frac{\mu_1(k)}{p_1(k)}\} + G_1(-ik) + \frac{id_0}{2}]e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k},$$

$arg(k) = 0, \psi_1(ik)$ is analytic in $\mathbb{C}^+$

$$\begin{aligned}&= \frac{1}{2\pi}\int_{0}^{\infty}[J_1(ik)\frac{\mu_1(k)}{p_1(k)} + G_1(-ik) + \frac{id_0}{2}]e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k}+ \\ &- \frac{1}{2\pi}\int_{L^{++}} J_1(ik)\frac{1}{p_1(k)}\psi_1(ik)e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k}.\end{aligned} \tag{1.171}$$

Note that $L^{++} = \{(i\infty, 0) \cup (0, \infty)\}$ denotes anticlockwise boundary of first quadrant of the complex k-plane. Now consider

$$\frac{1}{2\pi}\int_{l_2} h_2(k)e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k}$$

from equation (1.139) insert the value of $h_2(k)$ in the above equation to get

$$\begin{aligned}&= \frac{1}{2\pi}\int_{l_2}[\frac{J_2(k)}{\bar{J}_2(k)}\{\bar{J}_1(-ik)\psi_1(ik) + E(k)\bar{J}_3(-ik)\psi_3(ik) - \bar{G}(k)\} \\ &+ G_2(k) + i\frac{E(k)d_1 - d_0}{2}]e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k},\ arg(k) = \frac{\pi}{2}, \\ &= \frac{1}{2\pi}\int_{0}^{i\infty}[-\frac{J_2(k)}{\bar{J}_2(k)}\bar{G}(k) + G_2(k) + i\frac{E(k)d_1 - d_0}{2}]e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k}+ \\ &\frac{1}{2\pi}\int_{0}^{i\infty}\frac{J_2(k)}{\bar{J}_2(k)}[\bar{J}_1(-ik)\psi_1(ik) + E(k)\bar{J}_3(-ik)\psi_3(ik)]e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k}.\end{aligned} \tag{1.172}$$

Since $\psi_1(ik)$ and $\psi_3(ik)$ are analytic functions in $\mathbb{C}^+$, and the integrand in the 2nd integral in equation (1.172) has zeroes only in $\mathbb{C}^-$, integrand in the second integral is analytic in $\mathbb{C}^+$. Hence the application of Cauchy's theorem and Jordan's lemma implies that the value of 2nd integral in equation(1.172) is zero. Hence, equation (1.172) becomes

$$\begin{aligned}&\frac{1}{2\pi}\int_{l_2} h_2(k)e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k}\\&=\frac{1}{2\pi}\int_0^{i\infty}[-\frac{J_2(k)}{\bar{J}_2(k)}\bar{G}(k)+G_2(k)+i\frac{E(k)d_1-d_0}{2e^{i\frac{\pi}{2}}}]e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k}.\end{aligned} \tag{1.173}$$

Now consider

$$\frac{1}{2\pi}\int_{l_3} h_3(k)e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k}$$

from equation(1.144) insert the value of $h_3(k)$ in the above equation to get

$$=\frac{1}{2\pi}\int_{l_3} E(k)[-J_3(ik)\psi_3(-ik)+G_3(-ik)-\frac{id_1}{2}]e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k},\ arg(k)=\pi,$$

from equation (1.169) insert the value of $\psi_3(-ik)$ in the above equation to get

$$\begin{aligned}&=\frac{1}{2\pi}\int_{l_3} E(k)[-J_3(ik)\{\frac{1}{p_3(k)}\psi_3(ik)-\frac{\mu_3(k)}{p_3(k)}\}+G_3(-ik)-\frac{id_1}{2}]e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k},\\&arg(k)=\pi,\quad \psi_3(ik)\text{ is analytic in }\mathbb{C}^+.\end{aligned} \tag{1.174}$$

Equation (1.174) can be written as

$$\begin{aligned}\frac{1}{2\pi}\int_{l_3} h_3(k)e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k}&=\frac{1}{2\pi}\int_0^{-\infty}E(k)[G_3(-ik)-\frac{id_1}{2}]e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k}+\\&\frac{1}{2\pi}\int_0^{-\infty}E(k)J_3(ik)\frac{\mu_3(k)}{p_3(k)}e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k}+\\&\frac{1}{2\pi}\int_{L^{+-}}-E(k)J_3(ik)\frac{1}{p_3(k)}\psi_3(ik)e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k}\\\frac{1}{2\pi}\int_{l_3} h_3(k)e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k}&=-\frac{1}{2\pi}\int_{-\infty}^0 E(k)[G_3(-ik)-\frac{id_1}{2}]e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k}\\&-\frac{1}{2\pi}\int_{-\infty}^0 E(k)J_3(ik)\frac{\mu_3(k)}{p_3(k)}e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k}+\\&\frac{1}{2\pi}\int_{L^{-+}}E(k)J_3(ik)\frac{1}{p_3(k)}\psi_3(ik)e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k}.\end{aligned} \tag{1.175}$$

Note that $L^{+-} = \{(i\infty, 0) \cup (0, -\infty)\}$ denotes the clockwise drawn boundary of the 2nd quadrant of the complex k-plane, and $L^{-+} = \{(-\infty, 0) \cup (0, i\infty)\}$ denotes the anticlockwise drawn boundary of 2nd quadrant of the complex k-plane. Use the values of the integral expressions from equations (1.171), (1.173), (1.175) in equation (1.170) to get

$$q(z, \bar{z}) = I_0 + I_1 + I_2 + I_3, \text{ where} \tag{1.176}$$

$$\begin{aligned} I_0 = & \frac{1}{2\pi}\int_0^\infty [J_1(ik)\frac{\mu_1(k)}{p_1(k)} + G_1(-ik) + \frac{id_0}{2}]e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k} \\ & + \frac{1}{2\pi}\int_0^{i\infty} [-\frac{J_2(k)}{\bar{J}_2(k)}\bar{G}(k) + G_2(k) + i\frac{E(k)d_1 - d_0}{2e^{i\frac{\pi}{2}}}]e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k} \\ & - \frac{1}{2\pi}\int_{-\infty}^0 E(k)[G_3(-ik) - \frac{id_1}{2}]e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k}, \end{aligned} \tag{1.177}$$

$$I_1 = -\frac{1}{2\pi}\int_{L^{++}} J_1(ik)\frac{1}{p_1(k)}\psi_1(ik)e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k}, \tag{1.178}$$

$$I_2 = -\frac{1}{2\pi}\int_{-\infty}^0 E(k)J_3(ik)\frac{\mu_3(k)}{p_3(k)})e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k}, \tag{1.179}$$

$$I_3 = \frac{1}{2\pi}\int_{L^{-+}} E(k)J_3(ik)\frac{1}{p_3(k)}\psi_3(ik)e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k}. \tag{1.180}$$

Now to find the solution $q(z, \bar{z})$ from equation (1.176), $I_0, I_1, I_2,$ and I_3 are required to be evaluated. Note that I_0 and I_2 are independent of the unkknown analytic functions $\psi_1(ik)$ and $\psi_3(ik)$, where as I_1 and I_3 contain the unknown functions $\psi_1(ik)$ and $\psi_3(ik)$ which are analytic in $\mathbb{C}^+$. So, we will first evaluate I_1 and I_3, to check whether we can avoid the solution of scalar RHPs (1.166) and (1.167). Use the values of $J_1(ik), J_3(ik), p_1(k), p_3(k)$ from equations (1.136), (1.145), (1.163) and (1.164) in equations (1.178) and (1.180), and simplify to get

$$I_1 = \frac{1}{4\pi}\int_{L^{++}} \frac{(k^2 + \beta^2 + k\gamma_1)(k^2 - \beta^2 - ik\gamma_2)}{k^2(k^2 - \beta^2 + ik\gamma_2)}\psi_1(ik)e^{ikz+(\frac{\beta^2}{ik})\bar{z}}dk, \tag{1.181}$$

$$I_3 = \frac{1}{4\pi}\int_{L^{-+}} \frac{(k^2 - \beta^2 - ik\gamma_2)(k^2 + \beta^2 - k\gamma_3)}{k^2(k^2 - \beta^2 + ik\gamma_2)}E(k)\psi_3(ik)e^{ikz+(\frac{\beta^2}{ik})\bar{z}}dk. \tag{1.182}$$

Equations (1.181) and (1.182) can be expressed as

$$I_1 = \frac{1}{4\pi}\int_{L^{++}} \frac{(k^2+\beta^2+k\gamma_1)(k^2-\beta^2-ik\gamma_2)}{k^2 S(k)}\psi_1(ik)e^{ikz+(\frac{\beta^2}{ik})\bar{z}}dk, \quad (1.183)$$

$$I_3 = \frac{1}{4\pi}\int_{L^{-+}} \frac{(k^2-\beta^2-ik\gamma_2)(k^2+\beta^2-k\gamma_3)}{k^2 S(k)}E(k)\psi_3(ik)e^{ikz+(\frac{\beta^2}{ik})\bar{z}}dk. \quad (1.184)$$

Note that $S(k) = k^2+ik\gamma_2-\beta^2$, and $L^{++} = \{(i\infty, 0)\cup(0,\infty)\}$, $\quad L^{-+} = \{(-\infty, 0)\cup (0, i\infty)\}$ denote the positively oriented boundaries of the first and second quadrants of the complex k-plane. To find the values of I_1 and I_3, zeroes of $S(k) = k^2+ik\gamma_2-\beta^2$ are needed.

- Case-(a): If $\gamma_2 = 2|\beta|$, then the zeroes of $S(k)$ are $k_{1,2} = -i\frac{\gamma_2}{2}$, $\gamma_2 > 0$.

- Case-(b): If $\gamma_2 > 2|\beta|$, then the zeroes of $S(k)$ are $k_{1,2} = i(\frac{-\gamma_2}{2} \pm \frac{\sqrt{p}}{2})$, $\gamma_2 > 0$, $p = \gamma_2^2 - 4\beta^2 > 0$.

- Case-(c): If $\gamma_2 < 2|\beta|$, then the zeroes of $S(k)$ are $k_{1,2} = -i\frac{\gamma_2}{2} \pm \frac{\sqrt{p_1}}{2}$, $\gamma_2 > 0$, $p_1 = 4\beta^2 - \gamma_2^2 > 0$.

To find the value of I_1 in case-(a), it is observed that in this case the zeroes of $S(k)$ i.e. $k_{1,2} \in \mathbb{C}^-$. Note that $\psi_1(ik)$ is analytic and bounded in $\mathbb{C}^+$, so, it is analytic and bounded in the region defined by $x \geq 0,\ y \geq 0$, $\frac{1}{k^2}e^{ikz+\frac{\beta^2}{ik}\bar{z}}$ is analytic and bounded in the region defined by $x \geq 0,\ y \geq 0$. Also, the second degree polynomials in the numerator of the integrand are analytic and bounded in the region $x \geq 0,\ y \geq 0$. This implies that the integrand in I_1 is analytic and bounded in the region defined $x \geq 0,\ y \geq 0$. Hence application of Cauchy's theorem and Jordan's lemma to I_1 defined by equation (1.183), gives the following result.

$$I_1 = 0.$$

To find the value of I_1 in case-(b) and case-(c), it is observed that in these cases, the only possibility for the zeroes of $S(k)$ is $k_{1,2} \in \mathbb{C}^-$. Then using the same reasoning

as in case-(a), the result is

$$I_1 = 0.$$

To find the value of I_3 in case-(a), note that in this case the zeroes of $S(k)$ i.e. $k_{1,2} \in \mathbb{C}^-$. It is observed that $\psi_3(ik)$ is analytic and bounded in $\mathbb{C}^+$, so, it is analytic and bounded in the region defined by $x \leq 0,\ y \geq 0,\ \frac{1}{k^2}E(k)e^{ikz+\frac{\beta^2}{ik}\bar{z}} = \frac{1}{k^2}e^{(k+\frac{\beta^2}{k})a+ikz+\frac{\beta^2}{ik}\bar{z}}$ is analytic and bounded in the region defined by $x \leq 0,\ y \geq 0$. Also, the second degree polynomials in the numerator of the integrand are analytic and bounded in the region $x \leq 0,\ y \geq 0$. This implies that the integrand in I_3 is analytic and bounded in the region defined by $x \leq 0,\ y \geq 0$. Hence application of Cauchy's theorem and Jordans lemma to I_3 defined by equation (1.184), gives the following result:

$$I_3 = 0.$$

To find the value of I_3 in case-(b) and case-(c), it is observed that in these cases, the only possibility for the zeroes of $S(k)$ is $k_{1,2} \in \mathbb{C}^-$. Then using the same reasoning as in case-(a), the result is

$$I_3 = 0.$$

Note that I_0 and I_2 are independent of $\psi_1(ik)$ and $\psi_3(ik)$, and I_1 and I_3 which involve $\psi_1(ik)$ and $\psi_3(ik)$ have values equal to zero. Hence, in the case of impedance boundary conditions, there is no need to solve the scalar RHPs given by equations (1.166) and (1.167). Use the values of I_1 and I_3 in equation (1.176) to get

$$q(z, \bar{z}) = I_0 + I_2, \tag{1.185}$$

where I_0 and I_2 are given by equations (1.177) and (1.179) shown below.

$$
\begin{aligned}
I_0 = & \frac{1}{2\pi}\int_0^{\infty}[J_1(ik)\frac{\mu_1(k)}{p_1(k)} + G_1(-ik) + \frac{id_0}{2}]e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k} \\
& + \frac{1}{2\pi}\int_0^{i\infty}[-\frac{J_2(k)}{\bar{J}_2(k)}\bar{G}(k) + G_2(k) + i\frac{E(k)d_1 - d_0}{2e^{i\frac{\pi}{2}}}]e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k} \\
& - \frac{1}{2\pi}\int_{-\infty}^{0} E(k)[G_3(-ik) - \frac{id_1}{2}]e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k}, \\
I_2 = & -\frac{1}{2\pi}\int_{-\infty}^{0} E(k)J_3(ik)\frac{\mu_3(k)}{p_3(k)})e^{ikz+(\frac{\beta^2}{ik})\bar{z}}\frac{dk}{k}.
\end{aligned} \tag{1.186}
$$

$J_1(k), J_2(k), J_3(k)$ and $E(k)$ are defined by equations (1.136), (1.140), (1.145) and (1.142). Equations (1.137), (1.141), (1.146), (1.159) and (1.160) give expressions for $G_1(k)$, $G_2(k)$, $G_3(k)$, $\mu_1(k)$ and $\mu_3(k)$ respectively. Equation (1.185) gives the solution of the modified Helmholtz equation in a semi-infinite strip Ω subject to the impedance boundary conditions. The constants involved in I_0 and I_2 are obtained in the following way. Substitute $x = 0, y = 0$ in equation (1.64) to get $d_0 = q(0,0)$, and $x = 0, y = a$ in equation (1.64) to get $d_1 = q(0,a)$. Expressions defining d_0 and d_1 are given by

$$d_0 = \frac{1}{2\pi}\sum_{j=1}^{3}\int_{l_j} h_j(k)\frac{dk}{k}, \quad d_1 = \frac{1}{2\pi}\sum_{j=1}^{3}\int_{l_j} e^{-(k+\frac{\beta^2}{k})a}h_j(k)\frac{dk}{k}.$$

Note that $h_1(k), h_2(k), h_3(k)$ are given by equations (1.135), (1.139) and (1.144).

1.5 Helmholtz equation in a semi-infinite strip Ω

In this section we use the results developed by A. S. Fokas in [16], [17] and [18]. Consider the Helmholtz equation

$$(\partial_x^2 + \partial_y^2 + 4\beta^2)q(x,y) = g(x,y), \quad \beta \in \mathbb{R}, \quad (x,y) \in \Omega, \tag{1.187}$$

where Ω is a semi-infinite strip subject to the Poincare type boundary conditions shown in figure 1.4, with corners $z_1 = \infty$, $z_2 = 0$, $z_3 = ia$, $z_4 = \infty + ia$, $a > 0$. A Lax pair related to equation (1.187) is given in example 1.2.8 by equations (1.26)

and (1.27) with $\alpha = \beta^2$. Simultaneous spectral analysis of the Lax pair yields the following sectionally holomorphic function in complex k-plane.

$$\mu(z, \bar{z}, k) = \frac{1}{2\pi i} \sum_{i,j=1}^{3} \int_{L_{i,j}} Exp[i\acute{k}z + i\frac{\beta^2}{\acute{k}}\bar{z}] \frac{\rho_{i,j}(\acute{k})}{\acute{k} - k} d\acute{k} \tag{1.188}$$

$$\rho_{i,j}(k) = \int_{z_i}^{z_j} Exp[-ikz - i\frac{\beta^2}{k}\bar{z}](q_z(z, \bar{z})dz - i\frac{\beta^2}{k}q(z, \bar{z})d\bar{z}), \ i = j + 1 \tag{1.189}$$

$L_{i,j}$ are the curves formed by intersection of the sectors $\bar{S}_i$ and $\bar{S}_j$ defined by the equations

$$\tilde{S}_j = \{\lambda \in \mathbb{C}, arg(\lambda) \in [-arg(z_{j-1} - z_j), \pi - arg(z_{j+1} - z_j)\}, \ 2 \leq j < n, \tag{1.190}$$

$$\begin{aligned} \tilde{S}_i = \{\lambda \in \mathbb{C}, arg(\lambda) \in [-arg(z_{i-1} - z_i), \pi - arg(z_{i+1} - z_i)\}, 2 \leq j < n \\ i = j + 1, \end{aligned} \tag{1.191}$$

For the Helmholtz equation in a semi-infinite strip Ω, $\tilde{S}_1$ and $\tilde{S}_n$ are the half planes, obtained by using equations (1.47) and (1.48) and replacing k by λ.

$$\begin{aligned} \tilde{S}_1 &= \{\lambda \in \mathbb{C}, arg(\lambda) \in [-arg(z_2 - z_1), \pi - arg(z_2 - z_1)]\} \\ &= \{\lambda \in \mathbb{C}, arg(\lambda) \in [-\pi, 0]\} \end{aligned}$$

$$\begin{aligned} \tilde{S}_n &= \{\lambda \in \mathbb{C}, arg(\lambda) \in [-arg(z_{n-1} - z_n), \pi - arg(z_{n-1} - z_n)]\} \\ \tilde{S}_n &= \{\lambda \in \mathbb{C}, arg(\lambda) \in [-arg(z_3 - z_4), \pi - arg(z_3 - z_4)]\} \\ &= \{\lambda \in \mathbb{C}, arg(\lambda) \in [-\pi, 0]\} \end{aligned}$$

Contours and spectral functions for the Helmholtz equation in a semi-infinite strip Ω are shown in figure 1.7. The contours in figure 1.7 are calculated as follows:

$$\begin{aligned} \tilde{l_j} = \{k \in \mathbb{C} : [arg(k) = -arg(z_j - z_{j+1}), |k| > \beta] \cup \\ [arg(k) = \pi - arg(z_j - z_{j+1}), |k| < \beta]\}, \end{aligned}$$

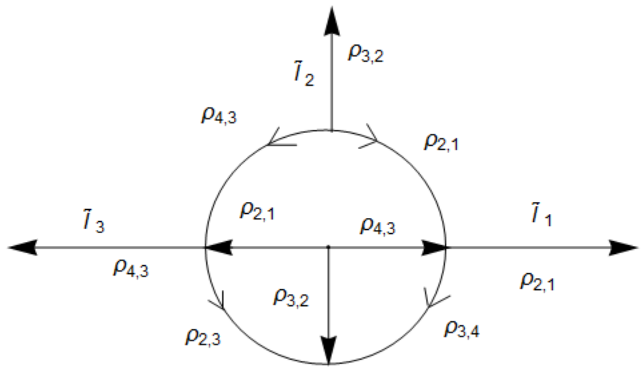

FIGURE 1.7. Contours and spectral functions for the Helmholtz equation in a semi-infinite strip Ω.

$$\begin{aligned}
\tilde{l_1} = \{k \in \mathbb{C} : &[arg(k) = -arg(z_1 - z_2), |k| > \beta] \cup \\
&[arg(k) = \pi - arg(z_1 - z_2), |k| < \beta]\} \\
= \{k \in \mathbb{C} : &[arg(k) = 0, |k| > \beta] \cup [arg(k) = \pi, |k| < \beta]\}, \\
\tilde{l_2} = \{k \in \mathbb{C} : &[arg(k) = -arg(z_2 - z_3), |k| > \beta] \cup \\
&[arg(k) = \pi - arg(z_2 - z_3), |k| < \beta]\} \\
= \{k \in \mathbb{C} : &[arg(k) = \frac{\pi}{2}, |k| > \beta] \cup [arg(k) = \frac{3\pi}{2}, |k| < \beta]\}, \\
\tilde{l_3} = \{k \in \mathbb{C} : &[arg(k) = -arg(z_3 - z_4), |k| > \beta] \cup \\
&[arg(k) = \pi - arg(z_3 - z_4), |k| < \beta]\} \\
= \{k \in \mathbb{C} : &[arg(k) = -\pi, |k| > \beta] \cup [arg(k) = 0, |k| < \beta]\}.
\end{aligned}$$

Note that L_j are the circular arcs formed by the intersection of ray $\tilde{l_j}$ with the circle $|k| = \beta$; if $\alpha_{j+1},\ b_i,\ \alpha_j$ are the points of intersection of the circle $|k| = \beta$ with the rays $\{\tilde{l}_{j+1}, |k| < \beta\}$, $\{\tilde{l_i}, |k| > \beta\}$, $\{\tilde{l_j}, |k| < \beta\}$ where b_i is between α_{j+1} and α_j. Then $\rho^{(j)}$ on $L_j = (b_i, \alpha_j)$ is $\rho_{i,j+1}$. Now S_j is found by the map $\lambda = (1 - \frac{\beta^2}{|k|^2})k$ from $\tilde{S}_j \to S_j$. To carry out analysis of the global relation, S_1 and S_n are required.

These are obtained through the map $\lambda = (1-\frac{\beta^2}{|k|^2})k:\ \tilde{S}_1 \to S_1;\ \tilde{S}_n \to S_n$.

$$S_1 = \{k \in \mathbb{C} : \{arg(k) \in [-arg(z_2 - z_1), \pi - arg(z_2 - z_1)], |k| > \beta\} \cup$$
$$\{arg(k) \in [\pi - arg(z_2 - z_1), 2\pi - arg(z_2 - z_1)], |k| < \beta\}\}$$
$$= \{k \in \mathbb{C} : \{arg(k) \in [-\pi, 0], |k| > \beta\} \cup$$
$$\{arg(k) \in [0, \pi], |k| < \beta\}\},$$
$$S_n = \{k \in \mathbb{C} : \{arg(k) \in [-\pi, 0], |k| > \beta\} \cup$$
$$\{arg(k) \in [0, \pi], |k| < \beta\}\}.$$

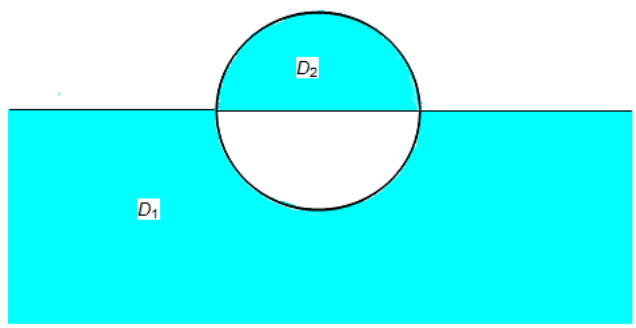

FIGURE 1.8. Domain of global relation for the Helmholtz equation in a semi-infinite strip Ω

Note that $S_1 = S_n$, hence $S = S_1 \cap S_n$ which is shown in figure 1.8. The global relation in this case becomes

$$\sum_{j=1}^{n-1} \rho_{j+1,j}(k) = 0,\ j = 1, 2, 3, \quad \forall \quad k \in S,$$

$$\rho_{2,1}(k) + \rho_{3,2}(k) + \rho_{4,3}(k) = 0.$$

Note that contour L is dividing the complex k-plane into the regions $S = D_1 \cup D_2$ and $\acute{S} = D_3 \cup D_4$ which are shown in figure 1.9. Following the unified transform

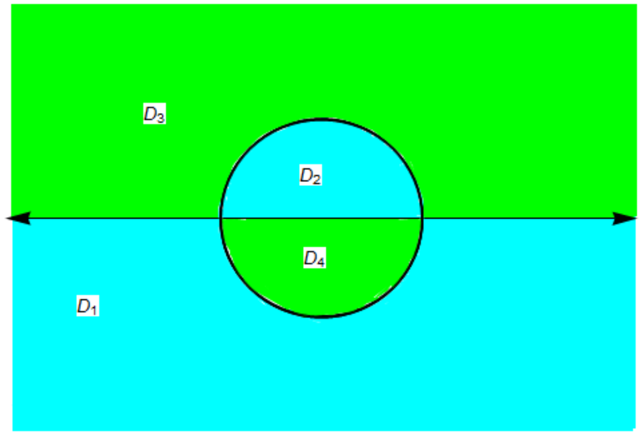

FIGURE 1.9. Contour and domain of analiticity of RHP for the Helmholtz equation in a semi-infinite strip.

method, to find the unknown boundary data, we need to form a RHP on the contour L, which should yield a function which is holomorphic in S and $\acute{S}$. In the literature, we do not have a method to deal with such type of a vector RHP.

1.6 Statement of problem and motivation

1.6.1 Statement of problem

We want to devise a method which can be used to find the solution of both the Helmholtz and modified Helmholtz equation in a semi-infinite strip not only subject to the Poincare type but the higher order boundary conditions also.

1.6.2 Motivation

Since introduction of the unified transform method in 1997, this method has been successfully applied to solve the Laplace and modified Helmholtz equations in some closed and open polygonal domains. Some of the applications of this method are discussed below. The unified transform method is used to discuss solution of the modified Helmholtz equation in a wedge [2]. This solution is used to find the explicit steady state of the diffusion-coalescence, on the half line, with trap source at the origin. The unified transform method is used to find solution of the modified Helmholtz equation in a triangular domain $0 \leq x \leq a - y \leq a$, subject to mixed

boundary conditions [1]. This solution is applied to the problem of diffusion-limited coalescence, in the segment $(-\frac{a}{2}, \frac{a}{2})$ with traps at the edges. The solution of the Laplace equation in a semi-infinite strip, the upper half complex plane, the first quadrant of the complex plane and a wedge, under different types of boundary conditions and classes of solutions, are discussed in [19]. The unified transform method is used to investigate in detail the solution of the modified Helmholtz and Laplace equations in a semi-infinite strip subject to the Poincare type and the Dirichlet boundary conditions [3]. In the case of the modified Helmholtz equation in a semi-infinite strip, it is exhibited that solution of the RHP formed along the real axis can be avoided in some cases. Such cases are referred as algebraic cases. It is shown that when solving the modified Helmholtz equation in a semi-infinite strip with the Poincare type boundary conditions, the problem is transformed to an order two vector RHP. Generally, it is not possible to find the closed form solution of an order two vector RHP. Also, it is shown that when the parameters involved in the boundary conditions satisfy certain algebraic relations, then the order two vector RHP is equivalent to two scalar RHPs or an order two triangular vector RHP. If closed form (integral representation) solutions of the scalar RHPs and order two triangular vector RHP are found, then the solution of the BVP of the modified Helmholtz equation in a semi-infinite strip Ω in the scalar and triangular cases, is found in closed form. The unified transform method is applied to find the solution of basic elliptic equations (Laplace, modified Helmholtz and Helmholtz equations) in an equilateral triangular domain subject to the Dirichlet boundary conditions, the oblique Robin boundary conditions and the Poincare type boundary conditions [13].

Recently, Antipov [4] used the Laplace transform with respect to two variables to solve a system of two Helmholtz equations coupled by impedance boundary conditions. That system models diffraction of an electromagnetic plane wave by a right-angled wedge. The main feature of the method [4] is that the parameter of the second Laplace transform is a function $\zeta(\eta)$, where η is the parameter of the first Laplace transform. The function ζ is a root of the characteristic polynomial of the ordinary differential operator that is the Laplace image of the Helmholtz operator. Here, we want to further develop this method which can be used to find the solution of both the Helmholtz and modified Helmholtz equations in a semi-infinite strip with the impedance boundary conditions and their generalizations.

There is a lot of literature related to applications of the Helmholtz equation in a semi-infinite strip subject to higher order boundary conditions in acoustics, fluid mechanics, marine technology and arctic engineering [36]. Solution of the Helmholtz equation in a semi-infinite strip subject to higher order boundary conditions is discussed in [26]. The problem under discussion arises in determination of the acoustic field generated by a point source in a plane semi-infinite wave guide with thin elastic walls, and also inside an infinite acoustic wave guide with thin elastic baffle. In this problem, the higher order boundary conditions exists due to the structure of the wave guide. It shows a good attempt to solve the problem but the derived solution depends on ansatz. Some applications of the Helmholtz equation in a semi-infinite strip subject to higher order boundary conditions are discussed in [27], [7]. There are a lot of physical situations that can be modeled in terms of propagation and scattering of acoustic waves in a wave guide with higher order boundary conditions [29]. Often such problems comprise of pipes or ducts with abrupt changes of material property or geometry, for example, in car

silencer designs, where there is a sudden change in cross sectional area, or when the bounding wall is lagged. The paper [29] investigates a class of problems in which the boundary conditions at the duct walls are not of the Dirichlet, Neumann or of impedance type, but these contain second or higher order derivative of the dependent variable. These type of boundary conditions are commonly found in models of fluid structural interactions, for example, membrane or plate boundaries, and in electromagnetic wave propagation. To use the mode matching technique, extra edge conditions imposed at points of discontinuity must be included because for these type of models eigen functions are not orthogonal. A new orthogonality relation is presented, for eigen functions involved for the general class of problems containing a scalar wave equation and higher order boundary conditions. The paper [29] also, sheds light on the process for taking into account the necessary edge conditions. By taking two specific examples from structural acoustics, which possess exact solution obtainable from other techniques, it is exhibited that the orthogonality relation permits mode matching to follow the same way as for simpler boundary conditions. Some techniques which are used to solve the problems in the field of fluid structure interactions involving a second order partial differential equation with higher order boundary conditions, are discussed in [31]. In particular, it considers the Laplace equation with higher boundary conditions in case of a semi-infinite strip $0 < x < \infty,$ and $0 < y < h$. Mode coupling relations are derived by utilizing the Fourier integral theorem and the expansion for the velocity potential in terms of corresponding eigen functions of the BVP. The symmetric wave potential, or the so called Green's function of the BVP of fluxural gravity wave maker is derived by utilizing expansion of the velocity potential. Then the expansion formulae for velocity potential are recovered by using integral form of the wave source potential indicating completeness of the eigen functions involved.

Oblique wave scattering due to cracks in a floating ice sheet in case of infinite depth is analyzed.

A boundary value problem for the Helmholtz equation which originates from context of the wave diffraction theory is investigated in [9]. In this paper, the Helmholtz equation in a strip Ω subject to higher order imperfect boundary conditions is studied by the view point of operator theory. Using the operator theoretical machinery, the physical problem is transformed in the language of operator theory, to study properties of certain types of operators. It involves Wiener Hopf and convolution type operators on finite intervals with semi-almost periodic Fourier symbol matrices. These operators are considered in Lebesgue and also Sobolev space because the original problem is considered in terms of Bessel potential spaces. In this work algebraic, operator and function theoretic features of operator theory are used in a constructive way. To formulate the problem some definitions and notations are given. $S(\mathbb{R}^n)$ denotes Schwartz space of rapidly decreasing functions, $\acute{S}(\mathbb{R}^n)$ denotes the dual space of tempered distributions on $\mathbb{R}^n$. The Bessel potential space is denoted by $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$ and is defined as:

$$H^s(\mathbb{R}^n) = \{\phi \in \acute{S}(\mathbb{R}^n) : \|\phi\|_{H^s(\mathbb{R}^n)} = \|F^{-1}(1+|\xi|^2)^{\frac{s}{2}} \cdot F\phi\|_{L^2(\mathbb{R}^n)} < +\infty\},$$

where $F = F_{x\to\xi}$ is Fourier transformation in R^n, and is defined as:

$$(F\phi)(\xi) = \int_{\mathbb{R}^n} e^{i\xi\cdot x}\phi(x)dx, \ \xi \in \mathbb{R}^n.$$

Let $\mathbb{D} \subset \mathbb{R}^n$ is a Lipschitz domain, $\tilde{H}^s(\mathbb{D})$ denotes the closed subspace of $H^s(\mathbb{R}^n)$. The elements of $\tilde{H}^s(\mathbb{D})$ have support in $\bar{\mathbb{D}}$ and $H^s(\mathbb{D})$ is the space of generalized functions which have extensions into $\mathbb{R}^n$ that belongs to $H^s(\mathbb{R}^n)$. Subspace topology is induced on $\tilde{H}^s(\mathbb{D})$, and norm of the quotient space $\frac{H^s(\mathbb{R}^n)}{\tilde{H}^s(\frac{\mathbb{R}^n}{\mathbb{D}})}$ is introduced on $H^s(\mathbb{D})$.

Note that

$$\mathbb{R}^n_{\pm} = \{x = (x_1, x_2, x_3, \ldots\ldots, x_{n-1}, x_n) \in \mathbb{R}^n, \pm x_n \geq 0\}.$$

Now the boundary transmission problem is formulated in the language of operator theory. For $n \in N_0$, N_0 denotes the set of non negative integers, properties of an element $u \in H^{1+\varepsilon}(\Omega)$ for some $\varepsilon \geq 0$, satisfying the Helmholtz equation, are analyzed. Consider the Helmholtz equation in a strip Ω

$$(\triangle + k^2)u(x, y) = 0, \quad (x, y) \in \Omega, \tag{1.192}$$

subject to the boundary conditions:

$$\begin{aligned} &u^+_{n+1} - ip^+u^+_n = h^+ \ 0n\ \Sigma, \text{ where } \Sigma \text{ denotes boundary of } \Omega, \\ &u^-_{n+1} - ip^-u^-_n = h^- \ 0n\ \Sigma. \end{aligned}$$

The wave number $k \in \mathbb{C}$ and the impedance parameter $p^{\pm} \in \mathbb{C}$ are given. Note that $h^{\pm} \in H^{-\frac{1}{2}-n+\varepsilon}(\Sigma)$ are arbitrary given elements and $u^{\pm}_n := (\frac{\partial^n u}{\partial y^n})|_{y=\pm 0}$ denote traces of $u(x, y)$ on the top and bottom of Σ, respectively. For $n = 0$ and $n = 1$, $u^{\pm}_n$ are the traditional Dirichlet and Neumann traces. The relations between the operators of the problem and new Wiener Hopf operators are established. Then these operator relations are used to investigate invertibility and the Fredholm properties of the operators related to the given problem. A problem of wave diffraction by a strip subject to higher order reactance boundary conditions by view point of integral equations is analyzed in [10]. The problem is formulated as a boundary transmission problem of the Helmholtz equation in a strip in the Bessel potential space $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$. By using integral equations and operator theoretical methods, many convolution type equations are constructed and related to the given problem. Solvability of the problem is discussed for a range of regularity orders of Bessel potential spaces. A problem of wave diffraction by a union of infinite

strips subject to higher order boundary conditions from the operator theory view point, is analyzed in [11]. It is investigated, under which conditions the operators associated with the problem possess the Fredholm property. To achieve the target, several operator extension methods are constructed between used convolution type operators.The Bessel potential space $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$ is used to formulate the problem and the operators, hence the Fredholm property is found for a set of regularity indices of $H^s(\mathbb{R}^n)$.

1.7 Order of dissertation

The rest of dissertation is ordered as follows:

- In chapter 2, a finite integral transformation and the Sturm Liouville problem method is used, to analyze the Helmholtz equation in a semi-infinite strip subject to the Poincare type boundary conditions. Then the Helmholtz equation in a semi-infinite strip subject to the impedance boundary conditions is analyzed. Later on, the Burniston-Siewert method to find zeroes of a transcendental equation in a complex plane is presented. This method gives us numerical as well as exact expressions for the zeroes of a transcendental equation.

- In chapter 3, using a special kind of interconnected Laplace transforms, and theory of RHPs, we have developed a new method to find solutions of both the Helmholtz and modified Helmholtz equations in a semi-infinite strip subject to the Poincare type boundary conditions and impedance boundary conditions. Some examples are solved by the new method and their results are compared by the finite integral transformation and the Sturm Liouville problem method introduced in chapter 2. This gives us verification of the new method. This newly developed method is quite efficient to solve both the

Helmholtz and modified Helmholtz equations in a semi-infinite strip subject to higher boundary conditions also.

- In chapter 4, an application of the new method to a physical model which generates a BVP of the Helmholtz equation in a semi-infinite strip subject to higher order boundary conditions is considered. Some properties of the solution are investigated.

Chapter 2

Finite integral transform method and Burniston-Siewert method

2.1 Finite integral transform method: Poincare type boundary conditions

Consider the Helmholtz equation

$$(\partial_x^2 + \partial_y^2 + k^2)q(x,y) = g(x,y), \quad Im(k) > 0, \quad (x,y) \in \Omega, \tag{2.1}$$

where Ω is a semi-infinite strip shown in figure 3.1, with corners $z_1 = \infty$, $z_2 = 0$, $z_3 = ia$, $z_4 = \infty + ia$, $a > 0$. Figure 2.1 shows the Poincare type boundary conditions along three sides of Ω.

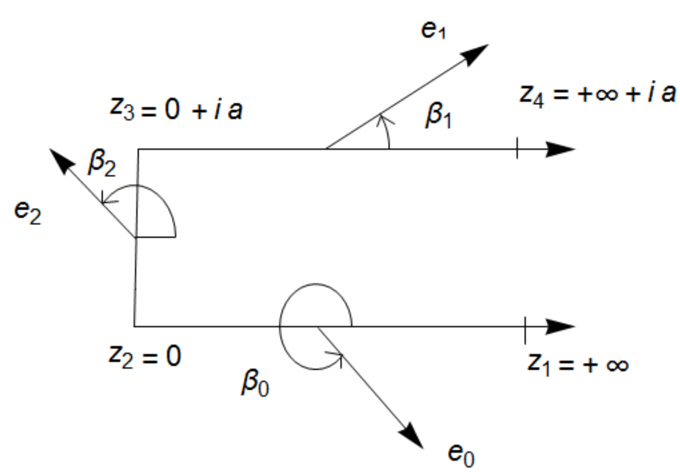

FIGURE 2.1. The Poincare type boundary conditions along the sides of Ω.

The Poincare type boundary conditions are

$$\left.\frac{\partial q}{\partial \nu}\right|_{e_j} + \mu_j q = g_j, \tag{2.2}$$

for $j=0,1,2$, $\frac{\partial q}{\partial \nu}\Big|_{e_j} = \bigtriangledown q \cdot e_j$ is the directional derivative in direction e_j specified by constant $\beta_j, j=0,1,2$, where $(0<\beta_1<\pi, \frac{\pi}{2}<\beta_2<3\frac{\pi}{2}, \pi<\beta_0<2\pi)$, μ_j is a real non negative constant, and g_j is a real valued function with appropriate smoothness and decay. The boundary conditions in equation (3.9) can be written as:

$$side1 : \cos\beta_0 q_x + \sin\beta_0 q_y + \mu_0 q = g_0(x), \quad 0<x<\infty, \quad y=0, \tag{2.3}$$

$$side2 : \cos\beta_2 q_y + \sin\beta_2 q_x + \mu_2 q = g_2(y), \quad x=0, \quad 0<y<a, \tag{2.4}$$

$$side3 : \cos\beta_1 q_x + \sin\beta_1 q_y + \mu_1 q = g_1(x), \quad 0<x<\infty, \quad y=a. \tag{2.5}$$

The functions $g_0(x)$ and $g_1(x)$ vanish at the points $x=0$ and $x=\infty$, $\sin\beta_j \neq 0, j=0,1,2$. To solve the given BVP of the Helmholtz equation in a semi-infinite strip Ω subject to the Poincare type boundary conditions by finite integral transform (FIT) method, we need to apply the finite integral transform to given BVP of the Helmholtz equation in a semi-infinite strip. This step generates a Sturm Liouville (SL) problem. Then we solve the SL problem to find related eigen values, and the kernel of the finite integral transform. So, to check validity of this method to the given BVP of Helmholtz equation in a semi-infinite strip, multiply equation (2.4) by the kernel $K_\lambda(y)$ of the finite integral transform, and integrate from 0 to a to get

$$\int_0^a g_2(y)K_\lambda(y)dy = \int_0^a \cos\beta_2 q_y(x,y)K_\lambda(y)dy + \int_0^a \sin\beta_2 q_x(x,y)K_\lambda(y)dy + \int_0^a \mu_2 q(x,y)K_\lambda(y)dy, \quad 0<y<a, \quad x=0.$$

Use definition of the finite integral transform in the above equation to get

$$g_{2\lambda} = \cos\beta_2 \int_0^a q_y(x,y)K_\lambda(y)dy + \sin\beta_2 \frac{\partial}{\partial x} q_\lambda(x) + \mu_2 q_\lambda(x), \quad 0<y<a, \quad x=0. \tag{2.6}$$

Now evaluate $\int_0^a q_y(x,y)K_\lambda(y)dy$.

$$\int_0^a q_y(x,y)K_\lambda(y)dy = K_\lambda(y)q(x,y)|_{y=0}^{y=a} - \int_0^a q(x,y)\frac{K_\lambda}{dy}(y)dy \tag{2.7}$$

Note that $\int_0^a q(x,y)\frac{K_\lambda}{dy}(y)dy \neq q_\lambda(x)$ because the finite integral transform is defined for the kernel $K_\lambda(y)$ not for $\frac{d}{dy}K_\lambda(y)$. Hence we cannot apply FIT method in the case of Helmholtz equation in a semi-infinte strip subject to Poincare type boundary conditions.

2.2 Impedance boundary conditions

To find solution of the Helmholtz equation in a semi-infinite strip subject to the impedance boundary conditions, substitute $\beta_1 = \frac{\pi}{2}, \beta_2 = \pi, \beta_0 = \frac{3\pi}{2}$ in equations (2.3), (2.4) and (2.5). Consider the Helmholtz equation

$$(\partial_x^2 + \partial_y^2 + k^2)q(x,y) = g(x,y), \quad Im(k) > 0, \quad (x,y) \in \Omega, \tag{2.8}$$

where Ω is a semi-infinite strip shown in figure 2.2, with the corners $z_1 = \infty$, $z_2 = 0$, $z_3 = ia$, $z_4 = \infty + ia$, $a > 0$. The impedance boundary conditions are

$$side1: -q_y(x,y) + \mu_0 q(x,y) = g_0(x), \quad 0 < x < \infty, \quad y = 0, \tag{2.9}$$

$$side2: -q_x(x,y) + \mu_2 q(x,y) = g_2(y), \quad x = 0, \quad 0 < y < a, \tag{2.10}$$

$$side3: q_y(x,y) + \mu_1 q(x,y) = g_1(x), \quad 0 < x < \infty, \quad y = a. \tag{2.11}$$

Figure 2.2 shows the impedance boundary conditions for Helmholtz equation in the semi-infinite strip Ω, along the sides of semi-infinte strip Ω. Note that $\mu_j, j = 0,1,2$, is a real non negative constant, and $g_j, j = 0,1,2$, is a real valued function with appropriate smoothness and decay. Let

$$q(x,y) = q^*(x,y) + q_*(x,y), \tag{2.12}$$

$$q_*(x,y) = (A_1 y + A_2)g_0(x) + (B_1 y + B_2)g_1(x). \tag{2.13}$$

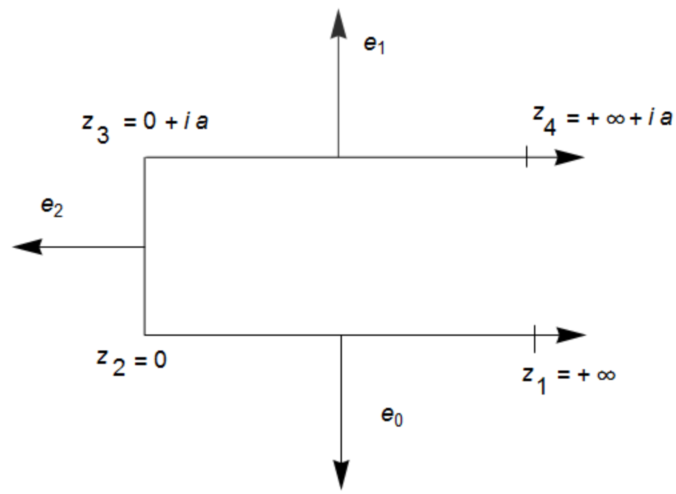

FIGURE 2.2. Impedance boundary condtions along sides of Ω.

Since $q(x, y)$ is a solution of the given BVP of the Helmholtz equation in a semi-infinite strip subject to the impedance boundary conditions, equation (2.12) indicates that the sum $q_*(x, y) + q^*(x, y)$ is also a solution of the given BVP. Hence boundary conditions defined by equations (2.9) and (2.11) become

$$side1 : -\frac{\partial q_*}{\partial y}(x, y) + \mu_0 q_*(x, y) = g_0(x), \quad 0 < x < \infty, \quad y = 0, \tag{2.14}$$

$$side3 : \frac{\partial q_*}{\partial y}(x, y) + \mu_1 q_*(x, y) = g_1(x), \quad 0 < x < \infty, \quad y = a. \tag{2.15}$$

Substitute the value of $q_*(x, y)$ from equation (2.13) in equations (2.14) and (2.15) to get

$$-(A_1 g_0(x) + B_1 g_1(x)) + \mu_0(A_2 g_0(x) + B_2 g_1(x)) = g_0(x),$$

$$(A_1 g_0(x) + B_1 g_1(x)) + \mu_1[(A_1 a + A_2) g_0(x) + (B_1 a + B_2) g_1(x)] = g_1(x).$$

Simplify the above equations to get

$$(-A_1 + \mu_0 A_2) g_0(x) + (-B_1 + \mu_0 B_2) g_1(x) = g_0(x), \tag{2.16}$$

$$[A_1 + \mu_1(aA_1 + A_2)] g_0(x) + [B_1 + \mu_1(B1a + B_2)] g_1(x)) = g_1(x). \tag{2.17}$$

Solve equations (2.16) and (2.17) to get

$$A_1 = \frac{-\mu_1}{\delta}, \quad A_2 = \frac{1 + a\mu_1}{\delta}, \quad B_1 = \frac{\mu_0}{\delta}, \quad B_2 = \frac{1}{\delta}, \quad \delta = \mu_0 + \mu_1 + a\mu_0\mu_1. \tag{2.18}$$

From equation (2.12), use $q(x, y) = q^*(x, y) + q_*(x, y)$ in the BVP defined by equations (2.8), (2.9), (2.10) and (2.11), then the BVP of Helmholtz equation in a semi-infinite strip subject to the impedance boundary conditions becomes

$$(\partial_x^2 + \partial_y^2 + k^2)q^*(x, y) = g^*(x, y), \quad Im(k) > 0, \quad (x, y) \in \Omega, \tag{2.19}$$

where Ω is the semi-infinite strip shown in figure 2.2, with the corners $z_1 = \infty$, $z_2 = 0$, $z_3 = ia$, $z_4 = \infty + ia$, $a > 0$. The impedance boundary conditions are

$$side1 : -\frac{\partial q^*}{\partial y}(x, y) + \mu_0 q^*(x, y) = 0, \quad 0 < x < \infty, \quad y = 0, \tag{2.20}$$

$$side2 : -\frac{\partial q^*}{\partial x}(x, y) + \mu_2 q^*(x, y) = g_2^*(y), \quad x = 0, \quad 0 < y < a, \tag{2.21}$$

$$side3 : \frac{\partial q^*}{\partial y}(x, y) + \mu_1 q^*(x, y) = 0, \quad 0 < x < \infty, \quad y = a. \tag{2.22}$$

Note that

$$\begin{aligned} g_2^*(y) = g_2(y) - [-\frac{\partial q_*}{\partial x}(x, y) + \mu_2 q_*(x, y)]_{x=0} = g_2(y) + (A_1 y + A_2) \\ (g_0'(0) - \mu_2 g_0(0)) + (B_1 y + B_2)(g_1'(0) - \mu_2 g_0(0)), \end{aligned} \tag{2.23}$$

$$\begin{aligned} g^*(x, y) = g(x, y) - (A_1 y + A_2)g_0''(x) - (B_1 y + B_2)g_1''(x) \\ - k^2(A_1 y + A_2)g_0 x - k^2(B_1 y + B_2)g_1(x). \end{aligned} \tag{2.24}$$

2.2.1 Kernel of the finite integral transform

We apply the finite integral transform to the BVP defined by equations (2.19), (2.20), (2.21) and (2.22), to obtain a SL problem. Then we solve that SL problem to find related eigen values, and the kernel of the finite integral transform. So,

multiply equation (2.19) by the kernel $K_\lambda(y)$ and integrate from 0 to a to get

$$g_\lambda(x) = (k^2 + \frac{d^2}{dx^2}) \int_0^a K_\lambda(y) q^*(x,y) dy + \int_0^a K_\lambda(y) \frac{\partial^2 q^*}{\partial y^2}(x,y) dy, \tag{2.25}$$

$$g_\lambda(x) = \int_0^a K_\lambda(y) g^*(x,y) dy, \quad g^*(x,y) \text{ is given by equation (2.24).} \tag{2.26}$$

Now evaluate $\int_0^a K_\lambda(y) \frac{\partial^2 q^*}{\partial y^2}(x,y)dy$ as follows.

$$\int_0^a K_\lambda(y) \frac{\partial^2 q^*}{\partial y^2}(x,y) dy = K_\lambda(y) \frac{\partial q^*}{\partial y}|_{y=0}^{y=a} - \int_0^a \frac{\partial q^*}{\partial y} \frac{d}{dy} K_\lambda(y) dy$$

$$= K_\lambda(y) \frac{\partial q^*}{\partial y}|_{y=0}^{y=a} - q^* \frac{d}{dy} K_\lambda(y)|_{y=0}^{y=a} + \int_0^a q^*(x,y) \frac{d^2}{dy^2} K_\lambda(y) dy. \tag{2.27}$$

Let

$$\frac{d^2}{dy^2} K_\lambda(y) = -\lambda^2 K_\lambda(y), \quad 0 < y < a. \tag{2.28}$$

The boundary conditions defined by equations (2.20) and (2.22) can be expressed as

$$\frac{\partial q*}{\partial y}(x,y) = \mu_0 q*(x,y), \quad y = 0, \tag{2.29}$$

$$\frac{\partial q*}{\partial y}(x,y) = -\mu_1 q*(x,y), \quad y = a. \tag{2.30}$$

Use equations (2.29), (2.30) to find

$$\begin{aligned} K_\lambda(y) \frac{\partial q^*}{\partial y}|_{y=0}^{y=a} - q^* \frac{d}{dy} K_\lambda(y)|_{y=0}^{y=a} &= [-K_\lambda(y)\mu_1 q^*(x,y) - \frac{d}{dy} K_\lambda(y) q^*(x,y)]_{y=a} \\ &\quad - [K_\lambda(y)\mu_0 q^*(x,y) - \frac{d}{dy} K_\lambda(y) q^*(x,y)]_{y=0}. \end{aligned} \tag{2.31}$$

Let

$$\mu_0 K_\lambda(y) = \frac{d}{dy} K_\lambda(y), \quad y = 0, \tag{2.32}$$

$$-\mu_1 K_\lambda(y) = \frac{d}{dy} K_\lambda(y), \quad y = a. \tag{2.33}$$

Substitute the values of $\mu_0 K_\lambda(y)$ and $\mu_1 K_\lambda(y)$ in equation (2.31), then use the resultant value, and the value of $\frac{d^2}{dy^2}K_\lambda(y)$ from equation (2.28) in equation (2.27) to get

$$\int_0^a K_\lambda(y)\frac{\partial^2 q^*}{\partial y^2}(x,y)dy = -\lambda^2\int_0^a q*(x,y)K_\lambda(y)dy = -\lambda^2 q_\lambda(x). \qquad (2.34)$$

Let the kernel of the finite integral transform $K_\lambda(y)$ solves the following SL problem which is obtained from equations (2.28), (2.32) and (2.33). The SL problem is

$$(\frac{d^2}{dy^2}+\lambda^2)K_\lambda(y) = 0, \quad 0 < y < a, \qquad (2.35)$$

$$\frac{d}{dy}K_\lambda(y) - \mu_0 K_\lambda(y) = 0, \quad y = 0, \qquad (2.36)$$

$$\frac{d}{dy}K_\lambda(y) + \mu_1 K_\lambda(y) = 0, \quad y = a. \qquad (2.37)$$

Now to find the transformed Helmholtz equation, substitute the value of $\int_0^a K_\lambda(y)\frac{\partial^2 q^*}{\partial y^2}(x,y)dy$ from equation (2.34), in equation (2.25), and simplify to get

$$\begin{aligned}&[\frac{d^2}{dx^2}+(k^2-\lambda^2)]q_\lambda(x) = g_\lambda(x), \quad 0 < x < \infty,\\ &[\frac{d^2}{dx^2}-\hat{\zeta}^2]q_\lambda(x) = g_\lambda(x), \quad 0 < x < \infty,\end{aligned} \qquad (2.38)$$

where $g_\lambda(x)$ is the finite integral transform of $g^*(x,y)$ and is given by equation (2.26). We fix a branch of the multi-valued function $\hat{\zeta} = \sqrt{\lambda^2 - k^2}$ by $Re(\hat{\zeta}) \geq 0$. Note that $\pm k$ are the branch points of the multi-valued funtion $\hat{\zeta}$, and its branch cut is shown in figure 2.3.

To transform the boundary condition along side 2 of the semi-infinite strip Ω, multiply equation (2.10) by the kernel $K_\lambda(y)$ and integrate from 0 to a to get

$$-\frac{d}{dx}q_\lambda(x) + \mu_2 q_\lambda(x) = g_{2\lambda}, \quad x = 0, \quad 0 < y < a, \quad q_\lambda(+\infty) = 0, \qquad (2.39)$$

$$g_{2\lambda} = \int_0^a g_2^*(y)K_\lambda(y)dy, \quad g_2^*(y) \text{ is given by equation } \quad (2.23). \qquad (2.40)$$

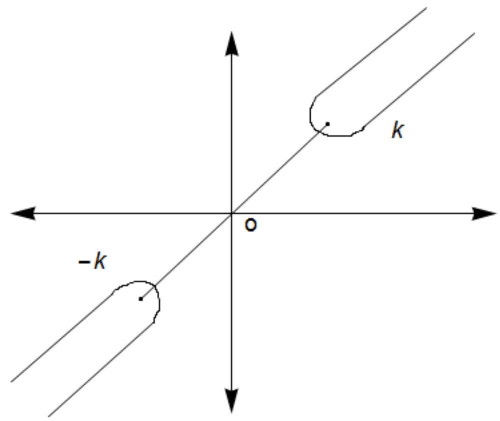

FIGURE 2.3. A branch cut of multi-valued function $\hat{\zeta}$

If $K_\lambda(y)$ solves the SL problem defined by equations (2.35), (2.36) and (2.37) then the transformed Helmholtz equation, and the boundary condition along side 2 of the semi-infinite strip Ω, are defined by equations (2.38) and (2.39), respectively. To find the solution of this SL problem, we need the solution of the 2nd order ordinary linear differential equation (2.35), and it is

$$K_\lambda(y) = C_1 \cos \lambda y + C_2 \sin \lambda y. \tag{2.41}$$

The Sturm Liouville theory for the boundary value problems implies that the eigen values λ_n of the given SL problem are non negative real numbers. It is verified that $\lambda_n = 0$ is not an eigen value of the given SL problem. We find eigen values when $\lambda_n > 0$. Equation (2.41) gives

$$K_\lambda(0) = C_1, \quad K_\lambda(a) = C_1 \cos \lambda a + C_2 \sin \lambda a, \quad K'_\lambda(0) = C_2 \lambda, \tag{2.42}$$

$$K'_\lambda(a) = -\lambda C_1 \sin \lambda a + C_2 \lambda \cos \lambda a. \tag{2.43}$$

Substitute values of $K'_\lambda(0), K_\lambda(0), K'_\lambda(a)$ and $K_\lambda(a)$ from equations (2.42) and (2.43) in the boundary conditions defined by equations (2.36) and (2.37) to get

$$0 = \lambda C_2 - \mu_0 C_1, \quad C_1 = \frac{\lambda C_2}{\mu_0}, \tag{2.44}$$

$$0 = C_1(-\lambda \sin \lambda a + \mu_1 \cos \lambda a) + C_2(\lambda \cos \lambda a + \mu_1 \sin \lambda a). \tag{2.45}$$

Substitute the value of C_1 from equation (2.44) in equation (2.45), and simplify to get

$$C_2[\lambda(-\lambda \sin \lambda a + \mu_1 \cos \lambda a) + \mu_0(\lambda \cos \lambda a + \mu_1 \sin \lambda a)] = 0. \tag{2.46}$$

For a non trivial solution $C_2 \neq 0$, so, equation (2.46) becomes

$$[\lambda(-\lambda \sin \lambda a + \mu_1 \cos \lambda a) + \mu_0(\lambda \cos \lambda a + \mu_1 \sin \lambda a)] = 0. \tag{2.47}$$

Simplify equation (2.47) to get

$$\tan \lambda a = \frac{\lambda(\mu_0 + \mu_1)}{\lambda^2 - \mu_0 \mu_1}. \tag{2.48}$$

2.2.2 1-dimensional boundary value problem

We will find roots of the transcendental equation (2.48) using the Burniston-Siewert method which will be explained in section 2.3. Let positive roots of the transcendental equation (2.48) are denoted by λ_n. Note that λ_n. are the eigen values of the SL problem defined by equations (2.35), (2.36) and (2.37). The corresponding eigen functions of the given SL problem are $K_{\lambda_n}(y)$, and are defined by

$$K_{\lambda_n}(y) = C_1 \cos \lambda_n y + C_2 \sin \lambda_n y. \tag{2.49}$$

From equation (2.44), use $C_1 = \frac{\lambda_n C_2}{\mu_0}$ in equation (2.49), and simplify to get

$$K_{\lambda_n}(y) = C_2[\frac{\lambda_n}{\mu_0} \cos \lambda_n y + \sin \lambda_n y]. \tag{2.50}$$

To make $\{K_{\lambda_n}\}_{n=0}^{\infty}$ an orthonormal system. Let $\parallel K_{\lambda_n}(y) \parallel = 1$. Use definition of norm, and simplify to get

$$\begin{aligned}
&C_2^2 \int_0^a [\frac{\lambda_n}{\mu_0} \cos \lambda_n y + \sin \lambda_n y]^2 dy = 1 \\
&C_2^2 \sigma_n^2 = 1, \quad C_2 = \frac{1}{\sigma_n}, \text{ where} \\
&\sigma_n = \sqrt{\int_0^a [\frac{\lambda_n}{\mu_0} \cos \lambda_n y + \sin \lambda_n y]^2 dy} \\
&\sigma_n = \frac{1}{2\sqrt{\lambda_n}\mu_0}[2\lambda_n(2\mu_0 \sin^2 4a\lambda_n + a^2\lambda_n^2 + a\mu_0^2) + (\lambda_n^2 - \mu_0^2)\sin 2a\lambda_n].
\end{aligned} \tag{2.51}$$

From equation (2.51) use the value of C_2 in equation (2.50) to get corresponding eigen functions of the SL problem defined by equations (2.35), (2.36) and (2.37).

$$K_{\lambda_n}(y) = \frac{1}{\sigma_n}[\frac{\lambda_n}{\mu_0} \cos \lambda_n y + \sin \lambda_n y], \quad \parallel K_{\lambda_n} \parallel = 1. \tag{2.52}$$

For the given BVP of the Helmholtz equation in a semi-infinite strip Ω, the kernel of the finite integral transform $K_{\lambda_n}(y)$ is given by equation (2.52). Hence corresponding to the eigen values λ_n and kernel $K_{\lambda_n}(y)$, the transformed Helmholtz equation and the boundary condition along side 2 of the semi-infinite strip Ω are

$$[\frac{d^2}{dx^2} - \hat{\zeta_n}^2] q_{\lambda_n}(x) = g_{\lambda_n}(x), \quad 0 < x < \infty, \text{ where} \tag{2.53}$$

$$g_{\lambda_n}(x) = \int_0^a K_{\lambda_n}(y) g^*(x,y) dy, \tag{2.54}$$

$$-\frac{d}{dx} q_{\lambda_n}(x) + \mu_2 q_{\lambda_n}(x) = g_{2\lambda_n}, \quad x = 0, \quad 0 < y < a, \quad q_{\lambda_n}(+\infty) = 0, \tag{2.55}$$

$$g_{2\lambda_n} = \int_0^a g_2^*(y) K_{\lambda_n}(y) dy. \tag{2.56}$$

Note that $\hat{\zeta_n} = \sqrt{\lambda_n^2 - k^2}$ is a multi-valued function. We fix a branch of it by $Re(\hat{\zeta_n}) \geq 0$, $q_{\lambda_n}(x) = \int_0^a K_{\lambda_n}(y) q^*(x,y) dy$ is the direct finite integral transform of

$q^*(x, y)$. The inverse finite integral transform of $q_{\lambda_n}(x)$ is recovered as

$$
\begin{aligned}
&q^*(x, y) = \sum_{n=0}^{\infty} C_n(x) K_{\lambda_n}(y), \text{ multiply by } K_{\lambda_m}(y) \text{ and integrate from 0 to a} \\
&\int_0^a q^*(x, y) K_{\lambda_m}(y) dy = \sum_{n=0}^{\infty} C_n(X) \int_0^a K_{\lambda_n}(y) K_{\lambda_m}(y) dy, \\
&\int_0^a q^*(x, y) K_{\lambda_m}(y) dy = \sum_{n=0}^{\infty} \delta_{nm} C_m(x), \quad \delta_{nm}, \text{ is Kroneckor's delta,} \\
&C_m(x) = \int_0^a q^*(x, y) K_{\lambda_m}(y) dy = q_{\lambda_m}(x).
\end{aligned}
\tag{2.57}
$$

To solve the 1-dimensional BVP defined by equations (2.53) and (2.55), we define

$$
U_0[F(x)] = (-\frac{d}{dx} + \mu_2) F(x)|_{x=0}, \quad F(+\infty) = 0, \tag{2.58}
$$

where U_0 is the functional of the boundary condition along the side $x = 0, 0 < y < a$ of the semi-infinite strip Ω. The Green's function of the 1-dimensional BVP is

$$
G(x, \xi) = \phi(x, \xi) - U_0[\phi(x, \xi)]\psi(x), \tag{2.59}
$$

where $\phi(x, \xi)$ is the fundamental function of differential the operator $L = \frac{d^2}{dx^2} - \hat{\zeta_n}^2$, and is defined by $\phi(x, \xi) = -\frac{1}{2\hat{\zeta_n}} e^{-\hat{\zeta_n}|x-\xi|}$. Note that $\psi(x)$ is the basis function satisfying the following properties:

1. $\psi(x)$ solves the 2nd order ordinary linear differential equation $L[\psi(x)] = 0$.

2. $U_0[\psi(x)] = [-\frac{d}{dx}\psi(x) + \mu_2\psi(x)]_{x=0} = 1$.

3. $\psi(+\infty) = 0$.

To find the basis function $\psi(x)$, general solution of the 2nd order ordinary linear differential equation $L[\psi(x)] = 0$ is given by

$$
\psi(x) = C_0 e^{\hat{\zeta_n} x} + C_1 e^{-\hat{\zeta_n} x}. \tag{2.60}
$$

The condition $\psi(x) \to 0$ as $x \to \infty$ is only satisfied when $C_0 = 0$. Hence, equation (2.60) becomes $\psi(x) = C_1 e^{-\hat{\zeta_n} x}$. To find value of C_1 use the condition

$$U_0[\psi(x)]_{x=0} = 1 \tag{2.61}$$

Apply the functional U_0 and simplify to get

$$\begin{aligned} &[-\frac{d}{dx}\psi(x) + \mu_2\psi(x)] = 1 \\ &-(-\hat{\zeta_n}C_1) + \mu_2 C_1 = 1. \end{aligned} \tag{2.62}$$

Simplify equation (2.62) to get

$$C_1 = \frac{1}{\mu_2 + \hat{\zeta_n}}. \tag{2.63}$$

From equation (2.63) use the value of C_1 in (2.60) to get $\psi(x)$.

$$\psi(x) = \frac{1}{\mu_2 + \hat{\zeta_n}} e^{-\hat{\zeta_n} x} \tag{2.64}$$

Now consider

$$\phi(x,\xi) = -\frac{1}{2\hat{\zeta_n}} e^{-\hat{\zeta_n}|x-\xi|}. \tag{2.65}$$

Apply the operator $\frac{\partial}{\partial x}$ to equation (2.65), and simplify to get

$$\partial_x\phi(x,\xi) = \frac{1}{2} e^{-\hat{\zeta_n}|x-\xi|} sgn(x-\xi). \tag{2.66}$$

We find the value of $U_0[\phi(x,\xi)]$ as follows:

$$\begin{aligned} U_0[\phi(x,\xi)] &= [(-\frac{d}{dx} + \mu_2)\phi(x,\xi)]_{x=0} \\ &= -\frac{d\phi}{dx}(x,\xi)|_{x=0} + \mu_2\phi(x,\xi)|_{x=0}. \end{aligned} \tag{2.67}$$

Now equation (2.67) becomes

$$\begin{aligned} U_0[\phi(x,\xi)] &= \frac{1}{2} e^{-\hat{\zeta_n}\xi} - \frac{\mu_2 e^{-\hat{\zeta_n}\xi}}{2\hat{\zeta_n}} \\ &= \frac{\hat{\zeta_n} - \mu_2}{2\hat{\zeta_n}} e^{-\hat{\zeta_n}\xi}. \end{aligned} \tag{2.68}$$

Substitute the values of $\phi(x,\xi), \psi(x)$ and $U_0[\phi(x,\xi)]$ from equations (2.65), (2.64) and (2.68) in equation (2.59) to get Green's function of the 1-dimensional BVP.

$$G(x,\xi) = -\frac{1}{2\hat{\zeta_n}} e^{-\hat{\zeta_n}|x-\xi|} - \frac{(\hat{\zeta_n} - \mu_2)}{2\hat{\zeta_n}(\hat{\zeta_n} + \mu_2)} e^{-\hat{\zeta_n}(x+\xi)} \tag{2.69}$$

Solution of the 1-dimensional BVP is

$$q_{\lambda_n}(x) = \int_0^\infty G(x,\xi) g_{\lambda_n}(\xi) d\xi + \psi(x) g_{2\lambda_n}. \tag{2.70}$$

Substitute the values of $\psi(x)$ and $G(x,\xi)$ from equations (2.64) and (2.69) respectively, in equation (2.70) to get

$$q_{\lambda_n}(x) = \int_0^\infty [-\frac{1}{2\hat{\zeta_n}} e^{-\hat{\zeta_n}|x-\xi|} - \frac{(\hat{\zeta_n} - \mu_2)}{2\hat{\zeta_n}(\hat{\zeta_n} + \mu_2)} e^{-\hat{\zeta_n}(x+\xi)}] g_{\lambda_n}(\xi) d\xi + \frac{1}{\hat{\zeta_n} + \mu_2} e^{-\hat{\zeta_n} x} g_{2\lambda_n}, \tag{2.71}$$

where $\hat{\zeta_n} = \sqrt{\lambda_n^2 - k^2}$, for $n = 0, 1, 2, 3, 4, ...$ To find the solution of the BVP defined by equations (2.19), (2.20), (2.21) and (2.22), suppose that $q^*(x,y)$ satisfies the SL problem given by equations (2.35), (2.36) and (2.37) for the eigen values λ_n, and $q^*(x,y) \in C^2(0,a)$ as a function of y, then the inverse finite integral transform given by relation (2.57) becomes

$$q^*(x,y) = \sum_{n=0}^{\infty} C_n(x) K_{\lambda_n}(y), \quad \text{where} \tag{2.72}$$

$$C_n(x) = \int_0^a q^*(x,y) K_{\lambda_n}(y) dy = q_{\lambda_n}(x). \tag{2.73}$$

From equation (2.71) use the value of $q_{\lambda_n}(x)$ in equation (2.73) to get

$$C_n(x) = \int_0^\infty [-\frac{1}{2\hat{\zeta_n}} e^{-\hat{\zeta_n}|x-\xi|} - \frac{(\hat{\zeta_n} - \mu_2)}{2\hat{\zeta_n}(\hat{\zeta_n} + \mu_2)} e^{-\hat{\zeta_n}(x+\xi)}] g_{\lambda_n}(\xi) d\xi + \frac{1}{\hat{\zeta_n} + \mu_2} e^{-\hat{\zeta_n} x} g_{2\lambda_n}. \tag{2.74}$$

Note that in this case, $q^*(x,y)$ defined by equation (2.72) is uniformly convergent on $(0,a)$.

2.2.3 Solution of the BVP of Helmholtz equation in a semi-infinite strip Ω: analysis; numerical results

Solution of the BVP of Helmholtz equation in a semi-infinite strip subject to the impedance boundary conditions, defined by equations (2.19), (2.20), (2.21) and (2.22), is found by substituting values of $C_n(x)$ and $K_{\lambda_n}(y)$ from equations (2.74) and (2.52) respectively, in equation (2.72). The required solution is given by

$$\begin{aligned} q^*(x,y) = \sum_0^\infty [-\frac{1}{2\hat{\zeta}_n}\int_0^\infty (e^{-\hat{\zeta}_n|x-\xi|} + \frac{(\hat{\zeta}_n-\mu_2)}{(\hat{\zeta}_n+\mu_2)} e^{-\hat{\zeta}_n(x+\xi)}) g_{\lambda_n}(\xi)d\xi + \\ \frac{1}{\hat{\zeta}_n+\mu_2} e^{-\hat{\zeta}_n x} g_{2\lambda_n}] \times \frac{1}{\sigma_n}[\frac{\lambda_n}{\mu_0}\cos\lambda_n y + \sin\lambda_n y], \end{aligned} \tag{2.75}$$

where σ_n is given by relation (2.51). Now solution of the given BVP of the Helmholtz equation in a semi infinite strip Ω defined by equations (2.8), (2.9), (2.10) and (2.11), is obtained by substituting values of $q_*(x,y)$ and $q^*(x,y)$ from equations (2.13) and (2.75) in equation (2.12). The required solution is

$$\begin{aligned} q(x,y) = \sum_0^\infty [-\frac{1}{2\hat{\zeta}_n}\int_0^\infty (e^{-\hat{\zeta}_n|x-\xi|} + \frac{(\hat{\zeta}_n-\mu_2)}{(\hat{\zeta}_n+\mu_2)} e^{-\hat{\zeta}_n(x+\xi)}) g_{\lambda_n}(\xi)d\xi + \frac{1}{\hat{\zeta}_n+\mu_2} e^{-\hat{\zeta}_n x} g_{2\lambda_n}] \\ \times \frac{1}{\sigma_n}[\frac{\lambda_n}{\mu_0}\cos\lambda_n y + \sin\lambda_n y] + (A_1 y + A_2)g_0(x) + (B_1 y + B_2)g_1(x), \end{aligned} \tag{2.76}$$

where σ_n is given by relation (2.51). The constants A_1, A_2, B_1, B_2 are given by relation (2.18). To discuss numerical results, we need to find the zeroes of the transcendental equation (2.47). In the next section, we give the Burniston-Siewert method, used to find all roots of a certain transcendental equation in a complex plane.

2.3 Burniston-Siewert method for solving certain transcendental equations

An elegant method to solve a certain type of transcendental equations is introduced in [8]. This method gives roots of a transcendental equation in closed form, which

is not possible by using any numerical technique. This method for finding roots of a transcendental equation is based on complex analysis, and ultimately requires a canonical solution of a RHP. The crux of this method is to establish a suitable RHP, and using several elementary properties of the resulting solution to deduce roots of the given transcendental equation. To explain this method, some terminology, and theorems are given below.

Definition 2.3.1. *Let f be a complex valued function defined on a closed set $S \subset \mathbb{C}$. We say that f satisfies Hölder's condition for a point $z_0 \in D$ if there exists constants $\mu, \nu > 0$ such that $|f(z) - f(z_0)| \leq \mu |z - z_0|^\gamma$ for all $z \in D$ sufficiently close to z_0. The constant γ is called exponent of Hölder's condition. If f satisfies the above inequality for all $z \in D$, then f is said to satisfy uniform Hölder's condition on D. For $0 < \gamma \leq 1$, all functions which satisfy the inequality (**??**) on D belong to Lipschitz class of order γ denoted by Lipγ.*

Note 2.3.1. *Any function belonging to Lipγ is uniformly continuous on D but converse is not true. In example 2.3.2 given below, $\phi(x)$ is uniformly continuous 0n $0 \leq x \leq \frac{1}{2}$ but does not satisfy Höder's condition for $0 \leq x \leq \frac{1}{2}$ and $0 < \gamma \leq 1$.*

Example 2.3.1. *Let $f(x) = \sqrt{x}$. It is evident that $f(x)$ satisfies the Hölder's condition for the exponent $\gamma = \frac{1}{2}$.*

Example 2.3.2. *Let* $\phi(x) = \frac{1}{\ln x}$, *for* $0 < x \leq \frac{1}{2}$ *and* $\phi(0) = 0$. *Note that* $\phi(x)$ *is continuous for* $0 \leq x \leq \frac{1}{2}$ *but* $|\phi(x) - \phi(0)| = \frac{1}{|\ln x|} > Ax^{\gamma}$ *because* $\lim_{x \to 0} x^{\gamma} \ln x = 0$ *for any* $\gamma > 0$, *and any values of* A *and* λ. *Hence* $\phi(x)$ *does not belong to class* $Lip\gamma$ *for* $0 < \gamma \leq 1$.

Definition 2.3.2. *Let* L *be a smooth, closed, and positively oriented contour in the plane of complex variables* z. *We denote the domain within contour* L *by* D^+, *and is called interior domain. The domain complement to* $D^+ + L$, *is called exterior domain denoted by* D^-. *If* $f(z)$ *is an analytic function for all* $z \in D^+$ *and continuous on* $D^+ + L$, *then by Cauchy's formula in the theory of complex variables,*

$$\frac{1}{2\pi i} \int_L \frac{f(\tau)}{\tau - z} d\tau = \begin{cases} f(z), & z \in D^+, \\ 0, & z \in D^-. \end{cases}$$

If $f(z)$ *is analytic in* D^- *and continuous on* $D^- + L$, *then*

$$\frac{1}{2\pi i} \int_L \frac{f(\tau)}{\tau - z} d\tau = \begin{cases} f(\infty), & z \in D^+, \\ -f(z) + f(\infty), & z \in D^-. \end{cases}$$

Definition 2.3.3. *Let* L *be a smooth, closed or open positively oriented contour. If* L *is closed then* D^+ *and* D^- *are the interior and exterior domains as defined in*

definition 2.3.2. If L is open then D^+ is on left side of L and D^- on right side of L, as one walks around the contour L. If $\tau \in L$ denotes complex coordinates of a point on L, and $\phi(\tau)$ is a continuous function of τ on L then the integral

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\phi(\tau)}{\tau - z}d\tau, \quad z \in \mathbb{C}\setminus L \tag{2.77}$$

is called a Cauchy integral. The function $\phi(\tau)$ is called the density ,and $\frac{1}{\tau - z}$ is called the kernel of $\Phi(z)$.

Definition 2.3.4. *Let L be a smooth closed contour, and $G(t)$ be a continuous function on L such that $G(t) \neq 0\ \forall t \in L$. Index of the function $G(t)$ w.r.t the counter L is defined as the change in argument of $G(t)$ divided by 2π when t traverses one round of L. If $[argG(t)]_L$ denotes change in argument of $G(t)$ when t traverses one round of L, then*

$$\chi = IndG(t) = \frac{1}{2\pi}[argG(t)]_L. \tag{2.78}$$

Since $lnG(t) = ln|G(t)| + iargG(t)$, and after having traversed the counter L, $|G(t)|$ returns to its original value hence $[argG(t)]_L = \frac{1}{i}[lnG(t)]_L$. So, equation (2.78) can be written as $\chi = IndG(t) = \frac{1}{2\pi i}[lnG(t)]_L$. Hence the index χ can be represented

by the following integral

$$\chi = IndG(t) = \frac{1}{2\pi}\int_L dargG(t) = \frac{1}{2\pi i}\int_L dlnG(t). \tag{2.79}$$

The integral in equation (2.79) is understood in Stieljes sense. For non vanishing continuous functions $F(t)$ and $G(t)$, we have the following observations:

1. *The indices of $F(t)$ and $G(t)$ on a closed contour L are always integers.*

2. $Ind[G(t)F(t)] = IndG(t) + IndF(t)$.

3. $Ind[\frac{G(t)}{F(t)}] = IndG(t) - IndF(t)$.

4. *Let $G(t)$ be the boundary value of a function $G(z)$ which is analytic inside or outside the closed contour L. Then its index is equal to the number of zeroes (counting multiplicities) of the function inside or outside the closed contour L, with negative sign.*

5. *Let $G(z)$ be a meromorphic function inside or outside a closed contour L. If Z denotes the number of zeroes of $G(z)$ inside the closed contour L, and P denotes number of poles inside the closed contour L, then*

$$\chi = IndG(t) = Z - P.$$

Theorem 2.3.1. *[23] Let L be a common smooth boundary of two domains D_1 and D_2, $f_1(z)$ and $F_2(z)$ be two analytic functions in domains D_1 and D_2 respectively. Suppose that for any point $t \in L$, $f_1^*(t) = \lim_{z\to t} f_1(z), \quad z \in D_1$ and $f_2^*(t) = \lim_{z\to t} f_2(z), \quad z \in D_2$, are continuous and $f_1^*(t) = f_2^*(t) \quad \forall t \in L$. Then the functions $f_1(z)$ and $f_2(z)$ are regarded as the analytic continuation of each other.*

Theorem 2.3.2. *[23] Let a function $f(z)$ be analytic in entire complex plane, except at the points $a_0 = \infty$, $a_k(k = 1, 2, 3, ..., n)$, where the function $f(z)$ has poles. Suppose that in vicinities of the poles $z = a_0 = \infty$ and $z = a_k$, the principal parts of the expansions of $f(z)$ have forms*

$$G_0(z) = C_1^0 z + C_2^0 z^2 + C_3^0 z^3 + ... + C_{c_n}^0 z_n^{n_0},$$

$$G(\frac{1}{z-a_k}) = \frac{C_1^k}{z-a_k} + \frac{C_2^k}{(z-a_k)^2} + ... + \frac{C_{m_k}^k}{(z-a_k)^{m_k}},$$

respectively. Then the function $f(z)$ is representable as a rational function $f(z) = C + G_0(z) + \sum_{k=1}^{n} G_k(\frac{1}{z-a_k})$. In the case, when $f(z)$ has only a pole of order m at ∞ then $f(z)$ has the form $f(z) = C_0 + C_1 z + ... + C_m z^m$.

Definition 2.3.5. *To define a RHP or a Privalov problem, let L be a positively oriented smooth contour (open or closed) dividing the plane of complex variables*

z into D^+ and D^- as in definition 2.3.3. Define $G(t)$ and $g(t)$ two continuous functions of position on the contour L which satisfy Holder's condition on L. Also, $G(t)$ and $g(t)$ are non zero functions for every point on the contour L. We want to find a sectionally analytic function

$$\Phi(z) = \begin{cases} \Phi^+(z), & z \in D^+ \\ \\ \Phi^-(z), & z \in D^- \cup \{\infty\} \end{cases}$$

which satisfies the following condition on the contour L

$$\Phi^+(t) = G(t)\Phi^-(t) \quad \forall \quad t \in L \text{ (homogeneous problem) or} \tag{2.80}$$

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t) \quad \forall \quad t \in L \text{ (non homogeneous problem).} \tag{2.81}$$

Note that

$$\Phi^+(t) = \lim_{z \to t} \Phi(z), \quad z \in D^+, \quad t \in L,$$

$$\Phi^-(t) = \lim_{z \to t} \Phi(z), \quad z \in D^-, \quad t \in L.$$

Lemma 2.3.1. *[24] If index of $G(t)$ is zero, then the general solution of the RHP defined by equation (2.80) is $\Phi(z) = C\Phi_0(z)$, where C is an arbitrary constant , and $\Phi_0(z) = e^{l(z)}$, $l(z) = \frac{1}{2\pi i}\int_L \frac{logG(t)}{t-z}dt$. Note that L is a smooth closed contour dividing the complex plane into D^+ and D^- as in definition 2.3.2.*

Theorem 2.3.3. *[24] Let* Γ *be a smooth closed positively oriented contour enclosing origin* O*. Suppose that* $G(t)$ *be a non vanishing function on* Γ *which belongs to* $Lip\gamma$ *for some* $0 < \gamma < 1$*. If the index* χ *of* $G(t)$ *is non negative, then the general solution of homogeneous RHP defined by equation (2.80) is given by*

$$\Phi(z) = \begin{cases} p(z)\Phi_1(z), & z \in D^+ \\ \\ z^{-n}\Phi_1(z), & z \in D^- \end{cases}$$

where, $p(z)$ *is a polynomial of degree* $\leq n$*, and*

$$\Phi_1(z) = e^{l_1(z)}, \quad l_1(z) = \frac{1}{2\pi i}\int_\Gamma \frac{log[t^{-n}G(t)]}{t-z}dt. \tag{2.82}$$

If $n < 0$*, the only solution of homogeneous RHP defined by equation (2.80) is a trivial solution.*

Theorem 2.3.4. *[24] Let* Γ *be the contour as defined in theorem 2.3.3. Suppose that* $G(t)$ *and* $g(t)$ *are non vanishing Hölder continuous functions on* Γ *for* $0 < \gamma < 1$*. If the index* χ *of* $G(t)$ *is a non negative integer, then the general solution*

of non homogeneous RHP defined by equation 2.81 is given by

$$\Phi(z) = \begin{cases} [p(z) + k(z)]e^{l_1(z)}, & z \in D^+ \\ \\ z^{-n}[p(z) + k(z)]e^{l_1(z)}, & z \in D^- \end{cases}$$

where, $p(z)$ is a polynomial of degree $\leq n$, and

$$\Phi_1(z) = e^{l_1(z)}, \quad l_1(z) = \frac{1}{2\pi i}\int_\Gamma \frac{log[t^{-n}G(t)]}{t-z}dt. \tag{2.83}$$

Note that $k(z)$ is defined as $k(z) = \frac{1}{2\pi i}\int_\Gamma \frac{g(t)e^{-l_1(t)}}{t-z}dt, \quad k(z) = O(z^n),$ as $z \to \infty$.

If $\chi = -1$, then the problem defined by equation (2.81) has precisely one solution. If $\chi < -1$, then the non homogeneous RHP defined by equation (2.81) has a solution only if $g(t)$ satisfies

$$\int_\Gamma t^h g(t)e^{-l_1^+(t)}dt = 0, \quad h = 0, 1, 2, 3, 4, ..., -n-2, \text{ where} \tag{2.84}$$

$$l_1^+(t) = l_1(z) \text{ as } z \to t \quad (t \in \Gamma \text{ is an interior point of } \Gamma, z \in D^+) \tag{2.85}$$

In either case, the solution of non homogeneous RHP defined by equation (2.81) (if it exists) is given by

$$\Phi(z) = \begin{cases} z^n[k_1(z) - k_1(0)]e^{l_1(z)}, & z \in D^+ \\ \\ [k_1(z) - k_1(0)]e^{l_1(z)}, & z \in D^- \end{cases}$$

$$k_1(z) = \frac{1}{2\pi i} \int_\Gamma \frac{t^{-n} g(t)}{\Phi_1^+(t)(t-z)} dt, \quad \Phi_1(z) = \Phi(z)[k(z)]^{-1}. \tag{2.86}$$

Theorem 2.3.5. *[24] Let $\Gamma : t = t(\tau), \alpha \leq \tau \leq \beta$ be a smooth open contour, and $G(t)$ be a non vanishing function of position on Γ satisfying Hölder's condition on Γ. The totality of solutions of homogeneous RHP (2.80) are given by*

$$\Phi_1(z) = s(z)\Phi_0(z), \text{ where } \Phi_0(z) = e^{l(z)}, \quad l(z) = \frac{1}{2\pi i} \int_\Gamma \frac{logG(t)}{t-z} dt. \tag{2.87}$$

Note that $s(z)$ is any analytic function with isolated singularities at the end points t_0 and t_1 of Γ, and having at most a pole at ∞.

Lemma 2.3.2. *[23] Let Γ is a smooth positively oriented open contour with end points a and b. Suppose that $\phi(t)$ satisfies Hölder's condition $\forall t \in \Gamma$ including the end points a and b. The Cauchy integral $\Phi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\phi(\tau)}{\tau - z} d\tau, \quad z \in \mathbb{C} \setminus \Gamma$ has the following behaviour near the end points a and b of contour Γ.*

1. *When $z \in \mathbb{C} \setminus \Gamma$ and $z \to a$, then $\Phi(z) = -\frac{\phi(a)}{2\pi i} ln(z-a) + \Phi_a(z)$.*

2. *When $z \in \mathbb{C} \setminus \Gamma$ and $z \to b$, then $\Phi(z) = \frac{\phi(b)}{2\pi i} ln(z-b) + \Phi_b(z)$.*

Note that $\Phi_a(z)$ and $\Phi_b(z)$ are bounded functions in vicinities of the respective end points, and tend to a definite value as $z \to a$ or b respectively, where $z \in \mathbb{C} \setminus \Gamma$.

So, the Cauchy type integral possess singularities of logarithmic type at end points of the contour which are determined by the values of $\Phi_a(a)$ *and* $\Phi_b(b)$.

Definition 2.3.6. *Let* $\Gamma : t = t(\tau), \alpha \leq \tau \leq \beta$, *be a simple contour such that* $t'(\tau) \in Lip\gamma$ *and* $t'(\tau) \neq 0 \forall \tau \in [\alpha, \beta]$ *and* $0 < \gamma < 1$. *Let* $\Phi(t)$ *be a function defined on* Γ *satisfying the following properties.*

i. $\Phi(z)$ *is an analytic function* $\forall z \in \mathbb{C} \setminus \Gamma$.

ii. $\Phi(z)$ *has at most a pole at* ∞.

iii. $\Phi(z)$ *is not required to be meromorphic at the end points* $t_0 = t(\alpha)$ *and* $t_1 = t(\beta)$ *of contour* Γ. *It is only required to be analytic at these end points. The behaviour of* $\Phi(z)$ *at the end points* t_0 *and* t_1 *should be pole like i.e. there exist real numbers* δ, ϵ *and* μ *such that , if* $|z - t_j|$ *is sufficiently small,* $z \in \mathbb{C} \setminus \Gamma$, *then* $\mu|z - t_j|^\delta \leq |\Phi(z)| \leq \mu|z - t_j|^\epsilon, \quad j = 0, 1$.

iv. $\Phi^+(t) = \lim_{z \to t} \Phi(z), \quad t \in \Gamma$ *is an interior point of* $\Gamma, \quad z \in D^+$,

$\Phi^-(t) = \lim_{z \to t} \Phi(z), \quad t \in \Gamma$ *is an interior point of* $\Gamma, \quad z \in D^-$,

exist and are non zero.

v. *Define* $a(t) = \frac{\Phi^+(t)}{\Phi^-(t)}, \quad \forall t \in \Gamma,$ *where* $\lim_{t\to t_0} a(t)$ *and* $\lim_{t\to t_1} a(t)$. *Note that* $loga(t)$ *(a branch of the multi-valued function* $Loga(t)$*) is defined* $\forall \quad t \in \Gamma$*, such that* $loga(t) \in Lip\gamma$ *for* $0 < \gamma < 1$.

Theorem 2.3.6. *[24] Let* $\Phi(z)$ *be a function satisfying the hypothesis given in definition 2.3.6. Then* $\Phi(z)$ *satisfies the following properties.*

a. *The function* $\Phi(z)$ *has finitely many zeroes.*

b. *A polynomial whose zeroes are exactly the zeroes of* $\Phi(z)$*, can be constructed rationally in terms of finitely many of the Laurent coefficients of* $\Phi(z)$ *at* ∞*, and finitely many of the quantities* $m_k = \frac{1}{2\pi i}\int_\Gamma t^k loga(t)dt, \quad k = 0, 1, 2, 3, ...$ *Note that* $loga(t)$ *denotes any continuous logarithm of* $a(t)$.

Proof. Since $\Phi(z)$ satisfies the conditions (i), (ii), (iv) and (v) given in definition 2.3.6, so, $\Phi(z)$ is a solution of the homogeneous RHP

$$\Phi^+(t) = a(t)\Phi^-(t) \quad \forall \quad t \in \Gamma, \tag{2.88}$$

where $a(t)$ is a non vanishing Hölder continuous function on Γ. Application of theorem 2.3.5 indicates that a special solution of the RHP defined by equation

(2.88) is given by $\Phi_0(z) = e^{g(z)}$, where $g(z)$ is the Cauchy integral

$$g(z) = \frac{1}{2\pi i}\int_\Gamma \frac{loga(t)}{t-z}dt. \tag{2.89}$$

Also, theorem 2.3.5 asserts that every solution of the RHP defined by equation (2.88), is of the form

$$\Phi(z) = s(z)\Phi_0(z) \tag{2.90}$$

where the function $s(z)$ has isolated singularities (at most) at the points t_0, t_1 and ∞. Conditions (ii) and (iii) in definition 2.3.6 imply that the singularities of $s(z)$ at these points are at most poles, and thus $s(z)$ in equation (2.90) is a rational function. Since $\Phi_0(z) \neq 0, z \in \mathbb{C} \setminus \Gamma$, equation (2.90) indicates that zeroes of $\Phi(z)$ are those of $s(z)$. Now identification of $s(z)$ is required. From above use value $\Phi_0(z) = e^{g(z)}$ in equation (2.90), and simplify to get

$$s(z) = \Phi(z)e^{-g(z)}. \tag{2.91}$$

Using equation (2.91) Laurent's series of $s(z)$ at ∞ can be calculated because the Laurent's series of $\Phi(z)$ is assumed to be known

$$\Phi(z) = \sum_{n=-l}^{\infty} a_n z^{-n}, \text{ where } l \text{ is the order of pole of } \Phi(z) \text{ at } \infty. \tag{2.92}$$

From equation (2.89) consider

$$-g(z)=\frac{1}{2z\pi i}\int_{\Gamma}\frac{loga(t)}{1-\frac{t}{z}}dt, \tag{2.93}$$

$$(1-\frac{t}{z})^{-1}=1+\frac{t}{z}+\frac{t^2}{z^2}+\frac{t^3}{z^3}+..., \tag{2.94}$$

$$\frac{1}{z}(1-\frac{t}{z})^{-1}=\frac{1}{z}+\frac{t}{z^2}+\frac{t^2}{z^3}+.... \tag{2.95}$$

Use value of $\frac{1}{z}(1-\frac{t}{z})^{-1}$ from equation (2.95) in equation(2.93) to get

$$-g(z)=\frac{1}{2\pi i}\int_{\Gamma}loga(t)dt[\frac{1}{z}+\frac{t}{z^2}+\frac{t^2}{z^3}+...]. \tag{2.96}$$

Let $m_n=\frac{1}{2\pi i}\int_{\Gamma}t^n loga(t)dt, \quad n=0,1,2,3,4,....$ Then equation (2.96) becomes

$$-g(z)=\sum_{n=0}^{\infty}\frac{m_n}{z^{n+1}}. \tag{2.97}$$

Hence, the Laurent's series of $e^{-g(z)}$ is

$$e^{-g(z)}=\sum_{n=0}^{\infty}b_n z^{-n}. \tag{2.98}$$

The coefficients b_n in equation (2.98) can be computed by comparing coefficients in the identity

$$\frac{d}{dz}(-e^{-g(z)})=\acute{g}(z)e^{-g(z)}. \tag{2.99}$$

Use values of $g'(z), g(z)$ and $e^{-g(z)}$ from equations (2.97) and (2.98) in equation (2.99) to get $\sum_{n=1}^{\infty}\frac{nb_n}{z^{n+1}}=\sum_{n=1}^{\infty}\frac{nm_{n-1}}{z^{n+1}}\sum_{j=0}^{\infty}\frac{b_j}{z^j}$. This equation yields the recur-

rence relation $b_0 = 1, \quad nb_n = nm_{n-1}b_0 + (n-1)m_{n-2}b_1 + \cdots + 1.m_0b_{n-1}, \quad n = 1, 2, 3, 4, ...$ Note that equations (2.92) and (2.98) give Laurent series expansion for $\Phi(z)$ and $e^{-g(z)}$ at ∞. Then using series representations of $\Phi(z)$ and $e^{-g(z)}$ from equations (2.92) and (2.98) respectively, in equation (2.91), and forming a Cauchy product, as many Laurent coefficients s_n can be constructed in the following expansion as one likes.

$$s(z) = \sum_{n=-l}^{\infty} s_n z^{-n} \tag{2.100}$$

Note that knowing finitely many coefficients in the expansion defined by equation (2.100) is not sufficient, however, to identify $s(z)$, even if $s(z)$ be a rational function. If, on the other hand, a bound for the order of poles of $s(z)$ were known, then identification can be made. Suppose that the order of the poles of $s(z)$ at t_j be at most $r_j, \quad j = 0, 1$. Since a rational function is the sum of its principal parts, so, $s(z)$ must be of the form

$$\begin{aligned} s(z) = \sum_{n=0}^{l} s_{-n} z^n (s_0 + \text{ principal part at } \infty) + \sum_{n=1}^{r_0} \frac{a_{0,n}}{(z-t_0)^n} (\text{principal part at } t_0) \\ + \sum_{n=1}^{r_1} \frac{a_{1,n}}{(z-t_1)^n} (\text{ principal part at } t_1). \end{aligned} \tag{2.101}$$

The coefficients $a_{j,n}$ in equation (2.101) are yet, unknown. However, one can expand the principal parts in equation (2.101) into their Laurent series at ∞ using the following formula.

$$\frac{1}{(z-t_j)^n} = z^{-n}(1-\frac{t_j}{z})^{-n} = z^{-n}\sum_{p=0}^{\infty}\frac{(n)_p}{p!}(\frac{t_j}{z})^p \tag{2.102}$$

Using expansion of $\frac{1}{(z-t_j)^n}$ from equation (2.102) in equation (2.101), it is evident that the coefficients of z^{-n} in the Laurent series of $s(z)$ at ∞ can be expressed as a linear combination of $a_{j,k}$, for all $k \leq n$. Now equating these expressions to s_n for $1 \leq n \leq r_0 + r_1$, a system of linear equations for $r_0 + r_1$ unknowns $a_{j,n}$ is obtained. This system of linear equations is used to determine the unknowns $a_{j,n}$, and hence $s(z)$. To complete the construction of $s(z)$, it is necessary to find bounds r_j for the order of poles of $s(z)$ at $t_j, \quad j = 0, 1$. Condition (iii) in definition 2.3.6 controls the behaviour of $\Phi(z)$ at these points. To study the behaviour of $\Phi_0^{-1}(z) = e^{-g(z)}$ near the end points t_0 and t_1, of Γ, lemma 2.3.2 is used. Application of lemma 2.3.2 gives the following result.

$$g(z) = \frac{1}{2\pi i}\int_\Gamma \frac{loga(t)}{t-z}dt \text{ behaves like } -\frac{1}{2\pi i}loga(t_0)log(z-t_0). \tag{2.103}$$

Use the above relation to find behavior of $e^{-g(z)}$ as $z \to t_0$. Note that

$$\text{for } z \to t_0, \quad z \in \mathbb{C} \setminus \Gamma, \quad e^{-g(z)} \sim \text{ constant } (z - t_0)^{\frac{1}{2\pi i} \log a(t_0)}. \tag{2.104}$$

$$\text{Similarly, for } z \to t_1, \quad z \in \mathbb{C} \setminus \Gamma, \quad e^{-g(z)} \sim \text{ constant } (z - t_1)^{-\frac{1}{2\pi i} \log a(t_1)}. \tag{2.105}$$

Note that Condition (iii), equation (2.91), relation (2.104), and condition (iii), equation (2.91), relation (2.105) gives following results.

$$|s(z)| \leq \text{ constant } |z - t_0|^{\epsilon + Re\frac{1}{2\pi i} loga(t_0)} \text{ as } z \to t_0 \tag{2.106}$$

$$|s(z)| \leq \text{ constant } |z - t_1|^{\epsilon - Re\frac{1}{2\pi i} loga(t_1)} \text{ as } z \to t_1 \tag{2.107}$$

Relations (2.106) and (2.107) indicate that the bounds for the order of poles at t_0 and t_1 are given by $r_0 = -[\epsilon + Re\frac{1}{2\pi i} loga(t_0)], \quad r_1 = -[\epsilon - Re\frac{1}{2\pi i} loga(t_1)]$. Note that Burniston-Siewert method also holds for a finite collection of non intersecting smooth arcs. □

2.4 Zeroes of the transcendental equation occurring in the solution of BVP of Helmholtz equation in a semi-infinite strip using FIT method

We apply the Burniston-Siewert method to find the zeroes of the following transcendental equation:

$$\tan \lambda a = \frac{\lambda(\mu_0 + \mu_1)}{\lambda^2 - \mu_0\mu_1}, \quad \mu_0, \mu_1 > 0. \tag{2.108}$$

While solving the BVP of the Helmholtz equation in a semi-infinite strip using FIT method, equation (2.108) occurs in chapter 2 subsection 2.2.1 page 59.

Let $\zeta = -i\lambda a,$ divide by ia to get $\lambda = \frac{i\zeta}{a}$. Then equation (2.108) becomes

$$\tan i\zeta = \frac{i\zeta(\mu_0+\mu_1)}{a(-\frac{\zeta^2}{a^2}-\mu_0\mu_1)}, \text{ simplify to get } \tan i\zeta = \frac{i\zeta(\mu_0+\mu_1)a}{-\zeta^2-\mu_0\mu_1 a^2}.$$

Use trigonometric and hyperbolic identities to get

$$tanh\zeta = -\frac{a\zeta(\mu_0+\mu_1)}{\zeta^2+\mu_0\mu_1 a^2}, \text{ definition of } tanh\zeta \text{ gives } \frac{e^{2\zeta}-1}{e^{2\zeta}+1} = -\frac{a\zeta(\mu_0+\mu_1)}{\zeta^2+\mu_0\mu_1 a^2}.$$

Cross multiply and simplify to get

$$(e^{2\zeta}-1)(\zeta^2+c^2) = -b\zeta(e^{2\zeta}+1), \quad c = a\sqrt{\mu_0\mu_1} > 0, \quad b = a(\mu_0+\mu_1) > 0,$$

$$e^{2\zeta}(\zeta^2+b\zeta+c^2) = \zeta^2-b\zeta+c^2, \quad e^{2\zeta} = \frac{\zeta^2-b\zeta+c^2}{\zeta^2+b\zeta+c^2},$$

$$\zeta = \frac{1}{2}Log\frac{\zeta^2-b\zeta+c^2}{\zeta^2+b\zeta+c^2} + \pi ik. \tag{2.109}$$

Using equation(2.109), we define the function $f(\zeta)$ as follows:

$$f(\zeta) = \frac{1}{2}Log\frac{\zeta^2-b\zeta+c^2}{\zeta^2+b\zeta+c^2} + \pi ik - \zeta. \tag{2.110}$$

Let $\alpha_0 = a\mu_0 > 0, \alpha_1 = a\mu_1 > 0$ and $\alpha_1 > \alpha_0$. Then equation (2.110) becomes

$$f(\zeta) = \frac{1}{2}Log\frac{(\zeta-\alpha_0)(\zeta-\alpha_1)}{(\zeta+\alpha_0)(\zeta+\alpha_1)} + \pi ik - \zeta, \quad k = 0, \pm1, \pm2, \pm3, \pm4, ... \tag{2.111}$$

Note that $Log\frac{(\zeta-\alpha_0)(\zeta-\alpha_1)}{(\zeta+\alpha_0)(\zeta+\alpha_1)}$ is a multi-valued function with branch points $\alpha_0, \alpha_1, -\alpha_0$ and $-\alpha_1$. The individual factor $Log(\zeta-\alpha_0)$ has the branch points α_0 and ∞, similar is true for the other factors $Log(\zeta-\alpha_1), Log(\zeta+\alpha_0)$ and $Log(\zeta+\alpha_1)$. We fix a branch of $f(\zeta)$ by drawing branch cuts in the complex plane as shown in figure 2.4.

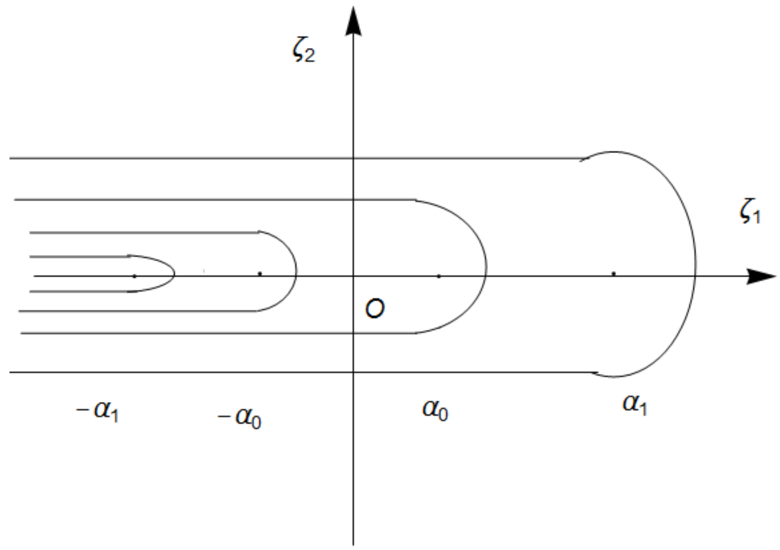

FIGURE 2.4. A branch cut for the multi-valued function $f(\zeta)$

We denote a single branch of $Log\frac{(\zeta-\alpha_0)(\zeta-\alpha_1)}{(\zeta+\alpha_0)(\zeta+\alpha_1)}$ by $log\frac{(\zeta-\alpha_0)(\zeta-\alpha_1)}{(\zeta+\alpha_0)(\zeta+\alpha_1)}$. For this single branch we have

$$\mathrm{Arg}(\zeta-\alpha_j)=\theta_j^+, \quad -\pi\le\theta_j^+\le\pi, \quad j=0,1,$$
$$\mathrm{Arg}(\zeta+\alpha_j)=\theta_j^-, \quad -\pi\le\theta_j^-\le\pi, \quad j=0,1.$$

For the single branch $log\frac{(\zeta-\alpha_0)(\zeta-\alpha_1)}{(\zeta+\alpha_0)(\zeta+\alpha_1)}$, equation (2.111) becomes

$$\begin{aligned} f(\zeta) &= \frac{1}{2}log\frac{(\zeta-\alpha_0)(\zeta-\alpha_1)}{(\zeta+\alpha_0)(\zeta+\alpha_1)}+\pi ik-\zeta, \quad k=0,\pm1,\pm2,\pm3,\pm4,... \\ &= \frac{1}{2}log|\frac{(\zeta-\alpha_0)(\zeta-\alpha_1)}{(\zeta+\alpha_0)(\zeta+\alpha_1)}|+\frac{i}{2}[arg(\zeta-\alpha_0)+arg(\zeta-\alpha_1)- \\ &\quad arg(\zeta+\alpha_0)-arg(\zeta+\alpha_1)]+\pi ik-\zeta, \quad k=0,\pm1,\pm2,\pm3,.... \end{aligned} \tag{2.112}$$

Equation (2.112) can be expressed as

$$\begin{aligned} f(\zeta) &= f_*(\zeta)+\frac{i}{2}[arg(\zeta-\alpha_0)+arg(\zeta-\alpha_1)-arg(\zeta+\alpha_0)-arg(\zeta+\alpha_1)] \\ &\quad +\pi ik-\zeta, \quad f_*(\zeta)=\frac{1}{2}log|\frac{(\zeta-\alpha_0)(\zeta-\alpha_1)}{(\zeta+\alpha_0)(\zeta+\alpha_1)}|. \end{aligned} \tag{2.113}$$

$$\text{For } \zeta = \zeta_1 \pm i0, \ -\infty < \zeta_1 < \alpha_1, \text{ we have } \theta_0^+ = \theta_1^+ = \theta_0^- = \theta_1^- = \pm\pi. \tag{2.114}$$

$$\text{Now equation (2.113) becomes } f(\zeta_1 \pm i0) = f_*(\zeta_1) + \pi ik - \zeta_1. \tag{2.115}$$

$$\text{For } \zeta = \zeta_1 \pm i0, \ -\alpha_1 < \zeta_1 < -\alpha_0, \text{ we have } \theta_0^+ = \theta_1^+ = \theta_0^- = \pm\pi, \theta_1^- = 0. \tag{2.116}$$

$$\text{Now equation (2.113) becomes } f(\zeta_1 \pm i0) = f_*(\zeta_1) \pm \frac{\pi}{2} i + \pi ik - \zeta_1. \tag{2.117}$$

$$\text{For } \zeta = \zeta_1 \pm i0, \ -\alpha_0 < \zeta_1 < \alpha_0, \text{ we have } \theta_0^+ = \theta_1^+ = \pm\pi, \theta_0^- = \theta_1^- = 0. \tag{2.118}$$

$$\text{Now equation (2.113) becomes } f(\zeta_1 \pm i0) = f_*(\zeta_1) \pm \pi i + \pi ik - \zeta_1. \tag{2.119}$$

$$\text{For } \zeta = \zeta_1 \pm i0, \ \alpha_0 < \zeta_1 < \alpha_1, \text{ we have } \theta_1^+ = \pm\pi, \theta_0^+ = \theta_0^- = \theta_1^- = 0. \tag{2.120}$$

$$\text{Now equation (2.113) becomes } f(\zeta_1 \pm i0) = f_*(\zeta_1) \pm \frac{\pi}{2} i + \pi ik - \zeta_1. \tag{2.121}$$

$$\text{For } \zeta = \zeta_1 \pm i0, \ \alpha_1 < \zeta_1 < \infty, \text{ we have } \theta_0^+ = \theta_1^+ = \theta_0^- = \theta_1^- = 0. \tag{2.122}$$

$$\text{Now equation (2.113) becomes } f(\zeta_1 \pm i0) = f_*(\zeta_1) + \pi ik - \zeta_1. \tag{2.123}$$

We denote the values of $f(\zeta)$ on the upper and lower edges of the contour $\Gamma = \Gamma_0 \cup \Gamma_1$ by $f^+(t)$ and $f^-(t)$ respectively, these values are determined by using equations (2.115), (2.117), (2.119), (2.121) and (2.123). So, the values of $f^+(t)$ and $f^-(t)$ are

$$f^+(t) = \phi(t) + \begin{cases} \pi i(k + \frac{1}{2}), & \alpha_0 < |t| < \alpha_1 = \Gamma_1 \\ \pi i(k+1), & |t| < \alpha_0 = \Gamma_0 \end{cases} \tag{2.124}$$

$$f^-(t) = \phi(t) + \begin{cases} \pi i(k - \frac{1}{2}), & \alpha_0 < |t| < \alpha_1 = \Gamma_1 \\ \pi i(k-1), & |t| < \alpha_0 = \Gamma_0. \end{cases} \tag{2.125}$$

$$\text{Note that } \phi(t) = f_*(t) - t = \frac{1}{2} log|\frac{(\zeta - \alpha_0)(\zeta - \alpha_1)}{(\zeta + \alpha_0)(\zeta + \alpha_1)}| - t. \tag{2.126}$$

$$\text{It is observed that } \phi(t) \to -\infty, \text{ as } t \to \alpha_0 \text{ or } t \to \alpha_1, \tag{2.127}$$

$$\phi(t) \to +\infty, \text{ as } t \to -\alpha_0 \text{ or } t \to -\alpha_1. \tag{2.128}$$

Now we define

$$a(t) = \frac{f^+(t)}{f^-(t)} = \frac{\phi(t) + \pi i(k + \frac{1}{2})}{\phi(t) + \pi i(k - \frac{1}{2})}, \quad t \in \Gamma_1 = [-\alpha_1, -\alpha_0] \cup [\alpha_0, \alpha_1], \tag{2.129}$$

$$a(t) = \frac{f^+(t)}{f^-(t)} = \frac{\phi(t) + \pi i(k + 1)}{\phi(t) + \pi i(k - 1)}, \quad t \in \Gamma_0 = [-\alpha_0, \alpha_0]. \tag{2.130}$$

Note that $f(\zeta)$ defined by equation (2.113) satisfies the five hypothesis given in definition 2.3.6.

Case-I. For $k = 0$, equations (2.129) and (2.130) give

$$\log a(t) = log \frac{\phi(t) + \frac{\pi i}{2}}{\phi(t) - \frac{\pi i}{2}}, \quad t \in \Gamma_1 = [-\alpha_1, -\alpha_0] \cup [\alpha_0, \alpha_1], \tag{2.131}$$

$$\log a(t) = log \frac{\phi(t) + \pi i}{\phi(t) - \pi i}, \quad t \in \Gamma_0 = [-\alpha_0, \alpha_0]. \tag{2.132}$$

Consider the segment $[-\alpha_1, -\alpha_0]$, using equation (2.131), we find that $|a(t)| = 1$, and

$$\begin{aligned} \text{Arg}a(t) &= Arg(\phi(t) + \frac{i\pi}{2}) - arg(\phi(t) - \frac{i\pi}{2}) \quad \text{using equation (2.128) to get} \\ &= 0 - 0 = 0. \end{aligned} \tag{2.133}$$

Using above results, $\log a(t) = \log|a(t)| + iarga(t) = 0,$ as $t \to -\alpha_1 + 0$ or $t \to -\alpha_0 - 0$. Now consider the segment $[\alpha_0, \alpha_1]$, using equation (2.131), we find that $|a(t)| = 1$, and

$$\begin{aligned} \text{Arg}a(t) &= arg(\phi(t) + \frac{i\pi}{2}) - arg(\phi(t) - \frac{i\pi}{2}) \quad \text{using equation (2.127) to get} \\ &= \pi - (-\pi) \\ &= 2\pi. \end{aligned} \tag{2.134}$$

Using above results, $\log a(t) = \log|a(t)| + iArga(t) = 2\pi i,$ as $t \to \alpha_0 + 0$ or $t \to \alpha_1 - 0$. Similarly, we can find behavoir of $\log a(t)$ at the end points of the segment $[-\alpha_0, \alpha_0]$.

TABLE 2.1. Values of log a(t)

	$-\alpha_1+0$	$-\alpha_0-0$	$-\alpha_0+0$	α_0-0	α_0+0	α_1-0
log a(t)	0	0	0	$2\pi i$	$2\pi i$	$2\pi i$

Table 2.1 shows values for $loga(t)$ for different values of t in the contours Γ_0 and Γ_1.

Case-II. For $k=1$, equations (2.129) and (2.130) give

$$loga(t) = log\frac{\phi(t)+\frac{3\pi i}{2}}{\phi(t)+\frac{\pi i}{2}}, \quad t \in \Gamma_1 = [-\alpha_1, -\alpha_0] \cup [\alpha_0, \alpha_1], \tag{2.135}$$

$$loga(t) = log\frac{\phi(t)+2\pi i}{\phi(t)}, \quad t \in \Gamma_0 = [-\alpha_0, \alpha_0]. \tag{2.136}$$

Table 2.2 shows values for $loga(t)$ for different values of t in the contours Γ_0 and Γ_1.

TABLE 2.2. Values of log a(t)

	$-\alpha_1+0$	$-\alpha_0-0$	$-\alpha_0+0$	α_0-0	$-\alpha_0+0$	α_1-0
log a(t)	0	0	0	0	0	0

Case-III. For $k=-1$, equations (2.129) and (2.130) give the same Table 2.2.

Case-IV. For $k>1$ or $k<-1$, equations (2.129) and (2.130) give the same Table 2.2.

Riemann Hilbert problem is:

$$f^+(t) = a(t)f^-(t), \quad t \in \Gamma = \Gamma_0 \cup \Gamma_1. \tag{2.137}$$

Application of theorem 2.3.5 indicates that a special solution of the RHP defined by equation (2.137) is given by $f_0(\zeta) = e^{g(\zeta)}$, where $g(\zeta)$ is the Cauchy integral given by

$$g(\zeta) = \frac{1}{2\pi i}\int_\Gamma \frac{loga(t)}{t-\zeta}dt. \tag{2.138}$$

The Burniston-Siewert method given by theorem 2.3.6 shows that our solution is of the following form.

$$f(\zeta) = s(\zeta)f_0(\zeta) \tag{2.139}$$

Note that $s(\zeta)$ is a rational function that may have poles at $\pm\alpha_0, \pm\alpha_1,$ and ∞, also, $f_0(\zeta) \neq 0$. Equation (2.139) indicates that zeroes of $f(\zeta)$ and $s(\zeta)$ are same. Equation (2.139) can be expressed as

$$s(\zeta) = f(\zeta)e^{-g(\zeta)}. \tag{2.140}$$

Analysis of the Cauchy integral (2.138) indicates that

$$-g(\zeta) \sim -\frac{1}{2\pi i} loga(\alpha_1) log(\zeta - \alpha_1), \text{ as } \zeta \to \alpha_1. \tag{2.141}$$

Now we consider the following cases:

a. $k = 0$

$$loga(t) = log\frac{\phi(t) + \frac{\pi i}{2}}{\phi(t) - \frac{\pi i}{2}} \quad \forall t \quad \in \Gamma_1 = (-\alpha_1, -\alpha_0) \cup (\alpha_0, \alpha_1) \tag{2.142}$$

Using relations defined by (2.127), (2.128) and (2.142), we notice that $loga(t) \sim 2\pi i,$ as $t \to \alpha_1$, hence relation (2.141) simplifies to

$$-g(\zeta) \sim -log(\zeta - \alpha_1), \text{ as } \zeta \to \alpha_1. \tag{2.143}$$

Relations (2.140) and (2.143) give the following result.

$$s(\zeta) \sim \text{constant}\frac{1}{\zeta - \alpha_1}, \text{ as } \zeta \to \alpha_1. \tag{2.144}$$

Since $loga(t) = 0$ for $t \to \pm\alpha_0, -\alpha_1$, using lemma 2.3.2 and equation (2.140), it is observed that $s(\zeta)$ has removable singularities at the points $\pm\alpha_0,$ and $-\alpha_1$. Note that $f(\zeta) \sim -\zeta$ as $\zeta \to \infty$, this implies that $s(\zeta) \sim -\zeta$ as $\zeta \to \infty$. (because $f(\zeta)$ and $s(\zeta)$ have same behavior as $\zeta \to \infty$.) Hence, application

of generalized Liouville's theorem 2.3.2 gives the following result.

$$\begin{aligned}
s(\zeta) &= -\zeta + \frac{c}{\zeta - \alpha_1} + c_1 \\
s(\zeta) &= -\zeta + \frac{c}{\zeta(1 - \frac{\alpha_1}{\zeta})} + c_1 \\
s(\zeta) &= -\zeta + \frac{c}{\zeta}[1 + \frac{\alpha_1}{\zeta} + \frac{\alpha_1^2}{\zeta^2} + \frac{\alpha_1^3}{\zeta^3} + \cdots] + c_1 \text{ as } \zeta \to \infty \\
s(\zeta) &= -\zeta + \frac{c}{\zeta} + c_1 + O(\frac{1}{\zeta^2}), \text{ as } \zeta \to \infty.
\end{aligned} \tag{2.145}$$

Equation (2.145) can be written as

$$s(\zeta) \sim -\zeta + \frac{c}{\zeta} + c_1, \text{ as } \zeta \to \infty. \tag{2.146}$$

Now, we judge the behavior of $f(\zeta)$ at ∞ , for $k = 0$ equation (2.113) can be expressed as

$$\begin{aligned}
f(\zeta) &= \frac{1}{2} log \frac{(1 - \frac{\alpha_0}{\zeta})(1 - \frac{\alpha_1}{\zeta})}{(1 + \frac{\alpha_0}{\zeta})(1 + \frac{\alpha_1}{\zeta}}) - \zeta \\
f(\zeta) &= \frac{1}{2}[log(1 - \frac{\alpha_0}{\zeta}) + log(1 - \frac{\alpha_1}{\zeta}) - log(1 + \frac{\alpha_0}{\zeta}) - log(1 + \frac{\alpha_1}{\zeta})] - \zeta \\
log(1 + \frac{\alpha_0}{\zeta}) &= \frac{\alpha_0}{\zeta} - \frac{\alpha_0^2}{2\zeta^2} + \frac{\alpha_0^3}{3\zeta^3} - \frac{\alpha_0^4}{4\zeta^4} + ... \text{ use this relation to get} \\
f(\zeta) &= \frac{1}{2}[(-\frac{\alpha_0}{\zeta} - \frac{\alpha_0^2}{2\zeta^2} - \frac{\alpha_0^3}{3\zeta^3} - ...) + (-\frac{\alpha_1}{\zeta} - \frac{\alpha_1^2}{2\zeta^2} - \frac{\alpha_1^3}{3\zeta^3} - ...) - \\
&\quad (\frac{\alpha_0}{\zeta} - \frac{\alpha_0^2}{2\zeta^2} + \frac{\alpha_0^3}{3\zeta^3} - \cdots) - (\frac{\alpha_1}{\zeta} - \frac{\alpha_1^2}{2\zeta^2} + \frac{\alpha_1^3}{3\zeta^3} - ...)] - \zeta, \text{ as } \zeta \to \infty.
\end{aligned} \tag{2.147}$$

Equation (2.147) can be expressed as

$$f(\zeta) = \frac{1}{\zeta}(-\alpha_0 - \alpha_1) + O(\frac{1}{\zeta^2}) - \zeta, \text{ as } \zeta \to \infty. \tag{2.148}$$

From equation (2.138), the Cauchy integral becomes

$$\begin{aligned}
-g(\zeta) &= -\frac{1}{2\pi i} \int_\Gamma \frac{loga(t)}{t - \zeta} dt, \quad t \in \Gamma = \Gamma_0 \cup \Gamma_1. \\
-g(\zeta) &= \frac{1}{2\zeta\pi i} \int_\Gamma loga(t)(1 - \frac{t}{\zeta})^{-1} dt = \sum_{n=0}^{\infty} \frac{1}{\zeta^{n+1}} \int_\Gamma loga(t) t^n dt.
\end{aligned} \tag{2.149}$$

Equation (2.149) can be expressed as

$$-g(\zeta)=\sum_{n=0}^{\infty}\frac{m_n}{\zeta^{n+1}},\quad m_n=\int_{\Gamma} loga(t)t^n dt$$

$e^{-g(\zeta)}$ can be expressed in series representation as

$$e^{-g(\zeta)}=e^{\sum_{n=0}^{\infty}\frac{m_n}{\zeta^{n+1}}}=e^{\frac{m_0}{\zeta}+\frac{m_1}{\zeta^2}+\cdots} \tag{2.150}$$

$$e^{-g(\zeta)}=1+\frac{m_0}{\zeta}+\frac{m_1}{\zeta^2}+\frac{m_2}{\zeta^3}+......+\frac{(\frac{m_0}{\zeta}+\frac{m_1}{\zeta^2}+\frac{m_2}{\zeta^3}+\cdots)^2}{2!}+....$$

$$e^{-g(\zeta)}=1+\frac{m_0}{\zeta}+\frac{m_1}{\zeta^2}+\frac{m_0^2}{2\zeta^2}+O(\frac{1}{\zeta^3}),\text{ as }\zeta\to\infty.$$

Substitute the values of $e^{-g(\zeta)}$ and $f(\zeta)$ from equations (2.150) and (2.148) in equation (2.140), to get

$$s(\zeta)=[\frac{1}{\zeta}(-\alpha_0-\alpha_1)+O(\frac{1}{\zeta^2})-\zeta][1+\frac{m_0}{\zeta}+\frac{m_1}{\zeta^2}+\frac{m_0^2}{2\zeta^2}+O(\frac{1}{\zeta^3})],$$

$$\text{as }\zeta\to\infty.\text{ Simplification gives} \tag{2.151}$$

$$s(\zeta)=[\frac{1}{\zeta}(-\alpha_0-\alpha_1)-\zeta-m_0-\frac{m_1}{\zeta}-\frac{m_0^2}{2\zeta}+O(\frac{1}{\zeta^2})],\text{ as }\zeta\to\infty.$$

Compare the expressions for $s(\zeta)$ from equations (2.145) and (2.151) to get $c=-\alpha_0-\alpha_1-m_1-\frac{m_0^2}{2},\quad c_1=-m_0$. Hence relation (2.146) becomes

$$s(\zeta)=-\zeta+\frac{c}{\zeta}-m_0,\quad c=-\alpha_0-\alpha_1-m_1-\frac{m_0^2}{2}. \tag{2.152}$$

Note that $s(\zeta)=0\Leftrightarrow\zeta^2+m_0\zeta-c=0$. Solve this equation to get two zeroes of the transcendental equation (2.108).

b. $k\neq 0$.

Without loss of generality, suppose that $k>0$. Consider $t\in\Gamma_1$, then

$$\log a(t)=\log|a(t)|+i\text{Arg a(t)},\text{ use equation (2.129) to get}$$

$$\log a(t)=\log|\frac{\phi(t)+\pi i(k+\frac{1}{2})}{\phi(t)+\pi i(k-\frac{1}{2})}|+i\text{Arg}[\frac{\phi(t)+\pi i(k+\frac{1}{2})}{\phi(t)+\pi i(k-\frac{1}{2})}]. \tag{2.153}$$

Equation (2.153) can be simplified to

$$\log a(t) = \frac{1}{2}\log[\frac{\phi(t)^2+\pi^2(k+\frac{1}{2})^2}{\phi(t)^2+\pi^2(k-\frac{1}{2})^2}] + i\mathrm{Arg}[\frac{\phi(t)^2+\phi(t)\pi i+\pi^2(k^2-\frac{1}{4})}{\phi(t)^2+\pi^2(k-\frac{1}{2})^2}]$$
$$\log a(t) = \frac{1}{2}\log[\frac{\phi(t)^2+\pi^2(k+\frac{1}{2})^2}{\phi(t)^2+\pi^2(k-\frac{1}{2})^2}] + i\arctan[\frac{\phi(t)\pi}{\phi(t)^2+\pi^2(k^2-\frac{1}{4})}]. \tag{2.154}$$

From relation (2.128), $\phi(t) \to +\infty,$ as $t \to -\alpha_0$ or $t \to -\alpha_1$. Hence equation (2.154) indicates that $\log a(t) \to 0$ when $\phi(t) \to +\infty,$ as $t \to -\alpha_0$ or $t \to -\alpha_1$. Now using the relations defined by (2.127) and (2.154), we have $\log a(t) \to 0$ when $\phi(t) \to -\infty,$ as $t \to \alpha_0$ or $t \to \alpha_1$. Similarly, we can show that, when $t \in \Gamma_0$, and $\zeta \to \pm\alpha_0,$ or $\pm\alpha_1,$ where $\zeta \in \mathbb{C}\setminus\Gamma$, then $\log a(t) \to 0$. Table 2.2 shows the similar results. Equation (2.111) can be expressed as

$$f(\zeta) = -\zeta + \pi ik + O(\frac{1}{\zeta}), \text{ as } \zeta \to \infty. \tag{2.155}$$

Equation (2.155) indicates that $f(\zeta)$ has a pole at $\zeta = \infty$. So, we concude that $s(\zeta)$ has a pole at ∞. ($f(\zeta)$ and $s(\zeta)$ have same behaviour as $\zeta \to \infty$.) We have already shown that $\log a(\zeta) \to 0,$ as $\zeta \to \pm\alpha_0,$ or $\pm\alpha_1,$ where $\zeta \in \mathbb{C}\setminus\Gamma$.

$$\text{Since } g(\zeta) = \frac{1}{2\pi i}\int_\Gamma \frac{loga(\tau)}{\tau-\zeta}d\tau, \quad \zeta \in \mathbb{C}\setminus\Gamma, \quad \Gamma = \Gamma_0 \cup \Gamma_1 \tag{2.156}$$

Equation (2.156) indicates that $g(\zeta)$ is vanishing at the end points $\pm\alpha_0, \pm\alpha_1$. Hence, $[f_0(\zeta)]^{-1} = e^{-g(\zeta)}$ is bounded at the end points $\pm\alpha_0, \pm\alpha_1$. So, from equation (2.140) $s(\zeta) = f(\zeta)e^{-g(\zeta)}$, we conclude that $s(\zeta)$ is bounded at the points $\pm\alpha_0, \pm\alpha_1$. (This means that $s(\zeta)$ has removable singularities at the points $\pm\alpha_0, \pm\alpha_1$.) Since $s(\zeta)$ has a pole at ∞, and is bounded at the points $\pm\alpha_0, \pm\alpha_1$. So, application of the generalized Liouville's theorem 2.3.2 gives

$$s(\zeta) = a\zeta + b. \tag{2.157}$$

From equation (2.150), $e^{-g(\zeta)}$ can be expressed as

$$e^{-g(\zeta)} = 1 + \frac{m_0}{\zeta} + O(\frac{1}{\zeta^2}). \tag{2.158}$$

From equations (2.158) and (2.155) use values of $e^{-g(\zeta)}$ and $f(\zeta)$ in equation (2.140) to get

$$\begin{aligned} s(\zeta) &= [-\zeta + \pi ik + O(\frac{1}{\zeta})][1 + \frac{m_0}{\zeta} + O(\frac{1}{\zeta^2})] \\ s(\zeta) &= -\zeta + \pi ik - m_0 + O(\frac{1}{\zeta}). \end{aligned} \tag{2.159}$$

Compare equations (2.157) and (2.159) to get $a = -1, \quad b = \pi ik - m_0$. Hence equation (2.157) becomes

$$\begin{aligned} &s(\zeta) = -\zeta + \pi ik - m_0, \quad m_0 = \frac{1}{2\pi i}\int_\Gamma loga(t)dt, \\ &m_0 = m_0(k), \quad k = 1, 2, 3, \cdots \end{aligned} \tag{2.160}$$

Using the Burniston-Siewert method, we have calculated the zeroes of the transcendental equation (2.108) for $\mu_0 = 2, \mu_1 = 3, a = 5$. These are shown in the tables 2.3 and 2.4.

TABLE 2.3: Zeroes of transcendental equation (2.108) when $k \geq 0$.

k	λ
0	0.5399633576
1	1.087479972
2	1.646783217
3	2.801213276
4	3.392922352
5	3.991706387

Continuation of Table 2.3	
k	λ
6	4.595979567
7	5.204505695
8	5.816345
9	6.430788563
10	7.047300816
11	7.665474543
12	8.284997332
13	8.905627065
14	9.527174117
15	10.14948842
16	10.77245001
17	11.39596204
18	12.01994564
19	12.64433599
20	13.26907944
21	13.89413119
22	14.5194536
23	15.14501483
24	15.77078779
25	16.39674927
26	17.02287932
27	17.64916065
28	18.27557826

Continuation of Table 2.3	
k	λ
29	18.90211903
30	19.52877147
31	20.15552546
32	20.78237206
33	21.40930335
34	22.03631226
35	22.6633925
36	23.29053843
37	23.91774498
38	24.54500761
39	25.17232219
40	25.799685
41	26.42709267
42	27.05454211
43	27.68203054
44	28.3095554
45	28.93711435
46	29.56470525
47	30.19232613
48	30.81997518
49	31.44765074
50	32.07535127
51	32.70307534

Continuation of Table 2.3	
k	λ
52	33.33082164
53	33.95858895
54	34.58637613
55	35.21418214
56	35.84200598
57	36.46984676
58	37.09770361
59	37.72557574
60	38.35346241

TABLE 2.4: Zeroes of transcendental equation (2.108) when $k \leq 0$.

k	λ
0	-0.5399633576
-1	-1.087479972
-2	-1.646783217
-3	-2.801213276
-4	-3.392922352
-5	-3.991706387
-6	-4.595979567
-7	-5.204505695
-8	-5.816345
-9	-6.430788563

Continuation of Table 2.4	
k	λ
-10	-7.047300816
-11	-7.665474543
-12	-8.284997332
-13	-8.905627065
-14	-9.527174117
-15	-10.14948842
-16	-10.77245001
-17	-11.39596204
-18	-12.01994564
-19	-12.64433599
-20	-13.26907944
-21	-13.89413119
-22	-14.5194536
-23	-15.14501483
-24	-15.77078779
-25	-16.39674927
-26	-17.02287932
-27	-17.64916065
-28	-18.27557826
-29	-18.90211903
-30	-19.52877147
-31	-20.15552546
-32	-20.78237206

Continuation of Table 2.4	
k	λ
-33	-21.40930335
-34	-22.03631226
-35	-22.6633925
-36	-23.29053843
-37	-23.91774498
-38	-24.54500761
-39	-25.17232219
-40	-25.799685
-41	-26.42709267
-42	-27.05454211
-43	-27.68203054
-44	-28.3095554
-45	-28.93711435
-46	-29.56470525
-47	-30.19232613
-48	-30.81997518
-49	-31.44765074
-50	-32.07535127
-51	-32.70307534
-52	-33.33082164
-53	-33.95858895
-54	-34.58637613
-55	-35.21418214

Continuation of Table 2.4	
k	λ
-56	-35.84200598
-57	-36.46984676
-58	-37.09770361
-59	-37.72557574
-60	-38.35346241

Mathematica programming is used to find the zeroes of the transcendental equation (2.108). These values are verified by applying the fixed point iteration method to the transcendental equation (2.108), and using matlab programming. Burniston-Siewert method gives numerical values for zeroes of the transcendental equation (2.108) along with closed form expressions for zeroes of the transcendental equation (2.108). It is not possible to get closed form expressions for zeroes of the transcendental equation (2.108) by applying any numerical technique.

Chapter 3

Riemann-Hilbert problem approach for Helmholtz equation in a semi-infinite strip

3.1 Preliminaries

Definition 3.1.1. *Let L_x denotes the Laplace transform operator w.r.t. $x \in (0, \infty)$. If $u(.,y) \in L^1(\mathbb{R}^+)$, then the Laplace transform of $u(x,y)$ w.r.t $x \in (0,\infty)$ is denoted by $\tilde{u}(\eta, y)$ and is defined by $\tilde{u}(\eta, y) = L_x[u(x,y)] = \int_0^\infty u(x,y)e^{i\eta x}dx, \eta \in \mathbb{C}$. Note that $\tilde{u}(\eta, y) \to 0$ as $\eta \to \infty$. If the inverse Laplace transform operator is denoted by L_x^{-1}, and $u(x,y)$ is continuous w.r.t. x on each finite interval $(0, A)$, $0 < A < \infty$, then the inverse Laplace transform of $\tilde{u}(\eta, y)$, is defined by*

$$u(x,y) = L_x^{-1}[\tilde{u}(\eta, y)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{u}(\eta, y)e^{-i\eta x}d\eta$$

Definition 3.1.2. *Let L_y denotes the Laplace transform operator w.r.t. $y \in (0, a), a > 0$. If $u(x, .) \in L^1(0, a)$, and $u(x,y) = 0, \forall y > a$, then the Laplace transform of $u(x,y)$ w.r.t $y \in (0,a)$ is denoted by $\hat{u}(x, i\lambda)$ and is defined by $\hat{u}(x, i\lambda) = L_y[u(x,y)] = \int_0^a u(x,y)e^{-\lambda y}dy, \lambda \in \mathbb{C}$. Note that $\hat{u}(x, i\lambda) \to 0$ as $\lambda \to \infty$. If the*

inverse Laplace transform operator is denoted by L_y^{-1}, *and* $u(x,y)$ *is continuous w.r.t.* y *on each finite interval* $(0,A)$, $0 < A < \infty$, *then the inverse Laplace transform of* $\hat{u}(x,i\lambda)$, *is defined by* $u(x,y) = L_y^{-1}[\hat{u}(x,i\lambda)] = \frac{1}{2\pi i}\int_\Gamma \hat{u}(x,i\lambda)e^{\lambda y}d\lambda$. *Note that* $\Gamma = (c - i\infty, c + i\infty)$, *where* $c = Re(\lambda)$ *and* $u(x,.) \in L^1(0,a)$, *and satisfies* $\int_o^\infty e^{-cx}|u(x,y)|dy < \infty$. Γ *is referred to as the Bromwich contour, and c is taken to the right of all the singularities in order to satisfy the above condition.*

Definition 3.1.3. *Let* L *be a smooth closed or open contour in the complex plane. If* L *is a positively oriented closed contour then* L *divides the complex plane in two parts namely* D^+ *and* D^-. *If* L *is an open contour then positive orientation of* L *means that if a person is walking on the contour* L *then* D^+ *is always on his left hand side while* D^- *is on his right hand side. Let* $G(t) = (G_{i,j})_{i,j}$ *be a non singular matrix on* L *and* $G_{i,j} \in H(L)$*(Holder continuous on* L*) and* $g(t) = (g_{i,j})_{i,j}$, $g_{i,j} \in H(L)$. *By the term vector RHP, we mean to find two vectors* $\Phi^+(z)$ *and* $\Phi^-(z)$ *analytic in* D^+ *and* D^- *respectively, such that their limiting values satisfy the boundary condition*

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad \forall t \in L.$$

Theorem 3.1.1. *[23] Let L be smooth positively oriented contour(open or closed), and $\phi(\tau)$ a function of position for all $\tau \in L$ which satisfies Holder's condition on L. If L is closed, then D^+ denotes domain interior to L, and D^- is complement to $D^+ + L$. If L is open, then D^+ is on the left side of L and D^- is on the right side of L as one walks along the contour L. Then the Cauchy type integral given by definition 2.3.3*

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\phi(\tau)}{\tau - z}d\tau, \quad z \in \mathbb{C} \setminus L \tag{3.1}$$

has limiting values $\Phi^+(t)$ and $\Phi^-(t)$ where

$$\Phi^+(t) = \lim_{z\to t}\Phi(z), \quad z \in D^+, t \in L, \text{ and } t \text{ is not an end point of } L, \tag{3.2}$$

$$\Phi^-(t) = \lim_{z\to t}\Phi(z), \quad z \in D^-, t \in L, \text{ and } t \text{ is not an end point of } L. \tag{3.3}$$

$\Phi^+(t)$ and $\Phi^-(t)$ are related to the singular integral $\frac{1}{2\pi i}\int_L \frac{\phi(\tau)}{\tau - t}d\tau, \quad t \in L$, and the density $\phi(t)$ of the singular integral through following relations:

$$\Phi^+(t) = \frac{1}{2}\phi(t) + \frac{1}{2\pi i}\int_L \frac{\phi(\tau)}{\tau - t}d\tau, \quad t \in L, \tag{3.4}$$

$$\Phi^-(t) = -\frac{1}{2}\phi(t) + \frac{1}{2\pi i}\int_L \frac{\phi(\tau)}{\tau - t}d\tau, \quad t \in L. \tag{3.5}$$

These are called the Sokhotski Plemelj formulae. By adding and subtracting equations (3.4) and (3.5), we get another form of the Sokhotski Plemelj formulas:

$$\Phi^+(t) + \Phi^-(t) = \frac{1}{\pi i}\int_L \frac{\phi(\tau)}{\tau - t} d\tau, \quad t \in L, \tag{3.6}$$

$$\Phi^+(t) - \Phi^-(t) = \phi(t), \quad t \in L. \tag{3.7}$$

3.2 Helmholtz equation in a semi-infinite strip subject to the Poincare type boundary conditions

Consider the Helmholtz equation

$$(\partial_x^2 + \partial_y^2 + k^2)q(x, y) = g(x, y), \quad Im(k) > 0, \quad (x, y) \in \Omega, \tag{3.8}$$

where Ω is a semi-infinite strip shown in figure 3.1 , with corners $z_1 = \infty$, $z_2 = 0$, $z_3 = ia$, $z_4 = \infty + ia$, $a > 0$. Figure 3.1 shows Poincare type boundary conditions along three sides of Ω .

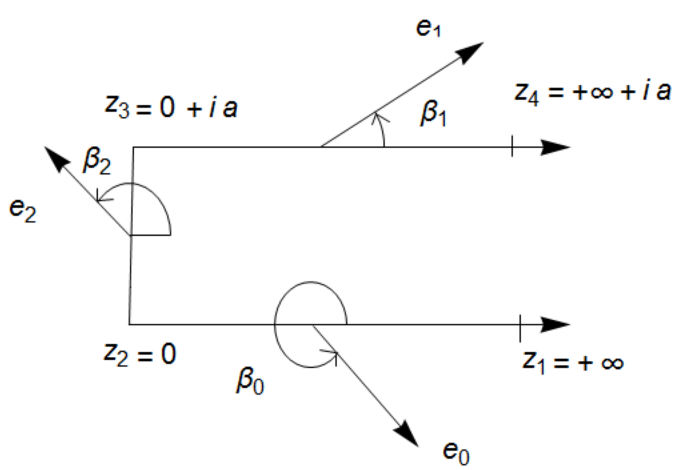

FIGURE 3.1. Poincare boundary conditions along sides of Ω.

The Poincare type boundary conditions are

$$\left.\frac{\partial q}{\partial \nu}\right|_{e_j} + \mu_j q = g_j, \tag{3.9}$$

where, for $j = 0, 1, 2$, $\left.\frac{\partial q}{\partial \nu}\right|_{e_j} = \bigtriangledown q \cdot e_j$ is the directional derivative in direction e_j specified by constant β_j, $\quad j = 0, 1, 2$ where, $(0 < \beta_1 < \pi, \quad \frac{\pi}{2} < \beta_2 < 3\frac{\pi}{2}, \quad \pi < \beta_0 < 2\pi)$, μ_j is a real non negative constant, and g_j is a real valued function with appropriate smoothness and decay. The boundary conditions in equation (3.9) can be written as:

$$side1 : \cos\beta_0 q_x + \sin\beta_0 q_y + \mu_0 q = g_0(x), \quad 0 < x < \infty, \quad y = 0, \tag{3.10}$$

$$side2 : \cos\beta_2 q_y + \sin\beta_2 q_x + \mu_2 q = g_2(y), \quad x = 0, \quad 0 < y < a, \tag{3.11}$$

$$side3 : \cos\beta_1 q_x + \sin\beta_1 q_y + \mu_1 q = g_1(x), \quad 0 < x < \infty, \quad y = a. \tag{3.12}$$

The functions $g_1(x)$, $g_3(x)$ vanish at the points $x = 0$ and $x = \infty$, $\sin\beta_j \neq 0$, $\quad j = 0, 1, 2$. Apply the the operator L_x from definition 3.1.1 to equation (3.8), to get

$$\int_0^\infty \partial_x^2 q(x,y) e^{i\eta x} dx + \partial_y^2 \int_0^\infty q(x,y) e^{i\eta x} dx + k^2 \int_0^\infty q(x,y) e^{i\eta x} dx = \int_0^\infty g(x,y) e^{i\eta x} dx. \tag{3.13}$$

Evaluate $\int_0^\infty \partial_x^2 q(x,y) e^{i\eta x} dx$.

$$\begin{aligned}
&\int_0^\infty \partial_x^2 q(x,y) e^{i\eta x} dx = e^{i\eta x} \partial_x q(x,y)|_{x=0}^{\infty} - \int_0^\infty \partial_x q(x,y) e^{i\eta x} (i\eta) dx, \\
&q(x,y) \in C^1(\overline{\Omega}) \cap C^2(\Omega) \text{ and } q(x,y)|_{x=\infty} = \partial_x q(x,y)|_{x=\infty} = 0, \\
&\qquad = -\partial_x q(0,y) - i\eta \int_0^\infty \partial_x q(x,y) e^{i\eta x} dx \text{ integrate by parts} \\
&\qquad = -\partial_x q(0,y) - i\eta [e^{i\eta x} q(x,y)|_{x=0}^{\infty} - \int_0^\infty q(x,y) i\eta e^{i\eta x} dx]
\end{aligned} \tag{3.14}$$

Use the property $q(x,y) \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ and $q(x,y)|_{x=\infty} = \partial_x q(x,y)|_{x=\infty} = 0$ and definition 3.1.1 in equation (3.14), to get

$$\int_0^\infty \partial_x^2 q(x,y) e^{i\eta x} dx = -\partial_x q(0,y) + i\eta q(0,y) - \eta^2 \int_0^\infty q(x,y) e^{i\eta x} dx. \tag{3.15}$$

Insert the value of $\int_0^\infty \partial_x^2 q(x,y)e^{i\eta x}dx$ in equation (3.8), and simplify to get

$$(\frac{d^2}{dy^2} - \zeta^2)\tilde{q}(\eta,y) = \frac{\partial}{\partial x}q(0,y) - i\eta q(0,y) + \tilde{g}(\eta,y). \tag{3.16}$$

Note that $\zeta = \sqrt{\eta^2 - k^2}$ is a multi-valued function. We fix a branch of it by $Re(\zeta) \geq 0$, and $\pm k$ are the branch points of this multi-valued function. The branch cut of this multi-valued function is shown in figure 3.2. Write equation (3.16) as

$$(\frac{d^2}{dy^2} - \zeta^2)\tilde{q}(\eta,y) = f(y), \quad f(y) = \partial_x q(0,y) - i\eta q(0,y) + \tilde{g}(\eta,y). \tag{3.17}$$

From definition 3.1.1 apply the the operator L_x to equation (3.10) to get

$$\int_0^\infty g_0(x)e^{i\eta x}dx = \cos\beta_0 \int_0^\infty \partial_x q(x,y)e^{i\eta x}dx + \sin\beta_0 \frac{d}{dy}\int_0^\infty q(x,y)e^{i\eta x}dx + \\ \mu_0 \int_0^\infty q(x,y)e^{i\eta x}dx, \quad 0 < x < \infty, \quad y = 0. \tag{3.18}$$

Integrate by parts and use definition 3.1.1

$$\tilde{g}_0(\eta) = \cos\beta_0[q(x,y)e^{i\eta x}|_{x=0}^{\infty} - \int_0^\infty q(x,y)e^{i\eta x}(i\eta x)dx] + \sin\beta_0 \frac{d}{dy}\tilde{q}(\eta,y) + \\ \mu_0\tilde{q}(\eta,y), \quad 0 < x < \infty, \quad y = 0.$$

Use the property $q(x,y) \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ and $q(x,y)|_{x=\infty} = \partial_x q(x,y)|_{x=\infty} = 0$, and simplify to get

$$\sin\beta_0 \frac{d}{dy}\tilde{q}(\eta,0) + (\mu_0 - i\eta\cos\beta_0)\tilde{q}(\eta,0) = \tilde{g}_0(\eta) + q(0,0)\cos\beta_0. \tag{3.19}$$

From definition 3.1.1 apply the the operator L_x to equation (3.12) to get

$$\int_0^\infty g_1(x)e^{i\eta x}dx = \cos\beta_1 \int_0^\infty \partial_x q(x,y)e^{i\eta x}dx + \sin\beta_1 \frac{d}{dy}\int_0^\infty q(x,y)e^{i\eta x}dx + \mu_1 \int_0^\infty q(x,y)e^{i\eta x}dx, \quad 0 < x < \infty, \quad y = a.$$

Integrate by parts and use definition 3.1.1

$$\tilde{g}_1(\eta) = \cos\beta_1[q(x,y)e^{i\eta x}|_{x=0}^{\infty} - \int_0^\infty q(x,y)e^{i\eta x}(i\eta)dx] + \sin\beta_1 \frac{d}{dy}\tilde{q}(\eta,y) + \mu_1\tilde{q}(\eta,y), \quad 0 < x < \infty, \quad y = a.$$

Use the property $q(x,y) \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ and $q(x,y)|_{x=\infty} = \partial_x q(x,y)|_{x=\infty} = 0$,

and simplify to get

$$\sin\beta_1 \frac{d}{dy}\tilde{q}(\eta,a) + (\mu_1 - i\eta\cos\beta_1)\tilde{q}(\eta,a) = \tilde{g}_1(\eta) + q(0,a)\cos\beta_1. \tag{3.20}$$

Now we define the functionals of the boundary conditions W_0 and W_1 as follows:

$$\begin{aligned} W_0[F(y)] &= \frac{d}{dy}F|_{y=0}\sin\beta_0 + F|_{y=0}(\mu_0 - i\eta\cos\beta_0), \\ W_1[F(y)] &= \frac{d}{dy}F|_{y=a}\sin\beta_1 + F|_{y=a}(\mu_1 - i\eta\cos\beta_1). \end{aligned} \tag{3.21}$$

Consider the homogeneous system of the Laplace transformed equations obtained

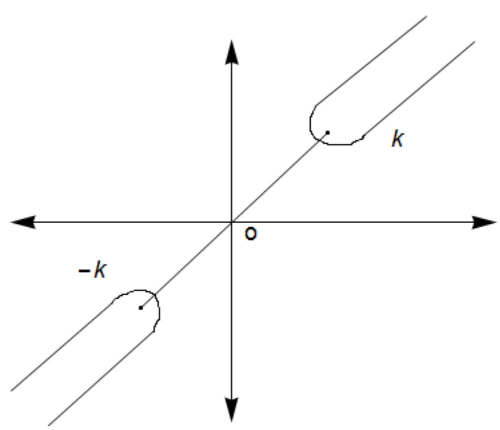

FIGURE 3.2. A branch cut for the multi-valued function ζ

from the above steps.

$$(\frac{d^2}{dy^2} - \zeta^2)\tilde{q}(\eta, y) = 0 \tag{3.22}$$

$$\sin\beta_0 \frac{d}{dy}\tilde{q}(\eta, 0) + (\mu_0 - i\eta\cos\beta_0)\tilde{q}(\eta, 0) = 0 \tag{3.23}$$

$$\sin\beta_1 \frac{d}{dy}\tilde{q}(\eta, a) + (\mu_1 - i\eta\cos\beta_1)\tilde{q}(\eta, a) = 0 \tag{3.24}$$

Green's function of the system defined by equations (3.22), (3.23) and (3.24) is

$$G(y, \xi) = \psi(y, \xi) - \sum_{j=0}^{1} W_j[\psi(y, \xi)]\psi_j(y), \tag{3.25}$$

where $\psi(y, \xi)$ is the fundamental function of the second order linear differential the operator $L = \frac{d^2}{dy^2} - \zeta^2$, and is defined by

$$\psi(y, \xi) = -\frac{1}{2\zeta}e^{-\zeta|y-\xi|}. \tag{3.26}$$

Note that $\psi_0(y)$ and $\psi_1(y)$ are the basis functions of the the operator L and satisfies the following properties: ($\{\psi_0(y), \psi_1(y)\}$ forms a basis of the solution space of the the operator L.)

$$L(\psi_j) = 0, \quad j = 0, 1 \text{ and } W_j[\psi_l] = \delta_{jl} \quad j, l = 0, 1. \tag{3.27}$$

$\psi_0(y), \psi_1(y)$ and their derivative are given below

$$\psi_0(y) = c_{00}\cosh\zeta y + c_{01}\sinh\zeta y, \quad \psi_0'(y) = \zeta c_{00}\sinh\zeta y + \zeta c_{01}\cosh\zeta y, \tag{3.28}$$

$$\psi_1(y) = c_{10}\cosh\zeta y + c_{11}\sinh\zeta y, \quad \psi_1'(y) = \zeta c_{10}\sinh\zeta y + \zeta c_{11}\cosh\zeta y. \tag{3.29}$$

$$\text{Let } \alpha_{00} = \sin\beta_0, \quad \alpha_{01} = \mu_0 - i\eta\cos\beta_0, \quad q(0, 0) = q_0, \tag{3.30}$$

$$\alpha_{10} = \sin\beta_1, \quad \alpha_{11} = \mu_1 - i\eta\cos\beta_1, \quad q(0, a) = q_1. \tag{3.31}$$

Hence equations (3.19) and (3.20) become

$$\alpha_{00}\frac{d}{dy}\tilde{q}(\eta, 0) + \alpha_{01}\tilde{q}(\eta, 0) = \tilde{g}_0(\eta) + q_0\cos\beta_0, \tag{3.32}$$

$$\alpha_{10}\frac{d}{dy}\tilde{q}(\eta, a) + \alpha_{11}\tilde{q}(\eta, a) = \tilde{g}_1(\eta) + q_1\cos\beta_1. \tag{3.33}$$

Apply the functionals of boundary condition W_0 and W_1 to $\psi_0(y)$.

$$W_0[\psi_0(y)] = \alpha_{00}\frac{d}{dy}\psi_0(y)|_{y=0} + \alpha_{01}\psi_0(y)|_{y=0} \tag{3.34}$$

Use equations (3.27) and (3.28 in the above equation to get

$$1 = \alpha_{00}\zeta c_{01} + \alpha_{01}c_{00}. \tag{3.35}$$

$$W_1[\psi_0(y)] = \alpha_{10}\frac{d}{dy}\psi_0(y)|_{y=a} + \alpha_{11}\psi_0(y)|_{y=a} \tag{3.36}$$

Use equations (3.27) and (3.28 in the above equation to get

$$0 = (\zeta\alpha_{10}\sinh\zeta a + \alpha_{11}\cosh\zeta a)c_{00} + (\zeta\alpha_{10}\cosh\zeta a + \alpha_{11}\sinh\zeta a)c_{01}. \tag{3.37}$$

Solve equations (3.35) and (3.37) to find values of c_{00} and c_{01}

$$c_{00} = -\frac{(\alpha_{10}\zeta\cosh\zeta a + \alpha_{11}\sinh\zeta a)}{d}, \tag{3.38}$$

$$c_{01} = \frac{\alpha_{10}\zeta\sinh\zeta a + \alpha_{11}\cosh\zeta a}{d}, \tag{3.39}$$

$$d = (\zeta^2\alpha_{00}\alpha_{10} - \alpha_{01}\alpha_{11})\sinh\zeta a + (\zeta\alpha_{00}\alpha_{11} - \alpha_{01}\zeta\alpha_{10})\cosh\zeta a. \tag{3.40}$$

Similarly, apply the functionals of boundary condition W_0 and W_1 on $\psi_1(y)$, to get

$$0 = \alpha_{00}\zeta c_{11} + \alpha_{01}c_{10} \tag{3.41}$$

$$1 = (\zeta\alpha_{10}\cosh\zeta a + \alpha_{11}\sinh\zeta a)c_{11} + (\zeta\alpha_{10}\sinh\zeta a + \alpha_{11}\cosh\zeta a)c_{10} \tag{3.42}$$

Solve equations (3.41) and (3.42) to find the values of c_{10} and c_{11}

$$c_{11} = -\frac{\alpha_{01}}{d}, \qquad c_{10} = \frac{\alpha_{00}\zeta}{d}. \tag{3.43}$$

Insert the values of c_{00}, c_{01}, c_{10} and c_{11} in equations (3.28) and (3.29) to get $\psi_0(y)$ and $\psi_1(y)$ given by

$$\psi_0(y) = -\frac{\alpha_{10}\zeta\cosh[(a-y)\zeta] + \alpha_{11}\sinh[(a-y)\zeta]}{d}, \tag{3.44}$$

$$\psi_1(y) = \frac{\alpha_{00}\zeta\cosh[\zeta y] - \alpha_{01}\sinh[\zeta y]}{d}. \tag{3.45}$$

Now we find $W_0[\psi](\xi)$ and $W_1[\psi](\xi)$.

$$W_0[\psi](\xi) = \alpha_{00}\frac{\partial}{\partial y}\psi(y,\xi)|_{y=0} + \alpha_{01}\psi(y,\xi)|_{y=0} \tag{3.46}$$

Use equation (3.26) to find

$$\begin{aligned}
\frac{\partial}{\partial y}\psi(y,\xi)|_{y=0} &= -\frac{1}{2\zeta}e^{-\zeta|y-\xi|}(-\zeta)Sgn(y-\xi)|_{y=0}\\
\frac{\partial}{\partial y}\psi(y,\xi)|_{y=0} &= \frac{1}{2}e^{-\zeta\xi}Sgn(-\xi), \quad 0<\xi<a\\
\frac{\partial}{\partial y}\psi(y,\xi)|_{y=0} &= -\frac{1}{2}e^{-\zeta\xi}\\
\psi(y,\xi)|_{y=0} &= -\frac{1}{2\zeta}e^{-\zeta\xi}.
\end{aligned} \tag{3.47}$$

Insert the values of $\frac{\partial}{\partial y}\psi(y,\xi)|_{y=0}$ and $\psi(y,\xi)|_{y=0}$ in equation (3.46) to get

$$W_0[\psi](\xi) = -\frac{e^{-\zeta\xi}}{2}(\alpha_{00} + \frac{\alpha_{01}}{\zeta}). \tag{3.48}$$

Now consider

$$W_1[\psi](\xi) = \alpha_{10}\frac{\partial}{\partial y}\psi(y,\xi)|_{y=a} + \alpha_{11}\psi(y,\xi)|_{y=a}. \tag{3.49}$$

Use equation (3.26) to find

$$\begin{aligned}
\frac{\partial}{\partial y}\psi(y,\xi)|_{y=a} &= -\frac{1}{2\zeta}e^{-\zeta|y-\xi|}(-\zeta)Sgn(y-\xi)|_{y=a}\\
\frac{\partial}{\partial y}\psi(y,\xi)|_{y=a} &= \frac{1}{2}e^{-\zeta|a-\xi|}Sgn(a-\xi), \quad 0<\xi<a\\
\frac{\partial}{\partial y}\psi(y,\xi)|_{y=0} &= \frac{1}{2}e^{-\zeta(a-\xi)}\\
\psi(y,\xi)|_{y=a} &= -\frac{1}{2\zeta}e^{-\zeta(a-\xi)}
\end{aligned} \tag{3.50}$$

Insert the values of $\frac{\partial}{\partial y}\psi(y,\xi)|_{y=a}$ and $\psi(y,\xi)|_{y=a}$ in equation (3.49) to get

$$W_1[\psi](\xi) = \frac{e^{-\zeta(a-\xi)}}{2}(\alpha_{10} - \frac{\alpha_{11}}{\zeta}). \tag{3.51}$$

Take the values of $\psi(y,\xi), \psi_0(y), \psi_1(y), W_0[\psi](\xi)$, and $W_1[\psi](\xi)$ from equations (3.26), (3.44), (3.45), (3.48) and (3.49) respectively, and insert in equation (3.25),

to get

$$G(y,\xi) = \frac{-e^{-\zeta|y-\xi|}}{2\zeta} - \frac{e^{-\zeta\xi}}{2d\zeta}(\alpha_{00}\zeta + \alpha_{01})(\alpha_{10}\zeta\cosh[(a-y)\zeta] + \alpha_{11}\sinh[(a-y)\zeta])$$
$$- \frac{e^{-\zeta(a-\xi)}}{2d\zeta}(\alpha_{10}\zeta - \alpha_{11})(\alpha_{00}\zeta\cosh[\zeta y] - \alpha_{01}\sinh[\zeta y]). \tag{3.52}$$

Now consider the following non homogeneous BVP consisting of equations (3.16), (3.32) and (3.33):

$$\begin{aligned}
&(\frac{d^2}{dy^2} - \zeta^2)\tilde{q}(\eta,y) = \frac{\partial}{\partial x}q(0,y) - i\eta q(0,y) + \tilde{g}(\eta,y),\\
&\alpha_{00}\frac{d}{dy}\tilde{q}(\eta,0) + \alpha_{01}\tilde{q}(\eta,0) = \tilde{g}_0(\eta) + q_0\cos\beta_0,\\
&\alpha_{10}\frac{d}{dy}\tilde{q}(\eta,a) + \alpha_{11}\tilde{q}(\eta,a) = \tilde{g}_1(\eta) + q_1\cos\beta_1.
\end{aligned} \tag{3.53}$$

Solution of the non homogeneous BVP defined by equations labeled by (3.53) is

$$\tilde{q}(\eta,y) = \int_0^a G(y,\xi)f(\xi)d\xi + [\tilde{g}_0(\eta) + q_0\cos\beta_0]\psi_0(y) + [\tilde{g}_1(\eta) + q_1\cos\beta_1]\psi_1(y). \tag{3.54}$$

Insert $y = 0$ in equation (3.54), then from equations (3.52), (3.17), (3.44), and (3.45), find the values of $G(0,\xi), f(\xi), \psi_0(0)$ and $\psi_1(0)$, respectively, and insert these values in resulting equation to get

$$\tilde{q}(\eta,0) = \int_0^a G(0,\xi)f(\xi)d\xi + [\tilde{g}_0(\eta) + q_0\cos\beta_0]\psi_0(0) + [\tilde{g}_1(\eta) + q_1\cos\beta_1]\psi_1(0)$$
$$\tilde{q}(\eta,0) = \int_0^a [-\frac{1}{2\zeta}e^{-\zeta\xi} - \frac{e^{-\zeta\xi}(\alpha_{00}\zeta + \alpha_{01})}{2\zeta d}(\alpha_{10}\zeta\cosh[a\zeta] + \alpha_{11}\sinh[a\zeta])$$
$$- \frac{e^{-\zeta(a-\xi)}\alpha_{00}(\alpha_{10}\zeta - \alpha_{11})}{2d}] \times [\frac{\partial}{\partial x}q(0,\xi) - i\eta q(0,\xi) + \tilde{g}(\eta,\xi)]d\xi +$$
$$[\tilde{g}_0(\eta) + q_0\cos\beta_0](\frac{\alpha_{10}\zeta\cosh[a\zeta] + \alpha_{11}\sinh[a\zeta]}{-d})$$
$$+ [\tilde{g}_1(\eta) + q_1\cos\beta_1](\frac{\zeta\alpha_{00}}{d}). \tag{3.55}$$

Consider

$$\begin{aligned}
&\int_0^a [-\frac{1}{2\zeta}e^{-\zeta\xi} - \frac{e^{-\zeta\xi}}{2d\zeta}(\alpha_{00}\zeta+\alpha_{01})(\alpha_{10}\zeta\cosh[a\zeta]+\alpha_{11}\sinh[a\zeta])\\
&\times [\frac{\partial}{\partial x}q(0,\xi) - i\eta q(0,\xi)]d\xi\\
&= -\frac{1}{2\zeta}[1+\frac{(\alpha_{00}\zeta+\alpha_{01})(\alpha_{10}\zeta\cosh[a\zeta]+\alpha_{11}\sinh[a\zeta]}{d}]\\
&\times \int_0^a [e^{-\zeta\xi}\frac{\partial}{\partial x}q(0,\xi) - i\eta q(0,\xi)]d\xi \qquad (3.56)\\
&= -\frac{1}{2\zeta}[1+\frac{(\alpha_{00}\zeta+\alpha_{01})(\alpha_{10}\zeta\cosh[a\zeta]+\alpha_{11}\sinh[a\zeta]}{d}]\\
&\times (\frac{\partial}{\partial x} - i\eta)\int_0^a e^{i(i\zeta)\xi}q(0,\xi)d\xi \text{ use definition 3.1.2}\\
&= \Lambda_{11}(\zeta,\eta)(\frac{\partial}{\partial x} - i\eta)\hat{q}(0,i\zeta), \text{ where}
\end{aligned}$$

$$\Lambda_{11}(\zeta,\eta) = -\frac{1}{2\zeta}[1+\frac{(\alpha_{00}\zeta+\alpha_{01})(\alpha_{10}\zeta\cosh[a\zeta]+\alpha_{11}\sinh[a\zeta]}{d}]. \qquad (3.57)$$

Now consider

$$\begin{aligned}
&\int_0^a -\frac{e^{-\zeta(a-\xi)}}{2d}\alpha_{00}(\alpha_{10}\zeta-\alpha_{11}) \times [\frac{\partial}{\partial x}q(0,\xi) - i\eta q(0,\xi)]d\xi\\
&= -\frac{e^{-\zeta a}}{2d}\alpha_{00}(\alpha_{10}\zeta-\alpha_{11})(\frac{\partial}{\partial x} - i\eta)\int_0^a e^{\zeta\xi}q(0,\xi)d\xi \qquad (3.58)\\
&= -\frac{e^{-\zeta a}}{2d}\alpha_{00}(\alpha_{10}\zeta-\alpha_{11})(\frac{\partial}{\partial x} - i\eta)\int_0^a e^{i(-i\zeta)\xi}q(0,\xi)d\xi.
\end{aligned}$$

Use definition (3.1.2) in equation (3.58) to get

$$\begin{aligned}
&\int_0^a -\frac{e^{-\zeta(a-\xi)}}{2d}\alpha_{00}(\alpha_{10}\zeta-\alpha_{11}) \times [\frac{\partial}{\partial x}q(0,\xi) - i\eta q(0,\xi)]d\xi\\
&= \Lambda_{12}(\zeta,\eta)(\frac{\partial}{\partial x} - i\eta)\hat{q}(0,-i\zeta), \text{ where} \qquad (3.59)\\
&\Lambda_{12}(\zeta,\eta) = -\frac{e^{-\zeta a}}{2d}\alpha_{00}(\alpha_{10}\zeta-\alpha_{11}).
\end{aligned}$$

Use the integrals defined by equations (3.56) and (3.59) in equation (3.55), to get

$$\begin{aligned}
\tilde{q}(\eta,0) = {} & \Lambda_{11}(\zeta,\eta)\frac{\partial}{\partial x}\hat{q}(0,i\zeta) - i\eta\Lambda_{11}(\zeta,\eta)\hat{q}(0,i\zeta) + \Lambda_{12}(\zeta,\eta)\frac{\partial}{\partial x}\hat{q}(0,-i\zeta)\\
& - i\eta\Lambda_{12}(\zeta,\eta)\hat{q}(0,-i\zeta) + h_0(\zeta,\eta). \qquad (3.60)
\end{aligned}$$

Note that

$$
\begin{aligned}
h_0(\zeta,\eta) = & \int_0^a [-\frac{1}{2\zeta}e^{-\zeta\xi} - \frac{e^{-\zeta\xi}(\alpha_{00}\zeta+\alpha_{01})}{2\zeta d}(\alpha_{10}\zeta\cosh[a\zeta]+\alpha_{11}\sinh[a\zeta]) \\
& - \frac{e^{-\zeta(a-\xi)}\alpha_{00}(\alpha_{10}\zeta-\alpha_{11})}{2d}]\tilde{g}(\eta,\xi)d\xi + [\tilde{g}_1(\eta)+q_1\cos\beta_1](\frac{\zeta\alpha_{00}}{d}) \\
& + [\tilde{g}_0(\eta)+q_0\cos\beta_0](\frac{\alpha_{10}\zeta\cosh[a\zeta]+\alpha_{11}\sinh[a\zeta]}{-d}) \\
& = \Lambda_{11}(\zeta,\eta)\int_0^a e^{-\zeta\xi}\tilde{g}(\eta,\xi)d\xi + \Lambda_{12}(\zeta,\eta)\int_0^a e^{\zeta\xi}\tilde{g}(\eta,\xi)d\xi \\
& + [\tilde{g}_1(\eta)+q_1\cos\beta_1](\frac{\zeta\alpha_{00}}{d}) - [\tilde{g}_0(\eta)+q_0\cos\beta_0]\times \\
& \quad \frac{(\alpha_{10}\zeta\cosh[a\zeta]+\alpha_{11}\sinh[a\zeta])}{d} \\
& = \Lambda_{11}(\zeta,\eta)\hat{\tilde{g}}(\eta,i\zeta) + \Lambda_{12}(\zeta,\eta)\hat{\tilde{g}}(\eta,-i\zeta) + [\tilde{g}_1(\eta)+q_1\cos\beta_1](\frac{\zeta\alpha_{00}}{d}) \\
& - [\tilde{g}_0(\eta)+q_0\cos\beta_0]\frac{(\alpha_{10}\zeta\cosh[a\zeta]+\alpha_{11}\sinh[a\zeta]}{d}.
\end{aligned}
\tag{3.61}
$$

Replace η by $-\eta$ in equation (3.60) to get

$$
\begin{aligned}
\tilde{q}(-\eta,0) = & \Lambda_{11}(\zeta,-\eta)\frac{\partial}{\partial x}\hat{q}(0,i\zeta) + i\eta\Lambda_{11}(\zeta,-\eta)\hat{q}(0,i\zeta) + \Lambda_{12}(\zeta,-\eta)\frac{\partial}{\partial x}\hat{q}(0,-i\zeta) \\
& + i\eta\Lambda_{12}(\zeta,-\eta)\hat{q}(0,-i\zeta) + h_0(\zeta,-\eta).
\end{aligned}
\tag{3.62}
$$

Now insert $y=a$ in equation (3.54), then from equations (3.52), (3.17), (3.44), and (3.45), find the values of $G(0,\xi)$, $f(\xi)$, $\psi_0(0)$ and $\psi_1(0)$, respectively, and insert these values in the resulting equation to get

$$
\begin{aligned}
\tilde{q}(\eta,a) = & \int_0^a G(a,\xi)f(\xi)d\xi + [\tilde{g}_0(\eta)+q_0\cos\beta_0]\psi_0(a) + [\tilde{g}_1(\eta)+q_1\cos\beta_1]\psi_1(a) \\
\tilde{q}(\eta,a) = & \int_0^a [-\frac{1}{2\zeta}e^{-\zeta(a-\xi)} - \frac{e^{-\zeta(a-\xi)}}{2d\zeta}(\alpha_{10}\zeta-\alpha_{11})(\alpha_{00}\zeta\cosh[a\zeta]-\alpha_{01}\sinh[a\zeta]) \\
& - \frac{e^{-\zeta\xi}}{2d}\alpha_{10}(\zeta\alpha_{00}+\alpha_{01})] \times [\frac{\partial}{\partial x}q(0,\xi) - i\eta q(0,\xi) + \tilde{g}(\eta,\xi)]d\xi \\
& - [\tilde{g}_0(\eta)+q_0\cos\beta_0]\frac{\alpha_{10}\zeta}{d} + [\tilde{g}_1(\eta)+q_1\cos\beta_1]\frac{(\zeta\alpha_{00}\cosh[\zeta a]-\alpha_{01}\sinh[\zeta a])}{d}.
\end{aligned}
\tag{3.63}
$$

Consider

$$\begin{aligned}
&\int_0^a -\frac{e^{-\zeta\xi}}{2d}\alpha_{10}(\zeta\alpha_{00}+\alpha_{01}) \times [\frac{\partial}{\partial x}q(0,\xi) - i\eta q(0,\xi)]d\xi \\
&= \Lambda_{21}(\zeta,\eta)(\frac{\partial}{\partial x} - i\eta)\int_0^a e^{i(i\zeta)\xi}q(0,\xi)d\xi \text{ use definition 3.1.2} \\
&= \Lambda_{21}(\zeta,\eta)(\frac{\partial}{\partial x} - i\eta)\hat{q}(0,i\zeta) \\
&= \Lambda_{21}(\zeta,\eta)\frac{\partial}{\partial x}\hat{q}(0,i\zeta) - i\eta\Lambda_{21}(\zeta,\eta)\hat{q}(0,i\zeta), \text{ where} \\
&\Lambda_{21}(\zeta,\eta) = -\frac{1}{2d}\alpha_{10}(\zeta\alpha_{00}+\alpha_{01}).
\end{aligned} \tag{3.64}$$

Now consider

$$\begin{aligned}
&\int_0^a [-\frac{1}{2\zeta}e^{-\zeta(a-\xi)} - \frac{e^{-\zeta(a-\xi)}}{2d\zeta}(\alpha_{10}\zeta-\alpha_{11})(\alpha_{00}\zeta\cosh[a\zeta] - \alpha_{01}\sinh[a\zeta])] \\
&\quad \times [\frac{\partial}{\partial x}q(0,\xi) - i\eta q(0,\xi)]d\xi \\
&= \Lambda_{22}(\zeta,\eta)(\frac{\partial}{\partial x} - i\eta)\int_0^a e^{i(-i\zeta)\xi}q(0,\xi)d\xi \text{ use definition3.1.2} \\
&= \Lambda_{22}(\zeta,\eta)(\frac{\partial}{\partial x} - i\eta)\hat{q}(0,-i\zeta) \\
&= \Lambda_{22}(\zeta,\eta)\frac{\partial}{\partial x}\hat{q}(0,-i\zeta) - i\eta\Lambda_{22}(\zeta,\eta)\hat{q}(0,-i\zeta) \\
&\Lambda_{22}(\zeta,\eta) = -\frac{1}{2\zeta}e^{-\zeta a}[1 + \frac{(\alpha_{10}\zeta-\alpha_{11})}{d}(\alpha_{00}\zeta\cosh[a\zeta] - \alpha_{01}\sinh[a\zeta])].
\end{aligned} \tag{3.65}$$

Use the integrals defined by equations (3.64) and (3.65) in equation (3.63), to get

$$\begin{aligned}
\tilde{q}(\eta,a) &= \Lambda_{21}(\zeta,\eta)\frac{\partial}{\partial x}\hat{q}(0,i\zeta) - i\eta\Lambda_{21}(\zeta,\eta)\hat{q}(0,i\zeta) + \Lambda_{22}(\zeta,\eta)\frac{\partial}{\partial x}\hat{q}(0,-i\zeta) \\
&\quad - i\eta\Lambda_{22}(\zeta,\eta)\hat{q}(0,-i\zeta) + h_1(\zeta,\eta).
\end{aligned} \tag{3.66}$$

Note that

$$\begin{aligned}
h_1(\zeta,\eta) &= \int_0^a [-\frac{1}{2\zeta}e^{-\zeta(a-\xi)} - \frac{e^{-\zeta(a-\xi)}}{2d\zeta}(\alpha_{10}\zeta-\alpha_{11})(\alpha_{00}\zeta\cosh[a\zeta] - \alpha_{01}\sinh[a\zeta]) \\
&\quad - \frac{e^{-\zeta\xi}}{2d}\alpha_{10}(\zeta\alpha_{00}+\alpha_{01})] \times \tilde{g}(\eta,\xi)]d\xi - [\tilde{g}_0(\eta) + q_0\cos\beta_0](\frac{\alpha_{10}\zeta}{d}) \\
&\quad + [\tilde{g}_1(\eta) + q_1\cos\beta_1](\frac{\zeta\alpha_{00}\cosh[\zeta a] - \alpha_{01}\sinh[\zeta a]}{d}) \\
h_1(\zeta,\eta) &= \Lambda_{21}(\zeta,\eta)\hat{\tilde{g}}(\eta,i\zeta) + \Lambda_{22}(\zeta,\eta)\hat{\tilde{g}}(\eta,-i\zeta) - [\tilde{g}_0(\eta) + q_0\cos\beta_0]\frac{\alpha_{10}\zeta}{d} \\
&\quad + [\tilde{g}_1(\eta) + q_1\cos\beta_1]\frac{(\alpha_{00}\zeta\cosh[a\zeta] - \alpha_{01}\sinh[a\zeta])}{d}.
\end{aligned} \tag{3.67}$$

Replace η by $-\eta$ in equation (3.66) to get

$$\tilde{q}(-\eta,a) = \Lambda_{21}(\zeta,-\eta)\frac{\partial}{\partial x}\hat{q}(0,i\zeta) + i\eta\Lambda_{21}(\zeta,-\eta)\hat{q}(0,i\zeta) + \Lambda_{22}(\zeta,-\eta)\frac{\partial}{\partial x}\hat{q}(0,-i\zeta) + i\eta\Lambda_{22}(\zeta,-\eta)\hat{q}(0,-i\zeta) + h_1(\zeta,-\eta). \tag{3.68}$$

Equations (3.60), (3.62), (3.66) and (3.68) define a system of four equations for four unknowns described as:

$$\tilde{q}(\eta,0) - h_0(\zeta,\eta) = \Lambda_{11}(\zeta,\eta)\frac{\partial}{\partial x}\hat{q}(0,i\zeta) - i\eta\Lambda_{11}(\zeta,\eta)\hat{q}(0,i\zeta) + \Lambda_{12}(\zeta,\eta)\frac{\partial}{\partial x}\hat{q}(0,-i\zeta) - i\eta\Lambda_{12}(\zeta,\eta)\hat{q}(0,-i\zeta), \tag{3.69}$$

$$\tilde{q}(-\eta,0) - h_0(\zeta,-\eta) = \Lambda_{11}(\zeta,-\eta)\frac{\partial}{\partial x}\hat{q}(0,i\zeta) + i\eta\Lambda_{11}(\zeta,-\eta)\hat{q}(0,i\zeta) + \Lambda_{12}(\zeta,-\eta)\frac{\partial}{\partial x}\hat{q}(0,-i\zeta) + i\eta\Lambda_{12}(\zeta,-\eta)\hat{q}(0,-i\zeta), \tag{3.70}$$

$$\tilde{q}(\eta,a) - h_1(\zeta,\eta) = \Lambda_{21}(\zeta,\eta)\frac{\partial}{\partial x}\hat{q}(0,i\zeta) - i\eta\Lambda_{21}(\zeta,\eta)\hat{q}(0,i\zeta) + \Lambda_{22}(\zeta,\eta)\frac{\partial}{\partial x}\hat{q}(0,-i\zeta) - i\eta\Lambda_{22}(\zeta,\eta)\hat{q}(0,-i\zeta), \tag{3.71}$$

$$\tilde{q}(-\eta,a) - h_1(\zeta,-\eta) = \Lambda_{21}(\zeta,-\eta)\frac{\partial}{\partial x}\hat{q}(0,i\zeta) + i\eta\Lambda_{21}(\zeta,-\eta)\hat{q}(0,i\zeta) + \Lambda_{22}(\zeta,-\eta)\frac{\partial}{\partial x}\hat{q}(0,-i\zeta) + i\eta\Lambda_{22}(\zeta,-\eta)\hat{q}(0,-i\zeta). \tag{3.72}$$

Note that

$$\Lambda_{11}(\zeta,\eta) = -\frac{1}{2\zeta}[1 + \frac{(\alpha_{00}\zeta + \alpha_{01})(\alpha_{10}\zeta\cosh[a\zeta] + \alpha_{11}\sinh[a\zeta]}{d}], \tag{3.73}$$

$$\Lambda_{12}(\zeta,\eta) = -\frac{e^{-\zeta a}}{2d}\alpha_{00}(\alpha_{10}\zeta - \alpha_{11}), \tag{3.74}$$

$$\Lambda_{21}(\zeta,\eta) = -\frac{1}{2d}\alpha_{10}(\zeta\alpha_{00} + \alpha_{01}), \tag{3.75}$$

$$\Lambda_{22}(\zeta,\eta) = -\frac{1}{2\zeta}e^{-\zeta a}[1 + \frac{(\alpha_{10}\zeta - \alpha_{11})}{d}(\alpha_{00}\zeta\cosh[a\zeta] - \alpha_{01}\sinh[a\zeta])], \tag{3.76}$$

$$d = d(\zeta,\eta) = (\zeta^2\alpha_{00}\alpha_{10} - \alpha_{01}\alpha_{11})\sinh[a\zeta] + (\zeta\alpha_{00}\alpha_{11} - \alpha_{01}\zeta\alpha_{10})\cosh[a\zeta], \tag{3.77}$$

$$\alpha_{00} = \sin\beta_0 \quad \alpha_{01} = \mu_0 - i\eta\cos\beta_0, \quad \alpha_{10} = \sin\beta_1, \alpha_{11} = \mu_1 - i\eta\cos\beta_1, \tag{3.78}$$

$$
\begin{aligned}
h_0(\zeta,\eta) &= \Lambda_{11}(\zeta,\eta)\hat{\tilde{g}}(\eta,i\zeta) + \Lambda_{12}(\zeta,\eta)\hat{\tilde{g}}(\eta,-i\zeta) + [\tilde{g}_1(\eta) + q_1\cos\beta_1](\frac{\zeta\alpha_{00}}{d}) \\
&\quad - [\tilde{g}_0(\eta) + q_0\cos\beta_0]\frac{(\alpha_{10}\zeta\cosh[a\zeta] + \alpha_{11}\sinh[a\zeta]}{d} \\
h_1(\zeta,\eta) &= \Lambda_{21}(\zeta,\eta)\hat{\tilde{g}}(\eta,i\zeta) + \Lambda_{22}(\zeta,\eta)\hat{\tilde{g}}(\eta,-i\zeta) - [\tilde{g}_0(\eta) + q_0\cos\beta_0]\frac{\alpha_{10}\zeta}{d} \\
&\quad + [\tilde{g}_1(\eta) + q_1\cos\beta_1]\frac{(\alpha_{00}\zeta\cosh[a\zeta] - \alpha_{01}\sinh[a\zeta])}{d}.
\end{aligned}
\tag{3.79}
$$

To find the unknowns $\frac{\partial}{\partial x}\hat{q}(0,\pm i\zeta)$ and $\hat{q}(0,\pm i\zeta)$, write the system defined by equations (3.69), (3.70), (3.71) and (3.72) in the matrix form shown below.

$$
\Lambda(\zeta,\eta)\begin{bmatrix} \frac{\partial}{\partial x}\hat{q}(0,i\zeta) \\ \hat{q}(0,i\zeta) \\ \frac{\partial}{\partial x}\hat{q}(0,-i\zeta) \\ \hat{q}(0,-i\zeta) \end{bmatrix} = \begin{bmatrix} \tilde{q}(\eta,0) - h_0(\zeta,\eta) \\ \tilde{q}(-\eta,0) - h_0(\zeta,-\eta) \\ tildeq(\eta,a) - h_1(\zeta,\eta) \\ \tilde{q}(-\eta,a) - h_1(\zeta,-\eta) \end{bmatrix}, \eta\in\mathbb{R},\ \text{where} \tag{3.80}
$$

$$
\Lambda(\zeta,\eta) = \begin{bmatrix}
\Lambda_{11}(\zeta,\eta) & -i\eta\Lambda_{11}(\zeta,\eta) & \Lambda_{12}(\zeta,\eta) & -i\eta\Lambda_{12}(\zeta,\eta) \\
\Lambda_{11}(\zeta,-\eta) & i\eta\Lambda_{11}(\zeta,-\eta) & \Lambda_{12}(\zeta,-\eta) & i\eta\Lambda_{12}(\zeta,-\eta) \\
\Lambda_{21}(\zeta,\eta) & -i\eta\Lambda_{21}(\zeta,\eta) & \Lambda_{22}(\zeta,\eta) & -i\eta\Lambda_{22}(\zeta,\eta) \\
\Lambda_{21}(\zeta,-\eta) & i\eta\Lambda_{21}(\zeta,-\eta) & \Lambda_{22}(\zeta,-\eta) & i\eta\Lambda_{22}(\zeta,-\eta)
\end{bmatrix}. \tag{3.81}
$$

Note that $\Lambda_{11}(\zeta,\eta), \Lambda_{12}(\zeta,\eta), \Lambda_{21}(\zeta,\eta), \Lambda_{22}(\zeta,\eta), d, h_0(\zeta,\eta)$ and $h_1(\zeta,\eta)$ are given by equations (3.73), (3.74), (3.75), (3.76), (3.77) and (3.79). Solving the system defined by equation (3.80), the values of $\frac{\partial}{\partial x}\hat{q}(0,i\zeta), \hat{q}(0,i\zeta)$, $\frac{\partial}{\partial x}\hat{q}(0,-i\zeta)$ and $\hat{q}(0,-i\zeta)$ are:

$$
\begin{aligned}
\frac{\partial}{\partial x}\hat{q}(0,i\zeta) &= -\frac{\sin\beta_0(-\mu_1 + i\eta\cos\beta_1 + \zeta\sin\beta_1)}{d(p_1+p_2)}q_{ap} + \\
&\quad \frac{[d + (-\mu_1 + i\eta\cos\beta_1 + \zeta\sin\beta_1)(\zeta\cosh a\zeta\sin\beta_0 - (\mu_0 - i\eta\cos\beta_0)\sinh a\zeta)]q_{0p}}{d\zeta(p_1+p_2)} \\
&\quad - \frac{\sin\beta_0(-\mu_1 - i\eta\cos\beta_1 + \zeta\sin\beta_1)}{d_1(p_3+p_4)}q_{am} + \\
&\quad \frac{[d_1 + (-\mu_1 - i\eta\cos\beta_1 + \zeta\sin\beta_1)(\zeta\cosh a\zeta\sin\beta_0 - (\mu_0 + i\eta\cos\beta_0)\sinh a\zeta)]q_{0m}}{d_1\zeta(p_3+p_4)},
\end{aligned}
\tag{3.82}
$$

$$\hat{q}(0,i\zeta) = \frac{-i}{\eta}[\frac{\sin\beta_0(-\mu_1+i\eta\cos\beta_1+\zeta\sin\beta_1)}{d(p_1+p_2)}q_{ap}-$$
$$\frac{[d+(-\mu_1+i\eta\cos\beta_1+\zeta\sin\beta_1)(\zeta\cosh a\zeta\sin\beta_0-(\mu_0-i\eta\cos\beta_0)\sinh a\zeta)]q_{0p}}{d\zeta(p_1+p_2)}$$
$$-\frac{\sin\beta_0(-\mu_1-i\eta\cos\beta_1+\zeta\sin\beta_1)}{d_1(p_3+p_4)}q_{am}+$$
$$\frac{[d_1+(-\mu_1-i\eta\cos\beta_1+\zeta\sin\beta_1)(\zeta\cosh a\zeta\sin\beta_0-(\mu_0+i\eta\cos\beta_0)\sinh a\zeta)]q_{0m}}{d_1\zeta(p_3+p_4)}, \tag{3.83}$$

$$\frac{\partial}{\partial x}\hat{q}(0,-i\zeta) = e^{a\zeta}[-\frac{\sin\beta_1(\mu_0-i\eta\cos\beta_0+\zeta\sin\beta_0)}{d(p_1+p_2)}q_{0p}+$$
$$\frac{[d+(\mu_0-i\eta\cos\beta_0+\zeta\sin\beta_0)(\zeta\cosh a\zeta\sin\beta_1+(\mu_1-i\eta\cos\beta_1)\sinh a\zeta)]q_{ap}}{d\zeta(p_1+p_2)}$$
$$-\frac{\sin\beta_1(\mu_0+i\eta\cos\beta_0+\zeta\sin\beta_0)}{d_1(p_3+p_4)}q_{0m}+$$
$$\frac{[d_1+(\mu_0+i\eta\cos\beta_0+\zeta\sin\beta_0)(\zeta\cosh a\zeta\sin\beta_1+(\mu_1+i\eta\cos\beta_1)\sinh a\zeta)]q_{am}}{d_1\zeta(p_3+p_4)}, \tag{3.84}$$

$$\hat{q}(0,-i\zeta) = \frac{-i}{\eta}e^{a\zeta}[\frac{\sin\beta_1(\mu_0-i\eta\cos\beta_0+\zeta\sin\beta_0)}{d(p_1+p_2)}q_{0p}-$$

$$\frac{[d+(\mu_0-i\eta\cos\beta_0+\zeta\sin\beta_0)(\zeta\cosh a\zeta\sin\beta_1+(\mu_1-i\eta\cos\beta_1)\sinh a\zeta)]}{d\zeta(p_1+p_2)}q_{ap}$$

$$-\frac{\sin\beta_1(\mu_0+i\eta\cos\beta_0+\zeta\sin\beta_0)}{d_1(p_3+p_4)}q_{0m}+$$

$$\frac{[d_1+(\mu_0+i\eta\cos\beta_0+\zeta\sin\beta_0)(\zeta\cosh a\zeta\sin\beta_1+(\mu_1+i\eta\cos\beta_1)\sinh a\zeta)]}{d_1\zeta(p_3+p_4)}q_{am}]. \tag{3.85}$$

Note that $p_1, p_2, p_3, p_4, q_{0p}, q_{0m}, q_{ap}$ and q_{am} are:

$$p_1 = \frac{\sin\beta_0(\mu_0-i\eta\cos\beta_0+\zeta\sin\beta_0)\sin\beta_1(-\mu_1+i\eta\cos\beta_1+\zeta\sin\beta_1)}{d^2}, \tag{3.86}$$

$$p_2 = \frac{-[d+(-\mu_1+i\eta\cos\beta_1+\zeta\sin\beta_1)(\zeta\cosh[a\zeta]\sin\beta_0-(\mu_0-i\eta\cos\beta_0)\sinh[a\zeta])]}{d\zeta^2}$$
$$\times[d+(\mu_0-i\eta\cos\beta_0+\zeta\sin\beta_0)(\zeta\cosh[a\zeta]\sin\beta_1+(\mu_1-i\eta\cos\beta_1)\sinh[a\zeta])], \tag{3.87}$$

$$p_3 = \frac{\sin\beta_0(\mu_0 + i\eta\cos\beta_0 + \zeta\sin\beta_0)\sin\beta_1(-\mu_1 - i\eta\cos\beta_1 + \zeta\sin\beta_1)}{d_1^2}, \tag{3.88}$$

$$p_4 = \frac{-[d_1 + (-\mu_1 - i\eta\cos\beta_1 + \zeta\sin\beta_1)(\zeta\cosh[a\zeta]\sin\beta_0 - (\mu_0 + i\eta\cos\beta_0)\sinh[a\zeta])]}{d_1\zeta^2}$$
$$\times [d_1 + (\mu_0 + i\eta\cos\beta_0 + \zeta\sin\beta_0)(\zeta\cosh[a\zeta]\sin\beta_1 + (\mu_1 + i\eta\cos\beta_1)\sinh[a\zeta]),] \tag{3.89}$$

$$q_{0p} = \tilde{q}(\eta, 0) - h_0(\zeta, \eta), \quad q_{0m} = \tilde{q}(-\eta, 0) - h_0(\zeta, -\eta), \tag{3.90}$$

$$q_{ap} = \tilde{q}(\eta, a) - h_1(\zeta, \eta), \quad q_{am} = \tilde{q}(-\eta, a) - h_1(\zeta, -\eta). \tag{3.91}$$

Now apply the the operator L_y to the boundary condition along side 2 of semi-infinite strip Ω defined by equation (3.11), to get

$$\begin{aligned}\int_0^a g_2(y)e^{i(i\zeta)y}dy &= \cos\beta_2\int_0^a \frac{\partial}{\partial x}q(x,y)e^{i(i\zeta)y}dy + \sin\beta_2\int_0^a \frac{\partial}{\partial y}q(x,y)e^{i(i\zeta y)}dy \\ &+ \mu_2\int_0^a q(x,y)e^{i(i\zeta)}dy, \quad x=0, \quad 0<y<a, \text{ integrate by parts} \\ &= \cos\beta_2\frac{\partial}{\partial x}\int_0^a q(x,y)e^{i(i\zeta)y}dy + \sin\beta_2[e^{i(i\zeta)y}q(x,y)|_{y=0}^a \\ &- \int_0^a q(x,y)e^{i(i\zeta)y}(-\zeta)dy] + \mu_2\int_0^a q(x,y)e^{i(i\zeta)y}dy.\end{aligned} \tag{3.92}$$

Use definition 3.1.2 in equation (3.92) to get

$$\begin{aligned}\hat{g}_2(i\zeta) &= \cos\beta_2\frac{\partial}{\partial x}\hat{q}(0,i\zeta) + \sin\beta_2 e^{-\zeta a}q(0,a) - \sin\beta_2 q(0,0) \\ &+ \sin\beta_2\zeta\hat{q}(0,i\zeta) + \mu_2\hat{q}(0,i\zeta).\end{aligned} \tag{3.93}$$

Let $q(0,0) = q_0 \quad q(0,a) = q_1$. Then equation (3.93) becomes

$$\cos\beta_2\frac{\partial}{\partial x}\hat{q}(0,i\zeta) + (\mu_2 + \zeta\sin\beta_2)\hat{q}(0,i\zeta) = \hat{g}_2(i\zeta) + (q_0 - e^{-\zeta a}q_1)\sin\beta_2. \tag{3.94}$$

Replace ζ by $-\zeta$ in equation (3.94) to get

$$\cos\beta_2\frac{\partial}{\partial x}\hat{q}(0,-i\zeta) + (\mu_2 - \zeta\sin\beta_2)\hat{q}(0,-i\zeta) = \hat{g}_2(-i\zeta) + (q_0 - e^{\zeta a}q_1)\sin\beta_2. \tag{3.95}$$

Now we have a system of two equations for four unknowns $\frac{\partial}{\partial x}\hat{q}(0,\pm i\zeta)$ and $\hat{q}(0,\pm i\zeta)$ defined by equations (3.94) and (3.95). From equations (3.82), (3.83), (3.84), and

(3.85), use the values of unkowns in equations (3.94) and (3.95) to get

$$
\begin{aligned}
&[\cos\beta_2 k_{11} + (\mu_2 + \zeta\sin\beta_2)k_{21}]\tilde{q}(\eta,0) + [\cos\beta_2 k_{13} + (\mu_2 + \zeta\sin\beta_2)k_{23}]\tilde{q}(\eta,a) = \\
&-[\cos\beta_2 k_{12} + (\mu_2 + \zeta\sin\beta_2)k_{22}]\tilde{q}(-\eta,0) - [\cos\beta_2 k_{14} + (\mu_2 + \zeta\sin\beta_2)k_{24}]\tilde{q}(-\eta,a) \\
&+[\cos\beta_2 k_{11} + (\mu_2 + \zeta\sin\beta_2)k_{21}]h_0(\eta) + [\cos\beta_2 k_{12} + (\mu_2 + \zeta\sin\beta_2)k_{22}]h_0(-\eta) \\
&+[\cos\beta_2 k_{13} + (\mu_2 + \zeta\sin\beta_2)k_{23}]h_1(\eta) + [\cos\beta_2 k_{14} + (\mu_2 + \zeta\sin\beta_2)k_{24}]h_1(-\eta) \\
&+\hat{g}_2(i\zeta) + (q_0 - e^{-\zeta a}q_1)\sin\beta_2.
\end{aligned}
\tag{3.96}
$$

Simplify equation (3.96) to get

$$
\begin{aligned}
&a_{11}\tilde{q}(\eta,0) + a_{12}\tilde{q}(\eta,a) = b_{11}\tilde{q}(-\eta,0) + b_{12}\tilde{q}(-\eta,a) + a_{11}h_0(\eta) - b_{11}h_0(-\eta) \\
&+a_{12}h_1(\eta) - b_{12}h_1(-\eta) + \hat{g}_2(i\zeta) + (q_0 - e^{-\zeta a}q_1)\sin\beta_2.
\end{aligned}
\tag{3.97}
$$

$$
\begin{aligned}
&[\cos\beta_2 k_{31} + (\mu_2 - \zeta\sin\beta_2)k_{41}]\tilde{q}(\eta,0) + [\cos\beta_2 k_{33} + (\mu_2 - \zeta\sin\beta_2)k_{43}]\tilde{q}(\eta,a) = \\
&-[\cos\beta_2 k_{32} + (\mu_2 - \zeta\sin\beta_2)k_{42}]\tilde{q}(-\eta,0) - [\cos\beta_2 k_{34} + (\mu_2 - \zeta\sin\beta_2)k_{44}]\tilde{q}(-\eta,a) \\
&+[\cos\beta_2 k_{31} + (\mu_2 - \zeta\sin\beta_2)k_{41}]h_0(\eta) + [\cos\beta_2 k_{32} + (\mu_2 - \zeta\sin\beta_2)k_{42}]h_0(-\eta) \\
&+[\cos\beta_2 k_{33} + (\mu_2 - \zeta\sin\beta_2)k_{43}]h_1(\eta) + [\cos\beta_2 k_{32} + (\mu_2 - \zeta\sin\beta_2)k_{44}]h_1(-\eta) \\
&+\hat{g}_2(-i\zeta) + (q_0 - e^{\zeta a}q_1)\sin\beta_2.
\end{aligned}
\tag{3.98}
$$

Simplify equation (3.98) to get

$$
\begin{aligned}
&a_{21}\tilde{q}(\eta,0) + a_{22}\tilde{q}(\eta,a) = b_{21}\tilde{q}(-\eta,0) + b_{22}\tilde{q}(-\eta,a) + a_{21}h_0(\eta) - b_{21}h_0(-\eta) \\
&+a_{22}h_1(\eta) - b_{22}h_1(-\eta) + \hat{g}_2(-i\zeta) + (q_0 - e^{\zeta a}q_1)\sin\beta_2.
\end{aligned}
\tag{3.99}
$$

The coefficients $a_{11}, a_{12}, a_{21}, a_{22}, b_{11}, b_{12}, b_{21}, b_{22}$ are defined below.

$$
\begin{aligned}
a_{11} = \frac{i(\mu_2 + \zeta\sin\beta_2)}{d\zeta\eta(p_1+p_2)}&[d + (-\mu_1 + i\eta\cos\beta_1 + \zeta\sin\beta_1)\times \\
&(\zeta\cosh[a\zeta]\sin\beta_0 + (-\mu_0 + i\eta\cos\beta_0)\sinh[a\zeta])] \\
&+\frac{\cos\beta_2}{d\zeta(p_1+p_2)}[d + (-\mu_1 + i\eta\cos\beta_1 + \zeta\sin\beta_1)\times \\
&(\zeta\cosh[a\zeta]\sin\beta_0 + (-\mu_0 + i\eta\cos\beta_0)\sinh[a\zeta])]
\end{aligned}
\tag{3.100}
$$

$$a_{12} = -\frac{\cos\beta_2 \sin\beta_0(-\mu_1 + i\eta\cos\beta_1 + \zeta\sin\beta_1)}{d(p_1+p_2)} - \frac{i\sin\beta_0(-\mu_1 + i\eta\cos\beta_1 + \zeta\sin\beta_1)(\mu_2 + \zeta\sin\beta_2)}{d\eta(p_1+p_2)} \tag{3.101}$$

$$a_{21} = -\frac{e^{a\zeta}[\cos\beta_2(\mu_0 - i\eta\cos\beta_0 + \zeta\sin\beta_0)\sin\beta_1]}{d(p_1+p_2)} - \frac{[ie^{a\zeta}(\mu_0 - i\eta\cos\beta_0 + \zeta\sin\beta_0)\sin\beta_1(\mu_2 - \zeta\sin\beta_2)]}{d\eta(p_1+p_2)} \tag{3.102}$$

$$\begin{aligned} a_{22} = {} & \frac{ie^{a\zeta}(\mu_2 - \zeta\sin\beta_2)}{d\zeta\eta(p_1+p_2)}[d + (\mu_0 - i\eta\cos\beta_0 + \zeta\sin\beta_0)\times \\ & (\zeta\cosh[a\zeta]\sin\beta_1 + (\mu_1 - i\eta\cos\beta_1)\sinh[a\zeta])] \\ & + \frac{e^{a\zeta}\cos\beta_2}{d\zeta(p_1+p_2)}[d + (\mu_0 - i\eta\cos\beta_0 + \zeta\sin\beta_0)\times \\ & (\zeta\cosh[a\zeta]\sin\beta_1 + (\mu_1 - i\eta\cos\beta_1)\sinh[a\zeta])] \end{aligned} \tag{3.103}$$

$$\begin{aligned} b_{11} = {} & \frac{i(\mu_2 + \zeta\sin\beta_2)}{d1\zeta\eta(p_3+p_4)}[d_1 + (-\mu_1 - i\eta\cos\beta_1 + \zeta\sin\beta_1)\times \\ & (\zeta\cosh[a\zeta]\sin\beta_0 - (\mu_0 + i\eta\cos\beta_0)\sinh[a\zeta])] \\ & - \frac{\cos\beta_2}{d1\zeta(p_3+p_4)}[d_1 + (-\mu_1 - i\eta\cos\beta_1 + \zeta\sin\beta_1)\times \\ & (\zeta\cosh[a\zeta]\sin\beta_0 - (\mu_0 + i\eta\cos\beta_0)\sinh[a\zeta])] \end{aligned} \tag{3.104}$$

$$b_{12} = \frac{\cos\beta_2 \sin\beta_0(-\mu_1 - i\eta\cos\beta_1 + \zeta\sin\beta_1)}{d_1(p_3+p_4)} - \frac{i\sin\beta_0(-\mu_1 - i\eta\cos\beta_1 + \zeta\sin\beta_1)(\mu_2 + \zeta\sin\beta_2)}{d_1\eta(p_3+p_4)} \tag{3.105}$$

$$b_{21} = \frac{e^{a\zeta}\cos\beta_2(\mu_0 + i\eta\cos\beta_0 + \zeta\sin\beta_0)\sin\beta_1}{d_1(p_3+p_4)} - \frac{ie^{a\zeta}(\mu_0 + i\eta\cos\beta_0 + \zeta\sin\beta_0)\sin\beta_1(\mu_2 - \zeta\sin\beta_2)}{d_1\eta(p_3+p_4)} \tag{3.106}$$

$$\begin{aligned} b_{22} = {} & \frac{ie^{a\zeta}(\mu_2 - \zeta\sin\beta_2)}{d_1\zeta\eta(p_3+p_4)}[d_1 + (\mu_0 + i\eta\cos\beta_0 + \zeta\sin\beta_0)\times \\ & (\zeta\cosh[a\zeta]\sin\beta_1 + (\mu_1 + i\eta\cos\beta_1)\sinh[a\zeta])] \\ & - \frac{e^{a\zeta}\cos\beta_2}{d_1\zeta(p_3+p_4)}[d_1 + (\mu_0 + i\eta\cos\beta_0 + \zeta\sin\beta_0)\times \\ & (\zeta\cosh[a\zeta]\sin\beta_1 + (\mu_1 + i\eta\cos\beta_1)\sinh[a\zeta])]. \end{aligned} \tag{3.107}$$

Write the system of two equations defined by (3.97) and (3.99) in matrix form to get

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \tilde{q}(\eta,0) \\ \tilde{q}(\eta,a) \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \tilde{q}(-\eta,0) \\ \tilde{q}(-\eta,a) \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} h_0(\eta) \\ h_1(\eta) \end{bmatrix}$$
$$- \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} h_0(-\eta) \\ h_1(-\eta) \end{bmatrix} + \begin{bmatrix} \hat{g}_2(i\zeta) + (q_0 - e^{-\zeta a} q_1) \sin\beta_2 \\ \hat{g}_2(-i\zeta) + (q_0 - e^{\zeta a} q_1) \sin\beta_2 \end{bmatrix}, \quad \eta \in \mathbb{R}. \tag{3.108}$$

Equation (3.108)can be written as

$$A \begin{bmatrix} \tilde{q}(\eta,0) \\ \tilde{q}(\eta,a) \end{bmatrix} = B \begin{bmatrix} \tilde{q}(-\eta,0) \\ \tilde{q}(-\eta,a) \end{bmatrix} + A \begin{bmatrix} h_0(\eta) \\ h_1(\eta) \end{bmatrix} - B \begin{bmatrix} h_0(-\eta) \\ h_1(-\eta) \end{bmatrix}$$
$$+ \begin{bmatrix} \hat{g}_2(i\zeta) + (q_0 - e^{-\zeta a} q_1) \sin\beta_2 \\ \hat{g}_2(-i\zeta) + (q_0 - e^{\zeta a} q_1) \sin\beta_2 \end{bmatrix}, \text{ where} \tag{3.109}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}. \tag{3.110}$$

Note that $a_{11}, a_{12}, a_{21}, a_{22}, b_{11}, b_{12}, b_{21}, b_{22}$ are given by equations (3.100), (3.101), (3.102), (3.103), (3.104), (3.105), (3.106) and (3.107), respectively. Now multiply equation (3.109) from left side by A^{-1}, to get

$$\begin{bmatrix} \tilde{q}(\eta,0) \\ \tilde{q}(\eta,a) \end{bmatrix} = G(\eta) \begin{bmatrix} \tilde{q}(-\eta,0) \\ \tilde{q}(-\eta,a) \end{bmatrix} + \begin{bmatrix} h_0(\eta) \\ h_1(\eta) \end{bmatrix} - G(\eta) \begin{bmatrix} h_0(-\eta) \\ h_1(-\eta) \end{bmatrix}$$
$$+ A^{-1} \begin{bmatrix} \hat{g}_2(i\zeta) + (q_0 - e^{-\zeta a} q_1) \sin\beta_2 \\ \hat{g}_2(-i\zeta) + (q_0 - e^{\zeta a} q_1) \sin\beta_2 \end{bmatrix}, \quad \eta \in \mathbb{R}. \tag{3.111}$$

Equation (3.111) can be expressed as

$$\phi^+(\eta) = G(\eta)\phi^-(\eta) + F(\eta), \quad \eta \in \mathbb{R}, \text{ where} \tag{3.112}$$

$$\phi^+(\eta) = \begin{bmatrix} \phi_1^+(\eta) = \tilde{q}(\eta, 0) \\ \phi_2^+(\eta) = \tilde{q}(\eta, a) \end{bmatrix}, \quad \phi^-(\eta) = \begin{bmatrix} \phi_1^-(\eta) = \tilde{q}(-\eta, 0) \\ \phi_2^-(\eta) = \tilde{q}(-\eta, a) \end{bmatrix}, \tag{3.113}$$

$$F(\eta) = \begin{bmatrix} h_0(\eta) \\ h_1(\eta) \end{bmatrix} - G(\eta) \begin{bmatrix} h_0(-\eta) \\ h_1(-\eta) \end{bmatrix} + A^{-1} \begin{bmatrix} \hat{g}_2(i\zeta) + (q_0 - e^{-\zeta a} q_1) \sin\beta_2 \\ \hat{g}_2(-i\zeta) + (q_0 - e^{\zeta a} q_1) \sin\beta_2 \end{bmatrix}, \tag{3.114}$$

$$G(\eta) = A^{-1}B = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}. \tag{3.115}$$

Note that g_{11}, g_{12}, g_{21} and g_{22} are given as follows.

$$\begin{aligned}
g_{11} &= s_1 + s_2 \\
s_1 &= -\frac{\sin\beta_0(\mu_0 + i\eta\cos\beta_0 + \zeta\sin\beta_0)\sin\beta_1}{dd_1(-\mu_2 + i\eta\cos\beta_2 + \zeta\sin\beta_2)(p_3 + p_4)}\{-\eta\cos\beta_2 + i(\mu_2 - \zeta\sin\beta_2)\}\times \\
&\quad \{\eta\cos\beta_1 + i(\mu_1 - \zeta\sin\beta_1)\} \\
s_2 &= \frac{\{d_1 + (\mu_1 + i\eta\cos\beta_1 - \zeta\sin\beta_1)(-\zeta\cosh[a\zeta]\sin\beta_0 + (\mu_0 + i\eta\cos\beta_0)\sinh[a\zeta])\}}{dd_1\zeta^2(\mu_2 - i\eta\cos\beta_2 + \zeta\sin\beta_2)(p_3 + p_4)} \\
&\quad \times \{id + \zeta\cosh[a\zeta](\eta\cos\beta_0 + i(\mu_0 + \zeta\sin\beta_0))\sin\beta_1 + (i\mu_1 + \eta\cos\beta_1) \\
&\quad \times (\mu_0 - i\eta\cos\beta_0 + \zeta\sin\beta_0)\sinh[a\zeta]\} \times \{i(\mu_2 + \zeta\sin\beta_2) - \eta\cos\beta_2\}
\end{aligned} \tag{3.116}$$

$$\begin{aligned}
g_{12} &= s_3 + s_4 \\
s_3 &= \frac{\sin\beta_0(-\mu_1 - i\eta\cos\beta_1 + \zeta\sin\beta_1)}{dd_1\zeta(\mu_2 - i\eta\cos\beta_2 + \zeta\sin\beta_2)(p_3 + p_4)} \times \{\eta\cos\beta_2 - i(\mu_2 + \zeta\sin\beta_2)\} \\
&\quad \times \{id + \zeta\cosh[a\zeta](\eta\cos\beta_0 + i(\mu_0 + \zeta\sin\beta_0))\sin\beta_1 + (i\mu_1 + \eta\cos\beta_1) \\
&\quad \times (\mu_0 - i\eta\cos\beta_0 + \zeta\sin\beta_0)\sinh[a\zeta]\} \\
s_4 &= \frac{\{d_1 + (\mu_0 + i\eta\cos\beta_0 + \zeta\sin\beta_0)(\zeta\cosh[a\zeta]\sin\beta_1 + (\mu_1 + i\eta\cos\beta_1)\sinh[a\zeta])\}}{dd_1\zeta(-\mu_2 + i\eta\cos\beta_2 + \zeta\sin\beta_2)(p_3 + p_4)} \\
&\quad \times \sin\beta_0\{i(\mu_2 - \zeta\sin\beta_2) - \eta\cos\beta_2\} \times \{\eta\cos\beta_1 + i(\mu_1 - \zeta\sin\beta_1)\}
\end{aligned} \tag{3.117}$$

$$g_{21} = s_5 + s_6$$

$$s_5 = \frac{i \sin\beta_1(\mu_0 + i\eta\cos\beta_0 + \zeta\sin\beta_0)}{dd_1\zeta(-\mu_2 + i\eta\cos\beta_2 + \zeta\sin\beta_2)(p_3 + p_4)}\{-\eta\cos\beta_2 + i(\mu_2 - \zeta\sin\beta_2)\}$$

$$\times\{d + \zeta\cosh[a\zeta]\sin\beta_0(-\mu_1 + i\eta\cos\beta_1 + \zeta\sin\beta_1) + (\mu_0 - i\eta\cos\beta_0)$$

$$\times(\mu_1 - i\eta\cos\beta_1 - \zeta\sin\beta_1)\sinh[a\zeta]\}$$

$$s_6 = \frac{\{d_1 + (\mu_1 + i\eta\cos\beta_1 - \zeta\sin\beta_1)(-\zeta\cosh[a\zeta]\sin\beta_0 + (\mu_0 + i\eta\cos\beta_0)\sinh[a\zeta])\}}{dd_1\zeta(\mu_2 - i\eta\cos\beta_2 + \zeta\sin\beta_2)(p_3 + p_4)}$$

$$\times\{i\sin\beta_1(\mu_0 - i\eta\cos\beta_0 + \zeta\sin\beta_0\}\{i(\mu_2 + \zeta\sin\beta_2) - \eta\cos\beta_2\} \tag{3.118}$$

$$g_{22} = s_7 + s_8$$

$$s_7 = \frac{i\sin\beta_0(\mu_0 - i\eta\cos\beta_0 + \zeta\sin\beta_0)\sin\beta_1}{dd_1(\mu_2 - i\eta\cos\beta_2 + \zeta\sin\beta_2)(p_3 + p_4)}\{\eta\cos\beta_2 - i(\mu_2 + \zeta\sin\beta_2)\}\times \tag{3.119}$$

$$\{-\mu_1 - i\eta\cos\beta_1 + \zeta\sin\beta_1\}$$

$$s_8 = \frac{\{d_1 + (\mu_0 + i\eta\cos\beta_0 + \zeta\sin\beta_0)(\zeta\cosh[a\zeta]\sin\beta_1 + (\mu_1 + i\eta\cos\beta_1)\sinh[a\zeta])\}}{dd_1\zeta^2(-\mu_2 + i\eta\cos\beta_2 + \zeta\sin\beta_2)(p_3 + p_4)}$$

$$\times\{d + \zeta\cosh[a\zeta]\sin\beta_0(-\mu_0 + i\eta\cos\beta_1 + \zeta\sin\beta_1) + (\mu_0 - i\eta\cos\beta_0)$$

$$\times(\mu_1 - i\eta\cos\beta_0)(\mu_1 - i\eta\cos\beta_1 - \zeta\sin\beta_1)\sinh[a\zeta]\}\{\mu_2 - \zeta\sin\beta_2 - i\eta\cos\beta_2\} \tag{3.120}$$

Note that equation (3.112) describes an order two vector RHP. Generally, in the literature, we do not have a method to solve an order two vector RHP in closed form. But we can find the closed form solution of an order two vector RHP in some special cases i.e. scalar and triangular cases. In the next section, we will discuss a scalar case, in which the directional derivatives are normal to the boundary of the semi-infinite strip.

3.3 Impedance boundary conditions

We seek the solution of the Helmholtz equation in a semi-infinite strip subject to the impedance boundary conditions. Substitute $\beta_0 = 3\frac{\pi}{2}, \beta_1 = \frac{\pi}{2}$ and $\beta_2 = \pi$ in the equations (3.8), (3.10), (3.11) and (3.12). Then the BVP of the Helmholtz equation

in a semi-infinite strip subject to the impedance boundary conditions becomes

$$(\partial_x^2 + \partial_y^2 + k^2)q(x, y) = g(x, y), \quad Im(k) > 0, \quad (x, y) \in \Omega, \tag{3.121}$$

where Ω is a semi-infinite strip shown in figure 4.1 with the corners $z_1 = \infty$, $z_2 = 0$, $z_3 = ia$, $z_4 = \infty + ia$, $a > 0$. Figure 4.1 shows the impedance boundary conditions along three sides of Ω. The impedance boundary conditions are

$$side1 : -q_y(x, y) + \mu_0 q(x, y) = g_0(x), \quad 0 < x < \infty, \quad y = 0, \tag{3.122}$$

$$side2 : -q_x(x, y) + \mu_2 q(x, y) = g_2(y), \quad x = 0, \quad 0 < y < a, \tag{3.123}$$

$$side3 : q_y(x, y) + \mu_1 q(x, y) = g_1(x), \quad 0 < x < \infty, \quad y = a. \tag{3.124}$$

$\mu_j, j = 0, 1, 2$ is a real non negative constant. The functions $g_0(x)$, $g_1(x)$ are real valued, and vanish at the points $x = 0$ and $x = \infty$, $\sin \beta_j \neq 0, \quad j = 0, 1, 2.$ Application of the Laplace transform the operator L_x from definition 3.1.1 to the

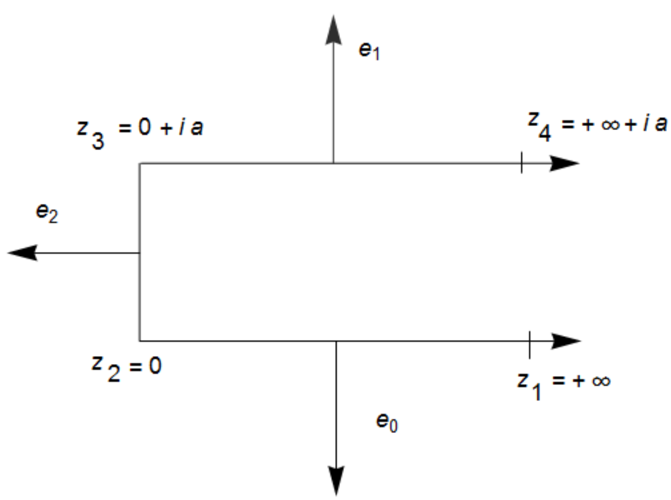

FIGURE 3.3. Impedance boundary condtions along sides of Ω.

Helmholtz equation (3.121) gives

$$(\frac{d^2}{dy^2} - \zeta^2)\tilde{q}(\eta, y) = f(y), \quad 0 < y < a, \text{ where} \tag{3.125}$$

$$f(y) = \partial_x q(0, y) - i\eta q(0, y) + \tilde{g}(\eta, y), \quad 0 < y < a. \tag{3.126}$$

Note that $\zeta = \sqrt{\eta^2 - k^2}$ is a multi-valued function. We fix a branch of it by $Re(\zeta) \geq 0$. The branch cut of this multi-valued function is shown in figure 3.2. Application of the Laplace transform operator L_x to the boundary conditions defined by equations (3.122) and (3.124), and $q(x, y)|_{x=\infty} = \partial_x q(x, y)|_{x=\infty} = 0$, gives the following results:

$$-\frac{d}{dy}\tilde{q}(\eta, 0) + \mu_0 \tilde{q}(\eta, 0) = \tilde{g}_0(\eta), \tag{3.127}$$

$$\frac{d}{dy}\tilde{q}(\eta, a) + \mu_1 \tilde{q}(\eta, a) = \tilde{g}_1(\eta). \tag{3.128}$$

Now we define the functionals of the boundary conditions W_0 and W_1 as follows:

$$W_0[F(y)] = -\frac{d}{dy}F|_{y=0} + \mu_0 F|_{y=0}, \tag{3.129}$$

$$W_1[F(y)] = \frac{d}{dy}F|_{y=a} + \mu_1 F|_{y=a}. \tag{3.130}$$

Equations (3.125), (3.127) and (3.128) are used to describe a homogeneous system of Laplace transformed equations written as

$$(\frac{d^2}{dy^2} - \zeta^2)\tilde{q}(\eta, y) = 0, \quad 0 < y < a, \tag{3.131}$$

$$-\frac{d}{dy}\tilde{q}(\eta, 0) + \mu_0 \tilde{q}(\eta, 0) = 0, \tag{3.132}$$

$$\frac{d}{dy}\tilde{q}(\eta, a) + \mu_1 \tilde{q}(\eta, a) = 0. \tag{3.133}$$

Now Green's function of the homogeneous system of the Laplace transformed equations (3.131), (3.132) and (3.133) is

$$G(y, \xi) = \psi(y, \xi) - \sum_{j=0}^{1} W_j[\psi(y, \xi)]\psi_j(y), \tag{3.134}$$

where, $\psi(y,\xi)$ is the fundamental function of the second order linear differential the operator $L = \frac{d^2}{dy^2} - \zeta^2$, and is defined by

$$\psi(y,\xi) = -\frac{1}{2\zeta}e^{-\zeta|y-\xi|}. \tag{3.135}$$

Note that $\psi_0(y)$ and $\psi_1(y)$ are the basis functions of the the operator L. These are found by using boundary functionals W_j, $j = 0, 1$ defined by equations (3.129) and (3.130) , and properties of the basis functions which are

$$L(\psi_j) = 0, \quad j = 0, 1 \text{ and } W_j[\psi_l] = \delta_{jl} \quad j, l = 0, 1. \tag{3.136}$$

Expressions for $\psi_0(y), \psi_1(y)$ are

$$\psi_0(y) = \frac{\zeta\cosh[(a-y)\zeta] + \mu_1\sinh[(a-y)\zeta]}{\Delta}, \tag{3.137}$$

$$\psi_1(y) = \frac{\zeta\cosh[\zeta y] + \mu_0\sinh[\zeta y]}{\Delta}, \text{ where} \tag{3.138}$$

$$\Delta = (\mu_0 + \mu_1)\zeta\cosh[a\zeta] + (\mu_0\mu_1 + \zeta^2)\sinh[a\zeta]. \tag{3.139}$$

To find the Green's function of the homogeneous BVP defined by equations (3.125), (3.127) and (3.128), use equation (3.134), follow the procedure given in section 3.2 pages 104, 105, 105 and 107. The Green's function is

$$\begin{aligned} G(y,\xi) = &-\frac{e^{-\zeta|y-\xi|}}{2\zeta} + \frac{\mu_0 - \zeta}{2\Delta\zeta}e^{-\zeta\xi}(\zeta\cosh[(a-y)\zeta] + \mu_1\sinh[(a-y)\zeta]) \\ &+ (\frac{\mu_1 - \zeta}{2\Delta\zeta})e^{-\zeta(a-\xi)}(\zeta\cosh[\zeta y] + \mu_0\sinh[\zeta y]). \end{aligned} \tag{3.140}$$

Solution of the non homogeneous BVP defined by equations (3.125), (3.127) and (3.128) is

$$\tilde{q}(\eta, y) = \int_0^a G(y,\xi)f(\xi)d\xi + \sum_{j=0}^{1}\tilde{g}_j(\eta)\psi_j(y). \tag{3.141}$$

Insert $y = 0$ in equation (3.141)to get

$$\tilde{q}(\eta, 0) = \int_0^a G(0,\xi)f(\xi)d\xi + \tilde{g}_0(\eta)\psi_0(0) + \tilde{g}_1(\eta)\psi_1(0) \tag{3.142}$$

From equations (3.140), (3.126), (3.137) and (3.138), find values of $G(0,\xi), f(\xi)$, $\psi_0(0)$ and $\psi_1(0)$, respectively, and insert these values in equation (3.142)to get

$$
\begin{aligned}
\tilde{q}(\eta,0) = \int_0^a [-\frac{e^{-\zeta\xi}}{2\zeta} + \frac{\mu_0-\zeta}{2\Delta\zeta} e^{-\zeta\xi}(\zeta\cosh[a\zeta]+\mu_1\sinh[a\zeta]) + (\frac{\mu_1-\zeta}{2\Delta})e^{-\zeta(a-\xi)}]\times \\
[\frac{\partial}{\partial x}q(0,\xi) - i\eta q(0,\xi) + \tilde{g}(\eta,\xi)]d\xi + \tilde{g}_0(\eta)\frac{(\zeta\cosh[a\zeta]+\mu_1\sinh[a\zeta])}{\Delta})+ \\
\tilde{g}_1(\eta)\frac{\zeta}{\Delta}
\end{aligned}
\tag{3.143}
$$

Using equation (3.143), and procedure in section 3.2 pages 107 and 109, we find that

$$
\begin{aligned}
\tilde{q}(\eta,0) = \Lambda_{11}(\zeta)\frac{\partial}{\partial x}\hat{q}(0,i\zeta) - i\eta\Lambda_{11}(\zeta)\hat{q}(0,i\zeta) + \Lambda_{12}(\zeta)\frac{\partial}{\partial x}\hat{q}(0,-i\zeta) \\
- i\eta\Lambda_{12}(\zeta)\hat{q}(0,-i\zeta) + h_0(\eta).
\end{aligned}
\tag{3.144}
$$

Note that

$$
\begin{aligned}
h_0(\eta) = \Lambda_{11}(\zeta)\hat{\tilde{g}}(\eta,i\zeta) + \Lambda_{12}(\zeta,\eta)\hat{\tilde{g}}(\eta,-i\zeta) + \tilde{g}_0(\eta)\frac{(\zeta\cosh[a\zeta]+\mu_1\sinh[a\zeta])}{\Delta} \\
+ \tilde{g}_1(\eta)\frac{\zeta}{\Delta},
\end{aligned}
\tag{3.145}
$$

$$
\Lambda_{11}(\zeta) = \frac{1}{2\zeta}[-1 + \frac{(\mu_0-\zeta)}{\Delta}(\zeta\cosh[a\zeta]+\mu_1\sinh[a\zeta])], \tag{3.146}
$$

$$
\Lambda_{12}(\zeta) = \frac{e^{-\zeta a}}{2\Delta}(\mu_1-\zeta), \tag{3.147}
$$

$$
\Delta = (\mu_0+\mu_1)\zeta\cosh[a\zeta] + (\mu_0\mu_1+\zeta^2)\sinh[a\zeta]. \tag{3.148}
$$

Replace η by $-\eta$ in equation (3.144) to get

$$
\begin{aligned}
\tilde{q}(-\eta,0) = \Lambda_{11}(\zeta)\frac{\partial}{\partial x}\hat{q}(0,i\zeta) + i\eta\Lambda_{11}(\zeta)\hat{q}(0,i\zeta) + \Lambda_{12}(\zeta)\frac{\partial}{\partial x}\hat{q}(0,-i\zeta) \\
+ i\eta\Lambda_{12}(\zeta)\hat{q}(0,-i\zeta) + h_0(-\eta).
\end{aligned}
\tag{3.149}
$$

Insert $y = a$ in equation (3.141), then from equations (3.140), (3.126), (3.137), and (3.138), find values of $G(a,\xi), f(\xi), \psi_0(a)$ and $\psi_1(a)$, respectively, and insert these

values in resulting equation to get

$$
\begin{aligned}
\tilde{q}(\eta,a) = & \int_0^a [-\frac{e^{-\zeta(a-\xi)}}{2\zeta} + \frac{(\mu_1-\zeta)}{2\Delta\zeta}e^{-\zeta(a-\xi)}(\zeta\cosh[a\zeta]+\mu_0\sinh[a\zeta]) \\
& + \frac{(\mu_0-\zeta)}{2\Delta}e^{-\zeta\xi}] \times [\frac{\partial}{\partial x}q(0,\xi) - i\eta q(0,\xi) + \tilde{g}(\eta,\xi)]d\xi + \tilde{g}_0(\eta)\frac{\zeta}{\Delta} \\
& + \tilde{g}_1(\eta)\frac{(\zeta\cosh[\zeta a]+\mu_0\sinh[\zeta a])}{\Delta}.
\end{aligned} \tag{3.150}
$$

Using equation (3.150), and procedure in section 3.2 pages 109 and 110, we find that

$$
\begin{aligned}
\tilde{q}(\eta,a) = & \Lambda_{21}(\zeta)\frac{\partial}{\partial x}\hat{q}(0,i\zeta) - i\eta\Lambda_{21}(\zeta)\hat{q}(0,i\zeta) + \Lambda_{22}(\zeta)\frac{\partial}{\partial x}\hat{q}(0,-i\zeta) \\
& - i\eta\Lambda_{22}(\zeta)\hat{q}(0,-i\zeta) + h_1(\eta), \text{ where}
\end{aligned} \tag{3.151}
$$

$$
\begin{aligned}
h_1(\eta) = & \Lambda_{21}(\zeta)\hat{\tilde{g}}(\eta,i\zeta) + \Lambda_{22}(\zeta,\eta)\hat{\tilde{g}}(\eta,-i\zeta) + \tilde{g}_1(\eta)\frac{(\zeta\cosh[a\zeta]+\mu_0\sinh[a\zeta])}{\Delta} \\
& + \tilde{g}_0(\eta)\frac{\zeta}{\Delta},
\end{aligned} \tag{3.152}
$$

$$
\Lambda_{21}(\zeta) = \frac{\mu_0-\zeta}{2\Delta}, \qquad \Lambda_{22}(\zeta) = \frac{e^{-\zeta a}}{2\zeta}(\zeta\cosh[a\zeta]+\mu_0\sinh[a\zeta]). \tag{3.153}
$$

Note that Δ is given by equation (3.148). Now replace η by $-\eta$ in equation (3.151) to get

$$
\begin{aligned}
\tilde{q}(-\eta,a) = & \Lambda_{21}(\zeta)\frac{\partial}{\partial x}\hat{q}(0,i\zeta) + i\eta\Lambda_{21}(\zeta)\hat{q}(0,i\zeta) + \Lambda_{22}(\zeta)\frac{\partial}{\partial x}\hat{q}(0,-i\zeta) \\
& + i\eta\Lambda_{22}(\zeta)\hat{q}(0,-i\zeta) + h_1(-\eta).
\end{aligned} \tag{3.154}
$$

Equations (3.143), (3.149), (3.150) and (3.154) describe the following system of four equations for four unknowns:

$$
\begin{aligned}
\tilde{q}(\eta,0) - h_0(\eta) = & \Lambda_{11}(\zeta)\frac{\partial}{\partial x}\hat{q}(0,i\zeta) - i\eta\Lambda_{11}(\zeta)\hat{q}(0,i\zeta) + \Lambda_{12}(\zeta)\frac{\partial}{\partial x}\hat{q}(0,-i\zeta) \\
& - i\eta\Lambda_{12}(\zeta)\hat{q}(0,-i\zeta),
\end{aligned} \tag{3.155}
$$

$$\tilde{q}(-\eta,0)-h_0(-\eta)=\Lambda_{11}(\zeta)\frac{\partial}{\partial x}\hat{q}(0,i\zeta)+i\eta\Lambda_{11}(\zeta)\hat{q}(0,i\zeta)+\Lambda_{12}(\zeta)\frac{\partial}{\partial x}\hat{q}(0,-i\zeta)$$
$$+\,i\eta\Lambda_{12}(\zeta)\hat{q}(0,-i\zeta),$$

$$\tilde{q}(\eta,a)-h_1(\eta)=\Lambda_{21}(\zeta)\frac{\partial}{\partial x}\hat{q}(0,i\zeta)-i\eta\Lambda_{21}(\zeta)\hat{q}(0,i\zeta)+\Lambda_{22}(\zeta)\frac{\partial}{\partial x}\hat{q}(0,-i\zeta)$$
$$-\,i\eta\Lambda_{22}(\zeta,\eta)\hat{q}(0,-i\zeta),$$

$$\tilde{q}(-\eta,a)-h_1(\eta)=\Lambda_{21}(\zeta)\frac{\partial}{\partial x}\hat{q}(0,i\zeta)+i\eta\Lambda_{21}(\zeta)\hat{q}(0,i\zeta)+\Lambda_{22}(\zeta)\frac{\partial}{\partial x}\hat{q}(0,-i\zeta)$$
$$+\,i\eta\Lambda_{22}(\zeta)\hat{q}(0,-i\zeta). \tag{3.156}$$

To find the unknowns $\frac{\partial}{\partial x}\hat{q}(0,\pm i\zeta)$ and $\hat{q}(0,\pm i\zeta)$, write the system defined by equations labeled by (3.155) and (3.156) in the matrix form shown below

$$\Lambda(\zeta)\begin{bmatrix}\frac{\partial}{\partial x}\hat{q}(0,i\zeta)\\ \hat{q}(0,i\zeta)\\ \frac{\partial}{\partial x}\hat{q}(0,-i\zeta)\\ \hat{q}(0,-i\zeta)\end{bmatrix}=\begin{bmatrix}\tilde{q}(\eta,0)-h_0(\eta)\\ \tilde{q}(-\eta,0)-h_0(-\eta)\\ \tilde{q}(\eta,a)-h_1(\eta)\\ \tilde{q}(-\eta,a)-h_1(-\eta)\end{bmatrix},\quad \eta\in\mathbb{R}, \tag{3.157}$$

(3.158)

$$\Lambda(\zeta)=\begin{bmatrix}\Lambda_{11}(\zeta) & -i\eta\Lambda_{11}(\zeta) & \Lambda_{12}(\zeta) & -i\eta\Lambda_{12}(\zeta)\\ \Lambda_{11}(\zeta) & i\eta\Lambda_{11}(\zeta) & \Lambda_{12}(\zeta) & i\eta\Lambda_{12}(\zeta\\ \Lambda_{21}(\zeta) & -i\eta\Lambda_{21}(\zeta) & \Lambda_{22}(\zeta) & -i\eta\Lambda_{22}(\zeta)\\ \Lambda_{21}(\zeta) & i\eta\Lambda_{21}(\zeta) & \Lambda_{22}(\zeta) & i\eta\Lambda_{22}(\zeta)\end{bmatrix}. \tag{3.159}$$

Note that $h_0(\eta), h_1(\eta), \Lambda_{11}(\zeta), \Lambda_{12}(\zeta), \Lambda_{21}(\zeta), \Lambda_{22}(\zeta)$ are given by equations (3.145), (3.152), (3.146), (3.147) and (3.153). Solving the system defined by equation (3.157),

the values of $\frac{\partial}{\partial x}\hat{q}(0,i\zeta), \hat{q}(0,i\zeta)$, $\frac{\partial}{\partial x}\hat{q}(0,-i\zeta)$ and $\hat{q}(0,-i\zeta)$ are

$$\begin{aligned}
\frac{\partial}{\partial x}\hat{q}(0,i\zeta) &= -\frac{-(q_{am}+q_{ap})(\zeta-\mu_1)+(q_{0m}+q_{0p})(\zeta+\mu_0)\cosh[a\zeta]}{2(\cosh[a\zeta]+\sinh[a\zeta])} \\
&\quad -\frac{(q_{0m}+q_{0p})(\zeta+\mu_0)\sinh[a\zeta]}{2(\cosh[a\zeta]+\sinh[a\zeta])} \\
\hat{q}(0,i\zeta) &= \frac{i\{(-q_{am}+q_{ap})(\zeta-\mu_1)+(q_{0m}-q_{0p})(\zeta+\mu_0)\cosh[a\zeta]\}}{2\eta(\cosh[a\zeta]+\sinh[a\zeta])} \\
&\quad +\frac{i(q_{0m}-q_{0p})(\zeta+\mu_0)\sinh[a\zeta]}{2\eta(\cosh[a\zeta]+\sinh[a\zeta])} \\
\frac{\partial}{\partial x}\hat{q}(0,-i\zeta) &= \frac{i\{(q_{0m}+q_{0p})(\zeta-\mu_0)-e^{a\zeta}(q_{am}+q_{ap})(\zeta+\mu_1)\}}{2\eta} \\
\hat{q}(0,-i\zeta) &= \frac{i\{(-q_{0m}+q_{0p})(\zeta-\mu_0)+e^{a\zeta}(q_{am}-q_{ap})(\zeta+\mu_1)\}}{2\eta}.
\end{aligned} \tag{3.160}$$

Note that $q_{0p}, q_{0m}, q_{ap}, q_{am}$ are:

$$q_{0p} = \tilde{q}(\eta,0) - h_0(\eta), \quad q_{0m} = \tilde{q}(-\eta,0) - h_0(-\eta), \tag{3.161}$$

$$q_{ap} = \tilde{q}(\eta,a) - h_1(\eta), \quad q_{am} = \tilde{q}(-\eta,a) - h_1(-\eta). \tag{3.162}$$

Now application of the the operator L_y to the boundary condition along side 2 of semi-infinite strip Ω, defined by equation (3.123), and use of definition 3.1.2 gives the following result.

$$-\frac{\partial}{\partial x}\hat{q}(0,i\zeta) + \mu_2\hat{q}(0,i\zeta) = \hat{g}_2(i\zeta) \tag{3.163}$$

Replace ζ by $-\zeta$ to get

$$-\frac{\partial}{\partial x}\hat{q}(0,-i\zeta) + \mu_2\hat{q}(0,-i\zeta) = \hat{g}_2(-i\zeta). \tag{3.164}$$

From equations labeled by (3.160), use values of $\frac{\partial}{\partial x}\hat{q}(0,\pm i\zeta)$ and $\hat{q}(0,\pm i\zeta)$ in equations (3.163) and (3.164) to get the following system of two equations:

$$\begin{aligned}
a_{11}\tilde{q}(\eta,0) + a_{12}\tilde{q}(\eta,a) &= b_{11}\tilde{q}(-\eta,0) + b_{12}\tilde{q}(-\eta,a) + a_{11}h_0(\eta) - b_{11}h_0(-\eta) \\
&\quad + a_{12}h_1(\eta) - b_{12}h_1(-\eta) + \hat{g}_2(i\zeta),
\end{aligned} \tag{3.165}$$

$$a_{21}\tilde{q}(\eta,0)+a_{22}\tilde{q}(\eta,a)=b_{21}\tilde{q}(-\eta,0)+b_{22}\tilde{q}(-\eta,a)+a_{21}h_0(\eta)-b_{21}h_0(-\eta)$$
$$+a_{22}h_1(\eta)-b_{22}h_1(-\eta)+\hat{g}_2(-i\zeta). \tag{3.166}$$

Note that

$$\begin{aligned}
a_{11} &= \frac{(\zeta+\mu_0)(\eta-i\mu_2)\cosh[a\zeta]+(\zeta+\mu_0)(\eta-i\mu_2)\sinh[a\zeta]}{2\eta(\cosh[a\zeta]+\sinh[a\zeta])},\\
a_{12} &= -\frac{(\zeta-\mu_1)(\eta-i\mu_2)}{2\eta(\cosh[a\zeta]+\sinh[a\zeta])},\\
a_{21} &= \frac{\eta(-\zeta+\mu_0)+i(\zeta-\mu_0)\mu_2}{2\eta},\\
a_{22} &= \frac{\eta e^{a\zeta}(\zeta+\mu_1)-ie^{a\zeta}(\zeta+\mu_1)\mu_2}{2\eta},
\end{aligned} \tag{3.167}$$

$$\begin{aligned}
b_{11} &= \frac{(\zeta+\mu_0)(\eta+i\mu_2)\cosh[a\zeta]+(\zeta+\mu_0)(\eta+i\mu_2)\sinh[a\zeta]}{2\eta(\cosh[a\zeta]+\sinh[a\zeta])},\\
b_{12} &= -\frac{(\zeta-\mu_1)(\eta+i\mu_2)}{2\eta(\cosh[a\zeta]+\sinh[a\zeta])},\\
b_{21} &= \frac{\eta(-\zeta+\mu_0)+i(-\zeta+\mu_0)\mu_2}{2\eta},\\
b_{22} &= \frac{\eta e^{a\zeta}(\zeta+\mu_1)+ie^{a\zeta}(\zeta+\mu_1)\mu_2}{2\eta}.
\end{aligned} \tag{3.168}$$

Write the system of two equations (3.165) and (3.166) in matrix form to get

$$A\begin{bmatrix}\tilde{q}(\eta,0)\\ \tilde{q}(\eta,a)\end{bmatrix}=B\begin{bmatrix}\tilde{q}(-\eta,0)\\ \tilde{q}(-\eta,a)\end{bmatrix}+A\begin{bmatrix}h_0(\eta)\\ h_1(\eta)\end{bmatrix}-B\begin{bmatrix}h_0(-\eta)\\ h_1(-\eta)\end{bmatrix}$$
$$+\begin{bmatrix}\hat{g}_2(i\zeta)\\ \hat{g}_2(-i\zeta)\end{bmatrix},\ \text{where} \tag{3.169}$$

$$A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix},\qquad B=\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}.$$

Note that $a_{11}, a_{12}, a_{21}, a_{22}, b_{11}, b_{12}, b_{21}$ and b_{22} are given by equations labeled by (3.167) and (3.168). Now multiply equation (3.169) from left side by A^{-1} to get

$$\begin{bmatrix} \tilde{q}(\eta,0) \\ \tilde{q}(\eta,a) \end{bmatrix} = G(\eta) \begin{bmatrix} \tilde{q}(-\eta,0) \\ \tilde{q}(-\eta,a) \end{bmatrix} + \begin{bmatrix} h_0(\eta) \\ h_1(\eta) \end{bmatrix} - G(\eta) \begin{bmatrix} h_0(-\eta) \\ h_1(-\eta) \end{bmatrix} \tag{3.170}$$

$$+ A^{-1} \begin{bmatrix} \hat{g}_2(i\zeta) \\ \hat{g}_2(-i\zeta) \end{bmatrix}, \text{ where } \eta \in \mathbb{R}.$$

Equation (3.170) can be expressed by

$$\phi^+(\eta) = G(\eta)\phi^-(\eta) + F(\eta), \quad \eta \in \mathbb{R}, \text{ where} \tag{3.171}$$

$$\phi^+(\eta) = \begin{bmatrix} \phi_1+(\eta) = \tilde{q}(\eta,0) \\ \phi_2+(\eta) = \tilde{q}(\eta,a) \end{bmatrix}, \quad \phi^-(\eta) = \begin{bmatrix} \phi_1-(\eta) = \tilde{q}(-\eta,0) \\ \phi_2-(\eta) = \tilde{q}(-\eta,a) \end{bmatrix} \tag{3.172}$$

$$F(\eta) = \begin{bmatrix} h_0(\eta) \\ h_1(\eta) \end{bmatrix} - G(\eta) \begin{bmatrix} h_0(-\eta) \\ h_1(-\eta) \end{bmatrix} + A^{-1} \begin{bmatrix} \hat{g}_2(i\zeta) \\ \hat{g}_2(-i\zeta) \end{bmatrix} \tag{3.173}$$

$$A^{-1} = \frac{1}{\Delta(\eta - i\mu_2)} \begin{bmatrix} e^{a\zeta}\eta(\zeta+\mu_1)\hat{g}_2(i\zeta) & \eta e^{-a\zeta}(\zeta-\mu_1)\hat{g}_2(-i\zeta) \\ \eta(\zeta-\mu_0)\hat{g}_2(i\zeta) & \eta(\zeta+\mu_0)\hat{g}_2(-i\zeta) \end{bmatrix} \tag{3.174}$$

$$G(\eta) = A^{-1}B = \begin{bmatrix} -\frac{\eta+i\mu_2}{\eta-i\mu_2} & 0 \\ 0 & -\frac{\eta+i\mu_2}{\eta-i\mu_2} \end{bmatrix}. \tag{3.175}$$

Δ is given from equation (3.148). Hence equation (3.173) can be expressed as

$$F(\eta) = \begin{bmatrix} h_0(\eta) \\ h_1(\eta) \end{bmatrix} - \begin{bmatrix} -\frac{\eta+I\mu_2}{\eta-I\mu_2} & 0 \\ 0 & -\frac{\eta+I\mu_2}{\eta-I\mu_2} \end{bmatrix} \begin{bmatrix} h_0(-\eta) \\ h_1(-\eta) \end{bmatrix}$$
$$+ \frac{1}{\Delta(\eta - i\mu_2)} \begin{bmatrix} e^{a\zeta}\eta(\zeta+\mu_1)\hat{g}_2(i\zeta) & \eta e^{-a\zeta}(\zeta-\mu_1)\hat{g}_2(-i\zeta) \\ \eta(\zeta-\mu_0)\hat{g}_2(i\zeta) & \eta(\zeta+\mu_0)\hat{g}_2(-i\zeta) \end{bmatrix} \begin{bmatrix} \hat{g}_2(i\zeta) \\ \hat{g}_2(-i\zeta) \end{bmatrix}. \tag{3.176}$$

Equation (3.176) shows the components of $F(\eta)$ are

$$\begin{aligned}F_1(\eta) &= h_0(\eta) + \frac{\eta + i\mu_2}{\eta - i\mu_2}h_0(-\eta) + \frac{\eta}{\Delta(\eta - i\mu_2)}[e^{a\zeta}(\zeta+\mu_1)\hat{g}_2(i\zeta) \\ &\quad + (\zeta-\mu_1)e^{-a\zeta}\hat{g}_2(-i\zeta)], \\ F_2(\eta) &= h_1(\eta) + \frac{\eta + i\mu_2}{\eta - i\mu_2}h_1(-\eta) + \frac{\eta}{\Delta(\eta - i\mu_2)}[(\zeta-\mu_0)\hat{g}_2(i\zeta) \\ &\quad + (\zeta+\mu_0)\hat{g}_2(-i\zeta)].\end{aligned} \tag{3.177}$$

Insert the value of $G(\eta)$ from equation (3.175) in equation (3.171), to obtain the following two scalar RHPs:

$$\phi_j^+(\eta) = -\frac{\eta + i\mu_2}{\eta - i\mu_2}\phi_j^-(\eta) + F_j(\eta), \quad \eta \in \mathbb{R}, \quad j = 1, 2, \tag{3.178}$$

$$\phi_1^\pm(\eta) = \tilde{q}(\pm\eta, 0), \quad \phi_2^\pm(\eta) = \tilde{q}(\pm\eta, a). \tag{3.179}$$

Note that $\phi_1^+(\eta), \phi_2^+(\eta)$ are analytic functions in the upper half η- complex plane, where as $\phi_1^-(\eta), \phi_2^-(\eta)$ are analytic functions in the lower half η-complex plane. These functions satisfy the following symmetry conditions:

$$\phi_j^+(\eta) = \phi_j^-(-\eta) \quad \forall \quad \eta \in \mathbb{C}^+, \quad j = 1, 2, \tag{3.180}$$

$$\phi_j^-(\eta) = \phi_j^+(-\eta) \quad \forall \quad \eta \in \mathbb{C}^-. \tag{3.181}$$

Due to this symmetry property, the scalar RHP defined by equation (3.171) is called a symmetric order two vector RHP. Now consider

$$\begin{aligned}\phi(\eta) &= \frac{h_j(-\eta)}{\eta - i\mu_2} + \frac{h_j(\eta)}{\eta + i\mu_2}, \quad j = 1, 2, \\ \phi(-\eta) &= \frac{h_j(\eta)}{-\eta - i\mu_2} + \frac{h_j(-\eta)}{-\eta + i\mu_2}, \\ \phi(-\eta) &= -(\frac{h_j(-\eta)}{\eta - i\mu_2} + \frac{h_j(\eta)}{\eta + i\mu_2}) = -\phi(\eta).\end{aligned} \tag{3.182}$$

Equation (3.182) indicates that $\phi(\eta)$ is an odd function in the variable η. Also note that $\frac{\eta}{(\eta^2+\mu_2^2)\Delta}[(\zeta-\mu_1)e^{-a\zeta}\hat{g}_2(-i\zeta) + e^{a\zeta}(\zeta+\mu_1)\hat{g}_2(i\zeta)]$ and $\frac{\eta}{(\eta^2+\mu_2^2)\Delta}[(\zeta-\mu_0)\hat{g}_2(i\zeta) + (\zeta+\mu_0)\hat{g}_2(-i\zeta)]$ are odd functions in η. Hence equations labeled by (3.177) imply

that $f_j(\eta) = \frac{F_j(\eta)}{\eta + i\mu_2}, j = 1, 2$ are odd functions in η. Now consider the Cauchy type integral of f_j.

$$\int_{-\infty}^{\infty} \frac{f_j(\tau)}{\tau - \eta} d\tau = \int_{-\infty}^{0} \frac{f_j(\tau)}{\tau - \eta} d\tau + \int_{0}^{\infty} \frac{f_j(\tau)}{\tau - \eta} d\tau$$

In first integral replace η by $-\eta$

$$= \int_{-\infty}^{0} \frac{f_j(\tau)}{\tau - \eta} d\tau + \int_{0}^{\infty} \frac{f_j(\tau)}{\tau - \eta} d\tau$$

use the property that $f_j(\tau)$ is odd, and simplify

$$\begin{aligned} &= \int_{0}^{\infty} \frac{f_j(\tau)}{\tau + \eta} d\tau + \int_{0}^{\infty} \frac{f_j(\tau)}{\tau - \eta} d\tau \\ &= \int_{0}^{\infty} \frac{2\tau f_j(\tau)}{\tau^2 - \eta^2} d\tau. \end{aligned} \tag{3.183}$$

Since $\dfrac{1}{\tau^2 - \eta^2} = -\dfrac{1}{\eta^2} + \dfrac{1}{\tau^2 - \eta^2} + \dfrac{1}{\eta^2}$

$$\frac{1}{\tau^2 - \eta^2} = -\frac{1}{\eta^2} + \frac{\tau^2}{\eta^2(\tau^2 - \eta^2)}, \text{ therefore}$$

$$\int_{-\infty}^{\infty} \frac{f_j(\tau)}{\tau - \eta} d\tau = -\frac{1}{\eta^2} \int_{0}^{\infty} 2\tau f_j(\tau) d\tau + \frac{1}{\eta^2} \int_{0}^{\infty} 2\tau^3 \frac{f_j(\tau)}{\tau^2 - \eta^2} d\tau.$$

Observe that $\tau f_j(\tau) = \dfrac{\tau F_j(\tau)}{\tau + i\mu_2} \in L_1(0, \infty)$ i.e.

$$\int_{0}^{\infty} 2\tau f_j(\tau) d\tau < \infty \Leftrightarrow f_j(\tau) \in L_1(0, \infty), \text{ hence}$$

$$\int_{-\infty}^{\infty} \frac{f_j(\tau)}{\tau - \eta} d\tau = O(\frac{1}{\eta^2}), \text{ as } \eta \to \infty.$$

Using the Sokhotski Plemelj formulae given in theorem 3.1.1 to the function $f_j(\eta) = \frac{F_j(\eta)}{\eta + i\mu_2}$, we have

$$f_j(\eta) = \frac{F_j(\eta)}{\eta + i\mu_2} = \psi_j^+(\eta) - \psi_j^-(\eta), \tag{3.184}$$

where $\psi_j^+(\eta)$ and $\psi_j^-(\eta)$ are the analytic functions in upper and lower half η-complex planes, respectively, and $\psi_j(\eta)$ is the Cauchy type integral defined by

$$\begin{aligned} \psi_j(\eta) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\tau - \eta} f_j(\tau) d\tau, \quad \forall \eta \in \mathbb{C} \setminus \mathbb{R}, \text{ use equation (3.183)} \\ &= \frac{1}{2\pi i} \int_{0}^{\infty} \frac{2\tau}{\tau^2 - \eta^2} f_j(\tau) d\tau. \end{aligned} \tag{3.185}$$

Use the value of $f_j(\tau)$ from equation (3.184) in equation (3.185), and simplify to get

$$\begin{aligned}\psi_j(\eta) &= \frac{1}{\pi i}\int_0^\infty \frac{\tau}{\tau^2-\eta^2}\frac{F_j(\tau)}{\tau+i\mu_2}d\tau,\\ \psi_j(\eta) &= \psi_j(-\eta).\end{aligned} \tag{3.186}$$

Let

$$\psi_j^+(\eta) = \lim_{z\to\eta}\psi_j(\eta), \quad z\in D^+, \text{ and } \psi_j^-(\eta) = \lim_{z\to\eta}\psi_j(\eta), \quad z\in D^-. \tag{3.187}$$

So, equation (3.186) satisfies the following realtions.

$$\psi_j^+(\eta) = \psi_j^-(-\eta), \quad \forall \quad \eta\in\mathbb{C}^+ \tag{3.188}$$

$$\psi_j^-(\eta) = \psi_j^+(-\eta), \quad \forall \quad \eta\in\mathbb{C}^- \tag{3.189}$$

Divide the scalar RHP defined by equation (3.178) by $\eta+i\mu_2$, and insert the value of $\frac{F_j(\eta)}{\eta+i\mu_2}$ from equation (3.184) in the resulting equation to get

$$\frac{\phi_j^+(\eta)}{\eta+i\mu_2} - \psi_j^+(\eta) = -\frac{\phi_j^-(\eta)}{\eta-i\mu_2} - \psi_j^-(\eta), \quad \forall \quad \eta\in\mathbb{R}. \tag{3.190}$$

Since $\frac{\phi_j^+(\eta)}{\eta+i\mu_2} - \psi_j^+(\eta)$ and $-\frac{\phi_j^-(\eta)}{\eta-i\mu_2} - \psi_j^-(\eta)$ are analytic functions in the upper and lower half η-complex planes, respectively, and satisfy (3.190), hence the theorem on analytic continuation 2.3.1 implies $\frac{\phi_j^+(\eta)}{\eta+i\mu_2} - \psi_j^+(\eta) = -\frac{\phi_j^-(\eta)}{\eta-i\mu_2} - \psi_j^-(\eta)$ is analytic everywhere in the η-complex plane, and we notice that it is vanishing at ∞. Hence by generalized Liouville's theorem 2.3.2, we have

$$\frac{\phi_j^+(\eta)}{\eta+i\mu_2} - \psi_j^+(\eta) = -\frac{\phi_j^-(\eta)}{\eta-i\mu_2} - \psi_j^-(\eta) = 0. \tag{3.191}$$

Equate each term to zero, and simplify to get

$$\phi_j^+(\eta) = (\eta+i\mu_2)\psi_j^+(\eta) = O(\frac{1}{\eta}) \text{ as } \eta\to\infty, \tag{3.192}$$

$$\phi_j^-(\eta) = -(\eta-i\mu_2)\psi_j^-(\eta) = O(\frac{1}{\eta}) \text{ as } \eta\to\infty. \tag{3.193}$$

It is observed that

$$\phi_j^-(-\eta) = -(-\eta - i\mu_2)\psi_j^-(-\eta) = (\eta + i\mu_2)\psi_j^-(-\eta), \text{ use equation (3.188) to get}$$

$$\phi_j^-(-\eta) = (\eta + i\mu_2)\psi_j^+(\eta)$$

$$\phi_j^-(-\eta) = \phi_j^+(\eta) \quad \forall \eta \in \mathbb{C}^+, \quad j = 1, 2. \tag{3.194}$$

Similarly

$$\phi_j^+(-\eta) = (-\eta + i\mu_2)\psi_j^+(-\eta) = -(\eta - i\mu_2)\psi_j^-(-\eta), \text{ use equation (3.189) to get}$$

$$\phi_j^+(-\eta) = -(\eta - i\mu_2)\psi_j^-(\eta)$$

$$\phi_j^+(-\eta) = \phi_j^-(\eta) \quad \forall \eta \in \mathbb{C}^-, \quad j = 1, 2. \tag{3.195}$$

Equations (3.194) and (3.195) describe the symmetry condition for the order two vector RHP defined by equation (3.171).

3.3.1 Formulae for reconstruction of solution of BVP of Helmholtz equation in a semi-infinite strip Ω

We can reconstruct the solution of the given BVP of the Helmholtz equation in a semi-infinite strip subject to the impedance boundary conditions, on the boundaries of the semi-infinite strip by using the inverse Laplace transforms given in definitions 3.1.1 and 3.1.2. To derive the relation for $q(x, 0)$, we consider the following inverse Laplace transform given in definition 3.1.1:

$$q(x, y_j) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{q}(\eta, y_j)e^{-i\eta x}d\eta, \quad j = 0, 1, \quad y_0 = 0, \quad y_1 = a, \quad a > 0. \tag{3.196}$$

For $y_0 = 0$, equation (3.196) gives the following result:

$$q(x, 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{q}(\eta, 0)e^{-i\eta x}d\eta. \tag{3.197}$$

The scalar RHPs defined by equation (3.178) gives

$$\tilde{q}(\eta, 0) = -\frac{\eta + i\mu_2}{\eta - i\mu_2}\tilde{q}(-\eta, 0) + F_1(\eta), \quad \eta \in \mathbb{R}. \tag{3.198}$$

Insert the value of $\tilde{q}(\eta,0)$ from equation (3.198) in equation (3.197) to get

$$\begin{aligned} q(x,0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}[-\frac{\eta+i\mu_2}{\eta-i\mu_2}\tilde{q}(-\eta,0)+F_1(\eta)]e^{-i\eta x}d\eta, \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}-\frac{\eta+i\mu_2}{\eta-i\mu_2}\tilde{q}(-\eta,0)e^{-i\eta x}d\eta+\frac{1}{2\pi}\int_{-\infty}^{\infty}F_1(\eta)e^{-i\eta x}d\eta. \end{aligned} \tag{3.199}$$

Since $\frac{\eta+i\mu_2}{\eta-i\mu_2}\tilde{q}(-\eta,0)e^{-i\eta x}$ is an analytic function in the lower half η-complex plane $\mathbb{C}^-$ and $x>0$, so, draw a closed contour $(R,-R)\cup C_R^-$ as shown in the figure 3.5. Application of the Cauchy's theorem gives the following result:

$$\int_{R}^{-R}-\frac{\eta+i\mu_2}{\eta-i\mu_2}\tilde{q}(-\eta,0)e^{-i\eta x}d\eta+\int_{c_R^-}-\frac{\eta+i\mu_2}{\eta-i\mu_2}\tilde{q}(-\eta,0)e^{-i\eta x}d\eta=0. \tag{3.200}$$

The integrand $\frac{\eta+i\mu_2}{\eta-i\mu_2}\tilde{q}(-\eta,0)e^{-i\eta x}$ in the second integral satisfies all the axioms of Jordan's lemma, so

$$\int_{C_R^-}-\frac{\eta+i\mu_2}{\eta-i\mu_2}\tilde{q}(-\eta,0)e^{-i\eta x}d\eta=0,\ \text{as } R\to\infty. \tag{3.201}$$

When $R\to\infty$ equations (3.200) and (3.201) imply

$$\begin{aligned} &\int_{R}^{-R}-\frac{\eta+i\mu_2}{\eta-i\mu_2}\tilde{q}(-\eta,0)e^{-i\eta x}d\eta=0 \text{ as } R\to\infty \text{ or} \\ &\int_{-\infty}^{\infty}-\frac{\eta+i\mu_2}{\eta-i\mu_2}\tilde{q}(-\eta,0)e^{-i\eta x}d\eta=0. \end{aligned} \tag{3.202}$$

Now equations (3.199) and (3.202) give the following result:

$$q(x,0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F_1(\eta)e^{-i\eta x}d\eta. \tag{3.203}$$

Now to derive the relation for $q(x,a)$, insert $y_1=a$, in equation (3.196) to get

$$q(x,a)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{q}(\eta,a)e^{-i\eta x}d\eta. \tag{3.204}$$

The scalar RHPs defined by equation (3.178) gives

$$\tilde{q}(\eta,a)=-\frac{\eta+i\mu_2}{\eta-i\mu_2}\tilde{q}(-\eta,a)+F_2(\eta),\quad \eta\in\mathbb{R}. \tag{3.205}$$

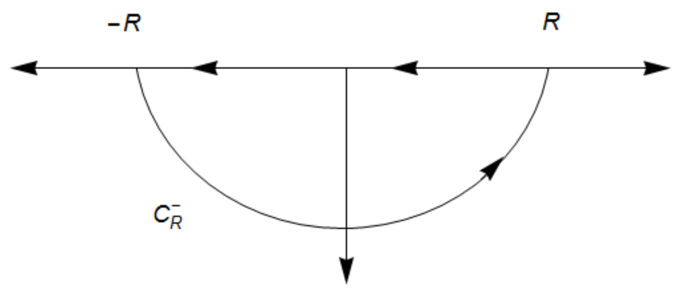

FIGURE 3.4. Contour to evaluate the integrals defining $q(x,0)$ and $q(x,a)$.

Insert the value of $\tilde{q}(\eta,a)$ from equation (3.205) in equation (3.204) to get

$$\begin{aligned} q(x,a) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}[-\frac{\eta+i\mu_2}{\eta-i\mu_2}\tilde{q}(-\eta,a)+F_2(\eta)]e^{-i\eta x}d\eta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}-\frac{\eta+i\mu_2}{\eta-i\mu_2}\tilde{q}(-\eta,a)e^{-i\eta x}d\eta+\frac{1}{2\pi}\int_{-\infty}^{\infty}F_2(\eta)e^{-i\eta x}d\eta. \end{aligned} \tag{3.206}$$

Since $\frac{\eta+i\mu_2}{\eta-i\mu_2}\tilde{q}(-\eta,a)e^{-i\eta x}$ is an anlytic function in the lower half η-complex plane $\mathbb{C}^-$ and $x>0$, so, we draw a closed contour $(R,-R)\cup C_R^-$ as shown in the figure 3.5. Application of the Cauchy's theorem gives the following result:

$$\int_{R}^{-R}-\frac{\eta+i\mu_2}{\eta-i\mu_2}\tilde{q}(-\eta,a)e^{-i\eta x}d\eta+\int_{c_R^-}-\frac{\eta+i\mu_2}{\eta-i\mu_2}\tilde{q}(-\eta,a)e^{-i\eta x}d\eta=0. \tag{3.207}$$

The integrand $\frac{\eta+i\mu_2}{\eta-i\mu_2}\tilde{q}(-\eta,a)e^{-i\eta x}$ in the second integral satisfies all the axioms of Jordan's lemma, so,

$$\int_{C_R^-}-\frac{\eta+i\mu_2}{\eta-i\mu_2}\tilde{q}(-\eta,a)e^{-i\eta x}d\eta=0,\ \text{as } R\to\infty. \tag{3.208}$$

When $R\to\infty$ equations (3.207) and (3.208) imply

$$\begin{aligned} &\int_{R}^{-R}-\frac{\eta+i\mu_2}{\eta-i\mu_2}\tilde{q}(-\eta,a)e^{-i\eta x}d\eta=0 \text{ as } R\to\infty \text{ or} \\ &\int_{-\infty}^{\infty}-\frac{\eta+i\mu_2}{\eta-i\mu_2}\tilde{q}(-\eta,a)e^{-i\eta x}d\eta=0. \end{aligned} \tag{3.209}$$

Now equations (3.206) and (3.209) give the following result:

$$q(x, a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_2(\eta) e^{-i\eta x} d\eta. \tag{3.210}$$

Note 3.3.1. *To evaluate the contour integrals defined by equations (3.203) and (3.210), the residue theory of complex variables is used. To apply this theory to evaluate these contour integrals, the integrands $F_1(\eta)e^{-i\eta x}$ and $F_2(\eta)e^{-i\eta x}$ should be meromorphic functions of η. Since η is related to the multi-valued function ζ through the relation $\zeta = \sqrt{\eta^2 - k^2}$, the integrands $F_1(\eta)e^{-i\eta x}$ and $F_2(\eta)e^{-i\eta x}$ should be even functions w.r.t ζ, to cancel out the effect of the branch cut in the η-complex plane, and make the integrands meromorphic functions of η in the η-complex plane.*

To find the solution along the side $x = 0, 0 < y < a$, of the semi-infinite strip Ω, use the inverse Laplace transform the operator L_y^{-1} given in definition 3.1.2. Using definition 3.1.2, consider

$$q(0, y) = \frac{1}{2\pi i} \int_{\Gamma} \hat{q}(0, i\zeta) e^{\zeta y} d\zeta, \quad 0 < y < a, \tag{3.211}$$

where $\hat{q}(0, i\zeta)$ is given by equation (3.160). To apply the residue theory of complex variables to evaluate the integral defined by equation (3.211), the integrand $\hat{q}(0, i\zeta)e^{\zeta y}$ should be a meromorphic function of ζ. Since ζ is a multi-valued function in the η-complex plane, and is related to η through the relation $\zeta = \sqrt{\eta^2 - k^2}$. So, we need the integrand $\hat{q}(0, i\zeta)e^{\zeta y}$ to be an even function w.r.t η, to cancel out the effect of branch cut in the η-complex plane, and make the integrand a meromorphic function of ζ in the η-complex plane.

To find solution of the BVP of the Helmholtz equation inside a semi-infinite strip Ω, we use the inverse Laplace transform operator L_x^{-1} given in definition 3.1.1. Consider

$$q(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{q}(\eta,y)e^{-i\eta x}d\eta. \tag{3.212}$$

The residue theory of complex variables is used to evaluate the integral defined by equation (3.212). To solve this integral, we need $\tilde{q}(\eta,y)$ which is given by equation (3.141). To apply the residue theory of complex variables, to evaluate the integral defined by equation (3.212), the integrand $\tilde{q}(\eta,y)e^{-i\eta x}$ in equation (3.212) should satisfy note 3.3.1. In the next section, we have considered a particular case to elaborate the procedure for re-construction of solution on the boundaries of semi-infinite strip Ω, of the given BVP of Helmholtz equation in a semi-infinite strip.

3.3.2 Solution of the BVP of Helmholtz equation in a semi-infinite strip Ω along the vertical boundary: $q(0,y)$

Case study

Example 3.3.1. *For the BVP defined by equations (3.121), (3.122), (3.123) and (3.124), let* $g_0(x) = g_1(x) = 0, g_2(y) = A$ *(constant* $), g(x,y) = 0$. *In this case equations (3.145) and (3.152) after simplification become* $h_0(\eta) = h_1(\eta) = 0$. *Insert these values in equations labeled by (3.177) to get*

$$F_1(\eta) = \frac{2A\eta(\zeta\sinh[a\zeta] + 2\mu_1\sinh^2[\frac{a\zeta}{2}])}{(\eta - i\mu_2)\zeta\Delta}, \tag{3.213}$$

$$F_2(\eta) = \frac{2A\eta(\zeta\sinh[a\zeta] + 2\mu_0\sinh^2[\frac{a\zeta}{2}])}{(\eta - i\mu_2)\zeta\Delta}, \quad \textit{where} \tag{3.214}$$

$$\Delta = (\mu_0 + \mu_1)\zeta\cosh[a\zeta] + (\mu_0\mu_1 + \zeta^2)\sinh[a\zeta]. \tag{3.215}$$

To find the solution along the side $x = 0, 0 < y < a$ *of the semi-infinite strip* Ω*, use definition 3.1.2, to get*

$$q(0, y) = \frac{1}{2\pi i} \int_{\Gamma} \hat{q}(0, i\zeta) e^{\zeta y} d\zeta, \quad \Gamma = (-i\infty, i\infty), \text{ where} \tag{3.216}$$

$\hat{q}(0, i\zeta)$ *is given by equation (3.160).*

$$\begin{aligned} \hat{q}(0, i\zeta) = & \frac{-i(\zeta + \mu_0) \cosh[a\zeta](h_0(-\eta) - h_0(\eta) + \tilde{q}(\eta, 0) - \tilde{q}(-\eta, 0))}{2\eta(\cosh[a\zeta] + \sinh[a\zeta])} - \\ & \frac{i(\zeta + \mu_0) \sinh[a\zeta](h_0(-\eta) - h_0(\eta) + \tilde{q}(\eta, 0) - \tilde{q}(-\eta, 0))}{2\eta(\cosh[a\zeta] + \sinh[a\zeta])} + \\ & \frac{i(\zeta - \mu_1)(h_1(-\eta) - h_1(\eta) + \tilde{q}(\eta, a) - \tilde{q}(-\eta, a))}{2\eta(\cosh[a\zeta] + \sinh[a\zeta])}. \end{aligned} \tag{3.217}$$

Insert $h_0(\eta) = h_1(\eta) = 0$ *in equation (3.217), and simplify to get*

$$\begin{aligned} \hat{q}(0, i\zeta) &= \frac{-i[(\zeta + \mu_0)(\tilde{q}(\eta, 0) - \tilde{q}(-\eta, 0) - (\zeta - \mu_1)(\tilde{q}(\eta, a) - \tilde{q}(-\eta, a))e^{-a\zeta}]}{2\eta} \\ &= i(\zeta - \mu_1) \frac{(\tilde{q}(\eta, a) - \tilde{q}(-\eta, a))e^{-a\zeta}}{2\eta} - i(\zeta + \mu_0) \frac{(\tilde{q}(\eta, 0) - \tilde{q}(-\eta, 0))}{2\eta}. \end{aligned} \tag{3.218}$$

Note that $\frac{\tilde{q}(\eta,0) - \tilde{q}(-\eta,0)}{2\eta}$ *is an even function of* η *in the* η*-complex plane because on upper side of the cut shown in the figure 3.2, we have*

$$\frac{\tilde{q}(\eta, 0) - \tilde{q}(-\eta, 0)}{2\eta} = \frac{\tilde{q}(\eta^+, 0) - \tilde{q}(-\eta^+, 0)}{2\eta} \tag{3.219}$$

η^+ is the value of η on the upper side of the cut. On the lower side of the cut shown in figure 3.2, we have

$$\frac{\tilde{q}(\eta,0)-\tilde{q}(-\eta,0)}{2\eta}=\frac{\tilde{q}(\eta^+,0)-\tilde{q}(-\eta^+,0)}{2\eta^+}. \tag{3.220}$$

Equations (3.219) and (3.220) prove the assertion that $\frac{\tilde{q}(\eta,0)-\tilde{q}(-\eta,0)}{2\eta}$ is an even function w.r.t to η. So, this function is continuous through the cut in η-complex plane. Since $\frac{\tilde{q}(\eta,0)-\tilde{q}(-\eta,0)}{2\eta}$ is continuous through the cut in the η- complex plane, and it is analytic everywhere in the η-complex plane except at a finite number of poles w.r.t ζ, hence it is a meromorphic function w.r.t ζ. We can continue analytically $\tilde{q}(-\eta,0)$ and $\tilde{q}(-\eta,a)$ into the plane $Im(\eta)>0$ by using the boundary condition of the RHP defined by equation (3.178), and is given as

$$\phi_j^+(\eta)=-\frac{\eta+i\mu_2}{\eta-i\mu_2}\phi_j^-(\eta)+F_j(\eta),\quad \eta\in\mathbb{R},\quad j=1,2. \tag{3.221}$$

Express the above equation in component form, and simplify to get

$$\tilde{q}(-\eta,0)=-\frac{\eta-i\mu_2}{\eta+i\mu_2}[\tilde{q}(\eta,0)-F_1(\eta)], \tag{3.222}$$

$$\tilde{q}(-\eta,a)=-\frac{\eta-i\mu_2}{\eta+i\mu_2}[\tilde{q}(\eta,a)-F_2(\eta)]. \tag{3.223}$$

Hence, equations (3.216), (3.218), (3.222) and (3.223) give the following result:

$$q(0,y) = \frac{1}{2\pi}\int_{\Gamma}[-(\zeta+\mu_0)(\frac{\tilde{q}(\eta,0)}{\eta+i\mu_2} - \frac{\eta-i\mu_2}{\eta+i\mu_2}\frac{F_1(\eta)}{2\eta}) \\ + e^{-a\zeta}(\zeta-\mu_1)(\frac{\tilde{q}(\eta,a)}{\eta+i\mu_2} - \frac{\eta-i\mu_2}{\eta+i\mu_2}\frac{F_2(\eta)}{2\eta})]e^{\zeta y}d\zeta \tag{3.224}$$

$$q(0,y) = \frac{1}{2\pi}\int_{\Gamma}[\frac{-(\zeta+\mu_0)}{\eta+i\mu_2}\tilde{q}(\eta,0) + \frac{\zeta-\mu_1}{\eta+i\mu_2}e^{-a\zeta}\tilde{q}(\eta,a) + \\ \frac{(\eta-i\mu_2)}{(\eta+i\mu_2)}(\zeta+\mu_0)\frac{F_1(\eta)}{2\eta} - e^{-a\zeta}(\zeta-\mu_1)\frac{\eta-i\mu_2}{\eta+i\mu_2}\frac{F_2(\eta)}{2\eta}]e^{\zeta y}d\zeta \tag{3.225}$$

Using the values of $F_1(\eta)$ and $F_2(\eta)$ from equations (3.213) and (3.214), consider the following expressions:

$$\begin{aligned}
\frac{\eta-i\mu_2}{\eta+i\mu_2}(\zeta+\mu_0)\frac{F_1(\eta)}{2\eta} &= \frac{\zeta+\mu_0}{2\eta}\times\frac{\eta-i\mu_2}{\eta+i\mu_2}[\frac{2A\eta(\zeta\sinh[a\zeta]+2\mu_1\sinh^2[\frac{a\zeta}{2}])}{(\eta-i\mu_2)\zeta\Delta}] \\
&= \frac{(\zeta+\mu_0)A}{\Delta\zeta(\eta+i\mu_2)}(\zeta\sinh[a\zeta]+2\mu_1\sinh^2[\frac{a\zeta}{2}]), \\
\frac{\eta-i\mu_2}{\eta+i\mu_2}(\zeta-\mu_1)e^{-a\zeta}\frac{F_2(\eta)}{2\eta} &= \frac{\zeta-\mu_1}{2\eta}\times\frac{\eta-i\mu_2}{\eta+i\mu_2}e^{-a\zeta}[\frac{2A\eta(\zeta\sinh[a\zeta]+2\mu_0\sinh^2[\frac{a\zeta}{2}])}{(\eta-i\mu_2)\zeta\Delta}] \\
&= \frac{(\zeta-\mu_1)A}{\Delta\zeta(\eta+i\mu_2)}e^{-a\zeta}(\zeta\sinh[a\zeta]+2\mu_0\sinh^2[\frac{a\zeta}{2}]).
\end{aligned} \tag{3.226}$$

Using equations labeled by (3.226), consider the following expressions:

$$-\frac{(\zeta+\mu_0)}{\eta+i\mu_2}\tilde{q}(\eta,0) + \frac{\eta-i\mu_2}{\eta+i\mu_2}(\zeta+\mu_0)\frac{F_1(\eta)}{2\eta} = \frac{(\zeta+\mu_0)}{\eta+i\mu_2}[-\tilde{q}(\eta,0) + \frac{A}{\Delta\zeta}\times \\ (\zeta\sinh[a\zeta]+2\mu_1\sinh^2[\frac{a\zeta}{2}])] \tag{3.227}$$

$$\frac{e^{-a\zeta}(\zeta-\mu_1)}{\eta+i\mu_2}\tilde{q}(\eta,a)-e^{-a\zeta}(\zeta-\mu_1)\frac{\eta-i\mu_2}{\eta+i\mu_2}\frac{F_2(\eta)}{2\eta}=\frac{(\zeta-\mu_1)}{\eta+i\mu_2}e^{-a\zeta}[\tilde{q}(\eta,a)-$$

$$\frac{A}{\Delta\zeta}\times(\zeta\sinh[a\zeta]+2\mu_0\sinh^2[\frac{a\zeta}{2}])]. \tag{3.228}$$

Use expressions labeled by (3.227) and (3.228) in equation (3.225), and simplify to get

$$q(0,y)=I_1(y)+I_2(y),\ \textit{where} \tag{3.229}$$

$$I_1(y)=\frac{1}{2\pi}\int_\Gamma\frac{(\zeta+\mu_0)}{\eta+i\mu_2}e^{\zeta y}[-\tilde{q}(\eta,0)+\frac{A}{\Delta\zeta}(\zeta\sinh[a\zeta]+2\mu_1\sinh^2[\frac{a\zeta}{2}])]d\zeta, \tag{3.230}$$

$$I_2(y)=\frac{1}{2\pi}\int_\Gamma\frac{(\zeta-\mu_1)}{\eta+i\mu_2}e^{(y-a)\zeta}[\tilde{q}(\eta,a)-\frac{A}{\Delta\zeta}(\zeta\sinh[a\zeta]+2\mu_0\sinh^2[\frac{a\zeta}{2}])]d\zeta. \tag{3.231}$$

To evaluate the integrals $I_1(y)$ and $I_2(y)$, we need the zeroes of

$$\Delta=(\mu_0+\mu_1)\zeta\cosh[a\zeta]+(\mu_0\mu_1+\zeta^2)\sinh[a\zeta]. \tag{3.232}$$

Now equation (3.230) can be expressed as

$$I_1(y)=-\frac{1}{2\pi}\int_\Gamma\frac{(\zeta+\mu_0)}{\eta+i\mu_2}e^{\zeta y}\tilde{q}(\eta,0)d\zeta+\frac{1}{2\pi}\int_\Gamma\frac{(\zeta+\mu_0)}{\eta+i\mu_2}e^{\zeta y}\times\frac{A}{\Delta\zeta}(\zeta\sinh[a\zeta]$$

$$+2\mu_1\sinh^2[\frac{a\zeta}{2}])d\zeta. \tag{3.233}$$

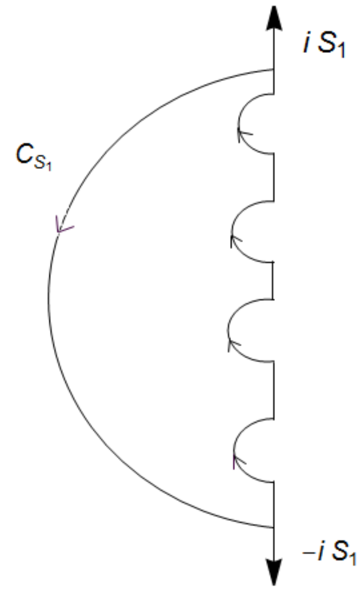

FIGURE 3.5. Contour to evaluate the integral $I_1(y)$ defining $q(0, y)$.

Note that $\Gamma = (-i\infty, i\infty)$. *Consider the contour* S_1 *constructed by removing the line segments* r_i *from a line segment* R. *Since* $y > 0$, *we enclose the contour* $(-iS_1, iS_1)$ *by drawing semi circular arcs each of radius* r_i *such that each singularity of the integrand in* $I_1(y)$ *is to be on the right side of* $(-iS_1, iS_1)$, *then we draw a semi-circle* C_{S_1} *of radius* S_1, *to the left side of* $(-iS_1, iS_1)$. *In this way, all singularities of the integrand in* $I_1(y)$ *are outside the closed contour* $C_{S_1} \cup (-iS_1, iS_1) \cup \gamma 1$, *as shown in figure 3.5. Note that* γ_1 *is the union of all small semi circles* C_{r_i} *of radii* r_i *as shown in figure 3.5. We observe that* $\tilde{q}(\eta, 0)$ *is an analytic function in* $\mathbb{C}^+$, *so, it is analytic in the region bounded by* $C_{S_1} \cup (-iS_1, iS_1) \cup \gamma_1$ *which is in* $\mathbb{C}^+$. *By using the boundary condition of the scalar RHP defined by equation (3.178),*

$\tilde{q}(\eta,0)$ can be analytically continued to $\mathbb{C}^-$, and hence it is analytic in the region in $\mathbb{C}^-$ which is enclosed in contour $C_{S_1}\cup(-iS_1,iS_1)\cup\gamma_1$. Hence, $\tilde{q}(\eta,0)$ is an analytic function in the region bounded by the closed contour $C_{S_1}\cup(-iS_1,iS_1)\cup\gamma_1$. Using Cauchy's theorem we have

$$\begin{aligned}&\int_{-iS_1}^{iS_1}\frac{(\zeta+\mu_0)}{\eta+i\mu_2}e^{\zeta y}\tilde{q}(\eta,0)d\zeta+\int_{C_{S_1}}\frac{(\zeta+\mu_0)}{\eta+i\mu_2}e^{\zeta y}\tilde{q}(\eta,0)d\zeta\\&+\int_{\gamma_1}\frac{(\zeta+\mu_0)}{\eta+i\mu_2}e^{\zeta y}\tilde{q}(\eta,0)d\zeta=0\end{aligned}\tag{3.234}$$

Simplify the above expression to get

$$\begin{aligned}&\int_{-iS_1}^{iS_1}\frac{(\zeta+\mu_0)}{\eta+i\mu_2}e^{\zeta y}\tilde{q}(\eta,0)d\zeta+\int_{C_{S_1}}\frac{(\zeta+\mu_0)}{\eta+i\mu_2}e^{\zeta y}\tilde{q}(\eta,0)d\zeta\\&+\sum_{i=1}^{n_0}\int_{C_{r_i}}\frac{(\zeta+\mu_0)}{\eta+i\mu_2}e^{\zeta y}\tilde{q}(\eta,0)d\zeta=0.\end{aligned}\tag{3.235}$$

Note that the integrand in the second integral in equation (3.235) satisfies all axioms of Jordan's lemma, so, for $S_1\to\infty$ the second integral is vanishing. Also, for each r_i, the third integral in equation (3.235) is vanishing because the integrand is analytic in each small semi circle C_{r_i}. When $S_1\to\infty$ and $r_i\to 0$ for each i, then equation (3.235) gives the following result.

$$\int_{-i\infty}^{i\infty}\frac{(\zeta+\mu_0)}{\eta+i\mu_2}e^{\zeta y}\tilde{q}(\eta,0)d\zeta=0.\tag{3.236}$$

Use value of the integral from equation (3.236) in equation (3.230) to get

$$I_1(y) = \frac{1}{2\pi}\int_{\Gamma}\frac{(\zeta+\mu_0)}{\eta+i\mu_2}e^{\zeta y}\frac{A}{\Delta\zeta}(\zeta\sinh[a\zeta]+2\mu_1\sinh^2[\frac{a\zeta}{2}])d\zeta. \tag{3.237}$$

On pages 140 and 142, we have used a procedure to evaluate $\int_{-i\infty}^{i\infty}\frac{(\zeta+\mu_0)}{\eta+i\mu_2}e^{\zeta y}\tilde{q}(\eta,0)d\zeta$.

Now, using that procedure on pages 140 and 142, for $I_1(y)$ *given by equation (3.237), we have*

$$\begin{aligned}
&\int_{-iS_1}^{iS_1}\frac{g(\zeta,\eta)}{2\pi}d\zeta+\int_{C_{S_1}}\frac{g(\zeta,\eta)}{2\pi}d\zeta+\int_{\gamma_1}\frac{g(\zeta,\eta)}{2\pi}d\zeta=0\\
&\int_{-iS_1}^{iS_1}\frac{g(\zeta,\eta)}{2\pi}d\zeta+\int_{C_{S_1}}\frac{g(\zeta,\eta)}{2\pi}d\zeta+\sum_{i=1}^{n_0}\int_{C_{r_i}}\frac{g(\zeta,\eta)}{2\pi}d\zeta=0. \qquad (3.238)\\
&g(\zeta,\eta)=\frac{(\zeta+\mu_0)}{\eta+i\mu_2}e^{y\zeta}\frac{A}{\Delta\zeta}(\zeta\sinh[a\zeta]+2\mu_1\sinh^2[\frac{a\zeta}{2}]).
\end{aligned}$$

Since the integrand $\frac{g(\zeta,\eta)}{2\pi}$ *in the 2nd integral in equation (3.238), satisfies all axioms of Jordan's lemma, hence* $\int_{C_{S_1}}\frac{g(\zeta,\eta)}{2\pi}d\zeta\to 0$ *as* $s_1\to\infty$. *When* $s_1\to\infty$, *and* $r_i\to 0$ *for each* i *then equation (3.238) becomes*

$$\begin{aligned}
&I_1(y)=-\frac{1}{2\pi}(-\pi i)[\sum_{n=1}^{\infty}[Residue|_{\zeta=\zeta_n^+,\eta=i\hat{\zeta_n^+}=i\hat{\zeta_n}}+Residue|_{\zeta=\zeta_n^-,\eta=i\hat{\zeta_n^-}=i\hat{\zeta_n}}]g(\zeta,\eta)),\\
&g(\zeta,\eta)=\frac{(\zeta+\mu_0)}{\eta+i\mu_2}e^{y\zeta}\frac{A}{\Delta\zeta}(\zeta\sinh[a\zeta]+2\mu_1\sinh^2[\frac{a\zeta}{2}]).
\end{aligned} \tag{3.239}$$

Equation (3.239) can be written as

$$I_1(y) = \frac{i}{2}\sum_{n=1}^{\infty} R_{n_1}, \ \ where \tag{3.240}$$

$$R_{n_1} = [Residue|_{\zeta=\zeta_n^+,\eta=i\hat{\zeta_n^+}=i\hat{\zeta_n}} + Residue|_{\zeta=\zeta_n^-,\eta=i\hat{\zeta_n^-}=i\hat{\zeta_n}}]g(\zeta,\eta). \tag{3.241}$$

To find R_{n_1}, we need the residue of $g(\eta,\zeta)$ at simple poles ζ_n^+ and ζ_n^-, for that purpose, let

$$g_1(\eta,\zeta) = \frac{A(\zeta+\mu_0)}{\zeta(\eta+i\mu_2)}e^{\zeta y}[\zeta\sinh[a\zeta] + 2\mu_1\sinh^2[\frac{a\zeta}{2}]], \tag{3.242}$$

$$g_2(\zeta) = \Delta = (\mu_0+\mu_1)\zeta\cosh[a\zeta] + (\mu_0\mu_1+\zeta^2)\sinh[a\zeta]. \tag{3.243}$$

We evaluate $g_1(\eta,\zeta)$ and $\frac{d}{d\zeta}g_2(\zeta)$ at simple poles $\zeta_n^+ = i\lambda_n$ and $\zeta_n^- = -i\lambda_n$ as follows.

$$\begin{aligned}
g_1(\eta,\zeta)|_{\zeta=\zeta_n^+=i\lambda_n,\eta=i\hat{\zeta_n^+}=i\hat{\zeta_n}} &= \frac{A(i\lambda_n+\mu_0)}{(\hat{\zeta_n}+\mu_2)\lambda_n}e^{i\lambda_n y}[\lambda_n\sin[a\lambda_n] + 2\mu_1\sin^2[\frac{a\lambda_n}{2}]] \\
g_1(\eta,\zeta)|_{\zeta=\zeta_n^-=-i\lambda_n,\eta=i\hat{\zeta_n^+}=i\hat{\zeta_n}} &= \frac{A(i\lambda_n-\mu_0)}{(\hat{\zeta_n}+\mu_2)\lambda_n}e^{-i\lambda_n y}[\lambda_n\sin[a\lambda_n] + 2\mu_1\sin^2[\frac{a\lambda_n}{2}]] \\
\Delta_0 = \frac{dg_2}{d\zeta}(\zeta)|_{\zeta=\zeta_n^+=i\lambda_n} &= \frac{dg_2}{d\zeta}(\zeta)|_{\zeta=\zeta_n^-=-i\lambda_n} = (\mu_0+\mu_1+a\mu_0\mu_1-a\lambda_n^2)\cos[a\lambda_n] \\
&\qquad - \lambda_n\sin[a\lambda_n](a\mu_0+a\mu_1+2).
\end{aligned} \tag{3.244}$$

Note that

$$Residue|_{\zeta=\zeta_n^+=i\lambda_n,\eta=i\hat{\zeta_n^+}=i\hat{\zeta_n}}g(\eta,\zeta)=\frac{g_1(\eta,\zeta)|_{\zeta=\zeta_n^+=i\lambda_n,\eta=i\hat{\zeta_n^+}=i\hat{\zeta_n}}}{\frac{dg_2}{d\zeta}(\zeta)|_{\zeta=\zeta_n^+=i\lambda_n}}$$
$$=\frac{A(i\lambda_n+\mu_0)}{\Delta_0(\hat{\zeta_n}+\mu_2)\lambda_n}e^{i\lambda_n y}[\lambda_n\sin[a\lambda_n]+2\mu_1\sin^2[\frac{a\lambda_n}{2}]]$$
$$Residue|_{\zeta=\zeta_n^-=-i\lambda_n,\eta=i\hat{\zeta_n^-}=i\hat{\zeta_n}}g(\eta,\zeta)=\frac{g_1(\eta,\zeta)|_{\zeta=\zeta_n^-=-i\lambda_n,\eta=i\hat{\zeta_n^-}=i\hat{\zeta_n}}}{\frac{dg_2}{d\zeta}(\zeta)|_{\zeta=\zeta_n^-=-i\lambda_n}}$$
$$=\frac{A(i\lambda_n-\mu_0)}{\Delta_0(\hat{\zeta_n}+\mu_2)\lambda_n}e^{-i\lambda_n y}[\lambda_n\sin[a\lambda_n]+2\mu_1\sin^2[\frac{a\lambda_n}{2}]] \tag{3.245}$$

Now equations labeled by (3.241) and (3.245) give the following result:

$$R_{n_1}=-i\frac{A(\lambda_n\sin[a\lambda_n]+2\mu_1\sin^2[\frac{a\lambda_n}{2}])}{\Delta_0\lambda_n(\hat{\zeta_n}+\mu_2)}[(-\lambda_n+i\mu_0)e^{i\lambda_n y}+(-\lambda_n-i\mu_0)e^{-i\lambda_n y}]. \tag{3.246}$$

Use the value of R_{n_1} *from equation (3.246) in equation (3.240) to get*

$$I_1(y)=\sum_{n=1}^{\infty}\frac{A(\lambda_n\sin[a\lambda_n]+2\mu_1\sin^2[\frac{a\lambda_n}{2}])}{2\Delta_0\lambda_n(\hat{\zeta_n}+\mu_2)}[(-\lambda_n+i\mu_0)e^{i\lambda_n y}+(-\lambda_n-i\mu_0)e^{-i\lambda_n y}]. \tag{3.247}$$

To evaluate $I_2(y)$*, write equation (3.231) as follows.*

$$I_2(y)=\frac{1}{2\pi}\int_\Gamma\frac{(\zeta-\mu_1)}{\eta+i\mu_2}e^{(y-a)\zeta}\tilde{q}(\eta,a)d\zeta-\frac{1}{2\pi}\int_\Gamma\frac{(\zeta-\mu_1)}{\eta+i\mu_2}e^{(y-a)\zeta}\frac{A}{\Delta\zeta}\times \tag{3.248}$$
$$(\zeta\sinh[a\zeta]+2\mu_0\sinh^2[\frac{a\zeta}{2}])d\zeta$$

Note that $\Gamma = (-i\infty, i\infty)$. *Consider the contour* R_1 *constructed by removing the line segments* r_i *from a line segment* R. *Since* $0 < y < a$, *so,* $y - a < 0$. *So, we enclose the contour* $(-iR_1, iR_1)$ *by drawing semi circular arcs each of radius* r_i *such that each singularity of the integrand in* $I_2(y)$ *is to the left side of* $(-iR_1, iR_1)$, *then we draw a semi-circle* C_{R_1} *of radius* R_1, *to the right side of* $(-iR_1, iR_1)$. *In this way, all singularities of the integrand in* $I_2(y)$ *are outside the closed contour* $C_{R_1} \cup (-iR_1, iR_1) \cup \gamma$, *as shown in the figure 3.6. Note that* γ *is the union of all small semi circles* C_{r_i} *of radii* r_i. *We observe that* $\tilde{q}(\eta, a)$ *is an analytic function in* $\mathbb{C}^+$, *so, it is analytic in the region bounded by* $C_{R_1} \cup (-iR_1, iR_1) \cup \gamma$ *which is in* $\mathbb{C}^+$. *By using boundary condition of the scalar RHP defined by equation (3.178),* $\tilde{q}(\eta, a)$ *can be analytically continued to* $\mathbb{C}^-$, *and hence it is analytic in the region in* $\mathbb{C}^-$ *which is enclosed in contour* $C_{R_1} \cup (-iR_1, iR_1) \cup \gamma$. *Hence,* $\tilde{q}(\eta, a)$ *is an analytic function in the region bounded by the closed contour* $C_{R_1} \cup (-iR_1, iR_1) \cup \gamma$. *Using Cauchy's theorem we have*

$$\begin{aligned}
&\int_{-iR_1}^{iR_1} \frac{(\zeta - \mu_1)}{\eta + i\mu_2} e^{(y-a)\zeta} \tilde{q}(\eta, a) d\zeta + \int_{C_{R_1}} \frac{(\zeta - \mu_1)}{\eta + i\mu_2} e^{(y-a)\zeta} \tilde{q}(\eta, a) d\zeta \\
&+ \int_{\gamma} \frac{(\zeta - \mu_1)}{\eta + i\mu_2} e^{(y-a)\zeta} \tilde{q}(\eta, a) d\zeta = 0.
\end{aligned} \tag{3.249}$$

Equation (3.249) becomes

$$\int_{-iR_1}^{iR_1} \frac{(\zeta-\mu_1)}{\eta+i\mu_2} e^{(y-a)\zeta}\tilde{q}(\eta,a)d\zeta + \int_{C_{R_1}} \frac{(\zeta-\mu_1)}{\eta+i\mu_2} e^{(y-a)\zeta}\tilde{q}(\eta,a)d\zeta$$
$$+\sum_{i=1}^{n_0}\int_{C_{r_i}} \frac{(\zeta-\mu_1)}{\eta+i\mu_2} e^{(y-a)\zeta}\tilde{q}(\eta,a)d\zeta = 0. \tag{3.250}$$

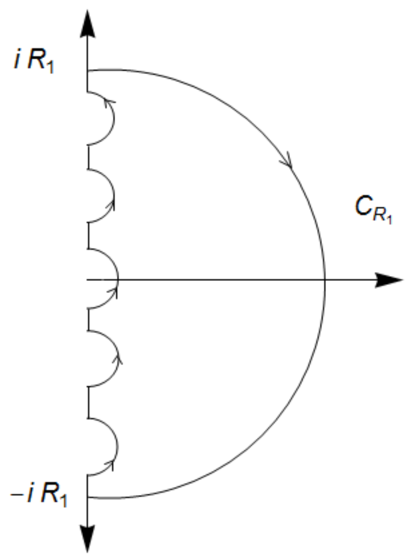

FIGURE 3.6. Contour to evalute the integral $I_2(y)$ defining $q(0,y)$.

Note that the integrand in the second integral in equation (3.250) satisfies all axioms of Jordan's lemma, so, for $R_1 \to \infty$ the second integral is vanishing. Also, for each r_i, the third integral in equation (3.250) is vanishing because the integrand is analytic in each small semi circle, C_{r_i}. When $R_1 \to \infty$ and $r_i \to 0$ for each i, then equation (3.250) becomes

$$\int_{-i\infty}^{i\infty} \frac{(\zeta-\mu_1)}{\eta+i\mu_2} e^{(y-a)\zeta}\tilde{q}(\eta,a)d\zeta = 0. \tag{3.251}$$

Use the value of the integral from equation (3.251) in equation (3.248) to get

$$I_2(y) = -\frac{1}{2\pi}\int_{\Gamma}\frac{(\zeta-\mu_1)}{\eta+i\mu_2}e^{(y-a)\zeta}\frac{A}{\Delta\zeta}(\zeta\sinh[a\zeta]+2\mu_0\sinh^2[\frac{a\zeta}{2}])d\zeta. \tag{3.252}$$

On the page (147), we have used a procedure to evaluate $\int_{-i\infty}^{i\infty}\frac{(\zeta-\mu_1)}{\eta+i\mu_2}e^{(y-a)\zeta}\tilde{q}(\eta,a)d\zeta$.

Now, using that procedure on page (147), for $I_2(y)$ *given by equation (3.252), we have*

$$\begin{aligned}
&\int_{-iR_1}^{iR_1}\frac{h(\zeta,\eta)}{2\pi}d\zeta+\int_{C_{R_1}}\frac{h(\zeta,\eta)}{2\pi}d\zeta+\int_{\gamma}\frac{h(\zeta,\eta)}{2\pi}d\zeta=0,\\
&\int_{-iR_1}^{iR_1}\frac{h(\zeta,\eta)}{2\pi}d\zeta+\int_{C_{R_1}}\frac{h(\zeta,\eta)}{2\pi}d\zeta+\sum_{i=1}^{n_0}\int_{C_{r_i}}\frac{h(\zeta,\eta)}{2\pi}d\zeta=0,\\
&h(\zeta,\eta)=\frac{(\zeta-\mu_1)}{\eta+i\mu_2}e^{(y-a)\zeta}\frac{A}{\Delta\zeta}(\zeta\sinh[a\zeta]+2\mu_0\sinh^2[\frac{a\zeta}{2}]).
\end{aligned} \tag{3.253}$$

Since the integrand $\frac{h(\zeta,\eta)}{2\pi}$ *in the 2nd integral in equation (3.253) satisfies all axioms of Jordan's lemma, hence* $\int_{C_{R_1}}\frac{h(\zeta,\eta)}{2\pi}d\zeta \to 0$ *as* $R_1 \to \infty$. *When* $R_1 \to \infty$, *and* $r_i \to 0$ *for each* i *then equation (3.253) becomes*

$$\begin{aligned}
&I_2(y)=\frac{1}{2\pi}(\pi i\sum_{n=1}^{\infty}[Residue|_{\zeta=\zeta_n^+,\eta=i\hat{\zeta_n^+}=i\hat{\zeta_n}}+Residue|_{\zeta=\zeta_n^-,\eta=i\hat{\zeta_n^-}=i\hat{\zeta_n}}]h(\zeta,\eta)),\\
&h(\zeta,\eta)=\frac{(\zeta-\mu_1)}{\eta+i\mu_2}e^{(y-a)\zeta}\frac{A}{\Delta\zeta}(\zeta\sinh[a\zeta]+2\mu_0\sinh^2[\frac{a\zeta}{2}]).
\end{aligned}$$

$$(3.254)$$

From equation (3.254) $I_2(y)$ can be written as

$$I_2(y) = \frac{i}{2}\sum_{n=1}^{\infty} R_{n_2}, \ \ where \tag{3.255}$$

$$R_{n_2} = [Residue|_{\zeta=\zeta_n^+,\eta=i\hat{\zeta_n^+}=i\hat{\zeta_n}} + Residue|_{\zeta=\zeta_n^-,\eta=i\hat{\zeta_n^-}=i\hat{\zeta_n}}]h(\zeta,\eta)). \tag{3.256}$$

To find R_{n_2}, we need the residue of $h(\eta,\zeta)$ at the simple poles ζ_n^+ and ζ_n^-, for that purpose, let

$$h_1(\eta,\zeta) = \frac{A(\zeta-\mu_1)}{\zeta(\eta+i\mu_2)}e^{(y-a)\zeta}[\zeta\sinh[a\zeta] + 2\mu_0\sinh^2[\frac{a\zeta}{2}]], \tag{3.257}$$

$$h_2(\zeta) = \Delta = (\mu_0+\mu_1)\zeta\cosh[a\zeta] + (\mu_0\mu_1+\zeta^2)\sinh[a\zeta]. \tag{3.258}$$

We evaluate $h_1(\eta,\zeta)$ and $\frac{d}{d\zeta}h_2(\zeta)$ at the simple poles $\zeta_n^+ = i\lambda_n$ and $\zeta_n^- = -i\lambda_n$ as follows.

$$\begin{aligned}
h_1(\eta,\zeta)|_{\zeta=\zeta_n^+=i\lambda_n,\eta=i\hat{\zeta_n^+}=i\hat{\zeta_n}} &= \frac{A(i\lambda_n-\mu_1)}{(\hat{\zeta_n}+\mu_2)\lambda_n}e^{i(y-a)\lambda_n}[\lambda_n\sin[a\lambda_n] + 2\mu_0\sin^2[\frac{a\lambda_n}{2}]] \\
h_1(\eta,\zeta)|_{\zeta=\zeta_n^-=-i\lambda_n,\eta=i\hat{\zeta_n^+}=i\hat{\zeta_n}} &= \frac{A(i\lambda_n+\mu_1)}{(\hat{\zeta_n}+\mu_2)\lambda_n}e^{-i(y-a)\lambda_n}[\lambda_n\sin[a\lambda_n] + 2\mu_0\sin^2[\frac{a\lambda_n}{2}]] \\
\Delta_0 = \frac{dh_2}{d\zeta}(\zeta)|_{\zeta=\zeta_n^+=i\lambda_n} = \frac{dg_2}{d\zeta}(\zeta)|_{\zeta=\zeta_n^-=-i\lambda_n} &= (\mu_0+\mu_1+a\mu_0\mu_1-a\lambda_n^2)\cos[a\lambda_n] \\
&\quad - \lambda_n\sin[a\lambda_n](a\mu_0+a\mu_1+2)
\end{aligned} \tag{3.259}$$

Note that

$$Residue|_{\zeta=\zeta_n^+=i\lambda_n,\eta=i\hat{\zeta_n^+}=i\hat{\zeta_n}}h(\eta,\zeta) = \frac{h_1(\eta,\zeta)|_{\zeta=\zeta_n^+=i\lambda_n,\eta=i\hat{\zeta_n^+}=i\hat{\zeta_n}}}{\frac{dg_2}{d\zeta}(\zeta)|_{\zeta=\zeta_n^+=i\lambda_n}}$$

$$= \frac{A(i\lambda_n-\mu_1)}{\Delta_0(\hat{\zeta_n}+\mu_2)\lambda_n}e^{i(y-a)\lambda_n}[\lambda_n\sin[a\lambda_n]+2\mu_0\sin^2[\frac{a\lambda_n}{2}]],$$

$$Residue|_{\zeta=\zeta_n^-=-i\lambda_n,\eta=i\hat{\zeta_n^-}=i\hat{\zeta_n}}h(\eta,\zeta) = \frac{g_1(\eta,\zeta)|_{\zeta=\zeta_n^-=-i\lambda_n,\eta=i\hat{\zeta_n^-}=i\hat{\zeta_n}}}{\frac{dg_2}{d\zeta}(\zeta)|_{\zeta=\zeta_n^-=-i\lambda_n}}$$

$$= \frac{A(i\lambda_n+\mu_1)}{\Delta_0(\hat{\zeta_n}+\mu_2)\lambda_n}e^{-i(y-a)\lambda_n}[\lambda_n\sin[a\lambda_n]+2\mu_0\sin^2[\frac{a\lambda_n}{2}]]. \tag{3.260}$$

Use the values of expressions in equations labeled by (3.260) in equation (3.256) to get

$$R_{n_2} = \frac{A(\lambda_n\sin[a\lambda_n]+2\mu_0\sin^2[\frac{a\lambda_n}{2}])}{\Delta_0\lambda_n(\hat{\zeta_n}+\mu_2)}[(i\lambda_n-\mu_1)e^{i(y-a)\lambda_n}+e^{-i(y-a)\lambda_n}(i\lambda_n+\mu_1)]. \tag{3.261}$$

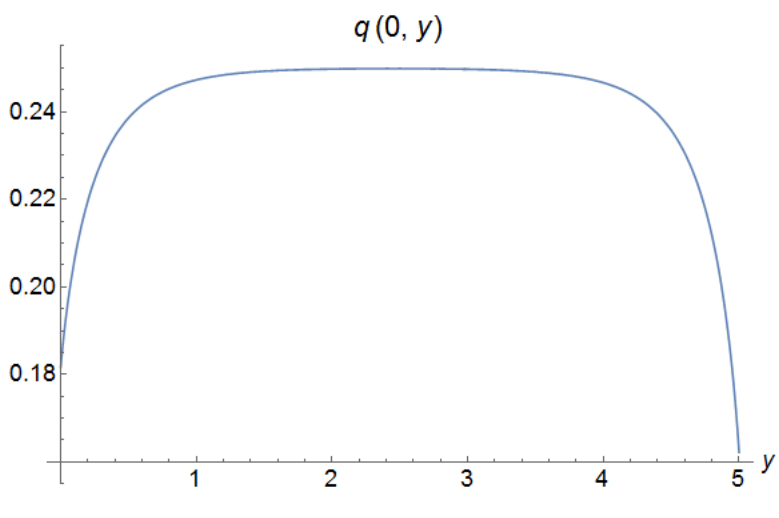

FIGURE 3.7. Solution along the boundary $x = 0, 0 < y < a$ of semi-infinite strip Ω.

Use the value of R_{n_2} from equation (3.261) in equation (3.255), and simplify to get

$$I_2(y) = \sum_{n=1}^{\infty} \frac{A(\lambda_n \sin[a\lambda_n] + 2\mu_0 \sin^2[\frac{a\lambda_n}{2}])}{2\Delta_0\lambda_n(\hat{\zeta_n} + \mu_2)}[(-\lambda_n - i\mu_1)e^{i(y-a)\lambda_n} + e^{-i(y-a)\lambda_n}(-\lambda_n + i\mu_1)]. \tag{3.262}$$

Use the values of $I_1(y)$ and $I_2(y)$ from equations (3.247) and (3.262), and insert in equation (3.229) to get

$$q(0,y) = \sum_{n=1}^{\infty}[\frac{A(\lambda_n \sin[a\lambda_n] + 2\mu_1 \sin^2[\frac{a\lambda_n}{2}])}{2\Delta_0\lambda_n(\hat{\zeta_n} + \mu_2)}\{(-\lambda_n + i\mu_0)e^{i\lambda_n y} + (-\lambda_n - i\mu_0)e^{-i\lambda_n y}\}$$

$$+ \frac{A(\lambda_n \sin[a\lambda_n] + 2\mu_0 \sin^2[\frac{a\lambda_n}{2}])}{2\Delta_0\lambda_n(\hat{\zeta_n} + \mu_2)}[(-\lambda_n - i\mu_1)e^{i(y-a)\lambda_n} + e^{-i(y-a)\lambda_n}(-\lambda_n + i\mu_1)]$$

$$q(0,y) = \sum_{n=1}^{\infty} \frac{A}{d_0}[d_1(d_{11}e^{i\lambda_n y} + d_{12}e^{-i\lambda_n y}) + d_2(d_{21}e^{i(y-a)\lambda_n} + d_{22}e^{-i(y-a)\lambda_n})]. \tag{3.263}$$

Note that

$$d_{11} = -\lambda_n + i\mu_0, \quad d_{12} = -(\lambda_n + i\mu_0), \quad d_{21} = -(\lambda_n + i\mu_1), \tag{3.264}$$

$$d_{22} = -\lambda_n + i\mu_1, \quad d_0 = 2\lambda_n(\hat{\zeta_n} + \mu_2)\Delta_0, \tag{3.265}$$

$$d_1 = \lambda_n \sin[a\lambda_n] + 2\mu_1 \sin^2[\frac{a\lambda_n}{2}], \quad d2 = \lambda_n \sin[a\lambda_n] + 2\mu_0 \sin^2[\frac{a\lambda_n}{2}], \tag{3.266}$$

$$\Delta_0 = (\mu_0 + \mu_1 + a\mu_0\mu_1 - a\lambda_n^2)\cos[a\lambda_n] - \lambda_n \sin[a\lambda_n](a\mu_0 + a\mu_1 + 2). \tag{3.267}$$

Now consider a particular case, in which $A = 1, \mu_0 = 2, \mu_1 = 3, \mu_2 = 2, a = 5, k = 2i$. *In this case the simple poles are* $\zeta = i\lambda_n, \zeta = -i\lambda_n$, *and correspondingly we have* $\eta = i\hat{\zeta_n^+} = i\hat{\zeta_n^-} = i\hat{\zeta_n} = i\sqrt{\lambda_n^2 - k^2}$. *Note that* λ_n *are given in Table 2.3 on page 90. Figure 3.7 shows the solution of the BVP of the Helmholtz equation in the semi-infinite strip subject to the impedance boundary conditions, along the side* $x = 0, 0 < y < a$.

3.3.3 Solution of BVP of the Helmholtz equation in a semi-infinite strip, along the horizontal boundaries: $q(x,0), q(x,a)$

Equation (3.203) gives the solution $q(x,0)$

$$q(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\eta)e^{-i\eta x}d\eta. \tag{3.268}$$

$F_1(\eta)$ *is given by equation (3.213), and observe that* $F_1(\eta)$ *can be written as* $F_1(\eta,\zeta)$. *Equation (3.213) shows that* $F_1(\eta,\zeta) = F_1(\eta,-\zeta)$. *So,* $F_1(\eta) = F_1(\eta,\zeta)$ *is an even function w.r.t.* ζ. *We know that* η *is related to the multi-valued function* ζ *through the relation* $\zeta = \sqrt{\eta^2 - k^2}$. *Since* $F_1(\eta) = F_1(\eta,\zeta)$ *is an even function w.r.t* ζ, *so, it cancels out effect of the branch cut on values of* $F_1(\eta) = F_1(\eta,\zeta)$ *in the* η-*complex plane, so, that* $F_1(\eta) = F_1(\eta,\zeta)$ *is continuous through the cut in* η-*complex plane. Hence* $F_1(\eta)$ *is a meromorphic function of* η *in* η-*complex plane. To evalu-*

ate the integral given by equation (3.268), we note that $x > 0$, so, we enclose the contour $(-R, R)$ by drawing a semi- circle C_R^- in the lower half η-complex plane. The poles of integrand $F_1(\eta)e^{-i\eta x}$ are found by solving the following transcendental equation:

$$\Delta = (\mu_0 + \mu_1)\zeta \cosh[a\zeta] + (\mu_0\mu_1 + \zeta^2)\sinh[a\zeta]. \tag{3.269}$$

By substituting $\zeta = -i\lambda_n$ in equation (3.269), zeroes of the resulting equation are found by "Burniston-Siewert method for solving certain transcendental equations" in section 2.4. Notice that the poles of the integrand $F_1(\eta)e^{-i\eta x}$ inside the closed contour $(-R, R) \cup C_R^-$, are $\zeta = -i\lambda_n$ and $\eta = -i\hat{\zeta_n}$, where $\hat{\zeta_n} = \sqrt{\lambda_n^2 - k^2}, Re(\hat{\zeta_n}) > 0$. Note that λ_n are given in table 2.3 on page 90 Now apply Cauchy's residue theorem to the integrand $F_1(\eta)e^{-i\eta x}$ in the region enclosed by the contour $(R, -R) \cup C_R^-$, to get

$$\int_R^{-R} F_1(\eta)e^{-i\eta x}d\eta + \int_{C_R^-} F_1(\eta)e^{-i\eta x}d\eta = 2\pi i \sum_{n=1}^{\infty} Residue \quad S(\eta,\zeta)|_{\zeta=-i\lambda_n, \eta=-i\hat{\zeta_n}}. \tag{3.270}$$

The integrand $F_1(\eta)e^{-i\eta x}$ in the first integral in equation (3.270)is satisfying all axiom's of Jordan's lemma, so, $\int_{C_R^-} F_1(\eta)e^{-i\eta x}d\eta \to 0$ as $R \to \infty$. So, application

of the limit $R \to \infty$ and Jordan's lemma to equation (3.270) results in the following equation.

$$\int_{-\infty}^{\infty} F_1(\eta) e^{-i\eta x} d\eta = -2\pi i \sum_{n=1}^{\infty} Residue \quad S(\eta, \zeta)|_{\zeta=-i\lambda_n, \eta=-i\hat{\zeta_n}} \tag{3.271}$$

$$S(\eta, \zeta) = \frac{2A\eta(\zeta \sinh[a\zeta] + 2\mu_1 \sinh[\frac{a\zeta}{2}]^2)e^{-i\eta x}}{\Delta(\eta - i\mu_2)\zeta} \tag{3.272}$$

Equations (3.268) and (3.271) give the following result:

$$q(x, 0) = -i \sum_{n=1}^{\infty} Residue \quad S(\eta, \zeta)|_{\zeta=-i\lambda_n, \eta=-i\hat{\zeta_n}}. \tag{3.273}$$

To find Residue $\quad S(\eta, \zeta)|_{\zeta=-i\lambda_n, \eta=-i\hat{\zeta_n}}$*, let*

$$S_1(\eta, \zeta) = \frac{2A\eta(\zeta \sinh[a\zeta] + 2\mu_1 \sinh[\frac{a\zeta}{2}]^2)e^{-i\eta x}}{(\eta - i\mu_2)\zeta}, \tag{3.274}$$

$$S_2(\zeta) = \Delta = (\mu_0 + \mu_1)\zeta \cosh[a\zeta] + (\mu_0\mu_1 + \zeta^2)\sinh[a\zeta], \zeta = \sqrt{\eta^2 - k^2}, Re(\zeta) > 0. \tag{3.275}$$

Use equation (3.274) to calculate

$$\begin{aligned} &S_1(\eta, \zeta)|_{\zeta=-i\lambda_n, \eta=-i\hat{\zeta_n}} \\ &= \frac{2A(-i\hat{\zeta_n})(-i\lambda_n \sinh[a(-i\lambda_n)] + 2\mu_1 \sinh[\frac{a(-i\lambda_n)}{2}]^2)e^{-i(-i\hat{\zeta_n})x}}{(-i\hat{\zeta_n} - \mu_2)(-i\lambda_n)} \\ &= \frac{-2iA\hat{\zeta_n}(\lambda_n \sin[a\lambda_n] + 2\mu_1 \sin[\frac{a\lambda_n}{2}]^2)e^{-\hat{\zeta_n}x}}{(\hat{\zeta_n} + \mu_2)\lambda_n}. \end{aligned} \tag{3.276}$$

Use equation (3.275) to calculate

$$\frac{dS_2}{d\eta}|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}} = \frac{dS_2}{d\zeta}\frac{d\zeta}{d\eta}|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}}$$

$$= [(\mu_0+\mu_1)\cos[a\lambda_n] + a(-\lambda_n^2+\mu_0\mu_1)\cos[a\lambda_n] - 2\lambda_n\sin[a\lambda_n] \tag{3.277}$$

$$- a\lambda_n(\mu_0+\mu_1)\sin[a\lambda_n]]\frac{\hat{\zeta_n}}{\lambda_n}$$

Equation (3.277) can be expressed as

$$\frac{dS_2}{d\eta}|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}} = \frac{\hat{\zeta_n}\Delta_0}{\lambda_n},\ \ \textit{where}$$

$$\Delta_0 = (\mu_0+\mu_1)\cos[a\lambda_n] + a(-\lambda_n^2+\mu_0\mu_1)\cos[a\lambda_n] - 2\lambda_n\sin[a\lambda_n] \tag{3.278}$$

$$- a\lambda_n(\mu_0+\mu_1)\sin[a\lambda_n].$$

Now Residue $\quad S(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}}$ *is found as follows.*

$$\textit{Residue}\quad S(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}} = \frac{S_1(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}}}{\frac{dS_2}{d\eta}|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}}}$$

$$= \frac{-2iA(\lambda_n\sin[a\lambda_n] + 2\mu_1\sin[\frac{a\lambda_n}{2}]^2)e^{-\hat{\zeta_n}x}}{\Delta_0(\hat{\zeta_n}+\mu_2)} \tag{3.279}$$

From equation (3.279) use value of Residue $\quad S(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}}$ *in equation (3.273), and simplify to get*

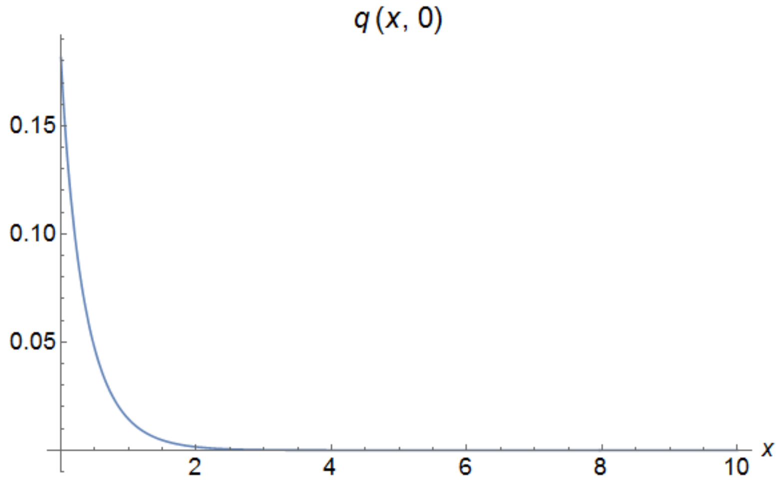

FIGURE 3.8. Solution along the boundary $0 < x < \infty, y = 0$, of semi-infinite strip Ω.

$$q(x,0) = -i\sum_{n=1}^{\infty} Residue \quad S(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}}$$

$$q(x,0) = \sum_{n=1}^{\infty} \frac{-2A(\lambda_n \sin[a\lambda_n] + 2\mu_1 \sin[\frac{a\lambda_n}{2}]^2)e^{-\hat{\zeta_n}x}}{\Delta_0(\hat{\zeta_n}+\mu_2)}, \ \textit{where} \tag{3.280}$$

$$\Delta_0 = (\mu_0+\mu_1)\cos[a\lambda_n] + a(-\lambda_n^2+\mu_0\mu_1)\cos[a\lambda_n] - 2\lambda_n \sin[a\lambda_n]$$

$$- a\lambda_n(\mu_0+\mu_1)\sin[a\lambda_n].$$

In the particular case, in which $A = 1, \mu_0 = 2, \mu_1 = 3, \mu_2 = 2, a = 5, k = 2i$, *simple poles of the integrand* $F_1(\eta)e^{-i\eta x}$ *are* $\zeta = -i\lambda_n$, *and correspondingly we have* $\eta = -i\hat{\zeta_n} = -i\sqrt{\lambda_n^2 - k^2}$. *Note that* λ_n *are given in Table 2.3 on page 90. Figure 3.8 shows the solution of the BVP of the Helmholtz equation in a semi-infinite strip subject to the impedance boundary conditions, along the side* $0 < x < \infty, y = 0$ *of*

the semi-infinite strip. Equation (3.210) gives the solution $q(x,a)$

$$q(x,a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_2(\eta)e^{-i\eta x}d\eta. \tag{3.281}$$

$F_2(\eta)$ *is given by equation (3.214), and observe that* $F_2(\eta)$ *can be written as* $F_2(\eta,\zeta)$*. Equation (3.214) shows that* $F_2(\eta,\zeta) = F_2(\eta,-\zeta)$*. So,* $F_2(\eta) = F_2(\eta,\zeta)$ *is an even function w.r.t.* ζ*. We know that* η *is related to the multi-valued function* ζ *through the relation* $\zeta = \sqrt{\eta^2 - k^2}$*. Since* $F_2(\eta) = F_2(\eta,\zeta)$ *is an even function w.r.t* ζ*, so, it cancels out effect of the branch cut on values of* $F_2(\eta) = F_2(\eta,\zeta)$ *in the* η*-complex plane, so, that* $F_2(\eta) = F_2(\eta,\zeta)$ *is continuous through the cut in* η*-complex plane. Hence* $F_2(\eta)$ *is a meromorphic function of* η *in* η*-complex plane. To evaluate the integral given by equation (3.281), we note that* $x > 0$*, so, we enclose the contour* $(R,-R)$ *by drawing a semi-circle* C_R^- *in lower half* η*-complex plane as shown in figure 3.4. In equation (3.281), the simple poles of integrand* $F_2(\eta)e^{-i\eta x}$ *inside the contour* $(R,-R)\cup C_R^-$*, are found as we did on page 153, and these are* $\zeta = -i\lambda_n$ *and* $\eta = -i\hat{\zeta_n}$*, where* $\hat{\zeta_n} = \sqrt{\lambda_n^2 - k^2}$*,* $\quad Re(\hat{\zeta_n}) > 0$*. Now apply Cauchy's residue theorem to integrand* $F_2(\eta)e^{-i\eta x}$ *in the region enclosed by contour*

$(R,-R)\cup C_R^-$, *to get*

$$\int_R^{-R} F_2(\eta)e^{-i\eta x}d\eta + \int_{C_R^-} F_2(\eta)e^{-i\eta x}d\eta = 2\pi i \sum_{n=1}^{\infty} Residue P(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}}, \tag{3.282}$$

$$P(\eta,\zeta) = F_2(\eta)e^{-i\eta x} = \frac{2A\eta(\zeta\sinh[a\zeta] + 2\mu_0\sinh[\frac{a\zeta}{2}]^2)e^{-i\eta x}}{\Delta(\eta - i\mu_2)\zeta}. \tag{3.283}$$

The integrand $F_2(\eta)e^{-i\eta x}$ *in first integral given by equation (3.282) is satisfying all axiom's of Jordan's lemma, so,* $\int_{C_R^-} F_2(\eta)e^{-i\eta x}d\eta \to 0$ *as* $R\to\infty$. *So, application of limit* $R\to\infty$ *and Jordan's lemma to equation (3.282) results in the following equation.*

$$\int_{-\infty}^{\infty} F_2(\eta)e^{-i\eta x}d\eta = -2\pi i \sum_{n=1}^{\infty} Residue \quad P(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}} \tag{3.284}$$

Use the value of the integral from equation (3.284)in equation (3.281) to get

$$q(x,a) = -i\sum_{n=1}^{\infty} Residue \quad P(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}}. \tag{3.285}$$

To find Residue $p(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}}$, *let*

$$P_1(\eta,\zeta) = \frac{2A\eta(\zeta\sinh[a\zeta] + 2\mu_0\sinh[\frac{a\zeta}{2}]^2)e^{-i\eta x}}{(\eta - i\mu_2)\zeta}, \tag{3.286}$$

$$P_2(\zeta) = \Delta = (\mu_0+\mu_1)\zeta\cosh[a\zeta] + (\mu_0\mu_1+\zeta^2)\sinh[a\zeta], \zeta = \sqrt{\eta^2-k^2}, Re(\zeta) > 0. \tag{3.287}$$

Use equation (3.286) to calculate

$$
\begin{aligned}
&P_1(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}} \\
&= \frac{2A(-i\hat{\zeta_n})(-i\lambda_n\sinh[a(-i\lambda_n)] + 2\mu_0\sinh[\frac{a(-i\lambda_n)}{2}]^2)e^{-i(-i\hat{\zeta_n})x}}{(-i\hat{\zeta_n}-\mu_2)(-i\lambda_n)} \\
&= \frac{-2iA\hat{\zeta_n}(\lambda_n\sin[a\lambda_n] + 2\mu_0\sin[\frac{a\lambda_n}{2}]^2)e^{-\hat{\zeta_n}x}}{(\hat{\zeta_n}+\mu_2)\lambda_n}.
\end{aligned}
\tag{3.288}
$$

Use equation (3.287) to calculate

$$
\begin{aligned}
&\frac{dP_2}{d\eta}|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}} = \frac{dS_2}{d\zeta}\frac{d\zeta}{d\eta}|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}} \\
&\frac{dP_2}{d\eta}|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}} = [(\mu_0+\mu_1)\cos[a\lambda_n] + a(-\lambda_n^2+\mu_0\mu_1)\cos[a\lambda_n] \\
&\qquad - 2\lambda_n\sin[a\lambda_n] - a\lambda_n(\mu_0+\mu_1)\sin[a\lambda_n]]\frac{\hat{\zeta_n}}{\lambda_n} \\
&\frac{dP_2}{d\eta}|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}} = \frac{\hat{\zeta_n}\Delta_0}{\lambda_n},\ \ \textit{where}
\end{aligned}
\tag{3.289}
$$

$$
\Delta_0 = (\mu_0+\mu_1)\cos[a\lambda_n] + a(-\lambda_n^2+\mu_0\mu_1)\cos[a\lambda_n] - 2\lambda_n\sin[a\lambda_n]
$$

$$
- a\lambda_n(\mu_0+\mu_1)\sin[a\lambda_n].
$$

Now Residue $P(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}}$ *is found as follows.*

$$
\begin{aligned}
&\textit{Residue}\quad P(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}} = \frac{P_1(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}}}{\frac{dP_2}{d\eta}|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}}} \\
&= \frac{-2iA(\lambda_n\sin[a\lambda_n] + 2\mu_0\sin[\frac{a\lambda_n}{2}]^2)e^{-\hat{\zeta_n}x}}{\Delta_0(\hat{\zeta_n}+\mu_2)}
\end{aligned}
\tag{3.290}
$$

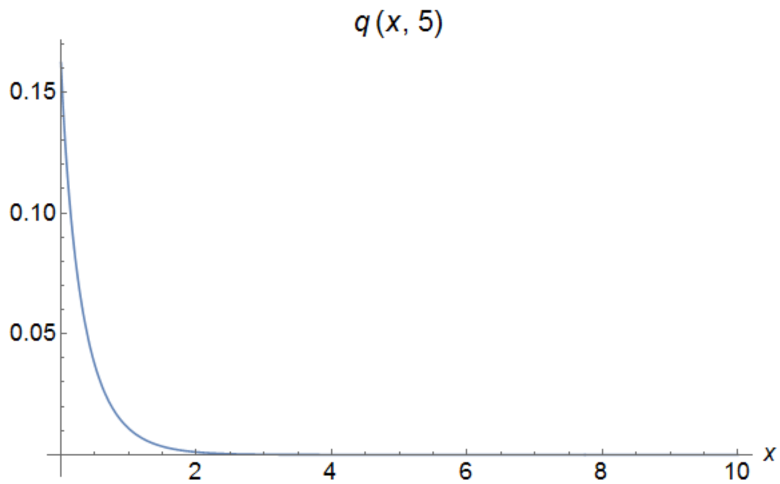

FIGURE 3.9. Solution along the boundary $0 < x < \infty, y = a$, of the semi-infinite strip Ω.

Use the value of Residue $\quad P(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}}$ *from equation (3.290) in equation (3.285) to get*

$$q(x,a) = -i\sum_{n=1}^{\infty} Residue \quad P(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}}$$

$$q(x,a) = \sum_{n=1}^{\infty} \frac{-2A(\lambda_n \sin[a\lambda_n] + 2\mu_0 \sin[\frac{a\lambda_n}{2}]^2)e^{-\hat{\zeta_n}x}}{\Delta_0(\hat{\zeta_n}+\mu_2)}, \quad where \tag{3.291}$$

$$\Delta_0 = (\mu_0+\mu_1)\cos[a\lambda_n] + a(-\lambda_n^2+\mu_0\mu_1)\cos[a\lambda_n] - 2\lambda_n \sin[a\lambda_n]$$

$$- a\lambda_n(\mu_0+\mu_1)\sin[a\lambda_n].$$

In the particular case, in which $A = 1, \mu_0 = 2, \mu_1 = 3, \mu_2 = 2, a = 5, k = 2i$, *simple poles of the integrand* $F_2(\eta)e^{-i\eta x}$ *are* $\zeta = -i\lambda_n$, *and correspondingly we have* $\eta = -i\hat{\zeta_n} = -i\sqrt{\lambda_n^2 - k^2}$. *Note that* λ_n *are given in Table 2.3 on page 90. Figure 3.9 shows the solution of the BVP of the Helmholtz equation in a semi-infinite strip*

subject to the impedance boundary conditions, along the side $0 < x < \infty, y = a$ *of the semi-infinite strip.*

Observation 3.3.1. *In the present case when* $g_0(x) = 0, g_1(x) = 0,\ g_2(y) = A$ *(constant),* $g(x, y) = 0$*, FIT method and the new method give the same solution on boundaries of the semi-infinite strip* Ω*. This gives a verification for the new method.*

3.3.4 Interior solution of BVP of Helmholtz equation in a semi-infinite strip Ω: $q(x, y)$

We know that the inverse transform defined by equation (3.212) can be used to find the solution $q(x, y)$ inside the semi infinite strip Ω. The residue theory of complex variables is used to evaluate $q(x, y)$ from equation (3.212), and the final expression for $q(x, y)$ contains a double series, that representation makes it harder for computational purposes. So, in the present case $g_0(x) = 0, g_1(x) = 0, g_2(y) = A$ (constant), $g(x, y) = 0$, we want to develop a formula to calculate $q(x, y)$ which is computationally more effective than the double series representation obtained by application of inverse transform defined by equation (3.212). In the present case, using FIT method (chapter 2), solution of the BVP of the Helmholtz equation inside a semi-infinite strip Ω is

$$\begin{aligned} q(x, y) = \sum_{n=1}^{\infty} \frac{A e^{-\hat{\zeta}_n x}}{(\mu_2 + \hat{\zeta}_n)\sigma_n^2 \mu_0^2 \lambda_n} [\mu_0(1 - \cos[a\lambda_n]) + \lambda_n \sin[a\lambda_n]] \times \\ [\lambda_n \cos[\lambda_n y] + \mu_0 \sin[\lambda_n y]]. \end{aligned} \tag{3.292}$$

The solution $q(0,y)$ along the side $x=0, 0<y<a$ of the semi-infinite strip Ω is

$$q(0,y)=\sum_{n=1}^{\infty}\frac{A}{(\mu_2+\hat{\zeta}_n)\sigma_n^2\mu_0^2\lambda_n}[\mu_0(1-\cos[a\lambda_n])+\lambda_n\sin[a\lambda_n]]\times \\ [\lambda_n\cos[\lambda_n y]+\mu_0\sin[\lambda_n y]]. \tag{3.293}$$

In the case when $g_0(x)=0, g_1(x)=0, g_2(y)=A$ (constant), $g(x,y)=0$, equations (3.292) and (3.293) reveal a relationship between $q(x,y)$ and $q(0,y)$. We observe that the nth term of the series solution of $q(x,y)$ defined by equation (3.292) can be obtained by multiplying nth term of the series solution of $q(0,y)$ given by equation (3.293) by $e^{-\hat{\zeta}_n x}$. Note that $\hat{\zeta}_n=\sqrt{\lambda_n^2-k^2}$, where λ_n are the eigen values corresponding to the eigen vector $K_{\lambda_n}(y)$ (The kernel of the finite integral transform in FIT method). Using the observation 3.3.1, we can exploit this property to find $q(x,y)$ by the new method. In the case when $g_0(x)=0, g_1(x)=0, g_2(y)=A$ (constant), $g(x,y)=0$, using the new method, equation (3.263) gives solution along the side $x=0, 0<y<a$ of semi-infinite strip Ω as follows:

$$q(0,y)=\sum_{n=1}^{\infty}\frac{A}{d_0}[d_1(d_{11}e^{i\lambda_n y}+d_{12}e^{-i\lambda_n y})+d_2(d_{21}e^{i(y-a)\lambda_n}+d_{22}e^{-i(y-a)\lambda_n})]. \tag{3.294}$$

Note that

$$d_{11}=-\lambda_n+i\mu_0, \quad d_{12}=-(\lambda_n+i\mu_0), \quad d_{21}=-(\lambda_n+i\mu_1), \tag{3.295}$$

$$d_{22}=-\lambda_n+i\mu_1, \quad d_0=2\lambda_n(\hat{\zeta}_n+\mu_2)\Delta_0, \tag{3.296}$$

$$d_1=\lambda_n\sin[a\lambda_n]+2\mu_1\sin^2[\frac{a\lambda_n}{2}], \quad d2=\lambda_n\sin[a\lambda_n]+2\mu_0\sin^2[\frac{a\lambda_n}{2}], \tag{3.297}$$

$$\Delta_0=(\mu_0+\mu_1+a\mu_0\mu_1-a\lambda_n^2)\cos[a\lambda_n]-\lambda_n\sin[a\lambda_n](a\mu_0+a\mu_1+2). \tag{3.298}$$

Hence, in the case $g_0(x)=0, g_1(x)=0, g_2(y)=A$ (constant), $g(x,y)=0$, using the new method, the solution inside the semi-infinite Ω is

$$q(x,y)=\sum_{n=1}^{\infty}\frac{Ae^{-\hat{\zeta}_n x}}{d_0}[d_1(d_{11}e^{i\lambda_n y}+d_{12}e^{-i\lambda_n y})+d_2(d_{21}e^{i(y-a)\lambda_n}+d_{22}e^{-i(y-a)\lambda_n})]. \tag{3.299}$$

In the given case, theoretically, the formula (3.299) to find the solution inside the semi-infinite Ω can be derived from the double series representation of $q(x, y)$ obtained from inverse transform defined by equation (3.212). In particular case, in which $A = 1, \mu_0 = 2, \mu_1 = 3, \mu_2 = 2, a = 5, k = 2i$, the simple poles of the integrand in the integral defining $q(x, y)$ given by equation (3.212), are $\zeta = -i\lambda_n$, and correspondingly we have $\eta = -i\hat{\zeta_n} = -i\sqrt{\lambda_n^2 - k^2}$. Note that λ_n are given in Table 2.3 on page 90. Figure 3.10 shows the solution $q(x, y)$ of the BVP of the Helmholtz equation inside a semi-infinite strip subject to the impedance boundary conditions.

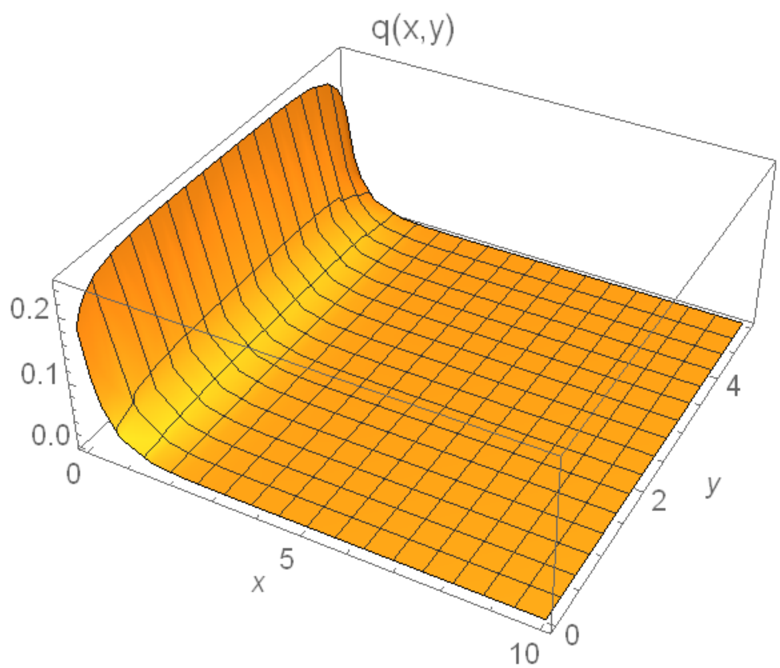

FIGURE 3.10. Solution of the BVP of the Helmholtz equation inside a semi-infinite strip Ω subject to the impedance boundary conditions.

Example 3.3.2. *Consider a particular case of the above example 3.3.1 for which*

$A = 1, \mu_0 = 2, \mu_1 = 3, \mu_2 = 2, a = 5, k = 3 + 2i$. *In this case the solution for the*

BVP of the Helmholtz along the boundaries of Ω and inside Ω are shown in the figures 3.11, 3.12, 3.13 and 3.14.

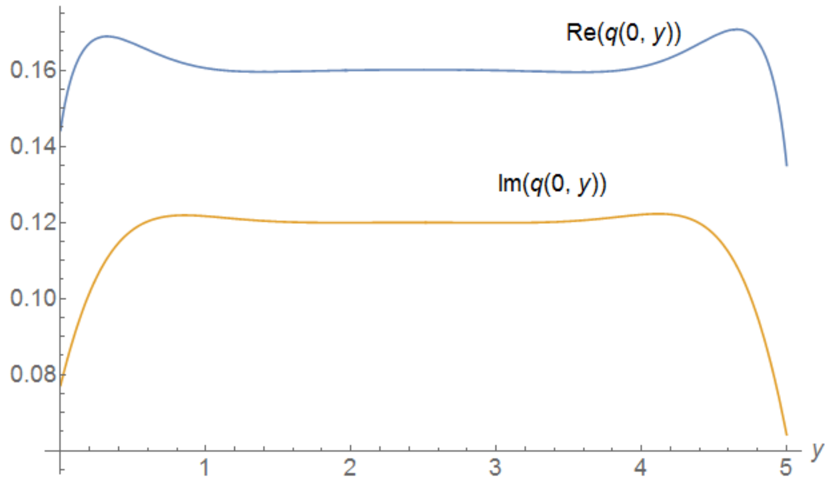

FIGURE 3.11. Solution along the boundary $x = 0, 0 < y < a$, of the semi-infinite strip Ω.

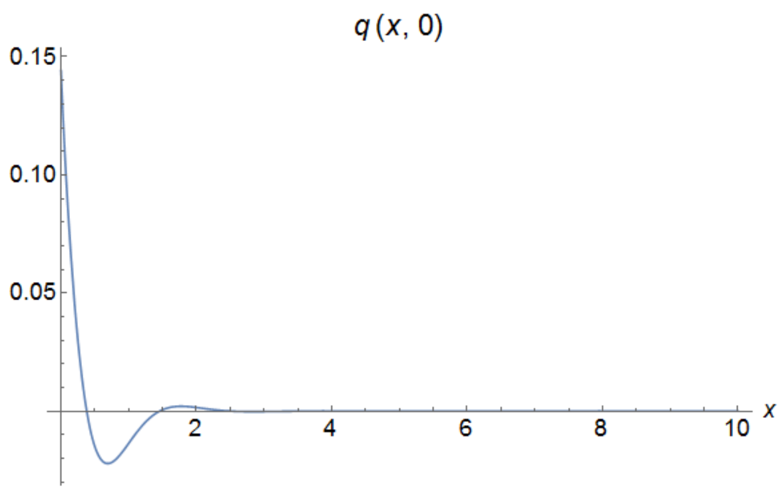

FIGURE 3.12. Solution along the boundary $0 < x < \infty, y = 0$, of the semi-infinite strip Ω.

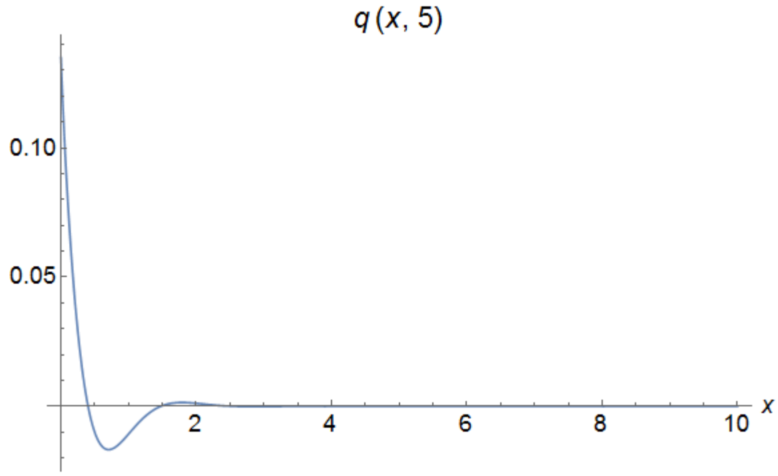

FIGURE 3.13. Solution along the boundary $0 < x < \infty, y = a$, of the semi-infinite strip Ω.

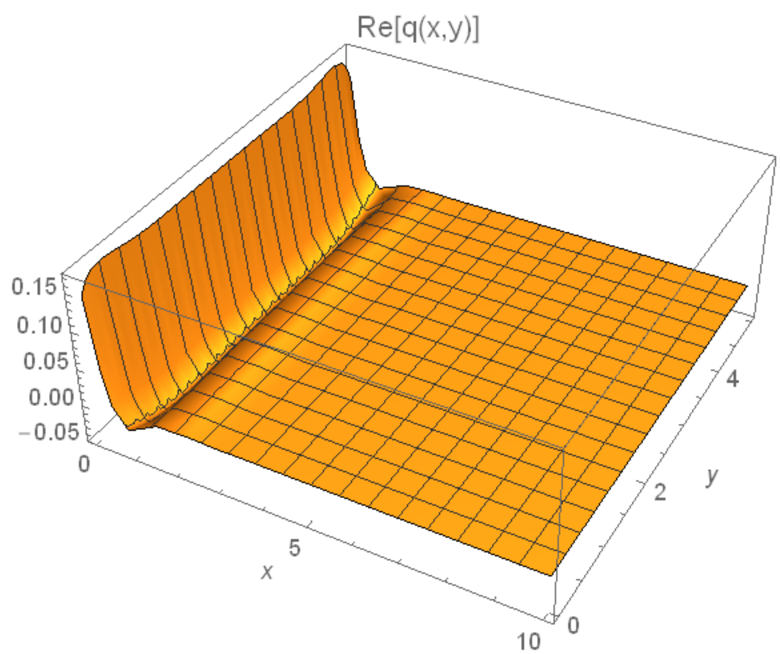

FIGURE 3.14. Solution of BVP of Helmholtz equation inside a semi-infinite strip Ω subject to impedance boundary conditions.

Example 3.3.3. *Consider a particular case of the above example 3.3.1 for which* $A = 1, \mu_0 = 2, \mu_1 = 3, \mu_2 = 2, a = 5, k = 4$. *In this case the solution for the BVP*

of the Helmholtz along the boundaries of Ω *and inside* Ω *are shown in the figures 3.15, 3.16, 3.17 and 3.18.*

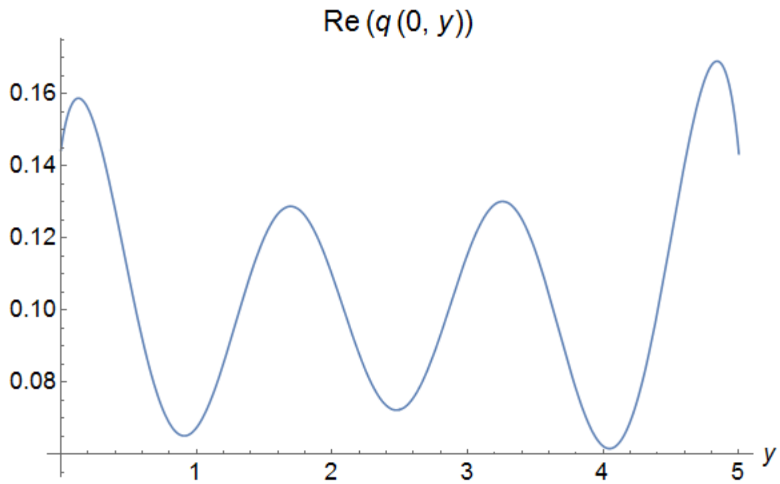

FIGURE 3.15. Solution along the boundary $x = 0, 0 < y < a$, of the semi-infinite strip Ω.

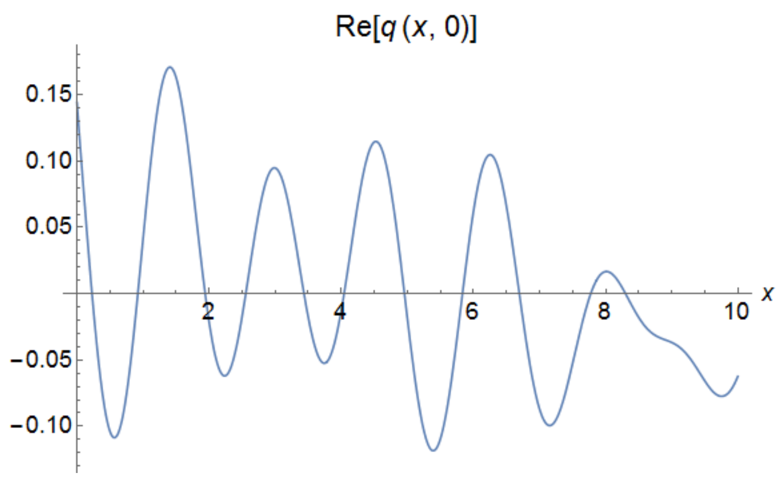

FIGURE 3.16. Solution along the boundary $0 < x < \infty, y = 0$, of the semi-infinite strip Ω.

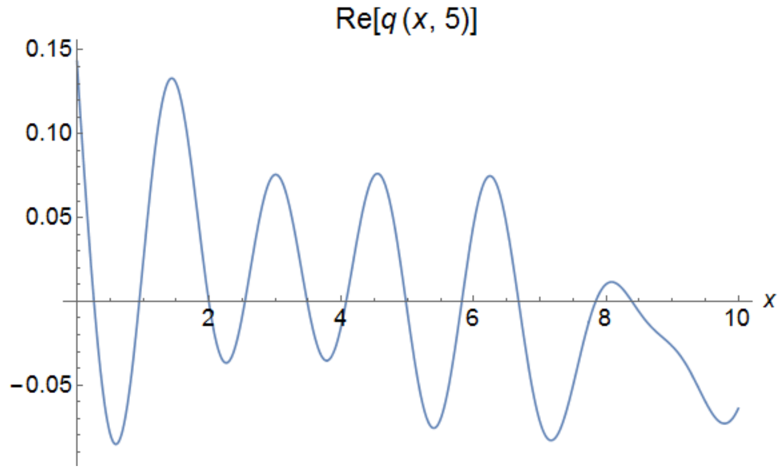

FIGURE 3.17. Solution along the boundary $0 < x < \infty, y = a$, of the semi-infinite strip Ω.

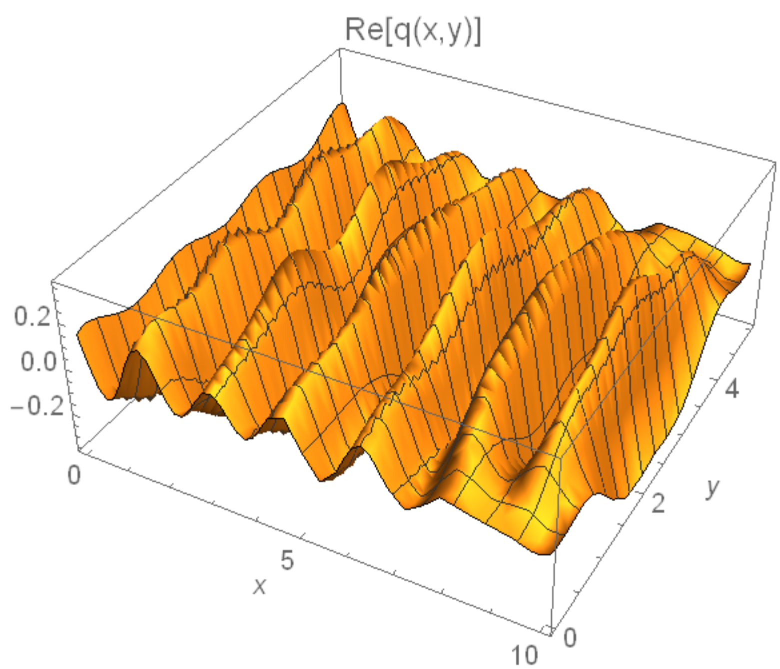

FIGURE 3.18. Solution of the BVP of the Helmholtz equation inside a semi-infinite strip Ω subject to the impedance boundary conditions.

Chapter 4

Boundary value problems of Helmholtz equation and higher order boundary conditions

In this chapter we have applied the new method to solve the BVP of the Helmholtz equation in a semi-infinite strip subject to the higher order boundary conditions.

4.1 Higher order boundary conditions

Consider the Helmholtz equation

$$(\partial_x^2 + \partial_y^2 + k^2)q(x,y) = g(x,y), \quad Im(k) > 0, \quad (x,y) \in \Omega, \tag{4.1}$$

where Ω is a semi-infinite strip shown in figure 4.1 with the corners $z_1 = \infty$, $z_2 = 0$, $z_3 = ia$, $z_4 = \infty + ia$, $a > 0$. Assume that along the sides S_1 and S_2, the impedance boundary conditions are imposed. The side S_0 is an infinite membrane clamped at the point (0,0) to the vertical side S_2. The higher order boundary condition along the side S_0 is derived from [28]. The boundary conditions along the three sides of Ω are:

$$S_0 : (\partial_{xx}^2 + k_0^2)\partial_y q(x,y) + \mu_0 q(x,y) = g_0(x), \quad 0 < x < \infty, \quad y = 0, \tag{4.2}$$

$$S_1 : q_y(x,y) + \mu_1 q(x,y) = g_1(x), \quad 0 < x < \infty, \quad y = a. \tag{4.3}$$

$$S_2 : -q_x(x,y) + \mu_2 q(x,y) = g_2(y), \quad x = 0, \quad 0 < y < a, \tag{4.4}$$

The functions $g_0(x)$, $g_1(x)$ are real valued, and vanish at the points $x = 0$ and $x = \infty$, $\sin\beta_j \neq 0$, $j = 0, 1, 2$.

Since the side S_0 is an infinite membrane, fixed at the point (0,0), so, there is no deflection (vertical displacement) at the point (0,0). This phenomena generates

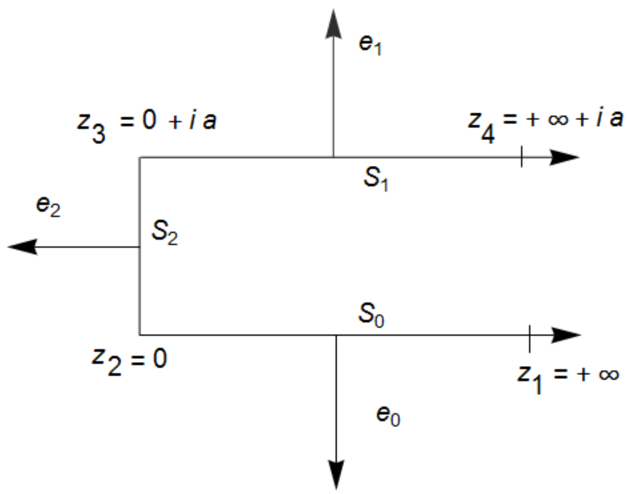

FIGURE 4.1. Impedance and higher order boundary condtions along the sides of Ω.

the following edge condition to get the unique solution of the given BVP.

$$\frac{\partial}{\partial y} q(x, 0) \to 0, \text{ as } x \to 0^+ \tag{4.5}$$

For this particular problem, to discuss the scattering of sound waves by the semi-infinite strip Ω, the parameters are selected in the following way. Note that $k_0 = \omega\sqrt{\frac{m}{T}}$, m is the mass per unit area, and T is the surface tension, so, $k_0 = k_{01} + ik_{02}$, $k_{01}, k_{02} > 0$. Now $k = \frac{\omega}{c}$ is the wave number, c is the sound speed in the fluid, and $\omega = \omega_1 + i\omega_2, \omega_1, \omega_2 > 0$ is the frequency. Hence $k = \frac{\omega}{c}$ indicates that $k = k_1 + ik_2, k_1, k_2 > 0$. Since $\mu_0 = \frac{\rho_0 \omega^2}{T}$, ρ_0 is the mean fluid density, this indicates that $\mu_0 = \mu_{01} + i\mu_{02}$, $\mu_{01}, \mu_{02} > 0$. Due to the impedance boundary conditions along the sides S_1 and $S_2, \mu_1, \mu_2 > 0$.

Application of the Laplace transform the operator L_x from definition 3.1.1 to the Helmholtz equation (4.1) gives

$$\left(\frac{d^2}{dy^2} - \zeta^2\right)\tilde{q}(\eta, y) = f(y), \quad 0 < y < a, \text{ where} \tag{4.6}$$

$$f(y) = \partial_x q(0, y) - i\eta q(0, y) + \tilde{g}(\eta, y), \quad 0 < y < a. \tag{4.7}$$

Note that $\zeta = \sqrt{\eta^2 - k^2}$ is a multi-valued function. We fix a branch of it by $Re(\zeta) \geq 0$. The branch cut of this multi-valued function is shown in figure 3.2. From definition 3.1.1 apply the the operator L_x to the boundary condition defined by equation (4.2) to get

$$\int_0^\infty (\partial^2_{xx} + k_0^2)\partial_y q(x,y)e^{i\eta x} + \mu_0 \tilde{q}(\eta, y) = \tilde{g}_0(\eta). \tag{4.8}$$

Consider the following integral

$$\int_0^\infty \partial^3_{xxy} q(x,y)e^{i\eta x}dx = e^{i\eta x}\partial^2_{xy} q(x,y)|_0^\infty - \int_0^\infty \partial^2_{xy} q(x,y) i\eta e^{i\eta x}dx. \tag{4.9}$$

Use property that $q(x,y) \in C^2(\Omega) \cap C^1(\overline{\Omega}) \cap C^3(S_0)$ and $q(x,y)|_{x=\infty} = 0$, $\partial_x q(x,y)|_{x=\infty} = 0$, $\partial^2_{xy} q(x,y)|_{x=\infty} = 0$, and integrate by parts to get

$$\begin{aligned}\int_0^\infty \partial^3_{xxy} q(x,y)e^{i\eta x}dx &= -\partial^2_{xy} q(0,0) - i\eta[e^{i\eta x}\partial_y q(x,y)|_0^\infty - \int_0^\infty \partial_y q(x,y) i\eta e^{i\eta x}dx] \\ &= -\partial^2_{xy} q(0,0) + i\eta\partial_y q(0,0) - \eta^2 \frac{d}{dy}\int_0^\infty q(x,y)e^{i\eta x}dx.\end{aligned} \tag{4.10}$$

Use the edge condition (4.5) and $\frac{\partial^2}{\partial x \partial y} q(0,0) = C_0$ in equation (4.10) to get

$$\int_0^\infty \partial^3_{xxy} q(x,y)e^{i\eta x}dx = -C_0 - \eta^2 \frac{d}{dy}\tilde{q}(\eta, 0). \tag{4.11}$$

Use value of the integral from equation (4.11) in equation (4.8), to get

$$\begin{aligned}&- C_0 - \eta^2 \frac{d}{dy}\tilde{q}(\eta,0) + k_0^2 \frac{d}{dy}\tilde{q}(\eta,0) + \mu_0\tilde{q}(\eta,0) = \tilde{g}_0(\eta) \\ &(-\eta^2 + k_0^2)\frac{d}{dy}\tilde{q}(\eta,0) + \mu_0\tilde{q}(\eta,0) = \tilde{g}_0(\eta) + C_0.\end{aligned} \tag{4.12}$$

Simplify the above equation (4.12) to get

$$(-\frac{d}{dy} + \tilde{\mu}_0)\tilde{q}(\eta,0) = \tilde{\rho}_0(\eta), \text{ where} \tag{4.13}$$

$$\tilde{\mu}_0 = \frac{\mu_0}{\eta^2 - k_0^2}, \quad \tilde{\rho}_0(\eta) = \frac{\tilde{g}_0(\eta) + C_0}{\eta^2 - k_0^2}. \tag{4.14}$$

From definition 3.1.1 apply the the operator L_x to the boundary condition defined by equation (4.3), and simplify to get

$$(\frac{d}{dy} + \mu_1)\tilde{q}(\eta, a) = \tilde{g}_1(\eta). \tag{4.15}$$

Now equations (4.6), (4.13) and (4.15) describe the following non homogeneous system of Laplace transformed equations

$$(\frac{d^2}{dy^2} - \zeta^2)\tilde{q}(\eta, y) = f(y), \quad 0 < y < a, \tag{4.16}$$

$$(-\frac{d}{dy} + \tilde{\mu}_0)\tilde{q}(\eta, 0) = \tilde{\rho}_0(\eta), \tag{4.17}$$

$$(\frac{d}{dy} + \mu_1)\tilde{q}(\eta, a) = \tilde{g}_1(\eta), \tag{4.18}$$

where $f(y), \tilde{\mu}_0$ and $\tilde{\rho}_0(\eta)$ are given by equations (4.7) and (4.14). Now compare the non homogeneous system of Laplace transformed equations (4.16), (4.17) and (4.18) with the non homogeneous system of Laplace transformed equations (3.125), (3.127) and (3.128), then the Green's function of the system defined by equations (4.16), (4.17) and (4.18) is given by replacing μ_0 with $\tilde{\mu}_0$ in equation (3.140), and is expressed as follows

$$\begin{aligned} G(y, \xi) = & -\frac{e^{-\zeta|y-\xi|}}{2\zeta} + \frac{\tilde{\mu}_0 - \zeta}{2\Delta\zeta} e^{-\zeta\xi}(\zeta \cosh[(a-y)\zeta] + \mu_1 \sinh[(a-y)\zeta]) \\ & + (\frac{\mu_1 - \zeta}{2\Delta\zeta})e^{-\zeta(a-\xi)}(\zeta \cosh[\zeta y] + \tilde{\mu}_0 \sinh[\zeta y]). \end{aligned} \tag{4.19}$$

Note that

$$\tilde{\mu}_0 = \frac{\mu_0}{\eta^2 - k_0^2}, \quad \Delta = (\tilde{\mu}_0 + \mu_1)\zeta \cosh[a\zeta] + (\tilde{\mu}_0\mu_1 + \zeta^2)\sinh[a\zeta]. \tag{4.20}$$

Then solution of the non homogeneous system of equations (4.16), (4.17) and (4.18) is obtained by replacing $\tilde{g}_0(\eta)$ with $\tilde{\rho}_0(\eta)$ in equation (3.141)

$$\tilde{q}(\eta, y) = \int_0^a G(y, \xi) f(\xi) d\xi + \tilde{\rho}_0(\eta)\psi_0(y) + \tilde{g}_1(\eta)\psi_1(y). \tag{4.21}$$

Insert $y = 0$ in equation (4.21) to get

$$\tilde{q}(\eta, 0) = \int_0^a G(0, \xi) f(\xi) d\xi + \tilde{\rho}_0(\eta)\psi_0(0) + \tilde{g}_1(\eta)\psi_1(0). \tag{4.22}$$

From equations (4.19), (4.7), (3.137) and (3.138), find the values of $G(0, \xi), f(\xi)$, $\psi_0(0)$ and $\psi_1(0)$, respectively, and insert these values in equation (4.22) to get

$$\begin{aligned}\tilde{q}(\eta, 0) = \int_0^a &[-\frac{e^{-\zeta\xi}}{2\zeta} + \frac{\tilde{\mu}_0 - \zeta}{2\Delta\zeta} e^{-\zeta\xi}(\zeta \cosh[a\zeta] + \mu_1 \sinh[a\zeta]) + (\frac{\mu_1 - \zeta}{2\Delta})e^{-\zeta(a-\xi)}]\times \\ &[\frac{\partial}{\partial x} q(0, \xi) - i\eta q(0, \xi) + \tilde{g}(\eta, \xi)]d\xi + \tilde{\rho}_0(\eta)\frac{(\zeta \cosh[a\zeta] + \mu_1 \sinh[a\zeta])}{\Delta})+ \\ &\tilde{g}_1(\eta)\frac{\zeta}{\Delta}.\end{aligned} \tag{4.23}$$

Using equation (4.23), and the procedure in section 3.2 pages 107 and 109, we find that

$$\begin{aligned}\tilde{q}(\eta, 0) = \Lambda_{11}(\zeta)\frac{\partial}{\partial x}\hat{q}(0, i\zeta) - i\eta\Lambda_{11}(\zeta)\hat{q}(0, i\zeta) + \Lambda_{12}(\zeta)\frac{\partial}{\partial x}\hat{q}(0, -i\zeta) \\ - i\eta\Lambda_{12}(\zeta)\hat{q}(0, -i\zeta) + h_0(\eta).\end{aligned} \tag{4.24}$$

Note that

$$\begin{aligned}h_0(\eta) &= \Lambda_{11}(\zeta)\hat{\tilde{g}}(\eta, i\zeta) + \Lambda_{12}(\zeta, \eta)\hat{\tilde{g}}(\eta, -i\zeta) + \tilde{\rho}_0(\eta)\frac{(\zeta \cosh[a\zeta] + \mu_1 \sinh[a\zeta])}{\Delta} \\ &\quad + \tilde{g}_1(\eta)\frac{\zeta}{\Delta}, \\ \Lambda_{11}(\zeta) &= \frac{1}{2\zeta}[-1 + \frac{(\tilde{\mu}_0 - \zeta)(\zeta \cosh[a\zeta] + \mu_1 \sinh[a\zeta])}{\Delta}] \\ \Lambda_{12}(\zeta) &= \frac{e^{-\zeta a}}{2\Delta}(\mu_1 - \zeta), \\ \Delta &= (\tilde{\mu}_0 + \mu_1)\zeta \cosh[a\zeta] + (\tilde{\mu}_0\mu_1 + \zeta^2)\sinh[a\zeta].\end{aligned} \tag{4.25}$$

Replace η by $-\eta$ in equation (4.24) to get

$$\begin{aligned}\tilde{q}(-\eta, 0) = \Lambda_{11}(\zeta)\frac{\partial}{\partial x}\hat{q}(0, i\zeta) + i\eta\Lambda_{11}(\zeta)\hat{q}(0, i\zeta) + \Lambda_{12}(\zeta)\frac{\partial}{\partial x}\hat{q}(0, -i\zeta) \\ + i\eta\Lambda_{12}(\zeta)\hat{q}(0, -i\zeta) + h_0(-\eta).\end{aligned} \tag{4.26}$$

Now insert $y = a$ in equation (4.21), from equations (4.19), (4.7), (3.137) and (3.138), find the values of $G(a, \xi), f(\xi)$, $\psi_0(0)$ and $\psi_1(0)$, respectively, then insert

these values in the resultant equation, and simplify to get the following:

$$\begin{aligned}\tilde{q}(\eta, a) = \Lambda_{21}(\zeta)\frac{\partial}{\partial x}\hat{q}(0, i\zeta) - i\eta\Lambda_{21}(\zeta)\hat{q}(0, i\zeta) + \Lambda_{22}(\zeta)\frac{\partial}{\partial x}\hat{q}(0, -i\zeta) \\ - i\eta\Lambda_{22}(\zeta)\hat{q}(0, -i\zeta) + h_1(\eta), \text{ where}\end{aligned} \tag{4.27}$$

$$\begin{aligned} h_1(\eta) &= \Lambda_{21}(\zeta)\hat{\tilde{g}}(\eta, i\zeta) + \Lambda_{22}(\zeta, \eta)\hat{\tilde{g}}(\eta, -i\zeta) + \tilde{g}_1(\eta)\frac{(\zeta\cosh[a\zeta] + \tilde{\mu}_0\sinh[a\zeta])}{\Delta} \\ &\quad + \tilde{\rho}_0(\eta)\frac{\zeta}{\Delta}, \\ \Lambda_{21}(\zeta) &= \frac{\tilde{\mu}_0 - \zeta}{2\Delta}, \qquad \Lambda_{22}(\zeta) = \frac{e^{-\zeta a}}{2\zeta}(\zeta\cosh[a\zeta] + \tilde{\mu}_0\sinh[a\zeta]). \end{aligned} \tag{4.28}$$

Note that $\tilde{\rho}_0(\eta)$ and Δ are given by equations (4.14) and (4.20), respectively. Now replace η by $-\eta$ in equation (4.27) to get

$$\begin{aligned}\tilde{q}(-\eta, a) = \Lambda_{21}(\zeta)\frac{\partial}{\partial x}\hat{q}(0, i\zeta) + i\eta\Lambda_{21}(\zeta)\hat{q}(0, i\zeta) + \Lambda_{22}(\zeta)\frac{\partial}{\partial x}\hat{q}(0, -i\zeta) \\ + i\eta\Lambda_{22}(\zeta)\hat{q}(0, -i\zeta) + h_1(-\eta).\end{aligned} \tag{4.29}$$

Equations (4.24), (4.26), (4.27) and (4.29) describe the following system of four equations for four unknowns:

$$\begin{aligned} \tilde{q}(\eta, 0) - h_0(\eta) &= \Lambda_{11}(\zeta)\frac{\partial}{\partial x}\hat{q}(0, i\zeta) - i\eta\Lambda_{11}(\zeta)\hat{q}(0, i\zeta) + \Lambda_{12}(\zeta)\frac{\partial}{\partial x}\hat{q}(0, -i\zeta) \\ &\quad - i\eta\Lambda_{12}(\zeta)\hat{q}(0, -i\zeta), \\ \tilde{q}(-\eta, 0) - h_0(-\eta) &= \Lambda_{11}(\zeta)\frac{\partial}{\partial x}\hat{q}(0, i\zeta) + i\eta\Lambda_{11}(\zeta)\hat{q}(0, i\zeta) + \Lambda_{12}(\zeta)\frac{\partial}{\partial x}\hat{q}(0, -i\zeta) \\ &\quad + i\eta\Lambda_{12}(\zeta)\hat{q}(0, -i\zeta), \\ \tilde{q}(\eta, a) - h_1(\eta) &= \Lambda_{21}(\zeta)\frac{\partial}{\partial x}\hat{q}(0, i\zeta) - i\eta\Lambda_{21}(\zeta)\hat{q}(0, i\zeta) + \Lambda_{22}(\zeta)\frac{\partial}{\partial x}\hat{q}(0, -i\zeta) \\ &\quad - i\eta\Lambda_{22}(\zeta, \eta)\hat{q}(0, -i\zeta), \\ \tilde{q}(-\eta, a) - h_1(\eta) &= \Lambda_{21}(\zeta)\frac{\partial}{\partial x}\hat{q}(0, i\zeta) + i\eta\Lambda_{21}(\zeta)\hat{q}(0, i\zeta) + \Lambda_{22}(\zeta)\frac{\partial}{\partial x}\hat{q}(0, -i\zeta) \\ &\quad + i\eta\Lambda_{22}(\zeta)\hat{q}(0, -i\zeta). \end{aligned} \tag{4.30}$$

Write the above system in matrix form, then solution of this system in terms of the unknowns $\frac{\partial}{\partial x}\hat{q}(0,i\zeta), \hat{q}(0,i\zeta)$, $\frac{\partial}{\partial x}\hat{q}(0,-i\zeta)$ and $\hat{q}(0,-i\zeta)$ is

$$\begin{aligned}\frac{\partial}{\partial x}\hat{q}(0,i\zeta) = &-\frac{-(q_{am}+q_{ap})(\zeta-\mu_1)+(q_{0m}+q_{0p})(\zeta+\tilde{\mu}_0)\cosh[a\zeta]}{2(\cosh[a\zeta]+\sinh[a\zeta])}\\ &-\frac{(q_{0m}+q_{0p})(\zeta+\tilde{\mu}_0)\sinh[a\zeta]}{2(\cosh[a\zeta]+\sinh[a\zeta])},\end{aligned} \tag{4.31}$$

$$\begin{aligned}\hat{q}(0,i\zeta) = &\frac{i\{(-q_{am}+q_{ap})(\zeta-\mu_1)+(q_{0m}-q_{0p})(\zeta+\tilde{\mu}_0)\cosh[a\zeta]\}}{2\eta(\cosh[a\zeta]+\sinh[a\zeta])}\\ &+\frac{i(q_{0m}-q_{0p})(\zeta+\tilde{\mu}_0)\sinh[a\zeta]}{2\eta(\cosh[a\zeta]+\sinh[a\zeta])},\\ \frac{\partial}{\partial x}\hat{q}(0,-i\zeta) = &\frac{i\{(q_{0m}+q_{0p})(\zeta-\tilde{\mu}_0))-e^{a\zeta}(q_{am}+q_{ap})(\zeta+\mu_1)\}}{2\eta},\\ \hat{q}(0,-i\zeta) = &\frac{i\{(-q_{0m}+q_{0p})(\zeta-\tilde{\mu}_0)+e^{a\zeta}(q_{am}-q_{ap})(\zeta+\mu_1)\}}{2\eta}.\end{aligned} \tag{4.32}$$

Note that $q_{0p}, q_{0m}, q_{ap}, q_{am}$ are:

$$q_{0p} = \tilde{q}(\eta,0)-h_0(\eta), \quad q_{0m} = \tilde{q}(-\eta,0)-h_0(-\eta), \tag{4.33}$$

$$q_{ap} = \tilde{q}(\eta,a)-h_1(\eta), \quad q_{am} = \tilde{q}(-\eta,a)-h_1(-\eta). \tag{4.34}$$

Now application of the operator L_y to the boundary condition along the side S_2 of the semi-infinite strip Ω, defined by equation (4.4), and use of definition 3.1.2 gives the following result:

$$-\frac{\partial}{\partial x}\hat{q}(0,i\zeta)+\mu_2\hat{q}(0,i\zeta) = \hat{g}_2(i\zeta). \tag{4.35}$$

Replace ζ by $-\zeta$ to get

$$-\frac{\partial}{\partial x}\hat{q}(0,-i\zeta)+\mu_2\hat{q}(0,-i\zeta) = \hat{g}_2(-i\zeta). \tag{4.36}$$

From equations labeled by (4.31) and (4.32), use the values of $\frac{\partial}{\partial x}\hat{q}(0,\pm i\zeta)$ and $\hat{q}(0,\pm i\zeta)$ in equations (4.35) and (4.36) to get the following system of two equations:

$$\begin{aligned}a_{11}\tilde{q}(\eta,0)+a_{12}\tilde{q}(\eta,a) = b_{11}\tilde{q}(-\eta,0)+b_{12}\tilde{q}(-\eta,a)+a_{11}h_0(\eta)-b_{11}h_0(-\eta)\\ +a_{12}h_1(\eta)-b_{12}h_1(-\eta)+\hat{g}_2(i\zeta),\end{aligned} \tag{4.37}$$

$$a_{21}\tilde{q}(\eta,0)+a_{22}\tilde{q}(\eta,a)=b_{21}\tilde{q}(-\eta,0)+b_{22}\tilde{q}(-\eta,a)+a_{21}h_0(\eta)-b_{21}h_0(-\eta) \\ +a_{22}h_1(\eta)-b_{22}h_1(-\eta)+\hat{g}_2(-i\zeta). \tag{4.38}$$

Note that $a_{11}, a_{12}, a_{21}, a_{22}, b_{11}, b_{12}, b_{21}$ and b_{22} are given by replacing μ_0 with $\tilde{\mu}_0$ in equations labeleled by (3.167) and (3.168). Write the system of two equations (4.37) and (4.38) in matrix form to get

$$A\begin{bmatrix}\tilde{q}(\eta,0)\\ \tilde{q}(\eta,a)\end{bmatrix}=B\begin{bmatrix}\tilde{q}(-\eta,0)\\ \tilde{q}(-\eta,a)\end{bmatrix}+A\begin{bmatrix}h_0(\eta)\\ h_1(\eta)\end{bmatrix}-B\begin{bmatrix}h_0(-\eta)\\ h_1(-\eta)\end{bmatrix} \\ +\begin{bmatrix}\hat{g}_2(i\zeta)\\ \hat{g}_2(-i\zeta)\end{bmatrix},\ \text{where} \tag{4.39}$$

$$A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix},\qquad B=\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}.$$

Now multiply equation (4.39) from left side by A^{-1} to get

$$\begin{bmatrix}\tilde{q}(\eta,0)\\ \tilde{q}(\eta,a)\end{bmatrix}=G(\eta)\begin{bmatrix}\tilde{q}(-\eta,0)\\ \tilde{q}(-\eta,a)\end{bmatrix}+\begin{bmatrix}h_0(\eta)\\ h_1(\eta)\end{bmatrix}-G(\eta)\begin{bmatrix}h_0(-\eta)\\ h_1(-\eta)\end{bmatrix} \\ +A^{-1}\begin{bmatrix}\hat{g}_2(i\zeta)\\ \hat{g}_2(-i\zeta)\end{bmatrix},\ \text{where } \eta\in\mathbb{R}. \tag{4.40}$$

Equation (4.40) can be expressed by

$$\phi^+(\eta)=G(\eta)\phi^-(\eta)+F(\eta),\quad \eta\in\mathbb{R},\ \text{where} \tag{4.41}$$

$$\phi^+(\eta)=\begin{bmatrix}\phi_1+(\eta)=\tilde{q}(\eta,0)\\ \phi_2+(\eta)=\tilde{q}(\eta,a)\end{bmatrix},\quad \phi^-(\eta)=\begin{bmatrix}\phi_1-(\eta)=\tilde{q}(-\eta,0)\\ \phi_2-(\eta)=\tilde{q}(-\eta,a)\end{bmatrix}, \tag{4.42}$$

$$F(\eta)=\begin{bmatrix}h_0(\eta)\\ h_1(\eta)\end{bmatrix}-G(\eta)\begin{bmatrix}h_0(-\eta)\\ h_1(-\eta)\end{bmatrix}+A^{-1}\begin{bmatrix}\hat{g}_2(i\zeta)\\ \hat{g}_2(-i\zeta)\end{bmatrix}, \tag{4.43}$$

$$A^{-1} = \frac{1}{\Delta(\eta - i\mu_2)} \begin{bmatrix} e^{a\zeta}\eta(\zeta+\mu_1)\hat{g}_2(i\zeta) & \eta e^{-a\zeta}(\zeta-\mu_1)\hat{g}_2(-i\zeta) \\ \eta(\zeta-\tilde{\mu}_0)\hat{g}_2(i\zeta) & \eta(\zeta+\tilde{\mu}_0)\hat{g}_2(-i\zeta) \end{bmatrix}, \tag{4.44}$$

$$G(\eta) = A^{-1}B = \begin{bmatrix} -\frac{\eta+i\mu_2}{\eta-i\mu_2} & 0 \\ 0 & -\frac{\eta+i\mu_2}{\eta-i\mu_2} \end{bmatrix}. \tag{4.45}$$

Δ is given from equation (4.20). Hence equation (4.43) can be expressed as

$$\begin{aligned} F(\eta) = &\begin{bmatrix} h_0(\eta) \\ h_1(\eta) \end{bmatrix} - \begin{bmatrix} -\frac{\eta+i\mu_2}{\eta-i\mu_2} & 0 \\ 0 & -\frac{\eta+i\mu_2}{\eta-i\mu_2} \end{bmatrix} \begin{bmatrix} h_0(-\eta) \\ h_1(-\eta) \end{bmatrix} \\ &+ \frac{1}{\Delta(\eta - i\mu_2)} \begin{bmatrix} e^{a\zeta}\eta(\zeta+\mu_1)\hat{g}_2(i\zeta) & \eta e^{-a\zeta}(\zeta-\mu_1)\hat{g}_2(-i\zeta) \\ \eta(\zeta-\tilde{\mu}_0)\hat{g}_2(i\zeta) & \eta(\zeta+\tilde{\mu}_0)\hat{g}_2(-i\zeta) \end{bmatrix} \begin{bmatrix} \hat{g}_2(i\zeta) \\ \hat{g}_2(-i\zeta) \end{bmatrix}. \end{aligned} \tag{4.46}$$

Equation (4.46) shows that the components of $F(\eta)$ are

$$\begin{aligned} F_1(\eta) &= h_0(\eta) + \frac{\eta+i\mu_2}{\eta-i\mu_2}h_0(-\eta) + \frac{\eta}{\Delta(\eta-i\mu_2)}[e^{a\zeta}(\zeta+\mu_1)\hat{g}_2(i\zeta) \\ &\quad + (\zeta-\mu_1)e^{-a\zeta}\hat{g}_2(-i\zeta)], \\ F_2(\eta) &= h_1(\eta) + \frac{\eta+i\mu_2}{\eta-i\mu_2}h_1(-\eta) + \frac{\eta}{\Delta(\eta-i\mu_2)}[(\zeta-\tilde{\mu}_0)\hat{g}_2(i\zeta) \\ &\quad + (\zeta+\tilde{\mu}_0)\hat{g}_2(-i\zeta)]. \end{aligned} \tag{4.47}$$

Insert the value of $G(\eta)$ from equation (4.45) in equation (4.41) to obtain the following two scalar RHPs:

$$\phi_j^+(\eta) = -\frac{\eta+i\mu_2}{\eta-i\mu_2}\phi_j^-(\eta) + F_j(\eta), \quad \eta\in\mathbb{R}, \quad j=1,2, \tag{4.48}$$

$$\phi_1^\pm(\eta) = \tilde{q}(\pm\eta, 0), \quad \phi_2^\pm(\eta) = \tilde{q}(\pm\eta, a). \tag{4.49}$$

Note that $\phi_1^+(\eta)$ and $\phi_2^+(\eta)$ are analytic functions in the upper half η-complex plane, where as $\phi_1^-(\eta)$, and $\phi_2^-(\eta)$ are analytic functions in the lower half η-complex plane. These functions satisfy the following symmetry conditions

$$\phi_j^+(\eta) = \phi_j^-(-\eta) \quad \forall \quad \eta\in\mathbb{C}^+, \quad \phi_j^-(\eta) = \phi_j^+(-\eta) \quad \forall \quad \eta\in\mathbb{C}^-. \tag{4.50}$$

Due to this symmetry property, the scalar RHP defined by equation (3.171) is called a symmetric order two vector RHP. Now consider

$$\begin{aligned}
\phi(\eta) &= \frac{h_j(-\eta)}{\eta - i\mu_2} + \frac{h_j(\eta)}{\eta + i\mu_2}, \quad j = 1, 2, \\
\phi(-\eta) &= \frac{h_j(\eta)}{-\eta - i\mu_2} + \frac{h_j(-\eta)}{-\eta + i\mu_2}, \\
\phi(-\eta) &= -\left(\frac{h_j(-\eta)}{\eta - i\mu_2} + \frac{h_j(\eta)}{\eta + i\mu_2}\right) = -\phi(\eta).
\end{aligned} \tag{4.51}$$

Equation (4.51) indicates that $\phi(\eta)$ is an odd function in the variable η. Also note that $\frac{\eta}{(\eta^2+\mu_2^2)\Delta}[(\zeta - \mu_1)e^{-a\zeta}\hat{g}_2(-i\zeta) + e^{a\zeta}(\zeta + \mu_1)\hat{g}_2(i\zeta)]$ and $\frac{\eta}{(\eta^2+\mu_2^2)\Delta}[(\zeta - \tilde{\mu}_0)\hat{g}_2(i\zeta) + (\zeta + \tilde{\mu}_0)\hat{g}_2(-i\zeta)]$ are odd functions in η. Hence equations labeled by (4.47) imply that $f_j(\eta) = \frac{F_j(\eta)}{\eta + i\mu_2}, j = 1, 2$ are odd functions in η. Using the procedure on pages 130, 131 and 132, following are proved.

1. The Cauchy type integral of f_j satisfies

$$\int_{-\infty}^{\infty} \frac{f_j(\tau)}{\tau - \eta} d\tau = O(\frac{1}{\eta^2}), \text{ as } \eta \to \infty, \quad j = 1, 2. \tag{4.52}$$

2.

$$\phi_j^+(\eta) = (\eta + i\mu_2)\psi_j^+(\eta) = O(\frac{1}{\eta}) \text{ as } \eta \to \infty, \quad j = 1, 2, \tag{4.53}$$

$$\phi_j^-(\eta) = -(\eta - i\mu_2)\psi_j^-(\eta) = O(\frac{1}{\eta}) \text{ as } \eta \to \infty. \tag{4.54}$$

Note that $\psi_j(z)$ is the Cauchy type integral

$$\psi_j(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{1}{\tau - z} f_j(\tau) d\tau, \quad \forall z \in \mathbb{C} \setminus \mathbb{R} \quad j = 1, 2. \tag{4.55}$$

3. $\phi_j^-(-\eta) = \phi_j^+(\eta), \quad \forall \eta \in \mathbb{C}^+, \quad \phi_j^+(-\eta) = \phi_j^-(\eta), \quad \forall \eta \in \mathbb{C}^-$, these are the symmetry conditions for the order two vector RHP defined by equation (4.41).

4.1.1 Solution of the BVP of the Helmholtz equation in a semi-infinite strip Ω subject to higher order boundary conditions, along the side S_2 of semi-infinite strip: $q(0, y)$

Case study

Example 4.1.1. *For the BVP defined by equations (4.1), (4.2), (4.4) and (4.3), let* $g_0(x) = g_1(x) = 0, g_2(y) = A$ *(constant* $), g(x, y) = 0$. *In this case equations (4.25) and (4.28) after simplification become*

$$h_0(\eta) = \frac{C_0(\zeta \cosh[a\zeta] + \mu_1 \sinh[a\zeta])}{(\eta^2 - k_0^2)\Delta}, \quad h_1(\eta) = \frac{C_0\zeta}{(\eta^2 - k_0^2)\Delta}. \tag{4.56}$$

From equation (4.56) use values of $h_0(\eta)$ *and* $h_1(\eta)$ *in equations labeled by (4.47), and simplify to get*

$$F_1(\eta) = \frac{2A\eta(\eta^2 - k_0^2)(\zeta \sinh[a\zeta] + 2\mu_1 \sinh^2[\frac{a\zeta}{2}]) + 2\eta C_0(\zeta \cosh[a\zeta] + \mu_1 \sinh[a\zeta])\zeta}{(\eta^2 - k_0^2)(\eta - i\mu_2)\Delta\zeta}, \tag{4.57}$$

$$F_2(\eta) = \frac{2A\eta(\zeta \sinh[a\zeta](\eta^2 - k_0^2) + 2\mu_0 \sinh^2[\frac{a\zeta}{2}]) + 2\zeta^2\eta C_0}{(\eta - i\mu_2)(\eta^2 - k_0^2)\zeta\Delta}, \tag{4.58}$$

$$\Delta = \frac{(\mu_0 + (\eta^2 - k_0^2)\mu_1)\zeta \cosh[a\zeta] + (\mu_0\mu_1 + (\eta^2 - k_0^2)\zeta^2) \sinh[a\zeta]}{\eta^2 - k_0^2}. \tag{4.59}$$

To find solution along the side S_2 *of semi-infinite strip* Ω, *use the expression given by equation (3.211)*

$$q(0, y) = \frac{1}{2\pi i} \int_\Gamma \hat{q}(0, i\zeta) e^{\zeta y} d\zeta, \quad \Gamma = (-i\infty, i\infty), \ \textit{where} \tag{4.60}$$

$\hat{q}(0, i\zeta)$ is given by equation (4.31).

$$\hat{q}(0,i\zeta) = \frac{-i(\zeta+\tilde{\mu}_0)\cosh[a\zeta](h_0(-\eta)-h_0(\eta)+\tilde{q}(\eta,0)-\tilde{q}(-\eta,0))}{2\eta(\cosh[a\zeta]+\sinh[a\zeta])} -$$
$$\frac{i(\zeta+\tilde{\mu}_0)\sinh[a\zeta](h_0(-\eta)-h_0(\eta)+\tilde{q}(\eta,0)-\tilde{q}(-\eta,0))}{2\eta(\cosh[a\zeta]+\sinh[a\zeta])} + \tag{4.61}$$
$$\frac{i(\zeta-\mu_1)(h_1(-\eta)+h_1(\eta)+\tilde{q}(\eta,a)-\tilde{q}(-\eta,a))}{2\eta(\cosh[a\zeta]+\sinh[a\zeta])}.$$

Since $\frac{h_0(-\eta)-h_0(\eta)}{2\eta} = \frac{h_1(-\eta)-h_1(\eta)}{2\eta} = 0$, equation 4.61 simplifies to

$$\hat{q}(0,i\zeta) = i(\zeta-\mu_1)\frac{(\tilde{q}(\eta,a)-\tilde{q}(-\eta,a))e^{-a\zeta}}{2\eta} - i(\zeta+\tilde{\mu}_0)\frac{(\tilde{q}(\eta,0)-\tilde{q}(-\eta,0))}{2\eta}. \tag{4.62}$$

It is proved on page 138 that $\frac{\tilde{q}(\eta,0)-\tilde{q}(-\eta,0)}{2\eta}$ and $\frac{\tilde{q}(\eta,a)-\tilde{q}(-\eta,a)}{2\eta}$ are analytic functions of ζ except at a finite number of poles in the η-complex plane. Hence these are meromorphic functions w.r.t. ζ. So, equation (4.61) indicates that $\hat{q}(0, i\zeta)$ is a meromorphic function of ζ. Note that we can continue analytically $\tilde{q}(-\eta, 0)$ and $\tilde{q}(-\eta, a)$ in the plane $Im(\eta) > 0$ by using boundary condition of the RHP defined by equation (4.48), and is given as $\phi_j^+(\eta) = -\frac{\eta+i\mu_2}{\eta-i\mu_2}\phi_j^-(\eta) + F_j(\eta), \quad \eta \in \mathbb{R}, \quad j = 0, 1.$

Express the above equation in component form, and simplify to get

$$\tilde{q}(-\eta,0) = -\frac{\eta-i\mu_2}{\eta+i\mu_2}[\tilde{q}(\eta,0)-F_1(\eta)], \tag{4.63}$$

$$\tilde{q}(-\eta,a) = -\frac{\eta-i\mu_2}{\eta+i\mu_2}[\tilde{q}(\eta,a)-F_2(\eta)]. \tag{4.64}$$

Hence, equations (4.60), (4.62), (4.63) and (4.64) give the following result.

$$q(0,y) = \frac{1}{2\pi}\int_\Gamma [\frac{-(\zeta+\tilde{\mu}_0)}{\eta+i\mu_2}\tilde{q}(\eta,0) + \frac{\zeta-\mu_1}{\eta+i\mu_2}e^{-a\zeta}\tilde{q}(\eta,a)+ \\ \frac{(\eta-i\mu_2)}{(\eta+i\mu_2)}(\zeta+\tilde{\mu}_0)\frac{F_1(\eta)}{2\eta} - e^{-a\zeta}(\zeta-\mu_1)\frac{\eta-i\mu_2}{\eta+i\mu_2}\frac{F_2(\eta)}{2\eta}]e^{\zeta y}d\zeta \tag{4.65}$$

Using values of $F_1(\eta)$ and $F_2(\eta)$ from equations (4.57) and (4.58), consider following expressions

$$-\frac{(\zeta+\tilde{\mu}_0)}{\eta+i\mu_2}\tilde{q}(\eta,0) + \frac{\eta-i\mu_2}{\eta+i\mu_2}(\zeta+\tilde{\mu}_0)\frac{F_1(\eta)}{2\eta} = \frac{(\zeta+\tilde{\mu}_0)}{\eta+i\mu_2}[-\tilde{q}(\eta,0)+ \\ \frac{A(\eta^2-k_0^2)(\zeta\sinh[a\zeta]+2\mu_1\sinh^2[\frac{a\zeta}{2}]) + C_0(\zeta\cosh[a\zeta]+\mu_1\sinh[a\zeta])\zeta}{\Delta\zeta(\eta^2-k_0^2)}],$$

$$\frac{e^{-a\zeta}(\zeta-\mu_1)}{\eta+i\mu_2}\tilde{q}(\eta,a) - e^{-a\zeta}(\zeta-\mu_1)\frac{\eta-i\mu_2}{\eta+i\mu_2}\frac{F_2(\eta)}{2\eta} = \frac{(\zeta-\mu_1)}{\eta+i\mu_2}e^{-a\zeta}[\tilde{q}(\eta,a)- \\ \frac{A(\zeta(\eta^2-k_0^2)\sinh[a\zeta]+2\mu_0\sinh^2[\frac{a\zeta}{2}]) + C_0\zeta^2}{\Delta(\eta^2-k_0^2)\zeta}]. \tag{4.66}$$

Use expressions labeled by (4.66) in equation (4.65), and simplify to get

$$q(0,y) = I_1(y) + I_2(y), \;\; where \tag{4.67}$$

$$I_1(y) = \frac{1}{2\pi}\int_\Gamma \frac{(\zeta+\tilde{\mu}_0)}{\eta+i\mu_2}e^{\zeta y}[-\tilde{q}(\eta,0) + \frac{1}{\Delta(\eta^2-k_0^2)\zeta}\{A(\eta^2-k_0^2)\times \\ (\zeta\sinh[a\zeta]+2\mu_1\sinh^2[\frac{a\zeta}{2}]) + C_0(\zeta\cosh[a\zeta]+\mu_1\sinh[a\zeta])\zeta\}]d\zeta, \tag{4.68}$$

$$I_2(y) = \frac{1}{2\pi}\int_\Gamma \frac{(\zeta-\mu_1)}{\eta+i\mu_2}e^{(y-a)\zeta}[\tilde{q}(\eta,a) - \frac{1}{\Delta(\eta^2-k_0^2)\zeta}\{A(\zeta(\eta^2-k_0^2)\sinh[a\zeta]+ \\ 2\mu_0\sinh^2[\frac{a\zeta}{2}] + C_0\zeta^2\}]d\zeta. \tag{4.69}$$

To evaluate the integrals $I_1(y)$ and $I_2(y)$, we need the zeroes of

$$\Delta = (\tilde{\mu}_0 + \mu_1)\zeta \cosh[a\zeta] + (\tilde{\mu}_0\mu_1 + \zeta^2)\sinh[a\zeta], \quad \tilde{\mu}_0 = \frac{\mu_0}{\eta^2 - k_0^2}. \tag{4.70}$$

Now, use the procedure on pages 140, 142 and 144 to get

$$I_1(y) = \frac{1}{2\pi}\int_\Gamma \frac{(\zeta + \tilde{\mu}_0)e^{\zeta y}}{\Delta(\eta + i\mu_2)(\eta^2 - k_0^2)\zeta}[A(\eta^2 - k_0^2)(\zeta\sinh[a\zeta] + 2\mu_1\sinh^2[\frac{a\zeta}{2}])+$$

$$C_0(\zeta\cosh[a\zeta] + \mu_1\sinh[a\zeta])\zeta]d\zeta, \tag{4.71}$$

$$I_1(y) = -\frac{1}{2\pi}(-\pi i)[\sum_{n=1}^{\infty}[Residue|_{\zeta=\zeta_n^+,\eta=i\hat{\zeta_n^+}=i\hat{\zeta_n}} + Residue|_{\zeta=\zeta_n^-,\eta=i\hat{\zeta_n^-}=i\hat{\zeta_n}}]g(\zeta,\eta)),$$

$$g(\zeta,\eta) = \frac{(\zeta + \tilde{\mu}_0)e^{\zeta y}}{\Delta(\eta + i\mu_2)(\eta^2 - k_0^2)\zeta}[A(\eta^2 - k_0^2)(\zeta\sinh[a\zeta] + 2\mu_1\sinh^2[\frac{a\zeta}{2}])+$$

$$C_0(\zeta\cosh[a\zeta] + \mu_1\sinh[a\zeta])\zeta]. \tag{4.72}$$

Equation (4.72) can be written as

$$I_1(y) = \frac{i}{2}\sum_{n=1}^{\infty} R_{n_1}, \ \ where \tag{4.73}$$

$$R_{n_1} = [Residue|_{\zeta=\zeta_n^+,\eta=i\hat{\zeta_n^+}=i\hat{\zeta_n}} + Residue|_{\zeta=\zeta_n^-,\eta=i\hat{\zeta_n^-}=i\hat{\zeta_n}}]g(\zeta,\eta). \tag{4.74}$$

To find R_{n_1}, we need the residue of $g(\eta,\zeta)$ at the simple poles ζ_n^+ and ζ_n^-, for that purpose, let

$$g_1(\eta,\zeta) = \frac{(\zeta+\tilde{\mu}_0)e^{\zeta y}}{(\eta+i\mu_2)(\eta^2-k_0^2)\zeta}[A(\eta^2-k_0^2)(\zeta\sinh[a\zeta]+2\mu_1\sinh^2[\frac{a\zeta}{2}])+ \\ C_0(\zeta\cosh[a\zeta]+\mu_1\sinh[a\zeta])\zeta], \quad \tilde{\mu}_0 = \frac{\mu_0}{\eta^2-k_0^2}, \tag{4.75}$$

$$g_2(\zeta) = \Delta = (\tilde{\mu}_0+\mu_1)\zeta\cosh[a\zeta] + (\tilde{\mu}_0\mu_1+\zeta^2)\sinh[a\zeta]. \tag{4.76}$$

We evaluate $g_1(\eta,\zeta)$ and $\frac{d}{d\zeta}g_2(\zeta)$ at the simple poles $\zeta_n^+ = i\lambda_n$ and $\zeta_n^- = -i\lambda_n$, then we get

$$Residue|_{\zeta=\zeta_n^+=i\lambda_n,\eta=i\hat{\zeta_n^+}=i\hat{\zeta_n}}g(\eta,\zeta) = \frac{g_1(\eta,\zeta)|_{\zeta=\zeta_n^+=i\lambda_n,\eta=i\hat{\zeta_n^+}=i\hat{\zeta_n}}}{\frac{dg_2}{d\zeta}(\zeta)|_{\zeta=\zeta_n^+=i\lambda_n}}$$
$$= e^{iy\lambda_n}\frac{(-ik_0^2\lambda_n - i\hat{\zeta}_n^2\lambda_n+\mu_0)}{\Delta_n}[C_0\lambda_n^2\cos[a\lambda_n]$$
$$-2A(k_0^2+\hat{\zeta_n}^2)\mu_1\sin^2[\frac{a\lambda_n}{2}] - \lambda_n(A(k_0^2+\hat{\zeta_n}^2)-C_0\mu_1)\sin[a\lambda_n]],$$
$$Residue|_{\zeta=\zeta_n^-=-i\lambda_n,\eta=i\hat{\zeta_n^-}=i\hat{\zeta_n}}g(\eta,\zeta) = \frac{g_1(\eta,\zeta)|_{\zeta=\zeta_n^-=-i\lambda_n,\eta=i\hat{\zeta_n^-}=i\hat{\zeta_n}}}{\frac{dg_2}{d\zeta}(\zeta)|_{\zeta=\zeta_n^-=-i\lambda_n}}$$
$$= e^{-iy\lambda_n}\frac{(ik_0^2\lambda_n + i\hat{\zeta}_n^2\lambda_n+\mu_0)}{\Delta_n}[-C_0\lambda_n^2\cos[a\lambda_n]$$
$$+2A(k_0^2+\hat{\zeta_n}^2)\mu_1\sin^2[\frac{a\lambda_n}{2}] + \lambda_n(A(k_0^2+\hat{\zeta_n}^2)-C_0\mu_1)\sin[a\lambda_n]],$$
$$\Delta_n = 2\lambda_n^2\mu_0(\hat{\zeta_n}+\mu_2)(\lambda_n\cosh[a\lambda_n]+\mu_1\sin[a\lambda_n]). \tag{4.77}$$

Now equations labeled by (4.74) and (4.77) give

$$R_{n_1} = \frac{e^{-iy\lambda_n}[i(1+e^{2iy\lambda_n})k_0^2\lambda_n + i(1+e^{2iy\lambda_n})\hat{\zeta}_n^2\lambda_n + \mu_0(1-e^{2iy\lambda_n})]}{\Delta_n}\times$$
$$[-C_0\lambda_n^2\cos[a\lambda_n] + 2A(k_0^2+\hat{\zeta_n}^2)\mu_1\sin^2[\frac{a\lambda_n}{2}] + \lambda_n(A(k_0^2+\hat{\zeta_n}^2) - C_0\mu_1)\sin[a\lambda_n]]. \tag{4.78}$$

Use the value of R_{n_1} from equation (4.78) in equation (4.73) to get

$$I_1(y) = \sum_{n=1}^{\infty}\frac{-e^{-iy\lambda_n}[(1+e^{2iy\lambda_n})k_0^2\lambda_n + (1+e^{2iy\lambda_n})\hat{\zeta}_n^2\lambda_n + i\mu_0(-1+e^{2iy\lambda_n})]}{2\Delta_n}\times$$
$$[-C_0\lambda_n^2\cos[a\lambda_n] + 2A(k_0^2+\hat{\zeta}_n^2)\mu_1\sin^2[\frac{a\lambda_n}{2}] + \lambda_n(A(k_0^2+\hat{\zeta}_n^2) - C_0\mu_1)\sin[a\lambda_n]]. \tag{4.79}$$

To evaluate $I_2(y)$, use the procedure on pages 146, 147, 149 and 150 to get

$$I_2(y) = -\frac{1}{2\pi}\int_\Gamma [\frac{(\zeta-\mu_1)e^{(y-a)\zeta}}{\Delta(\eta+i\mu_2)(\eta^2-k_0^2)\zeta}\{A(\zeta(\eta^2-k_0^2)\sinh[a\zeta]+ \tag{4.80}$$
$$2\mu_0\sinh^2[\frac{a\zeta}{2}] + C_0\zeta^2\}]d\zeta,$$

$$I_2(y) = -\frac{1}{2\pi}(-\pi i)\sum_{n=1}^{\infty}[Residue|_{\zeta=\zeta_n^+,\eta=i\hat{\zeta_n^+}=i\hat{\zeta_n}} + Residue|_{\zeta=\zeta_n^-,\eta=i\hat{\zeta_n^-}=i\hat{\zeta_n}}]h(\zeta,\eta),$$

$$h(\zeta,\eta) = \frac{(\zeta-\mu_1)e^{(y-a)\zeta}}{\Delta(\eta+i\mu_2)(\eta^2-k_0^2)\zeta}\{A(\zeta(\eta^2-k_0^2)\sinh[a\zeta]+$$
$$2\mu_0\sinh^2[\frac{a\zeta}{2}] + C_0\zeta^2\}. \tag{4.81}$$

We find that

$$I_2(y) = \sum_{n=1}^{\infty} \frac{e^{i(a-y)\lambda_n}(k_0^2+\hat{\zeta}_n^2)[(1+e^{2i(-a+y)\lambda_n})\lambda_n + i(-1+e^{2i(-a+y)\lambda_n})\mu_1]}{2\Delta_n}[C_0\lambda_n^2$$

$$+ A\mu_0(1-\cos[a\lambda_n]) - A(k_0^2+\hat{\zeta_n}^2)\lambda_n \sin[a\lambda_n])]. \tag{4.82}$$

$$q(0,y) = \sum_{n=1}^{\infty} \frac{-e^{-iy\lambda_n}[(1+e^{2iy\lambda_n})k_0^2\lambda_n + (1+e^{2iy\lambda_n})\hat{\zeta}_n^2\lambda_n + i\mu_0(-1+e^{2iy\lambda_n})]}{2\Delta_n}d_{1n}$$

$$+ \frac{e^{i(a-y)\lambda_n}dn[(1+e^{2i(-a+y)\lambda_n})\lambda_n + i(-1+e^{2i(-a+y)\lambda_n})\mu_1]}{2\Delta_n}d_{2n}. \tag{4.83}$$

Note that

$$d_{1n} = -C_0\lambda_n^2\cos[a\lambda_n] + 2Ad_n\mu_1\sin^2[\frac{a\lambda_n}{2}] + \lambda_n(Ad_n - C_0\mu_1)\sin[a\lambda_n], \tag{4.84}$$

$$d_{2n} = [C_0\lambda_n^2 + A\mu_0(1-\cos[a\lambda_n]) - Ad_n\lambda_n\sin[a\lambda_n])], \tag{4.85}$$

$$\Delta_n = 2\lambda_n^2\mu_0(\hat{\zeta}_n + \mu_2)(\lambda_n\cos[a\lambda_n] + \mu_1\sin[a\lambda_n]), \quad d_n = k_0^2 + \hat{\zeta_n}^2. \tag{4.86}$$

4.1.2 Solution of the BVP of the Helmholtz equation in a semi-infinite strip Ω subject to higher order boundary conditions, along the sides S_0 and S_1 of semi-infinite strip, and inside the semi-infinite strip: $q(x,0), q(x,a)$ and $q(x,y)$

Equation (3.203) gives solution $q(x,0)$

$$q(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\eta)e^{-i\eta x}d\eta. \tag{4.87}$$

$F_1(\eta)$ is given by equation (4.57), and note that $F_1(\eta)$ can be written as $F_1(\eta,\zeta)$. Equation (4.57) shows that $F_1(\eta,\zeta) = F_1(\eta,-\zeta)$. So, $F_1(\eta) = F_1(\eta,\zeta)$ is an even function w.r.t. ζ. We know that η is related to the multi-valued function ζ through the relation $\zeta = \sqrt{\eta^2 - k^2}$. Since $F_1(\eta) = F_1(\eta,\zeta)$ is an even function w.r.t ζ, so, it cancels out effect of the branch cut on values of $F_1(\eta) = F_1(\eta,\zeta)$ in the η-complex plane, so, that $F_1(\eta) = F_1(\eta,\zeta)$ is continuous through the cut in η-complex plane. Hence $F_1(\eta)$ is a meromorphic function of η in η-complex plane. To evaluate the integral given by equation (4.87), we follow the procedure on pages 153 and 154, to get

$$q(x,0) = -i\sum_{n=1}^{\infty} Residue \quad S(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}}, \ \ where \tag{4.88}$$

$$S(\eta,\zeta) = \frac{[p(\zeta\sinh[a\zeta] + 2\mu_1\sinh^2[\frac{a\zeta}{2}]) + 2\eta C_0(\zeta\cosh[a\zeta] + \mu_1\sinh[a\zeta])\zeta]e^{-i\eta x}}{\Delta(\eta^2-k_0^2)(\eta-i\mu_2)\zeta},$$

$$p = 2A\eta(\eta^2 - k_0^2).$$

To find $ResidueS(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}}$, let

$$S_1(\eta,\zeta) = \frac{[p(\zeta\sinh[a\zeta] + 2\mu_1\sinh^2[\frac{a\zeta}{2}]) + 2\eta C_0(\zeta\cosh[a\zeta] + \mu_1\sinh[a\zeta])\zeta]e^{-i\eta x}}{(\eta-k_0)(\eta-i\mu_2)\zeta}$$

$$S_2(\zeta) = \Delta(\eta+k_0) = \frac{(\mu_0 + \mu_1(\eta^2-k_0^2))\zeta\cosh[a\zeta] + (\mu_0\mu_1 + \zeta^2(\eta^2-k_0^2))\sinh[a\zeta]}{\eta-k_0}.$$

Now $ResidueS(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}}$ is found as follows.

$$
\begin{aligned}
Residue S(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}} &= \frac{S_1(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}}}{\frac{dS_2}{d\eta}|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}}} \\
Residue S(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}} &= \frac{2ie^{-x\hat{\zeta}_n}p_{1n}}{(\hat{\zeta}_n+\mu_2)\Delta_{0n}}
\end{aligned} \tag{4.89}
$$

From equation (4.89) use value of $Residue S(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}}$ *in equation (4.88), and simplify to get*

$$
q(x,0) = \sum_{n=1}^{\infty} \frac{2e^{-x\hat{\zeta}_n}p_{1n}}{(\hat{\zeta}_n+\mu_2)\Delta_{0n}} \tag{4.90}
$$

$p_{1n} = -C_0\lambda_n^2\cos[a\lambda_n] + 2Ad_n\mu_1\sin^2[\frac{a\lambda_n}{2}] + \lambda_n(Ad_n - C_0\mu_1)\sin[a\lambda_n],$ *where*

$\Delta_{0n} = (\mu_0 - d_n\mu_1 + a(d_n\lambda_n^2 + \mu_0\mu_1))\cos[a\lambda_n] +$

$\lambda_n(-a\mu_0 + (2 + a\mu 1)d_n)\sin[a\lambda_n], \quad d_n = \hat{\zeta}_n^2 + k_0^2.$

Note that to find the solution $q(x,0)$ *we need zeroes of* $S_2 = \Delta(\eta + k_0)$*. Simple poles of the integrand* $F_1(\eta)e^{-i\eta x}$ *are* $\zeta = -i\lambda_n$*, and correspondingly we have* $\eta = -i\hat{\zeta_n} = -i\sqrt{\lambda_n^2 - k^2}$*. To find the solution* $q(x,a)$ *along third side of semi-infinite strip* Ω*, equation (3.210) gives*

$$
q(x,a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_2(\eta)e^{-i\eta x}d\eta. \tag{4.91}
$$

$F_2(\eta)$ *is given by equation (4.58), and note that* $F_2(\eta)$ *can be written as* $F_2(\eta,\zeta)$*. Equation (4.58) shows that* $F_2(\eta,\zeta) = F_2(\eta,-\zeta)$*. So,* $F_2(\eta) = F_2(\eta,\zeta)$ *is an even*

function w.r.t. ζ. We know that η is related to the multi-valued function ζ through the relation $\zeta = \sqrt{\eta^2 - k^2}$. Since $F_2(\eta) = F_2(\eta, \zeta)$ is an even function w.r.t ζ, so, it cancels out effect of the branch cut on values of $F_2(\eta) = F_2(\eta, \zeta)$ in the η-complex plane. So, $F_2(\eta) = F_2(\eta, \zeta)$ is continuous through the cut in η-complex plane. Hence $F_2(\eta)$ is a meromorphic function of η in η-complex plane. To evaluate the integral given by equation (4.91), we follow the above procedure used to evaluate $q(x, 0)$, to get

$$q(x,a) = -i\sum_{n=1}^{\infty} Residue \quad H(\eta,\zeta)|_{\zeta=-i\lambda_n, \eta=-i\hat{\zeta_n}}, \ \ where \tag{4.92}$$

$$H(\eta,\zeta) = \frac{[2A\eta(\zeta\sinh[a\zeta](\eta^2-k_0^2)+2\mu_0\sinh^2[\frac{a\zeta}{2}])+2\zeta^2\eta C_0]e^{-i\eta x}}{(\eta-i\mu_2)(\eta^2-k_0^2)\zeta\Delta}, \tag{4.93}$$

$$\Delta = \frac{(\mu_0+(\eta^2-k_0^2)\mu_1)\zeta\cosh[a\zeta]+(\mu_0\mu_1+(\eta^2-k_0^2)\zeta^2)\sinh[a\zeta]}{\eta^2-k_0^2}. \tag{4.94}$$

To find Residue$H(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}}$, *let*

$$H_1(\eta,\zeta) = \frac{[2A\eta(\zeta\sinh[a\zeta](\eta^2-k_0^2)+2\mu_0\sinh^2[\frac{a\zeta}{2}])+2\zeta^2\eta C_0]e^{-i\eta x}}{(\eta-i\mu_2)(\eta-k_0)\zeta}, \tag{4.95}$$

$$\begin{aligned} H_2(\zeta) &= \Delta(\eta+k_0) \\ H_2(\zeta) &= \frac{(\mu_0+\mu_1(\eta^2-k_0^2))\zeta\cosh[a\zeta]+(\mu_0\mu_1+\zeta^2(\eta^2-k_0^2))\sinh[a\zeta]}{\eta-k_0}. \end{aligned} \tag{4.96}$$

Now $Residue H(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}}$ *is found as follows.*

$$\begin{aligned} Residue H(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}} &= \frac{H_1(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}}}{\frac{dH_2}{d\eta}|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}}} \\ Residue H(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}} &= \frac{2ie^{-x\hat{\zeta}_n}p_{2n}}{(\hat{\zeta}_n+\mu_2)\Delta_{0n}} \end{aligned} \tag{4.97}$$

From equation (4.97) use value of $Residue H(\eta,\zeta)|_{\zeta=-i\lambda_n,\eta=-i\hat{\zeta_n}}$ *in equation (4.92), and simplify to get*

$$\begin{aligned} q(x,a) &= \sum_{n=1}^{\infty}\frac{2e^{-x\hat{\zeta}_n}p_{2n}}{(\hat{\zeta}_n+\mu_2)\Delta_{0n}}, \\ p_{2n} &= -C_0\lambda_n^2 - A\mu_0(1-\cos[a\lambda_n]) + Ad_n\lambda_n\sin[a\lambda_n], \\ \Delta_{0n} &= (\mu_0 - d_n\mu_1 + a(d_n\lambda_n^2+\mu_0\mu_1))\cos[a\lambda_n]+ \\ &\quad \lambda_n(-a\mu_0+(2+a\mu 1)d_n)\sin[a\lambda_n], \quad d_n = \hat{\zeta}_n^2 + k_0^2. \end{aligned} \tag{4.98}$$

In chapter 3 we have verified that in the particular case $g_0(x)=0, g_1(x)=0$, $g_2(y)=A$ (constant), and $g(x,y)=0$, using new method the series representation for the solution $q(x,y)$ can be obtained by multiplying the nth term of the series representation of the solution $q(0,y)$ by $e^{-\hat{\zeta}_n x}$. From equation (4.83), the nth term of series representation of $q(0,y)$ is

$$\begin{aligned} a_n = [&\frac{-e^{-iy\lambda_n}[(1+e^{2iy\lambda_n})k_0^2\lambda_n + (1+e^{2iy\lambda_n})\hat{\zeta}_n^2\lambda_n + i\mu_0(-1+e^{2iy\lambda_n})]}{2\Delta_n}d_{1n} \\ &+ \frac{e^{i(a-y)\lambda_n}d_n[(1+e^{2i(-a+y)\lambda_n})\lambda_n + i(-1+e^{2i(-a+y)\lambda_n})\mu_1]}{2\Delta_n}d_{2n}], \end{aligned} \tag{4.99}$$

where d_{1n}, d_{2n}, d_n and Δ_n are given by equations (4.84), (4.85) and (4.86). Hence the formula to find the solution $q(x,y)$ of given BVP of the Helmholtz equation in

a semi-infinite strip Ω subject to higher order boundary conditions is

$$q(x,y) = \sum_{n=1}^{\infty}[\frac{-e^{-(x\hat{\zeta}_n+iy\lambda_n)}[(1+e^{2iy\lambda_n})k_0^2\lambda_n + (1+e^{2iy\lambda_n})\hat{\zeta}_n^2\lambda_n + i\mu_0(-1+e^{2iy\lambda_n})]}{2\Delta_n}$$
$$\times d_{1n} + \frac{e^{-(x\hat{\zeta}_n+i(y-a)\lambda_n)}d_n[(1+e^{2i(-a+y)\lambda_n})\lambda_n + i(-1+e^{2i(-a+y)\lambda_n})\mu_1]}{2\Delta_n}d_{2n}]. \tag{4.100}$$

4.1.3 Determination of the unknown constant C_0

To find the unknown constant C_0 use the Laplace transformed boundary condition along the side S_0, given by

$$(-\frac{d}{dy} + \tilde{\mu}_0)\tilde{q}(\eta,0) = \tilde{\rho}_0(\eta), \tag{4.101}$$

$$\tilde{\mu}_0 = \frac{\mu_0}{\eta^2-k_0^2}, \quad \tilde{\rho}_0(\eta) = \frac{\tilde{g}_0(\eta)+C_0}{\eta^2-k_0^2}. \tag{4.102}$$

For the given case study, $g_0(x) = g_1(x) = 0, g(x,y) = 0, g_2(y) = A$(constant), equation (4.102) gives $\tilde{\mu}_0 = \frac{\mu_0}{\eta^2-k_0^2}$ and $\tilde{\rho}_0(\eta) = \frac{C_0}{\eta^2-k_0^2}$. Apply the inverse Laplace transform to equation (4.101) to get

$$-\frac{d}{dy}q(x,0) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mu_0}{\eta^2-k_0^2}\tilde{q}(\eta,0)e^{-i\eta x}d\eta + \frac{1}{k_0}\sinh[k_0x]. \tag{4.103}$$

Apply $\lim_{x\to 0^+}$ to equation (4.103), and simplify to get

$$\int_{-\infty}^{\infty}\frac{\tilde{q}(\eta,0)}{\eta^2-k_0^2}d\eta = 0. \tag{4.104}$$

Since $\tilde{q}(\eta,0) = \phi_1^+(\eta)$ (solution of RHP (4.48)), so equation (4.104) becomes

$$\int_{-\infty}^{\infty}\frac{\phi_1^+(\eta)}{\eta^2-k_0^2}d\eta = 0. \tag{4.105}$$

To evaluate the integral given by equation (4.105) we construct the closed contour $\Gamma = [-R,R]\cup C_R^+$, where C_R^+ is a semi circle in the upper half η-complex plane, and $[-R,R]$ is a line segment. Using Cauchy's residue theorem we get

$$\begin{aligned}\int_{-R}^{R}\frac{\phi_1^+(\eta)}{\eta^2-k_0^2}d\eta + \int_{C_R^+}\frac{\phi_1^+(\eta)}{\eta^2-k_0^2}d\eta &= Residue|_{\eta=k_0}\quad \frac{\phi_1^+(\eta)}{\eta^2-k_0^2}\\ \int_{-R}^{R}\frac{\phi_1^+(\eta)}{\eta^2-k_0^2}d\eta + \int_{C_R^+}\frac{\phi_1^+(\eta)}{\eta^2-k_0^2}d\eta &= \frac{\phi_1^+(k_0)}{2k_0}.\end{aligned} \tag{4.106}$$

Since the integrand $\frac{\phi_1^+(\eta)}{\eta^2-k_0^2} = O(\frac{1}{\eta^3})$ as $\eta \to \infty$, so, $\lim_{R\to\infty} \int_{C_R^+} \frac{\phi_1^+(\eta)}{\eta^2-k_0^2} d\eta = 0$. Apply $\lim_{R\to\infty}$ to equation (4.106), use the previous result, and the value of integral defined by equation (4.105), to get $\phi_1^+(k_0) = 0$. Use this result in equation (4.53) to get

$$\psi_1(k_0) = 0. \tag{4.107}$$

Substitute $z = k_0$ in equation (4.55) to get

$$\psi_1(k_0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\tau - k_0} f_1(\tau) d\tau, \quad k_0 \in \mathbb{C} \setminus \mathbb{R}. \tag{4.108}$$

Use the value of $\psi_1(k_0)$ in equation (4.108), and simplify to get

$$\int_{-\infty}^{\infty} \frac{f_1(\tau)}{\tau - k_0} d\tau = 0. \tag{4.109}$$

Using equation (4.57) we find

$$\begin{aligned} f_1(\eta) = \frac{F_1(\eta)}{\eta + i\mu_2} = & \frac{2A\eta(\zeta \sinh[a\zeta] + 2\mu_1 \sinh^2[\frac{a\zeta}{2}])}{(\eta^2 + \mu_2^2)\Delta\zeta} + \\ & \frac{2\eta C_0(\zeta \cosh[a\zeta] + \mu_1 \sinh[a\zeta])}{(\eta^2 - k_0^2)(\eta^2 + \mu_2^2)\Delta}. \end{aligned} \tag{4.110}$$

Equation (4.59) gives the following expression for Δ

$$\Delta = \frac{(\mu_0 + (\eta^2 - k_0^2)\mu_1)\zeta \cosh[a\zeta] + (\mu_0\mu_1 + (\eta^2 - k_0^2)\zeta^2) \sinh[a\zeta]}{\eta^2 - k_0^2}. \tag{4.111}$$

To find the unknown constant C_0 we need to solve equation (4.109), for that purpose zeroes of Δ are found by solving transcendental equation (4.111). Use the value of $f_1(\tau)$ from equation (4.110) in equation (4.109) to get

$$C_0 = -\frac{I_1}{I_2}, \quad I_1 = \int_{-\infty}^{\infty} \frac{A\tau(\zeta \sinh[a\zeta] + 2\mu_1 \sinh^2[\frac{a\zeta}{2}])}{(\tau - k_0)(\tau^2 + \mu_2^2)\Delta\zeta} d\tau, \tag{4.112}$$

$$I_2 = \int_{-\infty}^{\infty} \frac{\tau(\zeta \cosh[a\zeta] + \mu_1 \sinh[a\zeta])}{(\tau - k_0)(\tau^2 - k_0^2)(\tau^2 + \mu_2^2)\Delta} d\tau. \tag{4.113}$$

To find the value of integral I_1, we construct the closed contour $\Gamma = [R, -R] \cup C_R^-$, where $[R, -R]$ is the line segment, and C_R^- is the semi circle in the lower half

η-complex plane. Apply Cauchy's residue theorem to the integrand in the closed contour Γ to get

$$\int_{R}^{-R} H(\tau)d\tau + \int_{C_R^-} H(\tau)d\tau = 2\pi i \sum_{n=1}^{\infty} Residue|_{\zeta=-i\lambda_n,\tau=-i\hat{\zeta}_n} H(\tau). \tag{4.114}$$

Note that $H(\tau) = \frac{A\tau(\zeta\sinh[a\zeta]+2\mu_1\sinh^2[\frac{a\zeta}{2}])}{(\tau-k_0)(\tau^2+\mu_2^2)\Delta\zeta} = O(\frac{1}{\tau^4}),$ as $\tau \to \infty$. Hence, $\lim_{R\to\infty}\int_{C_R^-} H(\tau)d\tau = 0$. Now apply $\lim_{R\to\infty}$, to equation (4.114), to get

$$I_1 = \int_{-\infty}^{\infty} H(\tau)d\tau = -2\pi i \sum_{n=0}^{\infty} Residue|_{\zeta=-i\lambda_n,\tau=-i\hat{\zeta}_n} H(\tau). \tag{4.115}$$

To find $\text{Residue}|_{\zeta=-i\lambda_n,\tau=-i\hat{\zeta}_n} H(\tau)$, let

$$H_1(\tau) = \frac{A\tau(\zeta\sinh[a\zeta] + 2\mu_1\sinh^2[\frac{a\zeta}{2}])}{(\tau-k_0)(\tau^2+\mu_2^2)\zeta}, \quad \zeta = \sqrt{\tau^2-k^2}, \tag{4.116}$$

$$H2(\tau) = \Delta = \frac{(\mu_0 + (\tau^2-k_0^2)\mu_1)\zeta\cosh[a\zeta] + (\mu_0\mu_1 + (\tau^2-k_0^2)\zeta^2)\sinh[a\zeta]}{\tau^2-k_0^2}, \tag{4.117}$$

$$d_n = \hat{\zeta}_n^2 + k_0^2. \tag{4.118}$$

We calculate

$$Residue|_{\zeta=-i\lambda_n,\tau=-i\hat{\zeta}_n} H(\tau) = \frac{A(k_0 - i\hat{\zeta}_n)[2\mu_1\sin^2[\frac{a\lambda_n}{2}] + \lambda_n\sin[a\lambda_n]]}{(\hat{\zeta}_n^2-\mu_2^2)\Delta_{0n}}, \tag{4.119}$$

$$\Delta_{0n} = [\mu_0 - d_n\mu_1 + a(\lambda_n^2 d_n + \mu_0\mu_1)]\cos[a\lambda_n] + \lambda_n[-a\mu_0 + (2+a\mu_1)d_n]\sin[a\lambda_n]. \tag{4.120}$$

Now equation (4.115) becomes

$$I_1 = \sum_{n=1}^{\infty} \frac{-2\pi A(ik_0 + \hat{\zeta}_n)[2\mu_1\sin^2[\frac{a\lambda_n}{2}] + \lambda_n\sin[a\lambda_n]]}{(\hat{\zeta}_n^2-\mu_2^2)\Delta_{0n}}, \tag{4.121}$$

where Δ_{0n} is given by equation (4.120). Now using the above procedure to find the integral I_2, we get

$$I_2 = \sum_{n=1}^{\infty} \frac{2\pi\lambda_n[\lambda_n\cos[a\lambda_n] + \mu_1\sin[a\lambda_n]]}{(-ik_0+\hat{\zeta}_n)(\hat{\zeta}_n^2-\mu_2^2)\Delta_{0n}}, \tag{4.122}$$

where Δ_{0n} is given by equation (4.120). To find the unknown constant C_0, substitute the values of I_1 and I_2 in equation (4.112). Insert the value of C_0 in equation (4.100), that gives solution of the given BVP of the Helmholtz equation in the semi-infinite strip Ω subject to higher order boundary conditions.

References

[1] D. ben-Avraham, A. S. Fokas, *The solution of the modified Helmholtz equation in a triangular domain and an application to diffusion-limited coalescence*, arXiv: cond-mat/0102175v1: 1-7, 2001.

[2] D. ben-Avraham, A. S. Fokas, *The solution of the modified Helmholtz equation in a wedge and an application to diffusion-limited coalescence*, arXiv: cond-mat/9906351v1: 1-11, 1999.

[3] Y. A. Antipov, A. S. Fokas, *The modified Helmholtz equation in a semi-infinite strip*, Math. Proc. Camb. Phil. Soc., 138: 339-365, 2005.

[4] Y. A. Antipov, *Diffraction of an obliquely incident electromagnetic wave by an impedance right-angled concave wedge*, Quart. J. Mech. Appl. Math. 67: 1-42, 2014.

[5] M. Ya. Antimirov, A. A. Kolyshkin, Rémi Vailancourt, Applied integral transforms, Americam mathematical society, Rhode Island, U.S.A., 1993.

[6] Mark J. Ablowitz, A. S. Fokas, Introduction and applications of Complex variables, Cambridge university press, Cambridge, 1997.

[7] B. P. Belinskii, D. P. Kouzov, V. D. Chel'tsova, *On acoustic wave diffraction by plates connected at right angle*, PMM J. Appl. Math. Mech. 37(2): 291-299, 1973.

[8] E. E. Burniston, C. E. Siewert, *The use of Riemann problems in solving a class of transcendental equations*, Proc. Camb. Phil. Soc., 73: 111-118, 1973.

[9] L. P. Castro, A. M. Simões, *The impedance problem of wave diffraction by a strip with higher order boundary conditions*, AIP Conf. Proc., 1561, 184-193, 2013.

[10] L. P. Castro, A. M. Simões, *Integral equation methods in problems of wave diffraction by a strip with higher order reactance conditions*, AIP Conf. Proc., 1493, 904-910, 2012.

[11] L. P. Castro, A. M. Simões, *Fredholm Analysis for a wave diffraction problem with higher order boundary conditions on a union of strips*, AIP Conf. Proc., 1186, 49-56, 2009.

[12] Lokenath Debnath, Integral transforms and their applications, CRC press, New York, 1995.

[13] G. Dassios, A. S. Fokas, *The basic elliptic equations in an equilateral domain*, Proc. Roy. Soc. Lond., 461(2061): 2721-2748, 2005.

[14] L. C. Evans, Partial differntial equations, Amer. Math. Soc., U.S.A., 2010.

[15] A. S. Fokas, *Lax pairs and a new spectral method for linear and integrable non linear PDEs*, Sel. Math. New ser., 4: 31-68, 1998.

[16] A. S. Fokas, *A unified transform method for solving linear and certain non linear PDEs*, Proc. Roy. Soc. Lond., 453(1962): 1411-1443, 1997.

[17] A. S. Fokas, *On the integrability of linear and non linear partial differential equations*, J. Math. Phy., 41: 4188-4237, 2000.

[18] A. S. Fokas, *Two dimensional linear partial differentail equations in a convex polygon*, Proc. Roy. Soc. Lond., 457(2006): 371-393, 2001.

[19] A. S. Fokas, A. A. Kapaev, *A Riemann-Hilbert problem approach to the Laplace equation*, J. Math. Anal. Appl., 251: 770-804, 2000.

[20] A. S. Fokas, M. Zyskin, *The fundamental differential form and boundary-value problems*, Q. J. Mech. Appl. Math., 55(3): 457-479, 2002.

[21] A. S. Fokas, *A new transform method for evolution partial differential equations*, IMA J. Appl. Math., 67: 559-590, 2002.

[22] A. S. Fokas, *From Green to Lax via Fourier transform*, Proc. sympos. Appl. Math., 65: 87-117, 2007

[23] F. D. Gakhov, Bounary value problems, Dover Publications, New york, 1990.

[24] P. Henrici, Applied and computational complex analysis volume III, John Wiley & sons, New York, 1988.

[25] Nikoloas I. Ioakimidis, *A unified Riemann-Hilbert approach to the analytical determination of zeroes of sectionally analytic functions*, J. Math. Anal. Appl. 129, 134-141, 1988.

[26] D. P. Kouzov, V. A. Pachin, *Diffraction of acoustic waves in a semi-infinite wave guide with elastic walls*, PMM J. Appl. Math. Mech. 40(1): 104-111, 1976.

[27] D. P. Kouzov, *Diffraction of a cylindrical hydroacoustic wave at the joint of two semi-infinite plates*, PMM J. Appl. Math. Mech. 33(2): 240-250, 1969.

[28] F.G. Leppington, *Accoustic scattering by membranes and plates with line constraints*, J. Sound Vib., 58(3), 319-332, 1978

[29] Jane B. Lawrie, I. David Abrahams, *An orthogonality relation for a class of problems with higher order boundary conditions; applications in sound-structure interaction*, Q. JI Mech. appl. Math. 52(2): 161-163, 1999.

[30] Jian-Ke Lu, Boundary value problems for analytic functions, World Sci. Publ. Co. Pte. Ltd. Singapore, 1994.

[31] S. R. Manam, J. Bhattacharjee, T. Sahoo, *Expansion formulae in wave structure interaction problems*, Proc. Roy. Soc. Lond., 462: 263-287, 2005.

[32] M. A. Naimark, Linear differential operators, George G. Harpap & Co. Ltd., London, 1968.

[33] A. N. Norris, A. V. Osipov, *Structural and acoustical wave interaction at a wedge-shaped junction of fluid loaded plates*, J. acout. Soc. Am., 101(2), 867-876, 1997.

[34] E. M. Stein, R. Shakarchi, Real anaysis: Measure theory, integration & Hilbert spaces, Princeton university press, New Jersey, 2005.

[35] A. M. Simões *A transmission problem for the Helmholtz equation with higher order boundary conditions*, the operators and Matrices, 9(1): 203-233, 2015.

[36] T. Sahoo, Mathematical techniques for wave interaction with flexible structures, CRC press, New York, 2013.

[37] C. E. Siewert, J. S. Phelps III, *On solutions of a transcendental equation basic to the theory of vibrating plates*, Siam J. Math. anal., 10(1): 105-111. 1979.

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