Sigma-Compact Subsets of Hyperspaces.

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# TABLE OF CONTENTS

ACKNOWLEDGMENTS ......................................... ii
ABSTRACT .................................................. iv

**CHAPTER 1. PRELIMINARIES**

1.1. Hyperspaces ............................... 1
1.2. Infinite-dimensional topology ............ 3
1.3. Past and present results ................  6

**CHAPTER 2. THE COMPACT ABSORPTION PROPERTY IN HYPERSPACES OF PEANO CONTINUA**

2.1. An alternate version of the CAP ............10
2.2. Growth hyperspaces of Peano continua .... 12
2.3. Main results .................................14
2.4. Applications .................................17

**CHAPTER 3. GEOMETRIC VERSIONS OF THE FINITE-DIMENSIONAL COMPACT ABSORPTION PROPERTY**

3.1. A cellular version ...........................23
3.2. A polyhedral version ........................29

**CHAPTER 4. THE FINITE-DIMENSIONAL COMPACT ABSORPTION PROPERTY IN HYPERSPACES OF GRAPHS**

4.1. Finite-dimensional growth hyperspaces . . 35
4.2. Hyperspaces generated by triangulations . . 39
4.3. Main results .................................43

BIBLIOGRAPHY .........................................  .  47
VITA ......................................................50
ABSTRACT

For $X$ a non-degenerate Peano continuum, it is known that the hyperspace $2^X$ of all non-empty closed subsets of $X$ with the topology induced by the Hausdorff metric is homeomorphic to the Hilbert cube $Q$, and that the hyperspace $C(X)$ of all subcontinua of $X$ is homeomorphic to $Q$ if $X$ contains no free arc (i.e., no arc whose boundary in $X$ is contained in its set of endpoints). In this dissertation we identify dense $\sigma$-compact subsets of $2^X$ or $C(X)$ having one of two positional properties.

A closed subset $A$ of $Q$ is a $\mathbb{Z}$-set if for each $\varepsilon > 0$ there exists a map $f:Q \to Q\setminus A$ with $d(f, id) < \varepsilon$. A subset $M$ of $Q$ is a capset (f-d capset, respectively) if $M$ can be expressed as $\bigcup_{i=1}^{\infty} M_i$, where

1) for each $i$, $M_i \subseteq M_{i+1}$ and $M_i$ is a $\mathbb{Z}$-set (finite-dimensional $\mathbb{Z}$-set, respectively) in $Q$, and

2) for each $\varepsilon > 0$, positive integer $j$, and compact subset (compact finite-dimensional subset, respectively) $K$ of $X$, there exists a positive integer $k$ and embedding $f:K \to M_k$ such that $f|K \cap M_j = id$ and $d(f, id) < \varepsilon$.

In either case, $Q\setminus M$ is homeomorphic to separable Hilbert space $l^2$.

$N \subseteq 2^X$ has the inclusion property if $F \in N$ whenever $F \subseteq 2^X$ and $F \supseteq E$, for some $E \in N$. More generally, $S \subseteq 2^X$
has the *growth property* if $F \in \mathcal{J}$ whenever $F \in 2^X$ and, for some $E \in \mathcal{J}$, $F \supseteq E$ and each component of $F$ meets $E$.

Two main results are obtained. The first generalizes earlier work by Nelly Kroonenberg and by D. W. Curtis. The second employs new techniques for recognizing f-d capsets.

**Theorem.** If $X$ is a Peano continuum, $2$ is a copy of $Q$ in $2^X$, and $\mathcal{J}$ is a dense $\sigma$-Z-set in $2$ with the growth property, then $\mathcal{J}$ is a capset in $2$ provided $X$ contains no free arc or $\mathcal{J}$ has the inclusion property.

**Theorem.** If $\Gamma$ is a finite connected graph, $2$ is a copy of $Q$ in $2^\Gamma$, $\mathcal{J}$ is a dense union of countably many finite-dimensional Z-sets in $2$, and $\mathcal{J}$ has the growth property, then $\mathcal{J}$ is an f-d capset in $2$.

As a corollary of the first theorem we show that the hyperspace of all closed subsets with non-empty interior in $X$ is a capset in $2^X$ if $X$ is a non-degenerate Peano continuum, and the hyperspace of all subcontinua with non-empty interior in $X$ is a capset in $C(X)$ if additionally $X$ contains no free arc.

As a corollary of both theorems we show that the hyperspace of all closed subsets of $X$ with finitely many components is a capset in $2^X$ if $X$ is a non-degenerate Peano continuum containing no free arc, an f-d capset if $X$ is a finite, non-degenerate, connected graph.
CHAPTER 1
PRELIMINARIES

1.1. Hyperspaces.

For a given topological space $X$, a hyperspace of $X$ is a space whose elements are compact subsets of $X$ and whose topology arises naturally from the topology on $X$. The universal hyperspace of $X$ is $2^X$, the collection of all non-empty compact subsets of $X$. The finite or Vietoris topology on $2^X$ has as a subbase the family of all collections of the form $\{F \in 2^X | F \subseteq G\}$ or $\{F \in 2^X | F \cap G \neq \emptyset\}$, where $G$ is any open set in $X$. Intuitively, this means that two elements of $2^X$ are close if, as subsets of $X$, they are contained in small neighborhoods of one another.

In fact, if $X$ is a compact metric space with metric $d$, $2^X$ can be given the Hausdorff metric $\rho$ defined by

$$\rho(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\}$$

$$= \inf\{\epsilon > 0 | A \subseteq U_\epsilon(B) \text{ and } B \subseteq U_\epsilon(A)\},$$

where $d(x, F) = \inf_{y \in F} d(x, y)$ and $U_\epsilon(F) = \{x \in X | d(x, F) < \epsilon\}$. The topology induced by $\rho$ is exactly the Vietoris topology.

The topology on $2^X$ is natural in the sense that $2^X$ contains a subspace $F_1(X)$ homeomorphic to $X$ under the obvious correspondence between singleton subsets of $X$ and elements of $X$; this correspondence is actually an isometry.
when X is a compact metric space. The most extensively studied subspace of $2^X$ is $C(X)$, the hyperspace of all sub-continua of X. Also of interest are the **containment hyperspaces** $2^X_A = \{ F \in 2^X | F \supseteq A \}$ and $C_A(X) = C(X) \cap 2^X_A$, and the **intersection hyperspaces** $2^X(A) = \{ F \in 2^X | F \cap A \neq \emptyset \}$ and $C(X;A) = C(X) \cap 2^X(A)$, where $A \in 2^X \setminus \{X\}$.

$\mathcal{N} \subseteq 2^X$ has the **inclusion property** if $F \in \mathcal{N}$ whenever $F \in 2^X$ and $F \supseteq E$, for some $E \in \mathcal{N}$. More generally, $\mathcal{J} \subseteq 2^X$ has the **growth property** if $F \in \mathcal{J}$ whenever $F \in 2^X$ and, for some $E \in \mathcal{J}$, $F \supseteq E$ and each component of F meets E. Under the standard assumption that X is connected, X is an element of each hyperspace with the growth property. A hyperspace that has the inclusion property or the growth property and is closed in $2^X$ is said to be an **inclusion hyperspace** or a **growth hyperspace**, respectively. $2^X_A$ and $2^X(A)$ are examples of inclusion hyperspaces. $C(X)$, $C_A(X)$, and $C(X;A)$ are growth hyperspaces which lack the inclusion property. $\mathcal{J}(X)$, the hyperspace of all compact subsets with non-empty interior in X, has the inclusion property but is not closed in $2^X$. Both the inclusion property and the growth property are preserved under unions, intersections, and the closure operator.

Of fundamental importance in this dissertation are **finitely-generated** inclusion and growth hyperspaces. If $A \in 2^X$, then $2^X_A$ is the smallest inclusion hyperspace containing the element A, and

$$G_A(X) = \{ F \in 2^X | F \supseteq A \text{ and each component of } F \text{ meets } A \}$$
is the smallest growth hyperspace containing the element \( A \); we say that \( 2^X \) and \( G_A(X) \) are the inclusion hyperspace and the growth hyperspace, respectively, generated by the set \( A \). For \( \{A_1, \ldots, A_n\} \subseteq 2^X \), \( \bigcup_{i=1}^n 2^{A_i} \) and \( \bigcup_{i=1}^n G_{A_i}(X) \) are the inclusion hyperspace and the growth hyperspace, respectively, generated by the collection \( \{A_1, \ldots, A_n\} \).

Work of J. L. Kelley [21] shows that any growth hyperspace of a Peano continuum is an absolute retract for the class of metrizable spaces. Using earlier results of Schori and West [26], [27], Curtis and Schori [15] have proved that \( 2^X \) is homeomorphic to the Hilbert cube if and only if \( X \) is a non-degenerate Peano continuum, which is exactly the situation in this dissertation.

1.2. Infinite-dimensional topology.

The standard Hilbert cube \( Q = J^\omega \) is the product of countably many copies of the interval \( J = [-1,1] \); the metric on \( Q \) is defined by \( d(x,y) = \sum_{i=1}^\infty 2^{-i} |x_i - y_i| \). The subset \( s = (-1,1)^\omega \) is called the pseudo-interior of \( Q \); it is directly homeomorphic to \( R^\omega \), the product of countably many copies of \( R \), which is homeomorphic to separable Hilbert space \( l^2 \). \( B(Q) = Q \setminus s \) is called the pseudo-boundary of \( Q \). An endslice of \( Q \) is a subset of the form \( \{x \in Q | x_i = 1\} \) or \( \{x \in Q | x_i = -1\} \), for some \( i \). \( B(Q) \) is the union of allendslices. Both \( B(Q) \) and \( s \) are dense in \( Q \) and nowhere locally compact, but \( B(Q) \) is \( \sigma \)-compact while \( s \) is not.

A crucial notion in the study of infinite-dimensional
topology is R. D. Anderson's "Property Z" [2]. A closed subset A of a space X is a Z-set in X if there are arbitrarily small maps f:X → X\A, i.e., if for each open cover U of X there exists f:X → X\A such that for each x ∈ X, both x and f(x) belong to some U ∈ U; when X is a compact metric space, this is just the requirement that for each ε > 0 there exist a map f:X → X\A with d(f, id) < ε. This positional property is preserved by homeomorphisms of pairs. In Q, Z-sets have topological infinite codimension: a closed subset A of Q is a Z-set if and only if there is a homeomorphism of Q onto itself taking A onto a set with infinite deficiency, i.e., a set whose projection in infinitely many coordinates is [0]. Z-sets in Q can also be characterized as the closed subsets A for which there is a homeomorphism of Q onto itself taking A into s.

By a σ-Z-set in X we mean a union of countably many Z-sets in X. It can be shown that a closed σ-Z-set in a complete metric space X is a Z-set in X. Thus, any closed subset of such X which lies in a σ-Z-set in X is a Z-set in X, and S will be a σ-Z-set in X if S is an Fσ in X and there are arbitrarily small maps f:X → X\S.

The σ-Z-sets of interest here are those with the compact absorption property (CAP) or the finite-dimensional compact absorption property (F-D CAP). These notions were introduced by Anderson [3] and have been used by Chapman [6] in the study of infinite-dimensional manifolds. A subset M of a metric space X has the CAP (F-D CAP, respectively) if
M can be expressed as \( \bigcup_{i=1}^{\infty} M_i \), where

1) for each \( i \), \( M_i \subseteq M_{i+1} \) and \( M_i \) is a compact Z-set (compact finite-dimensional Z-set, respectively) in \( X \), and

2) for each \( \epsilon > 0 \), positive integer \( j \), and compact (compact finite-dimensional, respectively) subset \( K \) of \( X \), there exists a positive integer \( k \) and embedding \( f:K \to M_k \) such that \( f|K \cap M_j = \text{id} \) and \( d(f, \text{id}) < \epsilon \);

in such case \( M \) is said to be a capset (f-d capset, respectively) in \( X \).

The property of being an (f-d) capset is preserved by homeomorphisms of pairs. Moreover, if \( M \) and \( M' \) are any two (f-d) capsets in \( Q \), then there exist arbitrarily small homeomorphisms \( h:(Q,M) \to (Q,M') \). It should be noted that a capset in \( Q \) can not also be an f-d capset since an f-d capset is countable-dimensional (\( \sigma \)-finite-dimensional) while a capset in \( Q \) must contain a copy of \( Q \), which is not countable-dimensional (see [20, p.49]).

\( B(Q) \) is a capset in \( Q \), so the pseudo-boundary of \( Q \) is topologically characterized by the CAP. Thus we refer to the complement of any capset in \( Q \) as a pseudo-interior of \( Q \). Not only is the complement of any capset in \( Q \) homeomorphic to \( l^2 \), but so is the complement of any f-d capset. Consequently, hyperspaces homeomorphic to \( l^2 \) can be found by identifying (f-d) capsets in hyperspaces homeomorphic to \( Q \).
1.3. **Past and present results.**

The statement that $2^X$ is homeomorphic to $Q$ for all non-degenerate Peano continua does not remain true if $2^X$ is replaced by $C(X)$. The simplest counterexample is $C(I) \cong I^2$. It is easy to see that whenever $X$ contains a free arc (i.e., an arc whose boundary is contained in its set of endpoints), $C(X)$ is necessarily two-dimensional at some point. The presence or absence of free arcs in $X$ is crucial in computing $C(X)$ and other hyperspaces. In fact, $C(X) \cong Q$ if and only if $X$ is a non-degenerate Peano continuum containing no free arc.

For any Peano continuum $X$ and proper closed subset $A$, $2^X_A$ and $2^X(A)$ are each homeomorphic to $Q$, as are $C_A(X)$ and $C(X;A)$ if $X$ contains no free arc [15],[16]. $2^X(A)$ is a Z-set in $2^X$ and $C(X;A)$ is a Z-set in $C(X)$ if and only if $A$ is locally non-separating in $X$ (i.e., $U \setminus A$ is non-empty and connected whenever $U$ is a non-empty, connected, open subset of $X$) [12]. $2^X_A$ is a Z-set in $2^X$ if and only if $A$ is not a finite set of local cut points in $X$ [10].

Free arcs are again a matter of concern in characterizing growth hyperspaces homeomorphic to $Q$. A theorem of Curtis [11] has the following corollary:

Let $\mathcal{J}$ be a non-trivial growth hyperspace of a Peano continuum $X$, such that either $X$ contains no free arc or $\mathcal{J}$ is an inclusion hyperspace. Then $\mathcal{J} \cong Q$ if and only if $\mathcal{J} \setminus \{X\}$ is contractible, if and only if $\{X\}$ is a Z-set in $\mathcal{J}$. 
The requirement that $X$ have no free arcs or $\mathcal{J}$ be an inclusion hyperspace reflects the inherent problem of making $\mathcal{J}$ big enough: if $F$ is the closure of the complement of a free arc in $X$, then $G_F(X)$ is one- or two-dimensional while $2^X_F$ is infinite-dimensional. The trade-off between the alternate hypotheses "no free arc" and "inclusion property" will also be apparent in the results of this dissertation.

Those results concern the identification of certain $(f-d)$ capsets in $2^X$ in terms of the geometry bestowed upon $2^X$ by $X$. The first examples of capsets in hyperspaces were obtained by Kroonenberg [23].

(1.1) **Theorem.** The hyperspace of all Z-sets in $Q$ is a pseudo-interior of $2^Q$, and the hyperspace of all connected Z-sets in $Q$ is a pseudo-interior of $C(Q)$.

(1.2) **Theorem.** The hyperspace of all closed zero-dimensional subsets of $I$ is a pseudo-interior of $2^I$.

(1.3) **Theorem.** The hyperspace of all Cantor sets in $I$ is a pseudo-interior of $2^I$.

(1.4) **Theorem.** $2^S$ and $C(s)$ are pseudo-interiors of $2^Q$ and $C(Q)$, respectively.

Curtis [12] generalized this last example by finding conditions under which a non-compact space $X$ has a Peano compactification $\tilde{X}$ such that $2^X$ is a pseudo-interior of $2^{\tilde{X}}$. What is actually needed is for the remainder $\tilde{X}\backslash X$ to
be "nice" (locally non-separating in $\mathcal{X}$), which is the case when $X$ has a metric $d$ with property $S$, i.e., $X$ has finite covers of connected open sets of diameter less than $\epsilon$, for each $\epsilon > 0$.

(Note that a space which admits a metric with property $S$ is necessarily separable and locally connected.) Curtis has proved the following (and its converse).

(1.5) **Theorem.** If $X$ is a connected, topologically complete, nowhere locally compact space which admits a metric with property $S$, then $X$ has a Peano compactification $\overline{X}$ such that $(2^{\overline{X}}, 2^X) \cong (C(\overline{X}), C(X)) \cong (Q, s)$.

In Chapter 2, we extend the techniques employed by Kroonenberg and by Curtis. The most general result combines two theorems, one for each of the alternate hypotheses mentioned previously.

(2.10) **Theorem.** If $X$ is a Peano continuum, $\mathcal{J}$ is a growth hyperspace of $X$ which is homeomorphic to $Q$, and $\mathcal{J}$ is a dense $\sigma$-Z-set in $\mathcal{J}$ with the growth property, then $\mathcal{J}$ is a capset in $\mathcal{J}$ provided $X$ contains no free arc or $\mathcal{J}$ (and therefore $2^X$) has the inclusion property.

Results (1.1), (1.4), (1.5), and significant generalizations of (1.2) and (1.3) can be obtained as corollaries of (2.10). In addition, we show that when $X$ is a non-degenerate Peano continuum containing no free arc, the hyperspace of all closed subsets of $X$ with finitely many components is a capset in $2^X$. 
In Chapter 3, methods are developed for recognizing f-d capsets. In Chapter 4, the structure of certain finite-dimensional growth hyperspaces is studied to obtain the following analogue of (2.10).

(4.11) Theorem. If $\Gamma$ is a finite connected graph, $\mathcal{J}$ is a growth hyperspace of $\Gamma$ which is homeomorphic to $Q$, $\mathcal{J}$ is a dense union of countably many finite-dimensional Z-sets in $\mathcal{J}$, and $\mathcal{J}$ has the growth property, then $\mathcal{J}$ is an f-d capset in $\mathcal{J}$. 
2.1. An alternate version of the CAP.

The verification of the compact absorption property for subsets of $\mathbb{Q}$ is greatly simplified by using a tower of copies of $\mathbb{Q}$ and two well-known properties of $\mathbb{Q}$. The first one follows from the classic result that for any separable metric space $K$, the space of embeddings of $K$ into $\mathbb{Q}$ is dense in the space of mappings of $K$ into $\mathbb{Q}$ (see [20, p.64]). The second is Barit's estimated version [4],[5] of a theorem of Anderson [1], which in turn has its roots in the work of Klee [22].

(2.1) Mapping Replacement Theorem for $\mathbb{Q}$. Given $\varepsilon > 0$ and $g:K \rightarrow \mathbb{Q}$, where $K$ is compact, there exists an embedding $h:K \rightarrow \mathbb{Q}$ with $d(g,h) < \varepsilon$ and $h(K)$ a $Z$-set in $\mathbb{Q}$.

(2.2) Homeomorphism Extension Theorem for $\mathbb{Q}$. Each homeomorphism $f:K \rightarrow K'$ between $Z$-sets in $\mathbb{Q}$ can be extended to an ambient homeomorphism $\bar{f}:\mathbb{Q} \rightarrow \mathbb{Q}$, and if $d(f,\text{id}_K) < \varepsilon$, $\bar{f}$ can be chosen so that $d(\bar{f},\text{id}_{\mathbb{Q}}) < \varepsilon$.

The following characterization of capsets in $\mathbb{Q}$ is due to Kroonenberg [23]. The proof is included here since it is the motivation for the development in Chapter 3 of new
characterizations of f-d capsets.

(2.3) **CAP Lemma.** A subset $M$ of $Q$ has the compact absorption property if $M$ can be expressed as $\bigcup_{i=1}^{\infty} M_i$, where

1) for each $i$, $M_i \cong Q$ and $M_i$ is a Z-set in both $M_{i+1}$ and $Q$, and

2) for each $\epsilon > 0$, there exists an integer $\ell$ and map $g:Q \to M_\ell$ with $d(g, \text{id}) < \epsilon$.

**Proof.** For each $\epsilon > 0$, positive integer $j$, and compact subset $K$ of $Q$, it is necessary to find an embedding $f$ as specified in condition 2) of the original definition. By hypothesis, there exists an integer $k$ and map $g:Q \to M_k$ such that $d(g, \text{id}_Q) < \epsilon/4$. By (2.1), there is an embedding $h:Q \to M_k$ such that $d(h, g) < \epsilon/4$ and $h(Q)$ is a Z-set in $M_k \cong Q$. Thus, $d(h, \text{id}_Q) < \epsilon/2$ and $d(h^{-1}, \text{id}_{h(Q)}) < \epsilon/2$. We can apply (2.2) to $M_k$ to get a homeomorphism $h':M_k \to M_k$ extending $h^{-1}|h(K \cap M_j)$ such that $d(h', \text{id}_{M_k}) < \epsilon/2$. Then $f = h' \circ h:K \to M_k$ is the desired embedding of $K$ into $M_k$ which is the identity on $K \cap M_j$ and which is within $\epsilon$ of the identity. \(\square\)

An immediate corollary of (2.3) is that

$$\Sigma = \{ x \in Q \mid \sup_{i>0} |x_i| < 1 \}$$

is a capset in $Q = J^\omega$ since it can be expressed as

$$\bigcup_{i=1}^{\infty} [-1 + 1/i, 1 - 1/i]^\omega.$$
(2.4) **The Standard Reduction.** It can be shown that a $\sigma$-Z-set in $Q$ which contains a capset is itself a capset. Thus, to demonstrate that a $\sigma$-Z-set $S$ in $Q$ is actually a capset, it suffices to exhibit a tower of Hilbert cubes contained in $S$ which satisfies the conditions of the CAP Lemma. It is likewise true that a union of countably many finite-dimensional Z-sets in $Q$ is an f-d capset if it contains an f-d capset.

2.2. **Growth hyperspaces of Peano continua.**

In the present chapter, certain hyperspaces with the growth property are shown to be capsets in $2^X$. To apply the CAP Lemma, it is necessary to construct hyperspaces homeomorphic to $Q$. Crucial in this regard is the following lemma, which is a consequence of Corollary 5.2 of [11].

(2.5) **Lemma.** Let $X$ be a non-degenerate Peano continuum and let $A_1, \ldots, A_n$ be non-empty closed subsets of $X$ such that $\bigcup_{i=1}^n A_i \neq X$. Then $\bigcup_{i=1}^n 2^{X_{A_i}} \simeq Q$, and

$$\bigcup_{i=1}^n G_{A_i}(X) \simeq Q$$

if $X$ contains no free arc.

A Peano continuum $X$ can be given a convex metric $d$ so that for each pair of distinct points $x, y \in X$ there exists an arc in $X$ between $x$ and $y$ which is isometric to the interval $[0, d(x,y)]$. Henceforth we assume $d$ is a convex metric when $X$ is a Peano continuum. For each $t \geq 0$, we define an expansion map $\eta_t : 2^X \rightarrow 2^X$ by
\[ \eta_t(A) = \eta(A, t) = \{ x \in X | d(x, A) \leq t \}. \]

If \( d \) is bounded by 1, then \( \eta: 2^X \times I \to 2^X \) is a contraction of \( 2^X \) to the element \( \{ x \} \). Furthermore, each growth hyperspace \( J \) is invariant under any expansion map; said another way, \( \eta(J \times I) \subseteq J \). This is because expanding a set \( A \) via a convex metric does not create any new components, i.e., each component of \( \eta_t(A) \) contains a component of \( A \).

To apply the CAP Lemma, it is also necessary to know when one hyperspace is a Z-set in another. Expansion maps are useful in some cases, for example, in showing \( F_1(X) \) is a Z-set in \( 2^X \). To show certain containment hyperspaces are Z-sets, Curtis and Schori constructed a different type of map in the proof of the following lemma \([16, \text{Lemma 5.4}]\).

\((2.6) \) Lemma. Let \( A \) be a closed set with non-empty interior in a non-degenerate Peano continuum \( X \). Then for each \( \epsilon > 0 \), there exists a map \( g: 2^X \to 2^X \setminus 2^X_A \) such that \( d(g, \text{id}) < \epsilon \) and \( g(F) \setminus A = F \setminus A \) for each \( F \in 2^X \). If \( A \) contains no free arc in \( X \), then \( g[C(X)] \subseteq C(X) \setminus C_A(X) \).

The preceding lemma can be generalized to show that certain finitely-generated growth hyperspaces and inclusion hyperspaces are Z-sets in \( 2^X \) and in other hyperspaces.

\((2.7) \) Lemma. Let \( A_1, \ldots, A_n \) be closed subsets with non-empty interiors in a non-degenerate Peano continuum \( X \). Then for each \( \epsilon > 0 \), there exists a map \( g: 2^X \to 2^X \setminus \bigcup_{i=1}^n 2^X_{A_i} \) such that \( d(g, \text{id}) < \epsilon \) and \( g(F) \setminus \bigcup_{i=1}^n A_i = F \setminus \bigcup_{i=1}^n A_i \) for each
If \( P \in 2^X \). Moreover, if \( B_1, \ldots, B_m \) are closed subsets of \( X \) contained in \( X \setminus \bigcup_{i=1}^{n} A_i \), then \( \bigcup_{j=1}^{m} 2^X_{B_j} \) is invariant under \( g \), and \( \bigcup_{j=1}^{m} G_{B_j}(X) \) is invariant under \( g \) provided \( X \) contains no free arc.

**Proof.** The map \( g \) is the composition of \( g_1, \ldots, g_n \), where each map \( g_i:2^X \to 2^X_{A_i} \) is obtained via (2.6). The desired invariance will follow from the invariance of \( 2^X_B \) and \( G_B(X) \), where \( B = B_1 \). If \( F \supseteq B \), then \( B \subseteq X \setminus \bigcup_{i=1}^{n} A_i \) and \( g(F) \setminus \bigcup_{i=1}^{n} A_i = F \setminus \bigcup_{i=1}^{n} A_i \) implies \( g(F) \cap B = F \cap B = B \). Therefore, \( g[2^X_B] \subseteq 2^X_B \).

In the case when \( X \) contains no free arc, each map \( g_i \) is defined in the proof of Lemma 5.4 of [16] by first defining \( r:X \to C(X) \) and then lifting \( r \) to a map on \( 2^X \) in the obvious way: \( g_i(F) = r[F] = \bigcup_{a \in F} r(a) \). Consequently, \( g_i \) commutes with set summation. So \( g_i \) not only maps each connected element to a connected element, but actually respects components: if \( \{F_\alpha | \alpha \in \mathcal{A}\} \) is the set of components of \( F \), then \( g_i(F) = \bigcup_{\alpha \in \mathcal{A}} g_i(F_\alpha) \), and each \( g_i(F_\alpha) \) is connected and meets \( B \) if \( F_\alpha \) does. Therefore, \( g_i[G_B(X)] \subseteq G_B(X) \), and the same is true of the composition \( g \). \( \square \)

### 2.3. Main results.

Two separate theorems are stated, one for "no free arc" and the other for "inclusion property"; these are then generalized and combined into one theorem.
(2.8) Theorem. If \( X \) is a non-degenerate Peano continuum containing no free arc and \( \mathcal{J} \) is a dense \( \sigma \)-\( Z \)-set in \( 2^X \) with the growth property, then \( \mathcal{J} \) is a capset in \( 2^X \).

Proof. It suffices to construct a tower \( \mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \cdots \) in \( \mathcal{J} \) which satisfies the conditions of the CAP Lemma. Each \( \mathcal{V}_i \) will be a finitely-generated growth hyperspace which, being a closed subset of \( 2^X \) contained in a \( \sigma \)-\( Z \)-set in \( 2^X \), is automatically a \( Z \)-set in \( 2^X \).

The generating collection for \( \mathcal{V}_1 \) is chosen so that \( \mathcal{V}_1 \) is homeomorphic to \( \mathcal{Q} \) and \( e_1 = \eta_{2^{-1}} \) maps \( 2^X \) into \( \mathcal{V}_1 \). Let \( X_i = [x_1, \ldots, x_m] \) be a \( (2^{-i-1}) \)-net in \( X \) and let \( D_1, \ldots, D_n \) be the non-empty subsets of \( X_i \). Choose a positive number \( \epsilon_i \) which is less than \( d(x_j, x_k)/2 \) whenever \( x_j \neq x_k \). When \( i > 1 \), we impose on \( \epsilon_i \) an additional condition (determined by \( \mathcal{V}_{i-1} \)) which will be stated when it is argued that \( \mathcal{V}_{i-1} \) is a \( Z \)-set in \( \mathcal{V}_i \). For \( j = 1, \ldots, n \), let \( B_j = \eta(D_j, \epsilon_i) \).

Each \( B_j \) is an element of \( \mathcal{J} \); because \( \mathcal{J} \) is dense in \( 2^X \), there exists \( B_j^* \in \mathcal{J} \) such that \( \rho(D_j, B_j^*) < \epsilon_i \), so \( B_j^* \subseteq B_j \) and each component of \( B_j \) must meet \( B_j^* \). Thus, \( \mathcal{V}_1 = \bigcup_{j=1}^n G_{B_j}(X) \) is contained in \( \mathcal{J} \). By (2.5), \( \mathcal{V}_1 \) is homeomorphic to \( \mathcal{Q} \) since \( \bigcup_{j=1}^n B_j = \eta(X_i, \epsilon_i) \neq X \) by our choice of \( \epsilon_i \).

To see that \( e_i : 2^X \to \mathcal{V}_i \), note that for each \( F \in 2^X \), each component of \( e_i(F) \) contains a ball of radius \( 2^{-i} \) (with respect to the convex metric \( d \)) which in turn must contain the ball of radius \( 2^{-i-1} \) centered at some \( x_k \in X_i \).
If \( D_j \) is the set of all such \( x_k \), then \( B_j = \eta(D_j, \epsilon_1) \) \( \subset \eta(D_j, 2^{-i-1}) \subset e_i(F) \) and each component of \( e_i(F) \) meets \( B_j \), so \( e_i(F) \in G_{A_j}(X) \subset \mathcal{K}_i \).

It remains to show that \( \mathcal{K}_{i-1} \) is a \( z \)-set in \( \mathcal{K}_i \) when \( i > 1 \). Let \( \{ B_1', \ldots, B_n' \} \) be the generating collection for \( \mathcal{K}_{i-1} \). If \( C_1, \ldots, C_m' \) are the components of \( \eta(X_{i-1}, \epsilon_{i-1}) \), then each component of each \( B_j' \) is some \( C_k \). Choose \( \epsilon_i \) small enough so that each \( C_k \setminus \eta(X_i, \epsilon_i) \) has non-empty interior, and set \( A_k = \text{cl}(C_k \setminus \eta(X_i, \epsilon_i)) \). By (2.7), given \( \epsilon > 0 \), there exists a map \( g: 2^X \to 2^X \setminus \bigcup_{j=1}^m 2^{A_j} \) such that \( d(g, \text{id}) < \epsilon \) and for any closed set \( B \subset X \setminus \bigcup_{j=1}^m A_j \), \( g[G_B(x)] \subset G_B(x) \).

Since \( B_1, \ldots, B_n \subset \eta(X_i, \epsilon_i) \subset X \setminus \bigcup_{j=1}^m A_j \), \( g[\mathcal{K}_i] \subset \mathcal{K}_i \), and \( \bigcup_{j=1}^m G_{B_j}(x) \subset \bigcup_{j=1}^m 2^{A_j} \) implies \( g: \mathcal{K}_i \to \mathcal{K}_i \setminus \mathcal{K}_{i-1} \). \( \square \)

(2.9) \textbf{Theorem.} If \( X \) is a non-degenerate Peano continuum and \( \mathcal{J} \) is a \( \sigma \)-\( z \)-set in \( 2^X \) with the inclusion property, then \( \mathcal{J} \) is a capset in \( 2^X \).

The proof is omitted since it so closely parallels the preceding proof. A tower of finitely-generated inclusion hyperspaces is constructed instead of a tower of finitely-generated growth hyperspaces. The details simplify somewhat since it is not necessary to keep track of components when dealing with the inclusion property.
(2.10) **Theorem.** If $X$ is a Peano continuum, $\mathcal{J}$ is a growth hyperspace of $X$ which is homeomorphic to $\mathbb{Q}$, and $\mathcal{J}_i$ is a dense $\sigma$-$\mathcal{Z}$-set in $\mathcal{J}$ with the growth property, then $\mathcal{J}_i$ is a capset in $\mathcal{J}$ provided $X$ contains no free arc or $\mathcal{J}$ (and therefore $\mathcal{J}_i$) has the inclusion property.

**Proof.** The proofs of (2.8) and (2.9) are unaffected by replacing $2^X$ with $\mathcal{J}$. $\mathcal{X}_i$ is closed in $2^X$ and contained in a $\sigma$-$\mathcal{Z}$-set in $\mathcal{J}$ and therefore is a $\mathcal{Z}$-set in $\mathcal{J}$. $\mathcal{X}_i$ is still homeomorphic to $\mathbb{Q}$, and $e_i$ certainly maps $\mathcal{J}$ into $\mathcal{X}_i$. □

It should be noted that only the fact that $\mathcal{J}$ is homeomorphic to $\mathbb{Q}$ was used above. $\mathcal{J}$ is automatically a growth hyperspace since it is the closure of a hyperspace with the growth property. The potential candidates for $\mathcal{J}$ include $C(X)$, $2^X_A$, $C_A(X)$, $2^X(A)$, $C(X;A)$, etc.

### 2.4. Applications.

The following three corollaries generalize an example of Kroonenberg [23] already cited: the hyperspace of all closed zero-dimensional subsets of $I$ is a pseudo-interior of $2^I$.

(2.11) **Corollary.** If $X$ is a non-degenerate Peano continuum, then $\mathcal{B}(X) = \{ F \in 2^X \mid \text{int } F = \emptyset \}$ is a pseudo-interior of $2^X$. If additionally $X$ contains no free arc, then $\mathcal{B}(X) \cap C(X)$ is a pseudo-interior of $C(X)$. 

Proof. Let $\{x_1, x_2, \ldots\}$ be a dense sequence in $X$. For each $i$, let $\mathcal{J}_i$ be the inclusion hyperspace generated by the collection $[\eta([x_1], 1/i), \ldots, \eta([x_i], 1/i)]$. By (2.6), each $\mathcal{J}_i$ is a Z-set in $2^X$, and $\mathcal{J}_i \cap C(X)$ is a Z-set in $C(X)$ if $X$ contains no free arc. The hyperspace $\mathcal{J}(X)$ of all closed subsets with non-empty interior in $X$ has the inclusion property. Thus, $\mathcal{J}(X) = \bigcup_{i=1}^{\infty} \mathcal{J}_i$ is a capset in $2^X$, and $\mathcal{J}(X) \cap C(X)$ is a capset in $C(X)$ if $X$ contains no free arc. □

(2.12) Corollary. If $X$ is a non-degenerate Peano continuum, then $\mathcal{J}(X) = \{ F \in 2^X \mid \dim F = 0 \}$ is a pseudo-interior of $2^X$.

Proof. $\mathcal{J}(X)$ consists of the closed, totally disconnected subsets of $X$. For each $i$, let $\mathcal{J}_i$ be the inclusion hyperspace of all closed subsets having a component with diameter at least $1/i$. So $\mathcal{J}(X) = \bigcup_{i=1}^{\infty} \mathcal{J}_i$ is an $F_0$ whose complement in $2^X$ is $\mathcal{J}(X)$. Curtis [14] has exhibited arbitrarily small maps of $2^X$ into the hyperspace $\mathcal{J}(X)$ of all finite subsets of $X$. Since $\mathcal{J}(X) \subseteq \mathcal{J}(X)$, $\mathcal{J}(X)$ is a $\sigma$-Z-set in $2^X$ containing the capset $\mathcal{J}(X)$, and therefore is itself a capset in $2^X$. □

(2.13) Corollary. If $X$ is a non-degenerate Peano continuum, then $\mathcal{J}(X) = \{ F \in 2^X \mid F$ is a Cantor set $\}$ is a pseudo-interior for $2^X$.

Proof. The Cantor sets are the perfect subsets of $X$
of dimension zero, so $\emptyset(X) \subseteq \partial(X)$. Since the latter is a pseudo-interior of $2^X$, it suffices to show that the complement $\partial(X) = 2^X \setminus \emptyset(X)$ is a $\sigma$-$Z$-set in $2^X$. Since $\partial(X)$ is the union of $d(X)$ and the hyperspace $\chi(X)$ of all closed subsets having an isolated point, we need only show that $\chi(X)$ is a $\sigma$-$Z$-set in $2^X$. There are arbitrarily small maps $\eta_\varepsilon : 2^X \to 2^X \setminus \chi(X)$ since, for any $\varepsilon > 0$ and $F \in 2^X$, $\eta_\varepsilon(F)$ can have no isolated points. For each $i$, let

$$\chi_i = \{ F \in 2^X \mid d(x, F \setminus \{x\}) \geq 1/i, \text{ for some } x \in F \}. $$

Then $\chi(X) = \bigcup_{i=1}^{\infty} \chi_i$ is an $F_\sigma$ in $2^X$, and therefore a $\sigma$-$Z$-set in $2^X$. □

The next two results were originally obtained by Kroonenberg [23].

(2.14) **Corollary.** $\bar{\rho}(Q) = \{ F \in 2^Q \mid F \text{ is a } Z \text{-set in } Q \}$ is a pseudo-interior of $2^Q$ and $\bar{\rho}(Q) \cap C(Q)$ is a pseudo-interior of $C(Q)$.

**Proof.** $\sigma(Q) = 2^Q \setminus \bar{\rho}(Q)$ is an $F_\sigma$ in $2^Q$ since it can be expressed as $\bigcup_{i=1}^{\infty} \sigma_i$, where

$$\sigma_i = \{ F \in 2^Q \mid d(f, \text{id}) > 1/n, \text{ for all } f : Q \to Q \setminus F \}. $$

There exist arbitrarily small maps $g : Q \to s$ which induce arbitrarily small maps $\bar{g} : (2^Q, C(Q)) \to (2^s, C(s))$. Since $2^s \subseteq \bar{\rho}(Q)$, $\sigma(Q)$ is a $\sigma$-$Z$-set in $2^Q$. Since $\sigma(Q)$ has the inclusion property, $\sigma(Q)$ is a capset in $2^Q$ and $\sigma(Q) \cap C(Q)$ is a capset in $C(Q)$. □

(2.15) **Corollary.** $(2^Q, 2^s) \sim (C(Q), C(s)) \sim (Q, s)$. 
Proof. Each endslice $\mathcal{W}$ of $\mathcal{Q}$ is a Z-set in $\mathcal{Q}$. It follows that $2^\mathcal{Q}(\mathcal{W})$ is a Z-set in $2^\mathcal{Q}$ and $C(\mathcal{Q}; \mathcal{W})$ is a Z-set in $C(\mathcal{Q})$. The union of all such $2^\mathcal{Q}(\mathcal{W})$ is $2^\mathcal{Q}\setminus 2^S$, which has the inclusion property. Thus, $2^\mathcal{Q}\setminus 2^S$ is a capset in $2^\mathcal{Q}$ and $C(\mathcal{Q})\setminus C(S)$ is a capset in $C(\mathcal{Q})$. □

The following generalization of (2.15) is due to Curtis [12].

(2.16) Corollary. If $\mathcal{X}$ is a connected, topologically complete, nowhere locally compact space which admits a metric with property $S$, then $\mathcal{X}$ has a Peano compactification $\tilde{\mathcal{X}}$ such that $(2^{\tilde{\mathcal{X}}}, 2^{\mathcal{X}}) \cong (C(\tilde{\mathcal{X}}), C(\mathcal{X})) \cong (\mathcal{Q}, S)$.

Proof. It can be shown that $\mathcal{X}$ has a Peano compactification $\tilde{\mathcal{X}}$ such that the remainder $\tilde{\mathcal{X}}\setminus \mathcal{X}$ is locally non-separating in $\tilde{\mathcal{X}}$. $\tilde{\mathcal{X}}\setminus \mathcal{X}$ can then be expressed as $\bigcup_{i=1}^{\infty} F_i$ where each $F_i$ is a closed, locally non-separating set in $\tilde{\mathcal{X}}$. For each $i$, $2^{\tilde{\mathcal{X}}}(F_i)$ is a Z-set in $2^{\tilde{\mathcal{X}}}$ and $C(\tilde{\mathcal{X}}; F_i)$ is a Z-set in $C(\tilde{\mathcal{X}})$. Since $\tilde{\mathcal{X}}\setminus \mathcal{X}$ is dense in $\tilde{\mathcal{X}}$, it is easy to see that $2^{\tilde{\mathcal{X}}}\setminus 2^{\mathcal{X}} = \bigcup_{i=1}^{\infty} 2^{\tilde{\mathcal{X}}}(F_i)$ is dense in $2^{\tilde{\mathcal{X}}}$ and likewise $C(\tilde{\mathcal{X}})\setminus C(\mathcal{X}) = \bigcup_{i=1}^{\infty} C(\tilde{\mathcal{X}}; F_i)$ is dense in $C(\tilde{\mathcal{X}})$ and $2^{\tilde{\mathcal{X}}}\setminus 2^{\mathcal{X}}$ and $C(\tilde{\mathcal{X}})\setminus C(\mathcal{X})$ each have the growth property; these are capsets in $2^{\tilde{\mathcal{X}}}$ and $C(\tilde{\mathcal{X}})$, respectively, since $\tilde{\mathcal{X}}$ contains no free arc. □

(2.17) Remark. In the original proof by Curtis and
Schori[15] that \( 2^X \approx \mathbb{Q} \) for any non-degenerate Peano continuum \( X \), \( 2^X \) was represented as the inverse limit of a sequence of hyperspaces \( 2^{\Gamma_1} \), where each \( \Gamma_1 \) is a finite connected graph contained in \( X \). The inverse limit is a subset of the product \( \prod_{i=1}^\infty 2^{\Gamma_i} \). If this product is given the metric \( \tilde{\rho} \) defined by \( \tilde{\rho}(x,y) = \sum_{i=1}^\infty 2^{-i} \rho_i(x_i,y_i) \), where each \( \rho_i \) is bounded by 1, then for each projection map \( \pi_i \), \( \tilde{\rho}(\pi_i, \text{id}) \leq 2^{-i} \). In particular, given \( \epsilon > 0 \), there exists a finite connected graph \( \Gamma \subset X \) and map \( f: 2^X \to 2^\Gamma \) such that \( \rho(f, \text{id}) < \epsilon \).

(2.18) Proposition. For each non-degenerate Peano continuum \( X \) and positive integer \( n \), the hyperspace \( C^n(X) \) of all closed subsets of \( X \) with at most \( n \) components is a \( Z \)-set in \( 2^X \).

Proof. Given \( \epsilon > 0 \), there exists a finite connected graph \( \Gamma \subset X \) and map \( f: 2^X \to 2^\Gamma \) such that \( \rho(f, \text{id}) < \epsilon/3 \).

Let \( e: 2^\Gamma \to 2^\Gamma \) be the expansion map \( \eta_{\epsilon/3} \). For each \( F \in 2^\Gamma \), each component of \( e(F) \) has diameter at least \( 2\epsilon/3 \). If \( \Delta \) is a fixed triangulation of \( \Gamma \) such that the length of each segment of \( \Gamma \) is less than \( \epsilon/3 \), then \( e(F) \) must contain some segment.

Say \( A_1, \ldots, A_m \) are the segments of \( \Gamma \). Subdivide each \( A_i \) into \( n \) subsegments \( A_{i1}, \ldots, A_{in} \). For each \( j = 1, \ldots, n \), pick an arc \( B_{ij} \subset A_{ij} \). By (2.7), there is a map \( g: 2^\Gamma \to 2^\Gamma \setminus \bigcup_{i=1}^m \bigcup_{j=1}^n 2^{\Gamma_{B_{ij}}} \) such that \( \rho(g, \text{id}) < \epsilon/3 \) and
$g(E) \setminus B = E \setminus B$ for each $E \in 2^\Gamma$, where $B = \bigcup_{i=1}^m \bigcup_{j=1}^n B_{ij}$. If $E$ contains a segment $A_k$, then $g(E) \cap A_k$ has at least $n+1$ components. Since $A_k$ is not a cycle in $\Gamma$, $g(E)$ has at least $n+1$ components. Thus, $g \circ e : 2^\Gamma \to 2^\Gamma \setminus C^n(\Gamma)$ and $g \circ e \circ f : 2^X \to 2^X \setminus C^n(X)$ with $d(g \circ e \circ f, \text{id}) < \epsilon$. □

Since each $C^n(X)$ is a growth hyperspace, we have the following consequence of (2.10).

(2.19) Corollary. If $X$ is a non-degenerate Peano continuum containing no free arc, then the hyperspace $\bigcup_{n=1}^\infty C^n(X)$ of all closed subsets with finitely many components is a capset in $2^X$. 
CHAPTER 3

GEOMETRIC VERSIONS OF THE

FINITE-DIMENSIONAL COMPACT ABSORPTION PROPERTY

3.1. **A cellular version.**

The parallel definitions of the compact absorption property and the finite-dimensional compact absorption property suggest that theorems pertaining to one property should have analogues pertaining to the other. While such is often the case, the development of a more usable characterization of the F-D CAP reveals dramatic differences between the two properties.

If we attempt to mimic the CAP Lemma, using a tower of finite-dimensional cubes in place of a tower of Hilbert cubes, two potential difficulties are immediately apparent. First, $I^n$ does not exhibit the nice homogeneity that $Q$ does. (The Homeomorphism Extension Theorem for $Q$ is also referred to as the Z-set homogeneity of $Q$.) Second, $I^n$ does not have the "universal appeal" that $Q$ has. In using (2.3) to verify that a subset of $Q$ has the CAP, copies of $Q$ are easy to come by; for instance, there are countless hyperspaces with different geometric descriptions which are all homeomorphic to $Q$. On the other hand, there are infinitely many topologically distinct finite-dimensional spaces of interest (in particular, among hyperspaces). Thus, a faithful analogue of the CAP Lemma may be too restrictive to
be of much use.

The first problem is overcome in this section by finding suitable finite-dimensional analogues of the Mapping Replacement and Homeomorphism Extension Theorems for $Q$. The next section addresses the second problem, anticipating the structure of the particular finite-dimensional hyper-spaces to be encountered in Chapter 4.

The analogue of (2.1) needed is the classic result that for any $m$-dimensional separable metric space $K$, the space of embeddings of $K$ into $I^{2m+1}$ is dense in the space of mappings of $K$ into $I^{2m+1}$ (see [20, p.56]).

(3.1) Finite-Dimensional Mapping Replacement Theorem. Given $\epsilon > 0$ and $g: K \rightarrow I^{2m+1}$, where $K$ is a separable metric space of dimension $m$, there exists an embedding $h: K \rightarrow I^{2m+1}$ with $d(g, h) < \epsilon$.

In contrast to the theorem for $Q$, it may be impossible to have $h(K)$ be a $Z$-set in $I^{2m+1}$. In fact, each $Z$-set in $I^n$ must be contained in $I^n$, the combinatorial boundary. To determine what hypothesis should correspond to the requirement in the CAP Lemma that $M_i$ be a $Z$-set in $M_{i+1}$, we examine Barit's proof in [5] of the Homeomorphism Extension Theorem for $Q$. The construction of the homeomorphism $H: Q \rightarrow Q$ extending a given homeomorphism $h: K_1 \rightarrow K_2$ between $Z$-sets hinges on the fact that $K_1$ and $K_2$ can be taken to lie in $s$ and to have infinite deficiency. The infinitely many "free" coordinates provide the room needed
to build the extension. Anderson observes in [3] that Barit's proof has a finite-dimensional corollary. Essentially, a homeomorphism $h: K_1 \rightarrow K_2$ between compact subsets of $\mathbb{R}^m$ can be extended if enough extra coordinates are provided. The following is a finite-dimensional adaptation of Barit's theorem and proof.

(3.2) **Finite-Dimensional Homeomorphism Extension**

**Theorem.** Let $h: K_1 \rightarrow K_2$ be a homeomorphism between compact subsets of $\mathbb{R}^m$ such that $d(h, \text{id}) < \epsilon$. Then there exists a homeomorphism $H: \mathbb{R}^{2m} \rightarrow \mathbb{R}^{2m}$ such that $H$ extends $h$ and $d(H, \text{id}) < \epsilon$.

**Proof.** We consider $\mathbb{R}^{2m}$ as $\mathbb{P}_1^m \times \mathbb{P}_2^m$ with $K_1, K_2 \subset \mathbb{P}_1^m$; the homeomorphism we extend is $h \times \text{id}: K_1 \times \{\bar{0}\} \rightarrow K_2 \times \{\bar{0}\}$.

Suppose $d(h, \text{id}) \leq \epsilon_1 < \epsilon_1 + 2\delta < \epsilon$. For $i = 1, 2$, choose homeomorphisms $f_i: K_i \rightarrow K \subset D = \{y \in \mathbb{P}_2^m \mid d(y, \bar{0}) \leq \delta\}$ so that $f^{-1}_2 \circ f_1 = h$, and let $\Gamma_i = \{(f_i^{-1}, y) \mid y \in K\}$ be the graph of $f_i$. 

$$
\begin{array}{c}
\{\bar{0}\} \times \mathbb{P}_2^m \\
\{\bar{0}\} \times K \\
\end{array}
\begin{array}{c}
\Gamma_1 \\
\Gamma_2 \\
\end{array}
\begin{array}{c}
(id, f_1) \\
\downarrow \\
(id, f_2)^{-1} \\
\end{array}
\begin{array}{c}
K_1 \times \{\bar{0}\} \\
K_2 \times \{\bar{0}\} \\
\mathbb{P}_1^m \times \{\bar{0}\} \\
\end{array}
\begin{array}{c}
h \times \text{id} \\
\end{array}$$
The composition of the maps

\[ (\text{id}, f_1) \xrightarrow{(\text{id}, f_2)} \Gamma_1 \xrightarrow{h \times \text{id}} \Gamma_2 \xrightarrow{(\text{id}, f_2)^{-1}} K_2 \times \{\overline{0}\} \]

is the map to be extended. The desired homeomorphism \( H \) will be the composition of the extensions

\[ F_1 \times F_2 \xrightarrow{F_1} R_1 \times F_2 \xrightarrow{G} F_1 \times F_2 \xrightarrow{F_2^{-1}} F_1 \times F_2 \]

to be constructed.

First, define \( g: K \longrightarrow R_1^m \) by \( g(y) = f_2^{-1}(y) - f_1^{-1}(y) \). By Dugundji's Extension Theorem (see [20, p.188]), there is an extension \( \tilde{g}: R_2^m \longrightarrow F_1^m \) whose image is contained in the convex hull of the image of \( g \). Since \( d(h, \text{id}) \leq \epsilon_1 \), the image \( g \) is contained in the set \( E = \{ x \in F_1^m \mid d(x, \overline{0}) \leq \epsilon_1 \} \), which is convex. Hence the image of \( \tilde{g} \) is contained in \( E \), i.e., \( d(\tilde{g}(y), \overline{0}) \leq \epsilon_1 \) for all \( y \in F_2^m \).

For \( i = 1, 2 \), \( f_i: K_i \longrightarrow F_i^m \) has its image in the convex set \( D \). Again by Dugundji's Extension Theorem, there exist extensions \( \tilde{f}_i: F_i^m \longrightarrow F_2^m \) whose images are contained in \( D \). Thus, \( d(\tilde{f}_i(x), 0) \leq \delta \) for all \( x \in F_i^m \).

Define \( G: F_1^m \times F_2^m \longrightarrow F_1^m \times F_2^m \) by \( G(x, y) = \langle x + \tilde{g}(y), y \rangle \). This is a homeomorphism and \( d(G, \text{id}) \leq \epsilon_1 \) because \( d(\tilde{g}(y), \overline{0}) \leq \epsilon_1 \) for all \( y \in F_2^m \). Moreover, if \( f_i^{-1}(y), y \in \Gamma_i \)

\[ G(f_i^{-1}(y), y) = \langle f_i^{-1}(y) + f_2^{-1}(y) - f_1^{-1}(y), y \rangle = \langle f_2^{-1}(y), y \rangle, \]

so \( G \) extends \( h \times \text{id}: \Gamma_1 \longrightarrow \Gamma_2 \).

We also extend \( \tilde{f}_i \) to a homeomorphism \( \tilde{F}_i \) of \( R_1^m \times R_2^m \) by defining \( \tilde{F}_i(x, y) = \langle x, y + \tilde{f}_i(x) \rangle \). Since \( d(\tilde{F}_i(x), 0) \leq \delta \)
for all $x \in F^m_1$, we have $d(F^m_1, \text{id}) \leq \delta$. Whenever $x \in K_i$, $F^m_1(x, \delta) = \langle x, \delta + f^1_i(x) \rangle = \langle x, f^1_i(x) \rangle$, so $F^m_1$ extends $(\text{id}, f^1_i): K_i \times \{\delta\} \rightarrow \Gamma_i$. The composition $H = F^{-1}_2 \circ G \circ F_1$ is an extension of $h \times \text{id}: K_i \times \{\delta\} \rightarrow K_2 \times \{\delta\}$ which satisfies $d(h, \text{id}) \leq \epsilon_1 + 2\delta < \epsilon$. □

In the preceding proof, $R^m_1 \times R^m_2 \cong (0,1)^{2n} = \mathbb{R}^{2m}$ corresponds geometrically to $L^2_1 \times L^2_2 \cong s_1 \times s_2$ in the proof of the infinite-dimensional theorem. It appears that the condition to be imposed on the position of $M_i$ in $M_{i+1}$ should correspond to the position of $F^m$ in $R^{2m}$. Actually, what is needed is $(E_2, E_1) \cong (R^{2n}, F^n)$ and $M_i \subset E_1 \subset E_2 \subset M_k$, for some $E_1, E_2, n, k$.

In the case when $M_i \cong T^{m_i}$ for each $i$, this can be achieved by simply requiring that $M_i$ be contained in $M_{i+1}$. We may assume that $M_i \subset F^0$ for some face $F$ of $M_{i+1}$. Let $n = \text{dim} F$. For each $j$, $M_j \times I$ embeds in $M_{j+1}$ with the image of $M_j \times \{0\}$ being $M_j$. So there exists an embedding $\varphi: F \times I^n \times I^n \rightarrow M_{i+2n}$ such that $\varphi(F \times \{0\} \times \{0\}) = F$. Let $L = \varphi[F \times \{0\} \times I^n] \cup \varphi[F \times I^n \times \{0\}]$. Then $(L^0, F^0) \cong (R^{2n}, R^n)$ and $M_i \subset F^0 \subset L^0 \subset M_{i+2n}$.

(3.3) **First F-D CAP Lemma.** A subset $M$ of $Q$ has the finite-dimensional compact absorption property if $M$ can be expressed as $\bigcup_{i=1}^{\infty} M_i$, where

1) for each $i$, $M_i \cong T^{m_i}$ for some $m_i$, $M_i$ is a $z$-set in $Q$, and $M_i \subset M_{i+1}$, and

2) for each $\epsilon > 0$, there exists an integer $l$ and map
Proof. For each \( \epsilon > 0 \), positive integer \( j \), and \( m \)-dimensional compact subset \( K \) of \( Q \), we must exhibit a positive integer \( k \) and embedding \( h: K \rightarrow M_k \) such that \( h|K \cap M_j = \text{id} \) and \( d(h, \text{id}) < \epsilon \). Choose \( l > \max\{j, 2m\} \) and \( g_1: Q \rightarrow M_l \) such that \( d(g_1, \text{id}) < \epsilon/4 \). Since \( M_l \approx I^{m_l} \) and \( m_l \geq l \geq 2m + 1 \), by (3.1) there exists an embedding \( g: K \rightarrow M_l \) such that \( d(g, g_1) < \epsilon/4 \). Thus, \( d(g, \text{id}) < \epsilon/2 \) and \( d(g^{-1}, \text{id}) < \epsilon/2 \).

By previous comments, there exist positive integers \( k \) and \( n_k \), and sets \( E_1 \) and \( E_2 \) such that \( M_k \subseteq E_1 \subseteq E_2 \subseteq M_k \) and \( (E_2, E_1) \approx (F^{2m_k}, R^{m_k}) \). Since \( g^{-1}|g[K \cap M_j] \) is a homeomorphism between compact subsets of \( E_1 \) such that \( d(g^{-1}, \text{id}) < \epsilon/2 \), by (3.2) there exists a homeomorphism \( f: E_2 \rightarrow E_2 \) extending \( g^{-1}|g[K \cap M_j] \) with \( d(f, \text{id}) < \epsilon/2 \). Then \( f \circ g \) is an embedding of \( K \) into \( M_k \) which leaves \( K \cap M_j \) fixed and which is within \( \epsilon \) of the identity. \( \Box \)

As a corollary we have that \( \tau = \bigcup_{i=1}^{\infty} (I^i \times \{0, 0, \ldots\}) \) is an \( f \)-d capset in \( I^\omega \). Also, in the standard Hilbert cube, \( \sigma = \{x \in \Sigma \mid x_i = 0 \text{ for all but finitely many } i\} \) is an \( f \)-d capset since it can be expressed as \( \bigcup_{i=2}^{\infty} ([1+1/i, 1-1/i]^i \times [0, 1-1/i] \times \{0, 0, \ldots\}) \).

To generalize the First F-D CAP Lemma, we observe that a tower is an \( f \)-d capset if it "sandwiches" a tower which is an \( f \)-d capset. This trivial observation yields the following lemma.
(3.4) Second F-D CAP Lemma. A subset $M$ of $Q$ has the finite-dimensional compact absorption property if $M$ can be expressed as $\bigcup_{i=1}^{\infty} M_i$, where

1) for each $i$, $M_i$ is a Z-set in $Q$,

2) for each $\varepsilon > 0$, there exists a positive integer $l$ and map $g: Q \rightarrow M_i$ such that $d(g, id) < \varepsilon$, and

3) for each $i$, there exists $D_i \sim I^m_i$ such that $M_i \subset D_i \subset D_i \subset M_n_i$, for some $m_i, n_i$.

Proof. Set $N_i = D_k(i)$, where $k(1) = 1$ and $k(i+1) = n_k(i)$. Then the tower $N_1 \subset N_2 \subset \ldots$ satisfies the conditions of the First F-D CAP Lemma since, for each $i$,

$$N_i = D_k(i) \subset M_{n_k(i)} \subset \hat{D}_{n_k(i)} = \hat{D}_k(i+1) = N_{i+1}.$$ 

Therefore, $\bigcup_{i=1}^{\infty} M_i = \bigcup_{i=1}^{\infty} N_i$ is an f-d capset in $Q$. □

As a corollary we note that

$$\{x \in \ell^2 \mid \sum_{i=1}^{n} (ix_i)^2 = 1, \text{ for some } n\}$$

is an f-d capset in the ellipsoidal Hilbert cube

$$E = \{x \in \ell^2 \mid \sum_{i=1}^{\infty} (ix_i)^2 \leq 1\}$$

since the former can be written as $\bigcup_{j=1}^{\infty} E_j$, where

$$E_j = \{x \in \ell^2 \mid \sum_{i=1}^{j} (ix_i)^2 = 1\}$$

is a $(j-1)$-sphere which is the combinatorial boundary of the $j$-hemisphere $E_{j+1}^+ \sim I^j$.

3.2. A polyhedral version.

In light of the Second F-D CAP Lemma, f-d capsets can be identified by finding conditions on the position of $M_i$ in $M_i+1$ which imply the existence of $D_i$ as in condition 3) above. Intuitively, one would expect to be able to build
such a $D_i$ in $M_{n_i}$ if locally $M_i$ has sufficiently high co-dimension in $M_{n_i}$. One way to obtain local codimension $m$ is to require that $M_i \times I^m$ can be embedded in $M_{n_i}$ so that the image of $M_i \times \{0\}$ is $M_i$. We say that $\varphi$ is an $m$-fringe over $K$ in $L$ if $\varphi : K \times I^m \to L$ is an embedding such that $K = \varphi[K \times \{0\}]$; we call $K$ the base of $\varphi$.

The characterization of the F-D CAP that will serve our needs is derived from the Second F-D CAP Lemma by showing that condition 3) of that lemma follows from the hypothesis that each $M_i$ is a contractible polyhedron which is the base of a 1-fringe in $M_i'$, for some $i'$. The proof of this uses some techniques from simple-homotopy theory (see Cohen [8]) and the theory of regular neighborhoods (see Rourke and Sanderson [25]).

Let $K$ be a subpolyhedron of $L$. We say $K \searrow L$ (there is an elementary expansion from $K$ to $L$) and $L \nearrow K$ (there is an elementary collapse from $L$ to $K$) if $L = K \cup A$, where $A$ is a simplex of $L$, $K \cap A$ is an $n$-ball in $K$, and $(A, K \cap A) \approx (I^{n+1}, I^n)$ for some $n$. If there is a finite sequence $K \nearrow K_1 \nearrow \ldots \nearrow L$, we say $K \nearrow L$ ($K$ expands to $L$) and $L \searrow K$ ($L$ collapses to $K$). Two polyhedra are said to have the same simple-homotopy type if one can be obtained from the other by a finite sequence of elementary expansions or collapses or both.

It is a standard result that if $K$ and $L$ have the same simple-homotopy type, then there exists a polyhedron $P$ such that $K \nearrow P \searrow L$. A contractible polyhedron $K$ may fail
to be collapsible, but it does have the simple-homotopy type of a point, and therefore $K/P \not\sim \{\text{point}\}$, for some $P$.

(3.5) Proposition. If $K$ is a contractible polyhedron, then there exists $D \approx I^m$ such that $K=K \times \{\bar{0}\} \subset D \subset K \times I^n$, for some $m,n$.

Proof. With $K/P \not\sim \{\text{point}\}$ as in the preceding comments, view $P$ as sitting in $P^m$ for some $m$. $K/P$ can be expressed as $K=K_1 \not\sim K_2 \not\sim \ldots \not\sim K_j=\bar{P}$. Each elementary expansion $K_i \not\sim K_{i+1}$ can be realized using a factor $I_{i+1}$, so we can assume $K \times \{\bar{0}\} \subset P \subset K \times I^j$.

Let $D$ be a regular neighborhood of $P$. $D \approx I^m$ since $P$ is collapsible. Furthermore, $D \not\sim P$, which can be expressed as $P=P_1 \not\sim P_2 \not\sim \ldots \not\sim P_k=\bar{D}$. So as before, $D$ lies in a copy of $P \times I^k$. Thus, $K \times \{\bar{0}\} \subset D \subset K \times I^{j+k}$. □

(3.6) Third $F-D$ CAP Lemma. A subset $M$ of $Q$ has the finite-dimensional compact absorption property if $M$ can be expressed as $\bigcup_{i=1}^{\infty} M_i$, where

1) for each $i$, $M_i$ is a $Z$-set in $Q$,

2) for each $\epsilon > 0$, there exists a positive integer $l$ and map $g:Q \to M_l$ with $d(g,\text{id}) < \epsilon$, and

3) for each $i$, $M_i$ is a contractible polyhedron which is the base of a $l$-fringe in $M_i$, for some $i'$.

Proof. By (3.5), for each $i$, there exists $D_i \approx I^{m_i}$ such that $M_i \subset D_i \subset M_{n_i}$, for some $m_i,n_i$. Let $\varphi$ be a $l$-fringe over $M_{n_i}$ in some $M_i$. Then $D_i^l=\varphi[D_i \times I] \approx I^{m_i+l}$.
and $M_i \subset D_i \subset D_j \subset M_j$. Thus the tower satisfies the conditions of the Second F-D CAP Lemma. □

It turns out that the hyperspaces of interest in Chapter 4 are contractible polyhedra, and fringes over the simplices of one of these (in some other one) arise quite naturally. However, given $\mathcal{X}$ from among these hyperspaces, there is no apparent way in terms of $X$ to define a 1-fringe over all of $\mathcal{X}$. The following theorem demonstrates that the existence of an abundance of limited fringes may imply the existence of a total fringe.

(3.7) Theorem. Let $K \subset X_1 \subset X_2 \subset \ldots$ be a tower with $K$ a collapsible polyhedron. Suppose there is a collection $\mathcal{F}$ each element of which is an $m$-fringe over $A$ in $X_n$, for some integers $m, n$ and simplex $A$ of $K$, and suppose that $\mathcal{F}$ satisfies the following conditions:

1. for each simplex $A$ of $K$, the 0-fringe $id_A$ belongs to $\mathcal{F}$,  
2. if $\varphi:A \times I^m \longrightarrow X_n$ belongs to $\mathcal{F}$ and $B$ is a face of $A$, then $\varphi|B \times I^m$ also belongs to $\mathcal{F}$, and  
3. if $\{\varphi_1, \ldots, \varphi_j\} \subset \mathcal{F}$ and, for some integer $k$ and simplex $A$ of $K$, the base of each $\varphi_i$ is contained in $A$ and $F = \bigcup_{i=1}^j \text{im} \varphi_i \subset X_k$, then there exists a 1-fringe $\psi:F \times I \longrightarrow X_{k+1}$ such that $\text{im} \psi \cap X_k = F$ and, for each $\varphi_i:A_i \times I^{m_i} \longrightarrow X_{n_i}$, the fringe $\psi \ast \varphi_i:A_i \times I^{m_i+1} \longrightarrow X_{k+1}$ defined by $\psi \ast \varphi_i(a; t_1, \ldots, t_{m_i+1}) = \psi(\varphi_i(a; t_1, \ldots, t_m), t_{m_i+1})$
belongs to \( \mathfrak{I} \).

Then there exists a 1-fringe \( \xi \) over \( K \) in some \( X_\ell \).

**Proof.** Suppose \([\text{point}] = K_1 \xi K_2 \xi \cdots \xi K_s = K\). We inductively construct a 1-fringe \( \xi_r \) over \( K_r \) in \( X_{X_r} \), for \( r = 1, \ldots, s \), and take \( \xi \) to be \( \xi_s \). Let \( \xi_1 \in \mathfrak{I} \) be some 1-fringe over \( K_1 \) in \( X_{X_1} \). Suppose we have already defined a 1-fringe \( \xi_r \) over \( K_r \) such that \( \text{im} \xi_r \cap K = K_r \), and for each simplex \( C \) in \( K_r \) there exists \( [\chi_1, \ldots, \chi_m] \in \mathfrak{I} \) such that \( \xi_r [C \times I] \subseteq \bigcup_{i=1}^m \text{im} \chi_i \) and the base of each \( \chi_i \) is contained in \( C \).

\( K_{r+1} \) is obtained from \( K_r \) by attaching a simplex \( A \) at a ball \( B \). Then there exists \( \varphi_1, \ldots, \varphi_m \in \mathfrak{I} \) such that \( \xi_r [B \times I] \subseteq \bigcup_{i=1}^m \text{im} \varphi_i \) and each \( \varphi_i \) has its base contained in \( B \). Let \( \varphi_{m+1} \) be the 0-fringe \( \text{id}_A \) and say \( F = \bigcup_{i=1}^{m+1} \text{im} \varphi_i \) is contained in \( X_K \). By hypothesis, there exists a fringe \( \psi : F \times I \to X_{K+1} \) such that \( \text{im} \psi \cap X_K = F \) and \( \psi_i = \psi \varphi_i \) belongs to \( \mathfrak{I} \) for each \( i = 1, \ldots, m+1 \). Let \( E \) denote \( \xi_r [B \times I] \).

\( (A, B) \sim (B \times I, B \times \{0\}) \)

so \( \psi [(E \cup A) \times I] \) can be reparametrized as \( A \times I \) so that \( E = B \times I \).

This means a 1-fringe \( \psi \) can be defined over \( A \) so as to extend \( \xi_r [B \times I] \). Thus, \( \xi_r \) can be extended to a 1-fringe \( \xi_{r+1} \) over \( K_{r+1} \), and \( \text{im} \xi_{r+1} \cap K = K_{r+1} \) since \( \text{im} \psi \cap K = A \).
It remains to show that for each simplex $C$ in $K_{r+1}$ there exists $\chi_1, \ldots, \chi_m \in \mathcal{C}$ such that $\xi_{r+1}[C \times I]$ is contained in $\bigcup_{i=1}^{m} \text{im} \chi_i$ and the base of each $\chi_i$ is contained in $C$. Actually it is only necessary to show this for the simplex $A$ and its free face $A'$ since these are the only simplices of $K_{r+1}$ which are not in $K_r$. By construction, $\xi_{r+1}[A \times I] = \psi'[A \times I] = \psi[(E \cup A) \times I] \subseteq \bigcup_{i=1}^{m+1} \text{im} \psi_i$, where $\psi'$, $\psi$, and $\psi_i$ are as above. Likewise, $\xi_{r+1}[A' \times I] = \psi'[A' \times I] \subseteq \bigcup_{i=1}^{m'} \text{im} \psi'_i$, where each $\psi'_i$ is a restriction of some $\psi_i'$, and therefore an element of $\mathcal{C}$. □
CHAPTER 4

THE FINITE-DIMENSIONAL

COMPACT ABSORPTION PROPERTY

IN HYPERSPACES OF GRAPHS

4.1. Finite-dimensional growth hyperspaces.

In Chapter 2, certain capsets in $2^X$ were identified by constructing a tower of growth hyperspaces. In this chapter, certain f-d capsets are identified in like manner. We will not be confronted with the alternate hypotheses "no free arc" and "inclusion property" that were needed to guarantee that the elements of the tower in Chapter 2 were Hilbert cubes. Rather, an abundance of free arcs is needed to provide enough finite-dimensional growth hyperspaces. Our attention is therefore restricted to hyperspaces of a finite, non-degenerate, connected graph $\Gamma$.

Unlike the previous situation, all the finitely-generated growth hyperspaces in the tower to be constructed are topologically distinct. In this section, a number of examples are provided to develop the techniques for analyzing their structure. Kelley noted in [21] that for a finite connected graph $\Gamma$, $C(\Gamma)$ is a polyhedron. Duda [17],[18] has studied the structure of $C(\Gamma)$ in detail. The simplest cases of $C(\Gamma)$ serve as a model for the parametrization of polyhedral growth hyperspaces.
(4.1) Example. $C(I)$ is homeomorphic to the triangle in $F^2$ with vertices $\langle 0,1 \rangle$, $\langle 1,1 \rangle$, $\langle 0,0 \rangle$ via the correspondence given by $F \leftrightarrow \langle \min F, \max F \rangle$. In this representation, $F(I)$ is the hypotenuse.

An alternate representation is the triangle with vertices $\langle 0,0 \rangle$, $\langle 1,1 \rangle$, $\langle \frac{1}{2},1 \rangle$, where $F$ corresponds to $\langle \text{midpt } F, \text{diam } F \rangle$. $F(I)$ appears as the base of the triangle.

(4.2) Example. Let $D^2$ be the closed unit disk in $F^2$, and let $S^1 = \dot{D}^2$. Each $F \in C(S^1) \backslash \{S^1\}$ can be characterized by its geodesic midpoint $m(F)$ and geodesic diameter (i.e., arc length) $\delta(F)$. The correspondence between $F$ and $(1 - \delta(F)/2\pi) \cdot m(F)$ gives a homeomorphism between $C(S^1)$ and $D^2$ with $F(I)(S^1) = S^1$.

Another approach is to obtain the space $S^1 \times [0,2\pi]$ by supposing $m(F)$ and $\delta(F)$ to be independent, and then identify the rim $S^1 \times \{2\pi\}$ to a point, namely $S^1 \in C(S^1)$. In this case, $F(I)(S^1)$ is $S^1 \times \{0\}$.

(4.3) Example. Let $T$ be the triod consisting of three segments $A$, $B$, $C$ meeting at the vertex $v$. Each
subcontinuum containing the point $v$ can be characterized by its endpoints in $A$, in $B$, and in $C$. Any other subcontinuum is either contained in $A$, in $B$, or in $C$. So $C(T)$ is the 3-cube $C_{\{v\}}(T)$ with the triangular flanges $C(A)$, $C(B)$, $C(C)$ attached along the edges meeting at the vertex $\{v\}$. The union of those three edges is $F_1(T)$.

If $E \in C(T)$, then $G_E(T)$ is a subset of $C(T)$; $C_{\{v\}}(T)$ in the preceding example is such a hyperspace. Even if $E \in 2^X$ has finitely many components, $G_E(T)$ is easier to parametrize than $C(T)$ since it is "anchored" at the element $E$; all other elements of $G_E(T)$ can be identified by how they have "grown" from $E$. Actually there are two types of growth to be parametrized: outward growth and gap-closing growth.

(4.4) Example. With $T$ as above, $G_A(T) \approx I_B \times I_C$, where $I_B$ is a copy of the segment $B$ which describes growth in $B$ (and likewise for $I_C$). We give the endpoints of $I_B$ the labels "UB" and "UB" to indicate no growth in $B$ (i.e., $F \cap B = \{v\}$) and inclusive growth in $B$ (i.e., $F \cap B = B$), respectively.
(4.5) Example. Trisect \( I \) as \( \alpha \cup \beta \cup \gamma \). To compute \( G_{\alpha \cup \gamma}(I) \), note that it is possible to grow outward from \( \alpha \) toward \( \gamma \) ("positive growth") or outward from \( \gamma \) toward \( \alpha \) ("negative growth"). However, the positive and negative growth are not independent, for simultaneous growth in both directions may close the gap \( \beta^0 \) between \( \alpha \) and \( \gamma \). In fact, there are infinitely many paths from "\( \cup \emptyset \)" to "\( \cup \beta \)". We represent this by a triangle in \( \mathbb{R}^2 \). The (\(+\))-coordinate measures growth to the right in \( \beta \) and the (\(-\))-coordinate measures growth to the left. The hypotenuse is dotted to indicate that it is identified to a point, in this case "\( \cup \beta \)" = \( I \in G_{\alpha \cup \gamma}(I) \).

(4.6) Example. With \( T \) as before, trisect \( A \) as \( A_1 \cup A_2 \cup A_3 \).

Consider \( G_{A_1 \cup A_3}(T) \). Starting from \( A_1 \cup A_3 \), growth in \( A_2 \), growth in \( B \), and growth in \( C \) are independent. Therefore, \( G_{A_1 \cup A_3}(T) \) can be represented as
\[
\left( \begin{array}{c}
U \emptyset \\
U A_2
\end{array} \right) \times \left( \begin{array}{c}
U \emptyset \\
U B
\end{array} \right) \times \left( \begin{array}{c}
U \emptyset \\
U C
\end{array} \right) = I^4.
\]
(4.7) Example. With \( G_A(T) \approx I^2 \) and \( G_{A_1UA_3}(T) \approx I^4 \) as above, \( G_A(T) \subset G_{A_1UA_3}(T) \). While the dimension of \( C(\Gamma) \) is determined by \( \Gamma \), the dimension of \( G_E(\Gamma) \) also depends on the number of components of \( E \), and therefore can be arbitrarily large. In comparing \( G_{A_1UA_3}(T) \) to \( G_A(T) \), it is apparent that the introduction of the gap \( A_2^0 \) not only generates a hyperspace of higher dimension, but actually results in \( G_A(T) \) having a natural 2-fringe in \( G_{A_1UA_3}(T) \).

4.2. Hyperspaces generated by triangulations.

For a tower of finitely-generated growth hyperspaces to be dense in \( 2^\Gamma \), the members of the tower must get successively closer to \( 2^\Gamma \) (with respect to the Hausdorff metric on \( 2^{2^\Gamma} \)). This will certainly happen if the generating collections themselves approximate \( 2^\Gamma \). An obvious choice for a generating collection is the collection \( \Delta \) of all segments of \( \Gamma \) with respect to some triangulation (which we also call \( \Delta \)). Successively finer triangulations generate growth hyperspaces successively closer to \( 2^\Gamma \). Moreover, the position of one such hyperspace in another can be studied in terms of "gaps" and the associated fringes.

Let \( \Delta \) be a triangulation of \( \Gamma \). By \( D \subset \Delta \) we mean that \( D \) is a non-empty subcollection of segments. Then \( J_\Delta(\Gamma) = \bigcup_{D \subset \Delta} G_{\cup D}(\Gamma) \) is the growth hyperspace of \( \Gamma \) generated by the triangulation \( \Delta \). \( F \in J_\Delta(\Gamma) \) if and only if each component of \( F \) contains some segment of \( \Gamma \) relative to \( \Delta \).
Viewing $J_\Delta(\Gamma)$ as the union of singly-generated growth hyperspaces does not give a convenient description of its structure since each $G_{\cup D}(\Gamma)$ is fairly complex, and intersections among these can be equally bad. A more organized picture results from decomposing $J_\Delta(\Gamma)$ into restricted-growth hyperspaces. For $D \subseteq \Delta$, define $R_D(\Delta)$ to be 

$\{F \in G_{\cup D}(\Gamma) | \text{each component of } F \cap \delta \text{ meets } \cup D, \text{ for each } \delta \in \Delta\}$. 

When the triangulation is understood, $R_D$ may be written for $R_D(\Delta)$. Let $D^*$ denote the collection of all segments which meet some segment of $D$. For elements of $R_D$, not only is growth from $\cup D$ limited to adjacent segments, but also growth from one segment of $D^* \setminus D$ to another is prohibited. So $R_D$ may be a proper subset of $G_{\cup D}(\cup D^*)$.

(4.8) **Example.** Let $\Delta = \{\alpha, \beta, \gamma, \delta\}$ be a partition of $I$ as shown. The polyhedron $J_\Delta(I)$ is diagrammed on the next page by means of restricted-growth hyperspaces. To clarify that picture, $G_{\alpha \delta}(I)$ is shown here. For convenience we write $\alpha \delta$ for $\alpha \cup \delta$, $<\alpha \delta>$ for $R_{\{\alpha, \delta\}}(\Delta)$, etc. Notice that points of $J_\Delta(I)$ need absolute labels.
Of particular interest in this example is the point I, which appears to be the center of the action. \( J_\Delta(I) \) is most complex at I and less complex farther away from I. In fact, \( J_\Delta(I) \) can be collapsed to I by collapsing (partially) the cells \( R_D \), starting with those farthest from I. The collapses are performed according to the scheme at right.
Theorem. If $\Delta$ is a triangulation of a finite connected graph $\Gamma$, then $\mathcal{J}_\Delta(\Gamma)$ is a collapsible polyhedron.

Proof. For $D \subseteq \Delta$, $D^* \setminus D = \{a_1, \ldots, a_j, b_1, \ldots, b_k\}$, where each $a_i$ intersects $UD$ in only one point and each $b_i$ intersects $UD$ in two points. $R^+_D$ can be parametrized by the allowable growth from $UD$ in each segment of $D^* \setminus D$, with growth in distinct segments being independent. Thus, $R^+_D = \Pi \{P_\gamma \mid \gamma \in D^* \setminus D\}$, where $P_\gamma$ parametrizes growth in the segment $\gamma$. For each $a_i$, $P_{a_i}$ is the segment $[U0, Ua_i]$ which we identify with $[0,1]$. For each $b_i$, $P_{b_i}$ is the triangle in which the hypotenuse is identified to the point "$Ub_i$"; in this case "$U0$" corresponds to $\langle 0,0 \rangle$ and "$Ub_i$" corresponds to $\langle 1,1 \rangle$ under some fixed reparametrization of $P_{b_i}$ as $I^2$.

Let $R^+_D = \bigcup \{R^+_D \cap R_E \mid D \subseteq E \subseteq D^*\}$. $R^+_D$ is actually the union of the $R^+_D \cap R_E$ for which $E = D \cup \{\delta\}$, for some $\delta \in D^* \setminus D$. For all elements of such an $R^+_D \cap R_E$, the $\delta$-coordinate must be 1 if $\delta$ meets $UD$ in one point, or $\langle 1,1 \rangle$ if $\delta$ meets $UD$ in two points; the other coordinates are unrestricted. So $R^+_D$ is a union of faces of $R_D$ which contain the point $I = \langle 1,1,\ldots,1 \rangle$. Note that $R^+_D$ is being viewed as $I^{j+2k}$.

If $W$ is an $n$-face of $R_D$ containing $I$, let $W^1$ be the union of the $n$ $(n-1)$-faces of $W$ containing $I$. Clearly $W \setminus W^1$; in particular, $R_D \setminus R_D^1$. $R_D^+$ may be
smaller than \( R^1_D \). However, each face \( W \) of \( R_D \) which lies in \( R^1_D \) but not \( R^+_D \) can be collapsed to \( W^1 \), and collapsing all such faces in this way, in order of decreasing dimension, effects a collapse of \( R^1_D \) to \( R^+_D \). The collapses \( R_D \setminus R^+_D \) can be performed for all \( D \) in order of increasing cardinality of \( D \) to obtain a collapse of \( J^D(\Gamma) \) to \( R_{\Delta^D} = \{ \Gamma \} \).

While the collection of restricted-growth hyperspaces does not constitute a simplicial complex, one can easily be obtained from the said collection by subdividing each \( R_D \) into simplices and including all faces of those simplices. Certainly \( J^D(\Gamma) \) collapses to the element \( \Gamma \) with respect to that simplicial complex. \( \square \)

4.3. Main results.

The Third F-D CAP Lemma and (3.7) were developed with the structure of the hyperspaces \( J^D(\Gamma) \) in mind to obtain the following analogue of (2.8).

(4.10) Theorem. If \( \Gamma \) is a finite, non-degenerate, connected graph, \( J \) is a dense union of countably many finite-dimensional \( Z \)-sets in \( 2^\Gamma \), and \( J \) has the growth property, then \( J \) is an f-d capset in \( 2^\Gamma \).

Proof. It suffices to construct \( \mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i \subset J \) such that the tower \( \mathcal{V}_1 \subset \mathcal{V}_2 \subset \ldots \) satisfies the conditions of the Third F-D CAP Lemma. For each \( i \), we take \( \mathcal{V}_i = J^D_{\Delta^D_i}(\Gamma) \), where \( \Delta^D_{i+1} \) is obtained from \( \Delta^D_i \) by subdividing each segment of \( \Delta^D_i \) into five equal parts. By (4.9), this will produce a
tower of collapsible polyhedra.

For any triangulation $\Delta$, let $\mu = \min\{\text{diam} \delta | \delta \in \Delta\}$, $M = \{\text{midpt} \delta | \delta \in \Delta\}$, and $J = \{S \subseteq M | S \neq \emptyset\}$. Since $J$ is dense in $2^\Gamma$, for each $S \in J$, there exists $F_S \in J$ such that $\rho(S, F_S) < \mu/3$. $F_S$ must be contained in $\bigcup D_S$, where $D_S = \{\delta \in \Delta | \delta \cap S \neq \emptyset\}$. $\bigcup D_S \in G_{F_S}(\Gamma) \subseteq J$ implies $G_{\bigcup D_S}(\Gamma) \subseteq J$. Therefore, $J(\Gamma) \subseteq J$, and $J(\Gamma)$ is a $Z$-set in $J$ since $J$ is a $C$-$Z$-set in $2^\Gamma$ by hypothesis.

If we assume that each segment of $\Delta_1$ has length 1, then each segment of $\Delta_i$ has length $5^{-i+1}$. Given $\varepsilon > 0$, choose $l$ so that $5^{-l+1} < \varepsilon/2$. The expansion map $\eta_\varepsilon$ has its image in $J(\Gamma)$ since, for each $F \in 2^\Gamma$, each component of $\eta_\varepsilon(F)$ must contain some segment of $\Delta_l$. So for each $\varepsilon > 0$, there exists an integer $l$ and map $g: 2^\Gamma \to \mathcal{K}_l$ with $\rho(g, \text{id}) < \varepsilon$.

It remains to show that for each $i$, there exists a 1-fringe over $\mathcal{K}_i$ in some $\mathcal{K}_i$. This is accomplished by verifying the conditions of (3.7). For a simplex $\sigma$ of $\mathcal{K}_i$, let $R_D(\Delta_i)$ be the smallest restricted-growth hyperspace (with respect to $\Delta_i$) containing $\sigma$. Then $A \in \sigma$ implies $A \supseteq \bigcup D_\sigma$. If $\delta \in D_\sigma$, $\delta$ is partitioned in $\Delta_{i+1}$ as $\{\delta(1), \ldots, \delta(5)\}$; likewise $\delta(3)$ is partitioned in $\Delta_{i+2}$ as $\{\delta(3,1), \ldots, \delta(3,5)\}$, etc. Of the $5^m$ subsegments of $\delta$ with respect to $\Delta_{i+m}$, let $\delta^m$ be the middle segment $\delta(3,3,\ldots,3)$. Define a 1-fringe $\varphi_{\delta,m}: \sigma \times I \to \mathcal{K}_{i+m}$ by introducing equal gaps centered in $\delta^{m-1}(2)$ and $\delta^{m-1}(4)$;
the gaps in $\delta \cap \varphi_{\delta,m}(A,t)$ each have length $5^{-i-m+1} t$.

Given positive integers $m_1 < m_2 < ... < m_j$ and $\delta_1, ..., \delta_j \in D_{\sigma}$, derive $\varphi_{\delta_1, ..., \delta_j, m_1, ..., m_j} : \sigma \times I^j \rightarrow \mathcal{X}_{i+m_j}$ from $\varphi_{\delta_1, m_1}, ..., \varphi_{\delta_j, m_j}$ in the obvious way. Let $\hat{\mathfrak{d}}_{\sigma}$ consist of the $\varphi_{\delta_1, ..., \delta_j, m_1, ..., m_j}$ for all possible choices of $j$, $m_1 < ... < m_j$, and $\delta_1, ..., \delta_j \in D_{\sigma}$, together with the 0-fringe $id_{\sigma}$. Take $\hat{\mathfrak{d}}$ to be the union of the $\hat{\mathfrak{d}}_{\sigma}$ for all simplices $\sigma$ of $\mathcal{X}_i$.

The collection $\hat{\mathfrak{d}}$ satisfies condition 2) of (3.7) since $D_{\sigma} \subseteq D'_{\sigma}$ for any face $\sigma'$ of $\sigma$. To verify condition 3), suppose $\{\varphi_1, ..., \varphi_k\} \subseteq \hat{\mathfrak{d}}$ with the base of each $\varphi_j$ contained in $\sigma$ and $\mathfrak{f} = \bigcup_{j=1}^k \text{im} \varphi_j \subseteq \mathcal{X}_{i+n}$. Then for each $\delta \in D_{\sigma}$, no $F \in \mathfrak{f}$ has a gap in the subsegment $\delta^n$. Thus we can define a 1-fringe $\hat{\psi} : \mathfrak{f} \times I \rightarrow \mathcal{X}_{i+n+1}$ by introducing equal gaps centered in $\delta^n(2)$ and $\delta^n(4)$; in other words, $\hat{\psi}$ is defined in the same way as $\varphi_{\delta,n+1}$, but with a larger base.

Since no element of $\mathcal{X}_{i+n}$ can have a component which is a proper subset of $\delta^n(2) \cup \delta^n(3) \cup \delta^n(4)$, $\text{im} \hat{\psi} \cap \mathcal{X}_{i+n} = \mathfrak{f}$. Furthermore, if $\varphi_j = \varphi_{\delta_1, ..., \delta_j, m_1, ..., m_j}$, then

$\hat{\psi} \varphi_j = \varphi_{\delta_1, ..., \delta_j, \delta: m_1, ..., m_j, n+1}$ belongs to $\hat{\mathfrak{d}}$.

Hence, for each $i$, there exists a 1-fringe over $\mathcal{X}_i$ in some $\mathcal{X}_i$, and the theorem follows from the Third F-D CAP Lemma. □
The following relative version of (4.10) is an immediate corollary of the proof. See the proof of (2.10).

(4.11) Theorem. If \( \Gamma \) is a finite connected graph, \( 2 \) is a growth hyperspace of \( \Gamma \) which is homeomorphic to \( Q \), \( J \) is a dense union of countably many finite-dimensional \( Z \)-sets in \( 2 \), and \( J \) has the growth property, then \( J \) is an \( f-d \) capset in \( 2 \).

(4.12) Corollary. If \( \Gamma \) is a finite, non-degenerate, connected graph, then \( \bigcup_{n=1}^{\infty} C^n(\Gamma) \) is an \( f-d \) capset in \( 2^\Gamma \).

Proof. For each \( n > 1 \), \( C^n(\Gamma) \setminus C^{n-1}(\Gamma) \) is the hyperspace of all closed subsets with exactly \( n \) components, which is locally homeomorphic to \( [C(\Gamma)]^n \). \( C^1(\Gamma) = C(\Gamma) \) is finite-dimensional, and \( C^n(\Gamma) = C^{n-1}(\Gamma) \cup [C^n(\Gamma) \setminus C^{n-1}(\Gamma)] \), so \( C^n(\Gamma) \) is finite-dimensional by induction. Each \( C^n(\Gamma) \) is a \( Z \)-set in \( 2^\Gamma \). \( \square \)

(4.13) Remark. For a non-degenerate Peano continuum \( X \), the hyperspace of all closed subsets with infinitely many components is homeomorphic to \( \ell^2 \) if either

1) \( X \) contains no free arc, by (2.19), or
2) \( X \) is a graph, by (4.12).

Since these two cases are poles apart, it is reasonable to conjecture that the conclusion holds in general. In fact, Curtis [14] has recently verified this.
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