Coloring Graphs Drawn with Crossings

Daniel Allen Guillot
Louisiana State University and Agricultural and Mechanical College, dguil16@math.lsu.edu

COLORING GRAPHS DRAWN WITH CROSSINGS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
Daniel A. Guillot
B.S., Louisiana State University, 2008
M.S., Louisiana State University, 2010
August 2014

Acknowledgments

First and foremost, I must thank my advisor, Bogdan Oporowski, for his support and guidance throughout my studies as a graduate student. His insight has been invaluable, his ability to know where to find answers to my questions has been uncanny, and his patience has been incredible. I would also like to thank the LSU Department of Mathematics faculty and staff for providing an outstanding environment to work and learn in. Many thanks to my fellow graduate students and friends who have been a great source of support and encouragement throughout the years. Though this list is far from exhaustive, I would especially like to thank Derek, Dennis, Adam, Irina, Amber, Joel, Laura, Jesse, Tyler, Simon, and Kimberly.

Thanks to Will Dickinson and the rest of the faculty in the GVSU REU program. My experience at Grand Valley provided a great foundation to build my graduate research upon, and it was one of the things that helped keep me going when things got rough in the early years of my graduate studies. I am also grateful to the LSU LA-STEM Research Scholars Program, the HHMI Professors Program, and the great staff at OSI. Dr. Warner, Monica, Lisa, Karin, and Melissa all played a major role in setting me on the path to a graduate degree and giving me many of the tools necessary to succeed.

A big thank you to my parents, David and Midge, for their love and support throughout the years. They have always encouraged my studies and have provided me with many opportunities to grow and learn. Finally, thanks to my fiancé Luke for his support and patience throughout my studies.

Table of Contents

Acknowledgments ii

List of Tables iv

List of Figures v

Abstract vi

Chapter 1: Introduction 1
- 1.1 Basic Concepts 1
- 1.2 Discharging Method 3
- 1.3 Overview of Results 4

Chapter 2: Proper Colorings 7
- 2.1 Introduction 7
- 2.2 Proofs 8
- 2.3 Finding Minimal Counterexamples 13

Chapter 3: Cyclic Colorings 19
- 3.1 Introduction 19
- 3.2 Non-Planar Graphs 20
- 3.3 Relation to Proper Colorings 26

Chapter 4: Total Colorings 28
- 4.1 Introduction 28
- 4.2 Results 29

Chapter 5: List Colorings 31
- 5.1 Introduction 31

Chapter 6: Graphs Drawn on Generalized Pseudosurfaces 33

References 41

Vita 43

List of Tables

2.1 When Conjecture 2.11 Holds 15

6.1 Minimum Charge of $W(P)$ 37

List of Figures

2.1 The graph $W(\Sigma)$. 11

2.2 Identifying two vertices of $W(\Sigma)$, one of which is incident with the 4-face. . . 12

2.3 Identifying two vertices of $W(\Sigma)$ not incident with the 4-face. 13

3.1 A $3x3$ grid embedded in a torus. 20

3.2 A reduction of a face with two adjacent vertices of degree 4. 22

3.3 A reduction of a face with consecutive vertices of degree $4, 5, 4$. 23

3.4 A reduction of a face with consecutive vertices of degree $4, 5, 5$. 23

3.5 A reduction of a face with consecutive vertices of degree $5, 4, 5$. 24

3.6 A cluster of crossings and its neighbors. 27

3.7 η' applied to the cluster in Figure 3.6 27

Abstract

This dissertation will examine various results for graph colorings. It begins by introducing some basic graph theory concepts, focusing on those ideas relevant to graph embeddings, and by introducing terminology to allow a formal discussion of drawings of graphs. Chapter 2 focuses on results for proper colorings of graphs with good drawings, using a previous result from Král and Stacho [KS10] as inspiration. Chapter 3 expands on the ideas of Chapter 2 and focuses on cyclic colorings of embedded graphs. Chapters 5 and 6 examine results for total and list colorings, respectively, of drawings of graphs. Finally, Chapter 6 introduces generalized pseudosurfaces and examines results for proper and cyclic colorings of graphs embedded in generalized pseudosurfaces.

Chapter 1
Introduction

1.1 Basic Concepts

This dissertation details various ways of coloring graphs drawn on surfaces. Here, a *graph* G is a triple $G = (V, E, \mathcal{I})$ where V is a finite set of *vertices*, E is a finite set of *edges*, and the set of incidence relations $\mathcal{I}$ is a a subset of $V \times E$ in which each edge is in relation with exactly one or two vertices. Two vertices are *adjacent* or *neighbors* if they are in relation with the same edge. The number of edges incident with a vertex v of G is its *degree* and is denoted by $d_G(v)$ or $d(v)$. For this dissertation, the set of vertices and edges of a graph G will be denoted by $V(G)$ and $E(G)$, respectively, or, simply by V and E if no ambiguities ensue. A graph H is a *subgraph* of G if the vertices of H are a subset of the vertices of G, the edges of H are a subset of the edges of G, and the incidence relations of H are induced by the incidence relations of G. A graph H is an *induced subgraph* of G if $E(H)$ consists of all of the elements of $E(G)$ whose incident vertices are in $V(H)$.

Given two vertices u and v an edge between u and v is denoted by uv, when no ambiguities arise. An edge incident with only one vertex is a *loop.* If $e_1 = uv$ and $e_2 = uv$ are two different edges between the same vertices, then e_1 and e_2 are said to be *parallel.* A graph without loops or parallel edges is *simple.* A *walk* in a graph is a sequence $v_0e_0v_1e_1 \dots e_{n-1}v_n$ where $e_i = v_iv_{i+1}$. Such a walk has *length n.* A *path* is a walk in which no vertices or edges repeat. A *cycle* is a walk in which $v_0 = v_n$ and no other elements are repeated. A graph is *connected* if there is a path between any two vertices of the graph. The *distance* between two vertices is the number of edges in the shortest path between them. In this dissertation, only loopless, connected graphs are considered. A *k-coloring* of a graph G is a labeling $f : V(G) \to \{1, 2, \dots, k\}$. The labels are called *colors.*

A *surface* is a compact Hausdorff space that is locally homeomorphic to the unit disk in $\mathbb{R}^2$. The following is a well-known result from topology. Any surface is homeomorphic to a sphere, a sphere with a finite number k of handles added (an orientable surface), or a sphere with a finite number n of crosscaps added (a nonorientable surface). The *genus* of a surface Σ is given by k if Σ is orientable or by n if Σ is nonorientable. The *Euler characteristic* of Σ is given by $\varepsilon(\Sigma) = 2 - 2k$ if Σ is orientable or $\varepsilon(\Sigma) = 2 - n$ if Σ is nonorientable.

Below is a brief review of the essential concepts of graph embeddings in surfaces, but the reader is referred to [MT01] for details and deeper insights. The notation and terminology of [MT01] will be extended to allow discussion of graphs drawn on surfaces. The formal description of graph drawings conforms to the usual conventions: edges may cross, but no edge may cross a vertex, and no three edges may cross at the same point.

An embedding of a graph G in a surface Σ is described through an *embedding scheme*, which consists of a set $\pi = \{\pi_v \mid v \in V(G)\}$ where π_v is a cyclic permutation of the edges incident with v, and a set $\lambda = \{\lambda_e \mid e \in E(G)\}$ where $\lambda_e \in \{-1, 1\}$ is the signature of the edge e. Let $W = v_1 e_1 v_2 e_2 \dots v_k e_k v_1$ be a closed walk, determined by the first edge $e_1 = v_1 v_2$, with the requirement that for each i the following holds: $\pi_{v_i}(e_i) = e_{i+1}$ if an even number of edges with signature -1 have been traversed, or $\pi_{v_i}^{-1}(e_i) = e_{i+1}$ if an odd number of edges with signature -1. Here $e_{k+1} = e_1$. Such a walk is called a π-*facial walk*. This, in particular, implies that this dissertation will only consider 2-cell embeddings, that is, embeddings in which all faces are homeomorphic to open disks. Given an embedding of a graph, the surface it is embedded in can be constructed by gluing a disk in each face. The Euler characteristic of an embedded graph is given by $\varepsilon(G) = |V(G)| - |E(G)| + |F(G)|$, where $F(G)$ denotes the set of faces of G. Note that the Euler characteristics of an embedded graph and its associated surface are equal. A proof of this can be found in [MT01].

Suppose G is a graph embedded in Σ and X is a subset of $V(G)$ such that every element of X has degree 4 in G. Two edges incident with a vertex x in X are x-*opposite* if they are non-consecutive in the cyclic permutation π_x. A cycle C of G is an X*-cycle* if for every element x of $V(C) \cap X$, it contains x-opposite edges. The pair (G, X) is a *drawing* on Σ if every X-cycle of G contains at least one vertex not in X, and every pair of x-opposite edges have the same signature. Suppose (G, X) is a drawing on Σ, a vertex x is in X, and the edges xv_1 and xv_3 are x-opposite, as are xv_2 and xv_4. The operation of *crossing edges at* x consists of deleting x and adding edges v_1v_3 and v_2v_4. Let G_X denote the graph obtained from G by repeatedly crossing edges at elements of X, until no elements of X remain. Then (G, X) is called a *drawing* of G_X on Σ; the elements of X are the *crossings* of G_X; and the edges of G_X that are not edges of G are the *crossed edges*. Two crossings of G_X are *independent* if their distance in G is at least 3, and are *dependent* otherwise. A drawing (G, X) of G_X is *good* if the crossings are all pairwise independent.

Note that if an X-cycle of G has only one vertex outside of X, the graph G_X has a loop. Since loops are not considered in this dissertation, the only drawings considered are those in which each X-cycle has at least two vertices not in X.

1.2 Discharging Method

Some of the results of this dissertation depend on the discharging method or discharging technique. This method was used to prove the Four Color Theorem [AH77] [AHK77], and is useful for many other problems involving graph embeddings. A brief overview of this technique is provided here for the reader's convenience.

A *charge* is a real number assigned to an element of a graph. Often, a charge is assigned to the vertices, edges, faces, or some combination of the previous three elements. Start by assigning certain elements an initial charge. Then, compute the total charge assigned to all of the elements of the graph. This is usually done using some global characteristic of the graph, often the Euler characteristic. Next, perform a *discharge*: move the charge between elements of the graph according to a list of rules that preserve the total charge of the graph.

Finally, compute the total charge of the graph a second time, using a different means than the first time. Often this is done using particular structures in the graph. Compare the two values of the total charge and draw some conclusions.

As an example, a simple discharging argument can be used to prove the following.

Proposition 1.1. *If G is plane triangulation with minimum degree at least 5, then G has a vertex of degree 5 adjacent to either another vertex of degree 5 or one of degree 6.*

Proof. Note that since G is a triangulation, then G is connected. Suppose that G is a counterexample to Proposition 1.1, that is, every vertex adjacent to a vertex of degree 5 has degree at least 7. Give each vertex v an initial charge $c(v) = d(v) - 6$ and each face f an initial charge $c(f) = 2(d^*(f) - 3)$. Observe that $c(f) = 0$ for every face, since G is a triangulation. The total charge of the graph is

$$\sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in F(G)} (2(d^*(f) - 3) = 2|E| - 6|V| + 2(4|E| - 6|F|),$$

which can be rewritten as $-6(|V| - |E| + |F|) = (-6)(2) = -12$. Perform discharging of G according to the following rule: every vertex of degree 5 sends a charge of $\frac{-1}{5}$ to each of its neighbors.

If $d(v) = 5$, then the new charge of v is $c'(v) = 0$, since no two vertices of degree 5 are adjacent to each other. If $d(v) = 6$, then $c'(v) = 0$, since $c(v) = 0$ and v is not incident to a vertex of degree 5. If $d(v) \geq 7$, then $c'(v) \geq \frac{-1}{5}\lfloor\frac{d(v)}{2}\rfloor + d(v) - 6 \geq 0$. Every vertex and every face has charge at least 0 and, thus the total charge of G is at least 0, which is a contradiction. Thus, no counterexample to Proposition 1.1 exists. $\square$

1.3 Overview of Results

The following is an overview of the major results of this dissertation. Note that relevant terminology will be introduced in later chapters. Proper colorings of graphs with good drawings are examined in Chapter 2. This dissertation will show that if a graph G has a good drawing on a surface Σ, other than the sphere, then it is $(h(\Sigma)+1)$-colorable. Moreover, if G does not contain $K_{h(\Sigma)+1}$ as a subgraph, then it is $h(\Sigma)$-colorable. This chapter concludes by examining when $K_{h(\Sigma)+1}$ has an embedding in Σ.

In Chapter 3 cyclic colorings of graphs embedded in surfaces are discussed. Using the techniques of Azarija et al. [AEK$^+$12] this dissertation proves that if G is a graph with an embedding with face-width at least 3 and maximum facial degree Δ^* on a surface Σ, with $\varepsilon(\Sigma) < 0$, in which all faces of degree greater than 3 are at least distance 3 apart, then G can be cyclically colored with $\max\{h(\Sigma)+1, \Delta^*+1\}$ colors. There is also a brief discussion of the relationship between proper colorings of graphs drawn with crossings and cyclic colorings of embedded graphs.

Chapter 4 investigates total colorings of graphs drawn with crossings. Behzad [Beh65] and Vizing [Viz68] conjectured that for a graph G the total chromatic number of G, denoted $\chi_t(G)$, is bounded above by $\Delta + 2$. This dissertation will show, using a result of Jendrol and Voss [JV00], that this conjecture is true for a graph embedded in a surface Σ with $\varepsilon(\Sigma) < 0$ and $\Delta(G) \geq 4h(\Sigma) - 5$. This result is also extended to graphs with good drawings, showing that the conjecture holds for graphs with good drawings on Σ and $\Delta(G) \geq 4h(\Sigma) - 4$.

List colorings of graphs drawn with crossings are discussed in Chapter 5. This dissertation shows that if G is a planar graph in which each pair of crossings are at least distance 5 apart, then G is 6-choosable. It also shows that if G has exactly one crossing, then G is 5-choosable. Using a result of Dirac [Dir56] and Ringel [Rin55] it is shown that if G is a graph drawn on a surface Σ other than the sphere with each pair of crossings at least distance 5 apart, and G does not contain $K_{h(\Sigma)+1}$, then G is $h(\Sigma)$-choosable.

Finally, in Chapter 6 results for graphs drawn on generalized pseudosurfaces are discussed. An overview of generalized pseudosurfaces is presented, following the notation and terminology of Heidema [Hei78]. Using the Euler characteristic of generalized pseudosurfaces, results similar to those for proper colorings and cyclic colorings are established for graphs embedded in generalized pseudosurfaces. Regarding proper colorings, it is shown that a graph embedded in a generalized pseudosurface P with no component having Euler characteristic greater than -1 has chromatic number at most $h(P)$, unless G contains $K_{h(P)+1}$ as a subgraph. For cyclic colorings, the results from Chapter 3 are extended directly to generalized pseudosurfaces.

Chapter 2
Proper Colorings

2.1 Introduction

A *proper coloring* of a graph G is a coloring of the vertices of G such that adjacent vertices receive distinct colors. If G has a proper k-coloring, then G is *k-colorable*. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest number of colors necessary to properly color G. For a surface Σ, its *Heawood number* $h(\Sigma)$ is defined as

$$h(\Sigma) = \left\lfloor \frac{7 + \sqrt{49 - 24\varepsilon(\Sigma)}}{2} \right\rfloor,$$

where $\varepsilon(\Sigma)$ is the Euler characteristic of Σ.

The following is a well-known result.

Theorem 2.1. *If G is a graph embedded in a surface Σ, then $\chi(G) \leq h(\Sigma)$.*

The proof of this result came in two parts: in 1890, Heawood [Hea90] proved it for all surfaces other than the sphere, while the proof for the sphere was published in 1977 by Appel, Haken, and Koch [AH77, AHK77], as a solution of the celebrated Four Color Problem. Ringel [Rin74] (jointly with Youngs) later showed that, given a surface Σ that is not a Klein Bottle, the complete graph $K_{h(\Sigma)}$ can be embedded in Σ, thereby proving that the bound in Theorem 2.1 is tight for all such Σ. Moreover, if Σ is a Klein Bottle, then no graph embedded in Σ requires more than 6 colors, while $h(\Sigma) = 7$.

Král and Stacho [KS10] showed the following:

Theorem 2.2. *If a graph has a good drawing on the sphere, then it is 5-colorable.*

The following result is a direct extension of Theorem 2.2 to all other surfaces.

Theorem 2.3. *If a graph has a good drawing on a surface Σ, other than the sphere, then it is $(h(\Sigma) + 1)$-colorable.*

The following, similar, theorem is the main result of this chapter.

Theorem 2.4. *If a graph G has a good drawing on a surface Σ, other than the sphere, and G does not contain $K_{h(\Sigma)+1}$ as a subgraph, then $\chi(G) \leq h(\Sigma)$.*

Proofs of Theorems 2.3 and 2.4 appear in Section 2.2. The remainder of this section contains the terminology and notation that will be used further in the dissertation.

A graph is a *minimal counterexample* to a proposition if it fails the proposition, but every graph on fewer vertices satisfies it. An *n-face* is a face of an embedding whose facial-walk has length n. Such a face f has degree $d^*(f) = n$. Also, while parallel edges may be allowed, for this dissertation it is required that all embeddings have a minimum facial degree of 3. A graph embedded in a surface is a 3-4-*tiling* if each face has degree three or four, and a 3-4-tiling is *good* if no vertex is incident with two 4-faces.

2.2 Proofs

The following theorem of Heawood [Hea90] will be useful in proving Theorem 2.4. It is well known, but a proof is included for the reader's convenience.

Proposition 2.5. *If G is a graph with an embedding in a surface Σ that is not the sphere and if the minimum facial degree of G is at least 3, then G has a vertex of degree at most $h(\Sigma)-1$.*

Proof. Let $\varepsilon = \varepsilon(\Sigma)$, and let δ denote the minimum degree of G. From the definition of the Euler characteristic and the fact that every face of G has degree at least 3, it follows that $2|E| \geq -3|V| + 3|E| + 3\varepsilon$, which be rewritten as $|E| \leq 3|V| - 3\varepsilon$. Clearly, $\delta|V| \leq 2|E|$, and so $\delta|V| \leq 6|V| - 6\varepsilon$, thus $(\delta - 6)|V| \leq -6\varepsilon$. If $\varepsilon > 0$, then $\delta < 6$, and hence the result holds for the projective plane. For a surface that is neither the sphere nor the projective plane, $\delta - 6$ is non-negative, and so $|V| \geq \delta + 1$ implies that

$$(\delta - 6)(\delta + 1) \leq -6\varepsilon(\Sigma).$$

Rewriting, this becomes

$$\delta^2 - 5\delta - 6 + 6\varepsilon(\Sigma) \leq 0,$$

which, when solved for δ, implies that

$$\delta \leq \frac{5 + \sqrt{49 - 24\varepsilon(\Sigma)}}{2},$$

which means that

$$\delta \leq h(\Sigma) - 1.$$

□

Proposition 2.5 yields a simple proof of Theorem 2.3.

Proof of Theorem 2.3. Suppose G is a minimal counterexample to Theorem 2.3. Note that G has a vertex v of degree at most $h(\Sigma)$, since there is a matching in G that can be deleted to obtain a graph embedded in Σ, which by Proposition 2.5 has minimum degree at most $h(\Sigma) - 1$. Now consider the graph $G - v$. Since G is a minimal counterexample, $G - v$ has an $(h(\Sigma) + 1)$-coloring. This coloring can be extended to an $(h(\Sigma) + 1)$-coloring of G, since $d(v) \leq h(\Sigma) - 1$. Hence no minimal counterexample exists. □

The main goal of this section is to prove Theorem 2.4. This theorem will be reformulated slightly, to utilize various results regarding graph embeddings. Given a drawing (G, X) of a graph G_X on a surface Σ, define $\eta(G)$ to be the graph obtained from G by first adding edges so that the resulting graph is a triangulation (all faces have degree 3) of Σ in which all elements of X retain degree 4, and then deleting X. Then, clearly, $\eta(G)$ is a 3-4-tiling whose 4-faces correspond to the crossings of G_X, and if the drawing (G, X) is good, then so is the 3-4-tiling $\eta(G)$. Note that this may create parallel edges when forming $\eta(G)$, depending on the original drawing of G_X, but such edges do not affect the chromatic number. Given a 3-4-tiling G of a surface Σ, let $\zeta(G)$ be the drawing on Σ obtained by adding a crossing in each 4-face of G. Given an embedding of a graph G, a *cyclic coloring* of G is a coloring of the vertices of G such that any two vertices incident with the same face receive distinct colors. Note that this means that this dissertation does not consider embeddings in which a vertex occurs multiple times in a facial walk. If $\eta(G)$ can be cyclically $h(\Sigma)$-colored, then the original graph G can be properly $h(\Sigma)$-colored. Thus, the following theorem has Theorem 2.4 as an immediate corollary.

Theorem 2.6. *Suppose that G is a good 3-4-tiling of a surface Σ other than the sphere. Then, either G is cyclically $h(\Sigma)$-colorable or $\zeta(G)$ contains $K_{h(\Sigma)+1}$ as a subgraph.*

Theorem 2.6 will be proven by examining the following strengthening of it.

Conjecture 2.7. *Suppose that G is a good 3-4-tiling of a surface Σ other than the sphere. Then G is cyclically $h(\Sigma)$-colorable.*

The following are structural results for any possible minimal counterexamples to Conjecture 2.7. Given an embedding of a graph G, *triangulate* G by adding edges to G until every face is a 3-face. Proposition 2.5 implies that a graph G embedded in a surface Σ other than the sphere has a vertex of degree at most $h(\Sigma) - 1$.

Proposition 2.8. *Suppose G is a minimal counterexample to Conjecture 2.7. Then any vertex v of G with $d(v) \leq h(\Sigma) - 1$ is incident with a 4-face.*

Proof. Suppose G is a minimal counterexample to Conjecture 2.7 and v is a vertex such that $d(v) \leq h(\Sigma) - 1$ and v is not incident with a 4-face. Construct a new embedded graph G' from the embedding of G by deleting v and triangulating the resulting face. It follows from the minimality of G that G' is cyclically $h(\Sigma)$-colorable. The degree of v in G is at most $h(\Sigma) - 1$, so clearly the coloring of G' can be extended to a coloring of G by selecting a color for v that is not among the colors of its neighbors. □

Thus, Propositions 2.5 and 2.8 imply that any minimal counterexample to Conjecture 2.7 requires a vertex v incident with a 4-face such that $d(v) \leq h(\Sigma) - 1$.

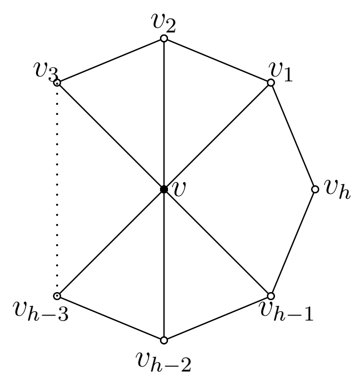

Figure 2.1: The graph $W(\Sigma)$.

Proposition 2.9. *If G is a minimal counterexample to Conjecture 2.7, then every vertex incident with a 4-face has degree at least $h(\Sigma) - 1$.*

Proof. Suppose G is a minimal counterexample to Conjecture 2.7 and v is a vertex such that $d(v) \leq h(\Sigma) - 2$, and v is incident with a 4-face. Construct a new graph G' from G by deleting v and triangulating the resulting face in such a way that the remaining three vertices of the 4-face incident with v are in the same 3-face. Note that G' satisfies the assumptions of Conjecture 2.7. Since G is a minimal counterexample, G' is cyclically $h(\Sigma)$-colorable. The degree of v in G is at most $h(\Sigma) - 2$, so the coloring of G' can be extended to a coloring of G by selecting a color for v that is not among the colors of its neighbors nor the color of the remaining vertex incident with the same 4-face as v. $\square$

Thus far, Propositions 2.5, 2.8 and 2.9 imply that any minimal counterexample to Conjecture 2.7 must have a vertex of degree $h(\Sigma) - 1$ incident with a 4-face. A *k-wheel* consists of a k-cycle, called the rim, and one other vertex, called the hub, with each vertex of the rim adjacent to the hub. Consider the graph obtained from an $h(\Sigma)$-wheel by deleting exactly one edge between the hub and rim (Figure 2.1). Such a graph is called $W(\Sigma)$. A graph G *contains $W(\Sigma)$ as a configuration* if G contains $W(\Sigma)$ as a subgraph and v and v_h are also nonadjacent in G.

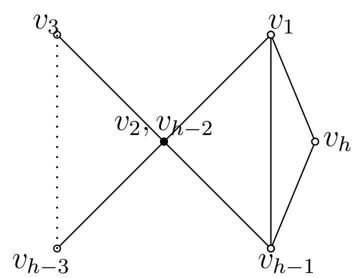

Figure 2.2: Identifying two vertices of $W(\Sigma)$, one of which is incident with the 4-face.

Proposition 2.10. *If G is a minimal counterexample to Conjecture 2.7 containing $W(\Sigma)$ (Figure 2.1) as a configuration, then the vertices v_i and v_j are incident with the same face for each pair of distinct i and j in $\{1, 2, \ldots, h\}$, where $h = h(\Sigma)$.*

Proof. Suppose G is a minimal counterexample to Conjecture 2.7 and suppose that the vertices v_i and v_j of G are not incident with the same face. First, suppose that neither vertex is v_h. Consider the graph G' obtained from G by identifying the vertices v_i and v_j, deleting v, deleting one edge from any new pair of parallel edges, and triangulating the resulting faces in such a way that v_1, v_{h-1}, and v_h are incident with the same face (Figure 2.2). The minimality of G implies that G' is cyclically $h(\Sigma)$-colorable.

Now instead suppose that the vertices v_i and v_h are not incident with the same face. Construct the graph G' from G as before, by deleting v, identifying v_i and v_h, and triangulating the resulting faces (Figure 2.3). The minimality of G implies that G' is cyclically $h(\Sigma)$-colorable.

In both cases, the $h(\Sigma)$ cyclic coloring of G' can be extended to one for G. The vertices on the rim of $W(\Sigma)$ only $h(\Sigma) - 1$ colors, so v can be colored with the remaining color. $\square$

Next is the proof of Theorem 2.6.

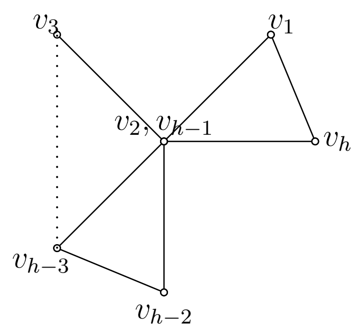

Figure 2.3: Identifying two vertices of $W(\Sigma)$ not incident with the 4-face.

Proof of Theorem 2.6. Suppose G is a minimal counterexample to Theorem 2.6. Then G is also a minimal counterexample to Conjecture 2.7. Proposition 2.5 implies that G has a vertex v of degree at most $h(\Sigma) - 1$, which Proposition 2.8 implies must be incident with a 4-face, and which, by Proposition 2.9, has degree exactly $h(\Sigma) - 1$. Since G is a good 3-4-tiling, each other face incident with v has degree 3, so G contains $W(\Sigma)$ as a configuration. Proposition 2.10 implies that each pair of vertices in the rim of $W(\Sigma)$ are either adjacent or in the same 4-face, which implies that that $\zeta(G)$ contains $K_{h(\Sigma)+1}$ as a subgraph; which is a contradiction. □

2.3 Finding Minimal Counterexamples

A natural next question is: when does $K_{h(\Sigma)+1}$ have a good drawing on Σ? Consider the following equivalent formulation of Conjecture 2.7.

Conjecture 2.11. *If a graph G has a good drawing on a surface Σ other than the sphere, then $\chi(G) \leq h(\Sigma)$.*

Note that Conjecture 2.11 implies Conjecture 2.7, which can be seen by applying η to G. Similarly, the converse can be seen by applying ζ.

In proving Theorem 2.6 it was shown that the only possible minimal counterexample to Conjecture 2.7 is $K_{h(\Sigma)+1}$. Define

$$\phi(\Sigma) = \frac{-2h(\Sigma)^2 + 12h(\Sigma) + 14}{12}.$$

Some easy calculations regarding the number of faces of a graph embedded in a given surface narrow down the potential counterexamples to Conjecture 2.11.

Proposition 2.12. *Given a surface Σ, if*

$$\varepsilon(\Sigma) > \phi(\Sigma),$$

then Conjecture 2.11 holds.

Proof. Suppose that $K_{h(\Sigma)+1}$ has a good drawing on Σ. Then there is a set A of at most $\frac{h(\Sigma)+1}{4}$ edges that can be deleted from $K_{h(\Sigma)+1}$ so that $G = K_{h(\Sigma)+1} \backslash A$ is a an embedding. This follows from the fact that the crossings are independent, so there are at most $\frac{h(\Sigma)+1}{4}$ crossings. Deleting one of the crossed edges from each pair of crossed edges yields a graph with an embedding on Σ. Euler's formula implies that the number of faces of G is

$$\varepsilon(\Sigma) + (h(\Sigma) + 1) - \left(\frac{h(\Sigma)^2 + h(\Sigma)}{2} - \frac{h(\Sigma) + 1}{4}\right) = \varepsilon(\Sigma) + \frac{2h(\Sigma)^2 - 3h(\Sigma) - 5}{4}.$$

Since each face of G has degree at least 3, then

$$|F(G)| \leq \frac{2|E(G)|}{3} = \frac{4h(\Sigma)^2 + 3h(\Sigma) - 1}{12}.$$

If

$$\frac{2|E(G)|}{3} < \varepsilon(\Sigma) + \frac{2h(\Sigma)^2 - 3h(\Sigma) - 5}{4},$$

then G cannot be embedded in Σ, since the number of edges does not support the required number of faces necessary to satisfy the Euler characteristic. This occurs when

$$\varepsilon(\Sigma) > \frac{-2h(\Sigma)^2 + 12h(\Sigma) + 14}{12} = \phi(\Sigma).$$

$\square$

Proposition 2.12 establishes some cases in which Conjecture 2.11 holds. For example, it holds for the surfaces Σ with Euler characteristic -3 and -5. Table 2.1 provides an analysis of surfaces Σ of Euler characteristic at least -80, their Heawood numbers, the value of $\phi(\Sigma)$, an evaluation of $\varepsilon(\Sigma) - \phi(\Sigma)$, and whether or not it is known if Conjecture 2.11 holds for Σ.

Table 2.1: When Conjecture 2.11 Holds

$\varepsilon(\Sigma)$	$h(\Sigma)$	$\phi(\Sigma)$	$\varepsilon(\Sigma) - \phi(\Sigma)$	Conjecture 2.11 Holds?
1	6	7/6	- 1/6	Unknown
0	7	0/1	0/1	Unknown
-1	7	0/1	- 1/1	Unknown
-2	8	- 3/2	- 1/2	Unknown
-3	9	- 10/3	1/3	Yes
-4	9	- 10/3	- 2/3	Unknown
-5	10	- 11/2	1/2	Yes
-6	10	- 11/2	- 1/2	Unknown
-7	10	- 11/2	- 3/2	Unknown
-8	11	- 8/1	0/1	Unknown
-9	11	- 8/1	- 1/1	Unknown
-10	12	- 65/6	5/6	Yes
-11	12	- 65/6	- 1/6	Unknown
-12	12	- 65/6	- 7/6	Unknown
-13	13	- 14/1	1/1	Yes
-14	13	- 14/1	0/1	Unknown
-15	13	- 14/1	- 1/1	Unknown
-16	13	- 14/1	- 2/1	Unknown

Continued on next page

Table 2.1 – *Continued from previous page*

$\varepsilon(\Sigma)$	$h(\Sigma)$	$\phi(\Sigma)$	$\varepsilon(\Sigma) - \phi(\Sigma)$	Conjecture 2.11 Holds?
-17	14	- 35/2	1/2	Yes
-18	14	- 35/2	- 1/2	Unknown
-19	14	- 35/2	- 3/2	Unknown
-20	15	- 64/3	4/3	Yes
-21	15	- 64/3	1/3	Yes
-22	15	- 64/3	- 2/3	Unknown
-23	15	- 64/3	- 5/3	Unknown
-24	16	- 51/2	3/2	Yes
-25	16	- 51/2	1/2	Yes
-26	16	- 51/2	- 1/2	Unknown
-27	16	- 51/2	- 3/2	Unknown
-28	16	- 51/2	- 5/2	Unknown
-29	17	- 30/1	1/1	Yes
-30	17	- 30/1	0/1	Unknown
-31	17	- 30/1	- 1/1	Unknown
-32	17	- 30/1	- 2/1	Unknown
-33	18	- 209/6	11/6	Yes
-34	18	- 209/6	5/6	Yes
-35	18	- 209/6	- 1/6	Unknown
-36	18	- 209/6	- 7/6	Unknown
-37	18	- 209/6	- 13/6	Unknown
-38	19	- 40/1	2/1	Yes
-39	19	- 40/1	1/1	Yes

Continued on next page

Table 2.1 – *Continued from previous page*

$\varepsilon(\Sigma)$	$h(\Sigma)$	$\phi(\Sigma)$	$\varepsilon(\Sigma) - \phi(\Sigma)$	Conjecture 2.11 Holds?
-40	19	- 40/1	0/1	Unknown
-41	19	- 40/1	- 1/1	Unknown
-42	19	- 40/1	- 2/1	Unknown
-43	19	- 40/1	- 3/1	Unknown
-44	20	- 91/2	3/2	Yes
-45	20	- 91/2	1/2	Yes
-46	20	- 91/2	- 1/2	Unknown
-47	20	- 91/2	- 3/2	Unknown
-48	20	- 91/2	- 5/2	Unknown
-49	21	- 154/3	7/3	Yes
-50	21	- 154/3	4/3	Yes
-51	21	- 154/3	1/3	Yes
-52	21	- 154/3	- 2/3	Unknown
-53	21	- 154/3	- 5/3	Unknown
-54	21	- 154/3	- 8/3	Unknown
-55	22	- 115/2	5/2	Yes
-56	22	- 115/2	3/2	Yes
-57	22	- 115/2	1/2	Yes
-58	22	- 115/2	- 1/2	Unknown
-59	22	- 115/2	- 3/2	Unknown
-60	22	- 115/2	- 5/2	Unknown
-61	22	- 115/2	- 7/2	Unknown
-62	23	- 64/1	2/1	Yes

Continued on next page

Table 2.1 – *Continued from previous page*

$\varepsilon(\Sigma)$	$h(\Sigma)$	$\phi(\Sigma)$	$\varepsilon(\Sigma) - \phi(\Sigma)$	Conjecture 2.11 Holds?
-63	23	- 64/1	1/1	Yes
-64	23	- 64/1	0/1	Unknown
-65	23	- 64/1	- 1/1	Unknown
-66	23	- 64/1	- 2/1	Unknown
-67	23	- 64/1	- 3/1	Unknown
-68	24	- 425/6	17/6	Yes
-69	24	- 425/6	11/6	Yes
-70	24	- 425/6	5/6	Yes
-71	24	- 425/6	- 1/6	Unknown
-72	24	- 425/6	- 7/6	Unknown
-73	24	- 425/6	- 13/6	Unknown
-74	24	- 425/6	- 19/6	Unknown
-75	25	- 78/1	3/1	Yes
-76	25	- 78/1	2/1	Yes
-77	25	- 78/1	1/1	Yes
-78	25	- 78/1	0/1	Unknown
-79	25	- 78/1	- 1/1	Unknown
-80	25	- 78/1	- 2/1	Unknown

Beyond this, trying to determine, in general, whether or not Conjecture 2.11 holds for a given surface becomes a problem similar to finding a minimum genus embedding of K_n, which was a notably difficult and lengthy problem to solve [Rin55].

Chapter 3
Cyclic Colorings

3.1 Introduction

The cyclic chromatic number relies, in part, on the size of the largest face of the embedding of G. The maximum facial degree of G is denoted by $\Delta^*(G)$ or simply Δ^*, and the cyclic chromatic number of a graph G will be denoted by $\chi_{\text{cyc}}(G)$ or χ_{cyc}. Note that if the embedding of G is a triangulation, then a cyclic coloring of G is equivalent to a proper coloring. Two distinct vertices are *cyclic neighbors* or are *cofacial* if they are incident with the same face. The *cyclic degree* of a vertex is the number of its cyclic neighbors.

Significant work has been done finding an upper bound for the cyclic chromatic number of planar graphs. Ore and Plummer [OP69] proved the following theorem for plane graphs.

Theorem 3.1. *Let G be a plane graph and let Δ^* be the maximum facial degree of G. Then $\chi_{\text{cyc}}(G) \leq 2\Delta^*$.*

For planar graphs, Borodin [Bor92] showed the following improvement over Theorem 3.1.

Theorem 3.2. *If G is a connected plane graph with maximum facial degree Δ^*, then $\chi_{\text{cyc}} \leq q(\Delta^*)$ for all $\Delta^* \geq 5$, where $q(5) = 9$, $q(6) = 11$, $q(7) = 12$, and $q(\Delta^*) = 2\Delta^* - 3$ for $\Delta^* \geq 8$.*

Note that if G is a plane triangulation, then $\chi_{\text{cyc}}(G) \leq 4$ is equivalent to the Four Color Theorem [AH77] [AHK77]. Borodin [Bor84] also showed that if G is a planar graph with $\Delta^*(G) = 4$, then $\chi_{\text{cyc}}(G) \leq 6$.

Azarija et al. [AEK$^+$12] proved the following theorem.

Theorem 3.3. *If G is a plane graph whose largest face has size Δ^* and all faces of size larger than 3 are pairwise independent, then G has a cyclic coloring with $\Delta^* + 1$ colors.*

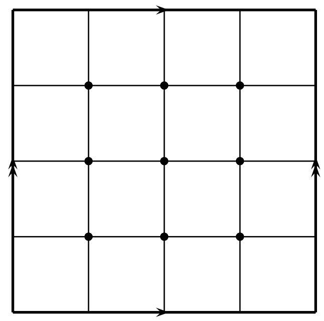

Figure 3.1: A $3x3$ grid embedded in a torus.

3.2 Non-Planar Graphs

A similar question can be asked about the cyclic chromatic number of graphs embedded in surfaces other than the sphere. For example, a three-by-three grid embedded in the natural way on a torus (Figure 3.1) has a cyclic chromatic number of 9, since each vertex is cofacial with every other vertex.

For a graph G embedded in a surface Σ with $\varepsilon(\Sigma) \leq 0$, a better upper bound on the cyclic chromatic number can be obtained, if restrictions on G similar to those in Theorem 3.3, along with an extra restriction on the face-width of G, are made. The *face-width* of an embedding of G is the smallest number of closed faces whose union contains a non-contractible cycle. The following lemma establishes an upper bound on the charge of a graph embedded in Σ in terms of $h(\Sigma)$.

Lemma 3.4. *Suppose Σ is a surface with $h(\Sigma) = h$. Then $-6\varepsilon(\Sigma) \leq h^2 - 5h - 8$*

Proof. By definition

$$h(\Sigma) \leq \frac{7 + \sqrt{49 - 24\varepsilon(\Sigma)}}{2} < h(\Sigma) + 1.$$

Rewriting yields

$$\frac{h^2 - 7h}{6} \leq -\varepsilon(\Sigma) < \frac{h^2 - 5h - 6}{6},$$

where $h = h(\Sigma)$. Since $-\varepsilon(\Sigma)$ must be an integer, then

$$-\varepsilon(\Sigma) \leq \frac{h^2 - 5h - 8}{6}.$$

□

The following theorem, which is similar to Theorem 3.3, establishes an upper bound on χ_{cyc} for a particular category of graphs.

Theorem 3.5. *Suppose a graph G has an embedding with face-width at least 3 and maximum facial degree Δ^* on a surface Σ with $\varepsilon(\Sigma) \leq 0$ such that all faces of size greater than 3 are at least distance 2 apart. Then G can be cyclically colored with $n = \max\{h(\Sigma) + 1, \Delta^* + 1\}$ colors.*

This is shown by first proving some structural results for any minimal counterexample to this theorem.

Lemma 3.6. *Suppose G is a minimal counterexample to Theorem 3.5. If f is a face incident with vertex v, then $d(v) + d^*(f) > n + 2$.*

Proof. Assume that G is a minimal counterexample and suppose that v is a vertex in one of the faces f of G with $d(v) + d^*(f) - 2 \leq n \leq \Delta^* + 1$. If f is a face of degree at least 4, then consider the graph G' obtained from G by deleting v. Observe that the face resulting from deleting v has degree $d(v) + d^*(f) - 3$. Since $d(v) + d^*(f) - 2 \leq \Delta^* + 1$, then $d(v) + d^*(f) - 3 \leq \Delta^*$. So G' is a smaller graph satisfying the assumptions of Theorem 3.5 and is cyclically n-colorable. This coloring can be extended to G by coloring v with the remaining color.

If f has degree 3, then $d(v) < n$. Consider the graph G' obtained from G by deleting v and triangulating the resulting face. Since G is minimal, G' has a cyclic n-coloring. Extend this coloring of G' to a cyclic n-coloring of G by coloring v with a color not among its neighbors. In either case, we obtain a contradiction. □

Lemma 3.6 has a few ramifications worth noting. First, any minimal counterexample to Theorem 3.5 has minimum vertex degree at least 4. If $\varepsilon(\Sigma) \leq 0$, then $h(\Sigma) \geq 7$, and hence $n \geq 8$, so this lemma also implies that in a minimal counterexample the smallest degree of a vertex in a 4-face is 7, and the smallest degree of a vertex in a 5-face is 6

Lemma 3.7 ([AEK+12])**.** *In a minimal counterexample to Theorem 3.5, no face of degree at least* 5 *contains two adjacent vertices of degree* 4*.*

Proof. Suppose G is a minimal counterexample to Theorem 3.5, that f is a face with $d^*(f) = l \geq 5$, and that f contains two adjacent vertices, v_1 and v_2, of degree 4. Let $v_1, \ldots, v_l$ be the vertices incident with f, listed in order and let u be the common neighbor of v_1 and v_l. Construct a new graph G' from G by deleting v_1 and v_2, adding an edge between v_l and v_3 and triangulating the face of degree 5 incident with u by adding edges incident with u (Figure 3.2). Since G is a minimal counterexample, G' has a cyclic coloring with n colors. Extend this coloring to G in the following way. Let a be the color assigned to u. If the color a is not assigned to any of the vertices $v_3, \ldots, v_l$, then color v_2 with a. Otherwise, color v_2 with any available color. Such a color exists since v_2 has at most $\Delta^* + 1$ cyclic neighbors, and the vertex v_1 has no color. Thus, there are at most Δ^* restrictions on the color of v_2. Note that u and at least one vertex of f are both colored a. Therefore there are also at most Δ^* restrictions on the color of v_1, so there is a cyclic n-coloring of G. ☐

Figure 3.2: A reduction of a face with two adjacent vertices of degree 4.

Lemma 3.8 ([AEK+12]). *In a minimal counterexample to Theorem 3.5, no face of degree at least* 5 *contains three consecutive vertices with degrees* $4,5,4$ *or* $4,5,5$ *or* $5,4,5$.

Proof. Suppose G is a minimal counterexample to Theorem 3.5, that f is a face of G with $d^*(f) = l$ and with vertices $v_1, \ldots, v_l$. Further suppose that v_1, v_2, v_3 have degrees $4,5,4$ or $4,5,5$ or $5,4,5$, respectively, and let u be the common neighbor of v_1 and v_2. Consider the graph G' obtained from G in the following way. Delete the vertices v_1, v_2, v_3, add an edge between v_l and v_4, and triangulate the new face incident with u with edges incident with u. Figures 3.3, 3.4, and 3.5 illustrate the reductions for each of these cases. The minimality of G implies that G' is cyclically n-colorable.

Figure 3.3: A reduction of a face with consecutive vertices of degree $4,5,4$.

Figure 3.4: A reduction of a face with consecutive vertices of degree $4,5,5$.

Extend the cyclic coloring of G' to a cyclic coloring of G as follows. Let a be the color assigned to u. If the color a is not assigned to any of the vertices $v_4, \ldots, v_l$, color v_3 with a. Otherwise, color v_3 with any available color. Such a color exists since the cyclic degree of v_3 is at most $\Delta^* + 2$ and two of its cyclic neighbors are uncolored. Next, color the remaining vertex of v_1 and v_2 whose degree is 5. Since this vertex has cyclic degree at most $\Delta^* + 2$ and

Figure 3.5: A reduction of a face with consecutive vertices of degree $5, 4, 5$.

two of its cyclic neighbors are colored a and one is uncolored, such a color exists. Finally, color the remaining vertex of degree 4 with a color not among its cyclic neighbors. Such a color exists since the cyclic degree of this vertex is at most $\Delta^* + 1$ and two of its cyclic neighbors are colored a. Thus G can be cyclically n-colored.

$\square$

Thus, any minimal counterexample to Theorem 3.5 has the following structure:

- If f is a face incident with vertex v, then $d(v) + d^*(f) > n + 2$ (Lemma 3.6).
- No face contains two adjacent vertices of degree 4 (Lemma 3.7).
- No face of degree at least 5 contains three consecutive vertices of degree $4, 5, 4$ or $4, 5, 5$ or $5, 4, 5$ (Lemma 3.8).

Given a set A of vertices, let $\nu(A)$ denote the set of neighbors of A. That is, $\nu(A)$ consists of the vertices of G not in A that are incident with a vertex in A.

Proof of Theorem 3.5. Suppose G is a minimal counterexample. Assign G a charge as follows: the charge of a vertex is $c(v) = d(v) - 6$ and the charge of a face is $c(f) = 2(d^*(f) - 3)$. The total charge of the graph is given by $-6(|V| - |E| + |F|) = -6\varepsilon(\Sigma)$. Given the structure G from Lemmas 3.6, 3.7, and 3.8, note the following. Any vertex of degree 4 or 5 must be incident with a face of degree at least 6. This implies that any vertex not incident with a face of degree greater than 3 contributes a charge of at least 0. If f is a face of degree 4 or 5, then all of the vertices incident with f have charge at least 0, so f and its incident vertices contribute a charge of at least 0. If f is a face of degree at least 6, then the sum of the charge of f and its incident vertices is at least 0, since this total charge is at least $-d^*(f) + 2(d^*(f) - 3) = d^*(f) - 6$. Since the face-width of G is at least 3, the minimum degree is at least 4, the faces of degree greater than 4 are distance at least 2 apart, and the largest face has a vertex of degree at least 6, then G contains at least $\Delta^* + 4$ vertices of degree at least $h(\Sigma) - 1$.

First, consider the case when $\Delta^* + 1 > h(\Sigma)$. In this case G contains at least $h(\Sigma) + 3$ vertices of degree at least $h(\Sigma) - 1$. These have a total charge of

$$(h(\Sigma) - 1)(h(\Sigma) + 3) = h(\Sigma)^2 + 2h(\Sigma) - 3,$$

which is clearly greater than the $h(\Sigma)^2 - 5h(\Sigma) - 8$ upper bound of the charge associated with any graph embedded in Σ (Lemma 3.4). The remaining faces of degree greater than 4 and their incident vertices have charge at least 0, and any vertices incident with only 3-face contribute a charge of at least 0. This implies that the charge of G is too large, and G cannot be embedded in Σ. In this case there is a contradiction.

It remains to examine the case when $n = h(\Sigma) + 1$. Again, observe that each face and its vertices together contribute a non-negative charge to the graph. Suppose f is a face of maximum degree and its set incident vertices is A. The vertices of A must have degree at least $h(\Sigma) - \Delta^* + 3$ (Lemma 3.6). Any vertex in $\nu(A)$ must have degree at least $h(\Sigma) + 1$ and a minimum charge of $\Delta^*(h(\Sigma) - \Delta^* + 3)(h(\Sigma) - 5)$. Further recall that the charge of G is bounded above by $h(\Sigma)^2 - 5h(\Sigma) - 6$. If it can be shown that

$$\Delta^*(h(\Sigma) - \Delta^* + 3)(h(\Sigma) - 5) - h(\Sigma)^2 + 5h(\Sigma) + 6 > 0, \tag{3.1}$$

then this implies that the charge of the graph is too large, and no minimal counterexample exists.

Rewrite the left hand side of Inequality 3.1 to obtain the following:

$$(5 - h(\Sigma))(\Delta^*)^2 + (h(\Sigma)^2 - 2h(\Sigma) - 15)\Delta^* - (h(\Sigma)^2 - 5h(\Sigma) - 6). \tag{3.2}$$

Note that for a fixed $h(\Sigma)$, this is a quadratic in Δ^* with $5 \leq \Delta^* \leq h(\Sigma) - 1$. It remains to show that this is positive when $\Delta^* = 5$ and when $\Delta^* = h(\Sigma) - 1$.

When $\Delta^* = 5$, Equation 3.2 becomes

$$4h(\Sigma)^2 - 30h(\Sigma) + 56,$$

which is positive when $h(\Sigma) \geq 7$. If $\Delta^* = h(\Sigma) - 1$, then

$$3h(\Sigma)^2 - 19h(\Sigma) + 26,$$

which is also positive when $h(\Sigma) \geq 7$. Thus, the minimum charge of G always exceeds the charge of a graph on Σ, and no minimal counterexample exists. □

3.3 Relation to Proper Colorings

There is a natural relationship between coloring graphs drawn with crossings and the cyclic colorings of embedded graphs. Given a drawing (G, X) on a surface Σ, a *cluster* C is a maximal subset of X such that the subgraph of G induced by C is connected.

Given a cluster C, define the degree of C to be $d_*(C) = |\nu(C)|$. The maximum size of a cluster in a drawing of G is denoted $\Delta_*(G)$. Given a drawing of a graph G on a surface Σ, define $\eta'(G)$ to be the graph embedded in Σ obtained in the following way. For each cluster C in the drawing of G, delete C. Next, add edges so that $\nu(C)$ is a facial cycle. In Figures 3.6 and 3.7, the black vertices correspond to a cluster C, while the white vertices correspond to $\nu(C)$. Figure 3.7 shows these first two steps performed on the cluster in Figure 3.6. Finally, add edges until each face other than those that correspond to clusters are all 3-faces.

Theorem 3.9. *Suppose G has a drawing on a surface Σ with $\varepsilon(\Sigma) \leq 0$ such that the drawing has following properties:*

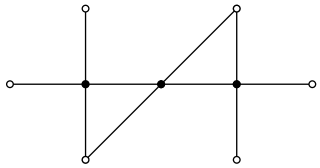

Figure 3.6: A cluster of crossings and its neighbors.

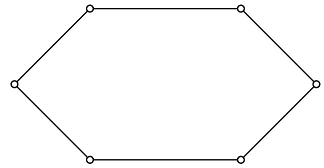

Figure 3.7: η' applied to the cluster in Figure 3.6

1. *Each pair of clusters is at least distance* 5 *apart.*

2. *Given a cluster C, the subgraph induced by $C \cup \nu(C)$ contains only contractible cycles.*

Then $\chi(G) \leq \max\{h(\Sigma) + 1,\, \Delta_* + 1\}$.

Proof. Consider the embedding $\eta'(G)$. The clusters of G correspond to faces of size greater than 3 in $\eta'(G)$. Since the clusters of G are at least distance 5 apart, then the faces of size greater than 3 are at least distance 3 apart. In G, the graph induced by $C \cup \nu(C)$ contains no non-contractible cycles, so $\eta'(G)$ has face-width 3. Now apply Theorem 3.5 to $\eta'(G)$. This cyclic coloring extends to a proper coloring of G. $\square$

Chapter 4
Total Colorings

4.1 Introduction

A *total coloring* of a graph G is a coloring of $V(G) \cup E(G)$ such that any two adjacent or incident elements receive different colors. The total chromatic number of G is denoted by $\chi_{\text{tot}}(G)$ or χ_{tot}.

Behzad [Beh65] and Vizing [Viz68] independently conjectured that $\chi_{\text{tot}}(G)$ is bounded above by $\Delta(G)+2$, where $\Delta(G)$ denotes the maximum degree of G. For $\Delta(G) = 3$, Rosenfield [Ros71] and Vijayaditya [Vij71] showed that the conjecture holds. Kostochka [Kos77] proved the conjecture for $\Delta(G) \leq 5$. Borodin [Bor89] showed that for planar graphs the conjecture is true for $\Delta(G) \notin \{6, 7, 8\}$. Zhao [Zha99] showed that for a graph G embeddable in a surface Σ with $\varepsilon(\Sigma) \geq 0$ and $\Delta(G) \geq 8$ the conjecture holds, as well as for the case when $\varepsilon(\Sigma) \leq 0$ and $\Delta(G) \geq \frac{20}{9}(3 - \varepsilon(\Sigma)) + 1$.

Jendrol and Voss [JV00] proved the following. Here, a *k-path* denotes a path through k vertices.

Theorem 4.1. *Suppose Σ is a surface with $\varepsilon(\Sigma) < 0$. Every graph G embedded in Σ that has a k-path contains a k-path such that each vertex has degree at most $k\left\lfloor \frac{5+\sqrt{49-24\varepsilon(\Sigma)}}{2} \right\rfloor = k(h(\Sigma) - 1)$.*

The main result of this chapter relies on applying this theorem to a 2-path, or edge. Thus, consider in following Corollary to Theorem 4.1.

Corollary 4.2. *Given a graph G embedded in a surface Σ with $\varepsilon(\Sigma) < 0$, there are two adjacent vertices each with degree at most $4(h(\Sigma) - 1)$.*

Proof. Note that since G is embedded in Σ and $\varepsilon(\Sigma) < 0$, then $|E(G)| > 0$, and hence G contains an edge. Theorem 4.1 implies that G contains an edge e such that each vertex incident with it has degree at most $2(h(\Sigma) - 1)$. $\square$

4.2 Results

The proof of the following theorem utilizes Corollary 4.2.

Theorem 4.3. *Suppose G is a graph embedded in a surface Σ such that $\varepsilon(\Sigma) < 0$ and $\Delta(G) \geq 4h(\Sigma) - 5$. Then $\chi_{\text{tot}}(G) = \Delta(G) + 1$.*

Proof. The proof follows the techniques of Borodin [Bor89]. Suppose G is a minimal counterexample with respect to $|E(G)|$. Corollary 4.2 implies that G has an edge $e = uv$ such that $d(u) \leq 2(h(\Sigma) - 1)$ and $d(v) \leq 2(h(\Sigma) - 1)$. Then $d(u) + d(v) \leq 4(h(\Sigma) - 1)$. Without loss of generality, assume $d(u) \leq \frac{\Delta(G)}{2}$. The graph obtained by deleting e can be totally colored with $\Delta(G) + 1$ colors. Extend this coloring to a total coloring of G in the following way. Color e with a color that does not appear on any edges adjacent to it and is also different from the color of vertex v. Since the sum of the degrees of u and v is at most $4h(\Sigma) - 4$, there are at most $4h(\Sigma) - 6$ other edges adjacent to e. Thus, there are at most $4h(\Sigma) - 5 = \Delta(G)$ restrictions and $\Delta(G) + 1$ colors. If the vertices u and v receive the same color, then recolor u with a color not used on its neighboring vertices and edges, which use at most $\Delta(G)$ colors. $\square$

Note that this is a stronger result than Zhao's when $\varepsilon(\Sigma) \leq -22$. This also allows for the following corollary.

Corollary 4.4. *Let G have a good drawing on a surface Σ with $\varepsilon(\Sigma) < 0$ and $\Delta(G) \geq 4h(\Sigma) - 5$. Then $\chi_{\text{tot}}(G) \leq \Delta(G) + 3$.*

Proof. Given a good drawing of G, consider the graph G' obtained from the drawing by deleting one edge from every pair of crossed edges, and then extend the drawing of G to an embedding of G'. This yields two cases. If $\Delta(G') = \Delta(G)$, then totally color G' with $\Delta(G)+1$ colors. Extend this total coloring to one of G by coloring each deleted edge with one new color, and using a second new color to recolor a vertex incident with a deleted edge, as necessary. Thus, $\chi_{\text{tot}}(G) = \Delta(G) + 3$. If $\Delta(G') \leq \Delta(G) - 1$, then repeat the process for the previous case and observe that $\chi_{\text{tot}}(G) = \Delta(G) + 2$. □

The following theorem allows for a better upper bound on χ_{tot} by slightly altering the bound on $\Delta(G)$.

Theorem 4.5. *Suppose G is a graph with a good drawing on a surface Σ such that $\varepsilon(\Sigma) < 0$ and $\Delta(G) \geq 4h(\Sigma) - 4$. Then $\chi_{\text{tot}}(G) = \Delta(G) + 1$.*

Proof. Consider the graph G' obtained from G by deleting one edge from every pair of crossed edges and extend the drawing of G to an embedding of G'. Corollary 4.2 implies that G' contains two adjacent vertices, each with degree at most $2(h(\Sigma) - 1)$. Thus, in G these two adjacent vertices have degrees at most $2(h(\Sigma) - 1)$ and $2(h(\Sigma) - 1) + 1$. Repeat the proof of Theorem 4.3 to show that $\chi_{\text{tot}}(G) = \Delta(G) + 1$. □

Chapter 5
List Colorings

5.1 Introduction

Let each vertex of a graph G have an associated set, called a *list*. A *list-coloring* is a proper coloring of G in which each vertex receives a color from its list. A graph G is *n-choosable* if it has a list-coloring for every set of lists of size n assigned to its vertices..

Thomassen proved the following theorem.

Theorem 5.1. *Every planar graph is* 5*-choosable.*

Thomassen's result is actually stronger, showing that if G is planar and one vertex has a fixed color, that is it has a list of size one, and every other vertex has list size five, then G is list-colorable.

Lemma 5.2. *Suppose G has a good drawing in the plane in which each crossing is at least distance 5 apart. Then G is 6-choosable.*

Proof. For each crossing, let the vertex v be incident with a crossed edge and fix an element a in its list. Delete v and delete the element a from the lists of all of the neighbors of v. Repeat this process for each crossing. The resulting graph is planar and is list-colorable, since every planar graph is 5-choosable (Theorem 5.1). This list coloring can be extended to G by coloring each deleted vertex with the color not used to color its neighbors. □

As a note, the assumption that G has a good drawing is stronger than is necessary. Instead assume that for each cluster of crossings in the drawing there is an edge in G whose deletion results in the drawing having no crossings. The above proof can then be applied to such graphs by deleting such edges.

Lemma 5.3. *Suppose G is a graph drawn on the plane with exactly one crossing. Then G is 5-choosable.*

Proof. Let v be a vertex incident with one of the crossed edges. Fix an element a in the list associated with v. Consider the graph G' obtain by deleting v and removing the element a from the lists of the neighbors of v. The graph G' is planar and, therefore, list colorable [Tho94], and so G is 5-choosable. $\square$

An analog to Lemma 5.2 can be constructed for graphs drawn on surfaces. This uses the following Theorem, due to Dirac [Dir56] and Ringel [Rin55].

Theorem 5.4. *If G embeds in a non-planar surface Σ, then G can be $(h(\Sigma)-1)$-list-colored, unless G contains $K_{h(\Sigma)}$.*

Using this theorem, the following can be shown.

Lemma 5.5. *Suppose G has a good drawing on a non-planar surface Σ such that each pair of crossings is at least distance 5 apart. Then G is $h(\Sigma)$-choosable, unless the graph G' obtained by deleting one crossed edge from each crossing contains $K_{h(\Sigma)}$.*

Proof. For each crossing let the vertex v be incident with a crossed edge and fix an element a in its list. Delete v and delete the element a from the lists of all of the neighbors of v. Repeat this process for each crossing. The resulting graph embeds in Σ and is $(h(\Sigma)-1)$-choosable, unless it contains $K_{h(\Sigma)}$ (Theorem 5.4). This list coloring can be extended to a list coloring of G by coloring each deleted vertex with the color not used to color its neighbors. $\square$

Chapter 6
Graphs Drawn on Generalized Pseudosurfaces

Some of the previous proofs rely primarily on the Heawood and Euler formulas. It is therefore of interested to explore generalized pseudosurfaces, which have similar formulas for graphs embedded in them.

More precisely, adopting the terminology of [Hei78], let $\Sigma_1, \ldots, \Sigma_k$ be pairwise disjoint surfaces and let $\{X_1, \ldots, X_t\}$ be a collection of pairwise disjoint subsets of $\bigcup\limits_{i=1}^{k} \Sigma_i$ with $1 < |X_j| < \infty$ for all $j = 1, \ldots, t$. Define and equivalence relation $\sim$ by $x \sim y$ if and only if $x = y$ or $x, y \in X_j$ for some j. If under the equivalence relation the space $\bigcup\limits_{i=1}^{k} \Sigma_i / \sim$ is a connected topological space, then it is called a *generalized pseudosurface* or a *k-component pseudosurface*. A 1-component pseudosurface is called simply a *pseudosurface*. Note that a generalized pseudosurface depends only on the underlying surfaces $\Sigma_1, \ldots, \Sigma_k$ and the set $\{X_1, \ldots, X_t\}$. Thus, a generalized pseudosurface is denoted by $P(\bigcup\limits_{i=1}^{k} \Sigma_i; \{X_1, \ldots, X_t\})$. The set of points where a generalized pseudosurface is not locally homeomorphic to an open disk are called *singular points*.

An embedding of a graph in a generalized pseudosurface P is described through a pseudo-embedding scheme. This pseudo-embedding scheme consists of a the set $\{\lambda_e | e \in E(G)\}$ as in the embedding scheme, and a set $\{\pi_v | v \in V(G)\}$ where π_v is a set of disjoint cyclic permutations consisting of the edges incident with v. Faces are defined for a pseudo-embedding schemes in the same way as embedding schemes. Note that for a graph embedded in a generalized pseudosurface, each singular point of P corresponds to a vertex that has more than one cyclic permutation associated with it.

Let P be a generalized pseudosurface. The Euler characteristic of a generalized pseudosurface is given by $\varepsilon(P) = \sum_{i=1}^{k} \varepsilon(\Sigma_i) - \sum_{j=1}^{t} (|X_j| - 1)$ [Hei78]. Moreover, $|V| - |E| + |F| = \varepsilon(P)$ [Pet71]. In particular, if $h(P) = \left\lfloor \frac{7+\sqrt{49-24\varepsilon(P)}}{2} \right\rfloor$, this implies the following.

Theorem 6.1. *If G is a graph embedded in a generalized pseudosurface P with no sphere component, and if the minimum facial degree of G is 3, then G has a vertex of degree at most $h(P) - 1$.*

Proof. Suppose G is embedded in a generalized pseudosurface P with no sphere component. Then $\varepsilon(P) < 2$. Let $\varepsilon = \varepsilon(P)$, and let δ denote the minimum degree of G. From the definition of the Euler characteristic and the fact that every face of G has degree at least 3, it follows that $2|E| \geq -3|V| + 3|E| + 3\varepsilon$, which can be rewritten as $|E| \leq 3|V| - 3\varepsilon$. Clearly, $\delta|V| \leq 2|E|$, and so $\delta|V| \leq 6|V| - 6\varepsilon$, thus $(\delta - 6)|V| \leq -6\varepsilon$. If $\varepsilon > 0$, then $\delta < 6$, and hence the result holds for the a generalized pseudosurface with $\varepsilon = 1$. If $\varepsilon(P) \leq 0$, then $\delta - 6$ is non-negative, and so $|V| \geq \delta + 1$ implies that $(\delta - 6)(\delta + 1) \leq -6\varepsilon(\Sigma)$, which, when solved for δ, implies that

$$\delta \leq \frac{5 + \sqrt{49 - 24\varepsilon(P)}}{2},$$

which means that

$$\delta \leq h(\Sigma) - 1.$$

□

The following is a Heawood-type result for generalized pseudosurfaces.

Theorem 6.2. *Suppose G is a graph embedded in a generalized pseudosurface P that has no sphere component. Then $\chi(G) \leq h(P)$.*

Lemma 6.3. *Suppose G is a graph that has the following property: G and every graph obtained from G by deleting a set of vertices has a vertex of degree at most $k - 1$. Then G is k-colorable.*

Proof. Suppose G is a minimal counterexample to Lemma 6.3 and let v be a vertex with $d(v) \leq k-1$. Consider the graph $G' = G - v$. The minimality of G implies that G' is k-colorable, and this coloring of G' can be extended to a k-coloring of G by assigning v a color not among its neighbors. □

Proof of Theorem 6.2. Let G be a graph embedded on a generalized pseudosurface P with no sphere component. Then by Theorem 6.1, G contains a vertex v of degree at most $h(P)-1$. Moreover, note that if $G' = G - v$, then G' embeds on a generalized pseudosurface P' with no sphere component that has $\varepsilon(P') \geq \varepsilon(P)$. Thus, G' has a vertex of degree at most $h(P)-1$ (Theorem 6.1). Then Lemma 6.3 implies that G has a proper k-coloring.

□

The following theorem and its associated conjecture are similar to Theorem 2.6 and Conjecture 2.7.

Theorem 6.4. *If G is a graph with a good 3-4-tiling in a generalized pseudosurface P with each component of P having Euler characteristic at most -2, then either G is $h(P)$-colorable or $\zeta(G)$ contains $K_{h(P)+1}$ as a subgraph.*

Conjecture 6.5. *If G is a graph with a good 3-4-tiling in a generalized pseudosurface P with each component of P having Euler characteristic at most -2, then G is $h(P)$-colorable.*

Proof of Theorem 6.4. Assume that G does not contain $K_{h(P)+1}$ as a subgraph. Let $\sigma(P)$ denote the number of singular points of P. Proceed by induction on $(\sigma(P), |V(G)|)$. Observe that when $\sigma(P) = 0$, Theorem 2.6 implies that Theorem 6.4 holds. Suppose $\sigma(P) > 0$ and $|V(G)| > h$.

If G contains a vertex v with $d(v) \leq h-1$ that is not incident with a 4-face, then consider the graph G' obtained from G by deleting v and adding edges to triangulate the resulting face(s). If v is a singular point, then G' embeds in a generalized pseudosurface P' with $\sigma(P') < \sigma(P)$, otherwise G' is a smaller graph embedded in P. In either case, induction implies that the resulting graph is cyclically $h(P)$-colorable. Extend this coloring to a cyclic coloring of G by coloring v with a color not among its neighbors.

If G contains a vertex v with $d(v) \leq h-2$ that is incident with a 4-face, then consider the graph G' obtained from G by deleting v and adding edges to triangulate the resulting face(s) so that the remaining three vertices of the 4-face are cofacial. If v is a singular point, then G' embeds in a generalized pseudosurface P' with $\sigma(P') < \sigma(P)$, otherwise G' is a smaller graph embedded in P. In either case, induction implies that the resulting graph is cyclically $h(P)$-colorable. Extend this coloring to a cyclic coloring of G by coloring v with a color not among its cyclic neighbors.

Now suppose that every vertex of G not incident with a 4-face has degree at least h, and every vertex incident with a 4-face has degree at least $h-1$. Theorem 6.1 implies that there is a vertex v incident with a 4-face and with $d(v) = h-1$. This implies that G contains $W(P)$ as a configuration. Now suppose that $V(G) \neq V(W(P))$. Then G contains $h(P)+1$ vertices of degree at least $h-1$. Charge G by giving each vertex v a charge of $c(v) = d(v)-6$ and each face a charge of $c(f) = 2(d^*(f)-3)$. The vertices of G contribute a charge of at least $(h+2)(h-7)$. Moreover, every 4 vertices contribute an extra charge of at least 2, since either these vertices are in a 4-face or each have degree of h and hence charge of $h-6$, and any remaining vertices that cannot be in a 4-face have charge of $h-6$. Let $V_4 = |V(G)| \pmod 4$. Then G has a minimum charge of

$$(h+2)(h-7) + 2\left\lfloor \frac{|V(G)|}{4} \right\rfloor + V_4.$$

Table 6.1 gives a list of these values when $\varepsilon(P) \geq 19$. If $\varepsilon(P) \leq 20$, then $h(P) \geq 15$ and G has a charge of at least $(h(P)+2)(h(P)-7)+8 = h(P)^2 - 5h(P) - 6$. In either of these cases, the minimum charge of G exceeds the upper bound on the charge of G established in Lemma 3.4. Thus $V(G) = V(W(P))$, which implies that $\chi(G) \leq h(P)$, or G is isomorphic to $K_{h(P)+1}$.

□

Table 6.1: Minimum Charge of $W(P)$

$\varepsilon(P)$	$h(P)$	Minimum Charge of $W(P)$	$-6\varepsilon(P)$	Minimum Charge $-6\varepsilon(P)$
-2	8	15	12	3
-3	9	28	18	10
-4	9	28	24	4
-5	10	43	30	13
-6	10	43	36	7
-7	10	43	42	1
-8	11	58	48	10
-9	11	58	54	4
-10	12	77	60	17
-11	12	77	66	11
-12	12	77	72	5
-13	13	98	78	20
-14	13	98	84	14
-15	13	98	90	8
-16	13	98	96	2
-17	14	121	102	19
-18	14	121	108	13
-19	14	121	114	7

The following is the generalized pseudosurface equivalent of Conjecture 2.7.

Conjecture 6.6. *If G is a graph with a good 3-4-tiling in a generalized pseudosurface P with no sphere component, then G is $h(P)$-colorable.*

An analog of Theorem 3.5 for generalized pseudosurfaces can also be proven.

Theorem 6.7. *Suppose G is a graph with an embedding on a pseudosurface P in which every component has Euler characteristic at most 0 with the following properties:*

1. *All faces of size greater than 3 are pairwise distance at least 2 apart.*

2. *The embedding has face-width at least 3.*

3. *No singular point is incident with a face of degree greater than 3.*

Then G can be cyclically colored with $n = \max\{h(P)+1, \Delta^*(G)+1\}$ colors.

This can be proven by establishing results similar to those in Chapter 3.

Lemma 6.8. *Suppose G is a minimal counterexample to Theorem 6.7. If f is a face incident with a vertex v, then $d(v) + d^*(f) > n + 2$.*

Proof. Assume that G is a minimal counterexample and suppose that v is a vertex in one of the faces f of G with $d(v) + d^*(f) - 2 \leq n \leq \Delta^* + 1$. If f is a face of degree at least 4, then consider the graph G' obtained from G by deleting v. Observe that the face resulting from deleting v has degree $d(v) + d^*(f) - 3$. Since $d(v) + d^*(f) - 2 \leq \Delta^* + 1$, then $d(v) + d^*(f) - 3 \leq \Delta^*$. So G' is a smaller graph satisfying the assumptions of Theorem 3.5 and is cyclically n-colorable. This coloring can be extended to G by coloring v with the remaining color.

If f has degree 3, then $d(v) < n$. Consider the graph G' obtained from G by deleting v and triangulating the resulting face(s). Since G is minimal, G' has a cyclic n-coloring. Extend this coloring of G' to a cyclic n-coloring of G by coloring v with a color not among its neighbors. In either case, we obtain a contradiction. □

Lemma 6.9. *In a minimal counterexample to Theorem 6.7, no face of degree at least 5 contains two adjacent vertices of degree 4.*

Proof. Suppose G is a minimal counterexample to Theorem 3.5, that f is a face with $d^*(f) = l \geq 5$, and that f contains two adjacent vertices, v_1 and v_2 of degree 4. Let $v_1, \ldots, v_l$ be the vertices incident with f, listed in order and let u be the common neighbor of v_1 and v_l. Construct a new graph G' from G by deleting v_1 and v_2, adding an edge between v_l and v_3 and triangulating the face of degree 5 incident with u by adding edges incident with u. Note that G' satisfies the criteria of Theorem 6.7, since this operation does not decrease the distance between faces of degree greater than 3, nor does it decrease the distance between singular points and faces of degree greater than 3. Since G is a minimal counterexample, G' has a cyclic coloring with n colors. Extend this coloring to G in the following way. Let a be the color assigned to u. If the color a is not assigned to any of the vertices $v_3, \ldots, v_l$, then color v_2 with a. Otherwise, color v_2 with any available color. Such a color exists since v_2 has at most $\Delta^* + 1$ cyclic neighbors, and the vertex v_1 has no color. Thus, there are at most Δ^* restrictions on the color of v_2. Note that u and at least one vertex of f are both colored a. Therefore there are also at most Δ^* restrictions on the color of v_1, so G has a cyclic $(\Delta^* + 1)$-coloring. □

Lemma 6.10. *In a minimal counterexample to Theorem 6.7, no face of degree at least* 5 *contains three consecutive vertices with degrees* 4, 5, 4 *or* 4, 5, 5 *or* 5, 4, 5.

Proof. Suppose G is a minimal counterexample to Theorem 3.5, that f is a face of G with $d^*(f) = l$ and with vertices $v_1, \ldots, v_l$. Let v_1, v_2, v_3 have degrees 4, 5, 4 or 4, 5, 5 or 5, 4, 5 and let u be the common neighbor of v_1 and v_2. Consider the graph G' obtained from G in the following way. Delete the vertices v_1, v_2, v_3, add an edge between v_l and v_4, and triangulate the new face with edges incident with u. Note that G' satisfies the criteria of Theorem 6.7, since this operation does not decrease the distance between faces of degree greater than 3, nor does it decrease the distance between singular points and faces of degree greater than 3. The minimality of G implies that G' is cyclically n-colorable.

Extend the cyclic coloring of G' to a cyclic coloring of G as follows. Let a be the color assigned to u. If the color a is not assigned to any of the vertices $v_4, \ldots, v_l$, color v_3 with a. Otherwise, color v_3 with any available color. Such a color exists since the cyclic degree of v_3 is at most $\Delta^* + 2$ and two of its cyclic neighbors are uncolored. Next, color the remaining vertex of v_1 and v_2 whose degree is 5. Since this vertex has cyclic degree at most $\Delta^* + 2$ and two of its cyclic neighbors are colored a and one is uncolored, such a color exists. Finally, color the remaining vertex of degree 4. Such a color exists since the cyclic degree of this vertex is at most $\Delta^* + 1$ and two of its cyclic neighbors are colored a. Thus G can be cyclically n colored.

$\square$

Using these lemmas, Theorem 6.7 can be shown by following the proof of Theorem 3.5.

References

[AEK+12] Jernej Azarija, Rok Erman, Daniel Král, Matjaž Krnc, and Ladislav Stacho, *Cyclic colorings of plane graphs with independent faces*, European Journal of Combinatorics **33** (2012), no. 3, 294 – 301, Topological and Geometric Graph Theory.

[AH77] Kenneth Appel and Wolfgang Haken, *Every planar map is four colorable. I. Discharging*, Illinois J. Math. **21** (1977), no. 3, 429–490. MR 0543792 (58 #27598a)

[AHK77] Kenneth Appel, Wolfgang Haken, and John Koch, *Every planar map is four colorable. II. Reducibility*, Illinois J. Math. **21** (1977), no. 3, 491–567. MR 0543793 (58 #27598b)

[Beh65] Mehdi Behzad, *GRAPHS AND THEIR CHROMATIC NUMBERS*, ProQuest LLC, Ann Arbor, MI, 1965, Thesis (Ph.D.)–Michigan State University. MR 2615420

[Bor84] O. V. Borodin, *Solution of the Ringel problem on vertex-face coloring of planar graphs and coloring of* 1*-planar graphs*, Metody Diskret. Analiz. (1984), no. 41, 12–26, 108. MR 832128 (87e:05067)

[Bor89] ______, *On the total coloring of planar graphs*, J. Reine Angew. Math. **394** (1989), 180–185. MR 977440 (89m:05049)

[Bor92] Oleg V. Borodin, *Cyclic coloring of plane graphs*, Discrete Mathematics **100** (1992), no. 13, 281 – 289.

[Dir56] G. A. Dirac, *Map colour theorems related to the Heawood colour formula*, J. London Math. Soc. **31** (1956), 460–471. MR 0081467 (18,408a)

[Hea90] Percy John Heawood, *Map colour theorem*, Quarterly J. Math. **24** (1890), 332–338.

[Hei78] Clare E. Heidema, *Generalized pseudosurface embeddings of graphs and associated block designs*, Theory and applications of graphs (Proc. Internat. Conf., Western Mich. Univ., Kalamazoo, Mich., 1976), Springer, Berlin, 1978, pp. 255–270. Lecture Notes in Math., Vol. 642. MR 0491228 (58 #10492)

[JV00] Stanislav Jendrol' and Heinz-Juergen Voss, *Light subgraphs of multigraphs embedded in compact 2-manifolds*, 6th International Conference on Graph Theory (Marseille, 2000), Electron. Notes Discrete Math., vol. 5, Elsevier, Amsterdam, 2000, p. 4 pp. (electronic). MR 1969953

[Kos77] A. V. Kostochka, *The total coloring of a multigraph with maximal degree* 4, Discrete Math. **17** (1977), no. 2, 161–163. MR 0453576 (56 #11838)

[KS10] Daniel Král and Ladislav Stacho, *Coloring plane graphs with independent crossings*, J. Graph Theory **64** (2010), no. 3, 184–205. MR 2674491 (2011i:05077)

[MT01] Bojan Mohar and Carsten Thomassen, *Graphs on surfaces*, Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, Baltimore, MD, 2001. MR 1844449 (2002e:05050)

[OP69] Oystein Ore and Michael D. Plummer, *Cyclic coloration of plane graphs*, Recent Progress in Combinatorics (Proc. Third Waterloo Conf. on Combinatorics, 1968), Academic Press, New York, 1969, pp. 287–293. MR 0252275 (40 #5496)

[Pet71] Wayne S Petroelje, *Imbedding graphs in pseudosurfaces*, Ph.D. thesis, Western Michigan University, 1971.

[Rin55] Gerhard Ringel, *Wie man die geschlossenen nichtorientierbaren flchen in möglichst wenig dreiecke zerlegen kann*, Mathematische Annalen **130** (1955), no. 4, 317–326 (German).

[Rin74] ______, *Map color theorem*, Springer-Verlag, New York, 1974, Die Grundlehren der mathematischen Wissenschaften, Band 209. MR 0349461 (50 #1955)

[Ros71] M. Rosenfeld, *On the total coloring of certain graphs*, Israel J. Math. **9** (1971), 396–402. MR 0278995 (43 #4721)

[Tho94] C. Thomassen, *Every planar graph is 5-choosable*, Journal of Combinatorial Theory, Series B **62** (1994), no. 1, 180 – 181.

[Vij71] N. Vijayaditya, *On total chromatic number of a graph*, J. London Math. Soc. (2) **3** (1971), 405–408. MR 0285447 (44 #2665)

[Viz68] V. G. Vizing, *Some unsolved problems in graph theory*, Uspehi Mat. Nauk **23** (1968), no. 6 (144), 117–134. MR 0240000 (39 #1354)

[Zha99] Yue Zhao, *On the total coloring of graphs embeddable in surfaces*, J. London Math. Soc. (2) **60** (1999), no. 2, 333–343. MR 1724810 (2001m:05101)

Vita

Daniel A. Guillot was born in 1986 in Alexandria, Louisiana. He graduated high school in 2004 from the Louisiana School for Math, Science, and the Arts. He finished his undergraduate studies at Louisiana University in May 2008. He earned a master of science degree in mathematics from LSU in May 2010. He is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2014.