Superposition coding based co-operative diversity schemes

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SUPERPOSITION CODING BASED CO-OPERATIVE DIVERSITY SCHEMES

A Thesis

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Abstract

With a prefixed decoding order at receiver end, superposition coding approach intended for degraded broadcast channels was proposed to improve the average throughput in a point-to-point wireless channel. In this thesis, we first rigorously show the optimal decoding strategy when two Gaussian codebooks are superposed together to send source information to its receiver. Our results rigorously prove the optimality of performing successive interference cancelation at receiver end. Decoding failure at relays imposes a bottle neck to improving overall throughput in relay channels with decode-and-forward relaying strategy. We then extend this framework for a single link to relay channels to optimize the overall throughput using superposition coding and joint decoding at transmitters and receivers, respectively. Numerical results reveal considerable gains of our proposed schemes compared against traditional approach.

Chapter 1
Introduction

In the past few decades, advancement of wireless communication techniques has enabled human beings to communicate more efficiently than ever. That is evidenced by the emerging wireless communication means such as cellular systems and wireless LANs. As expected, wireless channel places fundamental limitations [5] on the performance of wireless communication systems. The transmission path between a transmitter and its receiver can vary from a simple line-of-sight to the one that is severely obstructed by buildings, mountains and foliage. Unlike wired channels that are more stationary and predictable, radio channels are extremely random and do not offer easy analysis. Modeling the radio channel has historically been one of the most difficult part of wireless communication system design., and is typically done in statistical fashion, based on measurements made specifically for an intended communication system or spectrum allocation.

Small scale fading, or simply fading, is used to describe the rapid fluctuations of the amplitude, phases or multipath delays of radio signal over a short period of time or travel distance, so that large-scale path loss can be ignored. Fading is caused by interference between two or more versions of transmitted signals which arrive at destination at different time instances. These signals, called multipath signals will combine at receiver antenna to form a resultant signal which vary widely in amplitude and phase, depending on the distribution of intensity and relative propogation time of the waves and the bandwidth of the transmitted signal. The most important effects of multi-path fading on wireless channels are:

- Rapid changes in signal strength over a small travel distance or time interval.

- Random frequency modulation due to varying Doppler shifts on different multipath signals.

- Time dispersion caused by multipath propagation delays.

Multipath fading has a great impact on wireless system performance in term of reliability and throughput. To achieve reliable transmission under a throughput requirement, fading in wireless channels should be carefully handled.

Diversity technique [5], i.e. transmitting the same information through independent channels, is an effective means to combat detrimental effects of multiphath fading. The whole idea of diversity is based on a simple observation that when a receiver is supplied with two or more copies of transmitted signals through independent faded channels, the probability that all these received signals will be in deep fade simultaneously is considerably low. For example, let p be the probability that any one signal will fade below a certain point then p^L is the probability that all L independently fading replicas of the same signal will fade below the same point. There are several ways in which we can supply receiver with L copies of the signal. We can exploit temporal, frequency and/or spacial domain diversities which correspond to channel coding, equalization and multiple antenna techniques, respectively to improve the reliability and throughput.

Equalization is to "reverse" the frequency selective fading channel to exploit frequency domain diversity. Spatial diversity can be exploited using multiple antennas at transmitter and/or receiver side to construct the so-called multiple input and multiple output (MIMO) channel. While traditional channel coding with interleaving is essentially to copy information messages across independent time slots to explore the temporal diversity.

In this thesis, we are interested in exploiting spatial and temporal diversity to improve throughput in relay channels. Next, we review in details of the relay channels, as well as the superposition coding approach for broadcast channels.

1.1 Relay Channel

Relay channel was first introduced by Van der Meulen [6], [7] [8]. He has introduced a three terminal communication in which all the terminals co-operate with each other to optimize the transmission procedure. Cover and El Gamal[1] have done a groundbreaking research in this three terminal channels. The basic model of the scheme considered by them consists of a source, relay and terminal and is given by Fig 1.1 In this figure, terminal 'A' stands for Source terminal, terminal 'B'

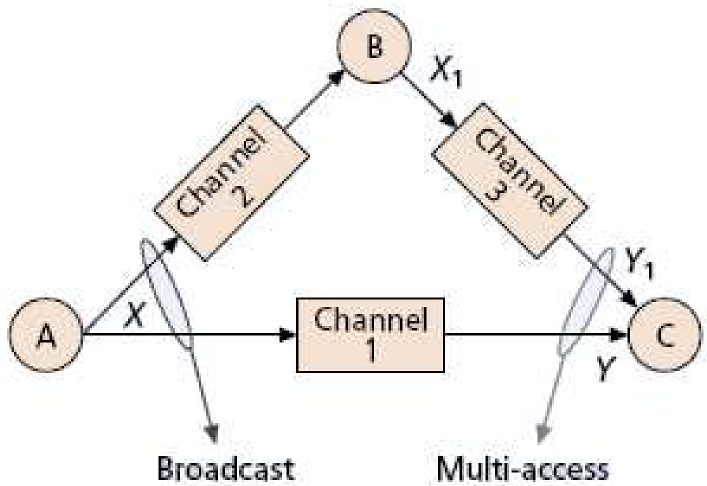

FIGURE 1.1. Model of relay channel proposed by Cover and El Gamal in [1]. This figure was adapted from [2]

acts as a relay terminal and terminal 'C' is the destination or receiver terminal to which data was intended. It was assumed that all nodes operate in the same band, so the system can be decomposed into a broadcast channel from the viewpoint of source and multi-access channel from destination or receiver viewpoint. They have determined channel capacities [1] for Gaussian relay and certain discrete relay channels. Moreover they have developed the lower bound to the capacity of a more general relay channel. This model forms the basis for the relay model of this thesis.

The best application of this relay model is in cooperative communication. In cooperative wireless communication, we are concerned with a wireless network, of the cellular or ad-hoc variety, where the wireless users, may increase their effective quality of service via cooperation. In this kind of communication, each user is assumed to transmit data and act as cooperative agent i.e. relay for another user as given in Fig 1.2. Cooperation leads to interesting trade-offs in code rates and

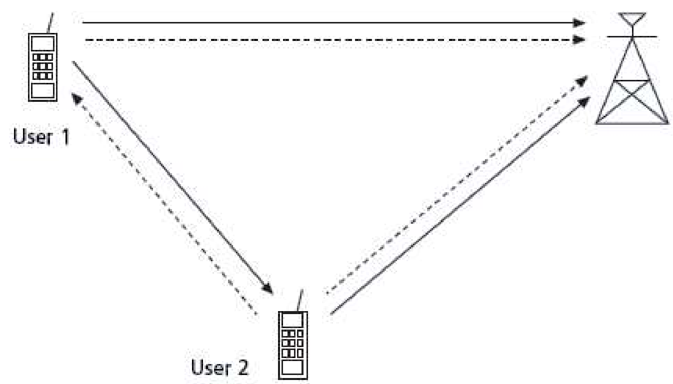

FIGURE 1.2. In cooperate communication, each user acts as both source and relay. This figure was adapted from [2]

transmit power[2].

Laneman has further studied the cooperative relay channel model given by Fig 1.1 and developed low-complexity cooperative diversity protocols. In [3], total degrees of freedom is divided into orthogonal slots to all transmitting terminals. Fig 1.3 illustrates the channel allocation for an time division approach with two terminals. Laneman studied different methods of cooperative signalling methods for which he developed efficient protocols in [3]. They are:

Amplify and Forward This method is a simple cooperative signaling. The user acting as a relay will just amplify the received signal from source user.

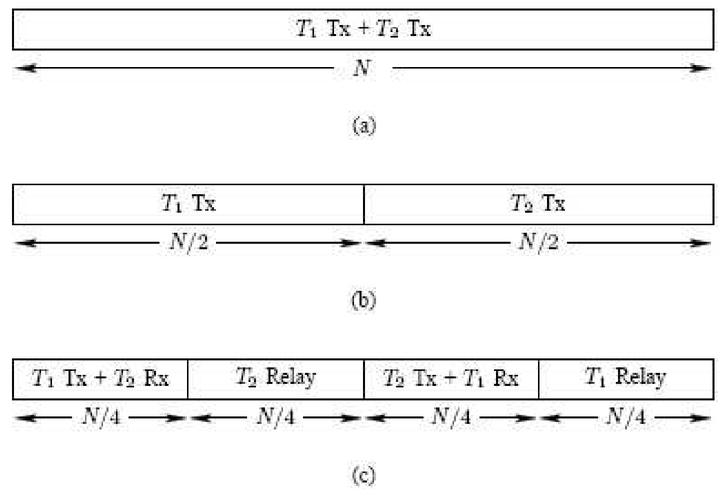

FIGURE 1.3. Time-division channel allocation for (a) direct transmission with interference, (b) orthogonal direct transmission, and (c) orthogonal cooperative diversity. This figure was adapted from [3]

Decode and Forward In this method, the user acting as relay will try to decode the message from source. If it succeeds in decoding, then the relay will retransmit the data. Laneman has considered repetition coding at relay. This means that relay will use the same kind of coding strategy as that of source.

Selection Relaying This relaying scheme corresponds to adaptive versions of amplify-and-forward and decode-and-forward, both of which fall back to direct transmission if the relay cannot decode. In this scheme relay will only transmit the data if destination cannot decode the message bits from source.

Further research has been done in relay channels in the recent past. Kramer and Wijngaarden [9] have proposed a multi-access channel in which different sources communicate with a common receiver with the help of relay. Laneman and Wornell have developed and analyzed the space-time coded cooperative diversity protocols in [10].

1.2 Broadcast Channel

In broadcast channels [11], information is sent simultaneously from one source to several receivers. The application include broadcasting of information to a crowd, or broadcasting TV information from a transmitter to multiple receivers in an area. A simple broadcast channel with one transmitter and two receiver is depicted in Fig 1.4 where (W_1, W_2) are the information bits intended for the two users in a

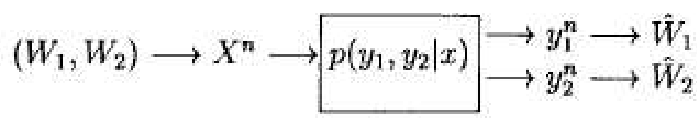

FIGURE 1.4. Simple Broadcast Channel. This figure was adapted from [4]

broadcast channel, X^n is the broadcast channel code transmitted by transmitter to two users simultaneously, $p(y_1, y_2|x)$ is the conditional probability distribution function of channel outputs given channel input and $\hat{W}_1$ and $\hat{W}_2$ are the decoded messages by the individual users separately. Refer to [4] for a good survey of broadcast channel information theory.

The coding techniques that are traditionally used for transmission between single transmitter and single receiver will not yield good results for broadcast channels. We need a novel coding technique in which the data intended for atleast two users are embedded in one code word. The encoding strategy should ensure that the data intended for a user can be treated as a noise to all other users. Superposition coding approach was developed to meet such demand [11]

1.2.1 Superposition Coding

Superposition coding in Broadcast channel was first studied by Cover in [11]. According to Cover the throughput of a broadcast channel can be increased by superimposing high-rate information on low-rate information. In a broadcast channel, a single transmission is sent to a number of users simultaneously, each of which has different channel quality. Superposition codebook can be visualized as some clouds with different centers. The code intended for the degraded channel chose a center and code intended for the non-degraded channel chose a point in the cloud surrounding the center. The good user will attain more data than the bad user. Thus different users will have different error protection. The superposition coding is closely related to multilevel coding[12] and Unequal Error Protection [13]. Bergmans and Cover [14] extended superposition coding to superposition coded modulation. Extensive research has been done on superposition coding schemes in recent past. Wang and Orchard [15] designed superposition coded modulation scheme using shaping techniques to reduce the interference between the fine-level code and the coarse-level code. Sun [16] has implemented Superposition turbo-coding scheme that performs within 1 dB of the capacity region boundary of the degraded broadcast channel at a bit-error rate of 10^{-5}.

Shamai[17], [18] and Liu[19] further studied the implementation of superposition coding in compound channels to enhance average throughput. The message to be transmitted is encoded using multi-level superposition coding and all layers of messages are transmitted over the channel simultaneously. In this approach, they treated channel states as the virtual users of a degraded broadcast channel, which means that the decoding order has been prefixed for the destination, i.e. successive interference cancelation is assumed at the receiver end. But this approach confines the achievable rates to the corner points of the capacity region. We need to find a better scheme in which all the rate pairs of the capacity region are considered.

1.3 Thesis Organization

The thesis is organized as follows. Chapter 2 gives the motivation behind this thesis and introduces the system model as well as the problem formulations. We start from analyzing the proposed scheme for direct transmissions without relay nodes in Chapter 3 and then devote the main efforts in investigating this idea for relay channels in Chapter 4 for both repetition and independent coding based decode-and-forward strategies with and without independent power allocation factor at relay. Numerical results are presented and discussed in Chapter 5. Finally, conclusions are drawn in Chapter 6.

Chapter 2
Motivation and System Model

2.1 Motivation

Channel quality plays an important role in determining the average throughput of wireless systems. If channel quality is too bad, transmitted signals will be corrupted and the receiver terminal will not be able to decode the data. Hence we need to have an adaptive transmitter which reduces the rate when the channel quality is bad. Fading[20] is the main factor that effects channel quality in networks such as ad-hoc wireless networks. In these networks, the transmitter will have no information about the channel in which it was transmitting. Hence the use of adaptive antennas is not possible. Consider a compound channel in which Channel State Information (CSI) is available at the receiver. These channels are characterized by slow fading which is a model of slowly varying channel characteristics. Cover[11] suggested that compound channels can be viewed as a broadcast channel. Different state of compound channel can be viewed as virtual users of broadcast channel. Cover[11] proposed to use superposition coding of broadcast channels in compound channel for better performance.

In this thesis, we first extend the idea of superposition coding to relay channels to exploit cooperative diversity gain in wireless networks [3]. More importantly, we develop the optimal way in encoding and decoding by changing our perspectives in applying this superposition coding based idea. We visualize the channel from multi-access channel perspective. The two layer data of the channel are treated as data from two different users rather than data transmitted to two users. This means that neither of the channel is degraded to other. This facilitates destination a freedom in the decoding order of rates. In other words, the optimal rates of the channel are not calculated over the corner points of the capacity region but over all the possible rates of the channel.

2.2 System Model

The relay channel model we assume is similar as that of [3]. In this model, we have a transmitting terminal, a relay terminal and a destination terminal as shown in figure 2.1. Throughout this thesis, we use the subscripts 's' for the source, 'r' for the relay and 'd' for the destination terminals. The wireless channel between every pair of terminals is frequency non-selective with fading coefficient $h_{i,j}$ which captures the effects of path-loss, shadowing, and fading of the (i, j) channel, where $i \in \{s, r\}$ and $j \in \{r, d\}$. It is assumed $h_{i,j}$ remains constant over an entire transmission period and is independent across different pairs of nodes.

The available time slot of frame is divided into two sub-frames which range over $\left[1 \ldots \frac{N}{2}\right]$ and $\left[\frac{N}{2} + 1 \ldots N\right]$, respectively. The two sub-frames are allocated to Source and Relay to transmit data. We assume the terminals perform half-duplex operation. Source terminal will transmit the data in the first sub-frame whereas

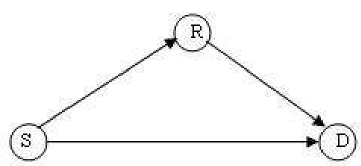

FIGURE 2.1. Illustration of system model with terminal S transmitting the information, terminal R acting as relay and terminal D the destination

relay and destination terminals will be in listening mode. In the second sub-frame, relay will transmit the data and destination terminal will be in listening mode.

The transmitter in our model transmits signals using two-level superposition coding [11] i.e. we will have two sub-channels and hence two signals $x_i^{(1)}[n]$ and

$x_i^{(2)}[n]$ for each sub-channel with rates R_1 and R_2 respectively. If the total power available at the transmitter is P, then the power allocated to the second level i.e. $x_i^{(2)}[n]$ is αP and hence the power allocated for the first level is $\overline{\alpha}P$, where $\alpha \epsilon [0,1]$, $\overline{\alpha} = 1-\alpha$ and $i \in \{s,r\}$. We encode the signals using independent gaussian codebook i.e. $x_i^{(1)}[n]$ and $x_i^{(2)}[n]$ are zero mean i.i.d complex Gaussian random variables with variance $\overline{\alpha}P$ and αP, respectively.

Over the first sub-frame for $n = 1, \ldots, N/2$, the received signals at destination and relay are [1]:

$$y_{s,d}[n] = h_{s,d} \left[\ x_s^{(1)}[n] + x_s^{(2)}[n] \ \right] + z_d[n] \tag{2.1}$$

$$y_{s,r}[n] = h_{s,r} \left[\ x_s^{(1)}[n] + x_s^{(2)}[n] \ \right] + z_r[n], \tag{2.2}$$

respectively, where $y_{s,d}[n]$ and $y_{s,r}[n]$ are the destination and relay received signals and $x_s^{(1)}[n]$ and $x_s^{(2)}[n]$ are the two signals transmitted by the source terminal using the superposition coding.

In the second sub-frame for $n = \frac{N}{2}+1, \ldots, N$, the received signal at the destination from relay is:

$$y_{r,d}[n] = h_{r,d} \left[\ x_r^{(1)}[n] + x_r^{(2)}[n] \ \right] + z_d[n] \tag{2.3}$$

where $x_r^{(1)}[n]$, $x_r^{(2)}[n]$ are the two signals transmitted by the relay. We have assumed that relay performs the decode-and-forward method i.e. relay decode the data from source in the first half of the frame and re-transmits the data in the second half of the frame to the destination terminal if decoding is successful.

In eqs (2.1)-(2.3), $z_j[n]$ captures the effects of receiver noise and other forms of interference in the system. We assume that the fading coefficients $h_{i,j}$ are accurately measured by receiver, but not known to its transmitting terminal. Statistically,

[1] For further understanding of channel model, refer to [3]

we model $h_{i,j}$ as zero-mean independent, circularly-symmetric complex Gaussian random variables with variance 1 and $z_j[n]$ as zero-mean mutually independent, circularly-symmetric, complex Gaussian random sequence with variance N_o. We denote the signal-to-noise ratio as $\text{SNR} = \frac{P}{N_o}$, where P is the total power available at source and relay for transmission.

2.3 Objective

The objective of this thesis is to find the optimal rate allocation vector $\overrightarrow{R} = [R_1, R_2]$ and power allocation factor α to maximize the average throughput $\widetilde{R} = R_1 \cdot Pr_1 + R_2 \cdot Pr_2$, where Pr_1 and Pr_2 are the probabilities of decoding correctly the superposed signals, respectively

Chapter 3
Direct Transmission

To develop the idea of multiple access channel based decoding of superposed signals without explicitly specifying rate functions and decoding order, we first study the optimal rate and power allocations for direct transmissions without relay node. In this model, the received signal is

$$y_d[n] = h_{s,d} \left[\; x_s^{(1)}[n] + x_s^{(2)}[n] \; \right] + z_d[n] \tag{3.1}$$

for $n = 1, \ldots, N$, where $x_s^{(1)}[n]$ and $x_s^{(2)}[n]$ are two signals transmitted by the source using superposition coding. The superimposed signals can be viewed as two users of a two-user multi-access channel in which the receiver employs maximum likelihood (ML) joint decoding of $x_s^{(1)}[n]$ and $x_s^{(2)}[n]$. For a given fading factor $h_{s,d}$, the receiver can calculate the instantaneous two-user multiple access channel capacity region, and determine if R_1 and R_2 are both achievable, i.e. this rate pair (R_1, R_2) is in the capacity region, or only one of them is achievable. Thus the possible outcomes at the destination are :

- Destination can decode the entire message.
- Destination can decode the first level alone.
- Destination can decode the second level alone.
- None of them is achievable.

The average throughput of the channel is determined by the mutual information between the different levels of transmitted and received signals. The mutual information between the received signals and the first level of the transmitted signal is given by:

$$I_D^{(1)} = \log_2 \left[1 + \frac{\overline{\alpha}\text{SNR}|h_{s,d}|^2}{1 + \alpha\text{SNR}|h_{s,d}|^2} \right] \quad (3.2)$$

The mutual information between received signals and the second level of the transmitted signal given that first level is known is given by:

$$I_D^{(2|1)} = \log_2 \left[1 + \alpha\text{SNR}|h_{s,d}|^2 \right] \quad (3.3)$$

The mutual information between received signals and the first level of the transmitted signal given that second level is known is given by:

$$I_D^{(1|2)} = \log_2 \left[1 + \overline{\alpha}\text{SNR}|h_{s,d}|^2 \right] \quad (3.4)$$

It can be shown[1] the valid range of α is $[0, \frac{2^{R_2}-1}{2^R-1}]$, where $R = R_2 + R_1$. The capacity region for this range of α at a given channel realization of $\{h_{s,d}\}$ is given in Figure 3.1. The valid rate pairs of this scheme will be in regions A_1, A_{2a} and A_{2b} as shown

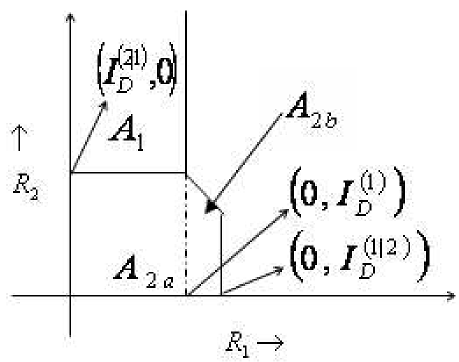

FIGURE 3.1. The caption should be "Instantaneous Capacity region for direct transmission scheme under a give $h_{s,d}$.

in the figure. Notice that rate pairs are fixed but the capacity region changes with time. Hence the rate pairs which are in A_1 at this time instance might be in A_{2a} in other time instance. Thus we need to consider the probability of rate pairs to be

[1] Refer to Appendix A for the proof

in a particular region when analyzing the average throughput. The corner points of the capacity regions are also shown in the figure. The mutual information $I_D^{(1)}$, $I_D^{(2|1)}$ and $I_D^{(1|2)}$ are given by (3.2)-(3.4) respectively. The destination will decode the first level alone if the following conditions are satisfied:

$$R_1 < I_D^{(1)} \tag{3.5}$$

$$R_2 > I_D^{(2|1)} \tag{3.6}$$

In other words, the rate pairs in the region A_1 will ensure destination to decode the first level alone. The rate pairs in A_{2a} and A_{2b} will ensure destination to decode the entire message. But we have proved in Appendix 6.2 that the rate pairs can never be in region A_{2b} under the given range of α. Hence the valid range of rate pairs will be in vertical column. This is exactly the same region when destination performs the successive interference cancelation method suggested by Liu[19] to decode the message. Thus the destination starts with decoding the first level and removes it if successful and then goes on to decode the second level message. Therefore, we have rigorously proved that fixing decoding order is an optimal way to achieve the maximum average throughput of a direct link channel when superposition coding approach is employed and multiple access channel perspective is taken in decoding the source message.

The probability with which destination can decode the first level is given by:

$$P_D^{(1)} = \Pr\begin{bmatrix} R_1 < I_D^{(1)} \& \\ R_2 > I_D^{(2|1)} \end{bmatrix} = \Pr\begin{bmatrix} |h_{s,d}|^2 > H_1 \& \\ |h_{s,d}|^2 < L_2 \end{bmatrix}$$

$$\Rightarrow P_D^{(1)} = e^{-H_1} - e^{-L_2} \tag{3.7}$$

where

$$H_1 = \frac{2^{R_1} - 1}{\text{SNR}(1 - \alpha 2^{R_1})} \tag{3.8}$$

$$L_2 = \frac{2^{R_2} - 1}{\alpha \text{SNR}} \tag{3.9}$$

The probability with which destination decodes the entire message is given by:

$$\begin{aligned} P_D^{(2)} &= \Pr \begin{bmatrix} R_2 < I_D^{(2|1)} \& \\ R_1 < I_D^{(1)} \end{bmatrix} = \Pr \left[R_2 < I_D^{(2|1)} \right] \\ &= \Pr \left[R_2 < \log_2 \left[1 + \alpha \text{SNR} |h_{s,d}|^2 \right] \right] \\ &= \Pr \left[|h_{s,d}|^2 > L_2 \right] \\ \Rightarrow P_D^{(2)} &= e^{-L_2} \end{aligned} \tag{3.10}$$

Hence the average throughput using direct transmission is given by:

$$\begin{aligned} \widetilde{R} &= R_1 \cdot P_D^{(1)} + R \cdot P_D^{(2)} \\ &= R_1 \cdot \left(e^{-H_1} - e^{-L_2} \right) + R \cdot e^{-L_2} \end{aligned} \tag{3.11}$$

The necessary conditions for optimizing the average throughput are given by:

$$\frac{\partial \widetilde{R}}{\partial R_1} = e^{-H_1} \left[1 - \frac{\overline{\alpha} R_1 2^{R_1} \ln(2)}{(1 - \alpha 2^{R_1})^2 \text{SNR}} \right] = 0 \tag{3.12}$$

$$\frac{\partial \widetilde{R}}{\partial R_2} = e^{-L_2} \left[1 - \frac{R_2 2^{R_2} \ln(2)}{\alpha \text{SNR}} \right] = 0 \tag{3.13}$$

$$\frac{\partial \widetilde{R}}{\partial \alpha} = \frac{H_1 e^{-H_1} R_1 2^{R_1}}{1 - \alpha 2^{R_1}} + \frac{R_2 e^{L_2} L_2}{\alpha} = 0 \tag{3.14}$$

For given α, the optimal values of R_2 is given by:

$$R_2^{opt} = \frac{L\big(\alpha \text{SNR}\big)}{\log_e 2} \tag{3.15}$$

where the Lambert's W function $w = L(x)$ satisfy $we^w = x$ [21]. And the optimum value of R_1 is the solution of the equation

$$R_1 2^{R_1} \ln(2)\ \overline{\alpha} = \text{SNR} \Bigg(1 + \alpha^2 2^{2R_1} - 2\alpha 2^{R_1} \Bigg) \tag{3.16}$$

The optimal α to maximize the average throughput is attained numerically by searching over the range $[0, \frac{2^{R_2}-1}{2^R-1}]$.

Chapter 4
Relay Network

We extend our theory of dual rates to relay networks to study the improvements over single link. For comparison purpose we investigate the performance of single rate in relay scheme.

4.1 Single Rate Relay Channel Network

The signals received over the first part of frame for single rate scheme are given by (2.1) and (2.2) for $n = 1 \ldots \frac{N}{2}$. The received signals at destination over the second part of frame is given by:

$$y_{r,d}[n] = h_{r,d}\; x_r[n] + z_d[n] \tag{4.1}$$

for $n = \frac{N}{2} + 1 \ldots N$, where $x_r[n]$ is the transmitted signal by the relay. We assume that relay performs repetition coding scheme when it re-encodes the data to destination. Depending on the decision of relay, we have two cases.

CASE 1

When relay decodes the message, the signals received at the destination over the second part of the message is given by (4.1). The received signals at the destination over the first part of the frame is given by (2.1). The mutual information between the received signals at relay and source message over the first half of the frame is given by:

$$I_S^{(s,r)} = \frac{1}{2}\log_2\left[1 + |h_{s,r}|^2 \mathrm{SNR}\right] \tag{4.2}$$

Now, the probability with which relay is able to decode the message is given by:

$$P_S^{(1)} = \Pr\left[R < I_S^{(s,r)}\right]$$

$$\Rightarrow P_S^{(1)} = e^{-\frac{2^{2R}-1}{\text{SNR}}} \tag{4.3}$$

The mutual information between the received signals and transmitted signals from the source and relay is given by:

$$I_S^{(1)} = \frac{1}{2}\log_2\left[1 + \left(|h_{s,d}|^2 + |h_{r,d}|^2\right)\text{SNR}\right] \tag{4.4}$$

Average throughput of the network given relay transmitted the entire message is given by:

$$\begin{aligned} \widetilde{R_S^{(1)}} &= R \cdot \left[\Pr\left[R < I_S^{(1)}\right]\right] \\ \Rightarrow \widetilde{R_S^{(1)}} &= R \cdot \left[1 + \frac{2^{2R}-1}{\text{SNR}}\right] \cdot e^{-\frac{2^{2R}-1}{\text{SNR}}} \end{aligned} \tag{4.5}$$

CASE 2

When relay is unable to decode the message, then relay is incapable of sending any message to the destination in the second part of the frame. The probability with which relay is unable to decode the message is given by:

$$P_S^{(2)} = 1 - P_S^{(1)} = 1 - e^{-\frac{2^{2R}-1}{\text{SNR}}} \tag{4.6}$$

The mutual information between the received signal and transmitted signal from the source is given by:

$$I_S^{(2)} = \frac{1}{2}\log_2\left[1 + |h_{s,d}|^2\text{SNR}\right] \tag{4.7}$$

The average throughput when relay is unable to decode the message is given by:

$$\begin{aligned} \widetilde{R_S^{(2)}} &= R \cdot \Pr\left[R < I_S^{(2)}\right] \\ \Rightarrow \widetilde{R_S^{(2)}} &= R \cdot e^{-\frac{2^{2R}-1}{\text{SNR}}} \end{aligned} \tag{4.8}$$

Thus the average throughput for a single rate relay scheme is given by:

$$R_S = P_S^{(1)} \cdot \widetilde{R_S^{(1)}} + P_S^{(2)} \cdot \widetilde{R_S^{(2)}} \tag{4.9}$$

For a given SNR, the optimum value of 'R' to maximize the average throughput will be the solution of the following nonlinear equation:

$$e^{-\frac{2^{2x}-1}{\text{SNR}}}\left[\text{SNR}\left(2^{2x}+2x2^{2x}\ln(2)-1\right)-4x2^{2x}\ln(2)(1+2^{2x})\right]$$
$$+\text{SNR}\left(\text{SNR}+2x2^{2x}\ln(2)\right)=0 \qquad (4.10)$$

4.2 Dual Rate Relay Channel Network

we extend our investigation of using superposition coding from single link to relay channels. The general equation for signals at the end of first half of frame is given by (2.1), (2.2) and for the second half of the frame is given by (2.3). Subject to outcomes of decoding at the relay node, we can compute the instantaneous capacity region of the equivalent two-user multiple access channel between the source and its destination, where two users could be equipped with virtual 2-antenna array if one of the level is decodable at the relay. Depending on the outcome of relay's decision, we have three cases. They are:

CASE 1

When the relay can decode complete message sent by the transmitter terminal. The probability with which relay decodes the entire message is given by:

$$P^{(1)}=Pr\left[|h_{s,r}|^2>\frac{2^{2R_2}-1}{\alpha\text{SNR}}\right] \qquad (4.11)$$

The received signals at the destination are given as:

$$y_{s,d}[n]=h_{s,d}\left[x_s^{(1)}[n]+x_s^{(2)}[n]\right]+z_d[n] \qquad (4.12)$$

$$y_{r,d}[n]=h_{r,d}\left[x_r^{(1)}[n]+x_r^{(2)}[n]\right]+z_d[n] \qquad (4.13)$$

CASE 2

When the relay can decode only one level from the message sent by the transmitter terminal. The probability with which relay decodes the first level alone is given by:

$$P^{(2)}=Pr\left[\frac{2^{2R_1}-1}{(1-\alpha 2^{2R_1})\text{SNR}}<|h_{s,r}|^2<\frac{2^{2R_2}-1}{\alpha\text{SNR}}\right] \qquad (4.14)$$

The received signal at the destination are given as:

$$y_{s,d}[n] = h_{s,d}\left[x_s^{(1)}[n] + x_s^{(2)}[n]\right] + z_d[n] \tag{4.15}$$

$$y_{r,d}[n] = h_{r,d}\left[x_r^{(1)}[n]\right] \qquad + z_d[n] \tag{4.16}$$

CASE 3

The relay cannot decode any level of the message sent by the transmitter terminal. The Probability with which relay decodes nothing is given by:

$$P^{(3)} = Pr\left[|h_{s,r}|^2 < \frac{2^{2R_1} - 1}{(1 - \alpha 2^{2R_1})\text{SNR}}\right] \tag{4.17}$$

The received signal at the destination are thus given by:

$$y_{s,d}[n] = h_{s,d}\left[x_s^{(1)}[n] + x_s^{(2)}[n]\right] + z_d[n] \tag{4.18}$$

$$y_{r,d}[n] = 0 \tag{4.19}$$

Throughout the remaining of this thesis, we have used mutual information between the received signal at the destination terminal and the various levels of transmitted signal. The general notation used for mutual information is $I_X^{(i-j|k)}$. The subscript 'X' stands for the type of scheme used at the relay terminal and $X \in \{R, I, R_\beta, I_\beta\}$, where 'R' stands for repetition coding as discussed in Section 4.2.1, 'I' stands for Independent coding as discussed in Section 4.2.2, 'R_β' stands for Repetition coding with relay using an independent power allocation factor 'β' as discussed in Section 4.2.3 and 'I_β' stands for Independent coding with relay using an independent power allocation factor 'β' as discussed in Section 4.2.4. The variable 'i' in superscript corresponds to the decision made at relay. As discussed above, we have three cases for each decision made at relay. Hence $i \in \{1, 2, 3\}$. The variable 'k' in the superscript indicates that destination already decoded the 'k'th level of message. The absence of this variable indicates that destination has no knowledge of any of the levels. The variable 'j' in superscript indicates the level of the signal to which mutual information is being calculated i.e. $I_X^{(i-j)}$ is the mutual information between the received signal at the destination and the j^{th} level of the message. Since we are utilizing the 2-level superposition coding scheme, the probable values of j are $\{0, 1, 2\}$. Here '0' stands for the entire message. Therefore $I_R^{(i-j|k)}$ stands for mutual information between the received signal and the j^{th} level of transmitted signal given that destination decoded k^{th} level for the Case-i at relay where relay uses Repetition coding scheme.

Now the relaying strategies we considered are as follows:

4.2.1 Repetition Coding Based Scheme

Relay uses the same coding method as that of source to encode the decoded message and the same power allocation factor 'α'. Hence relay allocates $\overline{\alpha}P$ power to the first level and αP power to the second level where P is the total power available to relay. Therefore, we have $x_r^{(1)} = x_s^{(1)}$ and $x_r^{(2)} = x_s^{(2)}$. Average throughput is evaluated under three cases depending on the decision at relay as discussed before. The three cases are:

CASE 1

The matrix form representation of received signals by the destination is given by:

$$\begin{bmatrix} y_{s,d}[n] \\ y_{r,d}[n+\frac{N}{2}] \end{bmatrix} = \begin{bmatrix} h_{s,d} & h_{s,d} \\ h_{r,d} & h_{r,d} \end{bmatrix} \begin{bmatrix} x_s^{(1)}[n] \\ x_s^{(2)}[n] \end{bmatrix} + \begin{bmatrix} z_{s,d}[n] \\ z_{r,d}[n+\frac{N}{2}] \end{bmatrix} \tag{4.20}$$

for $n = 1 \ldots \frac{N}{2}$. This can also be written as:

$$\mathbf{y} = A\mathbf{x} + n \tag{4.21}$$

where:

$$\mathbf{y} = \begin{bmatrix} y_{s,d}[n] \\ y_{r,d}[n+\frac{N}{2}] \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} h_{s,d} & h_{s,d} \\ h_{r,d} & h_{r,d} \end{bmatrix} \quad \mathbf{A^{(1)}} = \begin{bmatrix} h_{s,d} \\ h_{r,d} \end{bmatrix} \quad \mathbf{A^{(2)}} = \begin{bmatrix} h_{s,d} \\ h_{r,d} \end{bmatrix} \tag{4.22}$$

$$\mathbf{x} = \begin{bmatrix} x_s^{(1)}[n] \\ x_s^{(2)}[n] \end{bmatrix} \quad \mathbf{x^{(1)}} = \begin{bmatrix} x_s^{(1)}[n] \end{bmatrix} \quad \mathbf{x^{(2)}} = \begin{bmatrix} x_s^{(2)}[n] \end{bmatrix} \quad \mathbf{n} = \begin{bmatrix} z_{s,d}[n] \\ z_{r,d}[n + \frac{N}{2}] \end{bmatrix} \tag{4.23}$$

The procedure for calculation of the entropy of these kind of signals is given by Telatar in [22]. The entropy of the received signal is thus given by:

$$H(\mathbf{y}) = \log_2 \det |(\pi e)\, E(\mathbf{y} \cdot \mathbf{y}^*)| \tag{4.24}$$

Where E(.) is the expectation and $\mathbf{y}^*$ is the complex conjugate of $\mathbf{y}$. This can be simplified to:

$$H(\mathbf{y}) = \log_2 \det |(\pi e)\, (\mathbf{A}\mathbf{E}(\mathbf{x} \cdot \mathbf{x}^*)\mathbf{A}^* + \mathbf{E}(\mathbf{n} \cdot \mathbf{n}^*))|$$

$$= \log_2 \det \left| \begin{bmatrix} h_{s,d} & h_{s,d} \\ h_{r,d} & h_{r,d} \end{bmatrix} \begin{bmatrix} \overline{\alpha} P & 0 \\ 0 & \alpha P \end{bmatrix} \begin{bmatrix} h_{s,d}^* & h_{r,d}^* \\ h_{s,d}^* & h_{r,d}^* \end{bmatrix} + \begin{bmatrix} N_o & 0 \\ 0 & N_o \end{bmatrix} \right| + \log_2 (\pi e)^2$$

$$= \log_2 \det \begin{vmatrix} |h_{s,d}|^2 P + N_o & h_{s,d} h_{r,d}^* P \\ h_{s,d} h_{r,d}^* P & |h_{r,d}|^2 P + N_o \end{vmatrix} + \log_2 (\pi e)^2$$

$$H(\mathbf{y}) = \log_2 \left[(N_o \pi e)^2 \left(1 + (|h_{s,d}|^2 + |h_r, d|^2) \mathrm{SNR} \right) \right] \tag{4.25}$$

Where SNR $= \frac{P}{N_o}$ is the signal-to-noise ration of the received signal. Now the entropy of received signal given the first level is known at receiver is given by:

$$H\left(\mathbf{y}|\mathbf{x^{(1)}}\right) = \log_2 \det \left| \pi e \left(\mathbf{A^{(2)}} \mathbf{E}(\mathbf{x^{(2)}} \cdot \mathbf{x^{(2)*}}) \mathbf{A^{(2)*}} + E(\mathbf{n} \cdot \mathbf{n}^*) \right) \right|$$

This can be simplified to:

$$H\left(\mathbf{y}|\mathbf{x^{(1)}}\right) = \log_2 \det \left| \begin{bmatrix} h_{s,d} \\ h_{r,d} \end{bmatrix} \alpha P \begin{bmatrix} h_{s,d}^* & h_{r,d}^* \end{bmatrix} + \begin{bmatrix} N_o & 0 \\ 0 & N_o \end{bmatrix} \right| + \log_2 (\pi e)^2$$

$$H\left(\mathbf{y}|\mathbf{x^{(1)}}\right) = \log_2 \left[(N_o \pi e)^2 \left(1 + \alpha \left(|h_{s,d}|^2 + |h_{r,d}|^2 \right) \mathrm{SNR} \right) \right] \tag{4.26}$$

Similarly the entropy of the received signal given that first level is known is given by:

$$H\left(\mathbf{y}|\mathbf{x^{(2)}}\right) = \log_2 \left[(N_o \pi e)^2 \left(1 + \overline{\alpha} \left(|h_{s,d}|^2 + |h_{r,d}|^2 \right) \mathrm{SNR} \right) \right] \tag{4.27}$$

The entropy of the received signal when both levels are known to receiver is given by:

$$H(\mathbf{y}|\mathbf{x}) = \log_2 \det |\pi e E(\mathbf{n} \cdot \mathbf{n}^*)|$$
$$H(\mathbf{y}|\mathbf{x}) = \log_2 (N_o \pi e)^2 \tag{4.28}$$

From these values of entropies, we can calculate the mutual information of the system. The mutual information between the received signal and the first level of the transmitted signal from relay and source is given by

$$\begin{aligned} I_R^{(1-1)} &= \frac{1}{2}\left[H(\mathbf{y}) - H\left(\mathbf{y}|\mathbf{x}^{(1)}\right)\right] \\ &= \frac{1}{2}\log_2\left[1 + \frac{\left[|h_{s,d}|^2 + |h_{r,d}|^2\right]\overline{\alpha}\mathrm{SNR}}{1 + \left[|h_{s,d}|^2 + |h_{r,d}|^2\right]\alpha\mathrm{SNR}}\right] \end{aligned} \tag{4.29}$$

The factor '$\frac{1}{2}$' is introduced because the channel utilizes only one half of the available time slot for transmission. In other words, the transmitter will transmit data over one half of the transmission frame. The mutual information between the received signal and the second level of the transmitted signal from relay and source when the first level is already decoded is given by:

$$\begin{aligned} I_R^{(1-2|1)} &= \frac{1}{2}\left[H\left(\mathbf{y}|\mathbf{x}^{(1)}\right) - H(\mathbf{y}|\mathbf{x})\right] \\ &= \frac{1}{2}\log_2\left[1 + \alpha\mathrm{SNR}\left(|h_{s,d}|^2 + |h_{r,d}|^2\right)\right] \end{aligned} \tag{4.30}$$

Now, the average throughput of relay network when relay transmits the entire message is given by[1]

$$\widetilde{R_R^{(2)}} = R_1 \cdot Pr\begin{bmatrix} R_1 < I_R^{(1-1)} \& \\ R_2 > I_R^{(1-2|1)} \end{bmatrix} + R \cdot Pr\left[R_2 < I_R^{(1-2|1)}\right] \tag{4.31}$$

$$= R_1 \cdot \left((1 + H_1^*)e^{-H_1^*} - (1 + L_2^*)e^{-L_2^*}\right)\left(1 - e^{-L_2^*}\right)$$
$$+ R \cdot e^{-L_2^*} \tag{4.32}$$

[1]Refer Proposition 3 in Appendix B for the proof

where:

$$H_1^* = \frac{2^{2R_1} - 1}{(1 - \alpha 2^{2R_1})\,\mathrm{SNR}} \tag{4.33}$$

$$L_2^* = \frac{2^{2R_2} - 1}{\alpha \mathrm{SNR}} \tag{4.34}$$

CASE 2

The matrix form representation of the received signal for this case is given by:

$$\mathbf{y} = A\mathbf{x} + n \tag{4.35}$$

where:

$$\mathbf{y} = \begin{bmatrix} y_{s,d}[n] \\ y_{r,d}[n + \frac{N}{2}] \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} h_{s,d} & h_{s,d} \\ h_{r,d} & 0 \end{bmatrix} \quad \mathbf{A^{(1)}} = \begin{bmatrix} h_{s,d} \\ h_{r,d} \end{bmatrix} \quad \mathbf{A^{(2)}} = \begin{bmatrix} h_{s,d} \\ 0 \end{bmatrix} \tag{4.36}$$

$$\mathbf{x} = \begin{bmatrix} x_s^{(1)}[n] \\ x_s^{(2)}[n] \end{bmatrix} \quad \mathbf{x^{(1)}} = \begin{bmatrix} x_s^{(1)}[n] \end{bmatrix} \quad \mathbf{x^{(2)}} = \begin{bmatrix} x_s^{(2)}[n] \end{bmatrix} \quad \mathbf{n} = \begin{bmatrix} z_{s,d}[n] \\ z_{r,d}[n + \frac{N}{2}] \end{bmatrix} \tag{4.37}$$

The entropy of the received signal can be calculated as:

$$H(\mathbf{y}) = \log_2 \det |\pi e E(\mathbf{y} \cdot \mathbf{y}^*)| \tag{4.38}$$

Where E(.) is the expectation and $\mathbf{y}^*$ is the complex conjugate of $\mathbf{y}$. This can be simplified to:

$$\begin{aligned} H(\mathbf{y}) &= \log_2 \det |\pi e\,(\mathbf{AE}(\mathbf{x} \cdot \mathbf{x}^*)\mathbf{A}^* + \mathbf{E}(\mathbf{n} \cdot \mathbf{n}^*))| \\ &= \log_2 \det \left| \begin{bmatrix} h_{s,d} & h_{s,d} \\ h_{r,d} & 0 \end{bmatrix} \begin{bmatrix} \overline{\alpha} P & 0 \\ 0 & \alpha P \end{bmatrix} \begin{bmatrix} h_{s,d}^* & h_{r,d}^* \\ h_{s,d}^* & 0 \end{bmatrix} + \begin{bmatrix} N_o & 0 \\ 0 & N_o \end{bmatrix} \right| + \log_2 (\pi e)^2 \\ &= \log_2 \det \begin{vmatrix} |h_{s,d}|^2 P + N_o & h_{s,d} h_{r,d}^* P \\ h_{s,d} h_{r,d}^* P & \overline{\alpha} |h_{r,d}|^2 P + N_o \end{vmatrix} + \log_2 (\pi e)^2 \end{aligned}$$

$$H(\mathbf{y}) = \log_2 \left[(N_o \pi e)^2 \left(1 + (|h_{s,d}|^2 + \overline{\alpha} |h_{r,d}|^2)\mathrm{SNR} + |h_{s,d}|^2 |h_{r,d}|^2 \alpha \overline{\alpha} \mathrm{SNR}^2\right) \right] \tag{4.39}$$

Where $\text{SNR} = \frac{P}{N_o}$ is the signal-to-noise ration of the received signal. Now the entropy of received signal given the first level is known at receiver is given by:

$$H\left(\mathbf{y}|\mathbf{x^{(1)}}\right) = \log_2 \det \left|\pi e \left(\mathbf{A^{(2)}}\mathbf{E}(\mathbf{x^{(2)}} \cdot \mathbf{x^{(2)*}})\mathbf{A^{(2)*}} + E(\mathbf{n} \cdot \mathbf{n}^*)\right)\right|$$

This can be simplified to:

$$\begin{aligned} H\left(\mathbf{y}|\mathbf{x^{(1)}}\right) &= \log_2 \det \left| \begin{bmatrix} h_{s,d} \\ 0 \end{bmatrix} \alpha P \begin{bmatrix} h_{s,d}^* & 0 \end{bmatrix} + \begin{bmatrix} N_o & 0 \\ 0 & N_o \end{bmatrix} \right| + \log_2 (\pi e)^2 \\ &= \log_2 \left[(N_o \pi e)^2 \left(1 + \alpha |h_{s,d}|^2 \text{SNR}\right)\right] \end{aligned} \tag{4.40}$$

The entropy of received signal when the second level is know to receiver is given by:

$$\begin{aligned} H\left(\mathbf{y}|\mathbf{x^{(2)}}\right) &= \log_2 \det \left|\pi e \left(\mathbf{A^{(1)}}\mathbf{E}(\mathbf{x^{(1)}} \cdot \mathbf{x^{(1)*}})\mathbf{A^{(1)*}} + E(\mathbf{n} \cdot \mathbf{n}^*)\right)\right| \\ &= \log_2 \det \left| \begin{bmatrix} h_{s,d} \\ h_{r,d} \end{bmatrix} \overline{\alpha} P \begin{bmatrix} h_{s,d}^* & h_{r,d}^* \end{bmatrix} + \begin{bmatrix} N_o & 0 \\ 0 & N_o \end{bmatrix} \right| + \log_2 (\pi e)^2 \\ H\left(\mathbf{y}|\mathbf{x^{(2)}}\right) &= \log_2 \left[(N_o \pi e)^2 \left(1 + \overline{\alpha}\left(|h_{s,d}|^2 + |h_{r,d}|^2\right) \text{SNR}\right)\right] \end{aligned} \tag{4.41}$$

The mutual information between received signal and the first level of the signal from source and relay is given by:

$$\begin{aligned} I_R^{(2-1)} &= \frac{1}{2}\left[H(\mathbf{y}) - H\left(\mathbf{y}|\mathbf{x^{(1)}}\right)\right] \\ &= \frac{1}{2}\log_2 \left[1 + \frac{\left(|h_{s,d}|^2 + |h_{r,d}|^2\right)\overline{\alpha}\text{SNR} + |h_{s,d}|^2 |h_{r,d}|^2 \alpha\overline{\alpha}\text{SNR}^2}{1 + \alpha|h_{s,d}|^2\text{SNR}}\right] \end{aligned} \tag{4.42}$$

The mutual information between received signals at the destination and second level of the source, given first level is decoded is given by:

$$\begin{aligned} I_R^{(2-2|1)} &= \frac{1}{2}\left[H\left(\mathbf{y}|\mathbf{x^{(1)}}\right) - H(\mathbf{y}|\mathbf{x})\right] \\ &= \frac{1}{2}\log_2 \left[1 + \alpha\text{SNR}|h_{s,d}|^2\right] \end{aligned} \tag{4.43}$$

It can be proved that destination has to decode the message using successive interference cancelation method when relay transmits only one level. Now, the average throughput of relay network when relay transmits the first level given by[2]:

$$\widetilde{R_R^{(2)}} = R_1 \cdot Pr \begin{bmatrix} R_1 < I_R^{(2-1)} \& \\ R_2 > I_R^{(2-2|1)} \end{bmatrix} + R \cdot Pr \left[R_2 < I_R^{(2-2|1)} \right] \tag{4.44}$$

CASE 3

The received signals at the destination are given by (4.18), (4.19).This is similar to direct transmission. The matrix form representation of the received signal for this case is given by:

$$\mathbf{y} = A\mathbf{x} + n \tag{4.45}$$

where:

$$\mathbf{y} = \begin{bmatrix} y_{s,d}[n] \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} h_{s,d} & h_{s,d} \end{bmatrix} \quad \mathbf{A^{(1)}} = \begin{bmatrix} h_{s,d} \end{bmatrix} \quad \mathbf{A^{(2)}} = \begin{bmatrix} h_{s,d} \end{bmatrix} \tag{4.46}$$

$$\mathbf{x} = \begin{bmatrix} x_s^{(1)}[n] \\ x_s^{(2)}[n] \end{bmatrix} \quad \mathbf{x^{(1)}} = \begin{bmatrix} x_s^{(1)}[n] \end{bmatrix} \quad \mathbf{x^{(2)}} = \begin{bmatrix} x_s^{(2)}[n] \end{bmatrix} \quad \mathbf{n} = \begin{bmatrix} z_{s,d}[n] \end{bmatrix} \tag{4.47}$$

The entropy of the received signal can be calculated as:

$$H(\mathbf{y}) = \log_2 \det |\pi e E(\mathbf{y} \cdot \mathbf{y}^*)| \tag{4.48}$$

Where E(.) is the expectation and $\mathbf{y}^*$ is the complex conjugate of $\mathbf{y}$. This can be simplified to:

$$\begin{aligned} H(\mathbf{y}) &= \log_2 \det |\pi e \left(\mathbf{AE}(\mathbf{x} \cdot \mathbf{x}^*)\mathbf{A}^* + \mathbf{E}(\mathbf{n} \cdot \mathbf{n}^*) \right)| \\ &= \log_2 \det \left| \begin{bmatrix} h_{s,d} & h_{s,d} \end{bmatrix} \begin{bmatrix} \overline{\alpha} P & 0 \\ 0 & \alpha P \end{bmatrix} \begin{bmatrix} h_{s,d}^* \\ h_{s,d}^* \end{bmatrix} + \begin{bmatrix} N_o \end{bmatrix} \right| + \log_2 (\pi e) \\ &= \log_2 \det \left| |h_{s,d}|^2 P + N_o \right| + \log_2 (\pi e) \end{aligned}$$

[2]Refer Proposition 3 in Appendix B for the proof

$$H(\mathbf{y}) = \log_2 \left[(N_o \pi e)\left(1 + |h_{s,d}|^2 \text{SNR}\right)\right] \tag{4.49}$$

Where $\text{SNR} = \frac{P}{N_o}$ is the signal-to-noise ration of the received signal. Now the entropy of received signal given the first level is known at receiver is given by:

$$H\left(\mathbf{y}|\mathbf{x}^{(1)}\right) = \log_2 \det \left|\pi e \left(\mathbf{A}^{(2)}\mathbf{E}(\mathbf{x}^{(2)} \cdot \mathbf{x}^{(2)*})\mathbf{A}^{(2)*} + E(\mathbf{n} \cdot \mathbf{n}^*)\right)\right|$$

This can be simplified to:

$$\begin{aligned} H\left(\mathbf{y}|\mathbf{x}^{(1)}\right) &= \log_2 \det \left|\pi e \left(h_{s,d}\ \alpha P\ h^*_{s,d} + N_o\right)\right| \\ &= \log_2 \left[(N_o \pi e)\left(1 + \alpha |h_{s,d}|^2 \text{SNR}\right)\right] \end{aligned} \tag{4.50}$$

The entropy of received signal when the second level is know to receiver is given by:

$$\begin{aligned} H\left(\mathbf{y}|\mathbf{x}^{(2)}\right) &= \log_2 \det \left|\pi e \left(\mathbf{A}^{(1)}\mathbf{E}(\mathbf{x}^{(1)} \cdot \mathbf{x}^{(1)*})\mathbf{A}^{(1)*} + E(\mathbf{n} \cdot \mathbf{n}^*)\right)\right| \\ &= \log_2 \det \left|h_{s,d}\ \overline{\alpha} P\ h^*_{s,d} + N_o\right| + \log_2 (\pi e) \\ &= \log_2 \left[(N_o \pi e)\left(1 + \overline{\alpha} |h_{s,d}|^2 \text{SNR}\right)\right] \end{aligned} \tag{4.51}$$

The mutual information between received signal and the first level of the signal from source is given by:

$$\begin{aligned} I_R^{(3-1)} &= \frac{1}{2}\left[H(\mathbf{y}) - H\left(\mathbf{y}|\mathbf{x}^{(1)}\right)\right] \\ &= \frac{1}{2}\log_2 \left[1 + \frac{\overline{\alpha}\text{SNR}|h_{s,d}|^2}{1 + \alpha \text{SNR}|h_{s,d}|^2}\right] \end{aligned} \tag{4.52}$$

The mutual information between received signals at the destination and second level of the source given the first level is decoded is given by:

$$\begin{aligned} I_R^{(3-2|1)} &= \frac{1}{2}\left(H\left(\mathbf{y}|\mathbf{x}^{(1)}\right) - H(\mathbf{y}|\mathbf{x})\right) \\ &= \frac{1}{2}\log_2 \left[1 + \alpha \text{SNR}|h_{s,d}|^2\right] \end{aligned} \tag{4.53}$$

Now, the average throughput of relay network when relay transmits nothing is similar to direct transmission and is given by

$$\widetilde{R_R^{(3)}} = R_1 \cdot Pr \begin{bmatrix} R_1 < I_R^{(3-1)} \& \\ R_2 > I_R^{(3-2|1)} \end{bmatrix} + R \cdot Pr \left[R_2 < I_R^{(3-2|1)} \right] \tag{4.54}$$

$$= R_1 \cdot \left(e^{-H_1^*} - e^{-L_2^*} \right) + R \cdot e^{-L_2^*} \tag{4.55}$$

The average throughput of the network if relay uses the repetition coding scheme is given by:

$$R_R^{\Sigma} = P^{(1)} \cdot \widetilde{R_R^{(1)}} + P^{(2)} \cdot \widetilde{R_R^{(2)}} + P^{(3)} \cdot \widetilde{R_R^{(3)}} \tag{4.56}$$

Where $P^{(1)}$, $P^{(2)}$ and $P^{(3)}$ are given by 4.11, 4.14 and 4.17 respectively. There is no closed form for this equation. The optimal values of average throughput, as well as corresponding R_1, R_2, α can be attained using the standard convex optimization method.

4.2.2 Independent Coding Based Scheme

Relay employs independent codebook than the source with power scaling factor α after it successively decodes the corresponding source packets. This is applied to increase the spectral efficiency. Similar to repetition coding based relay scheme, independent coding based scheme has three cases depending on the decision made by relay over the first half of frame.

CASE 1

The matrix form representation of received signals by the destination is given by:

$$\begin{bmatrix} y_{s,d}[n] \\ y_{r,d}[n + \frac{N}{2}] \end{bmatrix} = \begin{bmatrix} h_{s,d} & h_{s,d} & 0 & 0 \\ 0 & 0 & h_{r,d} & h_{r,d} \end{bmatrix} \begin{bmatrix} x_s^{(1)}[n] \\ x_s^{(2)}[n] \\ x_r^{(1)}[n] \\ x_r^{(2)}[n] \end{bmatrix} + \begin{bmatrix} z_{s,d}[n] \\ z_{r,d}[n + \frac{N}{2}] \end{bmatrix} \tag{4.57}$$

for $n = 1 \dots \frac{N}{2}$. This can also be written as:

$$\mathbf{y} = \mathbf{A} \cdot \mathbf{x} + n = \mathbf{A^{(1)}}\mathbf{x^{(1)}} + \mathbf{A^{(2)}}\mathbf{x^{(2)}} + n \tag{4.58}$$

where:

$$\mathbf{y} = \begin{bmatrix} y_{s,d}[n] \\ y_{r,d}[n+\frac{N}{2}] \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} h_{s,d} & h_{s,d} & 0 & 0 \\ 0 & 0 & h_{r,d} & h_{r,d} \end{bmatrix} \quad \mathbf{A^{(1)}} = \begin{bmatrix} h_{s,d} & 0 \\ 0 & h_{r,d} \end{bmatrix} \tag{4.59}$$

$$\mathbf{A^{(2)}} = \begin{bmatrix} h_{s,d} & 0 \\ 0 & h_{r,d} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_s^{(1)}[n] \\ x_s^{(2)}[n] \\ x_r^{(1)}[n] \\ x_r^{(2)}[n] \end{bmatrix} \quad \mathbf{x^{(1)}} = \begin{bmatrix} x_s^{(1)}[n] \\ x_r^{(1)}[n] \end{bmatrix} \tag{4.60}$$

$$\mathbf{x^{(2)}} = \begin{bmatrix} x_s^{(2)}[n] \\ x_r^{(2)}[n] \end{bmatrix} \quad \mathbf{n} = \begin{bmatrix} z_{s,d}[n] \\ z_{r,d}[n+\frac{N}{2}] \end{bmatrix} \tag{4.61}$$

The entropy of the received signal can be calculated as:

$$H(\mathbf{y}) = \log_2 \det |\pi e E(\mathbf{y} \cdot \mathbf{y}^*)| \tag{4.62}$$

Where E(.) is the expectation and $\mathbf{y}^*$ is the complex conjugate of $\mathbf{y}$. This can be simplified to:

$$\begin{aligned} H(\mathbf{y}) &= \log_2 \det |\pi e \left(\mathbf{A}\mathbf{E}(\mathbf{x} \cdot \mathbf{x}^*)\mathbf{A}^* + \mathbf{E}(\mathbf{n} \cdot \mathbf{n}^*)\right)| \\ &= \log_2 \det \left| \pi e \left[\begin{bmatrix} h_{s,d} & h_{s,d} & 0 & 0 \\ 0 & 0 & h_{r,d} & h_{r,d} \end{bmatrix} \begin{bmatrix} \overline{\alpha}P & 0 & 0 & 0 \\ 0 & \alpha P & 0 & 0 \\ 0 & 0 & \overline{\alpha}P & 0 \\ 0 & 0 & 0 & \alpha P \end{bmatrix} \begin{bmatrix} h_{s,d}^* & 0 \\ h_{s,d}^* & 0 \\ 0 & h_{r,d}^* \\ 0 & h_{r,d}^* \end{bmatrix} + \begin{bmatrix} N_o & 0 \\ 0 & N_o \end{bmatrix} \right] \right| \\ &= \log_2 \det \pi e \begin{vmatrix} |h_{s,d}|^2 P + N_o & 0 \\ 0 & |h_{r,d}|^2 P + N_o \end{vmatrix} \end{aligned}$$

$$H(\mathbf{y}) = \log_2 \left[(N_o \pi e)^2 \left(1 + |h_{s,d}|^2 \text{SNR}\right) \left(1 + |h_r, d|^2 \text{SNR}\right) \right] \tag{4.63}$$

Where $\text{SNR} = \frac{P}{N_o}$ is the signal-to-noise ration of the received signal. Now the entropy of received signal given the first level is known at receiver is given by:

$$H\left(\mathbf{y}|\mathbf{x}^{(1)}\right) = \log_2 \det \left| \pi e \left(\mathbf{A}^{(2)} \mathbf{E}(\mathbf{x}^{(2)} \cdot \mathbf{x}^{(2)*}) \mathbf{A}^{(2)*} + E(\mathbf{n} \cdot \mathbf{n}^*) \right) \right|$$

This can be simplified to:

$$\begin{aligned} H\left(\mathbf{y}|\mathbf{x}^{(1)}\right) &= \log_2 \det \pi e \left\| \begin{bmatrix} h_{s,d} & 0 \\ 0 & h_{r,d} \end{bmatrix} \begin{bmatrix} \alpha P & 0 \\ 0 & \alpha P \end{bmatrix} \begin{bmatrix} h^*_{s,d} & 0 \\ 0 & h^*_{r,d} \end{bmatrix} + \begin{bmatrix} N_o & 0 \\ 0 & N_o \end{bmatrix} \right\| \\ &= \log_2 \left[(N_o \pi e)^2 \left(1 + \alpha |h_{s,d}|^2 \text{SNR}\right) \left(1 + \alpha |h_{r,d}|^2 \text{SNR}\right) \right] \end{aligned} \tag{4.64}$$

Similarly the entropy of the received signal given that first level is known is given by:

$$H\left(\mathbf{y}|\mathbf{x}^{(2)}\right) = \log_2 \left[(N_o \pi e)^2 \left(1 + \overline{\alpha} |h_{s,d}|^2 \text{SNR}\right) \left(1 + \overline{\alpha} |h_{r,d}|^2 \text{SNR}\right) \right] \tag{4.65}$$

The entropy of the received signal when both levels are known to receiver is given by:

$$\begin{aligned} H\left(\mathbf{y}|\mathbf{x}\right) &= \log_2 \det |\pi e E(\mathbf{n} \cdot \mathbf{n}^*)| \\ &= \log_2 \left(N_o \pi e\right)^2 \end{aligned} \tag{4.66}$$

The mutual information between the received signal and the first level of the transmitted signal from relay and source is given by:

$$\begin{aligned} I_I^{(1-1)} &= \frac{1}{2} \left(H\left(\mathbf{y}\right) - H\left(\mathbf{y}|\mathbf{x}^{(1)}\right) \right) \\ &= \frac{1}{2} \log_2 \left[\frac{\left(1 + |h_{s,d}|^2 \text{SNR}\right) \left(1 + |h_{r,d}|^2 \text{SNR}\right)}{\left(1 + \alpha |h_{s,d}|^2 \text{SNR}\right) \left(1 + \alpha |h_{r,d}|^2 \text{SNR}\right)} \right] \end{aligned} \tag{4.67}$$

The mutual information between the received signal and the second level of the transmitted signal with first level decoded is given by:

$$\begin{aligned} I_I^{(1-2|1)} &= \frac{1}{2}\left(H\left(\mathbf{y}|\mathbf{x^{(1)}}\right) - H\left(\mathbf{y}|\mathbf{x}\right)\right) \\ &= \frac{1}{2}\log_2\left[\left(1+\overline{\alpha}\text{SNR}|h_{s,d}|^2\right)\left(1+\overline{\alpha}\text{SNR}|h_{r,d}|^2\right)\right] \end{aligned} \tag{4.68}$$

It can be proved[3] that destination has to perform the successive interference cancelation method to decode the message. Hence the average throughput of relay network when relay transmits the entire message is given by:

$$\widetilde{R_I^{(1)}} = R_1 \cdot Pr\begin{bmatrix} R_1 < I_I^{(1-1)} \& \\ R_2 > I_I^{(1-2|1)} \end{bmatrix} + R \cdot Pr\left[R_2 < I_I^{(1-2|1)}\right] \tag{4.69}$$

CASE 2

Relay will transmit the first level of the message using a Gaussian codebook independent of that of source. The received signals at the destination are given by (4.15), (4.16). The matrix form representation of received signals by the destination is given by:

$$\begin{bmatrix} y_{s,d}[n] \\ y_{r,d}[n+\frac{N}{2}] \end{bmatrix} = \begin{bmatrix} h_{s,d} & h_{s,d} & 0 & 0 \\ 0 & 0 & h_{r,d} & 0 \end{bmatrix} \begin{bmatrix} x_s^{(1)}[n] \\ x_s^{(2)}[n] \\ x_r^{(1)}[n] \\ 0 \end{bmatrix} + \begin{bmatrix} z_{s,d}[n] \\ z_{r,d}[n+\frac{N}{2}] \end{bmatrix} \tag{4.70}$$

for $n = 1 \ldots \frac{N}{2}$. This can also be written as:

$$\mathbf{y} = \mathbf{A}\cdot\mathbf{x} + n = \mathbf{A^{(1)}}\mathbf{x^{(1)}} + \mathbf{A^{(2)}}\mathbf{x^{(2)}} + n \tag{4.71}$$

[3]Refer to Proposition 4 in Appendix B for the proof

where:

$$\mathbf{y} = \begin{bmatrix} y_{s,d}[n] \\ y_{r,d}[n+\frac{N}{2}] \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} h_{s,d} & h_{s,d} & 0 & 0 \\ 0 & 0 & h_{r,d} & 0 \end{bmatrix} \quad \mathbf{A^{(1)}} = \begin{bmatrix} h_{s,d} & 0 \\ 0 & h_{r,d} \end{bmatrix} \tag{4.72}$$

$$\mathbf{A^{(2)}} = \begin{bmatrix} h_{s,d} & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_s^{(1)}[n] \\ x_s^{(2)}[n] \\ x_r^{(1)}[n] \\ 0 \end{bmatrix} \quad \mathbf{x^{(1)}} = \begin{bmatrix} x_s^{(1)}[n] \\ x_r^{(1)}[n] \end{bmatrix} \tag{4.73}$$

$$\mathbf{x^{(2)}} = \begin{bmatrix} x_s^{(2)}[n] \\ 0 \end{bmatrix} \quad \mathbf{n} = \begin{bmatrix} z_{s,d}[n] \\ z_{r,d}[n+\frac{N}{2}] \end{bmatrix} \tag{4.74}$$

The entropy of the received signal can be calculated as:

$$H(\mathbf{y}) = \log_2 \det |\pi e E(\mathbf{y} \cdot \mathbf{y}^*)| \tag{4.75}$$

Where E(.) is the expectation and $\mathbf{y}^*$ is the complex conjugate of $\mathbf{y}$. This can be simplified to:

$$\begin{aligned} H(\mathbf{y}) &= \log_2 \det |\pi e \left(\mathbf{AE}(\mathbf{x} \cdot \mathbf{x}^*)\mathbf{A}^* + \mathbf{E}(\mathbf{n} \cdot \mathbf{n}^*)\right)| \\ &= \log_2 \det \pi e \left| \begin{bmatrix} h_{s,d} & h_{s,d} & 0 & 0 \\ 0 & 0 & h_{r,d} & 0 \end{bmatrix} \begin{bmatrix} \overline{\alpha}P & 0 & 0 & 0 \\ 0 & \alpha P & 0 & 0 \\ 0 & 0 & \overline{\alpha}P & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_{s,d}^* & 0 \\ h_{s,d}^* & 0 \\ 0 & h_{r,d}^* \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} N_o & 0 \\ 0 & N_o \end{bmatrix} \right| \\ &= \log_2 \det \pi e \begin{vmatrix} |h_{s,d}|^2 P + N_o & 0 \\ 0 & \overline{\alpha}|h_{r,d}|^2 P + N_o \end{vmatrix} \\ &= \log_2 \left[(N_o \pi e)^2 \left(1 + |h_{s,d}|^2 \mathrm{SNR}\right) \left(1 + \overline{\alpha}|h_{r,d}|^2 \mathrm{SNR}\right)\right] \end{aligned} \tag{4.76}$$

Where $\mathrm{SNR} = \frac{P}{N_o}$ is the signal-to-noise ration of the received signal. Now the entropy of received signal given the first level is known at receiver is given by:

$$H\left(\mathbf{y}|\mathbf{x^{(1)}}\right) = \log_2 \det \left|\pi e \left(\mathbf{A^{(2)}E}(\mathbf{x^{(2)}} \cdot \mathbf{x^{(2)*}})\mathbf{A^{(2)*}} + E(\mathbf{n} \cdot \mathbf{n}^*)\right)\right|$$

$$H\left(\mathbf{y}|\mathbf{x^{(1)}}\right) = \log_2\left[(N_o \pi e)^2 \left(1 + \alpha |h_{s,d}|^2 \mathrm{SNR}\right)\right] \tag{4.77}$$

Now the entropy of received signal given the second level is known at receiver is given by:

$$H\left(\mathbf{y}|\mathbf{x^{(2)}}\right) = \log_2 \det \left|\pi e \left(\mathbf{A}^{(1)}\mathbf{E}(\mathbf{x}^{(1)} \cdot \mathbf{x}^{(1)*})\mathbf{A}^{(1)*} + E(\mathbf{n} \cdot \mathbf{n}^*)\right)\right|$$

This can be simplified to:

$$\begin{aligned} H\left(\mathbf{y}|\mathbf{x^{(2)}}\right) &= \log_2 \det \pi e \left| \begin{bmatrix} h_{s,d} & 0 \\ 0 & h_{r,d} \end{bmatrix} \begin{bmatrix} \overline{\alpha} P & 0 \\ 0 & \overline{\alpha} P \end{bmatrix} \begin{bmatrix} h^*_{s,d} & 0 \\ 0 & h^*_{r,d} \end{bmatrix} + \begin{bmatrix} N_o & 0 \\ 0 & N_o \end{bmatrix} \right| \\ &= \log_2\left[(N_o \pi e)^2 \left(1 + \overline{\alpha} |h_{s,d}|^2 \mathrm{SNR}\right)\left(1 + \overline{\alpha} |h_{r,d}|^2 \mathrm{SNR}\right)\right] \end{aligned} \tag{4.78}$$

The entropy of the received signal when both levels are known to receiver is given by:

$$\begin{aligned} H\left(\mathbf{y}|\mathbf{x}\right) &= \log_2 \det \left|\pi e E(\mathbf{n} \cdot \mathbf{n}^*)\right| \\ &= \log_2 \left(N_o \pi e\right)^2 \end{aligned} \tag{4.79}$$

The mutual information between received signals at the destination and first level from the source and relay is given by:

$$\begin{aligned} I_I^{(2-1)} &= \frac{1}{2}\left(H\left(\mathbf{y}\right) - H\left(\mathbf{y}|\mathbf{x^{(1)}}\right)\right) \\ &= \frac{1}{2}\log_2\left[\frac{\left(1 + |h_{s,d}|^2 \mathrm{SNR}\right)\left(1 + \overline{\alpha} |h_{r,d}|^2 \mathrm{SNR}\right)}{1 + \alpha |h_{s,d}|^2 \mathrm{SNR}}\right] \end{aligned} \tag{4.80}$$

The mutual information between received signal and the second level of the signal from source given that first level is decoded is given by:

$$\begin{aligned} I_I^{(2-2|1)} &= \frac{1}{2}\left(H\left(\mathbf{y}|\mathbf{x^{(1)}}\right) - H\left(\mathbf{y}|\mathbf{x}\right)\right) \\ &= \frac{1}{2}\log_2\left[1 + \alpha |h_{s,d}|^2 \mathrm{SNR}\right] \end{aligned} \tag{4.81}$$

Similar to repetition coding, it can be proved[4] that destination has to perform successive interference cancelation to decode the message if relay transmits only first level. Hence, the average throughput of relay network when relay transmits only the first level of message is given by:

$$\widetilde{R_I^{(2)}} = R_1 Pr \begin{bmatrix} R_1 < I_I^{(2-1)} \& \\ R_2 > I_I^{(2-2|1)} \end{bmatrix} + RPr\left[R_2 < I_I^{(2-2|1)}\right] \qquad (4.82)$$

CASE 3

The average throughput for this case is same as that for repetition coding scheme and is given by (4.54). Hence $R_I^{(3)} = R_R^{(3)}$

The average throughput of the network when relay uses independent coding scheme is given by:

$$R_I^{\Sigma} = P^{(1)} \cdot \widetilde{R_I^{(1)}} + P^{(2)} \cdot \widetilde{R_I^{(2)}} + P^{(3)} \cdot \widetilde{R_I^{(3)}} \qquad (4.83)$$

Where $P^{(1)}$, $P^{(2)}$ and $P^{(3)}$ are given by 4.11, 4.14 and 4.17 respectively. There is no closed form for this equation. The optimal values of average throughput, as well as corresponding R_1, R_2, α can be attained using the numerical methods.

The power allocation factor for source and relay are the same in both the repetition coding and independent coding scheme methods as discussed above. Since source-destination and relay-destination channels are independent, the relay has the option not to use the same power allocation factor deployed at the source. Next, we investigate impact of overall throughput when relay employs an independent power allocation factor β in decode-and-forward strategies.

4.2.3 Repetition Coding Based Scheme with 'β'

In this subsection, we consider a similar repetition coding scheme as discussed in Subsection 4.2.1 except for relay uses an independent power allocation factor 'β'.

[4]Refer to Proposition 4 in Appendix B for the proof of this statement

Hence relay allocates $\overline{\beta}P$ power to the first level and βP power to the second level where P is the total power available to relay. Therefore, we have $x_r^{(1)} = \sqrt{(\bar{\beta}|\bar{\alpha})}x_s^{(1)}$ and $x_r^{(2)} = \sqrt{\beta|\alpha}x_s^{(2)}$. Average throughput is evaluated under three cases depending on the decision at relay as discussed before. The three cases are:

CASE 1

The matrix form representation of received signals by the destination is given by:

$$\begin{bmatrix} y_{s,d}[n] \\ y_{r,d}[n+\frac{N}{2}] \end{bmatrix} = \begin{bmatrix} h_{s,d} & h_{s,d} & 0 & 0 \\ 0 & 0 & h_{r,d} & h_{r,d} \end{bmatrix} \begin{bmatrix} x_s^{(1)}[n] \\ x_s^{(2)}[n] \\ x_r^{(1)}[n] \\ x_r^{(2)}[n] \end{bmatrix} + \begin{bmatrix} z_{s,d}[n] \\ z_{r,d}[n+\frac{N}{2}] \end{bmatrix} \tag{4.84}$$

for $n = 1 \dots \frac{N}{2}$. This can also be written as:

$$\mathbf{y} = \mathbf{A} \cdot \mathbf{x} + n = \mathbf{A^{(1)}}\mathbf{x^{(1)}} + \mathbf{A^{(2)}}\mathbf{x^{(2)}} + n \tag{4.85}$$

where:

$$\mathbf{y} = \begin{bmatrix} y_{s,d}[n] \\ y_{r,d}[n+\frac{N}{2}] \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} h_{s,d} & h_{s,d} & 0 & 0 \\ 0 & 0 & h_{r,d} & h_{r,d} \end{bmatrix} \quad \mathbf{A^{(1)}} = \begin{bmatrix} h_{s,d} & 0 \\ 0 & h_{r,d} \end{bmatrix} \tag{4.86}$$

$$\mathbf{A^{(2)}} = \begin{bmatrix} h_{s,d} & 0 \\ 0 & h_{r,d} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_s^{(1)}[n] \\ x_s^{(2)}[n] \\ x_r^{(1)}[n] \\ x_r^{(2)}[n] \end{bmatrix} \quad \mathbf{x^{(1)}} = \begin{bmatrix} x_s^{(1)}[n] \\ x_r^{(1)}[n] \end{bmatrix} \tag{4.87}$$

$$\mathbf{x^{(2)}} = \begin{bmatrix} x_s^{(2)}[n] \\ x_r^{(2)}[n] \end{bmatrix} \quad \mathbf{n} = \begin{bmatrix} z_{s,d}[n] \\ z_{r,d}[n+\frac{N}{2}] \end{bmatrix} \tag{4.88}$$

The entropy of the received signal can be calculated as:

$$H(\mathbf{y}) = \log_2 \det |\pi e E(\mathbf{y} \cdot \mathbf{y}^*)| \tag{4.89}$$

Where E(.) is the expectation and $\mathbf{y}^*$ is the complex conjugate of $\mathbf{y}$. This can be simplified to:

$$H(\mathbf{y}) = \log_2 \det \left|\pi e \left(\mathbf{AE}(\mathbf{x}\cdot\mathbf{x}^*)\mathbf{A}^* + \mathbf{E}(\mathbf{n}\cdot\mathbf{n}^*)\right)\right|$$

$$= \log_2 \det \pi e \left| \begin{bmatrix} h_{s,d} & h_{s,d} & 0 & 0 \\ 0 & 0 & h_{r,d} & h_{r,d} \end{bmatrix} \begin{bmatrix} \overline{\alpha}P & 0 & \sqrt{\overline{\alpha}\overline{\beta}}P & 0 \\ 0 & \alpha P & 0 & \sqrt{\alpha\beta}P \\ \sqrt{\overline{\alpha}\overline{\beta}}P & 0 & \overline{\beta}P & 0 \\ 0 & \sqrt{\alpha\beta}P & 0 & \beta P \end{bmatrix} \begin{bmatrix} h^*_{s,d} & 0 \\ h^*_{s,d} & 0 \\ 0 & h^*_{r,d} \\ 0 & h^*_{r,d} \end{bmatrix} + \begin{bmatrix} N_o & 0 \\ 0 & N_o \end{bmatrix} \right|$$

$$= \log_2 \det \pi e \begin{vmatrix} |h_{s,d}|^2 P + N_o & h_{s,d} h_{r,d}{}^* P \left[\sqrt{\overline{\alpha}\overline{\beta}} + \sqrt{\alpha\beta}\right] \\ h_{s,d}{}^* h_{r,d} P \left[\sqrt{\overline{\alpha}\overline{\beta}} + \sqrt{\alpha\beta}\right] & |h_{r,d}|^2 P + N_o \end{vmatrix}$$

$$H(\mathbf{y}) = \log_2 \left[(N_o \pi e)^2 \left(1 + \left(|h_{s,d}|^2 + |h_r, d|^2\right) \mathrm{SNR} + |h_{s,d}|^2 |h_{r,d}|^2 \mathrm{SNR}^2 \left(\sqrt{\alpha\overline{\beta}} - \sqrt{\beta\overline{\alpha}}\right)^2 \right) \right] \tag{4.90}$$

Where SNR $= \frac{P}{N_o}$ is the signal-to-noise ration of the received signal. Now the entropy of received signal given the first level is known at receiver is given by:

$$H\left(\mathbf{y}|\mathbf{x^{(1)}}\right) = \log_2 \det \left|\pi e \left(\mathbf{A}^{(2)}\mathbf{E}(\mathbf{x^{(2)}} \cdot \mathbf{x}^{(2)*})\mathbf{A}^{(2)*} + E(\mathbf{n}\cdot\mathbf{n}^*)\right)\right|$$

This can be simplified to:

$$H\left(\mathbf{y}|\mathbf{x^{(1)}}\right) = \log_2 \det \pi e \left| \begin{bmatrix} h_{s,d} & 0 \\ 0 & h_{r,d} \end{bmatrix} \begin{bmatrix} \alpha P & \sqrt{\alpha\beta}P \\ \sqrt{\alpha\beta}P & \beta P \end{bmatrix} \begin{bmatrix} h^*_{s,d} & 0 \\ 0 & h^*_{r,d} \end{bmatrix} + \begin{bmatrix} N_o & 0 \\ 0 & N_o \end{bmatrix} \right|$$

$$= \log_2 \left[(N_o\pi e)^2 \left(1 + \left(\alpha|h_{s,d}|^2 + \beta|h_{r,d}|^2\right) \mathrm{SNR}\right)\right] \tag{4.91}$$

Similarly the entropy of received signal given that first level is known is given by:

$$H\left(\mathbf{y}|\mathbf{x^{(2)}}\right) = \log_2 \left[(N_o\pi e)^2 \left(1 + \left(\overline{\alpha}|h_{s,d}|^2 + \overline{\beta}|h_{r,d}|^2\right) \mathrm{SNR}\right)\right] \tag{4.92}$$

The entropy of the received signal when both levels are known to receiver is given by:

$$\begin{aligned} H\left(\mathbf{y}|\mathbf{x}\right) &= \log_2 \det |\pi e E(\mathbf{n}\cdot\mathbf{n}^*)| \\ &= \log_2 \left(N_o \pi e\right)^2 \end{aligned} \tag{4.93}$$

The mutual information between the received signals at the destination and two levels of transmitted signals from relay and source is given by:

$$\begin{aligned} I_{R_\beta}^{(1-0)} &= \frac{1}{2}\left(H\left(\mathbf{y}\right) - H\left(\mathbf{y}|\mathbf{x}\right)\right) \\ &= \frac{1}{2}\log_2\left[1 + \mathrm{SNR}\left(|h_{s,d}|^2 + |h_{r,d}|^2\right)\right] \end{aligned} \tag{4.94}$$

The mutual information between the received signal and the first level of the transmitted signal from relay and source is given by:

$$\begin{aligned} I_{R_\beta}^{(1-1)} &= \frac{1}{2}\left(H\left(\mathbf{y}\right) - H\left(\mathbf{y}|\mathbf{x}^{(\mathbf{1})}\right)\right) \\ &= \frac{1}{2}\log_2\left[\frac{1 + \mathrm{SNR}\left(|h_{s,d}|^2 + |h_{r,d}|^2 + |h_{s,d}|^2|h_{r,d}|^2\left(\sqrt{\overline{\alpha}\beta} - \sqrt{\alpha\overline{\beta}}\right)^2\mathrm{SNR}\right)}{1 + \mathrm{SNR}\left[\alpha|h_{s,d}|^2 + \beta|h_{r,d}|^2\right]}\right] \end{aligned} \tag{4.95}$$

The mutual information between the received signal and the first level of the transmitted signal from relay and source when decoding of second level is successful is given by:

$$\begin{aligned} I_{R_\beta}^{(1-1|2)} &= \frac{1}{2}\left(H\left(\mathbf{y}|\mathbf{x}^{(\mathbf{2})}\right) - H\left(\mathbf{y}|\mathbf{x}\right)\right) \\ &= \frac{1}{2}\log_2\left[1 + \mathrm{SNR}\left(\overline{\alpha}|h_{s,d}|^2 + \overline{\beta}|h_{r,d}|^2\right)\right] \end{aligned} \tag{4.96}$$

The mutual information between the received signal and the second level of the transmitted signal from relay and source is given by:

$$I_{R_\beta}^{(1-2)} = \frac{1}{2}\left(H\left(\mathbf{y}\right) - H\left(\mathbf{y}|\mathbf{x}^{(\mathbf{2})}\right)\right)$$

$$I_{R_\beta}^{(1-2)} = \frac{1}{2}\log_2\left[\frac{1+\text{SNR}\left(|h_{s,d}|^2+|h_{r,d}|^2+|h_{s,d}|^2|h_{r,d}|^2\left(\sqrt{\overline{\alpha}\beta}-\sqrt{\overline{\beta}\alpha}\right)^2\text{SNR}\right)}{1+\text{SNR}\left(\overline{\alpha}|h_{s,d}|^2+\overline{\beta}|h_{r,d}|^2\right)}\right] \tag{4.97}$$

The mutual information between the received signal and the second level of the transmitted signal from relay and source when the first level is already decoded is given by:

$$\begin{aligned} I_{R_\beta}^{(1-2|1)} &= \frac{1}{2}\left(H\left(\mathbf{y}|\mathbf{x}^{(1)}\right)-H\left(\mathbf{y}|\mathbf{x}\right)\right) \\ &= \frac{1}{2}\log_2\left[1+\text{SNR}\left(\alpha|h_{s,d}|^2+\beta|h_{r,d}|^2\right)\right] \end{aligned} \tag{4.98}$$

Now, the average throughput of relay network when relay transmits the entire message is given by:

$$\begin{aligned} \widetilde{R_k^{(j)}} = R_1\cdot\left(Pr\begin{bmatrix} R_1 < I_k^{(j-1)}\& \\ R_2 > I_k^{(j-2|1)}\end{bmatrix}\right) + R_2\cdot\left(Pr\begin{bmatrix} R_1 > I_k^{(j-1|2)}\& \\ R_2 < I_k^{(j-2)}\end{bmatrix}\right) \\ +R\cdot\left(Pr\begin{bmatrix} R_1 < I_k^{(j-1|2)}\& \\ R_2 < I_k^{(j-2|1)}\& \\ R < I_k^{(j-0)}\end{bmatrix}\right) \end{aligned} \tag{4.99}$$

with $j=1$ and $k=R_\beta$. There is no closed form solution for this equation.

CASE 2

The matrix form representation of received signals by the destination is given by:

$$\begin{bmatrix} y_{s,d}[n] \\ y_{r,d}[n+\frac{N}{2}]\end{bmatrix} = \begin{bmatrix} h_{s,d} & h_{s,d} & 0 & 0 \\ 0 & 0 & h_{r,d} & 0\end{bmatrix}\begin{bmatrix} x_s^{(1)}[n] \\ x_s^{(2)}[n] \\ x_r^{(1)}[n] \\ 0\end{bmatrix} + \begin{bmatrix} z_{s,d}[n] \\ z_{r,d}[n+\frac{N}{2}]\end{bmatrix} \tag{4.100}$$

for $n = 1 \ldots \frac{N}{2}$. This can also be written as:

$$\mathbf{y} = \mathbf{A} \cdot \mathbf{x} + n = \mathbf{A^{(1)}} \mathbf{x^{(1)}} + \mathbf{A^{(2)}} \mathbf{x^{(2)}} + n \tag{4.101}$$

where:

$$\mathbf{y} = \begin{bmatrix} y_{s,d}[n] \\ y_{r,d}[n + \frac{N}{2}] \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} h_{s,d} & h_{s,d} & 0 & 0 \\ 0 & 0 & h_{r,d} & 0 \end{bmatrix} \quad \mathbf{A^{(1)}} = \begin{bmatrix} h_{s,d} & 0 \\ 0 & h_{r,d} \end{bmatrix} \tag{4.102}$$

$$\mathbf{A^{(2)}} = \begin{bmatrix} h_{s,d} & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_s^{(1)}[n] \\ x_s^{(2)}[n] \\ x_r^{(1)}[n] \\ 0 \end{bmatrix} \quad \mathbf{x^{(1)}} = \begin{bmatrix} x_s^{(1)}[n] \\ x_r^{(1)}[n] \end{bmatrix} \tag{4.103}$$

$$\mathbf{x^{(2)}} = \begin{bmatrix} x_s^{(2)}[n] \\ 0 \end{bmatrix} \quad \mathbf{n} = \begin{bmatrix} z_{s,d}[n] \\ z_{r,d}[n + \frac{N}{2}] \end{bmatrix} \tag{4.104}$$

The entropy of the received signal can be calculated as:

$$H(\mathbf{y}) = \log_2 \det |\pi e E(\mathbf{y} \cdot \mathbf{y}^*)| \tag{4.105}$$

Where E(.) is the expectation and $\mathbf{y}^*$ is the complex conjugate of $\mathbf{y}$. This can be simplified to:

$$\begin{aligned} H(\mathbf{y}) &= \log_2 \det |\pi e \mathbf{A} \mathbf{E}(\mathbf{x} \cdot \mathbf{x}^*) \mathbf{A}^* + \mathbf{E}(\mathbf{n} \cdot \mathbf{n}^*)| \\ &= \log_2 \det \pi e \left| \begin{matrix} \begin{bmatrix} h_{s,d} & h_{s,d} & 0 & 0 \\ 0 & 0 & h_{r,d} & 0 \end{bmatrix} \begin{bmatrix} \overline{\alpha} P & 0 & \sqrt{\overline{\alpha}\overline{\beta}} P & 0 \\ 0 & \alpha P & 0 & 0 \\ \sqrt{\overline{\alpha}\overline{\beta}} P & 0 & \overline{\beta} P & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_{s,d}^* & 0 \\ h_{s,d}^* & 0 \\ 0 & h_{r,d}^* \\ 0 & 0 \end{bmatrix} \\ + \begin{bmatrix} N_o & 0 \\ 0 & N_o \end{bmatrix} \end{matrix} \right| \end{aligned}$$

$$= \log_2 \det \pi e \begin{vmatrix} |h_{s,d}|^2 P + N_o & h_{s,d} {h_{r,d}}^* \sqrt{\overline{\alpha}\overline{\beta}} P \\ {h_{s,d}}^* h_{r,d} \sqrt{\overline{\alpha}\overline{\beta}} P & |h_{r,d}|^2 \beta P + N_o \end{vmatrix}$$

$$H(\mathbf{y}) = \log_2 \left[(N_o \pi e)^2 \left(1 + \left(|h_{s,d}|^2 + |h_r, d|^2 \right) \mathrm{SNR} + |h_{s,d}|^2 |h_{r,d}|^2 \alpha \overline{\beta} \mathrm{SNR}^2 \right) \right] \tag{4.106}$$

Where $\mathrm{SNR} = \frac{P}{N_o}$ is the signal-to-noise ration of the received signal. Now the entropy of received signal when the first level is known at receiver is given by:

$$\begin{aligned} H\left(\mathbf{y}|\mathbf{x}^{(1)}\right) &= \log_2 \det \left| \pi e \left(\mathbf{A}^{(2)} \mathbf{E}(\mathbf{x}^{(2)} \cdot \mathbf{x}^{(2)*}) \mathbf{A}^{(2)*} + E(\mathbf{n} \cdot \mathbf{n}^*) \right) \right| \\ &= \log_2 \det \pi e \left| \begin{bmatrix} h_{s,d} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha P & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} h_{s,d}^* & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} N_o & 0 \\ 0 & N_o \end{bmatrix} \right| \\ &= \log_2 \left[(N_o \pi e)^2 \left(1 + \alpha |h_{s,d}|^2 \mathrm{SNR} \right) \right] \end{aligned} \tag{4.107}$$

Now the entropy of received signal given the second level is known at receiver is given by:

$$H\left(\mathbf{y}|\mathbf{x}^{(2)}\right) = \log_2 \det \left| \pi e \left(\mathbf{A}^{(1)} \mathbf{E}(\mathbf{x}^{(1)} \cdot \mathbf{x}^{(1)*}) \mathbf{A}^{(1)*} + E(\mathbf{n} \cdot \mathbf{n}^*) \right) \right|$$

This can be simplified to:

$$\begin{aligned} H\left(\mathbf{y}|\mathbf{x}^{(2)}\right) &= \log_2 \det \pi e \left| \begin{bmatrix} h_{s,d} & 0 \\ 0 & h_{r,d} \end{bmatrix} \begin{bmatrix} \overline{\alpha} P & \sqrt{\overline{\alpha}\overline{\beta}} P \\ \sqrt{\overline{\alpha}\overline{\beta}} P & \overline{\beta} P \end{bmatrix} \begin{bmatrix} h_{s,d}^* & 0 \\ 0 & h_{r,d}^* \end{bmatrix} + \begin{bmatrix} N_o & 0 \\ 0 & N_o \end{bmatrix} \right| \\ &= \log_2 \left[(N_o \pi e)^2 \left(1 + \left(\overline{\alpha} |h_{s,d}|^2 + \overline{\beta} |h_{r,d}|^2 \right) \mathrm{SNR} \right) \right] \end{aligned} \tag{4.108}$$

The entropy of the received signal when both levels are known to receiver is given by:

$$\begin{aligned} H\left(\mathbf{y}|\mathbf{x}\right) &= \log_2 \det |\pi e E(\mathbf{n} \cdot \mathbf{n}^*)| \\ &= \log_2 (N_o \pi e)^2 \end{aligned} \tag{4.109}$$

The mutual information between received signals at the destination and second level of the source, given first level is decoded is given by:

$$I_{R_\beta}^{(2-2|1)} = \frac{1}{2}\left(H\left(\mathbf{y}|\mathbf{x}^{(1)}\right) - H\left(\mathbf{y}|\mathbf{x}\right)\right)$$
$$= \frac{1}{2}\log_2\left[1 + \alpha \text{SNR}|h_{s,d}|^2\right] \tag{4.110}$$

The mutual information between received signal and the first level of the signal from source and relay is given by:

$$I_{R_\beta}^{(2-1)} = \frac{1}{2}\left(H\left(\mathbf{y}\right) - H\left(\mathbf{y}|\mathbf{x}^{(1)}\right)\right)$$
$$= \frac{1}{2}\log_2\left[\frac{1 + |h_{s,d}|^2\text{SNR} + |h_{r,d}|^2\overline{\beta}\text{SNR} + |h_{s,d}|^2|h_{r,d}|^2\alpha\overline{\beta}\text{SNR}^2}{1 + \alpha|h_{s,d}|^2\text{SNR}}\right] \tag{4.111}$$

It can be proved[5] that destination has to decode the message using successive interference cancelation method when relay transmits only one level. Now, the average throughput of relay network when relay transmits the first level given by (4.99) with $i = 2$, $k = R_\beta$ and can be reduced to:

$$\widetilde{R_{R_\beta}^{(2)}} = R_1 \cdot Pr\begin{bmatrix} R_1 < I_{R_\beta}^{(2-1)} \& \\ R_2 > I_{R_\beta}^{(2-2|1)} \end{bmatrix} + R \cdot Pr\left[R_2 < I_{R_\beta}^{(2-2|1)}\right] \tag{4.112}$$

CASE 3

When relay couldn't decode any of the levels, then the channel is exactly the same as in case 3 of Repetition coding scheme given in Subsection 4.2.1. Hence the average throughput for this case is $\widetilde{R_{R_\beta}^{(3)}} = \widetilde{R_R^{(3)}}$. Where $\widetilde{R_R^{(3)}}$ is given by (4.54).

The average throughput of the network if relay uses the repetition coding scheme is given by:

$$R_{R_\beta}^{\Sigma} = P^{(1)} \cdot \widetilde{R_{R_\beta}^{(1)}} + P^{(2)} \cdot \widetilde{R_{R_\beta}^{(2)}} + P^{(3)} \cdot \widetilde{R_{R_\beta}^{(3)}} \tag{4.113}$$

[5] Refer to Proposition 5 in Appendix B for proof

Where $P^{(1)}$, $P^{(2)}$ and $P^{(3)}$ are given by 4.11, 4.14 and 4.17 respectively. There is no closed form for this equation. The optimal values of average throughput, as well as corresponding R_1, R_2, α and β can be attained using the numerical methods.

4.2.4 Independent Coding Based Scheme with 'β'

Relay employs independent codebook than the source with power scaling factor β after it successively decodes the corresponding source packets.

CASE 1

The matrix form representation of received signals by the destination is given by:

$$\begin{bmatrix} y_{s,d}[n] \\ y_{r,d}[n+\frac{N}{2}] \end{bmatrix} = \begin{bmatrix} h_{s,d} & h_{s,d} & 0 & 0 \\ 0 & 0 & h_{r,d} & h_{r,d} \end{bmatrix} \begin{bmatrix} x_s^{(1)}[n] \\ x_s^{(2)}[n] \\ x_r^{(1)}[n] \\ x_r^{(2)}[n] \end{bmatrix} + \begin{bmatrix} z_{s,d}[n] \\ z_{r,d}[n+\frac{N}{2}] \end{bmatrix} \tag{4.114}$$

for $n = 1 \dots \frac{N}{2}$. This can also be written as:

$$\mathbf{y} = \mathbf{A} \cdot \mathbf{x} + n = \mathbf{A^{(1)}} \mathbf{x^{(1)}} + \mathbf{A^{(2)}} \mathbf{x^{(2)}} + n \tag{4.115}$$

where:

$$\mathbf{y} = \begin{bmatrix} y_{s,d}[n] \\ y_{r,d}[n+\frac{N}{2}] \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} h_{s,d} & h_{s,d} & 0 & 0 \\ 0 & 0 & h_{r,d} & h_{r,d} \end{bmatrix} \quad \mathbf{A^{(1)}} = \begin{bmatrix} h_{s,d} & 0 \\ 0 & h_{r,d} \end{bmatrix} \tag{4.116}$$

$$\mathbf{A^{(2)}} = \begin{bmatrix} h_{s,d} & 0 \\ 0 & h_{r,d} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_s^{(1)}[n] \\ x_s^{(2)}[n] \\ x_r^{(1)}[n] \\ x_r^{(2)}[n] \end{bmatrix} \quad \mathbf{x^{(1)}} = \begin{bmatrix} x_s^{(1)}[n] \\ x_r^{(1)}[n] \end{bmatrix} \tag{4.117}$$

$$\mathbf{x^{(2)}} = \begin{bmatrix} x_s^{(2)}[n] \\ x_r^{(2)}[n] \end{bmatrix} \quad \mathbf{n} = \begin{bmatrix} z_{s,d}[n] \\ z_{r,d}[n+\frac{N}{2}] \end{bmatrix} \tag{4.118}$$

The entropy of the received signal can be calculated as:

$$H(\mathbf{y}) = \log_2 \det |\pi e E(\mathbf{y} \cdot \mathbf{y}^*)| \tag{4.119}$$

Where E(.) is the expectation and $\mathbf{y}^*$ is the complex conjugate of $\mathbf{y}$. This can be simplified to:

$$\begin{aligned}
H(\mathbf{y}) &= \log_2 \det |\pi e \left(\mathbf{A}\mathbf{E}(\mathbf{x} \cdot \mathbf{x}^*)\mathbf{A}^* + \mathbf{E}(\mathbf{n} \cdot \mathbf{n}^*)\right)| \\
&= \log_2 \det \pi e \left| \begin{bmatrix} h_{s,d} & h_{s,d} & 0 & 0 \\ 0 & 0 & h_{r,d} & h_{r,d} \end{bmatrix} \begin{bmatrix} \overline{\alpha}P & 0 & 0 & 0 \\ 0 & \alpha P & 0 & 0 \\ 0 & 0 & \overline{\beta}P & 0 \\ 0 & 0 & 0 & \beta P \end{bmatrix} \begin{bmatrix} h^*_{s,d} & 0 \\ h^*_{s,d} & 0 \\ 0 & h^*_{r,d} \\ 0 & h^*_{r,d} \end{bmatrix} + \begin{bmatrix} N_o & 0 \\ 0 & N_o \end{bmatrix} \right| \\
&= \log_2 \det \pi e \begin{vmatrix} |h_{s,d}|^2 P + N_o & 0 \\ 0 & |h_{r,d}|^2 P + N_o \end{vmatrix} \\
&= \log_2 \left[(N_o \pi e)^2 \left(1 + |h_{s,d}|^2 \text{SNR}\right) \left(1 + |h_r, d|^2 \text{SNR}\right) \right]
\end{aligned} \tag{4.120}$$

Where $\text{SNR} = \frac{P}{N_o}$ is the signal-to-noise ration of the received signal. Now the entropy of received signal given the first level is known at receiver is given by:

$$H\left(\mathbf{y}|\mathbf{x}^{(1)}\right) = \log_2 \det \left|\pi e \left(\mathbf{A}^{(2)}\mathbf{E}(\mathbf{x}^{(2)} \cdot \mathbf{x}^{(2)*})\mathbf{A}^{(2)*} + E(\mathbf{n} \cdot \mathbf{n}^*)\right)\right|$$

This can be simplified to:

$$\begin{aligned}
H\left(\mathbf{y}|\mathbf{x}^{(1)}\right) &= \log_2 \det \pi e \left| \begin{bmatrix} h_{s,d} & 0 \\ 0 & h_{r,d} \end{bmatrix} \begin{bmatrix} \alpha P & 0 \\ 0 & \beta P \end{bmatrix} \begin{bmatrix} h^*_{s,d} & 0 \\ 0 & h^*_{r,d} \end{bmatrix} + \begin{bmatrix} N_o & 0 \\ 0 & N_o \end{bmatrix} \right| \\
&= \log_2 \left[(N_o \pi e)^2 \left(1 + \alpha |h_{s,d}|^2 \text{SNR}\right) \left(1 + \beta |h_{r,d}|^2 \text{SNR}\right) \right]
\end{aligned} \tag{4.121}$$

Similarly the entropy of received signal given that second level is known at receiver is given by:

$$H\left(\mathbf{y}|\mathbf{x}^{(2)}\right) = \log_2 \left[(N_o \pi e)^2 \left(1 + \overline{\alpha} |h_{s,d}|^2 \text{SNR}\right) \left(1 + \overline{\beta} |h_{r,d}|^2 \text{SNR}\right) \right] \tag{4.122}$$

The entropy of the received signal when both levels are known to receiver is given by:

$$\begin{aligned} H\left(\mathbf{y}|\mathbf{x}\right) &= \log_2 \det \left|\pi e E(\mathbf{n}\cdot\mathbf{n}^*)\right| \\ &= \log_2 \left(N_o \pi e\right)^2 \end{aligned} \tag{4.123}$$

The mutual information between the received signals at the destination and both the levels of transmitted signals from relay and source is given by:

$$\begin{aligned} I_{I_\beta}^{(1-0)} &= \frac{1}{2}\left(H\left(\mathbf{y}\right) - H\left(\mathbf{y}|\mathbf{x}\right)\right) \\ &= \frac{1}{2}\log_2\left[\left(1+|h_{s,d}|^2\mathrm{SNR}\right)\left(1+|h_{r,d}|^2\mathrm{SNR}\right)\right] \end{aligned} \tag{4.124}$$

The mutual information between the received signal and the first level of the transmitted signal from relay and source is given by:

$$\begin{aligned} I_{I_\beta}^{(1-1)} &= \frac{1}{2}\left(H\left(\mathbf{y}\right) - H\left(\mathbf{y}|\mathbf{x}^{(\mathbf{1})}\right)\right) \\ &= \frac{1}{2}\log_2\left[\frac{\left(1+|h_{s,d}|^2\mathrm{SNR}\right)\left(1+|h_{r,d}|^2\mathrm{SNR}\right)}{\left(1+\alpha|h_{s,d}|^2\mathrm{SNR}\right)\left(1+\beta|h_{r,d}|^2\mathrm{SNR}\right)}\right] \end{aligned} \tag{4.125}$$

The mutual information between the received signals and the first level from source and relay given that second level is decoded is given by:

$$\begin{aligned} I_{I_\beta}^{(1-1|2)} &= \frac{1}{2}\left(H\left(\mathbf{y}|\mathbf{x}^{(\mathbf{2})}\right) - H\left(\mathbf{y}|\mathbf{x}\right)\right) \\ &= \frac{1}{2}\log_2\left[\left(1+\overline{\alpha}|h_{s,d}|^2\mathrm{SNR}\right)\left(1+\overline{\beta}|h_{r,d}|^2\mathrm{SNR}\right)\right] \end{aligned} \tag{4.126}$$

The mutual information between the received signal and the second level of the transmitted signal from relay and source is given by:

$$\begin{aligned} I_{I_\beta}^{(1-2)} &= \frac{1}{2}\left(H\left(\mathbf{y}\right) - H\left(\mathbf{y}|\mathbf{x}^{(\mathbf{2})}\right)\right) \\ &= \frac{1}{2}\log_2\left[\frac{\left(1+|h_{s,d}|^2\mathrm{SNR}\right)\left(1+|h_{r,d}|^2\mathrm{SNR}\right)}{\left(1+\overline{\alpha}|h_{s,d}|^2\mathrm{SNR}\right)\left(1+\overline{\beta}|h_{r,d}|^2\mathrm{SNR}\right)}\right] \end{aligned} \tag{4.127}$$

The mutual information between the received signals and the second level from source and relay given that first level is decoded is given by:

$$I_{I_\beta}^{(1-2|1)} = \frac{1}{2}\left(H\left(\mathbf{y}|\mathbf{x}^{(1)}\right) - H\left(\mathbf{y}|\mathbf{x}\right)\right) \\ = \frac{1}{2}\log_2\left[\left(1+\alpha|h_{s,d}|^2\text{SNR}\right)\left(1+\beta|h_{r,d}|^2\text{SNR}\right)\right] \tag{4.128}$$

Now, the average throughput of relay network when relay transmits the entire message is given by (4.99) with $i = 1$ and $k = I_\beta$. There is no closed form solution for this equation.

CASE 2

Relay will re-transmit only the first level of the message with the independent coding to that of source. The received signals at the destination are given by (4.15), (4.16). The matrix form representation of received signals by the destination is given by:

$$\begin{bmatrix} y_{s,d}[n] \\ y_{r,d}[n+\frac{N}{2}] \end{bmatrix} = \begin{bmatrix} h_{s,d} & h_{s,d} & 0 & 0 \\ 0 & 0 & h_{r,d} & 0 \end{bmatrix} \begin{bmatrix} x_s^{(1)}[n] \\ x_s^{(2)}[n] \\ x_r^{(1)}[n] \\ 0 \end{bmatrix} + \begin{bmatrix} z_{s,d}[n] \\ z_{r,d}[n+\frac{N}{2}] \end{bmatrix} \tag{4.129}$$

for $n = 1 \ldots \frac{N}{2}$. This can also be written as:

$$\mathbf{y} = \mathbf{A}\cdot\mathbf{x} + n = \mathbf{A}^{(1)}\mathbf{x}^{(1)} + \mathbf{A}^{(2)}\mathbf{x}^{(2)} + n \tag{4.130}$$

where:

$$\mathbf{y} = \begin{bmatrix} y_{s,d}[n] \\ y_{r,d}[n+\frac{N}{2}] \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} h_{s,d} & h_{s,d} & 0 & 0 \\ 0 & 0 & h_{r,d} & 0 \end{bmatrix} \quad \mathbf{A}^{(1)} = \begin{bmatrix} h_{s,d} & 0 \\ 0 & h_{r,d} \end{bmatrix} \tag{4.131}$$

$$\mathbf{A}^{(2)} = \begin{bmatrix} h_{s,d} & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_s^{(1)}[n] \\ x_s^{(2)}[n] \\ x_r^{(1)}[n] \end{bmatrix} \quad \mathbf{x}^{(1)} = \begin{bmatrix} x_s^{(1)}[n] \\ x_r^{(1)}[n] \end{bmatrix} \tag{4.132}$$

$$\mathbf{x^{(2)}} = \begin{bmatrix} x_s^{(2)}[n] \\ 0 \end{bmatrix} \quad \mathbf{n} = \begin{bmatrix} z_{s,d}[n] \\ z_{r,d}[n+\frac{N}{2}] \end{bmatrix} \tag{4.133}$$

The entropy of the received signal can be calculated as:

$$H(\mathbf{y}) = \log_2 \det |\pi e E(\mathbf{y} \cdot \mathbf{y}^*)| \tag{4.134}$$

Where E(.) is the expectation and $\mathbf{y}^*$ is the complex conjugate of $\mathbf{y}$. This can be simplified to:

$$\begin{aligned}
H(\mathbf{y}) &= \log_2 \det |\pi e \left(\mathbf{A}\mathbf{E}(\mathbf{x} \cdot \mathbf{x}^*)\mathbf{A}^* + \mathbf{E}(\mathbf{n} \cdot \mathbf{n}^*)\right)| \\
&= \log_2 \det \pi e \left| \begin{bmatrix} h_{s,d} & h_{s,d} & 0 & 0 \\ 0 & 0 & h_{r,d} & 0 \end{bmatrix} \begin{bmatrix} \overline{\alpha}P & 0 & 0 & 0 \\ 0 & \alpha P & 0 & 0 \\ 0 & 0 & \overline{\beta}P & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_{s,d}^* & 0 \\ h_{s,d}^* & 0 \\ 0 & h_{r,d}^* \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} N_o & 0 \\ 0 & N_o \end{bmatrix} \right| \\
&= \log_2 \det \pi e \begin{vmatrix} |h_{s,d}|^2 P + N_o & 0 \\ 0 & \overline{\alpha}|h_{r,d}|^2 P + N_o \end{vmatrix} \\
&= \log_2 \left[(N_o \pi e)^2 \left(1 + |h_{s,d}|^2 \text{SNR}\right) \left(1 + \overline{\beta}|h_{r,d}|^2 \text{SNR}\right)\right]
\end{aligned} \tag{4.135}$$

Where SNR $= \frac{P}{N_o}$ is the signal-to-noise ration of the received signal. Now the entropy of received signal given the second level is known at receiver is given by:

$$H\left(\mathbf{y}|\mathbf{x^{(2)}}\right) = \log_2 \det \left|\pi e \left(\mathbf{A^{(1)}}\mathbf{E}(\mathbf{x^{(1)}} \cdot \mathbf{x^{(1)*}})\mathbf{A^{(1)*}} + E(\mathbf{n} \cdot \mathbf{n}^*)\right)\right|$$

This can be simplified to:

$$\begin{aligned}
H\left(\mathbf{y}|\mathbf{x^{(2)}}\right) &= \log_2 \det \pi e \left| \begin{bmatrix} h_{s,d} & 0 \\ 0 & h_{r,d} \end{bmatrix} \begin{bmatrix} \overline{\alpha}P & 0 \\ 0 & \overline{\beta}P \end{bmatrix} \begin{bmatrix} h_{s,d}^* & 0 \\ 0 & h_{r,d}^* \end{bmatrix} + \begin{bmatrix} N_o & 0 \\ 0 & N_o \end{bmatrix} \right| \\
&= \log_2 \left[(N_o \pi e)^2 \left(1 + \overline{\alpha}|h_{s,d}|^2 \text{SNR}\right) \left(1 + \overline{\beta}|h_{r,d}|^2 \text{SNR}\right)\right]
\end{aligned} \tag{4.136}$$

Now the entropy of received signal given the first level is known at receiver is given by:

$$H\left(\mathbf{y}|\mathbf{x^{(1)}}\right) = \log_2 \det \pi e \left|\mathbf{A^{(2)}}\mathbf{E}(\mathbf{x^{(2)}} \cdot \mathbf{x^{(2)*}})\mathbf{A^{(2)*}} + E(\mathbf{n} \cdot \mathbf{n}^*)\right|$$
$$= \log_2 \left[(N_o \pi e)^2 \left(1 + \alpha |h_{s,d}|^2 \mathrm{SNR}\right)\right] \tag{4.137}$$

The entropy of the received signal when both levels are known to receiver is given by:

$$H\left(\mathbf{y}|\mathbf{x}\right) = \log_2 \det \pi e \left|E(\mathbf{n} \cdot \mathbf{n}^*)\right|$$
$$= \log_2 \left(N_o \pi e\right)^2 \tag{4.138}$$

The mutual information between received signals at the destination and first level from the source and relay is given by:

$$I_{I_\beta}^{(2-1)} = \frac{1}{2}\left(H\left(\mathbf{y}\right) - H\left(\mathbf{y}|\mathbf{x^{(1)}}\right)\right)$$
$$= \frac{1}{2}\log_2 \left[\frac{\left(1 + |h_{s,d}|^2 \mathrm{SNR}\right)\left(1 + \overline{\beta}|h_{r,d}|^2 \mathrm{SNR}\right)}{1 + \alpha |h_{s,d}|^2 \mathrm{SNR}}\right] \tag{4.139}$$

The mutual information between received signal and the second level of the signal from source given that first level is decoded is given by :

$$I_{I_\beta}^{(2-2|1)} = \frac{1}{2}\left(H\left(\mathbf{y}|\mathbf{x^{(1)}}\right) - H\left(\mathbf{y}|\mathbf{x}\right)\right)$$
$$= \frac{1}{2}\log_2 \left[1 + \alpha |h_{s,d}|^2 \mathrm{SNR}\right] \tag{4.140}$$

Similar to repetition coding, it can be proved[6] that destination has to perform successive interference cancelation to decode the message if relay transmits only first level. Hence, the average throughput of relay network when relay transmits only the first level of message is given by:

$$\widetilde{R_{I_\beta}^{(2)}} = R_1 Pr \begin{bmatrix} R_1 < I_I^{(2-1)} \& \\ R_2 > I_I^{(2-2|1)} \end{bmatrix} + R Pr \left[R_2 < I_I^{(2-2|1)}\right] \tag{4.141}$$

[6]Refer to Proposition 6 in Appendix B for proof

CASE 3

In this case, relay couldn't transmit anything. Hence the average throughput for this case is $\widetilde{R_{I_\beta}^{(3)}} = \widetilde{R_I^{(3)}}$

The average throughput of the network when relay uses independent coding scheme is given by:

$$R_{I_\beta}^{\Sigma} = P^{(1)} \cdot \widetilde{R_{I_\beta}^{(1)}} + P^{(2)} \cdot \widetilde{R_{I_\beta}^{(2)}} + P^{(3)} \cdot \widetilde{R_{I_\beta}^{(3)}} \tag{4.142}$$

Where $P^{(1)}$, $P^{(2)}$ and $P^{(3)}$ are given by 4.11, 4.14 and 4.17 respectively. There is no closed form for this equation. The optimal values of average throughput, as well as corresponding R_1, R_2, α and β can be attained using the numerical methods.

Chapter 5
Results and Discussion

In this chapter, we compare the average throughput for schemes developed in Chapter 3-4. To make it clear, we partition the discussion into two parts. Section 5.1 talks about the average throughput in direct transmission channel and its comparison with that of Liu's[19] scheme. Section 5.2 discusses the comparison of average throughput of relay network channels where relay uses repetition and independent coding strategies. It also discusses the extension of Liu's scheme to relay network channel.

5.1 Direct Transmission

As discussed in Chapter 3 and proved in Appendix A, the valid range of α for the direct transmission chosen in this thesis is $\left[0, \frac{2^{R_2}-1}{2^{R}-1}\right]$. As discussed in Appendix A, destination has to perform successive interference cancelation to achieve maximum throughput in the direct transmission. We have also thoroughly proved that the optimal rates and power allocation in the proposed scheme is exactly the same as that of Liu's as proposed in [19]. The optimal value of average throughput along with the corresponding optimal values of R_1, R_2 and α are calculated as discussed in Chapter 3. The variation of fraction of power allocated for the first level by source with SNR is given in Figure 5.1. The variation of individual rates with SNR is given in Figure 5.2. The variation average throughput of the direct transmission with SNR is given in Figure 5.3. We can see from Fig 5.3, there isn't much improvements in throughput at low SNR region. This is because destination will have the same bottleneck when two-level superposition coding is used as that of a traditional approach. If we divide the power when SNR is low, the power

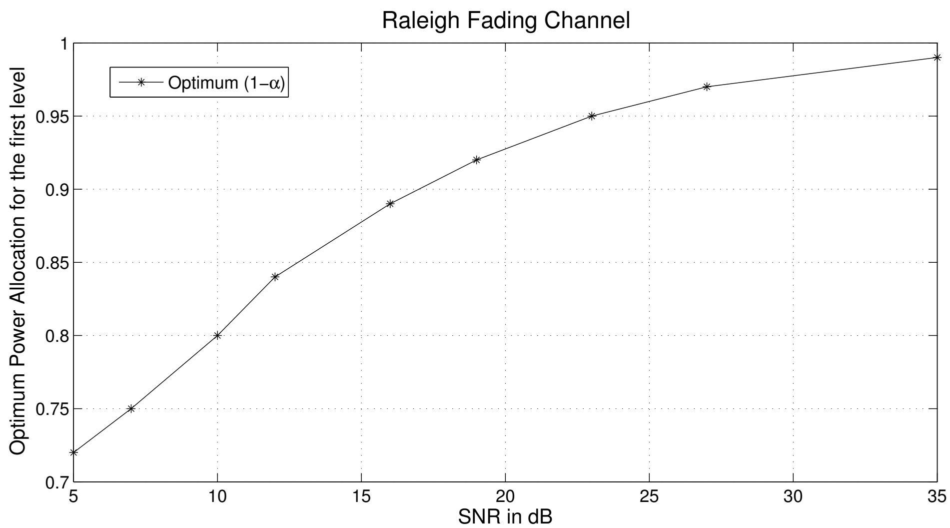

FIGURE 5.1. The value of optimum $\overline{\alpha}$, power allocation factor for the first level, with SNR in a Direct Transmission

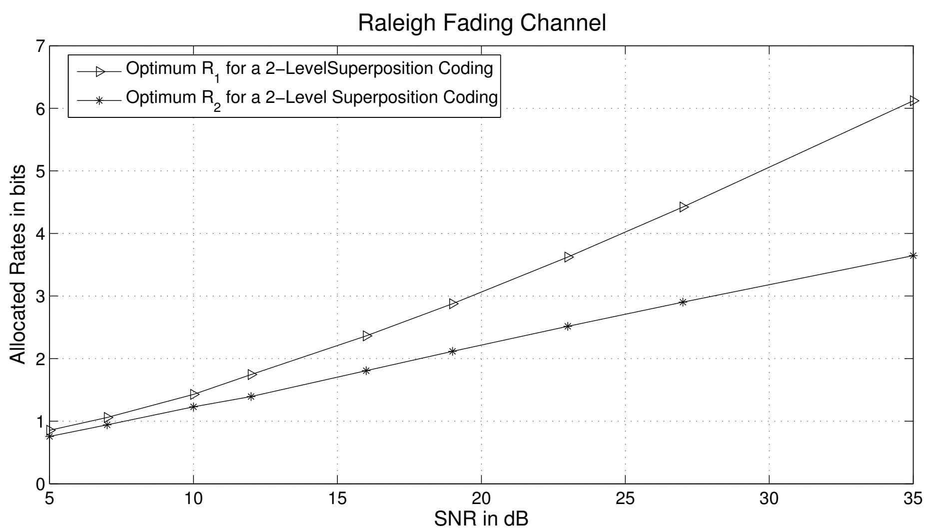

FIGURE 5.2. The value of rates R_1 and R_2 with SNR in a Direct Transmission

allocated to individual levels will still be low. The possible rates of the channel will be restricted and hence we cannot expect much improvements in this region.

We have also observed that the simulation results match exactly with Liu's results. Hence we can conclude that pre-fixing the order of decoding is indeed the optimal way to achieve the maximum throughput for a single link transmission without relays.

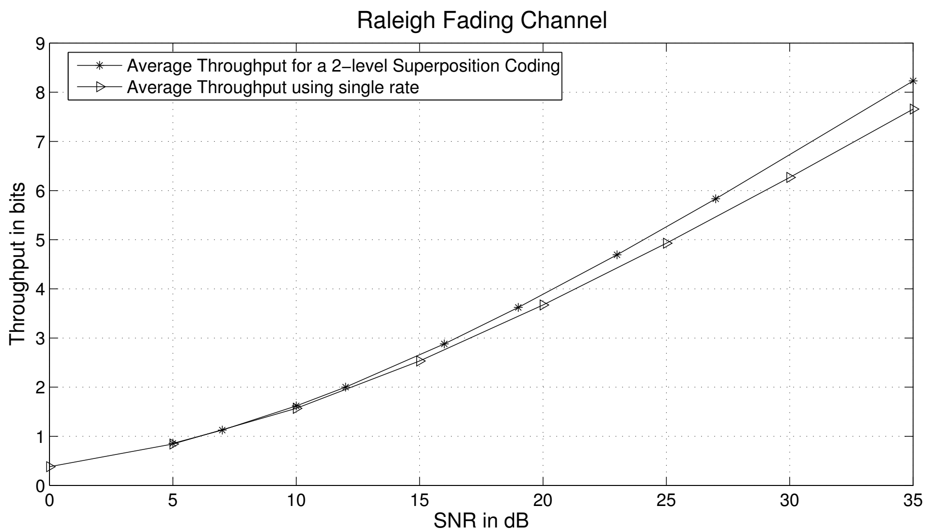

FIGURE 5.3. The value of average Throughput with SNR for Direct Transmission

5.2 Relay Networks

As discussed in Chapter 4 and proved in Appendix B, destination in relay networks has to perform the successive interference cancelation to achieve maximum average throughput when relay uses the same power allocation factor 'α'. When relay uses an independent power allocation factor 'β', we have proved in Proposition 5 and Proposition 6 that destination will perform successive interference cancelation when relay uses independent power allocation factor given that relay transmits first level to destination. The maximum average throughput and corresponding optimal rate allocation, power allocation factor are calculated using numerical methods as explained in Chapter 4.

For comparison purpose, we begin with relay networks which utilizes single rate. The optimal values of rate 'R' is determined from (4.10) and whence average throughput is determined from (4.9) and is plotted in the Figure 5.4.

Now, we extend our scheme of dual rates into relay networks and observe the improvements of average throughput. Figure 5.5 gives the comparison of average throughput of the different relay schemes such as single rate relay scheme, repe-

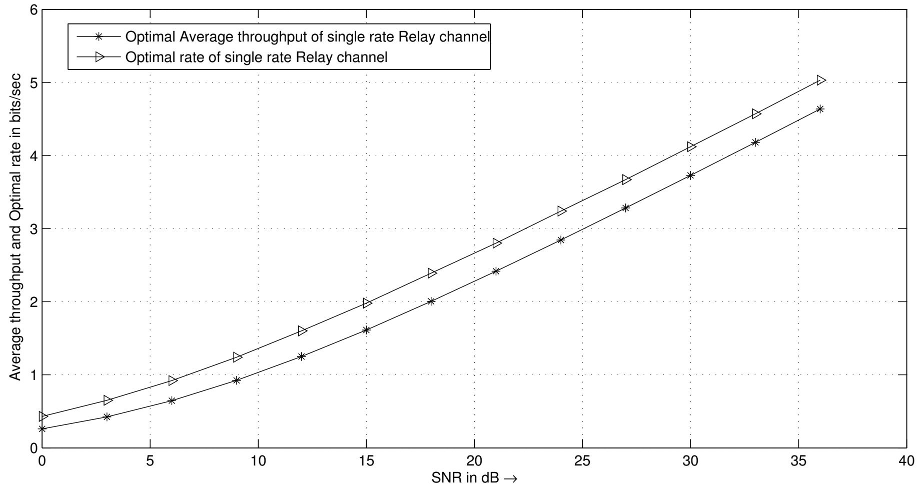

FIGURE 5.4. Plot of Optimal Values of Rate and Average Throughput of Relay Channel when Single Rate is employed

tition relay scheme and independent relay scheme with and without independent power allocation factor 'β' at relay at high SNR region. We observed the improvements given in Table 5.1 over single rate relay scheme by different dual rate schemes at high SNR region. Figure 5.6 gives the complete comparison curves. Figure 5.7

TABLE 5.1. Improvements of various dual rate relay schemes over single rate relay scheme

Dual Rate Relay Scheme	Improvements over Single Rate Relay Scheme in dB
Repetition Coding	1.5 dB
Independent Coding	2.1 dB
Repetition Coding with β	1.9 dB
Independent Coding with β	2.4 dB

compares the percentage of power allocated to first level of the message by source and relay when relay uses the same power allocation factor as that of source.

Table 5.2 gives the variations of power allocation factors 'α' and 'β' with SNR when relay employs the repetition coding scheme, where '$\overline{\alpha}$' is the ratio of power allocated to first level by the source and '$\overline{\beta}$' is the ratio of power allocated to first level by the relay. Table 5.3 gives the variations of power allocation factors 'α' and

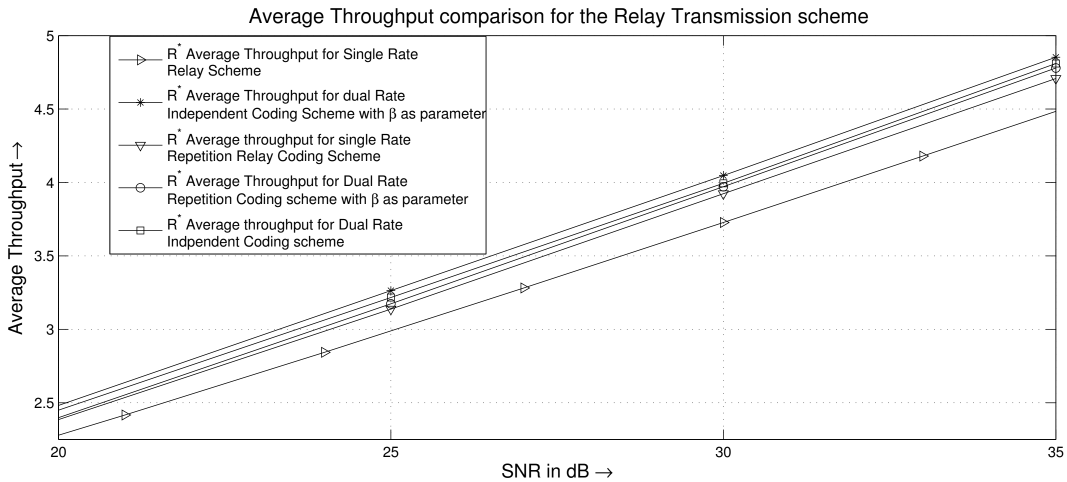

FIGURE 5.5. Comparison of different relay schemes: For clarity, we concentrated on high SNR region.

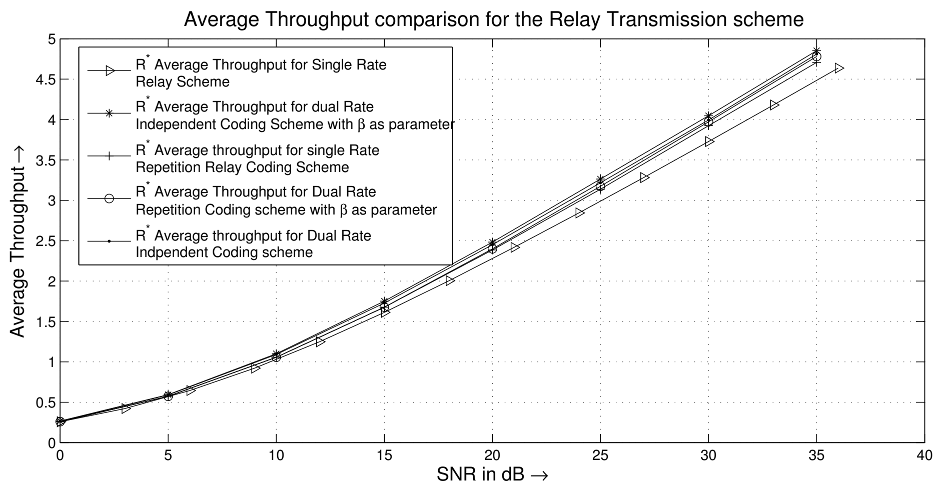

FIGURE 5.6. Comparison of Average throughput in Relay Channel where optimal rate is obtained by considering all rates of the capacity region

'β' with SNR when relay employs the independent coding scheme, where '$\overline{\alpha}$' is the ratio of power allocated to first level by the source and '$\overline{\beta}$' is the ratio of power allocated to first level by the relay. We have also extended the Liu's method to relay schemes. We have attained the same curves as that of ours. Figure 5.8 gives the comparison plot of throughput of relay channel when relay uses successive interference cancelation method with repetition and independent coding methods with independent power allocation factor 'β'. The detailed description of this pro-

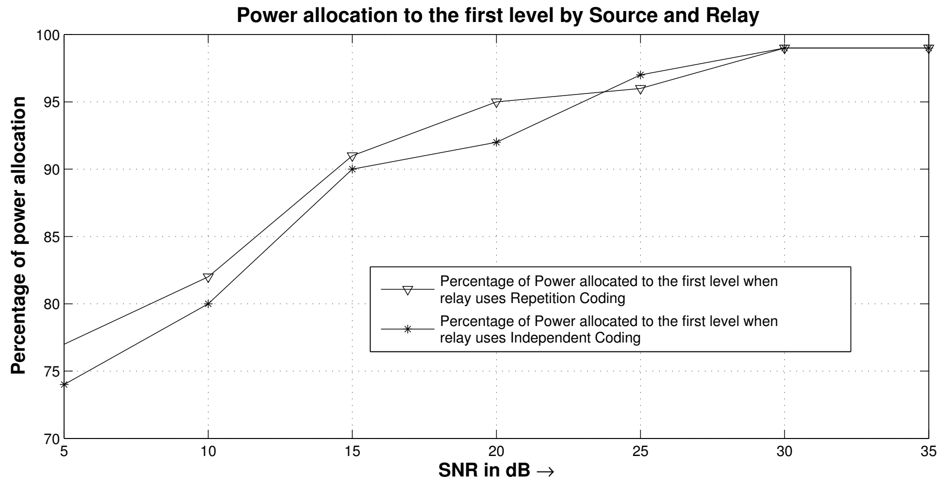

FIGURE 5.7. Comparison of percentage of power allocated to first level of the message i.e. $\overline{\alpha} \cdot 100$

TABLE 5.2. Variation of power allocation factors of source and relay with SNR when relay uses Repetition coding

SNR in dB	$\overline{\alpha}$	$\overline{\beta}$
0	0.78	0.79
5	0.77	0.76
10	0.82	0.81
15	0.91	0.89
20	0.91	0.75
30	0.96	0.81
30	0.97	0.81
35	0.99	0.93

TABLE 5.3. Variation of power allocation factors of source and relay with SNR when relay uses independent coding

SNR in dB	$\overline{\alpha}$	$\overline{\beta}$
0	0.75	0.54
5	0.77	0.5
10	0.89	0.7
15	0.9	0.42
20	0.95	0.42
25	0.98	0.54
30	0.99	0.22
35	0.99	0.38

cedure is given in Appendix C. Fig 5.9 gives the comparison of average throughput between our scheme and that of Liu's scheme applied to relay networks when relay

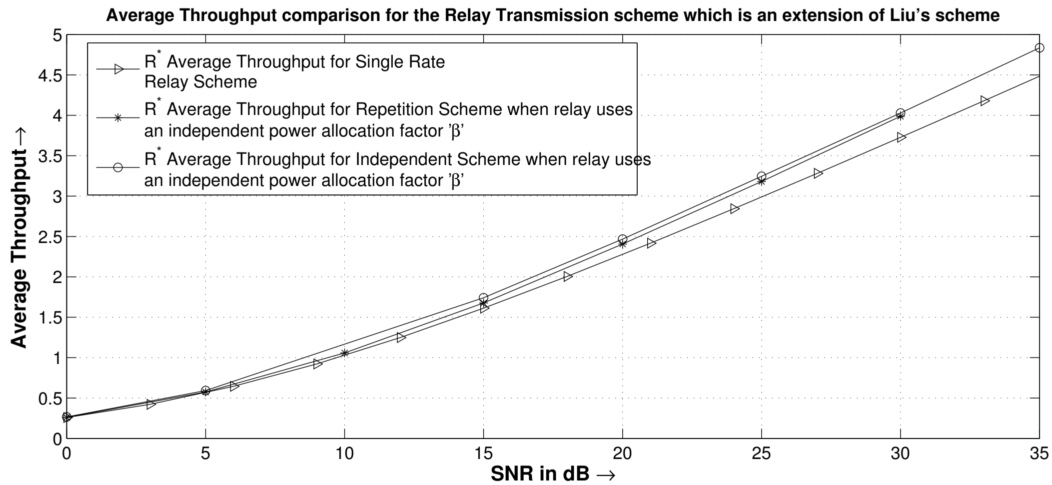

FIGURE 5.8. Comparison of Average throughput when relay uses Successive Interference cancelation method

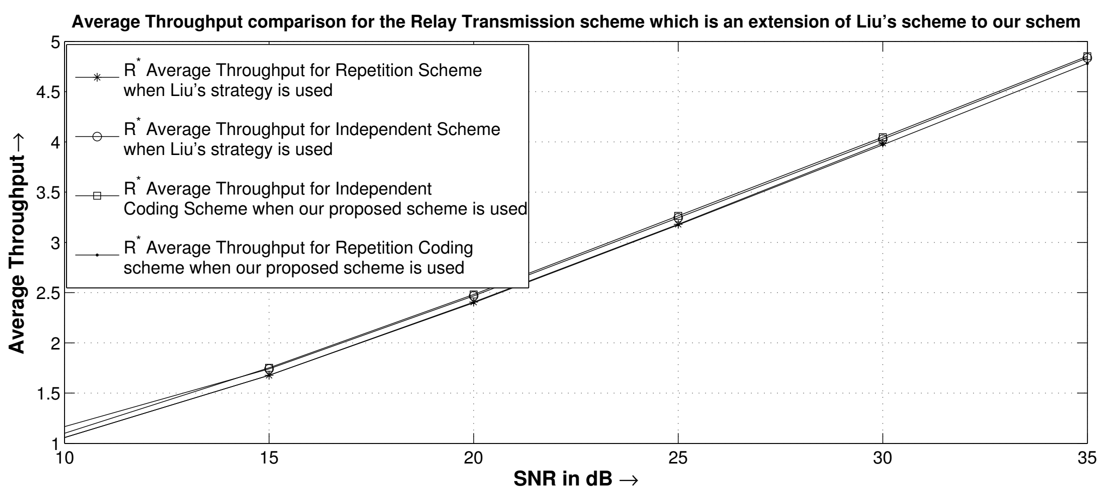

FIGURE 5.9. Comparison between Average throughput of Liu's strategy applied to relay network and our scheme

uses independent power allocation factor β. From the simulations, we have observed that probability for destination to decode second level alone, when relay uses independent power allocation factor β and if it transmits entire message over the second half, is negligible. We know that probability of destination to decode second level is zero when relay transmits first level alone or nothing. Thus the performance of joint decoding will nearly equal to successive interference cancelation method. Thus the improvements in average throughput when relay uses joint decoding will be comparable to improvements in average throughput when relay uses successive interference cancelation method.

Chapter 6
Conclusion and Future Work

6.1 Conclusion

To improve the overall throughput of relay networks, we propose to employ superposition coding based schemes across transmitting nodes and deploy multiple user joint decoding at receiver sides. We have seen the destination terminal from the perspective of multi-access channel rather than degraded user channel as proposed by Liu [19]. We have started with implementing our scheme of superposition coding to direct transmission. We have proved that pre-fixing decoding order is the optimal way of perform superposition coding. We then extended our proposed scheme of superposition to relay channels. We have then proved that when relay uses the same power allocation factor as the source does, i.e. $\beta = \alpha$, the optimal decoder strategy is always successive interference cancelation for both independent coding based and repetition coding based relaying schemes. Numerical results demonstrate the savings of up to 2.1 dB in the high SNR region of our proposed schemes over the traditional one with fixed single rate coding. We have also observed savings of up to 2.5 dB in high SNR region of our proposed scheme when relay uses independent power allocation factor over traditional approach.

6.2 Future Work

There is plenty of scope for further research in this field. We suggest some possible research problems as listed below:

- We have considered only one relay in our system model. In real wireless mobile world, where every mobile phone is a potential relay, study needs to be done when we have more than one relay. We need to develop effective protocols which involves more than one relays

- We have studied the implementation of relay channels from information theory point of view. We need to develop some practical coded modulation schemes, such as multi-level coding strategy, for relay channels to investigate the improvements on overall throughput.

- We have used decode-and-forward scheme at relay. We can develop protocols for other schemes like amplify-and-forward, Selection relaying and Incremental relaying as discussed by Laneman [3].

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Appendix A
Direct Transmission

In this appendix, we derive the valid ranges of α, power allocation factor, for direct transmission.

Proposition 1. *In a direct transmission, the best valid range of α is* $\left[0, \frac{2^{R_2}-1}{2^{R}-1}\right]$.

Proof. Consider a direct transmission with superposition coding as discussed in Chapter 3. Destination will be able to decode the first level alone if the following conditions are satisfied.

$$R_1 < \log_2\left[1 + \frac{|h_{s,d}|^2\overline{\alpha}\text{SNR}}{1 + |h_{s,d}|^2\alpha\text{SNR}}\right] \tag{1}$$

$$R_2 > \log_2\left[1 + |h_{s,d}|^2\alpha\text{SNR}\right] \tag{2}$$

These conditions can be simplified to:

$$H_1 < |h_{s,d}|^2 < L_2 \tag{3}$$

where: H_1, L_2 are given by (3.8) and (3.9) respectively and $R = R_1 + R_2$. Now, Destination will decode the second level alone if the following conditions are satisfied:

$$R_1 > \log_2\left[1 + \overline{\alpha}\text{SNR}|h_{s,d}|^2\right] \tag{4}$$

$$R_2 < \log_2\left[1 + \frac{\alpha\text{SNR}|h_{s,d}|^2}{1 + \overline{\alpha}\text{SNR}|h_{s,d}|^2}\right] \tag{5}$$

These conditions can be simplified to:

$$H_2 < |h_{s,d}|^2 < L_1 \tag{6}$$

where H_2, L_1 are given by:

$$H_2 = \frac{2^{R_2} - 1}{\left(1 - \overline{\alpha}2^{R_2}\right)\text{SNR}} \tag{7}$$

$$L_1 = \frac{2^{R_1} - 1}{\overline{\alpha}\text{SNR}} \tag{8}$$

For a given value of $\alpha \in [0,1]$, we have following inequalities:

$$
\begin{aligned}
& L_1 < H_1 && (9) \\
\Leftrightarrow & \frac{2^{R_1}-1}{\overline{\alpha}\mathrm{SNR}} < \frac{2^{R_1}-1}{(1-\alpha 2^{R_1})\,\mathrm{SNR}} \\
\Leftrightarrow & 1-\alpha 2^{R_1} < \overline{\alpha} \\
\Leftrightarrow & 2^{R_1} > 1
\end{aligned}
$$

Similarly we can state that

$$
L_2 < H_2 \tag{10}
$$

From (3), destination will have a chance to decode the first level alone if $H_1 < L_2$. This simplifies to:

$$
H_1 < L_2 \tag{11}
$$

$$
\frac{2^{R_1}-1}{(1-\alpha 2^{R_1})\,\mathrm{SNR}} < \frac{2^{R_2}-1}{\alpha\mathrm{SNR}}
$$

$$
\Rightarrow \qquad 0 < \alpha < \frac{2^{R_2}-1}{2^{R}-1} \tag{12}
$$

From (9)-(11), the mutual relation between L_1, L_2, H_1, H_2 when $\alpha \in \left[0, \frac{2^{R_2}-1}{2^{R}-1}\right]$ is:

$$
L_1 < H_1 < L_2 < H_2 \tag{13}
$$

From (6), destination will have a chance to decode the second level alone if $H_2 < L_1$. But from (13), this condition cannot be achieved under the assumed range of α. Hence destination cannot decode the second level alone when $\alpha \in \left[0, \frac{2^{R_2}-1}{2^{R}-1}\right]$. Destination will have chance to decode the second level alone when:

$$
H_2 < L_1 \tag{14}
$$

$$
\frac{2^{R_2}-1}{(1-\overline{\alpha}2^{R_2})\,\mathrm{SNR}} < \frac{2^{R_1}-1}{\overline{\alpha}\mathrm{SNR}}
$$

$$
\frac{2^{R}-2^{R_1}}{2^{R}-1} < \alpha < 1 \tag{15}
$$

Hence from (9), (10) and (14) when $\alpha \in \left[\frac{2^R - 2^{R_1}}{2^R - 1}, 1\right]$ we have:

$$L_2 < H_2 < L_1 < H_1 \tag{16}$$

From (3), we can say that destination cannot decode the first level alone when $\alpha \in \left[\frac{2^R - 2^{R_1}}{2^R - 1}, 1\right]$. This means that the two cases of α, $\left[0, \frac{2^{R_2} - 1}{2^R - 1}\right]$, $\left[\frac{2^R - 2^{R_1}}{2^R - 1}, 1\right]$, are mutually exclusive events. Now consider $\alpha \in \left[\frac{2^{R_2} - 1}{2^R - 1}, \frac{2^R - 2^{R_1}}{2^R - 1}\right]$. From (12), we can see that $H_1 > L_2$. Hence Destination cannot decode the first level alone. From (15), we can see that $H_2 > L_1$. Hence destination cannot decode the second level alone. This means that when $\alpha \in \left[\frac{2^{R_2} - 1}{2^R - 1}, \frac{2^R - 2^{R_1}}{2^R - 1}\right]$, destination can only decode the message as a whole, i.e. Destination will treat the received signal as a message with only one rate $R = R_1 + R_2$. Hence this range of α will not yield any improvements over single rate protocols. Thus without loss of generality we chose the range of α as $\left[0, \frac{2^{R_2} - 1}{2^R - 1}\right]$. □

Corollary 1.1. *In a Direct transmission with superposition coding, destination terminal can decode the entire message if the following inequality holds:*

$$|h_{s,d}|^2 \; > \; \frac{2^{R_2} - 1}{\alpha \; SNR} \quad \text{for a given } \alpha \; \in \; \left[0, \frac{2^{R_2} - 1}{2^R - 1}\right]$$

Proof. We have already proved that for a given α, destination cannot decode the second level alone. After successful decoding of first level, destination tries to decode the entire message. Hence the conditions for the successful decoding of entire message ar given by:

$$R_2 < \log_2 \left[1 + \alpha \mathrm{SNR} |h_{s,d}|^2\right] \tag{17}$$

$$R_1 < \log_2 \left[1 + \frac{|h_{s,d}|^2 \overline{\alpha} \mathrm{SNR}}{1 + \alpha \mathrm{SNR} |h_{s,d}|^2}\right] \tag{18}$$

These can be simplified as:

$$|h_{s,d}|^2 > L_2 \tag{19}$$

$$|h_{s,d}|^2 > H_1 \tag{20}$$

We have already proved that $L_2 > H_1$ for the given range of $\alpha \in \left[0, \frac{2^{R_2}-1}{2^R-1}\right]$. Hence (18) is satisfied if (17) is satisfied. But destination can decode the entire message if (17) is true. Hence the decoding order is fixed at destination for direct transmission and condition for decoding the entire message is:

$$|h_{s,d}|^2 > L_2 \tag{21}$$

□

For Relay channels, source-relay channel can be viewed as a direct transmission. Hence similar conditions can be applied to relay while decoding messages.

Appendix B
Relay Channel Scheme

In this appendix, we tried to prove for relay channels that decoding order has to be pre-fixed to achieve the maximum average throughput.

Proposition 2. *With the assumed range of* $\alpha \in \left[0, \frac{2^{2R_2}-1}{2^{2R}-1}\right]$*, the following inequalities hold:*

$$\max\left(L_1^*, L_2^*, M^*, H_1^*\right) = L_2^* \ \&\ 1 - \alpha 2^{2R_1} > 0$$

Where:

$$R = R_1 + R_2 \tag{22}$$

$$L_1^* = \frac{2^{2R_1}-1}{\overline{\alpha}SNR} \tag{23}$$

$$L_2^* = \frac{2^{2R_2}-1}{\alpha SNR} \tag{24}$$

$$M^* = \frac{2^{2R}-1}{SNR} \tag{25}$$

$$H_1^* = \frac{2^{2R_1}-1}{\left(1-\alpha 2^{2R_1}\right)SNR} \tag{26}$$

Proof. With the assumed range of $\alpha \in \left[0, \frac{2^{2R_2}-1}{2^{2R}-1}\right]$, we have:

$$\alpha < \frac{2^{2R_2}-1}{2^{2R}-1}$$

$$\Rightarrow \frac{2^{2R}-1}{\text{SNR}} < \frac{2^{2R_2}-1}{\alpha\text{SNR}}$$

$$\Rightarrow M^* < L_2^* \tag{27}$$

$$\alpha < \frac{2^{2R_2}-1}{2^{2R}-1}$$

$$\Rightarrow 2^{2R_1} + 2^{2R_2} - 2 < 2^{2R} - 1 < \frac{2^{2R_2}-1}{\alpha}$$

$$\Rightarrow \alpha < \frac{2^{2R_2}-1}{2^{2R_1}+2^{2R_2}-2}$$

$$\Rightarrow \frac{2^{2R_1}-1}{\overline{\alpha}\text{SNR}} < \frac{2^{2R_2}-1}{\alpha\text{SNR}}$$

$$\Rightarrow L_1^* < L_2^* \tag{28}$$

$$\alpha < \frac{2^{2R_2} - 1}{2^{2R} - 1}$$

$$\alpha 2^{2R} - \alpha < 2^{2R_2} - 1$$

$$\alpha 2^{2R_1} - \alpha < 2^{2R_1} - 1 - \alpha 2^{2R} + \alpha 2^{2R_1}$$

$$\frac{2^{2R_1} - 1}{(1 - \alpha 2^{2R_1})\,\mathrm{SNR}} < \frac{2^{2R_2} - 1}{\alpha \mathrm{SNR}}$$

$$H_1^* < L_2^* \tag{29}$$

From (27)-(29), we can see that $\max\left(L_1^*, L_2^*, M^*, H_1^*\right) = L_2^*$.

With the assumed range of $\alpha \in \left[0, \frac{2^{2R_2}-1}{2^{2R}-1}\right]$, we have:

$$\alpha < \frac{2^{2R_2} - 1}{2^{2R} - 1}$$

$$2^{2R_2}\left(1 - \alpha 2^{2R_1}\right) > \overline{\alpha} > 0$$

$$\Rightarrow 1 - \alpha 2^{2R_1} > 0 \tag{30}$$

□

Proposition 3. *Maximum average throughput can be achieved in relay channel with relay employing repetition coding if and only if decoding order is pre-fixed such that destination has to perform successive interference cancelation method to decode the two-layered message.*

Proof. Consider Relay channel with relay employing repetition coding as discussed in Section 4.2.1. Depending on relay's decision, we have three cases.

CASE 1

For this case, the entropies of received signal, $H(\mathbf{y}), H\left(\mathbf{y}|\mathbf{x}^{(1)}\right), H\left(\mathbf{y}|\mathbf{x}^{(2)}\right)$ and $H(\mathbf{y}|\mathbf{x})$ are given by (4.25)- (4.28) respectively. The mutual information between the received signals and both the layers of transmitted signal is given by:

$$I_R^{(1-0)} = \frac{1}{2}\left(H(\mathbf{y}) - H(\mathbf{y}|\mathbf{x})\right)$$

$$I_R^{(1-0)} = \frac{1}{2}\log_2\left[1 + \left(|h_{s,d}|^2 + |h_{r,d}|^2\right)\text{SNR}\right] \tag{31}$$

The mutual information between the received signals and the first level of the transmitted signal with second level decoded is given by:

$$\begin{aligned} I_R^{(1-1|2)} &= \frac{1}{2}\left(H\left(\mathbf{y}|\mathbf{x}^{(2)}\right) - H\left(\mathbf{y}|\mathbf{x}\right)\right) \\ &= \frac{1}{2}\log_2\left[1 + \frac{\left(|h_{s,d}|^2 + |h_{r,d}|^2\right)\overline{\alpha}\text{SNR}}{1 + \left(|h_{s,d}|^2 + |h_{r,d}|^2\right)\alpha\text{SNR}}\right] \end{aligned} \tag{32}$$

The mutual information between the received signals and the second level with the first level decoded is given by (4.30). Destination will be able to decode the entire message if the following conditions are satisfied.

$$R_1 < I_R^{(1-1|2)} \tag{33}$$

$$R_2 < I_R^{(1-2|1)} \tag{34}$$

$$R_1 + R_2 = R < I_R^{(1-0)} \tag{35}$$

The above conditions can be simplified to:

$$|h_{s,d}|^2 + |h_{r,d}|^2 > \max\left(L_2^*, L_1^*, M^*\right) \tag{36}$$

where L_1^*, L_2^* and M^* are given by (23), (24) and (25) respectively. From Proposition 2, we know that $\max\left(L_1^*, L_2^*, M^*\right) = L_2^*$. Hence the destination can decode the entire message if:

$$|h_{s,d}|^2 + |h_{r,d}|^2 > L_2^* \tag{37}$$

Destination can decode the second level alone if the following conditions are satisfied:

$$R_2 < I_R^{(1-2)} \tag{38}$$

$$R_1 > I_R^{(1-1|2)} \tag{39}$$

If (38) is satisfied, then (34) will always be satisfied. This means that destination will decode the entire message. In other words, destination cannot decode the second level alone. Decoding order is said to be fixed if the conditions to be satisfied for the destination to decode the entire message are:

$$R_2 < I_R^{(1-2|1)} \tag{40}$$

$$R_1 < I_R^{(1-1)} \tag{41}$$

The above conditions can be simplified to:

$$|h_{s,d}|^2 + |h_{r,d}|^2 > L_2^* \tag{42}$$

$$h_{s,d}|^2 + |h_{r,d}|^2 > H_1^* \tag{43}$$

From proposition 2, we know that $L_2^* > H_1^*$. Hence (41) will be satisfied if (40) is satisfied. But (40) is sufficient and necessary condition for destination to decode the entire message. Hence the decoding order is fixed if the relay transmits the entire message.

CASE 2

For this case, the entropies of received signal, $H(\mathbf{y}), H\left(\mathbf{y}|\mathbf{x}^{(1)}\right), H\left(\mathbf{y}|\mathbf{x}^{(2)}\right)$ are given by (4.39)- (4.41) respectively. The received signals at the destination when relay transmits the first level alone are given by (4.15), (4.16). The mutual information between the received signals and the entire transmitted message is given by:

$$\begin{aligned} I_R^{(2-0)} &= \frac{1}{2}\left(H(\mathbf{y}) - H(\mathbf{y}|\mathbf{x})\right) \\ &= \frac{1}{2}\log_2\left[1 + \text{SNR}\left(|h_{s,d}|^2 + \alpha|h_{r,d}|^2\right) + \alpha\overline{\alpha}|h_{s,d}|^2|h_{r,d}|^2\text{SNR}^2\right] \end{aligned} \tag{44}$$

The mutual information between the received signals and the first level of transmitted signal given that second level is decoded is given by:

$$\begin{aligned} I_R^{(2-1|2)} &= \frac{1}{2}\left(H\left(\mathbf{y}|\mathbf{x}^{(\mathbf{2})}\right) - H\left(\mathbf{y}|\mathbf{x}\right)\right) \\ &= \frac{1}{2}\log_2\left[1+\overline{\alpha}\mathrm{SNR}\left(|h_{s,d}|^2 + |h_{r,d}|^2\right)\right] \end{aligned} \tag{45}$$

The mutual information between the received signal and the second level alone given that first level is decoded is given by (4.43). Destination will decode the entire message if the following conditions are satisfied.

$$R_1 < I_R^{(2-1|2)} \tag{46}$$

$$R_2 < I_R^{(2-2|1)} \tag{47}$$

$$R < I_R^{(2)} \tag{48}$$

These can be simplified to:

$$|h_{s,d}|^2 + |h_{r,d}|^2 > L_1^* \tag{49}$$

$$|h_{s,d}|^2 > L_2^* \tag{50}$$

$$|h_{s,d}|^2 + \alpha|h_{r,d}|^2 + \alpha\overline{\alpha}|h_{s,d}|^2|h_{r,d}|^2\mathrm{SNR} > M^* \tag{51}$$

Under assumed range of α, we have already proved that $\max\left(L_1^*, L_2^*, M^*\right) = L_2^*$. Using the fact that the variables in these inequalities are all non-negative, we can clearly say that (50) is the necessary and sufficient condition for destination to decode the entire message. Destination will decode the second level alone if the following inequalities hold:

$$R_2 < I_R^{(2-2)} \tag{52}$$

$$R_1 > I_R^{(2-1|2)} \tag{53}$$

When (52) is satisfied, then (47) will always be satisfied. It means that destination can decode the entire message. In the other words, we can say that destination can never decode the second level alone. Decoding order is said to fixed if the conditions to be satisfied for destination to decode the entire message are given by:

$$R_2 < I_R^{(2-2|1)} \tag{54}$$

$$R_1 < I_R^{(2-1)} \tag{55}$$

The above inequalities can be simplified to:

$$|h_{s,d}|^2 > L_2^* \tag{56}$$

$$|h_{s,d}|^2 + \frac{|h_{r,d}|^2\overline{\alpha}}{1-\alpha 2^{2R_1}}\left(1+|h_{s,d}|^2\alpha\text{SNR}\right) > H_1^* \tag{57}$$

From Proposition 2, we have proved that $L_2^* > H_1^*$. Using the fact that all the expression in the equations are non-negative, we can clearly see that (55) is satisfied if (54) is true. But (54) is the necessary and sufficient condition for destination to decode the entire message. Hence the decoding order is fixed when relay transmits only first level.

CASE 3

For this case, the entropies of received signal, $H(\mathbf{y}), H\left(\mathbf{y}|\mathbf{x}^{(1)}\right), H\left(\mathbf{y}|\mathbf{x}^{(2)}\right)$ are given by (4.49)- (4.51) respectively. The received signals when relay cannot decode any of the levels are given by (4.18), (4.19). This is similar to direct transmission since relay doesn't transmit any signals. The mutual information between the received signals and both levels of transmitted signals is given by:

$$\begin{aligned} I_R^{(3-0)} &= \frac{1}{2}\left(H(\mathbf{y}) - H(\mathbf{y}|\mathbf{x})\right) \\ &= \frac{1}{2}\log_2\left[1+|h_{s,d}|^2\text{SNR}\right] \end{aligned} \tag{58}$$

The mutual information between the first level with second level decoded is given by:

$$
\begin{aligned}
I_R^{(3-1|2)} &= \frac{1}{2}\left(H\left(\mathbf{y}|\mathbf{x}^{(\mathbf{2})}\right) - H\left(\mathbf{y}|\mathbf{x}\right)\right) \\
&= \frac{1}{2}\log_2\left[1+|h_{s,d}|^2\overline{\alpha}\text{SNR}\right]
\end{aligned} \tag{59}
$$

The mutual information between the received signals and the second level with the first level decoded is given by (4.53). destination can decode the entire message if the following conditions are satisfied.

$$R_1 < I_R^{(3-1|2)} \tag{60}$$

$$R_2 < I_R^{(3-2|1)} \tag{61}$$

$$R < I_R^{(3)} \tag{62}$$

These can be simplified to:

$$|h_{s,d}|^2 > \max\left(L_1^*, L_2^*, M^*\right) \tag{63}$$

We have already shown that $\max\left(L_1^*, L_2^*, M^*\right) = L_2^*$. Hence the destination will decode the entire message when the following condition is satisfied:

$$|h_{s,d}|^2 > L_2^* \tag{64}$$

Destination will decode the second level alone if the following conditions are satisfied.

$$R_2 < I_R^{(3-2)} \tag{65}$$

$$R_1 > I_R^{(3-1|2)} \tag{66}$$

when (66) is satisfied, then (61) will always be satisfied. This means that destination will decode the entire message. In other words, destination cannot decode the second level alone. Decoding order is said to fixed if the conditions to be satisfied for the destination to decode the entire message is given by

$$R_2 < I_R^{(3-2|1)} \tag{67}$$

$$R_1 < I_R^{(3-1)} \tag{68}$$

These can be simplified to:

$$|h_{s,d}|^2 > L_2^* \tag{69}$$

$$|h_{s,d}|^2 > H_1^* \tag{70}$$

From proposition 2, we know that $L_2^* > H_1^*$. Hence (68) is satisfied if (67) is satisfied. But (67) is the necessary and sufficient condition for the destination to decode the entire message. Hence decoding order is fixed when relay transmits nothing.

Hence decoding order is fixed at destination when relay employs repetition coding. □

Proposition 4. *Maximum average throughput can be achieved in relay channel with relay employing independent coding if and only if decoding order is fixed such that destination has to perform successive interference cancelation method to decode the two-layered message.*

Proof. Consider Relay channel with relay employing independent coding as discussed in Section 4.2.2. Depending on relay's decision, we have three cases.

CASE 1

For this case, the entropies of received signal, $H(\mathbf{y}), H\left(\mathbf{y}|\mathbf{x^{(1)}}\right), H\left(\mathbf{y}|\mathbf{x^{(2)}}\right)$ are given by (4.120)- (4.122) respectively. The mutual information between the received signals and both the layers of transmitted signal is given by:

$$
\begin{aligned}
I_I^{(1-0)} &= \frac{1}{2}\left(H\left(\mathbf{y}\right) - H\left(\mathbf{y}|\mathbf{x}\right)\right) \\
&= \frac{1}{2}\log_2\left[\left(1+\mathrm{SNR}|h_{s,d}|^2\right)\left(1+\mathrm{SNR}|h_{r,d}|^2\right)\right]
\end{aligned} \tag{71}
$$

The mutual information between the received signals and the first level of the transmitted signal with second level decoded is given by:

$$
\begin{aligned}
I_I^{(1-1|2)} &= \frac{1}{2}\left(H\left(\mathbf{y}|\mathbf{x}^{(2)}\right) - H\left(\mathbf{y}|\mathbf{x}\right)\right) \\
&= \frac{1}{2}\log_2\left[\left(1+\overline{\alpha}\mathrm{SNR}|h_{s,d}|^2\right)\left(1+\overline{\alpha}\mathrm{SNR}|h_{r,d}|^2\right)\right]
\end{aligned} \tag{72}
$$

The mutual information between the received signals and the second level with the first level decoded is given by (4.68). Now the destination will be able to decode the entire message if the following conditions are satisfied.

$$R_1 < I_I^{(1-1|2)} \tag{73}$$

$$R_2 < I_I^{(1-2|1)} \tag{74}$$

$$R < I_I^{(1-0)} \tag{75}$$

The above conditions can be simplified to:

$$|h_{s,d}|^2 + |h_{r,d}|^2 + |h_{s,d}|^2|h_{r,d}|^2\overline{\alpha}\mathrm{SNR} > L_1^* \tag{76}$$

$$|h_{s,d}|^2 + |h_{r,d}|^2 + |h_{s,d}|^2|h_{r,d}|^2\alpha\mathrm{SNR} > L_2^* \tag{77}$$

$$|h_{s,d}|^2 + |h_{r,d}|^2 + |h_{s,d}|^2|h_{r,d}|^2\mathrm{SNR} > M^* \tag{78}$$

where L_1^*, L_2^* and M^* are given by (23), (24) and (25) respectively. With the assumed range of $\alpha \in \left[0, \frac{2^{2R_2}-1}{2^{2R}-1}\right]$, we have already proved that $\max\left(L_1^*, L_2^*, M^*\right) = L_2^*$. We also know that $\alpha < \overline{\alpha} < 1$. With these known facts, we can clearly see that destination can decode the entire message if:

$$|h_{s,d}|^2 + |h_{r,d}|^2 + |h_{s,d}|^2|h_{r,d}|^2\alpha\mathrm{SNR} > L_2^* \tag{79}$$

For the destination can decode the second level alone if the following conditions are satisfied:

$$R_2 < I_R^{(1-2)} \tag{80}$$

$$R_1 > I_R^{(1-1|2)} \tag{81}$$

If (80) is satisfied, then (74) will always be satisfied. This means that destination will decode the entire message. In other words, destination cannot decode the second level alone. Decoding order for the destination is said to be fixed if the conditions for destination to decode the entire message are given by:

$$R_2 < I_I^{(1-2|1)} \tag{82}$$

$$R_1 < I_I^{(1-1)} \tag{83}$$

These conditions are simplified as:

$$|h_{s,d}|^2 + |h_{r,d}|^2 + |h_{s,d}|^2 |h_{r,d}|^2 \alpha \mathrm{SNR} > L_2^* \tag{84}$$

$$|h_{s,d}|^2 + |h_{r,d}|^2 + |h_{s,d}|^2 |h_{r,d}|^2 \left(\alpha + \overline{\alpha}\right) \mathrm{SNR} > H_1^* \tag{85}$$

We proved that destination will decode the entire message if (82) is satisfied. we also proved that $L_2^* > H_1^*$. With these conditions in mind, we can clearly see that (83) will be satisfied. Hence the decoding order is fixed.

CASE 2

For this case, the entropies of received signal, $H(\mathbf{y})\,, H\left(\mathbf{y}|\mathbf{x}^{(1)}\right), H\left(\mathbf{y}|\mathbf{x}^{(2)}\right)$ are given by (4.76)- (4.78) respectively. The received signals at the destination when relay transmits the first level alone are given by (4.15), (4.16). The mutual information between the received signals and the entire transmitted message is given by:

$$I_I^{(2-0)} = \frac{1}{2}\left(H\left(\mathbf{y}\right) - H\left(\mathbf{y}|\mathbf{x}\right)\right)$$
$$I_I^{(2-0)} = \frac{1}{2}\log_2\left[\left(1+|h_{s,d}|^2\text{SNR}\right)\left(1+|h_{r,d}|^2\overline{\alpha}\text{SNR}\right)\right] \tag{86}$$

The mutual information between the received signals and the first level of transmitted signal given that second level is decoded is given by:

$$I_R^{(2-1|2)} = \frac{1}{2}\left(H\left(\mathbf{y}|\mathbf{x^{(2)}}\right) - H\left(\mathbf{y}|\mathbf{x}\right)\right)$$
$$= \frac{1}{2}\log_2\left[\left(1+|h_{s,d}|^2\overline{\alpha}\text{SNR}\right)\left(1+|h_{r,d}|^2\overline{\alpha}\text{SNR}\right)\right] \tag{87}$$

The mutual information between the received signal and the second level alone given that first level is decoded is given by (4.81). Destination will decode the entire message if the following conditions are satisfied.

$$R_1 < I_I^{(2-1|2)} \tag{88}$$

$$R_2 < I_I^{(2-2|1)} \tag{89}$$

$$R < I_I^{(2)} \tag{90}$$

These can be simplified to:

$$|h_{s,d}|^2 + |h_{r,d}|^2 + |h_{s,d}|^2|h_{r,d}|^2\overline{\alpha}\text{SNR} > L_1^* \tag{91}$$

$$|h_{s,d}|^2 > L_2^* \tag{92}$$

$$|h_{s,d}|^2 + |h_{r,d}|^2\overline{\alpha} + |h_{s,d}||h_{r,d}|^2\overline{\alpha}\text{SNR} > M^* \tag{93}$$

Under assumed range of α, we have already proved that $\max\left(L_1^*, L_2^*, M^*\right) = L_2^*$. With the known fact that the elements in these inequalities are all non-negative, we can clearly see that destination will decode the entire message if:

$$|h_{s,d}|^2 > L_2^* \tag{94}$$

Destination will decode the second level alone if the following inequalities hold:

$$R_2 < I_I^{(2-2)} \tag{95}$$

$$R_1 > I_I^{(2-1|2)} \tag{96}$$

When (95) is satisfied, then (89) will always be satisfied. It means that destination can decode the entire message. In the other words, we can say that destination can never decode the second level alone. Decoding order is said to be fixed if the conditions for destination to decode the entire level are given by:

$$R_2 < I_I^{(2-2|1)} \tag{97}$$

$$R_1 < I_I^{(2-1)} \tag{98}$$

These can be simplified as:

$$h_{s,d}|^2 > L_2^* \tag{99}$$

$$|h_{s,d}|^2 + \frac{\overline{\alpha}|h_{r,d}|^2}{1-\alpha 2^{2R_1}}\left(1+|h_{s,d}|^2\text{SNR}\right) > H_1^* \tag{100}$$

With the assumed range of α, we know that $L_2^* > H_1^*$ and all the expressions are non-negative. Hence we can clearly see that (98) will be satisfied if (97) is satisfied. Destination will decode the entire message if (97) is satisfied. Hence decoding order is fixed.

Case 3

The received signals when relay cannot decode any of the levels are given by (4.18), (4.19). This is similar to direct transmission since relay doesn't transmit any signals. Since relay doesn't transmit any level of the signal, Independent coding and Repetition coding schemes are exactly the same. Hence from Case 3 of the Repetition Coding, we can say that decoding order is fixed.

Hence Decoding order is fixed at destination when relay employs independent coding. □

Proposition 5. *The destination has to perform successive interference cancelation when relay uses repetition coding scheme with 'β' as power allocation factor given that relay transmits first level to destination.*

Proof. Consider relay scheme discussed in Section 4.2.3. When relay transmits only the first level to the destination, the received signal by the destination over the second part of the frame is given by (4.16). For this case, the entropies of received signal, $H(\mathbf{y}), H\left(\mathbf{y}|\mathbf{x^{(1)}}\right), H\left(\mathbf{y}|\mathbf{x^{(2)}}\right)$ are given by (4.106)- (4.108) respectively. The mutual information between the received signals and the second level given that first level is decoded is given by (4.110). The mutual information between received signal and the first level of the message given that second level is decoded is given by:

$$\begin{aligned} I_{R_\beta}^{(2-1|2)} &= \frac{1}{2}\left(H\left(\mathbf{y}|\mathbf{x^{(2)}}\right) - H(\mathbf{y}|\mathbf{x})\right) \\ &= \frac{1}{2}\log_2\left[1 + \left[\overline{\alpha}|h_{s,d}|^2 + \overline{\beta}|h_{r,d}|^2\right]\text{SNR}\right] \end{aligned} \tag{101}$$

The mutual information between the received signal and the transmitted signals from destination and relay is given by

$$\begin{aligned} I_{R_\beta}^{(2-0)} &= \frac{1}{2}\left(H(\mathbf{y}) - H(\mathbf{y}|\mathbf{x})\right) \\ &= \frac{1}{2}\log_2\left[1 + \left(|h_{s,d}|^2 + \overline{\beta}|h_{r,d}|^2\right)\text{SNR} + |h_{s,d}|^2|h_{r,d}|^2\alpha\overline{\beta}\text{SNR}^2\right] \end{aligned} \tag{102}$$

The mutual information between the received signals and the second level is given by:

$$\begin{aligned} I_{R_\beta}^{(2-2)} &= \frac{1}{2}\left(H(\mathbf{y}) - H\left(\mathbf{y}|\mathbf{x^{(2)}}\right)\right) \\ &= \frac{1}{2}\log_2\left[1 + \frac{\left(|h_{s,d}|^2\alpha + |h_{s,d}|^2|h_{r,d}|^2\alpha\overline{\beta}\text{SNR}\right)\text{SNR}}{1 + \left(\overline{\alpha}|h_{s,d}|^2 + \overline{\beta}|h_{r,d}|^2\right)\text{SNR}}\right] \end{aligned} \tag{103}$$

The destination will decode the entire message if the following conditions are satisfied:

$$R_1 < I_{R_\beta}^{(2-1|2)} \tag{104}$$

$$R_2 < I_{R_\beta}^{(2-2|1)} \tag{105}$$

$$R_1 + R_2 = R < I_{R_\beta}^{(2)} \tag{106}$$

The above inequalities can be simplified to:

$$|h_{s,d}|^2 + |h_{r,d}|^2 \frac{\overline{\beta}}{\overline{\alpha}} > L_1^* \tag{107}$$

$$|h_{s,d}|^2 > L_2^* \tag{108}$$

$$|h_{s,d}|^2 + \left(|h_{r,d}|^2\overline{\beta} + |h_{s,d}|^2|h_{r,d}|^2\alpha\overline{\beta}\text{SNR}\right)\text{SNR} > M^* \tag{109}$$

Where L_1^*, L_2^*, M^* are given by (23)-(25) respectively. From Proposition 2 and the fact that all the elements in the above inequalities are non-negative, we can clearly see that (108) is necessary and sufficient condition for the destination to decode the entire message. Now the destination will decode the second level alone if the following inequalities are satisfied:

$$R_2 < I_{R_\beta}^{(2-2)} \ \& \ R_1 > I_{R_\beta}^{(2-1|2)} \tag{110}$$

From the capacity region figure, we can observer that $I_{R_\beta}^{(2-2)} < I_{R_\beta}^{(2-2|1)}$ at all times. Therefore, we can say that if destination is capable of decoding the second level alone, then it is capable of decoding the entire message. In other words, we can say that destination can never decode the second level alone. Now the destination can use successive interference cancelation method to decode the message if the conditions required to be met for the destination to decode the entire message is given by:

$$R_2 < I_{R_\beta}^{(2-2|1)} \ \& \ R_1 < I_{R_\beta}^{(2-1)} \tag{111}$$

which can be simplified to:

$$|h_{s,d}|^2 > L_2^* \tag{112}$$

$$|h_{s,d}|^2 + \frac{\overline{\beta}|h_{r,d}|^2}{1-\alpha 2^{2R_1}}\left[1 + |h_{s,d}|^2 \alpha \mathrm{SNR}\right] > H_1^* \tag{113}$$

respectively. From Proposition 2 and the fact that all the elements in the inequalities are non-negative, we can say that (108) is necessary and sufficient condition for the destination to decode the entire message. Therefore the destination in our scheme will perform successive interference cancelation method to decode the message when relay performs the repetition coding scheme with 'β' as parameter given that relay transmits only first level to destination over the second half of the frame. □

Proposition 6. *The destination will perform successive interference cancelation method to decode the message when relay uses independent coding scheme with 'β' as parameter given that relay transmits first level to destination over the second part of the frame.*

Proof. Consider the Independent relay scheme with 'β' as parameter as discussed in Section 4.2.4. When relay decodes the first level alone, the received signal at the destination over the second part of the frame is given by (4.16). For this case, the entropies of received signal, $H(\mathbf{y}), H\left(\mathbf{y}|\mathbf{x^{(1)}}\right), H\left(\mathbf{y}|\mathbf{x^{(2)}}\right)$ are given by (4.106)-(4.108) respectively. The mutual information between the received signal and the first level given that second level is decoded is given by:

$$\begin{aligned} I_{I_\beta}^{(2-1|2)} &= \frac{1}{2}\left(H\left(\mathbf{y}|\mathbf{x^{(2)}}\right) - H(\mathbf{y}|\mathbf{x})\right) \\ &= \frac{1}{2}\log_2\left[\left(1+\overline{\alpha}|h_{s,d}|^2\mathrm{SNR}\right)\left(1+\overline{\beta}|h_{r,d}|^2\mathrm{SNR}\right)\right] \end{aligned} \tag{114}$$

The mutual information between the received signal and the second level given that first level is decoded is given by (4.140). The mutual information between the received signal and the transmitted signal by relay and source is given by

$$\begin{aligned} I_{I_\beta}^{(2-0)} &= \frac{1}{2}\left(H\left(\mathbf{y}\right) - H\left(\mathbf{y}|\mathbf{x}\right)\right) \\ &= \frac{1}{2}\log_2\left[\left(1+|h_{s,d}|^2\mathrm{SNR}\right)\left(1+\overline{\beta}|h_{r,d}|^2\mathrm{SNR}\right)\right] \end{aligned} \tag{115}$$

The mutual information between the received signal and the second level is given by:

$$\begin{aligned} I_{I_\beta}^{(2-2)} &= \frac{1}{2}\left(H\left(\mathbf{y}\right) - H\left(\mathbf{y}|\mathbf{x}^{(2)}\right)\right) \\ &= \frac{1}{2}\log_2\left[\frac{\left(1+|h_{s,d}|^2\mathrm{SNR}\right)\left(1+\overline{\beta}|h_{r,d}|^2\mathrm{SNR}\right)}{\left(1+\overline{\alpha}|h_{s,d}|^2\mathrm{SNR}\right)\left(1+\overline{\beta}|h_{r,d}|^2\mathrm{SNR}\right)}\right] \end{aligned} \tag{116}$$

The destination can decode the entire message if the following conditions are satisfied:

$$R_1 < I_{I_\beta}^{(2-1|2)} \;\&\; R_2 < I_{I_\beta}^{(2-2|1)} \tag{117}$$

$$R_1 + R_2 = R < I_{I_\beta}^{(2)} \tag{118}$$

which are simplified to:

$$|h_{s,d}|^2 + |h_{r,d}|^2\frac{\overline{\beta}}{\overline{\alpha}} + |h_{s,d}|^2|h_{r,d}|^2\overline{\beta}\mathrm{SNR} > L_1^* \tag{119}$$

$$|h_{s,d}|^2 > L_2^* \tag{120}$$

$$|h_{s,d}|^2 + |h_{r,d}|^2\overline{\beta}\left(1+|h_{s,d}|^2\mathrm{SNR}\right) > M^* \tag{121}$$

Where L_1^*, L_2^*, M^* are given by (23)-(25) respectively. From Proposition 2 and the fact that all the elements in the above inequalities are non-negative, we can clearly see that (120) is necessary and sufficient condition for the destination to decode the entire message. Now the destination will decode the second level alone if the following inequalities are satisfied:

$$R_2 < I_{I_\beta}^{(2-2)} \;\&\; R_1 > I_{I_\beta}^{(2-1|2)} \tag{122}$$

From the capacity region figure, we can observer that $I_{I_\beta}^{(2-2)} < I_{I_\beta}^{(2-2|1)}$ at all times. Therefore, we can say that if destination is capable of decoding the second level alone, then it is capable of decoding the entire message. In other words, we can say that destination can never decode the second level alone. Now the destination can use successive interference cancelation method to decode the message if the conditions required to be met for the destination to decode the entire message is given by:

$$R_2 < I_{I_\beta}^{(2-2|1)} \;\&\; R_1 < I_{I_\beta}^{(2-1)} \tag{123}$$

which can be simplified to:

$$|h_{s,d}|^2 > L_2^* \tag{124}$$

$$|h_{s,d}|^2 + \frac{\overline{\beta}|h_{r,d}|^2}{1-\alpha 2^{2R_1}}\left[1 + |h_{s,d}|^2 \mathrm{SNR}\right] > H_1^* \tag{125}$$

respectively. From Proposition 2 and the fact that all the elements in the inequalities are non-negative, we can say that (120) is necessary and sufficient condition for the destination to decode the entire message. Therefore the destination in our scheme will perform successive interference cancelation method to decode the message when relay performs the independent coding scheme with 'β' as parameter given that relay transmits only first level to destination over the second half of the frame. □

Appendix C
Extension of Liu's Strategy to Relay Networks

As discussed in Section 1.2.1, Liu [19] has implemented superposition coding in a direct transmission with the condition that destination has to perform successive interference cancelation scheme. From instantaneous capacity region of direct transmission given in Figure 3.1, we can see that Liu has restricted the rates under consideration to be in regions A_1 and A_{2a}. In his scheme, Liu has treated different states of the channel as degraded users of broadcast channel. In his scheme, the rates for the two levels are already fixed so is the decoding order. The rates for the two levels in Liu's scheme are given by:

$$R_1 = \log_2 \left[1 + \frac{\overline{\alpha} h_1 \text{SNR}}{1 + \alpha h_1 \text{SNR}} \right] \tag{126}$$

$$R_2 = \log_2 \left[1 + \alpha h_2 \text{SNR} \right] \tag{127}$$

where R_1 and R_2 are the rates for the two levels, $\overline{\alpha} = 1 - \alpha$ is the fraction of available power allocated to first level by the source and h_1 and h_2 are the channel characteristics. It is assumed that second level is degraded to the first level. Hence $h_1 < h_2$. This condition will yield a relationship between R_1, R_2 and α and is given as:

$$\begin{aligned} h_1 &< h_2 \\ \frac{2^{R_1} - 1}{(1 - \alpha 2^{R_1}) \text{SNR}} &< \frac{2^{R_2} - 1}{\alpha \text{SNR}} \\ \alpha &< \frac{2^{R_2} - 1}{2^{R_1 + R_2} - 1} \end{aligned} \tag{128}$$

This condition coincides with the valid range of α which we have proved in Proposition 1. We have also proved in Chapter 3 that we can attain maximal average throughput if and only if the destination uses successive interference cancelation method i.e. the valid rate pair (R_1, R_2) is in regions A_1 and A_{2a} of the capacity region for direct transmission.

Now we would like to extend his strategy to relay networks. The rates for the two levels in a Liu's strategy of relay channel can be given as:

$$R_1 = \frac{1}{2}\log_2\left[1 + \frac{\overline{\alpha} h_1^* \text{SNR}}{1 + \alpha h_1^* \text{SNR}}\right] \tag{129}$$

$$R_2 = \frac{1}{2}\log_2\left[1 + \alpha h_2^* \text{SNR}\right] \tag{130}$$

Where the factor $\frac{1}{2}$ is introduced due to the fact that the receiver uses only one half of the available bandwidth, h_1^* and h_2^* are the channel characteristics treating the three terminal relay channel as two terminal direct transmission channel. The second level is considered to be degraded to that of first level. Hence we have:

$$h_1^* < h_2^*$$

$$\frac{2^{2R_1} - 1}{(1 - \alpha 2^{2R_1})\,\text{SNR}} < \frac{2^{2R_2} - 1}{\alpha \text{SNR}}$$

$$\alpha < \frac{2^{2R_2} - 1}{2^{2R} - 1} \tag{131}$$

Where $R = R_1 + R_2$ is the total rate in the channel. This inequality is same as the condition we considered for relay channel in Chapter 4.2. We have proved[1] that the optimal approach to attain maximal throughput, when relay uses the same power allocation factor, is to restrict the rate pairs (R_1, R_2) so that destination has to perform successive interference cancelation method. Now we will implement the Liu's scheme to relay channel when relay uses an independent power allocation factor β. The received signals at destination, the probability with which relay decodes the two layers are given in Section 4.2. The only difference is that destination will first try to decode the first level. If it is successful, then it tries to decode the second level. Now consider the repetition coding strategy at relay.

[1] Refer to Proposition 3 and Proposition 4 in Appendix B for this proof

C.1 Repetition Coding Scheme

This kind of scheme is discussed in Section 4.2.3. Depending on the relay's decision, we have three cases as discussed.

Case 1

In this case, relay will transmit the entire message. Destination will decode the first level alone if the following conditions are satisfied.

$$R_1 < I_{R_\beta}^{(1-1)} \ \& \ R_2 > I_{R_\beta}^{(1-2|1)} \tag{132}$$

Where $I_{R_\beta}^{(1-1)}$ and $I_{R_\beta}^{(1-2|1)}$ are given by 4.95 and 4.98 respectively. The condition for the destination to decode the second level is given by:

$$R_1 < I_{R_\beta}^{(1-1)} \ \& \ R_2 < I_{R_\beta}^{(1-2|1)} \tag{133}$$

Thus the average throughput when relay transmits both the levels is given by:

$$\widehat{R_{R_\beta}^{(1)}} = R_1 \cdot Pr \begin{bmatrix} R_1 < I_{R_\beta}^{(1-1)} \& \\ R_2 > I_{R_\beta}^{(1-2|1)} \end{bmatrix} + R \cdot Pr \begin{bmatrix} R_1 < I_{R_\beta}^{(1-1)} \& \\ R_2 < I_{R_\beta}^{(1-2|1)} \end{bmatrix} \tag{134}$$

Case 2

When relay transmits only the first level to destination, we have already proved in Proposition 5 that destination will perform successive interference cancelation. Hence the average throughput for this case is given by 4.112 i.e. $\widehat{R_{R_\beta}^{(2)}} = \widetilde{R_{R_\beta}^{(2)}}$

Case 3

When relay was unable to transmit any data to destination, then decoding at destination will be equivalent to that of direct transmission. Hence the average throughput is given by 4.54 i.e. $\widehat{R_{R_\beta}^{(3)}} = \widetilde{R_{R_\beta}^{(3)}}$.

Thus the total average throughput when relay uses repetition coding is given by:

$$\widehat{R_{R_\beta}} = P^{(1)} \widehat{R_{R_\beta}^{(1)}} + P^{(2)} \widehat{R_{R_\beta}^{(2)}} + P^{(3)} \widehat{R_{R_\beta}^{(3)}} \tag{135}$$

where $P^{(1)}, P^{(2)}, P^{(3)}$ are given by 4.11, 4.14 and 4.17 respectively. We need to calculate the maximum average throughput over the values of h_1^*, h_2^* and α given the condition $h_1^* < h_2^*$.

Now consider the independent coding strategy at relay.

C.2 Independent Coding Scheme

This kind of scheme is discussed in Section 4.2.4. Depending on the relay's decision, we have three cases as discussed.

Case 1

In this case, relay will transmit the entire message. Destination will decode the first level alone if the following conditions are satisfied.

$$R_1 < I_{I_\beta}^{(1-1)} \ \& \ R_2 > I_{I_\beta}^{(1-2|1)} \tag{136}$$

Where $I_{I_\beta}^{(1-1)}$ and $I_{I_\beta}^{(1-2|1)}$ are given by 4.125 and 4.128 respectively. The condition for the destination to decode the second level is given by:

$$R_1 < I_{I_\beta}^{(1-1)} \ \& \ R_2 < I_{I_\beta}^{(1-2|1)} \tag{137}$$

Thus the average throughput when relay transmits both the levels is given by:

$$\widehat{R_{I_\beta}^{(1)}} = R_1 \cdot Pr \begin{bmatrix} R_1 < I_{I_\beta}^{(1-1)} \& \\ R_2 > I_{I_\beta}^{(1-2|1)} \end{bmatrix} + R \cdot Pr \begin{bmatrix} R_1 < I_{I_\beta}^{(1-1)} \& \\ R_2 < I_{I_\beta}^{(1-2|1)} \end{bmatrix} \tag{138}$$

Case 2

When relay transmits only the first level to destination, we have already proved in Proposition 6 that destination will perform successive interference cancelation. Hence the average throughput for this case is given by 4.141 i.e. $\widehat{R_{I_\beta}^{(2)}} = \widetilde{R_{I_\beta}^{(2)}}$

Case 3

When relay was unable to transmit any data to destination, then decoding at destination will be equivalent to that of direct transmission. Hence the average throughput is given by 4.54 i.e. $\widehat{R_{I_\beta}^{(3)}} = \widetilde{R_{I_\beta}^{(3)}}$.

Thus the total average throughput when relay uses repetition coding is given by:

$$\widehat{R_{I_\beta}} = P^{(1)}\widehat{R_{I_\beta}^{(1)}} + P^{(2)}\widehat{R_{I_\beta}^{(2)}} + P^{(3)}\widehat{R_{I_\beta}^{(3)}} \tag{139}$$

where $P^{(1)}, P^{(2)}, P^{(3)}$ are given by 4.11, 4.14 and 4.17 respectively. We need to calculate the maximum average throughput over the values of h_1^*, h_2^* and α given the condition $h_1^* < h_2^*$.

The maximum throughput and the corresponding optimal values of h_1^*, h_2^* and α can be attained by use of numerical methods. The comparison curve between the repetition and independent coding is given in Figure 5.8.

Vita

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