Witt Classification of Inner Product Spaces.

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WITT CLASSIFICATION OF INNER
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In this paper our goal has been to develop the algebraic machinery for the study of the Witt group \( W(k, I) \) of degree \( k \) mapping structures and the consequent generalization of Scharlau's exact sequence to our situation both for a field and the integers. The idea to do this, and the possibilities inherent in the program we have undertaken, comes from the author's dissertation advisor, Professor Conner. It is thus a pleasure to thank him for his help in this project without which this paper would never have been written. The author feels fortunate to have had the opportunity to study under Professor Conner, who should be thanked for his patience and understanding. Further, for many of the ideas herein, he should receive credit.

The author is also grateful to Professor Stoltzfus for numerous conversations throughout this project. His work on *Unraveling the Integral Knot Concordance Group* provided the foundation for much of the material found here. Thanks also go to Professors Cordes and Butts at Louisiana State University; to Professor A. Liulevicius at the University of Chicago; and to Professor A. Ross at Ohio State University for their interest in the author at various stages of his career.
Finally, the author wishes to express his gratitude to his wife, Hiroko, daughter, Amy, and parents Dr. and Mrs. Albert Warshauer all of whom have given their constant support and encouragement. To them, we dedicate this work.
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A Witt group $W(k,D)$ of degree $k$ maps is defined. This group consists of Witt equivalence classes of triples $(M,B,\ell)$, in which $B$ is a $D$-valued inner product defined on $M$ and $\ell$ is a map of degree $k$, meaning $B(\ell x,\ell y) = kB(x,y)$ for all $x$ and $y$ in $M$. This Witt group is computed for $D$ a field. $W(k,Z)$ is studied by a boundary sequence which relates $W(k,Z)$ to $W(k,Q)$.

The Scharlau transfer sequence $[Lm]$ is generalized to the Witt group of degree $k$ maps. An exact octagon is developed which computes the cokernel of the Scharlau transfer sequence. Exactness of the octagon over $Z$ is proved using the boundary sequence.
INTRODUCTION

Our object is to define and study a Witt group, $W(k, I)$, where $k$ is an integer, $D$ is the underlying ring, and $I$ is a $D$-module. Since $D$ is understood, it is not mentioned in the notation $W(k, I)$. This group consists of Witt equivalence classes of degree $k$ mapping structures. These are tuples $[M, B, \ell]$ where $M$ is a finitely generated $D$-module, $B$ is an $I$-valued inner product on $M$, and $\ell$ is a $D$-module homomorphism of $M$ which satisfies $B(\ell x, \ell y) = kB(x, y)$. In Chapter I we make the necessary definitions.

The first task is to compute $W(k, F)$, for $F$ a field. To this end we decompose $W(k, F)$ according to the characteristic polynomial of $\ell$. In Chapter II a study is made of the properties of these characteristic polynomials. This is used in Chapter III to compute the Witt group $W(k, F)$, for $F$ a field.

In Chapter IV an exact octagon is developed. This octagon is given by:
\[ W^+(k,F) \xrightarrow{S} W^+(k^2,F) \xrightarrow{} W^+(-k,F) \]

\[ W^-(k,F) \leftarrow W^-(k^2,F) \leftarrow W^-(k^2,F) \]

\[ A(F) \leftarrow A(F) \]

\[ W^-(k,F) \leftarrow W^-(k^2,F) \leftarrow W^-(k,F) \]

\[ W^+(k,F) \] denotes Witt equivalence classes with \( B \) symmetric. \( W^-(k,F) \) denotes Witt equivalence classes with \( B \) skew-symmetric. \( A(F) \) denotes Witt equivalence classes of pairs \([M,B]\) where \( B \) is not restricted by any symmetry requirements. The squaring map \( S \) is defined by : \([M,B,\ell] \xrightarrow{S} [M,B,\ell^2]\). This generalizes the Scharlau transfer map, \( t_* \), given in [Im 201]. In fact, we may view \( S \) as \( t_* \), the transfer, in the appropriate setting. The octagon then becomes the Scharlau transfer sequence with certain terms vanishing. Our approach enables us to interpret the kernel term in the transfer sequence and compute the cokernel of \( t_* \).

The preceding discussion is for \( F \) a field. In order to prove exactness in the octagon over \( Z \), where \( Z \) denotes the integers, for \( k = \pm 1 \), some algebraic number theory is required. A review of required results is given in Chapter \( V \). The reader is also referred to O'Meara, \textit{Introduction to Quadratic Forms}, [O'M], for a more thorough exposition.
Chapter VI discusses the rank, discriminant, and signature invariants. In terms of these Witt invariants one may compute the Hermitian Witt groups arising in the decomposition of $W(k, Q)$, for $Q$ the field of rational numbers. These results are standard, and are presented here for completeness.

Chapter VII is suggested by Conner. The cohomology groups $H^*(C_2; 0(E)^*)$ and $H^*(C_2; C(E))$, where $0(E)$ is the ring of integers in the algebraic number field $E$, and $C(E)$ is the ideal class group, are studied. Two other groups, Gen and Iso $(E/F)$, which naturally arise are defined and computed. The collection of these ideas is given in an exact hexagon which involves these different groups.

After these preliminaries, we are ready to relate $W(k, Z)$ to $W(k, Q)$. This is done by the boundary, $\partial$, exact sequence:

$$0 \rightarrow W(k, Z) \rightarrow W(k, Q) \rightarrow W(k, Q/Z) \rightarrow 0$$

For $k = \pm 1$, $\partial$ is onto. A study is also made of $\partial$ when $k$ is prime.

This gives the modus operandi we follow. A result
is proved over fields. In this instance, we develop an exact octagon which generalizes Scharlau's transfer sequence. We then prove this result for the integers \( \mathbb{Z} \) by using the \( \partial \) exact sequence above.

In Chapter IX, the terms in the octagon are studied in detail. In Chapter X, exactness in the octagon over \( \mathbb{Z} \) is proved.

The geometric motivation for studying this Witt group is as follows. Let \( M^{2n} \) be a closed, differentiable manifold of dimension \( 2n \), together with a map \( \ell \) of degree \( k \). The cup product pairing on the middle dimensional cohomology modulo torsion, followed by evaluation on the fundamental class, yields a bilinear pairing \( B \). By Poincaré duality, \( B \) is an inner product, \([V\ 159]\). Thus \( (M, B, \ell) \), with \( M = H^n(M^{2n}; \mathbb{Z})/\text{torsion} \), \( B \) as described, and \( \ell = \ell^* \) the induced map on cohomology, is a degree \( k \) mapping structure. For \( n \) even, \( B \) is symmetric. For \( n \) odd, \( B \) is skew-symmetric. Thus we have an algebraic classification of the above topological object.

For a further description of the history and topological significance of this Witt ring, the reader is referred to Alexander, Conner, Hamrick, [Odd Order Group]
Actions and Witt Classification of Inner Products

[A,C,H], and Stoltzfus, Unraveling the Integral Knot Concordance Group, [Sf]. Indeed, the present work carries on much of the work begun there.
CONVENTIONS

A complete list of symbols and notations, as well as an index, can be found in the back. We number theorems, propositions, and definitions consecutively in each chapter. We refer to a theorem in the same chapter as it is numbered. However, when referring to a theorem from another chapter, we use a Roman numeral to indicate the chapter from which the theorem is taken.

The end of a proof is designated by the symbol □. Occasionally, this symbol is also used alone, without a proof, to indicate that certain Lemmas or Propositions follow in a straightforward manner from the preceding, or that the proof is not difficult.
Chapter I  THE WITT RING

In this chapter we define the algebraic structures which are to be studied. Section 1 begins by describing the setting for our inner product spaces and their elementary properties. Since these inner product spaces need not be symmetric, we are led to examining a symmetry operator in Section 2. Section 3 shows how to construct new inner product spaces out of old. In particular, the operations of direct sum and tensor product are discussed. These operations later become addition and multiplication in the Witt ring.

An inner product space comprises part of the data of a degree k mapping structure which is defined in Section 4. A Witt equivalence relation is then defined on these structures.

Section 5 is concerned with selecting from each Witt equivalence class a certain "anisotropic" representative. In many instances this representative is unique.
1. **Inner products**

Let $D$ be a Dedekind Domain $[Z, S^{-1}]$ with involution $-$. Let $E$ be the quotient field of $D$. Then $-$ extends to a Galois automorphism of $E$, and we denote the fixed field of $-$ by $F$. It may happen that $F = E$; in fact this is precisely the case when $-$ is the trivial involution.

If $\mathfrak{p}$ is a prime ideal in $D$, then we write $\overline{\mathfrak{p}}$ to denote the image of $\mathfrak{p}$ under $-$. $\overline{\mathfrak{p}}$ is also prime, hence maximal, as these are equivalent over a Dedekind Domain. Let $I$ be a $-$ invariant fractional ideal in $D$. Since $I$ factors uniquely into a product of prime ideals, $I = \prod_{i=1}^{n} \mathfrak{p}_i^{n_i}$, $n_i \in \mathbb{Z}$, it follows that $\overline{I} \subseteq I$ if and only if $\overline{I} = I$.

In order to discuss inner products over $D$, we need to discuss the setting we will study. We shall be studying the $D$-module $\text{Hom}_D(M, K)$ where $M, K$ satisfy either:

(a) $M$ is a finitely generated torsion free $D$-module (and hence projective since $D$ is Dedekind), with $K = I$ a $-$ invariant frac-
tional ideal in $D$.

or

(b) $M$ is a finitely generated torsion $D$-module, with $K = E/I$ where $E$ is the quotient field of $D$ and $I$ is a $\sigma$-invariant fractional ideal in $D$.

We shall be specifically concerned with these two cases. Thus whenever we write $\text{Hom}_D(M,K)$ we shall assume $M$ and $K$ satisfy either (a) or (b) above. The $D$-module structure on $\text{Hom}_D(M,K)$ is now given by defining $df(x) = f(dx)$ for $d \in D$, $x \in M$, $f \in \text{Hom}_D(M,K)$.

**Definition 1.1** A $K$-valued inner product space $(M,B)$ over $D$ is a finitely generated $D$-module $M$, together with a non-singular mapping

$$B : M \times M \to K$$

satisfying $B(dx,y) = B(x,dy) = dB(x,y)$ for all $x, y \in M$, $d \in D$.

$B$ is linear in the first variable, conjugate linear in the second variable. Again, $M,K$ are assumed also to
satisfy one of the standard assumptions (a) or (b) above.

It is still necessary to say what it means for $B$ to be non-singular.

**Definition 1.2** The map $B : M \times M \to K$ is non-singular provided the adjoint map $\text{Ad}_R B : M \to \text{Hom}_D(M,K)$ is a $D$-module isomorphism. By $\text{Ad}_R B$ we are denoting the right adjoint map, namely $\text{Ad}_R B(x) = B(-,x)$.

The left adjoint map, $\text{Ad}_L B$ is similarly defined by $\text{Ad}_L B(x) = \overline{B(x,-)}$. We must conjugate in order to have $\text{Ad}_L B(x) \in \text{Hom}_D(M,K)$, i.e. to make $\text{Ad}_L B(x)$ $D$-linear.

We have left out any symmetry requirements on $B$. This is taken care of by:

**Definition 1.3** An inner product space $(M,B)$ is u Hermitian provided $B$ satisfies $B(x,y) = \overline{uB(y,x)}$ for all $x, y \in M$, $u$ fixed $u \in D$.

Since $B(x,y) = \overline{uB(x,y)}$, it follows that $uu = 1$ and $u$ is a unit in $D$ of norm 1 [S 60]. We see that $1$ Hermitian is the usual notion of Hermitian. $-1$ Hermitian corresponds to skew-Hermitian.

When the involution $-$ is trivial, meaning the identity,
we speak of 1 Hermitian as symmetric since
\( B(x,y) = B(y,x) \). Similarly, we define skew-symmetric,
and \( u \) symmetric in the case that \(-\) is the identity.

Let us now return to study the module \( M \) in case (a), namely \( M \) is a finitely generated projective \( D \)-
module.

We form the vector space \( M \otimes_D E = V \) over \( E \).

**Definition 1.4** The rank of a finitely generated tor-

sion free \( D \)-module \( M \) is the dimension of the vector space
\( M \otimes_D E \) over \( E \), \( E \) being the quotient field of \( D \).

Thus viewed \( M \) is a \( D \)-lattice in \( V \) \([0'M 209]\). Hence \( M \) splits as a direct sum \( M = \bigoplus_{i=1}^{n} A_i \), \( n = \text{rank } M \),
where each \( A_i \) is a fractional ideal in \( D \). In fact
\([0'M 212]\), there is the splitting:

\[
M = A_1 z_1 \oplus Dz_2 \oplus \ldots \oplus Dz_n
\]

where \( A_1 \) is fractional ideal in \( D \);
\[
A_2 = A_3 = \ldots = A_n = D.
\]

\( \{z_i\} \) is a basis for \( V \).

Since \( M \) splits as a sum of fractional ideals, and \( \text{Hom}_D \)
is additive over direct sums, we are reduced to
studying $\text{Hom}_D(A, I)$ where $A$ is a fractional ideal in $D$.

Lemma 1.5 Let $A$, $I$ be fractional ideals in $D$, with $I$ - invariant. Then the map

$$\tau : \frac{A^{-1} I}{\text{Hom}_D(A, I)}$$

given by

$$x \rightarrow \tau(x) \quad \text{with} \quad \tau(x)(c) = cx$$

is a $D$-module isomorphism. Here the $D$-module structure on $\text{Hom}_D(A, I)$ is as previously defined.

Proof: $\tau$ is a $D$-module homorphism since

$$\tau(dx)(c) = c \cdot \overline{dx} = (\overline{cd}) \cdot \overline{x} = \tau(x)(\overline{d} \cdot c) = [d, \tau(x)](c).$$

$\tau(x) \in \text{Hom}_D(A, I)$, since $\tau(x)$ is clearly $D$-linear.

We must show $\tau$ is an isomorphism.

(a) $\tau$ is 1-1: Suppose $\tau(x) = \tau(y)$. Then

$c\overline{x} = c\overline{y}$, for all $c \in A$. Let $c \neq 0$, and cancel to obtain $\overline{x} = \overline{y}$, hence $x = y$.

(b) $\tau$ is onto: Let $f \in \text{Hom}_D(A, I)$. We must show $f \in \text{image} \ \tau$. Tensoring with $E$, we extend $f$ to $\text{Hom}_D(E, E)$, where $E$ is the quotient field of $D$. 
Since the Lemma is clearly true for $E$, it follows that $f(c) = \bar{x}_0c$, for $\bar{x}_0 \in E$. But $\bar{x}_0c \in I$ for all $c \in A$. Hence $\bar{x}_0 \in A^{-1}I$, and $x_0 \in \bar{A}^{-1}I = A^{-1}I$. Therefore $f(c) = \tau(x_0)$, and $f \in \text{image } \tau$. □

**Theorem 1.6** Let $M$ be a finitely generated torsion free $D$ module, and $I$ a - invariant maximal ideal in $D$. Then there is a canonical $D$-module isomorphism

$$\phi: M \rightarrow \text{Hom}_D(\text{Hom}_D(M,I),I)$$

given by

$$\phi(x)(f) = \bar{f}(x) .$$

**Proof:** Recall again the module structure on $\gamma \in \text{Hom}_D(\text{Hom}_D(M,I),I)$) is given by : $(d\gamma)(f) = \gamma(d\cdot f)$, where $(d\bar{f})(x) = f(dx)$ . By the remarks immediately preceding Lemma 1.5, it suffices to prove the theorem for $M = A$ a fractional ideal.

We apply Lemma 1.5 twice to obtain an isomorphism $\phi$. $\phi: A = (A^{-1}I)I \rightarrow \text{Hom}_D(\bar{A}^{-1}I, I) \rightarrow \text{Hom}_D(\text{Hom}_D(A,I),I)$. $\phi$ is given by the composition. We have then:

$$\phi(x)(f) = \tau(x)(m) \text{ where } \tau(m) = f, \text{ so } f(x) = x \cdot \bar{m}$$

$$= \bar{x} \cdot m$$

$$= \bar{f}(x) \quad \text{as claimed.}$$
We again observe that \( \varphi \) is a \( D \)-module isomorphism since: 
\[
\varphi(dx)(f) = f(dx) = (\partial f)(x) = \varphi(x)(\partial f) = ((d \varphi(x))(f)).
\]

We now wish to establish this result in case (b), namely when \( M \) is a finitely generated torsion \( D \)-module, with \( K = E/I \), and \( I = \bar{I} \) as usual.

Just as a finitely generated projective \( D \)-module decomposes, so does a finitely generated torsion \( D \)-module decompose as a direct sum of its \( \mathfrak{p} \)-primary submodules. Each piece of \( \mathfrak{p} \)-primary torsion further splits as a direct sum of \( \mathfrak{p} \)-primary cyclic \( D \)-modules. [R-2 68,179] or [A,Mc 99]

Our notation for this is:

Let \( \mathfrak{p} \) be a maximal ideal in \( D \).

Then \( M(\mathfrak{p}) = M \) localized at \( \mathfrak{p} = \{ x \in M: \mathfrak{p}^n x = 0 \text{, some } n \} \)

\( = \mathfrak{p} \)-primary component of \( M \)

so \( M \approx \bigoplus_\mathfrak{p} M(\mathfrak{p}) \).

Each \( M(\mathfrak{p}) \approx \) direct sum of cyclic modules, with each cyclic module being \( D/\mathfrak{p}^i \), some \( i \in \mathbb{Z} \). Again, since \( \text{Hom}_D \) is
additive over finite direct sums, without loss of generality we may assume $M$ is a cyclic module, ie. $M = D/\mathfrak{p}^i$, some $i \in \mathbb{Z}$.

For $I$ a fractional ideal, we factor $I$ as $I = \mathfrak{p}_1 \cdots \mathfrak{p}_k$, where the $\mathfrak{p}_i$ are maximal ideals in $D$. Let $\pi_i$ be a uniformizer for $\mathfrak{p}_i$, [V 1.5]. Then, in the notation above, $I(\mathfrak{p}_i) = I$ localized at $\mathfrak{p} = IS^{-1}$ where $S = D-\mathfrak{p}$. Since $I = \mathfrak{p}_1 \cdots \mathfrak{p}_k$, it is clear then that

$$I(\mathfrak{p}_i) = \pi^{-j}D(\mathfrak{p}).$$

We can now simplify $K = E/I$.

**Lemma 1.7** $E/I = \oplus_{\mathfrak{p}} E/I(\mathfrak{p})$.

**Proof**: Define $f:E/I \rightarrow \oplus_{\mathfrak{p}} E/I(\mathfrak{p})$ by $e + I \mapsto \oplus_{\mathfrak{p}} (e + I(\mathfrak{p}))$. $f$ is clearly well-defined, and a homomorphism.

(a) $f$ is 1-1 : Suppose $f(e + I) = 0$. Then $e \in I(\mathfrak{p})$ for all $\mathfrak{p}$. Thus $e \in I$, by [0'M 46]. Hence, $f$ is 1-1 .

(b) $f$ is onto : Consider $\oplus_{\mathfrak{p}} (a_i + I(\mathfrak{p}_i)) \in \oplus_{\mathfrak{p}} E/I(\mathfrak{p})$.

We now apply the Strong Approximation Theorem, [0'M 42].
Letting \( | |_i \) denote the \( \mathfrak{a}_i \)-adic valuation, \([V \, 1.5]\), we can find \( x \in E \) with \( | x - a_i |_i < | I(\mathfrak{a}_i) |_i \), at the finite set of \( i \) when (a) \( a_i \notin I(\mathfrak{a}_i) \), or (b) \( I(\mathfrak{a}_i) \notin D(\mathfrak{a}_i) \); with \( | x |_i \leq 1 \) otherwise. It follows that \( f(x + I) = \bigoplus a_i + I(\mathfrak{a}_i) \), and \( f \) is onto.

Notice that the only summand of \( E/I \) with \( \mathfrak{a} \)-torsion is \( E/I(\mathfrak{a}) \). Thus, when \( M = D/\mathfrak{a}^i \), we may apply Lemma 1.7 to obtain an isomorphism:

\[
\text{Hom}_D(M, E/I(\mathfrak{a})) \cong \text{Hom}_D(M, E/I(\mathfrak{a})) .
\]

We further identify \( E/I(\mathfrak{a}) \cong E/\pi^j D(\mathfrak{a}) \cong E/D(\mathfrak{a}) \); \( I(\mathfrak{a}) = \pi^j D(\mathfrak{a}) \), where \( \pi \) is a uniformizer for \( \mathfrak{a} \) as before. The last isomorphism is multiplication by \( \pi^{-j} \).

The module structure of \( M \) lifts to \( D(\mathfrak{a}) \), and we have a \( D(\mathfrak{a}) \)-module isomorphism \( \tau : M(\mathfrak{a}) \rightarrow \text{Hom}_{D(\mathfrak{a})}(M(\mathfrak{a}), E/D(\mathfrak{a})) \). In order to define \( \tau \), it suffices to consider the case \( M(\mathfrak{a}) \cong D/\mathfrak{a}^i \), since \( M(\mathfrak{a}) \) is a direct sum of cyclic modules. Let \( x \in M \) be a generator for \( D/\mathfrak{a}^i \).

Then define \( \tau(x) \) by: \( \tau(x)(cx) = c/\pi^i \) where \( c \in D(\mathfrak{a}) \), \( x \in M(\mathfrak{a}) \). As in Lemma 1.5, \( \tau(dx)(cx) = \bar{d} \cdot c/\pi^i = (dT)(x)(cx) \) defines \( \tau \) over \( M \).

\( \tau \) is clearly an isomorphism. Combining these isomorphisms over all cyclic module summands of \( M \), we obtain:
Lemma 1.8 Let $M$ be a finitely generated torsion $D$-module. Then there is a canonical isomorphism $M \cong \text{Hom}_D(M,E/I)$.

Exactly as Theorem 1.6 was proved, we now have:

Theorem 1.9 Let $M$ be a finitely generated torsion $D$-module, $I$ a $D$-invariant fractional ideal. Then there is a canonical $D$-module isomorphism:

$$\varphi : M \rightarrow \text{Hom}_D(\text{Hom}_D(M,E/I),E/I)$$

given by

$$\varphi(x)(f) = \overline{f(x)}.$$

For the purposes of the next section, we also need the following propositions which describe the relationship of $\otimes$ to $\text{Hom}$.

Proposition 1.10 There is a canonical isomorphism $\psi : \text{Hom}_D(A,\text{Hom}_D(B,C)) \rightarrow \text{Hom}_D(A \otimes B,C)$, where $A,B,C$ are $D$-modules.

Proof: [R-1 25] Define $\psi$ by $(\psi f)(a \otimes b) = f(a)(b)$. The inverse of $\psi$ is given by $(\psi^{-1} g)(a)(b) = g(a \otimes b)$. \qed
Proposition 1.11  Let \( X_1, X_2, Y_1, Y_2 \) be finitely generated projective \( D \)-modules. Then
\[
\text{Hom}_D(X_1, Y_1) \otimes \text{Hom}_D(X_2, Y_2) \cong \text{Hom}_D(X_1 \otimes X_2, Y_1 \otimes Y_2).
\]
The isomorphism is given by:
\[
f \otimes g \mapsto f \cdot g, \quad \text{where} \quad (f \cdot g)(x_1 \otimes x_2) = f(x_1) \otimes g(x_2),
\]
and \( f \cdot g \) is extended to \( X_1 \otimes X_2 \) bilinearly.

Proof: For \( X_i, Y_j \) free, the assertion is clear using bases. Now, if \( X_i, Y_j \) are projective, then each is a direct summand of a free. \( \text{Hom} \) and \( \otimes \) are additive over finite direct sums. Hence the isomorphism for free splits into isomorphisms for the summands. \( \square \)
2. Constructing new inner products out of old

2.1 Direct Sums

Let \((M, B)\) and \((M_1, B_1)\) be two \(K\)-valued inner product spaces. The easiest way to construct a new inner product space is to form the sum.

\[
(M, B) \oplus (M_1, B_1) \cong (M \oplus M_1, B \oplus B_1)
\]

Here \((B \oplus B_1)(x, y) = B(x) + B_1(y)\). It is clear that \(B \oplus B_1\) is an inner product since the adjoint map

\[
\text{Ad}_R(B \oplus B_1) : M \oplus M_1 \to \text{Hom}_D(M \oplus M_1, K)
\]

splits as \(\text{Ad}_R B \oplus \text{Ad}_R B_1\).

2.2 Tensor Products

The next operation on an inner product space is \(\otimes\). Let \((M, B)\) and \((M_1, B_1)\) be two type (a) inner product spaces. In other words, \(M\) and \(M_1\) are both finitely generated projective \(D\)-modules. Assume that:

\[
B : M \times M \to Y_1 \quad \text{and} \quad B_1 : M_1 \times M_1 \to Y_2.
\]

We have the adjoint isomorphisms:
Taking the tensor product of these, we obtain by Proposition 1.11
\[ \text{Ad}_R(B \otimes B_1) = \text{Ad}_R(B) \otimes \text{Ad}_R(B_1) : M \otimes M_1 \rightarrow \text{Hom}_D(M, Y_1) \otimes \text{Hom}_D(M_1, Y_2) \]
\[ \approx \text{Hom}_D(M \otimes M_1, Y_1 \otimes Y_2). \]

This shows that the adjoint of $B \otimes B_1$ is an isomorphism, and hence $(M \otimes M_1, B \otimes B_1)$ is a $Y_1 \otimes Y_2$-valued inner product space. We can identify $Y_1 \otimes Y_2$ with the product of ideals, $Y_1 Y_2$.

### 2.3 Scaling an inner product

There is the operation of scaling an inner product. Let $(M, B)$ be a $K$-valued inner product space, with $d \in E^*$. $E^*$ denotes the units in $E$, which is $E - \{0\}$ since $E$ is a field.

Clearly, $(M, dB)$ is a $dK$-valued inner product space, where $(dB)(x, y) = d \cdot B(x, y)$. We may view this as a special case of tensor product, namely

$(M, B) \otimes (D, B_d) = (M, dB)$, where $B_d(x, y) = d \cdot x \cdot y$.

### 2.4 Tensoring with the quotient field of $D$
Given an inner product space \((M, B)\), with \(M\) of type (a), we can form \((M, B) \otimes_D E = (M \otimes_D E, B \otimes 1)\) where we now denote the extension of \(B\) to the quotient field by \(B \otimes 1\). When there is no confusion, we will write \(B \otimes 1 = B\).

2.5 The discriminant inner product space

Let \((M, B)\) be a \(D\)-valued inner product space, and suppose \(M \approx A \oplus_D \cdots \oplus_D D\) \(n\) factors total, where \(n\) is the rank of \(M\). We form the \(n^{th}\) exterior power, \(\wedge^n_M \approx_A\), with inner product \(\wedge^n_B : \wedge^n_M \times \wedge^n_M \to D\) defined by

\[
\wedge^n_B(x_1 \wedge \cdots \wedge x_n, y_1 \wedge \cdots \wedge y_n) = \text{determinant} \begin{pmatrix} a_{ij} \end{pmatrix},
\]

where the matrix \(\begin{pmatrix} a_{ij} \end{pmatrix}\) is given by \(a_{ij} = B(x_i, x_j)\).

In order to verify that this is an inner product, one again needs to check that the adjoint \(\text{Ad}_R(\wedge^n_B)\) is an isomorphism, \([B\;30]\). We then call \((\wedge^n_M, \wedge^n_B)\) the discriminant inner product space.

**Comment 2.6** The adjoint map of \(B\), \(\text{Ad}_R B\) or simply \(\text{Ad}_B\)

\(\text{Ad}_B : M \to \text{Hom}_D(M, I)\) is an isomorphism by hypothesis.

Taking the \(n^{th}\) exterior power,

\(\wedge^n(\text{Ad}_B) : \wedge^n_M \to \wedge^n(\text{Hom}_D(M, I))\) is an isomorphism. However,

\[
\text{Hom}_D(M, I) \approx \bigoplus_{i=1}^{n-1} \text{Hom}_D(D, I) \oplus \text{Hom}_D(A, I) \approx \bigoplus_{i=1}^{n-1} I \oplus A^{-1} I.
\]
Thus $\Lambda^n(\text{Hom}_D(M,I)) \approx I \otimes I \otimes \overline{A}^{-1}I$. However, $I \otimes I \otimes \overline{A}^{-1}I$ is not in general isomorphic to $\text{Hom}_D(\Lambda^n M, I)$, so that $\Lambda^n B$ is not in general non-singular.

However, for $I = D$, the Dedekind ring of integers, $\text{Ad}(\Lambda^n B) = \Lambda^n(\text{Ad} B) : \Lambda^n M \rightarrow \text{Hom}_D(\Lambda^n M, D)$ will be an isomorphism by the above, and $(\Lambda^n M, \Lambda^n B)$ is indeed an inner product space.

We note that $M$ is free as a $D$-module if and only if the ideal $A$ is principal. Thus the discriminant inner product space yields information about the structure of the $D$-module $M$.

We may apply the operation of $2.4$, tensoring with the field, to the discriminant inner product space. $\Lambda^n B \otimes 1$ is then multiplication by a fixed $x_0 \in E$.

Thus, associated with an inner product space $(M, B)$ is a pair $(x_0, A)$ where $A = \Lambda^n M$, and $x_0$ is as described.

This $x_0$ specifies the adjoint isomorphism, $\text{Ad}_R^\Lambda^n B : A \rightarrow \text{Hom}_D(A, D) = \overline{A}^{-1}D$, by $\Lambda^n B(a_1, a_2) = x_0 a_1 \overline{a}_2$, with $x_0$ unique in $F^*/\text{NE}^*$, where $F^*$ is the non-zero elements in the fixed field $F$ of $-$, and $\text{NE}^*$ denotes the multiplicative group of norms of elements in $E^*$. Hence, $x_0 A = \overline{A}^{-1}D$, i.e. $x_0 A \overline{A} = D$.

These pairs $(x_0, A)$, with $x_0 A \overline{A} = D$ form the additive generators for $H(0(E))$ as we shall discuss in VII 3. Here $D = 0(E)$ is the Dedekind ring of integers in $E$.
3. The symmetry operator

In Definitions 1.1 and 1.2 of an inner product space, and non-singular mapping, we made use of the right adjoint operator $\text{Ad}_R$. In this section, we relate the two adjoint operators $\text{Ad}_R$ and $\text{Ad}_L$.

Using Theorems 1.6 and 1.9, this is done as follows.

**Theorem 3.1** Let $(M, B)$ be an inner product space of either type (a) or (b). Let $B : M \times M \to K$ satisfy $B(dx, y) = B(x, dy) = dB(x, y)$. Then the right adjoint map $\text{Ad}_R B$ is an isomorphism if and only if the left adjoint map is.

**Proof:** Let $\varphi$ denote the canonical isomorphism $\varphi : M \to \text{Hom}_D(\text{Hom}_D(M, K), K)$ of Theorems 1.6 and 1.9, given by $\varphi(x)(f) = \overline{f(x)}$.

Assume $\text{Ad}_R B$ is an isomorphism. We can thus identify $M \approx \text{Hom}_D(M, K)$ via $\text{Ad}_R B$. Here $y \in M$ is identified with $B(-, y) \in \text{Hom}_D(M, K)$.

The isomorphism $\varphi$ is now given by:

$$\varphi : M \to \text{Hom}_D(\text{Hom}_D(M, K), K) \xrightarrow{\text{Ad}_R B} \text{Hom}_D(M, K)$$

$$\varphi_x(y) = B(x, y),$$
in other words $\varphi$ is $\text{Ad}_L B$. The converse follows similarly. \hfill \Box

Corollary 3.2 Let $(M,B)$ be an inner product space. Then we can define a unique $D$-linear isomorphism $s : M \to M$ by the equation $B(x,y) = B(y, sx)$.

Proof: $\text{Ad}_L B(x) = \overline{B(x,-)} \in \text{Hom}_D (M, K)$. Since $B$ is an inner product, we define $s(x)$ by:

$$\text{Ad}_R (sx) = \overline{B(x,-)} = B(-, sx).$$

$s$ is an isomorphism by Theorem 3.1. \hfill \Box

Notation 3.3 We shall reserve the letter $s$ for this map which is related to the symmetry of $B$ as described above. It is precisely this map $s$ which enables us to work with non-symmetric inner product spaces.

Let $N$ be a subspace of $M$. We say that $N$ is $s$ invariant provided $s(N) \subseteq N$.

Proposition 3.4 $N$ is $s$ invariant if and only if $sN = N$.

Proof: Sufficiency is clear.
In order to prove necessity, suppose \( s(N) \subseteq N \).

Then we can form an ascending chain of submodules of \( M \), \( T_i = \{ m \in M : s^i(m) \in N \} \), \( T_{i+1} \supseteq T_i \). Since \( D \) is Noetherian, and \( M \) is finitely generated, \( M \) is Noetherian, \([S \,47]\). Hence, this chain terminates.

Suppose \( T_i = T_N \) for \( i \geq N \).

Claim: \( T_0 = T_1 = \ldots = T_N \), and hence \( sN = N \).

It clearly suffices to show that \( T_i \neq T_{i+1} \) implies \( T_{i+1} \neq T_{i+2} \). Suppose then that \( T_i \neq T_{i+1} \). Let \( m_0 \in T_{i+1} - T_i \). Then \( s^{i+1}(m_0) \in N \) and \( s^i(m_0) \notin N \).

\( s \) is an isomorphism, so there exists \( m_1 \in M \) with \( sm_1 = m_0 \). Hence \( m_1 \in T_{i+2} \). If \( m_1 \in T_{i+1} \), \( m_0 \in T_i \), contradiction. Thus, \( m_1 \in T_{i+2} - T_{i+1} \). \( \square \)

We observe that Proposition 3.4 can be proved for any ring \( R \) in place of \( D \), with \( R \) not necessarily Noetherian. Namely, let \( s \) be an isomorphism \( s : M \to M \) of a finitely generated \( R \)-modules, with \( R \) not necessarily Noetherian.

Claim: If \( N \subseteq M \) is a submodule with \( sN \subseteq N \), then \( sN = N \).
We sketch the proof using a general form of Nakayama's Lemma: \( M = IM \) implies \( I + \text{Ann}M = R \), where \( \text{Ann}M \) denotes the annihilator of \( M \), provided \( M \) is finitely generated.

Consider the diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & M \\
\downarrow & & \downarrow \\
M/\text{N} & \rightarrow & M/\text{N} \rightarrow 0
\end{array}
\]

It suffices to show \( \hat{s} \) is 1-1. View \( M/\text{N} \) as an \( R[x] \)-module, where \( x \) acts as \( \hat{s} \). Suppose \( x \cdot m_0 = 0 \).

We want to show \( m_0 = 0 \). Let \( I = (x) \). Since \( \hat{s} \) is onto, \( M/\text{N} = I(M/\text{N}) \). Hence, by Nakayama, above, \( I + \text{Ann}(M/\text{N}) = R[x] \). Write \( 1 = rx + t \), where \( r \in R[x] \), \( t \in \text{Ann}(M/\text{N}) \).

Then \( 1 \cdot m_0 = rx \cdot m_0 + t \cdot m_0 = 0 \), so \( m_0 = 0 \).

Thus \( \hat{s} \) is 1-1. □

For a subspace \( N \) of \( M \), we define \( N_R = \{ v \in M : B(n, v) = 0 \text{ for all } n \in N \} \). Thus \( N_R \) is the kernel of \( \text{Ad}_R^B \) restricted to \( N \), \( \text{Ad}_R^B : M \rightarrow \text{Hom}_D(N, K) \). We call \( N_R \) the right orthogonal complement of \( N \).

The underlying inner product space is understood. Similarly,

\[ N_L = \{ v \in M : B(v, n) = 0 \text{ for all } n \in N \} \]

Now, let \( N \) be \( s \)-invariant. Since \( sN = N \), it follows that \( N_L = N_R \). We denote this common orthogonal
Note 3.5 \( N^\perp \) is only defined when \( N \) is \( s \) invariant.

Proposition 3.6 If \((M,B)\) is an inner product space of type (a), meaning \( M \) is torsion free, then \( N_L \) and \( N_R \) are direct summands of \( M \).

Proof: Consider the exact sequence:

\[
0 \rightarrow N_L \rightarrow M \rightarrow M/N_L \rightarrow 0.
\]

It suffices to show that \( M/N_L \) is torsion free, hence projective over the Dedekind domain \( D \). Then the sequence splits, and \( N_L \) is a summand.

Suppose to the contrary \( M/N_L \) has torsion. Then there exists \( x \in M, d \neq 0, d \in D \), with \( x \not\in N_L \) and \( dx \in N_L \). So \( B(dx,y) = dB(x,y) = 0 \) for all \( y \in N \).

Thus \( B(x,y) = 0 \) for all \( y \in N \) since \( D \) is a domain.

Hence \( x \in N_L \). Contradiction. Thus \( M/N_L \) is torsion free. \( \square \)

We have already remarked that if \( N \) is \( s \) invariant, then \( N_L = N_R \). The above shows that \( N_L \) and \( N_R \) are always
summands. These two conditions turn out to give the converse.

**Proposition 3.7** Let \((M, B)\) be an inner product space with \(M\) torsion free. Let \(N\) be a summand of \(M\). Then \(N\) is \(s\) invariant if and only if \(N_L = N_R\).

**Proof:** As observed before Note 3.5, necessity is clear. In order to prove sufficiency, consider the two exact sequences:

\[
0 \rightarrow N_L \rightarrow M \overset{\text{Ad}_B}{\rightarrow} \text{Hom}_D(N, K) \rightarrow 0.
\]

\[
0 \rightarrow (N_L)_R \rightarrow M \overset{\text{Ad}_B}{\rightarrow} \text{Hom}_D(N_L, K) \rightarrow 0.
\]

\(\text{Ad}_L B\) and \(\text{Ad}_R B\) are onto because \(N\) is a summand. Clearly we have rank \(N_L = \text{rank}(\text{Hom}_D(N_L, K))\). Thus, by the two sequences above, rank \((N_L)_R = \text{rank}(\text{Hom}_D(N, K)) = \text{rank} N\). However, \(B(n_{\ell}, n) = 0\) for all \(n_{\ell} \in N_L, n \in N\). Thus \(N \subseteq (N_L)_R\). Since \(N\) is a summand, ranks equal, it follows that \(N = (N_L)_R\). Similarly \(N = (N_R)_L\).

We now wish to show that \(sN \subseteq N = (N_L)_R\). So we compute \(B(n_{\ell}, sn) = \overline{B(n, n_{\ell})} = 0\) for all \(n_{\ell} \in N_L\), since \(N_L = N_R\) by hypothesis. Thus \(sN \subseteq (N_L)_R = N\) as desired. \(\square\)
Proposition 3.8 \( N = N_L \) if and only if \( N = N_R \).

Proof: Suppose \( N = N_L \). Then \( B(m,n) = 0 \) for all \( m, n \in N \). Thus \( n \in N_R \) and \( N \subseteq N_R \). By Proposition 3.6, \( N = N_L \) implies \( N \) is a summand. Clearly, as in Proposition 3.7, \( \text{rank } N = \text{rank } N_R \), so that \( N = N_R \).

The converse is similar. □

Theorem 3.1 stated that for an inner product space \((M,B)\), both \( \text{Ad}_R B \) and \( \text{Ad}_L B \) are isomorphisms. This enabled us to define the symmetry operator \( s \), with \( B(x,y) = B(y,sx) \). In a like manner, one can see:

Proposition 3.9 Fixing an inner product space \((M,B)\), let \( \ell : M \to M \), be a \( D \)-linear operator. Then there is a unique \( \ell^* : M \to M \), \( D \)-linear, with

\[
B(x, \ell y) = B(\ell^* x, y)
\]

Notation: \( \ell^* \) is called the adjoint operator of \( \ell \), not to be confused with the adjoint maps \( \text{Ad}_R B \), \( \text{Ad}_L B \) previously defined.

Proof: For fixed \( x \), we have the map \( B(x, \ell(-)) \in \text{Hom}_D(M,K) \). Since \( B \) is non-singular, \( \text{Ad}_L \) is an isomorphism and we can
find a unique $w \in M$ such that:

$$B(x, l(-)) = B(w, -) .$$

Define $l^* x = w$. Then $B(x, l y) = B(l^* x, y)$. $l^*$ is clearly a well-defined, $D$-linear map with the desired properties. $\text{Ad}_L$ being an isomorphism shows $l^*$ is unique. □

Thus there is the correspondence $l \rightarrow l^*$. This gives an anti-involution of the algebra of linear operators on $M$. For example, if $l, \tau \in \text{Hom}_D(M, M)$, then

$$B((l \tau)^* v, w) = B(v, (l \tau)_w)$$
$$= B(l^* v, \tau w)$$
$$= B(\tau^* l^* v, w) .$$

$B$ non-singular implies $(l \tau)^* = \tau^* l^*$. Likewise, an easy calculation shows that $(l^*)^* = l$. 
4. The Witt equivalence relation

**Definition 4.1** Let $k$ be given, $k \in D$. A degree $k$ mapping structure over $D$ is a triple $(M, B, \ell)$ satisfying:

(a) $(M, B)$ is an inner product space over $D$.

(b) $\ell : M \to M$ is a $D$-linear map satisfying

$$B(\ell x, \ell y) = kB(x, y) \text{ for all } x, y \in M.$$ 

$\ell$ is called a map of degree $k$. For all future considerations, we shall assume henceforth that $k \in \mathbb{Z}$.

In the case that $M$ is torsion free, and $k \neq 0$, it follows that $\ell$ is non-singular. To see this, suppose $\ell(x) = 0$. Then $B(\ell x, \ell y) = kB(x, y) = 0$.

Since $B$ has values in a $K = I$-invariant fractional ideal, we cancel $k$ and conclude $B(x, y) = 0$ for all $y \in M$. However, $B$ is non-singular, so that $x = 0$, and $\ell$ is 1-1.

The Witt equivalence relation for degree $k$ mapping structures comes from:

**Definition 4.2** A degree $k$ mapping structure $(M, B, \ell)$ is metabolic if there is a $D$-submodule $N \subseteq M$ satisfying:

(a) $N$ is $\ell$ invariant
(b) \( N \) is \( s \) invariant
(c) \( N = N^\perp \)

When \( (M, B, \ell) \) is metabolic, an \( N \) satisfying (a), (b) and (c) above will be called a metabolizer for \( M \). We shall also refer to the triple \( (M, B, \ell) \) as \( M \), when \( B \) and \( \ell \) are understood, and speak of \( M \) as being metabolic.

The operation of direct sum on inner product spaces extends to degree \( k \) mapping structures. The notation:
\[
(V, B, \ell) \oplus (W, B', \ell') = (V \oplus W, B \oplus B', \ell \oplus \ell').
\]
It is clear that \( \ell \oplus \ell' \) is of degree \( k \) with respect to \( B \oplus B' \).

At this point we can introduce a relation \( \sim \) on degree \( k \) mapping structures by:

\[
(V, B, \ell) \sim (W, B', \ell') \text{ when } (V \oplus W, B \oplus -B', \ell \oplus \ell')
\]
is metabolic. In what follows, we will show \( \sim \) is an equivalence relation, called the Witt equivalence relation. This agrees with the usual notion of Witt equivalence \([M-H]\) when no \( \ell \) is present.

Notation: \( W^+_{k,k}(K) \) denotes degree \( k \) mapping structures \( (M, B, \ell) \) modulo \( \sim \) with values in \( K \); together with the additional requirement that \( s = \text{id} \), the identity map, so \( B \) is symmetric. The underlying ring \( D \) is understood;
emphasis is given to the range of $B : M \times M \to K$.

Similarly, $W^{-1}(k,K)$ is Witt equivalence classes of triples $(M,B,\ell)$ having $B$ skew-symmetric.

When there is no $\ell$, so that we are taking inner product spaces modulo $\sim$, without condition (a) of 4.2, we denote the symmetric equivalence classes $W^+_{-1}(K)$, the skew-symmetric $W^{-1}_{-1}(K)$. Finally, when no $\ell$ is present, with no symmetry requirement at all placed on $B$, the resulting Witt group is denoted $A(K)$.

We have also defined the notion of $B$ being $u$-Hermitian. We write $H_u(K)$ to denote those Witt equivalence classes $[M,B]$ for which $B(x,y) = uB(y,x)$.

In summary, our notation is:

$W^+_{-1}$ : $B$ symmetric

$W^{-1}_{-1}$ : $B$ skew-symmetric

$A$ : no symmetry requirements on $B$ (B asymmetric)

$H_u$ : $B$ u-Hermitian

If we write $W^+_{-1}(K)$, we are thinking of pairs $(M,B)$; if we write $W^{-1}_{-1}(k,K)$ we are thinking of triples $(M,B,\ell)$ with $B(\ell x, \ell y) = kB(x,y)$. The $K$ means that $B : M \times M \to K$.

We write $A(k,K)$ to denote the group which consists of triples $(M,B,\ell)$, with no symmetry requirements on $B$. 
If we let \( k \) range over \( \mathbb{Z} \), we can form a graded ring, with multiplication defined by \( \otimes \),

\[
\mathbf{A}(k,K) \times \mathbf{A}(k',K') \to \mathbf{A}(kk',KK')
\]

\[
(M,B,\ell) \times (M',B',\ell') \to (M \otimes_D M', B \otimes B', \ell \otimes \ell')
\]

This follows from 2.2.

However, for this paper, we shall only be concerned with the Abelian group structure arising from direct sum. Our objective now is to show that \( \sim \) is an equivalence relation. \( \sim \) is clearly reflexive and symmetric. We must show \( \sim \) is transitive.

Again when \((M,B,\ell)\) is metabolic, we will say \( M \) is metabolic, the \( B,\ell \) being understood, and write \( M \sim 0 \).

The following proposition is clear.

**Proposition 4.3** \( \sim \) is transitive if and only if \( H \sim 0 \) and \( M \oplus H \sim 0 \) implies \( M \sim 0 \). \( \square \)

We call \( M \) stably metabolic if there exists \( H \sim 0 \) with \( M \oplus H \sim 0 \). We may then restate Proposition 4.3 as saying \( \sim \) is transitive if and only if stably metabolic implies metabolic.

**Comment:** Once we have shown that \( \sim \) is transitive, it is also clear that the following relation, \( \cong \), would
have yielded the same relation as ~. Define
\[ M_0 \cong M_1 \text{ if and only if there exists } H_0 \sim 0, H_1 \sim 0 \]
with \( M_0 \otimes H_0 \) isomorphic to \( M_1 \otimes H_1 \).

**Lemma 4.4** Suppose \( M \) is a finitely generated torsion free \( D \)-module. Then \( (M, B, \ell) \sim 0 \) over \( D \) if and only if \( (M, B, \ell) \otimes_{D} E \sim 0 \) over \( E \).

Here \( E \) is the quotient field of \( D \) (see 2.4).

**Proof:** Necessity is clear, for if \( N \) is a metabolizer for \( (M, B, \ell) \), then \( N \otimes_{D} E \) is a metabolizer for \( (M, B, \ell) \otimes_{D} E \).

**Sufficiency:** Note that \( M \) is embedded into \( M \otimes_{D} E \) as \( M \otimes 1 \). Suppose \( M \otimes_{D} E \) has metabolizer \( N = N^1 \). Let \( N_1 = N \cap (M \otimes 1) \subset M \otimes 1 \).

Claim: \( N_1 = N_1^1 \) in \( M \otimes 1 \approx M \), so that \( M \sim 0 \).

To begin with, \( N_1 \) is \( s, \ell \) invariant since \( N \) is.

It also is clear that \( N_1 \subset N_1^1 \). Conversely, if \( x \otimes 1 \in N_1^1 \), \( B(x \otimes 1, y \otimes 1) = 0 \) for all \( y \otimes 1 \in N_1 \).

However, if \( y \otimes r \in N \), then \( y \otimes 1 \in N_1 \), so
\[ B(x \otimes 1, y \otimes r) = \mathbb{F} B(x \otimes 1, y \otimes 1) = 0. \]
Hence \( x \otimes 1 \in N_1 = N \). Thus \( x \otimes 1 \in N_1 \), and \( N_1^1 \subset N_1 \).
Thus \( N_1 = N_1^1 \) and \( M \otimes 1 \simeq M \sim 0 \). □

Using Lemma 4.4, we see that to prove \( \sim \) is transitive in the case that \( M \) is torsion free, we can assume that \( D = E \) a field, and \( K = E \).

We thus assume for the rest of the proof of transitivity that \( K = E = D \) a field for \( M \) torsion free, \( K = E/D \) for \( M \) torsion as usual.

In either case, \( K \) is an injective \( D \)-module and we have:

Theorem 4.5 Let \((M, B)\) be an inner product space, with values in \( K = E \) or \( K = E/D \). If \( N \) is \( s \)-invariant, then \( N = (N^1)^1 \).

Remark 4.6 This is not true for \( K = I \) an arbitrary fractional ideal, or even \( K = D \).

Proof: We have the exact sequence

\[
0 \rightarrow N^1 \rightarrow M \xrightarrow{\text{Ad}^B_R} \text{Hom}_D(N,K) \rightarrow 0.
\]

\( \text{Ad}_R^B \) is onto since \( K \) is an injective \( D \)-module.

Applying the \( \text{Hom} \) functor, we obtain

\[
0 \rightarrow \text{Hom}_D(\text{Hom}_D(N,K), K) \rightarrow \text{Hom}_D(M, K) \rightarrow \text{Hom}_D(N^1, K) \rightarrow 0
\]
Again, \( \text{Ext}(\text{Hom}_D(N,K),K) = 0 \), since \( K \) is injective and the last map is onto.

We can identify \( \text{Hom}_D(\text{Hom}_D(N,K),K) \cong N \) by Theorem 1.6. This clearly yields the commutative diagram:

\[
\begin{array}{cccccc}
0 & \to & N & \to & \text{Hom}_D(M,K) & \to & \text{Hom}_D(N^\perp,K) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & \to & (N^\perp)^\perp & \to & M & \to & \text{Hom}_D(N^\perp,K) & \to & 0
\end{array}
\]

The inner product provides an isomorphism of the middle terms, so by diagram chase [M 50] the inclusion \( N \subseteq (N^\perp)^\perp \) is an isomorphism. □

**Lemma 4.7** For any two \( r, s \) invariant submodules \( R \) and \( S \) of \( M \), where \( (M,B) \) is an inner product space as above, we have:

1. \((R + S)^\perp = R^\perp \cap S^\perp\)
2. \(R^\perp + S^\perp = (R \cap S)^\perp\)

**Proof:** (1) follows from the definition of \( \perp \). To show (2), observe that

\[(R^\perp + S^\perp)^\perp = (R^\perp)^\perp \cap (S^\perp)^\perp = R \cap S\]
Thus, taking \( R^1 + S^1 = (R \cap S)^1 \). \( \square \)

Lemma 4.8 Let \((M, B)\) be an inner product space as above. Suppose that \(M \sim 0\) with metabolizer \(N\).

Let \(L \subseteq M\) satisfy \(L \subseteq L^1\). Then \(L + N \cap L^1 = (L + N \cap L^1)^1\).

Remark 4.9 This Lemma shows how to go from a metabolizer \(N\), and a subspace \(L \subseteq L^1\) to another metabolizer, namely, \(L + N \cap L^1\), which contains the self annihilating subspace \(L\).

Proof: The assumption that \(L\) is \(L^1\)'s invariant is understood, in order that \(L^1\) make sense.

We compute using Lemma 4.7:

\[
(L + (N \cap L^1))^1 = L^1 \cap (N \cap L^1)^1 = L^1 \cap (N^1 + (L^1)^1). \\
= L^1 \cap (N + L) \quad \text{since } N^1 = N \text{ and } (L^1)^1 = L \\
= (L^1 \cap N) + (L^1 \cap L) \quad \text{since } L \subseteq L^1 \\
= L + (N \cap L^1). \\
\]

Thus \(L + (N \cap L^1)\) is also a metabolizer. \( \square \)

Theorem 4.10 (Transitivity of \(\sim\)) Let \(H \sim 0\).

Then \(M \oplus H \sim 0\) if and only if \(M \sim 0\).
Proof: Sufficiency is clear.

Necessity: Let $N$ be a metabolizer for $M \oplus H$, and $H_0$ a metabolizer for $H$. We embed $H_0, H$ into $M \oplus H$ as $0 \oplus H_0, 0 \oplus H$ respectively. Notice that $0 \oplus H_0 \subseteq (0 \oplus H_0)^\perp$, so that by Lemma 4.8 we may rechoose $N$ such that $0 \oplus H_0 \subseteq N$.

We review our notation. $(M, B), (H, B')$ and $(M \oplus H, B \oplus B')$ are the inner product spaces. We will write elements in $M \oplus H$ as pairs $(x, y)$ with $x \in M, y \in H$.

Let $N_0 =$ projection of $N$ onto $M$

$$= \{ a \in M : (a, h) \in N \text{ for some } h \}.$$ 

Claim: $N_0$ is a metabolizer for $M$.

$N_0$ is clearly $\ell,s$ invariant since $N$ is, and projection commutes with $\ell,s$ on $M \oplus H$.

We first show that $N_0 \oplus H_0 = N$. If $(a, h) \in N$ we claim $h \in H_0$. Let $(0, h_1) \in 0 \oplus H_0 \subseteq N = N^\perp$. Then $(B \oplus B')(0, h), (0, h_1) = B(0, 0) + B'(h, h_1)$

$$= B(a, 0) + B'(h, h_1) = (B \oplus B')(a, h), (0, h_1) = 0$$ since $N = N^\perp$. Hence $B'(h, h_1) = 0$ for all $h_1 \in H_0$, so that $h \in H_0^\perp = H_0$ as claimed. Thus $N_0 \oplus H_0 = N$.

Clearly $N_0 \subseteq N_0^\perp$. Conversely, let $b \in N_0^\perp$. Then by computing as above $(b, 0) \in N^\perp = N$, so that $b \in N_0$. Hence $N_0 = N_0^\perp$ is a metabolizer for $M$. □
Thus ~ is an equivalence relation, and we may form the Witt group consisting of equivalence classes of triples \((M, B, \ell)\) modulo ~. \((M, B, \ell) \sim (M_1, B_1, \ell_1)\) provided 
\[(M \oplus M, B \oplus B_1, \ell \oplus \ell_1) \sim 0.\]

Notation: \([M, B, \ell]\) will denote the Witt equivalence class of \((M, B, \ell)\).
5. Anisotropic representatives

Our final goal of this chapter is to find a representative of each equivalence class. As long as \( K = E \) a field, or \( E/I \) in the torsion case, this representative is unique.

We begin by describing the representative we will obtain.

**Definition 5.1** A degree \( k \) mapping structure \((M, B, \ell)\) is anisotropic if for any \( s, \ell \) invariant D-submodule \( N \) of \( M \), \( N \cap N^\perp = 0 \).

**Theorem 5.2.** Every Witt equivalence class \([M, B, \ell]\) has an anisotropic representative.

We prove this theorem by way of a sequence of Lemmas which are of interest in their own right.

**Lemma 5.3** Let \( T \) be an \( s, \ell \) invariant D-submodule of \( M \), with \( T \subseteq T^\perp \). Then \( T^\perp/T \) inherits a quotient degree \( k \) mapping structure. \((T^\perp/T, B, \ell)\).

**Proof:** Let \([t]\) denote an element in \( T^\perp/T \).
Define $\bar{B}([t_1],[t_2]) = B(t_1,t_2)$, where $t_1,t_2$ are representatives of $[t_1],[t_2]$ respectively. $\bar{B}$ is clearly well-defined since $T$ is self-annihilating, i.e. $T \subset T^\perp$. It is likewise clear that $\bar{I}$, the induced map on $T^\perp/T$ is of degree $k$ with respect to $\bar{B}$, and well-defined.

We must show that $\bar{B}$ is an inner product, i.e. that $\text{Ad}_{\bar{B}} : T^\perp/T \to \text{Hom}_D(T^\perp/T,K)$ is an isomorphism.

Applying the functor $\text{Hom}_D(-,K)$ to the exact sequence:

$$0 \to T \to T^\perp \to T^\perp/T \to 0,$$

we obtain the embedding:

$$0 \to \text{Hom}_D(T^\perp/T,K) \to \text{Hom}_D(T^\perp,K).$$

Suppose $\bar{g} \in \text{Hom}_D(T^\perp/T,K) \to \tilde{g} \in \text{Hom}_D(T^\perp,K)$. We can lift $\tilde{g}$ to $g : M \to K$ since $T^\perp$ is a summand by Proposition 3.6 in the torsion free case, and since $K$ is injective in the torsion case.

$\text{Ad}_{\bar{B}} : M \to \text{Hom}_D(M,K)$ is an isomorphism. Hence $g = B(-,x)$. $g$ restricted to $T$ equals $0$, so $x \in T^\perp$. Thus $(x + t)$ gives the same $\tilde{g}$ for all $t \in T$. So we may read $x \in T^\perp/T$.

This procedure defines a map:

$$\text{Hom}_D(T^\perp/T,K) \to T^\perp/T, \text{ namely } \bar{g} \to \tilde{g} \to [x].$$

The inverse of this map is simply
\[ [x] \in T^1/T \quad \text{Ad}_R^B \quad B(-,[x]), \text{ Hence } \text{Ad}_R^B \text{ is an isomorphism, and } B \text{ is an inner product.} \]

\begin{lemma}
With the same hypotheses as in Lemma 5.3, \( M \otimes -T^1/T \) is metabolic.
\end{lemma}

\begin{proof}
In the torsion free case, by Lemma 4.4 we may assume \( K = E \) a field. Thus, in any case there is no loss of generality in assuming the hypotheses of Theorem 4.5, namely that \( K = E \) or \( K = E/I \). Consequently, for \( N \) an \( s,l \) invariant subspace of \( M \), we have \( N = (N^1)^\perp \).

We wish to show \( M \otimes -T^1/T \sim 0 \). So consider \( N = \{(x,x + T): x \in T^1\} \). \( N \) is an \( s,l \) invariant subspace, and clearly \( N \subset N^\perp \). Let \( (a,b + T) \in N^\perp \), with \( b \in T^1 \). We compute \( (B \otimes -B)(a,b + T), (x,x + T)) \equiv B(a,x) - B(b,x) = 0 \) for all \( (x,x + T) \in N \). Thus \( B(a-b,x) = 0 \) for all \( x \in T^1 \). Hence \( (a-b) \in (T^1)^\perp = T \), since by assumption Theorem 4.5 applies. So \([b] = [a] - [(a-b)] = [a] \) in \( T^1/T \), and

\( (a,b + T) = (a,a + T) \in N \).
Therefore \( N^\perp \subseteq N \), and \( N \) is a metabolizer for \( M \cong T^\perp /T \).

Lemma 5.4 shows then that \( M \sim T^\perp /T \) whenever \( T \subseteq T^\perp \). In the torsion free case, we can use that \( \text{rank } M < \infty \) to conclude, after successive applications of Lemmas 5.3 and 5.4, that \( M \sim M_0 \) where \( M_0 \) has no \( \ell \)-invariant subspace \( T \) with \( T \subseteq T^\perp \). In other words, \( M_0 \) is anisotropic.

For the torsion module case, we repeatedly apply Lemmas 5.3, 5.4 to obtain sequences:

\[
M \supset T^\perp \supset T^\perp_1 \supset \ldots \supset T^\perp_2 \supset T_1 \supset T
\]

Since \( M \) is Noetherian, the ascending chain condition implies that the sequence \( \{T^\perp_i\} \) terminates. Hence, it follows the the sequence \( \{T^\perp_i\} \) will also terminate.

Since both chains terminate, \( M \sim T^\perp_r /T_r \), with \( T^\perp_r /T_r \) having no \( \ell \)-invariant submodule \( T^\perp_{r+1} \) with \( T^\perp_{r+1} \subseteq T^\perp_{r+1} \). Thus \([M,B,\ell]\) has an anisotropic representative, \( T^\perp_r /T_r \), as claimed. This completes the proof of Theorem 5.2.

This anisotropic representative need not be unique for torsion free D-modules, \([M-H]\). However, for
K = E a field, or K = E/I, we shall see that it is.

**Theorem 5.5** As long as \([M, B, \ell] \in W(k, K)\) satisfies Theorem 4.5, i.e., for \(K = E\) or \(E/I\), every Witt equivalence class \([M, B, \ell]\) has a unique anisotropic representative up to isomorphism.

**Proof:** Suppose \((M, B, \ell) \sim (M', B', \ell')\), with \(M\) and \(M'\) both anisotropic. Let \(N \subseteq M \oplus M'\) be a metabolizer, with respect to \(B \oplus -B'\). We will show that \(N\) is the graph of an isomorphism \(f: M \to M'\) which satisfies

\[
B'(f(k), f(y)) = B(x, y), \quad \ell' \circ f = f \circ \ell \quad \text{and} \quad s' \circ f = f \circ s.
\]

Thus \(f\) is an isomorphism, between \((M, B, \ell)\) and \((M', B', \ell')\).

Let \(A = \{a \in M : \text{there exists } a_1 \in M' \text{ with } (a, a_1) \in N\}\).

Claim: For given \(a\), \(a_1\) is unique.

For suppose \((a, a_1)\) and \((a, a_2)\) \(\in N\). Then \((0, a_1 - a_2) \in N\). Consider the \(s', \ell'\) invariant subspace, \(M_1\), of \(M'\) generated by \(a_1 - a_2\). Since \(N\) is \(s \oplus s'\) and \(\ell \oplus \ell'\) invariant, this subspace \(M_1\) will have \((0, M_1) \subseteq N\). Hence \(M_1 \subseteq M_1^1\) since \(N = N^1\). This is a contradiction to \(M'\) being anisotropic unless \(a_1 - a_2 = 0\).
so that \( a_1 = a_2 \).

Similarly, let \( B = \{a_\perp \in M' : \text{there exists } a \in M \text{ with } (a, a_\perp) \in N \} \). As above, each \( a_\perp \in B \) has a unique \( a \in M \) with \( (a, a_\perp) \in N \). It follows that \( N \) is the graph of a \( 1-1 \) function \( f \).

We claim that \( A = M \) and \( B = M' \). To see this we show that \( A^\perp = 0 \) in \( M \), hence \( (A^\perp)^\perp = A = M \). So let \( a \in A^\perp \), and consider \( (a, 0) \in M \otimes M' \). Take any \( (x, y) \in N \). Then:

\[
(B \otimes -B')( (a, 0), (x, y)) = B(a, x) = 0
\]
since

\[
a \in A^\perp, \: x \in A.
\]

Thus \( (a, 0) \in N^\perp = N \). By the first claim, this implies \( a = 0 \). A similar argument shows \( B = M' \). It follows that \( f : M \to M' \) is an isomorphism.

Let \( (a, f(a)) \in N \). Then \( (la, l'f(a)) \in N \) since \( N \) is \( l \otimes l' \) invariant. Thus, by definition, \( (f \circ l)(a) = (l' \circ f)(a) \). Similarly, \( (f \circ s)(a) = (s' \circ f)(a) \).

Finally, consider \( (x, f(x)) \) and \( (y, f(y)) \in N \).

\[
(B \otimes -B')( (x, f(x)), (y, f(y))) = 0 , \text{ so } B(x, y) - B'(f(x), f(y)) = 0 , \text{ and } B(x, y) = B'(f(x), f(y)) \text{ as desired.} \]
Chapter II POLYNOMIALS

Given a Witt equivalence class, \([M,B,\ell]\) in \(W(k,F)\), we shall decompose it as \([M,B,\ell]\) = \(\oplus [M_i,B_i,\ell_i]\), according to the irreducible factors of the characteristic polynomial of \(\ell\). This is the object in Chapter III.

In this chapter, we lay the groundwork for the above decomposition. This involves a careful study of the characteristic and minimal polynomials of \(\ell\). These polynomials belong to \(K(F) = \{p(t) : p(t) \text{ is a monic polynomial with non-zero constant term, coefficients in } F \text{ a field}\}\). We assume throughout this section that we are working over a field \(F\).

On \(K(F)\) we define an involution \(T_k : K(F) \rightarrow K(F)\). The characteristic and minimal polynomials are shown to be \(T_k\) fixed.

When \(p(t)\) is irreducible and \(T_k\) fixed, we consider the field \(F[t,t^{-1}]/(p(t)) = F(\theta)\). It is shown that there is an induced involution of \(F(\theta)\) given by \(\overline{\theta} = k\theta^{-1}\).

This discussion provides the key ingredients for the computations to be made later.
Let \( (M,B) \) be an inner product space, and let \( \ell : M \to M \) be \( F \)-linear. Recall the adjoint, \( \ell^* \) of \( \ell \) is defined by the equation \( B(v,\ell w) = B(\ell^*v,w) \). \[ I 3.9 \]

**Lemma 1.1** \( \ell \) and \( \ell^* \) have the same characteristic polynomials, and the same minimal polynomials.

**Proof:** For any polynomial \( p(t) \),
\[
B(p(\ell^*)v,w) = B(v,p(\ell)w).
\]
Thus, \( p(\ell^*) = 0 \) if and only if \( p(\ell) = 0 \), since \( B \) is non-singular. The assertion about minimal polynomials follows.

Working over a field, we may view \( M \) as the space of \( n \times 1 \) column matrices and \( B \) as an \( n \times n \) matrix \( B' \), \[ VI 2 \]. \( B(v,w) = v^t \bar{w} \), where \( v^t \) denotes the transpose of \( v \), \( \bar{w} \) denotes the conjugate of \( w \),
\[
w = \left( \begin{array}{c} \bar{a}_1 \\ \vdots \\ \bar{a}_n \end{array} \right), \quad \bar{w} = \left( \begin{array}{c} \bar{a}_1 \\ \vdots \\ \bar{a}_n \end{array} \right).
\]
\( \ell \) is multiplication by an \( n \times n \) matrix \( L \). To simplify our notation, we identify \( B \) with its matrix and write \( B' = B \).

We compute,
\[
[(B^{-1}L^tB)v]^t \bar{w} = v^tBLB^{-1} \bar{w} = v^tB(L \bar{w}).
\]
It follows that \( L^* = B^{-1}L^tB = \) matrix of \( \ell^* \). Hence, letting \( \det \) denote the determinant, and \( I \) the \( n \times n \) identity matrix,
\[
\det(tI - B^{-1}L^tB) = (\det B)(\det (tI - B^{-1}L^tB))(\det B^{-1})
\]
\[ = \det (BtB^{-1} - BB^{-1}L^tBB^{-1}) \]
\[ = \det (tI - L^t) \]
\[ = \det (tI - L) . \]

The assertion for characteristic polynomials follows. □

Let \( \ell \) be a map of degree \( k \). Then \( \ell \) is non-singular, and \( \ell \) is related to \( \ell^* \) by:

**Lemma 1.2** If \( \ell \) has matrix \( L \), and \( \ell^* \) has matrix \( L^* \), then \( L^* = kL^{-1} \).

Proof: \( B(\ell^*v,w) = B(v,\ell w) = B(\ell \ell^{-1}v,\ell w) \)
\[ = kB(\ell^{-1}v,w) = B(k\ell^{-1}v,w) . \]
Again since \( B \) is non-singular, it follows that \( \ell^* = k\ell^{-1} \). □

**Proposition 1.3** Let \( \ell \) be a map of degree \( k \). Then both the minimal and characteristic polynomials of \( \ell \) satisfy:
\[ \text{degree } p(t)p(t^{-1}k) = a_0p(t) , \]
where \( a_0 = \text{constant term of } p(t) \).

Proof: Let \( \% (t) = \text{characteristic polynomial of } \ell \).
Since \( \ell \) is non-singular the constant term of \( \% (t) \) is non-
zero. (Of course the dimension of the vector space \( M \) is \( n \), the degree of \( \chi(t) \)).

\[ \ell^* = k\ell^{-1} \] by Lemma 1.2. Thus, by Lemma 1.1, \( \ell \) and \( k\ell^{-1} \) have the same characteristic polynomial. The identity \((-t^{-1}\ell)(tI-k\ell^{-1}) = (kt^{-1}I-L)\) yields

\[ \det(-t^{-1}\ell)\chi(t) = \chi(kt^{-1}). \]

However \( \det(-t^{-1}\ell) = (-1)^n t^{-n} \det L \)

\[ = (-1)^n t^{-n} (-1)^n a_0 \]

\[ = t^{-n} a_0 \quad , \quad a_0 = \text{constant term of } \chi(t). \]

Thus \( t^{-n} a_0 \chi(t) = \chi(kt^{-1}) \), so that \( t^n \chi(kt^{-1}) = a_0 \chi(t) \) as desired.

In order to check the result for the minimal polynomial \( p(t) \) of \( \ell \) we again use \( \ell^* = k\ell^{-1} \). Let degree \( p(t) = m \). By Lemma 1.1, \( p(k\ell^{-1}) = 0 \). Thus \( q(t) = a_0^{-1}t^m p(kt^{-1}) \) is a monic polynomial of degree \( m = \text{degree } p(t) \) with \( q(\ell) = 0 \). Hence \( p(t) = a_0^{-1}t^m p(kt^{-1}) \) as claimed.

We continue the study of these polynomials by letting \( K(F) = \{ p(t) : p(t) \text{ is a monic polynomial, with constant term } a_0 \neq 0 \} \). Here \( p(t) = \sum_{i=0}^{n} a_i t^i \). This is a cancellation semigroup with respect to multiplication of polynomials. Further, any polynomial in \( K(F) \) can be uniquely factored into a product of powers of irreducible polynomials in \( K(F) \).
For \( k \neq 0, k \in F^* \), we are led by Proposition 1.3 to introduce an automorphism of period 2 on \( K(F) \) by:

\[
T_k : p(t) \to t^{\deg p(t)} a_0^{-1} p(kt^{-1}) = (T_k p)(t).
\]

Proposition 1.3 then says that for a degree \( k \) mapping structure \((M,B,\ell)\), both the characteristic and minimal polynomial of \( \ell \) are \( T_k \) fixed.

**Lemma 1.4** A polynomial \( p(t) \) is fixed under \( T_k \) if and only if its coefficients satisfy

\[
a_j k^j = a_0 a_{n-j},
\]

for \( 0 \leq j \leq n = \deg p(t) \).

**Proof:** Clear by definition of \( T_k \). \( \Box \)

We note that \( a_0^2 = k^n \).

Thus, if \( p(t) \in K(F) \) is \( T_k \) fixed, exactly one of the following three cases applies.

**Type 1:** \( \deg p(t) = 2n \) and \( a_0 = k^n \).

Thus \( a_j = k^{n-j} a_{2n-j} \) for \( 0 \leq j \leq n \).

**Type 2:** \( \deg p(t) = 2n \) and \( a_0 = -k^n \). Assume \( \text{char } F \neq 2 \).

Thus \( a_j = -k^{n-j} a_{2n-j} \) for \( 0 \leq j \leq n \), so that \( a_n = -a_n = 0 \).
Note: There is no loss of generality in assuming characteristic $F \neq 2$ in this case.

**Type 3**: $\deg p(t) = 2d + 1$.

So $k^{2d+1} = a_0^2$, and $k = \left(\frac{a_0}{k^d}\right)^2$.

**Lemma 1.5** If $p(t) \in K(F)$ is $T_k$ fixed, of degree $2d + 1$, then $-a_0 k^{-d}$ is a root of $p(t)$.

**Proof:** Consider $p(-a_0 k^{-d})$. The $2j$ term is $a_{2j}(-a_0 k^{-d})^{2j} = a_{2j} k^j$.

However, this $2j$ term cancels with the $2(d-j)+1$ term since

$$a_{2(d-j)+1} (-a_0 k^{-d})^{2(d-j)+1}$$

$$= a_2(d-j)+1 (-a_0 k^{-d}) (-a_0 k^{-d})^{2(d-j)}$$

However

$$a_2(d-j)+1 a_0 = a_{2j} k^{2j}$$

and $(-a_0 k^{-d})^2 = k$.

So the above equals:

$$= -a_{2j} k^{2j} k^{-d} k^{d-j} = -a_{2j} k^j \quad \square$$

**Lemma 1.6** If $p(t) \in K(F)$ is of type 2, then $(t^2 - k)$ divides $p(t)$. (Characteristic $F \neq 2$)
**Proof:** For $0 \leq j < n$, $p(\sqrt[k]{k})$ will have $j^{th}$ term $a_j(\sqrt[k]{k})^j$, and $(2n-j)^{th}$ term $a_{2n-j}(\sqrt[k]{k})^{2n-j}$.

Further,

$$a_j(\sqrt[k]{k})^j = -k^{n-j}a_{2n-j}(\sqrt[k]{k})^j$$
$$= - (\sqrt[k]{k})^{2(n-j)}a_{2n-j}$$
$$= -a_{2n-j}(\sqrt[k]{k})^{2n-j},$$

and these terms cancel.

Since $\text{char } F \neq 2$, $a_n = 0$ and $\sqrt[k]{k}$ is a root of $p(t)$. Hence we can write $(t^2 - k)q(t) = p(t)$ over $F(\sqrt[k]{k})$. It is clear that $q(t) \in F[t]$, since $t^2 - k$ and $p(t)$ are.

\[ \square \]

Hence, irreducible polynomials in $K(F)$ which are $T_k$ fixed fall into the three following types.

**Type 1:** $\deg p(t) = 2n$ and $a_0 = k^n$

**Type 2:** $k \not\in F** and $t^2 - k = p(t)$, when $\text{char } F \neq 2$

**Type 3:** $k \in F** and $p(t) = t + \sqrt[k]{k}$

On $F[t, t^{-1}]$ we introduce the involution:

$t \rightarrow kt^{-1}$, $t^{-1} \rightarrow k^{-1}t$ Denote this by: $\gamma \rightarrow \overline{\gamma}$. See [VIII].
Let \( \gamma \in F[t, t^{-1}] \), say \( \gamma = \sum_{-m}^{n} A_j t^j \). Then \( \gamma = \gamma' \) if and only if \( n = m \) and we have \( A_{-j} = A_j k^j \) \( 0 \leq j \leq n \).

Suppose \( \gamma = \gamma' \) and \( A_n = 1 \). Then \( t^n \gamma = p(t) \) belongs to \( K(F) \) and is a \( T_k \) fixed polynomial of type 1. Conversely, any \( T_k \) fixed polynomial of type 1 can be written as \( t^n \gamma = p(t) \), for a unique \( \gamma = \gamma' \), where \( 2n = \text{degree } p(t) \).

Continuing, suppose \( p(t) \) is a \( T_k \) fixed polynomial of type 2. Then by Lemma 1.6, \( p(t) = (t^2 - k) q(t) \). However \( T_k \) is multiplicative, and \( t^2 - k \) is \( T_k \) fixed. It follows that \( q(t) \) is also a \( T_k \) fixed polynomial. \( q(t) \) has degree \( 2(n-1) \); constant term \( k^{n-1} \).

Hence \( p(t) = (t^2 - k) q(t) \) where \( q(t) \) is a \( T_k \) fixed polynomial of type 1, or \( q(t) = 1 \). So we can write \( p(t) = (t^2 - k) t^{n-1} \gamma = t^n (t - kt^{-1}) \gamma \) where \( \gamma = \gamma' \) and degree \( p(t) = 2n \).

Finally for type 3, let \( p(t) \) have constant term \( a_0 \). By Lemma 1.5, \( p(t) = (t + a_0 k^d) q(t) \). As above we show \( q(t) \) is a type 1 \( T_k \) fixed polynomial of degree \( 2d \) or \( q(t) = 1 \).

**Lemma 1.7** If \( p(t) \in K(F) \) is \( T_k \) fixed, then the principal ideal \( (p(t)) \subset F[t, t^{-1}] \) is invariant.
Proof: We take first the case when \( p(t) \) is of type 1, say \( p(t) = t^n \gamma \). It follows that

\[
(p(t)) = (\gamma) \quad \text{since} \quad t \text{ is a unit in } F[t, t^{-1}].
\]

But \( \gamma = \overline{\gamma} \), so that \( (p(t)) \) is \( \overline{\cdot} \)-invariant.

Next, let \( p(t) \) be of type 2. Then by the discussion before the Lemma, \( p(t) \) factors as \( p(t) = t^n (t - k t^{-1}) \gamma \), where \( \gamma = \overline{\gamma} \) and \( (t - k t^{-1}) = -(t - k t^{-1}) \). Clearly then,

\[
(p(t)) = ((t - k t^{-1}) \gamma) \quad \text{is} \quad \overline{\cdot} \text{-invariant}.
\]

For \( p(t) \) of type 3, \( p(t) = t^d (t + a_0 k^{-d}) \gamma \), with \( \gamma = \overline{\gamma} \). Now, we compute

\[
(t + a_0 k^{-d}) = k t^{-1} + a_0 k^{-d} = (t + a_0 k^{-d})(a_0 k^{-d} t^{-1}) \quad \text{since} \quad \left(\frac{a_0}{k^d}\right)^2 = k. \quad \text{However}
\]

\( a_0 k^{-d} t^{-1} \) is a unit in \( F[t, t^{-1}] \), which again yields that \( (p(t)) \) is \( \overline{\cdot} \)-invariant. □

We summarize this discussion. Let \( p(t) \) be a \( T_k \) fixed irreducible polynomial in \( K(F) \). Then there are three cases to consider.

- **Type 1:** \( F[t, t^{-1}] / (p(t)) = F(\overline{\theta}) \) is a simple algebraic extension of \( F \) together with a non-trivial involution \( \overline{\theta} = k \theta^{-1} \). Here \( \theta \) is identified with \( t \). If \( \overline{t} = t \), then \( k t^{-1} = t \) so that \( t^2 = k, t^2 - k = 0 \) and we are in type 2.
Type 2: \[ F[t, t^{-1}]/(p(t)) = F[t, t^{-1}]/(t^2 - k) = F(\sqrt{k}) \]
for the case that \( k \not\in F^{**} \). The induced involution is \( \sqrt{k} \to k(\sqrt{k})^{-1} = \sqrt{k} \), which is trivial. Note that this is not the involution \( \sqrt{k} \to -\sqrt{k} \).

Type 3: In this case, \( k \in F^{**} \), say \( f^2 = k \). The irreducible polynomial is \( p(t) = t + f \). The field \( F[t, t^{-1}]/(t + f) \cong F \), by identifying \( t \) with \( -f \).

The involution:
\[ -f \to k - f = \left( \frac{f + f}{-f} \right)^2 = -f \]
is trivial, and so is the extension.

Finally, in the type 1 situation when the involution \( - \) is non-trivial, we wish to describe the fixed field.

**Lemma 1.8** In the type 1 situation, the fixed field of \((F(\theta), -)\) is \( F(\theta + k\theta^{-1}) \).

**Proof:** There is the embedding \( F[x] \to F[t, t^{-1}] \) given by \( x \to t + kt^{-1} \). We claim that the image of \( F[x] \) is the subring of \( - \)-fixed elements.

Let \( \gamma \in F[t, t^{-1}] \), \( \gamma = \sum_{j=0}^{n} A_j t^j \) with \( A_{-j} = A_j k^j \), be a typical \( - \)-fixed element. Consider
\[ \gamma = A_n(t + kt^{-1})^n. \] This is still fixed, and can be written as
\[ \sum_{j=-n+1}^{n-1} B_j t^j. \] Continuing inductively,
\[ \gamma = \sum_{i=0}^{n} a_i (t + kt^{-1})^i = q(t + kt^{-1}) \] as claimed. \qed

Suppose \( A_n = 1 \) and \( \gamma \) is the image of a monic polynomial, \( q(x) \), in \( F[x] \).

**Claim:** If \( t^n \gamma = p(t) \) is irreducible, then so is \( q(x) \). For if \( q(x) \) factors as \( q(x) = q_1(x)q_2(x) \), with \( r = \text{degree } q_1(x) \), \( w = \text{degree } q_2(x) \), then
\[ p(t) = t^n \gamma = [t^r q_1(t + kt^{-1})] [t^w q_2(t + kt^{-1})], \]
so that \( p(t) \) also factors.

We may thus write \( F[x]/(q(x)) \) as the fixed elements in \( (F(\theta), -) \). Clearly, the minimal polynomial of \( \theta \) over \( F(\theta + k\theta^{-1}) \) is \( x^2 - (\theta + k\theta^{-1}) x + k \).
Chapter III  WITT GROUP OF A FIELD

We wish to compute the Witt group $W(k,F)$ for $F$ a field. This is done by decomposing $W(k,F)$ as a direct sum of groups $W(k,F;f)$ according to the irreducible factors $f(t)$ of the characteristic polynomial of $\ell$. We identify each group,

$$W(k,F;f) \cong W(k,F;F[t]/(f(t)))$$

by taking anisotropic representatives.

On $F[t]/(f(t))$ there is an induced involution by Chapter II. We prove a trace lemma which then enables us to compute these groups $W(k,F;F[t]/(f(t)))$. In this manner then we will have computed $W(k,F)$.

The trace lemma is then used in several cases to compute Witt groups. This computation is valuable for the ensuing chapters.
1. **Decomposition by characteristic polynomial**

Given a degree k mapping structure \((M, B, \ell)\), we may view \(M\) as a \(D[t]-\)module by defining the action of the indeterminate \(t\) to be the same as \(\ell\). By II.1.3, the characteristic polynomial of \(\ell\), \(p(t)\), is \(T_k\) fixed.

**Proposition 1.1** If \((M, B, \ell)\) is metabolic, and \(\ell\) has characteristic polynomial \(p(t)\), then \(p(t)\) factors as: \(p(t) = f(t) \cdot T_k f(t)\) for some monic polynomial \(f(t)\).

**Proof:** Let \(f(t)\) be the characteristic polynomial of \(\ell\) restricted to \(N\), where \(N\) is a metabolizer for \(M\).

We now make \(\text{Hom}_D(N, K)\) into a \(D[t]-\)module. This is done by defining the action of \(t\) by:

Let \(h \in \text{Hom}_D(N, K)\). Then \((t \cdot h)(n) = h(\ell^* n)\), where \(\ell^*\) is the adjoint of \(\ell\). Viewed thus, \(\text{Ad}_R B : M \to \text{Hom}_D(N, K)\) is a \(D[t]-\)module homomorphism since:

\[
\text{Ad}_R B(t \cdot m) = \text{Ad}_R B(\ell m) = B(-, \ell m) = B(\ell^* (-), m) = t \cdot B(-, m) = t \cdot \text{Ad}_R B(m),
\]
We thus obtain an exact sequence of $D[t]$-modules:

$$0 \rightarrow N \rightarrow M \xrightarrow{\text{Ad} \cdot B} \text{Hom}_D(N, K) \rightarrow 0$$

By definition of the action of $t$ on $\text{Hom}_D(N, K)$, its characteristic polynomial is simply that of $t^\star|_N$. We can see this by identifying $N$ with its dual space, $\text{Hom}_D(N, K)$. The action of $t$ on $N$ induced from the corresponding action of $t$ on $\text{Hom}_D(N, K)$ above is then $t^\star|_N$.

Note: In this section we are working over $F$ a field, so that $D = K = F$. We have used the notation $D, K$ to follow our previous conventions.

The question arises; what is the characteristic polynomial of $t^\star|_N$? To begin with, by Lemma II.1.2, $t^\star|_N = kI$ on $M$, hence all the more so on $N$. We write $L_1^\perp$ as the matrix of $\ell$ restricted to $N$. Then the matrix of $t^\star|_N$ is $kL_1^\perp$. Now $f(t) = \text{characteristic polynomial of } L = \det(tI - L_1)$. We compute

$$\det(tI - kL_1^\perp) = \det(-tI_1 + kI) \det(-L_1^{-1})$$

$$= \det(tI) \det(k^{-1}I - L_1) \det(-L_1^{-1})$$

$$= t^n \det(k^{-1}I - L_1) \cdot \det(-L_1^{-1})$$

$$= t^n f(k^{-1}) \cdot \det(-L_1^{-1})$$.
where \( n = \text{degree } f(t) = \text{dimension } N \). Here \( \det(-L_1^{-1}) \)
is a constant; from which it follows that \( \det(-L_1^{-1}) = a_0^{-1} \)
where \( a_0 \) is the constant term of \( f(t) \), and that
\[
T_k f(t) = \det(tI - kL_1^{-1}) = \text{characteristic polynomial of } L^*|_N.
\]
The exact sequence given, together with the computation given above then yields, by \([L-1 402]\) that
\[
p(t) = f(t) - T_k f(t) \text{ as claimed.} \quad \square
\]

We continue by forming \( GK(F) \), the Grothendieck

group associated to \( K(F) \). This is the free abelian
group generated multiplicatively by the irreducible

polynomials in \( K(F) \). \( T_k \) induces an automorphism of

period 2 on \( GK(F) \), so we can form
\[
H^2(C_2; GK(F)) \text{, denoted simply } H^2(k; K(F)) \text{.} \quad [M 122]
\]
This is identified as \( \{ f \in K(F) : T_k f = f \} \) modulo
\( \{ g \in K(F) : g = h \cdot T_k h \} \). This in turn is an \( F_2 \)-vector

space with a basis element for each \( T_k \) fixed irreducible

polynomial. We denote this basis by \( \mathcal{B} \).

**Lemma 1.2** The map \( \chi : W(k,F) \to H^2(k,K(F)) \) given

by: \( [M,B,t] \to \text{characteristic polynomial of } \mathcal{L} \) is an

epimorphism.
Proof: \( \gamma \) is well-defined by 1.1. To see that \( \gamma \) is onto it suffices to show that every \( p(t) \in \mathcal{P} \) is in the image of \( \gamma \).

Given \( p(t) \), consider \( V = F[t, t^{-1}]/(p(t)) = F(\theta) \). Let \( B \) be given by: \( B(x, y) = \text{trace}_{F(\theta)/F^\theta} x\bar{y} \), where \(-\) denotes the involution induced on \( F(\theta) \) by \( \theta \to k\theta^{-1} \).

\( B \) is symmetric since \( \text{trace} \, x\bar{y} = \text{trace} \, \bar{x}y \). \( B \) is clearly non-singular; one may apply the trace lemma 2.1 to be proved or prove it directly.

Define \( \ell : V \to V \) by \( x \to \theta x \). We compute:

\[
B(\ell x, \ell y) = B(\theta x, \theta y) = \text{trace}(\theta \theta x\bar{y}) = \text{trace}(k y\bar{x}) \\
= k B(x, y).
\]

Hence \([V, B, \ell] \in W(k, F)\). Since \( \theta \) satisfies \( p(t) \), the minimal polynomial of \( \ell \) is \( p(t) \). However, \( p(t) \) divides the characteristic polynomial of \( \ell \), and degree \( p(t) = \text{degree of characteristic polynomial} \). Hence \( p(t) = \text{characteristic polynomial of} \, \ell \). □

Given a Witt equivalence class \([M, B, \ell]\), \( M \) is a \( D[t] \)-module by identifying \( t \) with \( \ell \). We now wish to decompose \([M, B, \ell]\) according to the characteristic polynomial of \( \ell \). We begin with:
Lemma 1.3 Let \( p(t) \) be \( T_k \) fixed. Then we can factor
\[
p(t) = p_1 \cdots p_w q_1^{s_1} q_2^{s_2} \cdots q_k^{s_k},
\]
into irreducible factors, where \( q_i \) denotes \( T_k q_i \). In this decomposition, each \( p_i(t) \) is -invariant, i.e. \( T_k \) fixed, and the \( q_i \) are not -invariant. □

Lemma 1.4 Suppose \( (M,B,\ell) \) has characteristic polynomial \( p(t) = p_1(t) \cdot p_2(t) \), with \( p_1(t), p_2(t) \) relatively prime polynomials, which are both -invariant. Then \( (M,B,\ell) \cong (M_1,B_1,\ell_1) \oplus (M_2,B_2,\ell_2) \) where \( \ell_i \) has characteristic polynomial \( p_i(t) \).

Proof: Let \( M_i = \{ v \in M : p_i(\ell)(v) = 0 \} \), \( B_i = B \mid_{M_i} \), \( \ell_i = \ell \mid_{M_i} \).

Since \( (p_1,p_2) = 1 \), we can find polynomials \( r(t), s(t) \) with \( p_1(t)r(t) + p_2(t)s(t) = 1 \).

Remark 1.5: This statement is false over \( Z \), and is the reason the decomposition fails over \( Z \).

Hence, if \( v \in M \), then \( p_1(\ell)r(\ell)v + p_2(\ell)s(\ell)v = v \).

However \( p_1(\ell)r(\ell)v \in M_2 \) since \( p_1(\ell)p_2(\ell)v = 0 \). It follows that \( M = M_1 + M_2 \).
If \( v \in M_1 \cap M_2 \), by the above it is clear that \( v = 0 \). It follows that \( M \) is a direct sum of \( M_1, M_2 \), ie. \( M = M_1 \oplus M_2 \).

We next show that \( B = B_1 \oplus B_2 \), so that \( B_1 \) and \( B_2 \) are inner products.

To begin with consider \( p_2(\ell) : M_1 \to M_1 \cdot P_2(\ell) \) is clearly 1-1, and hence an isomorphism as we are working over a field. Thus, if \( v \in M_1 \), we may write
\[
v = p_2(\ell)v_1.\]
Let \( w \in M_2 \).

\[
B(v,w) = B(p_2(\ell)v_1,w) = B(v_1,p_2(\ell)^*w)
\]
\[
= \frac{1}{k^n} B(\ell^n v_1, a \cdot p_2(\ell)^*w)
\]
\[
= \frac{1}{k^n} B(\ell^n v_1, a_{0} p_2(\ell)w) = 0
\]
where \( a_0 = \) constant term of \( p_2(t) \), since \( p_2(t) \) is \( - \) invariant.

Thus \( B = B_1 \oplus B_2 \) as claimed.

Finally, we must show \( \ell_i : M_i \to M_i \). Note that
\[
p_i(\ell)(\ell v) = \ell(p_i(\ell)(v)) = \ell(0) = 0.\]
Hence \( \ell_i = \ell |_{M_i} \) maps \( M_i \to M_i \). This shows
\[(M, B, \ell) \approx (M_1, B_1, \ell_1) \oplus (M_2, B_2, \ell_2)\]
as claimed. □

**Lemma 1.6.** Suppose \((M, B, \ell)\) has characteristic polynomial \(p(t) = \text{characteristic polynomial of } \ell\), and \(p = \frac{r_1}{q_1} \cdot \frac{r_1}{\bar{q}_1}\), where \(q_1\) is irreducible. Then \((M, B, \ell) \sim 0\).

**Proof:** We are assuming \(q_1 \neq \bar{q}_1\). Since \((q_1, \bar{q}_1) = 1\), \(M\) will split as \(M = M_1 \oplus M_2\), as in 1.4. We must now examine \(B\).

Let \(v \in M_1\). As in 1.4, we may write \(v = q_1^r(\ell)(v_1)\). Let \(w \in M_1\).

\[
B(v, w) = B(q_1^r(\ell)v_1, q_1^r(\ell)w_1) = B(v_1, q_1^r(\ell)\bar{q}_1^r(\ell)w_1) = 0
\]
as before. Thus, \(B\) has matrix \(\begin{pmatrix} 0 & \cdot \\ \cdot & 0 \end{pmatrix}\). Since \(B\) is non-
singular, \(\dim M_1 = \dim M_2 = 1/2 \dim M\). As in 1.4, \(M_1\) is \(\ell\) invariant. It follows that \(M_1\) is a metabo-
lizer for \((M, B, \ell)\). □

We are almost ready to state the Decomposition Theorem.
First, some notation.

**Definition 1.7** Let $T$ be a multiplicative subset of $D[t]$. Then $W(k,K;T)$, respectively $A(k,K;T)$, denotes Witt equivalence classes in $W(k,K)$, $A(k,K)$, which are annihilated by $T$. In particular, when $T$ consists of non-negative powers of a fixed irreducible polynomial $f$, we shall use the notation $W(k,K;f)$.

**Theorem 1.8** (The Decomposition Theorem) For $F$ a field $W(k,F) \cong \bigoplus_{f \in \mathcal{F}} W(k,F;f)$ where $\mathcal{F}$ denotes the basis of $H^2(k;K(F))$ consisting of fixed irreducible polynomials.

**Proof:** Let $[M,B,\ell] \in W(k,F)$. Let $p(t)$ be the characteristic polynomial of $\ell$. By 1.3, we can factor $p$ as $p(t) = p_1^{r_1} \cdots p_w^{s_1} (q_1^{s_1} q_1^{-s_1}) \cdots (q_k^{s_k} q_k^{-s_k})$. By induction and 1.4, $[M, B, \ell] = \bigoplus_{f_1} [M_1, B_1, \ell_1]$, where each $\ell_1$ has characteristic polynomial $p_1^{r_1}$ or $(q_1^{s_1} q_1^{-s_1})$. By 1.6, those $[M_1, B_1, \ell_1]$ with characteristic polynomial $(q_1^{s_1} q_1^{-s_1})$ are Witt $\sim 0$. This defines a homomorphism $L : W(k,F) \to \bigoplus_{f \in \mathcal{F}} W(k,F;f)$.

We must show $L$ is well-defined. So suppose
[M \bigoplus B \oplus \ell] is metabolic. Then \((\bigoplus [M, B, \ell_i]) \sim 0\) where
\(\ell_i\) has characteristic polynomial \(p_i\). We need to show that if \([M_1, B_1, \ell_1] \bigoplus [M_2, B_2, \ell_2] \sim 0\), where
\(p_1(t)\) is the characteristic polynomial of \(\ell_1\), and
\(p_1(t)\) is relatively prime to \(p_2(t)\), then \(M_1 \sim 0\)
and \(M_2 \sim 0\).

We identify \(M_1\) with \(M_1 \oplus 0 \subset M_1 \oplus M_2\). Let \(H\) be a metabolizer for \(M_1 \oplus M_2\). Then \(H\) is 
\(\ell = \ell_1 \oplus \ell_2\)
invariant. Further, since \(p_2(\ell) = p_2(\ell_1) \oplus p_2(\ell_2)\), \(p_2(\ell) M \subset M_1\).

In fact, since \(p_1(t)\) and \(p_2(t)\) are relatively prime, it follows that \(p_2(\ell)\) is a 1-1 mapping:
\(H \cap M_1 \sim H \cap M_1\). Since we are working over a field, 
\(p_2(\ell)(H \cap M_1) = H \cap M_1\). We claim \((H \cap M_1) \subset (H \cap M_1)^\perp\) in 
\(M_1\) so that \(M_1 \sim 0\) and \(L\) is well-defined.

Clearly \(H \cap M_1\) is \(\ell_1\) invariant, and
\(H \cap M_1 \subset (H \cap M_1)^\perp\).

Let \(x \in (H \cap M_1)^\perp\). We must show \(x \in H \cap M_1\).

To begin with, note that if \((h_1, h_2) \in H\), then so is
\((p_2(\ell_1)h_1, 0)\) since \(H\) is \(\ell_1 \oplus \ell_2 = \ell\) invariant.
Further, since \(p_2(\ell_1)\) is an isomorphism on \(H \cap M_1\), it follows that \((h_1, 0) \in H\).

Now \((x, 0) = x \in (H \cap M_1)^\perp\). If \(h = (h_1, h_2) \in H\), it follows that \((h_1, 0) \in H \cap M_1\). Thus

\((B_1 \oplus B_2)((x, 0), (h_1, h_2)) = B((x, 0), (h_1, 0)) = 0\)
Hence \( x \in H^1 = H \), and \((H \cap M_1)^\perp \subset H \cap M_1\).

\( L \) is clearly onto by 1.2.

\( L \) is 1-1, since if \( \mathbb{Q}[M_1, B, \ell_1] \) has each \( M_1 \sim 0 \) then so too is \( \mathbb{Q}[M_1] \sim 0 \).

Let us give another interpretation of this isomorphism \( L \). Let \( f(t) \) be a \( T_k \) fixed irreducible polynomial, so \( f \in \mathcal{B} \). Let \( S = D[t] - (f(t)) \), and \([M, B, \ell] \in W(k, K)\). Then localizing with respect to \( S \), we obtain, 
\[(M(S), B_S, \ell_S)\]. Note that the adjoint map,

\[ \text{Ad}_{R_D} : M(S) \to (\text{Hom}_D(M, K))(S) = \text{Hom}_D(M(S), K(S)) \]

is an isomorphism. \( \text{Ad}_{R_D} \) is an isomorphism since localization is an exact functor, [A,Mc 39]. The second isomorphism follows from [B-2 II 2.7].

\( M \) is a torsion \( D[t] \)-module. Thus \( M(S) \) is annihilated by \( f^i(t) \), some \( i \). Hence \( (M(S), B_S, \ell_S) \in W(k, K; f) \).

We combine over all \( f \in \mathcal{B} \), to obtain exactly the \( L \) given in Theorem 1.8. Since \( L \) can be viewed as arising from localizing, we shall call \( L \) the localization homomorphism.

In fact, as long as we localize at all prime ideals in \( D \), or \( D[t] \), where \( M \) is a finitely generated torsion \( D \), or \( D[t] \)-module, we obtain such an \( L \).
Theorem 1.9 Let \( K = F/D \). By localizing at all prime ideals \( \mathfrak{q} \) in \( D \), we obtain an isomorphism:

\[
L : W(k, K) \rightarrow \bigoplus_{\mathfrak{q} \text{prime in } D} W(k, K(\mathfrak{q}); D(\mathfrak{q}))
\]

Here

\[
K(\mathfrak{q}) = (F/D)(\mathfrak{q}) = F/D(\mathfrak{q})
\]

Proof: Exactly as in 1.8. \qed

We should like to describe these pieces \( W(k, F; f) \). In order to do this, we need some further notation.

Definition 1.10 Let \( S \) be a \( D \)-algebra, finitely generated as a \( D \)-module. Then \( W(k, K; S) \) denotes Witt equivalence classes, \([M, B, \ell]\) in \( W(k, K) \) with a compatible \( S \)-module structure, meaning there exists \( r \in S \) with \( rm = \ell m \) for all \( m \in M \).

We shall be specifically interested in the case \( S = F[t, t^{-1}]/(f(t)) \), where \( f(t) \in \mathfrak{B} \). For this \( S \), observe that there is an inclusion

\[
j : W(k, F; S) \rightarrow W(k, F; f)
\]

Structures on the left are annihilated by \( f(t) \), those on the right are annihilated by some power of \( f \).
Proposition 1.11 j is an isomorphism.

Proof: j is clearly 1-1. Let (M, B, ℓ) be an anisotropic representative of a Witt equivalence class in \( W(k, F; f) \). Thus if \( N \subset M, N \neq 0 \) is \( ℓ \) invariant, then \( N \cap N^\perp = 0 \). It follows that \( M = N \oplus N^\perp \), and that \( (M, B, ℓ) = (N, B|, ℓ|) \oplus (N^\perp, B|, ℓ|) \), where \( B|, ℓ| \) denote the restrictions of \( B, ℓ \) to \( N, N^\perp \). This is standard linear algebra, see [H 157]. Continuing we can write \( M = N_1 \oplus N_2 \oplus \cdots \oplus N_r \) as a direct sum of inner product space \( (N_i, B_i, ℓ_i) \), where each \( N_i \) has no non-trivial \( ℓ \) invariant submodules. Such \( N_i \) are called irreducible.

Let \( T_i = \text{annihilator of } N_i \) in \( F[t] \). We want to show \( T_i \) is a maximal ideal in \( F[t] \). Suppose not. Then \( T_i \supseteq S_i \supseteq F[t] \), for some ideal \( S_i \).

Claim: \( S_i N_i \neq N_i \). For if \( S_i N_i = N_i \), we recall Theorem 76: [K-2 50] Let \( R \) be a ring, \( I \) an ideal in \( R, A \) a finitely generated \( R \)-module satisfying \( IA = A \). Then \( (1+y)A = 0 \) for some \( y \in I \).

It follows that \( (1+y)N_i = 0 \) for some \( y \in S_i \). Hence \( 1+y \in T_i \subseteq S_i \), so \( 1+y \in S_i \). Hence \( 1 \in S_i \). This contradicts \( S_i \not\subseteq F[t] \).
Thus $S_i N_i \neq N_i$.

$S_i N_i \neq 0$, since $S_i \neq T_i$. $S_i N_i$ is $t$ invariant, ie. $\ell$ invariant as we identify the action of $t$ with $\ell$, because $S_i$ is an ideal.

However, we have thus constructed a non-trivial $\ell$ invariant submodule of $N_i$. This contradicts $N_i$ being irreducible. It follows that $T_i$ is indeed a maximal ideal in $F[t]$. Thus $T_i = (f(t))$, and $j$ is onto. □

**Remark 1.12** Proposition 1.11 has shown the two notations given in 1.7 and 1.10 to be redundant. Nonetheless we shall use both. The notation $W(k, F; f)$ is used when we wish to stress the polynomial aspect of the mapping structure. $W(k, F; F[t]/f(t))$ is used when we wish to stress the module structure.

**Proposition 1.13** Let $K = F/D$. Then the inclusion

$$W(k, K; D/\mathfrak{p}) \xrightarrow{j} W(k, K(\mathfrak{p}); D(\mathfrak{p}))$$

is an isomorphism, where $\mathfrak{p}$ is a prime ideal in $D$.

**Proof:** Same as 1.11. □
Here \( W(k, K; D/\emptyset) \) denotes equivalence classes \([M, B, \ell]\) in which \( M \) has a \( D/\emptyset \) module structure. \( W(k, K(\emptyset); D(\emptyset)) \) denotes equivalence classes in which \( M \) has a \( D(\emptyset) \) module structure.

For \( F = \mathbb{Q} \), \( D = \mathbb{Z} \), \( D/\emptyset = \mathbb{F}_p \) a finite field. Since \( M \) is a vector space over \( \mathbb{F}_p \), \( B \) must take its values in the cyclic subgroup of \( \mathbb{Q}/\mathbb{Z} \) annihilated by \( p \). By the natural choice of generator for this subgroup, namely \( 1/p \), we have \( W(k, K; D/\emptyset) \approx W(k, \mathbb{F}_p) \).

**Proposition 1.14** \( A(F) \) decomposes as

\[
\bigoplus_{f \in \mathbb{G}} A(F; f) = \bigoplus_{f \in \mathbb{G}} A(F; F[t]/f(t))
\]

**Proof:** The proof is exactly like 1.11, where now \( t \) acts as \( s \), the symmetry operator. \( \square \)
2. The trace lemma

Given \( f(t) \in \mathcal{A} \), meaning \( f \) is irreducible, \( \mathbb{T}_k \) fixed, we form the field \( \mathbb{F}[t, t^{-1}]/(f(t)) = \mathbb{F}(\theta) \).

By II.1.7 \((f(t))\) is - invariant, so there is an induced involution on \( \mathbb{F}(\theta) \). This involution is non-trivial in the type 1 situation only. We aim now to identify explicitly the group \( W(k, \mathbb{F}; \mathbb{F}[t]/(f(t)) \). We begin with:

Lemma 2.1 (The trace lemma). Let \( R \) be a commutative ring with unit, \( A \) an \( R \)-algebra, \( E \) an \( A \)-module, and \( F \) an \( R \)-module.

Then there is the following correspondence:

Let \( \langle,\rangle : M \times M \to E \) be a non-singular bilinear form over \( A \).

Let \( t : E \to F \) be an \( R \)-linear map, which induces an isomorphism \( \wedge \)

\( t : E \to \text{Hom}_R(A, F) \), by

\( e \mapsto t(- \cdot e) = t(e) \).

Then the map \( \langle,\rangle : M \times M \to F \) is non-singular.

Conversely, if \( M \) is an \( A \)-module with non-singular form \( (,): M \times M \to F \), \((,): R \)-linear, then there is a
non-singular form $<,>: M \times M \rightarrow E$ with $t \circ <,> = (,)$. $<,>$ is A-linear.

Further, this correspondence preserves annihilators of submodules and the metabolic property provided the R-module structure of M lifts compatibly to A.

Proof: Part 1: Given $<,>: M \times M \rightarrow E$, and $t: E \rightarrow F$, we wish to show $(,) = t \circ <,>: M \times M \rightarrow F$ is non-singular.

Let $Ad_R^* : M \rightarrow \text{Hom}_R(M,F)$ denote the adjoint of $(,)$. We want to show $Ad_R^*$ is an isomorphism.

$Ad_R^*$ is 1-1: Let $m \neq 0$ be in M. We want to show $(-,m) \neq 0$. Since $<,>$ is non-singular, we can find $n \in M$ with $<n,m> \neq 0$. Now $<n,m> \in E$ and we have $\hat{t} : E \rightarrow \text{Hom}_R(A,F)$. Thus, since $t$ is an isomorphism, $t(-,n,m) \neq 0$. Let $a \in A$ have $t(a \cdot <n,m>) \neq 0$. $<,>$ is bilinear over A, so

$$a<n,m> = <an,m>$$

Hence,

$$t(a<n,m>) = t(<an,m>) = (an,m) \neq 0$$

Thus $(-,m) \neq 0$ as claimed, and $Ad_R^*$ is 1-1.
\textbf{Ad}_R \text{ is onto :} \text{ Let } f \in \text{Hom}_R(M,F). \text{ For each } m \in M, \text{ define an } R\text{-linear map } A \to F \text{ by } a \to f(am).

Since \( \hat{t} \) is an isomorphism, this map equals \( t(-f_0(m)) \) for some \( f_0(m) \in E \). Now \( f_0 \) defines an \( A \)-linear map \( f_0 : M \to E \). By non-singularity of \( <,> \) it follows that \( f_0(m) = <m,n_0> \) for some \( n_0 \in M \). Combining,

\[
f(m) = t(f_0(m)) = t(<m,n_0>) = (m,n_0),
\]

so that \( \text{Ad}_R \) is onto as claimed.

\textbf{Part 2 :} \text{ Let } M \text{ be an } A\text{-module, together with a non-singular } R\text{-linear form } (,): M \times M \to F. \text{ We need to define } <,> \text{ with } t<,> = (,).

Let \( (-,n_0) \in \text{Hom}_R(M,F) \). As before, for each \( m \in M \) we can define an \( R \)-linear map \( A \to F \) by \( a \to (am,n_0) \). Again \( E \cong \text{Hom}_R(A,F) \) implies \( (am,n_0) = t(afo(m)) \) for some unique \( f_0(m) \in E \). Now define \( <m,n_0> = f_0(m) \). Then by definition

\[
(m,n_0) = t(f_0(m)) = t(<m,n_0>).
\]

\( f_0 \) and \( <,> \) are clearly \( A \)-bilinear. We now must show \( <,> \) is non-singular.
Let \( Ad_R \) denote the adjoint of \( \langle, \rangle \), \( Ad_R : M \to \text{Hom}_A(M,E) \)

\( Ad_R \) is \( 1-1 \): Let \( m \neq 0 \) be in \( M \). We want to show \( \langle -, m \rangle \neq 0 \). By non-singularity of \( (,) \), we can find \( n \in M \) with \( (n,m) \neq 0 \). Hence \( \langle n,m \rangle \neq 0 \), else \( t(<m,n>) = 0 = (m,n) \). \( Ad_R \) is onto: Let \( f \in \text{Hom}_A(M,E) \).

Then \( (t \circ f) \in \text{Hom}_R(M,F) \). By the non-singularity of \( (,) \) there exists \( n_0 \in M \) such that \( (t \circ f)(m) = (m,n_0) \), for all \( m \in M \). Hence

\[
(t \circ f)(am) = t(f(am)) = t(af(m)) = (am,n_0),
\]

so that by definition of \( \langle, \rangle \) we have \( \langle m,n_0 \rangle = f(m) \), and \( Ad_R \) is onto.

The last statement of the theorem follows from the definitions.  \( \Box \)

We may extend Lemma 2.1 in a special case.

**Lemma 2.2** Suppose that \( A,E \) given in 2.1 have a compatible involution \(-\), meaning \( (ae) = \bar{a} \bar{e} \). Suppose also that \( t : E \to F \) satisfies \( t(e) = t(\bar{e}) \) for all \( e \in E \). Then the correspondence of Lemma 2.1 extends to a correspondence between Hermitian forms \( (M,\langle, \rangle) \) with values in \( E \), and symmetric forms \( (M,(,)) \) with values in \( F \) which have a compatible \( A \)-module structure, meaning
(ax, y) = (x, ay).

**Proof:** If \( \langle , \rangle \) is Hermitian,

\[
a \langle x, y \rangle = \langle ax, y \rangle = \langle x, ay \rangle = \langle ay, x \rangle = a \langle y, x \rangle.
\]

Now

\[
(x, y) = t(\langle x, y \rangle) = t(\langle y, x \rangle) = t(\langle y, x \rangle) = (y, x),
\]
so that \( (, ) \) is symmetric.

Also

\[
(ax, y) = t(\langle ax, y \rangle) = t(\langle x, ay \rangle) = (x, ay).
\]

Conversely, let \( (, ) \) be symmetric. Then

\[
t(a \langle x, y \rangle) = t(\langle ax, y \rangle) = (ax, y) = (y, ax)
\]

\[
= (ay, x) = t(\langle ay, x \rangle)
\]

\[
= t(\langle x, ay \rangle) = t(\langle y, x \rangle)
\]

However \( E \cong \text{Hom}_R(A, F) \) via \( t \), so

\[
\langle x, y \rangle = \langle y, x \rangle , \text{ and } \langle , \rangle \text{ is Hermitian}. \quad \square
\]

We recall the identification made at the beginning of Section 2, \( F[t]/(f(t)) = F(\theta) \). We are now ready to compute \( W(k, F; F(\theta)) \), where \( f(t) \) is \( T_k \) fixed and irreducible.

**Theorem 2.3** \( W(k, F; F(\theta)) \cong H(F(\theta)) \).
Proof: We apply 2.1 and 2.2, with \( R = F \), and \( A = E = F(\theta) \), \( t : E \to F \) is the trace homomorphism.

We must check the non-singularity condition on \( t \), namely that \( t : F(\theta) \to \text{Hom}_F(F(\theta), F) \) induces an isomorphism.

We must assume \( F(\theta) \) is a finite, separable extension of \( F \). Thus \( t(-x) \neq 0 \) for \( x \neq 0 \) \([L-1.211]\).
It follows that \( t \) is 1-1. However, \( F(\theta) \) and \( \text{Hom}_F(F(\theta), F) \) are vector spaces over \( F \) of the same dimension. Hence \( t \) is an isomorphism, so that we may apply 2.1 and 2.2. □

Comment: Clearly \( t(e) = t(\bar{e}) \), so that 2.2 applies.
In our identification, \([V, <, >] \in H(F(\theta))\) corresponds to \([V, (,), \lambda] \in W(k, F; F(\theta))\). Here \( t \circ <, > = (,) \). The map \( \lambda \) is recovered from Hermitian as multiplication by \( \theta \), \( \lambda v = \theta v \).

We shall reserve the term Hermitian for the case that the involution is non-trivial. Thus 2.3 is for type 1 polynomials.

For type 2 irreducible, \( f(t) = t^2 - k \). We then read Theorem 2.3 as:

\[
W(k, F; F[t]/((t^2 - k))) \cong W(F/\kappa)
\]
We may thus restate the decomposition theorem, 1.8, as

**Theorem 2.4**  If $k \not\in F^{**}$, then

$$W(k, F) \cong W(F(\sqrt{k})) \oplus \bigoplus_{f \in \mathfrak{B}} H(F[t]/(f(t)))$$

If $k \in F^{**}$,

$$W(k, F) \cong W(F) \oplus W(F) \oplus \bigoplus_{f \in \mathfrak{B}} H(F[t]/f(t)) \quad \Box.$$  

Remarks:

(1). The Hermitian terms $\oplus$ runs over all irreducible $T_k$ fixed polynomials in $K(F)$ of type 1. The same field $F[t]/(f(t)) = F(\theta) = F(\sigma) = F[t]/(g(t))$ may be repeated.

(2). If $k \in F^{**}$, the two Witt terms correspond to the two irreducible polynomials of type 3, $t + \sqrt{k}, t - \sqrt{k}$.

(3). If the characteristic of $F$ is 2, $k \in F^{**}$, so

$$W(k, F) = W(F) \oplus H(F(\theta))$$

since

$$t + \sqrt{k} = t - \sqrt{k}$$

in this case. Again, the field $F(\theta)$ may be repeated.
(4). This theorem equally applies to the skew case; simply write

$$W^e(k, F) \cong W^e(F(\sqrt[k]{/})) \oplus H^e(F(\theta)) \quad \epsilon = \pm 1.$$  

Our final goal of this section is to relate this discussion to the asymmetric case. Let \([M, B] \in A(F)\).

Recall the symmetry operator \(\sigma: M \rightarrow M\) satisfying \(B(x, y) = B(y, sx)\). Consequently \(B(x, y) = B(sx, sy)\), so that \(\sigma\) yields a map of degree 1 on \([M, B]\).

Again, there is the involution on \(F[t, t^{-1}]\) induced by \(T_k\), with \(k = 1\). This extends to \(F(\theta) = F[t, t^{-1}]/(f(t))\), where \(f\) is an irreducible \(T_k\)-fixed polynomial. The induced involution on \(F(\theta)\) is \(\theta \rightarrow \bar{\theta} = \theta^{-1}\).

Let \(S = F[t]/(f(t))\), and consider \(A(F; S)\). This denotes structures \([M, B]\), in which \(t\) acts as \(\sigma\) the symmetry operator. Since \(A(F) \cong \oplus A(F; S)\) by 1.14, we wish now to compute \(A(F, S)\). We do this in two ways.

**Theorem 2.5** Let \(f(t)\) be an irreducible \(T_k\)-fixed polynomial of type 1. (\(k = 1\)) Then

$$A(F; f) \cong H(F(\theta)) \cong H_0 F(\theta).$$
**Proof:** The idea in this computation is to apply the trace lemma. We may do this either using trace:

\[ F(\theta) \to F, \] or a scaled trace: \[ F(\theta) \to F. \]

**Part (a):** Using trace: \( t:F(\theta) \to F. \)

Let \( [M,\langle,\rangle] \in H_\theta F(\theta), \) and \( [M,(,)] \in A(F:f). \)

We need to show that \( \langle,\rangle \) is \( \theta \) Hermitian with values in \( F(\theta) \) if and only if \( (,) \) is asymmetric with values in \( F, \) satisfying \( (x,y) = (y,\theta x), \) with \( \theta \) being identified with \( \theta. \) We are of course applying the trace lemma with the map \( t = \text{trace } F(\theta)/F. \)

Let \( (,\) satisfy \( (x,y) = (y,\theta x). \) Then

\[
t(a\langle x,y \rangle) = t\langle ax,y \rangle = (ax,y) = (y,\theta ax)\]
\[
= (\theta a\overline{y},x) = t\langle a\theta \overline{y},x \rangle = t(a\theta \langle y,x \rangle)\]
\[
= t(a\theta \langle y,x \rangle).
\]

Hence, since \( t \) is an isomorphism, \( \langle x,y \rangle = \theta \overline{\langle y,x \rangle} \), so that \( \langle,\rangle \) is \( \theta \) Hermitian.

Conversely, suppose \( \langle,\rangle \) is \( \theta \) Hermitian. Then

\[
(x,y) = t\langle x,y \rangle = t(\theta \overline{\langle y,x \rangle}) = t(\overline{\theta}\langle y,x \rangle)\]
\[
= t\langle y,\theta x \rangle = (y,\theta x)
\]
as desired.
Thus $A(F;f) \cong H_\theta(F(\theta))$, where $F[t]/(f(t)) = F(\theta)$.

**Part (b):** Using the scaled trace. Since $\theta \tilde{\theta} = 1$, by Hilbert 90, there exist $u \in F(\theta)$ with $u\bar{u}^{-1} = \theta$. Let $t_1 : F(\theta) \to F$ be given by $x \mapsto \text{trace } x\bar{u}^{-1}$. It is clear that $t_1$ is an isomorphism. Again $t$ denotes trace $F(\theta)/F$.

Now suppose $[M,\langle , \rangle] \in H(F(\theta))$. Then

\[
\begin{align*}
(x, y) &= t_1 \langle x, y \rangle = t(\bar{u}^{-1} \langle x, y \rangle ) \\
&= t \langle x, u^{-1} y \rangle = t \langle u^{-1} y, x \rangle \\
&= t \langle u^{-1} y, x \rangle = t \langle y, \theta u^{-1} x \rangle \\
&= t \langle y, \theta u^{-1} x \rangle = t_1 \langle y, \theta x \rangle \\
&= (y, \theta x) .
\end{align*}
\]

Conversely, suppose $(x, y) = (y, \theta x)$. Then:

\[
\begin{align*}
t(a\langle x, y \rangle) &= t(ax, y) = t_1 \bar{u} \langle ax, y \rangle \\
&= t_1 \langle ax, uy \rangle = (ax, uy) \\
&= (uy, a\theta x) = (u\bar{u} y, ax) \\
&= (\bar{u} y, ax) = t_1 \bar{u} a \langle y, x \rangle \\
&= t (\bar{a} \langle y, x \rangle ) \\
&= t (a \langle y, x \rangle ).
\end{align*}
\]

Again, $t$ is non-singular, so $\langle x, y \rangle = \langle y, x \rangle$, and $\langle , \rangle$ is Hermitian. □
Note that we can choose \( u = \theta/(1 + \theta) \) so 
\[ \bar{u} = 1/(1 + \theta), \text{ and } \quad \bar{u}^{-1} = 1 + \theta. \] We give both identifications in this theorem since on certain occasions it is more convenient to think of Hermitian forms as giving \( A(F) \). The disadvantage is that we must use a scaled trace to make this identification.

We should also give the third identification. Namely, it follows from this theorem that

\[ H(F(\theta)) \cong H_\theta(F(\theta)) \text{ via } h : \langle , \rangle \to \langle , \rangle_1 \]

with \( h \) defined by

\[ \langle x, y \rangle_1 = \langle x, u^{-1}y \rangle \text{ where } uu^{-1} = \theta. \]

(a) If \( \langle , \rangle_1 \) is \( \theta \) Hermitian,

\[
\langle x, y \rangle = \langle x, uy \rangle_1 = \theta \frac{\langle uy, x \rangle_1}{\langle uy, x \rangle_1} = \langle \theta uy, x \rangle_1 = \langle u \theta^{-1} y, x \rangle_1 = \langle uy, x \rangle_1 = \langle y, ux \rangle_1 = \langle y, x \rangle,
\]

and \( \langle , \rangle \) is Hermitian.

(b) If \( \langle , \rangle \) is Hermitian,

\[
\langle x, y \rangle_1 = \langle x, u^{-1}y \rangle = \frac{\langle u^{-1}y, x \rangle}{\langle u^{-1}y, x \rangle} = \langle y, u^{-1}x \rangle = \theta \frac{\langle y, u^{-1}x \rangle}{\langle y, u^{-1}x \rangle} = \theta \frac{\langle y, u^{-1}x \rangle}{\langle y, u^{-1}x \rangle} = \theta \langle y, x \rangle_1
\]
so that $\langle , \rangle_1$ is $\theta$ Hermitian.

**Corollary 2.6** There is a commutative triangle of

\[
\begin{array}{ccc}
H(F(\theta)) & \xrightarrow{h} & H_{\theta}F(\theta) \\
\downarrow t_1 & & \downarrow t \\
A(F; F[t]/(f(t)) & &
\end{array}
\]

isomorphisms. $t$ is trace $F(\theta)/F$. $t_1$ is trace scaled by $u^{-1}$, where $u/\bar{u} = \theta$. $h : \langle , \rangle \rightarrow \langle , \rangle_1$, is defined by

$\langle x, y \rangle_1 = \langle x, u^{-1} y \rangle$. 

Thus, the decomposition theorem reads,

**Theorem 2.7**

\[
A(F) \cong \bigoplus_{f \in B} H_{\theta}(F(\theta)) \cong \bigoplus_{f \in B} H(F(\theta)).
\]

For $F = \mathbb{Q}$ the rationals, or $F = \mathbb{F}_p$ a finite field, Hermitian of $F(\theta)$ is well-known [Lh] and [M,H]. We shall use this computation in Chapter VI.
3. Computing Witt groups

We are interested in the group $W(k, \mathbb{Z})$. Let $[M, B, t] \in W(k, \mathbb{Z})$. Then we may view $M$ as a $\mathbb{Z}[t, t^{-1}]/(f(t))$ module, where $f(t)$ is the characteristic polynomial of $\ell$, and $t$ acts as $\ell$. As has been pointed out, for $S = \mathbb{Z}[t, t^{-1}]/(f(t))$, the decomposition theorem fails; $W(k, \mathbb{Z}) \neq W(k, \mathbb{Z}; S)$.

Later, we shall measure this failure, [VIII 6].

Our next task is to describe these pieces $W(k, \mathbb{Z}; S)$, for $S = \mathbb{Z}[\theta]$ above. Thus let $f$ be a monic, integral $T_k$ fixed irreducible polynomial, $S = \mathbb{Z}[\theta] = \mathbb{Z}[t, t^{-1}]/(f(t))$. We begin by describing the maximal ideals in $S$.

**Proposition 3.1** The maximal ideals of $S$ are of the form $\mathfrak{m} = (p, g(\theta))$, where $g$ is a monic integral polynomial whose mod $p$ reduction $\gamma$ is irreducible and $\gamma$ divides the mod $p$ reduction of $f$, denoted $\varphi$.

**Proof:** $S/\mathfrak{m}$ is clearly a finite field, indeed it embeds into $D/\varnothing$ where $D$ is the maximal order, and $\varnothing \cap S = \mathfrak{m}$.

Suppose $S/\mathfrak{m}$ lies over the prime field $F_p$. It follows that $p \in \mathfrak{m}$. Further, $S/\mathfrak{m}$ is generated by $\theta_1$, the image of $\theta$ in residue field.
Let $\gamma(t)$ be the monic irreducible polynomial over $F_p$ of $\theta_1$. Let $g(t)$ be a monic integral polynomial whose mod $p$ reduction is $\gamma(t)$. Then clearly $g(\theta) \in \mathfrak{m}$ and $(p, g(\theta)) = \mathfrak{m}$.

Since $g$ is irreducible mod $p$, $g$ is irreducible. Further $f(\theta) = 0$, so $\varphi(\theta_1) = 0$, and $\gamma$ divides $\varphi$ as claimed. □

Remark: $\mathfrak{m}$ is invariant under the involution - induced by $\bar{\theta} = k\theta^{-1}$ if and only if $S/\mathfrak{m}$ has a well-defined involution induced by $-$. Further, $S/\mathfrak{m}$ has involution - if and only if $\gamma(t)$, the irreducible polynomial of $\theta_1$ is $T_k$ fixed. For if $\gamma$ is $T_k$ fixed, we have already seen there is an involution induced on $F_p(\theta_1) \cong S/\mathfrak{m}$. Conversely, when there is the $-$ involution on $F_p(\theta_1)$, $\gamma(\theta_1) = \gamma(\bar{\theta}_1) = 0$. Equating coefficients (VIII 7) it follows easily that $\gamma$ is $T_k$ fixed.

We wish to apply the trace lemma to compute $W(k, Z; S)$. Thus we consider the inverse different of $S$,

$$\Delta^{-1}(S/Z) = I = \{x \in E : \text{trace}_{E/Q}(xS) \subseteq Z\}$$

Here $E$ is the quotient field of $S$.

Again, there is the $-$ involution on $E$, and $S$ is a $-$ invariant order. It follows that the inverse different $I$ is a $-$ invariant fractional ideal over $S$.

We may describe $I$ by Euler's theorem [A 92], namely $I = \mathbb{Z}(\theta)/f'(\theta)$, where $f'$ is the derivative of $f$. 
We wish to apply the trace lemma with: \( A = S \), \( E = \Delta^{-1}(S/Z) \), \( F = Z \), \( t : E \to F \), \( t = \text{trace } E/F \), \( R = Z \).

In order to do this, we must verify that
\[ t : \Delta^{-1}(S/Z) \xrightarrow{\sim} \text{Hom}_Z(S,Z) \]
is an isomorphism. To begin with, the map \( x \mapsto t(-x) \) is 1-1 since it is 1-1 on the quotient fields. Continuing, let \( h \in \text{Hom}_Z(S,Z) \).

Then \( h = t(-x_0) \), for \( x_0 \in E \), since \( t \) is an isomorphism on the field level. However \( h \big|_S \) is \( Z \)-valued, so that \( \text{trace}(x_0S) \in \mathbb{Z} \), and \( x_0 \in I \). Hence \( t \) is onto.

Thus, there is an isomorphism between \( I \)-valued Hermitian forms, and \( Z \)-valued symmetric forms with a compatible \( S \)-module structure, meaning
\[ (rx,y) = (x,ry) \]
for all \( r \in S \).

We state this as:

**Theorem 3.2** The trace lemma yields an isomorphism

\[ t_* : H(A^{-1}(S/Z)) \xrightarrow{\sim} W(k,Z;S) \]

The same result naturally holds for asymmetric, and we have:

**Theorem 3.3** The trace lemma yields an isomorphism

\[ H_\Theta(A^{-1}(S/Z)) \xrightarrow{\sim} A(Z;S) \]

Caution: Using a scaled trace may be impossible since if \( uu^{-1} = \theta \), \( u \in E \), and \( u \) may not even be in \( S \).
We next apply the trace lemma to the trace map
\[ t_\star : E/\Delta^{-1}(S/Z) \to Q/Z \], in other words \( t_\star \) is induced from trace \( E/Q \). There is the induced isomorphism:
\[ t_\star : E/\Delta^{-1}(S/Z) \to \text{Hom}_Z(E/\Delta^{-1}(S/Z), Q/Z). \]

Again, the trace lemma yields, (see Remark 3.9):

**Theorem 3.4** \[ H(E/\Delta^{-1}(S/Z)) \cong W(k, Q/Z; S) \]
\[ H_\partial(E/\Delta^{-1}(S/Z)) \cong A(Q/Z; S). \] □

While we are discussing \( W(k, Q/Z; S) \), we continue with the analog of the Decomposition Theorem 1.9.

**Theorem 3.5** \[ W(k, Q/Z; S) = \bigoplus_{\mathfrak{m} \in \mathcal{M}} W(k, Q/Z; S/\mathfrak{m}) \]
where \( \bigoplus \) runs over all \( S \)-invariant maximal ideals in \( S \).

**Proof:** The proof is exactly as before. Let \([M, B, t] \in W(k, Q/Z; S)\). We write this as \( \bigoplus [M_i, B_i, t_i] \), of irreducible modules. Thus \( A = S \)-annihilator of \( M_i = \{ r \in S : \text{ann} m = 0 \text{ for all } m \in M_i \} \), is a maximal ideal in \( S \). Now we check that \( A_i \) is \( S \)-invariant.

Observe \( 0 = B_i(ax, y) = B_i(x, ay) \) for all \( a \in A \), \( x, y \in M_i \).
Hence \( \bar{a} \in A_i \), for if \( \bar{a} \not\in A \); \( \bar{a}y \neq 0 \) for some \( y \in M_i \), and \( 0 = B(x, \bar{a}y) \) for all \( x \) contradicts the non-singularity of \( B_i \). The rest is as before to give the first isomorphism.

Now \( S/\mathfrak{m} \) is a finite field, with induced involution since \( \mathfrak{m} = \mathfrak{m} \). Of course, this may be the trivial involution. Any finitely generated \( S/\mathfrak{m} \)-module is a finite dimensional vector space whose underlying abelian group is \( p \) torsion, where \( p = \text{char } S/\mathfrak{m} \). The second isomorphism then follows by selecting a generator, say \( 1/p \) for the \( p \) torsion in \( \mathbb{Q}/\mathbb{Z} \). The last isomorphism follows by 2.1.

**Remark 3.6** A similar theorem holds for \( A(k, \mathbb{Q}/\mathbb{Z}; S) \).

**Remark 3.7** If the involution on \( S/\mathfrak{m} \) is trivial, the last term is actually \( W(S/\mathfrak{m}) \), Witt of a finite field, which is determined by the cardinality of \( S/\mathfrak{m} \). Let \( q = \) cardinality of \( S/\mathfrak{m} \).

(a) If \( q \equiv 1 \pmod{4} \) \( W(S/\mathfrak{m}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \)

(b) If \( q \equiv 0 \pmod{2} \) \( W(S/\mathfrak{m}) \cong \mathbb{Z}/2\mathbb{Z} \)

(c) If \( q \equiv 3 \pmod{4} \) \( W(S/\mathfrak{m}) \cong \mathbb{Z}/4\mathbb{Z} \)

If the involution on \( S/\mathfrak{m} \) is non-trivial, we have Hermitian of a finite field. Here rank is the only
invariant. \[M-H\ 117\]

**Remark 3.8** We have thus shown that \( H(E/I) \approx \bigoplus_{\mathfrak{m}=\mathfrak{m}} H(S/\mathfrak{m}) \).

In fact, this holds directly, when \( I = \mathfrak{I} \) is a \( \lambda \)-invariant fractional ideal, with \( S = D \) the underlying ring of integers in \( E \). For if \([M,B] \in H(E/I)\), with \( M \) a finitely generated torsion \( D \)-module, we take anisotropic representatives, decompose into irreducibles, etc. It follows that \([M,B] = \bigoplus_{i} [M_i,B_i]\), where the annihilator of \( M_i \), say \( \mathfrak{Q} \), is a \( \lambda \)-invariant maximal ideal in \( D \).

Thus \( B_i \) takes values in \( E/I(\mathfrak{Q}) \). We may identify this with a \( D/\mathfrak{Q} \)-valued form, \([VII\ 4]\). For 

\( I = \Delta^{-1}(D/\mathfrak{Z}) \), this was done in 3.5. In general, we embed \( D/\mathfrak{Q} \) into \( E/I(\mathfrak{Q}) \) by \( r + \mathfrak{Q} \sim_\rho r + I(\mathfrak{Q}) \), where \( \rho \) has valuation \( v(\mathfrak{Q}) - 1 \), \([VII\ 4]\), and obtain an isomorphism between Hermitian forms \([M,B]\) with values in \( D/\mathfrak{Q} \), and Hermitian forms \([M,B]\), where \( M \) is a \( D/\mathfrak{Q} \)-module, with values in \( E/I(\mathfrak{Q}) \).

**Remark 3.9** The Hermitian groups \( H(E/\Delta^{-1}(S/\mathfrak{Z})) \) are defined because \( E/\Delta^{-1}(S/\mathfrak{Z}) \) is an injective \( S \)-module, for \( S \) an order. One verifies this using the trace induced map from \( E/\Delta^{-1} \) to \( \mathbb{Q}/\mathbb{Z} \).
We wish now to study the squaring map
\[ S_\epsilon : W^\epsilon(k,K) \to W^\epsilon(k^2,K), \]
where \( K \) is a field, or Dedekind domain. \( S_\epsilon \) is defined by:
\[ [M,B,\ell] \to [M,B,\ell^2]. \]

Here \( \epsilon = +1 \) if \( B \) is symmetric; \( \epsilon = -1 \) if \( B \) is skew-symmetric. To begin with we study \( S_\epsilon \) for \( K = \mathbb{F} \), a field. We shall relate this to the case of \( K = \mathbb{Z} \) the integers in the ensuing chapters.

We shall derive an exact sequence involving the groups \( A(\mathbb{F}) \), and \( W^\epsilon(-k,\mathbb{F}) \). The octagon we obtain is:
\[
\begin{array}{ccccccccc}
W^1(k,F) & \to & W^1(k^2,F) & \to & W^1(-k,F) \\
\downarrow & & \downarrow & & \downarrow \\
A(\mathbb{F}) & & A(\mathbb{F}) & & A(\mathbb{F}) \\
\uparrow & & \uparrow & & \uparrow \\
W^{-1}(-k,F) & \to & W^{-1}(k^2,F) & \to & W^{-1}(k,F) \\
m_\epsilon : A(\mathbb{F}) \to W^\epsilon(k,F) \text{ is defined by:}
\end{array}
\]

\[ [M,B] \to [M \otimes M, B_\epsilon, \varphi_\epsilon] \text{ where } B_\epsilon((x,y),(z,w)) = B(x,w) + \epsilon B(z,y) \text{ and } \varphi_\epsilon(x,y) = (\epsilon k^{-1} y, x). \]
\( I_\varepsilon : \mathcal{W}^\varepsilon(k^2,F) \rightarrow \mathcal{W}^\varepsilon(-k,F) \) is defined by:
\[
[M, B, \ell] \rightarrow [M \oplus M, B \oplus -kB, \ell] \text{ where } \ell(x,y) = (\ell y, x).
\]

\( d_\varepsilon : \mathcal{W}^\varepsilon(-k,F) \rightarrow A(F) \) is defined by:
\[
[M, B, \ell] \rightarrow [M, B] \text{ where } B(x,y) = k^{-1}B(x, \ell y).
\]
This octagon is only defined for \( k \neq 0 \) in \( F \).

When \( k = 0 \), the maps do not all make sense.

**Remark 0.1** If \( k = 0 \), \( \mathcal{W}(k,F) = \mathcal{W}(F) \).

To see this, let \( [M, B, \ell] \in \mathcal{W}(k,F) \) and suppose \( (M, B, \ell) \) is anisotropic. We claim \( \ell = 0 \).

Consider the \( \ell \) invariant subspace of \( M \) generated by: \( \{ \ell M, \ell^2 M, \ldots, \ell^t M \} \), where \( t = \) degree of char polynomial of \( \ell \). This subspace is self annihilating since
\[ B(\ell x, \ell y) = kB(x,y) = 0 . \]
It follows that it must be 0 since \( (M, B, \ell) \) is anisotropic. Hence \( \ell M = 0 \) as claimed.

**Remark 0.2** For \( k = \pm 1 \), the maps given are well-defined over \( Z \). The key observation is that the image spaces actually are inner products.

For the map \( m_\varepsilon \), let \( [W, B] \in A(Z) \), with symmetry operator \( s \). Then \( B(x,y) = B(sx, sy) \), from which it follows that \( s^* = s^{-1} \), so that \( s^{-1} \) exists and \( s \) is non-singular. Thus \( m_\varepsilon \) makes sense over \( Z \), and \( \psi_\varepsilon \) is well-defined.
$s_\varepsilon$ and $I_\varepsilon$ are clearly well-defined.

Let $[M,B,\ell] \in W^\varepsilon(-k,Z)$. As above, $\ell$ is non-singular and $\overline{B}(x,y) = k^{-1}B(x,\ell y)$; so $\overline{B}$ is non-singular when $k=\pm 1$.

In Section 2, we shall prove exactness of this octagon over $F$ a field. The proof given does not work over $\mathbb{Z}$. We shall develop the machinery to study this problem in the following chapters.

In Section 1 we motivate the methods used in Section 2 by deriving the Scharlau transfer sequence. In fact, the exact octagon is a generalization of these maps. The Scharlau transfer sequence is an exact octagon with several terms vanishing. We shall prove:

The sequence below is exact:

$$
\begin{array}{cccccccc}
W^+1(F(a)) & \to & S & \to & W^+1(F) & \to & W^+1(-a,F,f) & \\
\downarrow & & & & & & & \\
W^+1(F) & & & & & & & \\
\downarrow & & & & & & & \\
W^+1(-a,F,f) & \to & 0 & \to & 0
\end{array}
$$

where $f(t) = t^2 - a$.

The term $W^\varepsilon(-a,F,f)$ with $f(t) = t^2 - a$ denotes Witt equivalence classes of triples $(M,B,\ell)$ with:

(a) $B \in$ symmetric $B : M \times M \to F$

(b) $B(\ell x,\ell y) = -aB(x,y)$
(c) \( \ell \) satisfies \( \ell^2 = a \). It follows that \( \ell^* = -\ell \).

This is consistent with our previous notation. By the trace lemma, \( W^\epsilon(-a, F; f) = H^\epsilon(F(\sqrt{a})) \).

The maps in this sequence are given by:

\[
d : W^{-1}(-a, F; f) \to W(F) \quad [M, B, \ell] \to [M, \overline{B}]
\]

where \( \overline{B}(x, y) = B(x, \ell^{-1}y) \)

\[
m : W(F) \to W(F(\sqrt{a})) \quad [M, B] \to [M, B] \otimes \mathbb{F}(\sqrt{a})
\]

\[
S : W(F(\sqrt{a})) \to W(F) \text{ is Scharlau's transfer}
\]

\[
[I : W(F) \to W^{+1}(-a, F; f) \quad [M, B] \to [M, B, \overline{B}, \ell]
\]

\[
\overline{B}(x, y) = B(x, u) - aB(y, v) = B \otimes -aB
\]

\[
\ell(x, y) = (ay, x)
\]

We shall begin with the case above, and discuss the Scharlau transfer sequence.
1. **Scharlau's transfer**

Let $F$ be a field, $a \notin F^{**}$. Then we can form $F(\sqrt{a})$.

Consider $W(F)$. This is Witt equivalence classes of pairs $[M, B]$, where $B : M \times M \rightarrow F$ is a symmetric inner product.

There is the map $m : W(F) \rightarrow W(F(\sqrt{a}))$ given by $[M, B] \mapsto [M, B] \otimes_F F(\sqrt{a})$. Likewise, there is a map $S : W(F(\sqrt{a})) \rightarrow W(F)$ given by $[M, B] \mapsto [M, \overline{B}]$ where

$\overline{B}(x, y) = t \circ B(x, y)$ with $t$ defined by $t(1) = 0$, $t(\sqrt{a}) = 1$. This map $S$ is called Scharlau's transfer. It arises from a scaled trace, namely

$$t(x) = \text{trace}_{F(\sqrt{a})/F} \left( \frac{\sqrt{a}}{2} \cdot x \right).$$

In [Lm 201] an exact sequence involving $m$ and $S$ is discussed. $W(F) \xrightarrow{m} W(F(\sqrt{a})) \xrightarrow{S} W(F)$.

We examine this sequence in a setting which will generalize to our situation. In so doing, we compute the cokernel of $S$.

To begin with, let us identify $M \otimes_F F(\sqrt{a})$ with $M \otimes M$ as a vector space over $F(\sqrt{a})$. Thus, if $\{v_i\}$ is a basis for $M$ over $F$, $\{(v_i, 0)\}_i$ will be a basis
for $M \otimes M$ over $F(\sqrt{a})$. We must be careful about scalar multiplication.

We are viewing $\sqrt{a}$ as $(0,1)$, so that

$$(1,0) \cdot (v_i,0) = (v_i,0)$$
$$(0,1) \cdot (v_i,0) = (0,v_i)$$
$$(1,0) \cdot (0,v_i) = (0,v_i)$$
$$(0,1) \cdot (0,v_i) = (av_i,0)$$

In other words, scalars from $F(\sqrt{a})$ are ordered pairs $(c,d)$, $c,d \in F$ to be identified with $c + d\sqrt{a}$.

They operate on $M \otimes M$ as described above.

Under this identification, we can view the map $m$ as defined by:

$$m : [M,B] \rightarrow [M \otimes M,B']$$

where

$$B'((x_1,y_1),(x_2,y_2)) = B(x_1,x_2) + aB(y_1,y_2)$$

$$+ [B(x_1,y_2) + B(x_2,y_1)]\sqrt{a}.$$ 

With these preliminaries, we proceed to define the maps involved in the transfer exact sequence.

Let $f(t) = t^2 - a$, and form $W^{-1}(-a,F;f)$.

The maps are defined by:

$$d : W^{-1}(-a,F;f) \rightarrow W(F) [M,B] \rightarrow [M,\overline{B}]$$

$$\overline{B}(x,y) = B(x,\ell^{-1}y)$$
\[ m : W(F) \to W(F^{\sqrt{a}}) \quad [M, B] \to [M, B] \otimes_F F(\sqrt{a}) \]

\[ S : W(F^{\sqrt{a}}) \to W(F) \quad [M, B] \to [M, \tilde{B}] \]

\[ \tilde{B}(x, y) = t \circ B(x, y) \]

\[ t = \text{scaled trace} \]

\[ I : W(F) \to W^{+1}(-a, F; f) \quad [M, B] \to [M \oplus M, \tilde{B}, \tilde{\gamma}] \]

\[ \tilde{B} = B \oplus -aB \]

\[ \tilde{\gamma}(x, y) = (ay, x) \]

We begin by studying the map \( d \).

\[ d : [M, B, \ell] \to [M, \tilde{B}], \tilde{B}(x, y) = B(x, \ell^{-1}y) \]

\( \tilde{B} \) is symmetric since

\[ \tilde{B}(x, y) = B(x, \ell^{-1}y) \]

\[ = -B(x, -\ell^{-1}y) \]

\[ = B(-\ell^{-1}y, x) \quad \text{since } B \text{ is skew-symmetric} \]

\[ = B(y, \ell^{-1}x) \quad \text{since } \ell^* = -\ell \]

\[ = \tilde{B}(y, x) \].

\[ \text{Lemma 1.1} \quad \ker d = 0. \]

Define \( I : W^{-1}(F) \to W^{-1}(-a, F; f) \) by

\[ [M, B] \to [M \oplus M, \tilde{B}, \tilde{\gamma}] \quad \text{where} \]

\[ \tilde{B}((x, y), (u, v)) = B(x, u) - aB(y, v) \]

\[ \tilde{\gamma}(x, y) = (ay, x). \]
I is clearly a well-defined group homomorphism. We shall show \( \ker d \subseteq \text{im } I \).

However, \( w^{-1}(F) = 0 \), so this will show that \( \ker d = 0 \).

Let \((M, B, \ell)\) be an anisotropic representative of a Witt class in \( w^{-1}(-a,F;f) \), with \( \bar{d}([M,B,\ell]) = 0 \). Let \( N \) be a metabolizer for \([M,\bar{B}]\).

Consider \([N,B_1]\), where \( B_1 = \bar{B}_N \), the restriction of \( B \) to \( N \). \( B_1 \) is non-singular since \((M,B,\ell)\) is anisotropic over a field, and \( N \) is a metabolizer for \( \bar{B} \). \( B_1 \) is a skew-symmetric inner product on \( N \).

Applying \( I \) we obtain

\[
[N \oplus N, \bar{B}_1, \ell], \quad \text{where } \bar{B}_1(x,y) = B_1 \oplus -aB_1.
\]

Define \( \gamma : N \oplus N \to M \) by \( (n_1, n_2) \mapsto n_1 + \ell n_2 \).

We shall show \( \gamma \) is an equivariant isomorphism, hence \([M,B,\ell] \in \text{im } I \), which completes the proof.

Since \( \dim(N \oplus N) = \dim M \), in order to show that \( \gamma \) is an isomorphism, it suffices to show \( \gamma \) is \( 1 - 1 \), since these are vector spaces. So suppose

\[
\gamma(n_1, n_2) = n_1 + \ell n_2 = 0.
\]

Then form \( W = \langle n_2, \ell n_2 \rangle \), the subspace generated by \( n_2 \) and \( \ell n_2 \). \( W \) is \( \ell \) invariant since
\[ \ell^2 = a : \text{ We compute:} \]

\[
B(n_2', n_2) = B(n_2', -\ell^{-1}n_1) \\
= \bar{B}(n_2, n_1) = 0 \quad \text{since } N = N^\perp \\
\text{with respect to } \bar{B} \\
B(n_2', \ell n_2) = B(-\ell^{-1}n_1, -n_1) \\
= -B(n_1, -\ell^{-1}n_1) = 0
\]

Similarly \( B(\ell n_2', \ell n_2) = 0 \), and we see that \( W \) is an \( \ell \)
invariant subspace of \( M \) with \( W \subset W^\perp \). This contradicts \( (M, B, \ell) \) being anisotropic unless \( W = 0 \). Thus \( n_2 = 0 \)
and since \( n_1 + \ell n_2 = 0 \), \( n_1 = 0 \). Hence \( \gamma \) is an
isomorphism.

The following computations show that \( \gamma \) is equivariant:

\[
\bar{B}_1((n_1, n_2), (n_1', n_2')) = B_1(n_1, n_1') - aB_1(n_2, n_2') \\
= B(n_1, n_1') + B(\ell n_2, \ell n_2') + \\
B(n_1', \ell n_2') + B(\ell n_2, n_1') \quad (\text{since } N = N^\perp \text{ with respect to } \bar{B}) \\
= B(n_1 + \ell n_2, n_1' + \ell n_2') \\
= B(\gamma(n_1, n_2), \gamma(n_1', n_2')) \\
\gamma \circ \gamma(n_1, n_2) = \gamma(an_2, n_1) = an_2 + \ell n_1 \\
\ell \circ \gamma(n_1, n_2) = \ell(n_1 + \ell n_2) = \ell n_1 + \ell^2 n_2 = \ell n_1 + an_2.
\]

Thus \( \gamma \) yields an equivariant isomorphism.
(N ⊗ N, B₁, l) → (M, B, l). It follows that [M, B, l] ∈ im I as was to be shown. □

The natural second step is to continue computing kernels. We could do this formally although it would amount to computing the kernel of the 0 mapping:

$$I : W^{-1}(F) → W^{-1}(-a, F; f),$$

where $W^{-1}(F) = 0$. We shall return to $I$ later, as this "same map" occurs at the end of our octagon.

Thus, we next study the cokernel of $d$, and the map $m : W(F) → W(F/v^2)$ as observed previously we may view $m$ as being defined by $[M, B, l] → [M ⊕ M, B']$ where

$$B'(x_1, y_1, x_2, y_2) = B(x_1, x_2) + aB(y_1, y_2)$$
$$+ [B(x_1, y_1) + B(x_2, y_2)]/a$$

Lemma 1.2 ker $m = im d$.

**Step 1:** $im d ⊆ ker m$. Suppose $[M, B, l] → [M, B'] → [M ⊕ M, B']$. Consider

$$N = \{(\ell v, v) : v ∈ M\} ⊂ M ⊕ M$$

$$B'(\ell v, v, \ell w, w) = B(\ell v, \ell w) + \ell B(v, w)$$
$$+ [B(\ell v, w) + B(\& w, v)]/a$$
$$= B(\ell v, w) + aB(v, l^{-1}w)$$
+ [B(ℓ v, ℓ⁻¹ w) + B(ℓ w, ℓ⁻¹ v)]/a

= -aB(v, ℓ⁻¹ w) + aB(v, ℓ⁻¹ w) + [B(v, -w) + B(-w, v)]/a

= 0

Thus N ⊆ N⁻¹, rank N = rank M, and
rank N + rank N⁻¹ = rank M ⊕ M.
Hence rank N⁻¹ = rank M = rank N, and N = N⁻¹, so

\[ [M ⊕ M, \overline{B}'] = 0. \]

Step 2: ker m ⊆ im d.

Let (M, B) be anisotropic. Suppose m[M, B] = 0.
Let N be a metabolizer for [M ⊕ M, B'].

Consider K = \{x ∈ M: (x, 0) ∈ N\}.

B'((x, 0), (y, 0)) = B(x, y) = 0 for all x, y ∈ N. Thus K ⊆ K⁻¹, so that K = 0 since (M, B) is anisotropic.

Similarly, \{x ∈ M: (0, x) ∈ N\} = 0, and we can conclude that N is the graph of a 1-1 function

ℓ: M → M, ie. N = \{(ℓx, x): x ∈ V\}.

B'((ℓx, x), (ℓy, y)) = B(ℓ x, ℓ y) + aB(x, y)

+ [B(ℓ x, y) + B(ℓ y, x)]/a = 0.

Hence B(ℓ x, ℓ y) = -aB(x, y).

B(ℓ x, y) = -B(ℓ y, x) = B(x, -ℓ y).
Thus $\ell$ is of degree $-a$, and $\ell^* = -\ell$. Further, 

$$-aB(x, y) = B(\ell x, \ell y) = -B(x, \ell^2 y)$$

so that $\ell^2 = a$.

Now we form the space $(M, B, \ell)$, where $B_1$ is defined by $B_1(x, y) = B(x, \ell y)$. In order to show $B_1$ is non-singular, we claim: $B_1(-, x) = 0$ implies $x = 0$.

Proof: $B_1(-, x) = 0 = B(-, \ell x)$, hence $\ell x = 0$ since $B$ is non-singular. Thus $x = 0$ since $\ell$ is $1-1$.

Since we are over a field, $Ad_{R_1} B_1$ implies $Ad_{R_1} B_1$ is an isomorphism, and $B_1$ is non-singular.

$B_1$ is skew-symmetric since:

$$B_1(x, y) = B(x, \ell y) = -B(\ell x, y) = -B_1(y, x).$$

$\ell$ is of degree $-a$ with respect to $B_1$ since:

$$B_1(\ell x, \ell y) = B(\ell x, \ell^2 y) = -aB(x, \ell y) = -aB_1(x, y).$$

Thus $[M, B_1, \ell] \in W^{-1}(-a, F; f)$, where $f(t) = t^{2-a}$.

Applying $d$ we obtain $[M, B_1]$.

$$B_1(x, y) = B_1(x, \ell^{-1} y) = B(x, y).$$

Hence $[M, B] \in \text{im} \, d$ as required.

Lemma 1.3 \quad ker S = im m.

Step 1: \quad im m \subseteq ker S.

Let $m([M, B]) = [M \oplus M, B']$. Applying $S$ we obtain
\[ [M \oplus M, t \circ B'] \]. Now consider \( M \oplus 0 \subset M \oplus M \). By definition of \( t \circ B' \), we have \( M \oplus 0 \subset (M \oplus 0)^{\perp} \). However, \( \dim (M \oplus 0) = \frac{1}{2} \dim (M \oplus M) \), so that \( M \oplus 0 \) is a metabolizer, and \( [M \oplus M, t \circ B'] = 0 \).

**Step 2:** \( \ker S \subseteq \text{im } m \).

Let \( (M,B) \) be anisotropic.

Suppose \( S[M,B] = [M,t \circ B] = 0 \). Recall \( t \) is the scaled trace of \( F(\sqrt{a}) \) over \( F \), \( \text{trace} (\frac{\sqrt{a}}{2a}) \). Let \( N \) be a metabolizer for \( [M,t \circ B] \).

If \( c + d/a \in F(\sqrt{a}) \), where \( c,d \in F \), we shall call \( c \) the \( F \)-part, \( d \) the \( \sqrt{a} \)-part of \( c + d/a \). \( t : F(\sqrt{a}) \to F \) is given by \( c + d/a \mapsto d \), projection to the \( \sqrt{a} \)-part.

Consider \( [N,B |] \in W(F) \). Applying \( m \), we obtain \( [N \oplus N,B'] \). Define \( \gamma : N \oplus N \to M \) by

\[
(n_1,n_2) \mapsto n_1 + \sqrt{a} n_2.
\]

We shall show that \( \gamma \) is an isomorphism of \( (N \oplus N,B') \) with \( (M,B) \), and hence \( \ker S \subseteq \text{im } m \).

**Comment:** In order that \( [N,B |] \in W(F) \), we should again check that \( B | \) is non-singular. This follows as in 1.1 since \( (M,B) \) is anisotropic.

An alternate proof can be given since \( \gamma \) is an equivariant isomorphism. Since \( B \) is non-singular, so is \( B' \), and hence so is \( B | \).

In order to show \( \gamma \) is an equivariant isomorphism,
we first show that $\gamma$ is 1-1, and hence an isomorphism as we are working over a field.

Suppose $\gamma \cdot (n_1', n_2') = n_1 + \sqrt{a}n_2 = 0$. Consider $\langle n_1 \rangle \subseteq M$. $B(n_1, n_1)$ has $/a$-part 0 since $N = N^+$ with respect to $\bar{B} = t \circ B$. However, $B(n_1, n_1) = B(n_1, -\sqrt{a}n_2) = -\sqrt{a}B(n_1, n_2)$. Observe that the $/a$-part of $B(n_1, n_2)$ is 0 also, since $n_1, n_2 \in N = N^+$. So $B(n_1, n_1) = -\sqrt{a}B(n_1, n_2)$, implying that the $/a$-part of $B(n_1', n_1')$ is 0 also. Hence $B(n_1', n_1') = 0$. This contradicts $(M, B)$ being anisotropic, unless $n_1 = n_2 = 0$.
Thus $\gamma$ is 1-1, and hence an isomorphism.

We must check that $\gamma$ is equivariant:

\[
B'((n_1, n_2), (n_1', n_2')) = B(n_1, n_1') + \sqrt{a}B(n_2, n_2') + [B(n_1, n_2') + B(n_1', n_2')]/a
\]

\[
= B(n_1 + \sqrt{a}n_2, n_1 + \sqrt{a}n_2')
\]

\[
= B(\gamma(n_1', n_2'), \gamma(n_1, n_2'))
\]

It follows that $(M, B) \sim (N \otimes N, B')$ as desired. □

Lemma 1.4 ker $I = \text{im } S$.

Step 1: $\text{im } S \subseteq \ker I$. Let $S[M, B] = [M, t \circ B]$.
Applying $I$ we obtain $[M \otimes M, (t \circ B), \tilde{i}]$. Let $N = \{(\sqrt{a}v, v) : v \in M\}$. $N$ is $\tilde{i}$ invariant since
\( \tilde{f}(av,v) = (av,\sqrt{av}) \). Further it is self-annihilating since

\[
(t \circ B)((av,v),(aw,w)) = (t \circ B)((av,\sqrt{av}),(aw,\sqrt{aw}))(t \circ B)(v,w) = 0
\]

Since \( \text{rank } N = \frac{1}{2} \text{rank } (M \otimes M) \), \( N = N^\perp \) and \( M \otimes M \sim 0 \).

**Step 2:** \( \text{ker } I \subseteq \text{im } S \).

Let \((M,B)\) be anisotropic.

Suppose \( I[M,B] = [M \otimes M,B,\tilde{l}] = 0 \). Let \( N \) be a metabolizer for \( M \otimes M \).

Let \( K = \{ x \in M : (x,0) \in N \} \). If \( x,y \in K \),

\[
\tilde{B}((x,0),(y,0)) = B(x,y) = 0.
\]

Since \((M,B)\) is anisotropic, \( K = \{ 0 \} \). Similarly, \( \{ x \in M : (0,x) \in N \} = 0 \).

Thus \( N \) is the graph of a 1 - 1 function \( \tilde{l} : M \rightarrow M \).

(We need \( F \) a field to conclude that \( \tilde{l} \) is an isomorphism of \( M \) onto \( M \).)

We may write \( N = \{ (\tilde{l}v,v) : M \rightarrow M \} \).

\[
\tilde{B}((\tilde{l}v,v),(\tilde{l}w,w)) = B(\tilde{l}v,\tilde{l}w) - aB(v,w) = 0
\]

Thus \( \tilde{l} \) is of degree \( a \).

\( N \) is \( \tilde{l} \) invariant, so that \((\tilde{l}v,v) \in N\) implies \((av,\tilde{l}v) \in N\). Hence \( \tilde{l}(\tilde{l}v) = av \), and \( \tilde{l}^2 = a \). Also

\[
\tilde{B}((av,\tilde{l}v),(\tilde{l}w,w)) = B(av,\tilde{l}w) - aB(\tilde{l}v,w) = 0
\]
Thus \( B(v, w) = B(\ell v, w) \). Hence \( \ell^* = \ell \).

\( N \) is already an \( F \)-vector space, with

\[ d(\ell v, v) = (d \ell(v), dv) \quad \text{for} \quad d \in F. \]

We make \( N \) into an \( F(\sqrt{a}) \)-vector space by defining

\[ \sqrt{a}(\ell v, v) = (av, \ell v) = \tilde{\ell}(\ell v, v). \]

Now define \( B_1 \) on \( N \) by \( B_1((\ell v, v), (\ell w, w)) \)
\[ = B(\ell v, w) + B(v, w) \sqrt{a} \] . \( B_1 \) is symmetric since \( B(v, \ell w) = B(\ell v, w) \). \( B_1 \) is clearly \( F(\sqrt{a}) \)-bilinear with the \( F(\sqrt{a}) \)-vector space structure defined above. \( B_1 \) is an inner product since \( B \) is.

Now we apply \( S \) to \([N, B_1] \) , \( S[N, B_1] = [N, t \circ B_1] \).
Then define \( \gamma : N \to M \) by \((\ell v, v) \mapsto v \).

\( B(\gamma(\ell v, v), \gamma(\ell w, w)) \)
\[ = B(v, w) = (t \circ B_1)((\ell v, v), (\ell w, w)). \]

Hence \( [N, t \circ B_1] \)
\[ = [M, B] \], and \([M, B] \in \text{im} S. \]

\underline{Lemma 1.5} \( I \) is onto.

We define: \( d : \ell^1(-a, F; f) \to \ell^{-1}(F) \) by

\[ [M, N, \ell] \to [M, \bar{B}] \]
where \( \bar{B}(x, y) = B(x, \ell^{-1}y) \)

\[ \bar{B}(y, x) = B(\ell^{-1}y, x) = -B(-\ell^{-1}y, x) = -B(x, \ell^{-1}y) \]
\[ = -\bar{B}(x, y). \]
Hence $\tilde{B}$ is skew-symmetric. (This of course is the same $d$ we have already defined. However, $B$ symmetric yields $\tilde{B}$ skew-symmetric). Note that $W^{-1}(F) = 0$, so that kernel $d = W^+(-a,F;f)$.

Let $[M,B,\ell] \in W^+(-a,F;f)$, so $[M,B,\ell] \in \text{kernel } d$.

Suppose $(M,B,\ell)$ is anisotropic. Since $[M,\tilde{B}] = 0$, we let $N$ be a metabolizer for $(M,\tilde{B})$. Then $N \cap \ell N = \{0\}$ since $(M,B,\ell)$ is anisotropic.

Consider $[N,B_1] \in W(F)$, where $B_1 = B^\perp$. Applying I we obtain $[N \oplus_N B_1,\tilde{\ell}]$. We shall show this is isomorphic to $(M,\tilde{B},\ell)$. From this it follows that I is onto.

Define $\gamma : N \oplus N \rightarrow M$ by $(n_1',n_2') \rightarrow n_1' + \ell n_2'$. Then $B(\gamma(n_1',n_2'),\gamma(n_1',n_2')) = B(n_1'+\ell n_2',n_1'+\ell n_2')$

$= B(n_1',n_1') + B(\ell n_2',\ell n_2') + B(n_1',\ell n_2') + B(\ell n_2',n_1')$

$= B(n_1',n_1') - aB(n_2',n_2')$

$= \tilde{B}_1((n_1',n_2'),(n_1',n_2'))$.

$\ell\gamma(n_1',n_2) = \ell(n_1'+\ell n_2) = \ell n_1 + \ell^2 n_2 = \ell n_1 + an_2$.

Thus $\gamma$ is equivariant. Again, $\gamma$ is an isomorphism since $N \cap \ell N = 0$. Hence $(M,B,\ell) \approx (N \oplus N,\tilde{B}_1,\tilde{\ell})$ as claimed.

$\square$
We have thus shown:

**Theorem 1.5** There is an exact octagon:

\[
\begin{array}{c}
\text{\(W^+ (F(\sqrt{a}))\)} \xrightarrow{\text{\(S\)}} \text{\(W^+ (F)\)} \xrightarrow{\text{\(m\)}} \text{\(W^+ (-a, F; f)\)} \\
\text{\(W^+ (F)\)} \xrightarrow{\text{\(d\)}} \text{\(0\)} \\
\text{\(W^+ (-a, F; f)\)} & \leftarrow \text{\(0\)} & \leftarrow \text{\(0\)}
\end{array}
\]

The map \(S\) is Scharlau's transfer, with the other maps as previously described. \(\qed\)
2. The exact octagon over a field.

The Scharlau transfer sequence has been derived by computing successive kernels. For degree k mapping structures, the maps of the Scharlau transfer sequence become

\[ d_\epsilon : W^\epsilon(-k,F) \to A(F) \]

\[ [M,B,\ell] \to [M,B] \quad \overline{B}(x,y) = k^{-1}B(x,\ell y) \]

\[ m_\epsilon : A(F) \to W^\epsilon(k,F) \]

\[ [M,B] \to [M \oplus M, B_\epsilon, \ell_\epsilon] \quad \text{where} \]

\[ B_\epsilon((x,y),(z,w)) = B(x,w) + \epsilon B(z,y) \]

\[ \ell_\epsilon(x,y) = (ek^{-1}y,x) \quad s \text{ is the symmetry operator for } B \]

\[ S_\epsilon : W^\epsilon(k,F) \to W^\epsilon(k^2,F) \]

\[ [M,B,\ell] \to [M,B,\ell^2] \]

\[ I_\epsilon : W^\epsilon(k^2,F) \to W^\epsilon(-k,F) \]

\[ [M,B,\ell] \to [M \oplus M, B \oplus -kB, \tilde{\ell}] \quad \tilde{\ell}(x,y) = (\ell y,x) \]

We shall prove:

**Theorem 2.1** The above maps combine to yield an exact octagon:
We begin with the map \( m_\varepsilon \). First, we observe that \( \ell_\varepsilon \) is of degree \( k \) with respect to \( B_\varepsilon \) since:

\[
B_\varepsilon (\ell_\varepsilon (x,y), \ell_\varepsilon (z,w)) = B_\varepsilon ((\varepsilon k s^{-1} y, x), (\varepsilon k s^{-1} w, z))
\]

\[
= B (\varepsilon k s^{-1} y, z) + \varepsilon B (\varepsilon k s^{-1} w, x) = k B (s^{-1} w, x) + \varepsilon k B (s^{-1} y, x)
\]

\[
= k B (x, w) + \varepsilon k B (z, y) = k [B (x, w) + \varepsilon B (z, y)] = k B_\varepsilon ((x, y), (z, w)).
\]

\( m_\varepsilon \) is well-defined, for if \( (M, B) \sim 0 \) has metabolizer \( N \) then \( m_\varepsilon (M, B) \) has metabolizer \( N \odot N \).

**Lemma 2.2** \( \ker S_\varepsilon = \text{im } m_\varepsilon \).

**Proof:** Suppose \( m_\varepsilon [M, B] = [M \oplus M', B_\varepsilon', \ell_\varepsilon] \).

Then \( M \oplus 0 \) is an \( \ell_\varepsilon^2 \) invariant subspace, equal to its own annihilator by the way \( B_\varepsilon \) is defined. Thus \( \text{im } m_\varepsilon \subseteq \ker S_\varepsilon \).
Conversely, suppose \([M,B,\ell] \in \ker S_\ell\).

Let \(N\) be an \(\ell^2\) invariant subspace of \(M\) with \(N = N^1\). \(N\) is a metabolizer for \(S_\ell[M,B,\ell] = [M,B,\ell^2]\).

We assume as usual \((M,B,\ell)\) is anisotropic.

Note: \(N \cap \ell N = 0\) This is seen as follows.
Let \(n \in N \cap \ell N\), and form \(N_1 = \langle n, \ell n, \ldots, \ell^{w-1} n \rangle\) where \(w\) is the degree of the minimal polynomial of \(\ell\).

Clearly \(N_1\) is \(\ell\) invariant, with \(N_1 \subset N^1\). Thus, since we took \((M,B,\ell)\) anisotropic, \(N_1 = 0\) and \(n = 0\).

For vector spaces then it follows that \(M = N \oplus \ell N\).

Define \(B_1\) on \(N\) by \(B_1(n_1,n_2) = B(n_1,\ell n_2)\). We must verify that \(B_1\) is an inner product. So consider the adjoint of \(B_1\), \(\text{Ad}_{R_1} B_1 : N \to \text{Hom}_F(N,F)\). Since we are working over a field, it suffices to show \(\text{Ad}_{R_1} B_1\) is \(1-1\).

We need \(\ell \not= 0\), so that \(\ell\) is non-singular. Suppose \(B_1(-,n_2) = 0 = B(-,\ell n_2)\). Then \(\ell n_2 \in N^1 = N\). However, by the note, \(N \cap \ell N = 0\). Thus \(\ell n_2 = 0\). Since \(\ell\) is non-singular \(n_2 = 0\). Hence \(B_1\) is non-singular.

We may thus form \([N,B_1] \in A(F)\). Applying \(m_\ell \in \epsilon\) we obtain \([N \oplus N, B_1, \ell e]\). Here

\[
B_1(n_1,n_2) = B(n_1,\ell n_2) = \epsilon B(\ell n_2,n_1) = \epsilon k B(n_2,\ell^{-1} n_1)
\]

\[
= B(n_2,\ell (ek\ell^{-2} n_1)) = B_1(n_2,ek\ell^{-2} n_1)
\]

Thus \(s = ek\ell^{-2}\) for \(B_1\), or \(\ell^2 = ek s^{-1}\).
It follows that \( l_\varepsilon(x,y) = (\varepsilon k s^{-1}y,x) = (l^2y,x) \)

Now define \( \gamma : N \oplus N \to M \) by \( (n_1,n_2) \mapsto n_1 + l n_2 \)

Since \( N \cap lN = 0 \), \( \gamma \) is an isomorphism. We now show that \( \gamma \) is equivariant, i.e., that \( (N \oplus N,B_1, l_\varepsilon) \cong (M,B,l) \)

It follows that \( \ker S_\varepsilon \subseteq \text{im} \, m_\varepsilon \) as desired. We compute:

\[
B_1((x,y),(z,w)) = B_1(x,w) + \varepsilon B_1(z,y) = B(x,\varepsilon w) + \varepsilon B(z,\varepsilon y)
\]

\[+ B(x,z) + B(\varepsilon y,\varepsilon w) \quad \text{(since)} \quad B(x,z) = B(\varepsilon y,\varepsilon w) = 0 \quad \text{as} \quad N = N^1 \]

\[= B(x,\varepsilon w + z) + B(\varepsilon y,\varepsilon w + z) \]

\[= B(x + \varepsilon y, z + \varepsilon w) = B(\gamma(x,y), \gamma(z,w)) .\]

Also, \( \gamma(l^2(n_1,n_2)) = \gamma(\varepsilon k s^{-1}n_2,n_1) = \gamma(l^2 n_2,n_1) \)

\[= l^2 n_2 + \varepsilon n_1 = l(\varepsilon n_2 + n_1) = l(\gamma(n_1,n_2)) . \]

\[\square\]

**Lemma 2.3** \( \ker I_\varepsilon = \text{im} S_\varepsilon . \)

**Proof:** Let \([M,B,l^2] \in \text{im} S_\varepsilon . \) We first show

\[I_\varepsilon([M,B,l^2]) = 0 . \]

\[I_\varepsilon[M,B,l] = [M \oplus M,B \oplus -kB,\ell^2] , \text{ where } (\ell^2)(x,y) = (l^2y,x) . \]

Let \( N = \{(x,\ell^{-1}x) : x \in M\} . \) \( N \) is \( (\ell^2) \) invariant.

Further, \( (B \oplus -kB)((x,\ell^{-1}x),(y,\ell^{-1}y)) = B(x,y) - kB(\ell^{-1}x,\ell^{-1}y) \)

\[= B(x,y) - B(x,y) = 0 \]

Thus \( N \subseteq N^1 , \text{ rank } N = (1/2) \text{ rank } M, \text{ so that } N = N^1 \)
and $M \otimes M \sim 0$.

Conversely, let $[M, B, \ell] \in \ker I_\epsilon$. So

$I_\epsilon [M, B, \ell] = [M \otimes M, B \otimes -kB, \ell] = 0$.

Let $N$ be a metabolizer for $M \otimes M$. $N$ is $\ell$ invariant.

Now assume $(M, B, X)$ is anisotropic. We claim that $N$ is the graph, $N = \{ (x, tx) : x \in M \}$, of a 1-1 function $t : M \to M$. Consider $K = \{ x \in M : (x, 0) \in N \}$.

If $(x, 0) \in N$, then $(0, x) \in N$ since $N$ is $\ell$ invariant. Thus $(tx, 0) \in N$, and $tx \in K$. Hence $K$ is $\ell$ invariant. However, if $x, y \in K$,

$B(x, y) = (B \otimes -kB)((x, 0), (y, 0)) = 0$, since $N = N^\perp$.

Thus $K \subseteq K^\perp$. This contradicts $(M, B, \ell)$ anisotropic unless $K = \{0\}$.

Similarly, $\{ x \in M : (0, x) \in N \} = 0$.

Thus $N$ is the graph of a 1-1 function $t : M \to M$ and we can write $N = \{ (x, tx) : x \in M \}$. $t$ maps $M$ onto $M$ since $\dim N = \dim M$.

On $N$ we define $B_\perp$, $\ell_\perp$ by:

$B_\perp ((x, tx), (y, ty)) = B(x, y)$.

$\ell_\perp (x, tx) = (\ell(tx), x) \quad \ell_\perp : N \to N$ since $N$ is $\ell$ invariant.

We compute:
\[ B_1(\ell_1(x, tx), \ell_1(y, ty)) = B_1((\ell(tx), x), (\ell(ty), y)) \]

\[ = B(\ell(tx), \ell(ty)) = k^2 B(tx, ty) . \]

However, \( (B \otimes -kB)((x, tx), (y, ty)) = B(x, y) - kB(tx, ty) = 0 \)
since \( N = N^\perp \). Thus, the above equals \( kB(x, y) = kB_1((x, tx), (y, ty)) . \)

In order to have \( [N, B_1, \ell_1] \in W^1(k, F) \), we still must show \( B_1 \) is an inner product. Again, it suffices to show that the adjoint is \( 1-1 \).

\[ \text{Ad}_{R^B_1} : N \rightarrow \text{Hom}_F(N, F) \]

\[ (x, tx) \mapsto B_1(-, (x, ty)) \]

Suppose \( B_1(-, (x, tx)) = 0 \) on \( N \). Then \( B(-, x) = 0 \)
for all \( y \), when \( (y, ty) \in N \). Since we are over a field, \( B(-, x) = 0 \) on \( M \). This contradicts \( (M, B, \ell) \)
anisotropic unless \( x = 0 \). Thus \( \text{Ad}_{R^B_1} \) is \( 1-1 \), and hence an isomorphism.

We thus consider \( [N, B_1, \ell_1] \). Define \( \gamma : (N, B_1, \ell_1^2) \rightarrow (M, B, \ell) \) by \( \gamma : (x, tx) \mapsto x \). \( \gamma \) is clearly an isomorphism. We claim \( \gamma \) is equivariant, so that

\[ S_\epsilon [N, B_1, \ell_1] = [M, B, \ell] \] as desired.

\[ B_1((x, tx), (y, ty)) = B(x, y) = B(\gamma(x, tx), \gamma(y, ty)) . \]

\[ \gamma \circ \ell_1^2(x, tx) = \gamma \circ \ell_1(\ell(tx), x) = \gamma \circ (\ell x, \ell(tx)) \]

\[ = \ell x = \ell \circ \gamma(x, tx) . \] \( \square \)
Lemma 2.4 \[ \text{ker } d_\varepsilon = \text{im } I_\varepsilon . \]

Let \( I_\varepsilon [M, B, \ell] = [M \oplus M, B \oplus -kB, \tilde{\ell}] \). Applying \( d_\varepsilon \)

we obtain \([M \oplus M, B \oplus -kB] \) where

\[
(B \oplus -kB)((x, y), (u, v)) = \left( \frac{B \oplus -kB}{k} \right)((x, y), \ell(u, v))
\]

\[= \frac{1}{k} (B \oplus -kB)((x, y), (\ell v, u)) .\]

In general, if \([M, B, \ell] \in W_\varepsilon (-k, F)\), applying \( d_\varepsilon \)

we obtain \([M, B] \) where

\[
\bar{B}(x, y) = \frac{1}{k} B(x, \ell y)
\]

\[= \frac{\varepsilon}{k} B(\ell y, x)\]

\[= -\varepsilon B(y, \ell^{-1}x)\]

\[= \frac{1}{k} B(y, \ell (\ell^{-2}(-\varepsilon kx)))\]

\[= \bar{B}(y, \ell^{-2}(-\varepsilon kx)) .\]

So \( s = -\varepsilon k\ell^{-2} \) is the symmetry operator for \( \bar{B} \).

Now consider \( N = M \oplus 0 \subset M \oplus M \). \( N \) is \( s = -\varepsilon k\ell^{-2} \)

invariant, rank \( N = \frac{1}{2} \text{rank } (M \oplus M) \) and

\[
(B \oplus -kB)((v, 0), (w, 0)) = \frac{1}{k} (B \oplus -kB)((v, 0), (0, w)) = 0 .
\]

Thus

\[(M \oplus M, B \oplus -kB) \sim 0 ,\]

with metabolizer \( N \).
Conversely, suppose \([M, B, \ell] \in \ker d_\ell\). Let \(N\) be an \(s = -\ell R^{-2}\) invariant subspace of \(M\) with \(N = N^1\) with respect to \(\bar{B}\), where \(\bar{B}(x, y) = \frac{1}{R}B(x, \ell y)\).

Define \(\gamma : N \oplus N \to M\) by \((n_1, n_2) \to n_1 + \ell n_2\). We assume \((M, B, \ell)\) is anisotropic.

Claim: \(\gamma\) is \(1-1\). Suppose \(n_1 + \ell n_2 = 0\). Then form \(K = \langle n_1, \ell n_1, \ldots, \ell^{r-1} n_1 \rangle\), where \(r =\) degree of the minimal polynomial of \(\ell\). Thus \(K\) is \(\ell\) invariant.

\[B(n_1, n_1) = B(n_1, -\ell n_2) = (-\ell) \left( \frac{1}{R}B(n_1, \ell n_2) \right) = -\ell B(n_1, n_2) = 0\]
as \(N = N^1\) with respect to \(\bar{B}\). By similar computations, it follows that \(K \subset K^1\). \(K\) is \(\ell\) invariant. This contradicts \((M, B, \ell)\) being anisotropic, unless \(K = \{0\}\). Thus \(\gamma\) is \(1-1\), and hence an isomorphism.

Consider \([N, B, \ell^2] \in \mathcal{W}(k^2, F)\). We must show \(B\) is an inner product. Again, it suffices to show \(\text{Ad}_\mathcal{R}B\) is \(1-1\). So suppose \(B(-, n) = 0\) on \(N\). Note that \(B(\ell N, n) = 0\) since \(N = N^1\). By the above \(N \oplus \ell N = M\), so \(B(-, n) = 0\) on \(M\). Since \(B\) is an inner product, \(n = 0\), and \(\text{Ad}_\mathcal{R}B\) is an isomorphism on \(N\).

We now apply \(I_\mathcal{C}\) to \([N, B, \ell^2]\). This yields \([N \oplus N, B \oplus -kB, \ell^2]\). Claim: \(\gamma\) provides an equivariant isomorphism \(\gamma : I_\mathcal{C}[N, B, \ell^2] \to [M, B, \ell]\). To see this, we compute:
\[(B \oplus -kB)((x, y), (u, v)) = B(x, u) - kB(y, v)\]
\[= B(x, u) + B(\ell y, \ell v) + B(x, \ell v) + B(\ell y, u)\]
\[= B(x+\ell y, u+\ell v) = B(\gamma(x, y), \gamma(u, v))\]
\[\gamma \circ \ell^2(x, y) = \gamma(\ell^2 y, x) = \ell^2 y + \ell x = \ell(\ell y + x) = \ell \circ \gamma(x, y)\]

**Lemma 2.5** \(\ker m_\varepsilon = \im d_{-\varepsilon}\).

Suppose \(d_{-\varepsilon}[M, B, \ell] = [M, \bar{B}]\). The associated symmetry operator for \(\bar{B}\) is \(s = -(\varepsilon)k\ell^{-2}\). Note: \(B\) is \(-\varepsilon\) symmetric.

Applying \(m_\varepsilon\) to \([M, \bar{B}]\), we obtain \([M \oplus M, \bar{B}_\varepsilon, \ell_\varepsilon]\) where \(\ell_\varepsilon(w_1, w_2) = (\varepsilon k\ell^{-1} w_2, w_1)\)
\[= (\ell^2 w_2, w_1)\]

Consider \(N \subseteq M \oplus M\) defined by
\(N = \{(x, \ell^{-1} x) : x \in M\}\). \(N\) is \(\ell_\varepsilon\) invariant. Further
\[\bar{B}_\varepsilon((x, \ell^{-1} x), (y, \ell^{-1} y)) = \bar{B}(x, \ell^{-1} y) + \varepsilon \bar{B}(y, \ell^{-1} x)\]
\[= \frac{1}{k} B(x, y) + \varepsilon \frac{1}{k} B(y, x)\]
\[= \frac{1}{k} B(x, y) + \varepsilon (-\varepsilon) \frac{1}{k} B(x, y) = 0\]

Thus \([M \oplus M, \bar{B}_\varepsilon, \ell_\varepsilon]\) = 0 and \(\im d_{-\varepsilon} \subseteq \ker m_\varepsilon\).

Conversely, suppose \(m_\varepsilon[M, B] = [M \oplus M, B_\varepsilon, \ell_\varepsilon] = 0\).

Let \(N\) be a metabolizer for \(M \oplus M\) above. Assume \((M, B)\) is anisotropic.

Consider \(K = \{x \in M : (x, 0) \in N\}\). Since \(N\) is \(\ell_\varepsilon\)
invariant, \((x,0) \in N\) implies \((0,x) \in N\) and 
\((\epsilon k s^{-1} x,0) \in N\). Also, \(N\) is \(\ell_\epsilon^{-1}\) invariant since 
\[
B_\epsilon((u,v),\ell_\epsilon^{-1}(x,y)) = \frac{1}{k}B_\epsilon(\ell_\epsilon(u,v),(x,y)) = 0
\]
for all \((u,v),(x,y) \in N\). Thus \((x,0) \in N\) implies 
\((\epsilon/k(sx),0) \in N\). \(N\) is a subspace, so \((sx,0) \in N\) 
whenever \((x,0) \in N\). Thus \(K\) is \(s\) invariant.

Further \(B(x,y) = B_\epsilon((x,0),(0,y)) = 0\) for all \(x,y \in K\).
This follows since \((y,0) \in N\) implies \(\ell_\epsilon(y,0) = (0,y) \in N\) 
also, and \(N = N^1\) with respect to \(B_\epsilon\). Thus \(K\) is 
an \(s\) invariant, self annihilating subspace of \(M\). This 
contradicts \(M\) being anisotropic unless \(K = 0\). Similarly 
\([x \in M: (0,x) \in N] = [0]\). It follows that \(N\) is 
the graph of a \(1-1\) function \(t : M \rightarrow M\), and we can write

\[
N = \{(w,tw) : w \in M\}.
\]

We now study this map \(t\). First, since \(N\) is \(\ell_\epsilon\) 
and \(\ell_\epsilon^{-1}\) invariant, if \((x,y) \in N\) then so is 
\[
\ell_\epsilon(x,y) = (\epsilon k s^{-1} y, x)
\]
and 
\[
\ell_\epsilon^{-1}(x,y) = (y, \epsilon/k sx) .
\]
Thus, if \((y,ty) \in N\), so is \((ty, \epsilon/k sy)\), so that 
\(t(ty) = \frac{\epsilon}{k} sy\). More simply, \(t^2 = \frac{\epsilon}{k} s\), or \(\epsilon k t^2 = s\).
Moreover, \( B_\varepsilon ((x, tx), (y, ty)) = 0 \)

\[
= B(x, ty) + \varepsilon B(y, tx) = B(ty, sx) + \varepsilon B(y, tx) = \\
= B(y, t^*sx) + \varepsilon B(y, tx) \quad \text{where} \quad t^* = \text{adjoint of } t \\
= B(y, (t^*s + \varepsilon t)x) = 0
\]

Since \( B \) is non-singular, \( t^*s + \varepsilon t = 0 \). Thus

\[
t^* = -\varepsilon ts^{-1} = (-\varepsilon t)(\varepsilon/k t^{-2}) = -\frac{t^{-1}}{k}.
\]

On \( M \), define an inner product \( B_1 \) by

\[
B_1(x, y) = k B(x, ty). \quad B_1 \text{ is non-singular since } B \text{ and } t \text{ are, as usual.}
\]

\( B_1 \) is \((-\varepsilon)\) symmetric since :

\[
B_1(x, y) = k B(x, ty) = k B(ty, sx) = k B(y, t^*sx) \\
= k B(y, -\varepsilon tx) = -\varepsilon k B(y, tx) = -\varepsilon B_1(y, x).
\]

Now consider \( (M, B_1, t^{-1}) \).

\[
B_1(t^{-1}x, t^{-1}y) = k B(t^{-1}x, y) = k B(x, t^*^{-1}y) \\
= k B(x, -kty) = (-k)(k)B(x, ty) \\
= (-k) B_1(x, y)
\]

Thus \( [M, B_1, t^{-1}] \in W^- \epsilon(-k, F) \).

Applying \( d_{-\varepsilon} \), we obtain \( [M, \bar{B}_1] \).

\[
\bar{B}_1(x, y) = k^{-1}B_1(x, t^{-1}y) = B(x, y) \quad \text{as desired.} \]
This completes the proof of the exact octagon over a field \( F \), when \( k \neq 0 \). The failure of this proof for a Dedekind domain, eg. \( Z \) the ring of integers, is the verification that the bilinear maps we are constructing are actually \( Z \)-inner products. In order to overcome this difficulty, we use a different approach. Namely, we study a boundary sequence relating \( W(k,Z) \) to \( W(k,Q) \). This boundary sequence then enables us to study the octagon over \( Z \).

First, however, we need to recall some algebraic number theory, and elementary facts about the Witt ring. This will prepare the boundary sequence which comes later.
Chapter V ALGEBRAIC NUMBER FIELDS

The purpose of this chapter is to establish notation, and recall some results from ring theory and algebraic number theory. The reader should also see O'Meara, [O'M] Introduction to Quadratic Forms, for a complete exposition.

1. Prime ideals

Let \( E \) be an algebraic number field, \( - \) an involution on \( E \). As in Chapter I, we let \( F \) be the fixed field of \(-\). The Dedekind rings of integers in \( E \) and \( F \) are denoted by \( \mathcal{O}(E) \) and \( \mathcal{O}(F) \) respectively. \( \mathcal{O}(F) = \mathcal{O}(E) \cap F \).

If \( \mathfrak{p} \) is a prime ideal in \( \mathcal{O}(E) \), then \( \mathfrak{p} = \mathfrak{p} \cap \mathcal{O}(F) \) will denote the corresponding prime ideal in \( \mathcal{O}(F) \).

Conversely, if \( \mathfrak{P} \) is a prime ideal in \( \mathcal{O}(F) \), by the Going Up Theorem, [A,Mc 63] there exists a prime ideal \( \mathfrak{p} \) in \( \mathcal{O}(E) \) with \( \mathfrak{p} \cap \mathcal{O}(E) = \mathfrak{P} \). In fact there may be several such prime ideals in \( \mathcal{O}(E) \) lying over \( \mathfrak{P} \). The answer is given by considering \( \mathfrak{p} \mathcal{O}(E) \), [S 71].

We factor \( \mathfrak{p} \mathcal{O}(E) = \prod_{i=1}^{g} \mathfrak{p}_i^{e_i} \). The \( \mathfrak{p}_i \) satisfy \( \mathfrak{p}_i \cap \mathcal{O}(E) = \mathfrak{p} \). Since the extension \( [E:F] \) is of degree \( 2 \), \( \sum_{i=1}^{g} e_if_i = 2 \), where \( f_i = [\mathcal{O}(E)/\mathfrak{p}_i : \mathcal{O}(F)/\mathfrak{P}] \)

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is the residue field degree. We thus obtain the following cases:

1.1 **Split** \( e = 1 \) \( f = 1 \) \( g = 2 \). In this case \( PO(E) = \mathfrak{p} \mathfrak{p} \) where \( \mathfrak{p} \) is a prime ideal in \( O(E) \) with \( \mathfrak{p} \neq \mathfrak{p} \). We say that \( \mathfrak{p} \) splits in this case.

We may examine the split case in terms of the local completion of \( F \) at \( \mathfrak{p} \), which we denote \( \tilde{F}(\mathfrak{p}) \).

Write \( E = F(\sqrt[\mathfrak{p}]{\mathfrak{p}}) \), and suppose \( \sqrt[\mathfrak{p}]{\mathfrak{p}} \) satisfies the irreducible polynomial \( p_{\sqrt[\mathfrak{p}]{\mathfrak{p}}/F}(x) = p(x) \). Then factor \( p(x) \) in \( \tilde{F}(\mathfrak{p})[x] \). The split case corresponds to \( p(x) = f_1(x) \cdot f_2(x) \). The prime spots \( \mathfrak{p}_i \) dividing \( \mathfrak{p} \) are determined by \( F \)-monomorphisms \( \gamma: E \rightarrow L \), where \( L \) is an algebraic closure of \( \tilde{F}(\mathfrak{p}) \). The \( - \) involution interchanges the prime spots, hence \( PO(E) = \mathfrak{p} \mathfrak{p} \).

\[ [E(\mathfrak{p}) : \tilde{F}(\mathfrak{p})] = 1 \]

is the local degree.

1.2 **Inert** \( e = 1 \) \( f = 2 \). In this case \( PO(E) = \mathfrak{p} \) a prime in \( O(E) \). \( \mathfrak{p} = \mathfrak{p} \), and we say \( \mathfrak{p} \) remains prime, or is inert.

1.3 **Ramified** \( e = 2 \) \( f = 1 \) \( PO(E) = \mathfrak{p}^2, \mathfrak{p} = \mathfrak{p} \) in this case also. We say \( \mathfrak{p} \) ramifies.
In both the inert and ramified cases, \( p(x) = p\sqrt{\sigma}/F(x) \) is irreducible in \( F(p)[x] \), and the local degree \([\tilde{E}(\sigma) : F(p)]\) equals 2.

This describes the situation for finite primes.

We next consider all embeddings \( \tau : F \rightarrow \mathbb{C} \), where \( \mathbb{C} \) is the complex numbers. If \( \tau : F \rightarrow \mathbb{R} \) we call \( \tau \) a real infinite prime. Otherwise \( \tau \) is called a complex infinite prime. We denote infinite primes by \( p_\infty \). Our only concern will be with real infinite primes.

Again, since \([E : F] = 2\), and the characteristic of these fields is 0 (not 2), we may write \( E = F(\sqrt{\sigma}) \), for \( \sigma \in F^* \) unique up to multiplication by a square in \( F^* \). For an infinite prime \( p_\infty \), there are two cases:

1.4 Split If \( p_\infty \) is complex infinite, \( p_\infty \) is split. If \( p_\infty \) is real infinite, and \( \sigma > 0 \) with respect to the ordering induced by \( p_\infty \), we again say \( p_\infty \) is split. In the case of a real split prime, \( p_\infty \), the ordering of \( F \) can be extended to \( E \) in two distinct ways.

1.5 Ramified If \( p_\infty \) is a real infinite prime, and \( \sigma < 0 \) with respect to the \( p_\infty \) induced ordering we say \( p_\infty \) is ramified. In this case, the \( p_\infty \) ordering of \( F \) can be uniquely extended to an embedding of \( E \) into \( \mathbb{C} \) in such a way that the imaginary part of \( \sqrt{\sigma} \) is positive.
Let \( \tau \) denote the extension of \( P_\infty \) to \( E \). Then \( \tau \) is equivariant with respect to complex conjugation. This means there is a commutative diagram:

\[
\begin{array}{ccc}
E & \xrightarrow{\tau} & C \\
\downarrow & & \downarrow \\
E & \xrightarrow{\tau} & C
\end{array}
\]

The map \( \tau : E \to E \) is the involution, \( \tau : C \to C \) is complex conjugation. There should be no confusion.

Associated to a finite prime \( P \) in \( O(F) \), or \( \varphi \) in \( O(E) \), is a discrete, non-Archimedean valuation \( | \cdot |_P \), respectively \( | \cdot |_{\varphi} \). \( \varphi \) lies over \( P \) if and only if \( | \cdot |_{\varphi} \) extends \( | \cdot |_P \).

Here \( | \cdot |_P \) is a homomorphism \( | \cdot |_{\varphi} : E^* \to \text{cyclic, multiplicative subgroup of } \mathbb{P}^+ \). We may form \( O_F(P) \) and \( O_E(\varphi) \), the local rings of integers associated to the finite primes \( P \) and \( \varphi \). Here

\[
O_E(\varphi) = \{ w \in E : |w|_{\varphi} \leq 1 \} .
\]

We also have

\[
O(E) = \bigcap_{\varphi} O_E(\varphi) .
\]

In the local ring of integers \( O_E(\varphi) \), associated to a prime \( \varphi \) there is a unique maximal ideal \( m(\varphi) = \{ w \in E : |w|_{\varphi} < 1 \} \). \( m(\varphi) \) is generated by \( \pi \), for some prime element \( \pi \in O_E(\varphi) \). Any such \( \pi \) genera-
ting \( m(\varphi) \) is called a local uniformizer. Two such \( \pi \) clearly have as quotient a local unit.

It is useful to think of the exponential version of \( |x|_\varphi \). Following [B,S 23], we denote this by \( v_\varphi \).

\( v_\varphi : E^* \to \mathbb{Z} \) is given by:

\[
v_\varphi(x) = n \ \text{means} \ |x|_\varphi = |\pi|^n.
\]

In terms of \( v_\varphi \), \( m(\varphi) = \{w \in E : v_\varphi(w) > 0\} \). If \( y \in F^* \),

\( v_\varphi(y) \) is the exponent to which the prime \( \varphi \) is raised in the factorization of the fractional ideal \( y \mathcal{O}(E) \).

If \( P \mathcal{O}(E) = \varphi \pi \) is split, then a local uniformizer \( \pi_P \in \mathcal{O}_F(P) \) is also a local uniformizer for both \( \mathcal{O}_E(\varphi) \) and \( \mathcal{O}_E(\varphi) \). Careful, this does not mean \( \varphi \) and \( \varphi \) induce the same valuation. On the contrary, \( \mathcal{O}_E(\varphi) \neq \mathcal{O}_E(\varphi) \). It only says \((\pi)\) in \( \mathcal{O}_E(\varphi) \) is the unique maximal ideal.

If \( P \mathcal{O}(E) = \varphi \) is inert, then a local uniformizer \( \pi_P \) for \( \mathcal{O}_F(P) \) is a local uniformizer for \( \mathcal{O}_E(\varphi) \) also.

If \( P \mathcal{O}(E) = \varphi^2 \) is ramified, then any local uniformizer \( \pi \) of \( \mathcal{O}_E(\varphi) \) will have norm \( \pi \varpi \), and \( \pi \varpi \) is a local uniformizer for \( \mathcal{O}_F(P) \). This follows since \( \varphi = \varphi \), hence \( \pi \) and \( \varpi \) are both local uniformizers for \( \mathcal{O}_E(\varphi) \).

So \( v_\varphi(\pi \varpi) = 2 = v_\varphi(\pi_P) \). Thus \( \pi \varpi \)
is a local uniformizer for $O_F(P)$, when $P$ ramifies.

For $y \in F^*$, we summarize:

(1) If $P$ splits, $P_0(E) = \vartheta$. $v_p(y) = v_{\vartheta}(y)$

(2) If $P$ is inert $P_0(E) = \vartheta$ $v_p(y) = v_{\vartheta}(y)$

(3) If $P$ ramifies $P_0(E) = \vartheta^2$ $2v_p(y) = v_{\vartheta}(y)$.

This is not true for $y \in E^*$.

Associated to a prime $\vartheta$, finite or infinite, lying over a prime $P$, is the extension of localized completions. The degree $[\widetilde{E}(\vartheta) : \widetilde{F}(\vartheta)]$ is denoted $n_{\vartheta}$. $n_{\vartheta} = 1$ if $\vartheta$ is split and 2 otherwise.
2. Hilbert symbols

We begin by recalling the theorem of Hasse.

**Theorem 2.1** Let $y \in F^*$. Then $y$ is a norm from $E^*$ if and only if $y \in \tilde{F}(\mathfrak{p})^*$ is a norm from $\tilde{F}(\mathfrak{p})^*$, for all $\mathfrak{p}$, finite and infinite in $E$. [0'M 186]

This condition is trivial over $P$ split, for then $\tilde{F}(P) = \tilde{E}(\mathfrak{p})$.

We should like to rephrase this in terms of Hilbert symbols. [0'M 169] We now state briefly the salient properties of these symbols.

If $a, \sigma \in F^*$, a symbol $(a, \sigma)_p$ is defined by:

$$(a, \sigma)_p = +1$$

if and only if $a$ is a norm from $\tilde{F}(P)(\mathcal{O})$ if and only if there exists $x, y \in \tilde{F}(P)$ satisfying:

$$ax^2 + \sigma y^2 = +1.$$

In terms of the prime ideals, we summarize.

2.2 If $P$ splits: $(a, \sigma)_P = +1$

2.3 If $P$ is inert: The local degree

$$n_\mathfrak{p} = [\tilde{E}(\mathfrak{p}) : \tilde{F}(P)] = 2.$$

By [0'M 169], every local unit is a local norm, and the local uniformizer $\tau \in \tilde{F}(P)$ is not a local norm.

In terms of Hilbert symbols, for $a \in F^*$, we have:
Let \( a = \pi^n v \).
\( \pi \) a local uniformizer
\( v \) a local unit

\[
(a, \sigma)_p = (\pi, \sigma)_p^n (v, \sigma)_p \quad \text{by properties to be listed}
\]

\[
= (\pi, \sigma)_p^n
\]

\[
= (-1)^n = (-1)^{p(a)}.
\]

2.4 If \( P \) is ramified: Again the local degree is 2. As we have seen, in this case we may pick a local uniformizer \( \pi_P \) of \( p \) to be a local norm, namely

\[
\pi_P = \pi_\varphi \pi_\varphi', \text{ where } \pi_\varphi \text{ is a uniformizer for } E(\varphi).
\]

We thus study the local units.

The residue field, \( \mathcal{O}_P(P)/m(P) \cong \mathcal{O}(F)/P \) is isomorphic to the completion \( \mathcal{O}_P(P)/\overline{m}(P) \). If \( u \) is a local unit, then for the following we denote by \( u_\parallel \) the image of \( u \) in the residue field, \( \mathcal{O}_P(P)/m(P) \).

**Claim:** For \( p \) ramified, a local unit \( u \) is a local norm, i.e. \( (u, \sigma)_p = +1 \) if and only if \( u_\parallel \) is a square in the residue field. (characteristic \( \neq 2 \)).

**Proof:** If \( u_\parallel \) is a square in the residue field, we may factor the polynomial \( t^2 - u_\parallel = f(t) \) in the residue field as \( (t + \sqrt{u_\parallel})(t - \sqrt{u_\parallel}) \).

We are assuming the characteristic of the residue field is not 2, so these two factors are relatively prime. Hence we may apply Hensel's Lemma, and conclude
that $t^2 - u$ factors in the completion $\bar{F}(P)$. Thus $u$ has a square root in $F(P)$, and $(u, \sigma)_P = +1$.

Conversely if $(u, \sigma)_P = +1$, then $u$ is a norm from $F(\mathfrak{p})$, say $x^2 = u$. We write $x$ as $x = w\pi^r$, for $w$ a local unit, $\pi$ a local uniformizer. Then $x^2 = w\pi^r = u$. Since $u$ is a local unit, $r = 0$. Thus $u$ is a square in $\bar{O}_E(\mathfrak{p})/\mathfrak{m}(\mathfrak{p})$, since the induced involution is trivial there.

However, $\bar{O}_E(\mathfrak{p})/\mathfrak{m}(\mathfrak{p}) = O_E(\mathfrak{p})/\mathfrak{m}(\mathfrak{p})$, so that $u_1$ is a square in the residue field.

2.5 If $P_\infty$ is infinite ramified: $(a, \tau)_\infty = -1$ if and only if $a < 0$ and $\tau < 0$. This is clear, as the completion with respect to $P_\infty$ is $R$.

We restate the Theorem of Hasse in terms of symbols.

**Theorem 2.6** $y \in F^*$ is a norm from $E^*$ if and only if $(y, \sigma)_P = +1$ at all primes $P$, finite and infinite.

We also list the important properties of the Hilbert symbols, in addition to the discussion above.

2.7 $(a, \sigma)_P = +1$ for almost all $P$ since at almost all $P$, $a$ and $\sigma$ are both units, [0'M 166]. Almost all means all but finitely many in this case.
2.8 **Hilbert Reciprocity**  \( \prod_{p} (a, \sigma)_p = +1 \)

2.9 **Realization:** If \( \epsilon(p) \in \mathbb{Z}^* \) is a function defined for all \( p \) satisfying:

1. \( \epsilon(p) = +1 \) if \( p \) splits
2. \( \epsilon(p) = +1 \) at almost all primes
3. \( \prod_{p} \epsilon(p) = +1 \)

then there is an \( f \in F^* \) with \( (f, \sigma)_p = \epsilon(p) \).

\[ \mathbb{Z}^* = \{ \pm 1 \} = \text{units in } \mathbb{Z} \]  [O'M 203]

**Note:** At non-split primes, \( n_{\sigma} = 2 \), and \( \sigma \) is not a square in \( \bar{F}(p) \).
Chapter VI: WITT INVARIANTS

We begin by considering \((M, B)\) a u Hermitian inner product space over a field \(E\) with involution \(-\) and fixed field \(F\). \(B: M \times M \to E\) satisfies:

\[ B(x, dy) = \overline{uB(dy, x)} = \overline{dB(x, y)} \quad \text{for} \quad d \in E, \ x, y \in M. \]

In Section 1, we discuss the rank mod 2 of \(M\) as a Witt group invariant. Next, we introduce the discriminant invariant, the Witt analog of the determinant for matrices. Thus, we review the matrix representation of \(B\) and diagonalization in order to define this invariant. Section 3 is concerned with the signature invariants, which arise from the real infinite ramified primes.

These invariants completely determine the Hermitian group \(H^+(E)\) for \(E\) an algebraic number field by Landherr's Theorem.

Notation: \(F_2 = \{0,1\} = \text{additive group of } \mathbb{Z} \text{ modulo 2.} \)

\((\text{field with two elements})\)

\(\mathbb{Z}^* = \{1,-1\} = \text{multiplicative group of units in } \mathbb{Z}. \)
1. Rank

Let \([M,B] \in H_u(E)\). We define the rank mod 2 of \([M,B]\), denoted \(\text{rk}[M,B]\), by:

\[
\text{rk}[M,B] = 0 \text{ if } [M:E] \text{ is even.}
\]
\[
= 1 \text{ if } [M,E] \text{ is odd.}
\]

Here \([M,E]\) is the rank of the vector space \(M\) over \(E\).

**Theorem 4.1** \(\text{rk}: H_u(E) \to F_2\) is a well-defined group homomorphism.

**Proof:** First we must show \(\text{rk}\) is well-defined. So let \([M,B] \in H_u(E)\) have \([M,B] = 0\). Then there is a metabolizer \(N \subset M\) with \(N = N^\perp\). This yields the exact sequence:

\[
\begin{align*}
0 & \to N^\perp \to M \to \text{Ad}_B^R \text{Hom}_E(N,E) \to 0 \\
\end{align*}
\]

Hence \(\text{rank } M = \text{rank } N^\perp + \text{rank}(\text{Hom}_E(N,E))\)

\[
= \text{rank } N^\perp + \text{rank } N
\]
\[
= \text{rank } N + \text{rank } N
\]
\[
= 2\text{rank } N.
\]

Thus \([M,B] = 0\) implies \(\text{rk}[M,B] = 0\). It follows that \(\text{rk}\) is well-defined. Clearly this is a group homomorphism.

**Caution:** \(\text{rk}\) is not a ring homomorphism.

**Corollary 4.2** \(\text{rk}: H_u(I) \to F_2\) defined as above is a well-defined group homomorphism. Here \(H_u(I)\) denotes
$\mathbb{I}$-valued non-Hermitian inner products on torsion free $\mathcal{D}$-modules.

**Proof:** Apply $I \ 4.4$. □
2. Diagonalization and the discriminant

In order to discuss the discriminant, we must establish some notation. We first pick a fixed basis, \( \{ e_1, \ldots, e_n \} \) for \( M \). Thus, if \( x \in M \), we write \( x = (a_1, \ldots, a_n) \) to mean \( x = \sum_{i=1}^{n} a_i e_i \), \( a_i \in E \).

Associated to the inner product \( B : M \times M \to E \), there is the matrix \( B' = (b_{ij}) \), where \( b_{ij} = B(e_i, e_j) \). If \( x = (a_1, \ldots, a_n) \), and \( y = (b_1, \ldots, b_n) \), then in terms of \( B' \) we have \( (a_1, \ldots, a_n)B'(\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}) = B(x, y) \); which we also write as \( B(x, y) = xB'y^t \). This follows since \( B \) is linear in the first variable and conjugate linear in the second.

Now \( [M, B] \in \text{H}_u(E) \). Thus \( B(e_i, e_j) = uB(e_j, e_i) \) so that \( b_{ij} = ub_{ji} \). It follows that \( B' \) satisfies \( B' = uB'^t \).

We now let \( \{ e_i^\# \}_{i=1}^{n} \) denote the dual basis to \( \{ e_i \} \).

\( e_i^\# : M \to E \) is defined on a basis of \( M \) by \( e_i^\#(e_j) = \delta_{ij} \), the Kronecker \( \delta \), and extended linearly to \( M \).

We consider the adjoint map of \( B \), \( \text{Ad}_R B : M \to \text{Hom}_E(M, E) \).

\( \text{Ad}_R B : e_i \to B(-, e_i) \). We express \( (\text{Ad}_R B)(e_i) \) as a linear combination of the \( \{ e_j^\# \} \). This yields

\[
(\text{Ad}_R B)e_i = \sum_{j=1}^{n} B(e_j, e_i)e_j^\#.
\]

We thus see that the matrix of the adjoint transformation, in terms of the bases \( \{ e_i \}, \{ e_i^\# \} \) is none other than \( B' = (B(e_i, e_j)) \).

We can thus state:
Proposition 2.1  Given a bilinear map $B: M \times M \rightarrow E$, the adjoint $Ad_R B: M \rightarrow \text{Hom}_E(M, E)$ is an isomorphism if and only if $(B(e_i^#, e_j^#))$ is an invertible matrix. □

Next, we wish to relate $u$ Hermitian to $1$ Hermitian.

**Proposition 2.2** $H_u(E) \cong H_1(E)$

**Proof:** Since $u$ satisfies $uu = 1$, by Hilbert's Theorem 90, we can find $x_1 \in E$ with $x_1^{-1} = u$. Then clearly $x_1$ yields an isomorphism: $H_u(E) \cong H_1(E)$, merely by scaling the inner product with $x_1^{-1}$. In other words,

$[M,B] \in H_u(E) \rightarrow [M,B_1] \in H_1(E)$

where $B_1(x,y) = (1/x_1)B(x,y)$. We must check $B_1$ is $1$ Hermitian.

$B_1(x,y) = (1/x_1)B(x,y) = (u/x_1)\overline{B(y,x)} = (1/x_1)\overline{B(y,x)} = B_1(y,x)$.

Conversely, if $[M,B_1] \in H_1(E)$, we must check that $B$ is $u$ Hermitian, where $B(x,y) = x_1B_1(x,y)$. We compute:

$B(x,y) = x_1B_1(x,y) = ux_1B_1(x,y) = \overline{ux_1B_1(y,x)} = \overline{u(x_1B_1(y,x))} = uB(x,y)$. □

Remark: We could choose $x_1 = 1 + u$.

When the characteristic of $E$ is not 2, we shall see that it is possible to choose a basis for $M$ so that the matrix $B'$ of $B$ is diagonalized. We prove this first for $[M,B] \in H_1(E)$. For $[M,B] \in H_u(E)$, we apply Proposition 2.2 and the above, observing that the isomorphism given in 2.2 preserves diagonalization.
Now let \([M,B] \in H(E)\). By the trace lemma, \(B_1\) defined on \(M\) by \(B_1(x,y) = \text{tr}_{E/F} \circ B(x,y)\) is a non-singular symmetric bilinear form on \(M\). Here \(\text{tr}_{E/F}\) denotes the trace map.

Since \(B_1(x,y) = (1/2)(B_1(x+y,x+y) - B_1(x,x) - B_1(y,y))\), and \(B_1 \neq 0\), it follows that there exists \(v \in M\) with \(B_1(v,v) \neq 0\). It follows that \(B(v,v) \neq 0\) also. Extend \(v\) to a basis of \(M\), \(\{v, v_2, \ldots, v_n\}\). Notice that

\[
\{v, v_2 - (B(v_2,v)/B(v,v))v, \ldots, v_n - (B(v_n,v)/B(v,v))v\}
\]

= \(\{w_i\}\) is also a basis for \(M\). The computations:

\[
B(v_i - (B(v_i,v)/B(v,v))v, v) = B(v_i, v) - B(v_i, v) = 0
= B(v, v_i - (B(v_i,v)/B(v,v))v)
\]

show that with respect to \(\{w_i\}\), the matrix of \(B\) looks like:

\[
\begin{pmatrix}
B(v,v) & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots
\end{pmatrix}
\]

Continuing, consider \(B_1(x,y)\), for \(y\) in the span of \(\{w_2, \ldots, w_n\}\), \(x \in M\). Again, since \(B_1\) is non-singular, we can find \(x\) with \(0 \neq B_1(x,y) = (1/2)(B_1(x+y,x+y) - B_1(x,x) - B_1(y,y))\). Write \(x = \sum_{i=1}^{n} a_i w_i\). Then it is clear that:

\[
B_1(x,y) = B_1(\sum_{i=2}^{n} a_i w_i, y) + B_1(a_1 w_1, y) = B_1(\sum_{i=2}^{n} a_i w_i, y).
\]

Thus, we can find \(v \in \langle w_2, \ldots, w_n \rangle\) with \(B(v,v) \neq 0\). Continuing inductively, we form \(\{w_1, v, \ldots\}\), and diagonalize \(B\). We may thus state:
Proposition 2.3  Given \([V,B] \in H^1(E)\), there is a basis for \(V\) which makes the matrix of \(B\) diagonal. (characteristic \(E \neq 2\))

Remark 2.4  This also holds for \(H_u(E)\) by applying 2.2.

Remark 2.5  As long as the involution \(-\), on \(E\) is non-trivial, we may prove 2.3 directly even if the characteristic of \(E\) is 2. In order to see this, we must show how to produce a vector \(v\) with \(B(v,v) \neq 0\). Suppose to the contrary that \(B(v,v) = 0\) for all \(v \in M\). Assuming that \(B\) is non-singular, so not identically 0, we can find \(v,w \in M\) with \(B(v,w) \neq 0\). However \(B(v + w,v + w) = 0 = B(v,w) + B(w,v)\).

Thus \(B(v,w) = -B(w,v)\). Hence, for any \(a \in E\),

\[
aB(v,w) = B(av,w) = -B(w,av) = -\bar{a}B(w,v).
\]

Since \(B(v,w) \neq 0\), \(a = \bar{a}\) for all \(a \in E\), and the involution on \(E\) is trivial. Contradiction.

Once we have such a vector \(v\), we proceed as in 2.3 to produce an orthogonal basis.

Remark 2.6  Thus, we see that 2.3 holds for \(H_u(E)\), provided we are not in the situation of a trivial involution or a field of characteristic 2. For \([M,B] \in W_u(E)\), where the characteristic of \(E\) is 2, we may write \(B\) as a direct sum of 1-dimensional forms and metabolic forms, \[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix},
\]

see [K-1 22].
Diagonalizing an inner product space \((M,B)\) means choosing a basis of \(M\) with respect to which the matrix of \(B\) is diagonal. In other words, \((M,B) \cong (M_1,B_1)\), where \(M_1\) is a 1-dimensional vector space. It is natural then to compare the matrices of \(B\) given by different choices of bases for \(M\).

Suppose that \(\{e_1,\ldots,e_n\}\) and \(\{f_1,\ldots,f_n\}\) are two bases of \(M\). We write \(E = \text{matrix of } B\) with respect to \(\{e_i\}\), and \(F = \text{matrix of } B\) with respect to \(\{f_j\}\).

We may express \(\{e_i\}\) in terms of \(\{f_j\}\). Suppose 
\[
e_i = \sum_{j=1}^{n} c_{ij} f_j,
\]
and let \(C = (c_{ij})\), \(C^t = \text{transpose of } C\).

**Proposition 2.7** \(E = CFC^t\)

**Proof:** In terms of \(\{e_i\}\), write 
\[
e_i = (0,\ldots,1,\ldots,0),
\]
e_j = \((0,\ldots,1,\ldots,0)\), and 
\[
e_i E e_j = e_i j = B(e_i,e_j).
\]
The \(ij\) component of \(CFC^t\) is likewise given by:
\[
(0,\ldots,1,\ldots,0) CF C^t \begin{pmatrix}
i \\
ingen \\
1 \\
0
\end{pmatrix}
\]
\[
= (c_{i1},\ldots,c_{in}) F \begin{pmatrix}
c_{j1} \\
\vdots \\
c_{jn}
\end{pmatrix}
\]
\[
= B(c_{i1} f_1 + \cdots + c_{in} f_n, c_{j1} f_1 + \cdots + c_{jn} f_n)
\]
\[
= B(e_i,e_j) = e_{ij} \text{ as above.} \quad \Box
\]
We would now like a Witt group invariant corresponding to the determinant of a matrix. This invariant should be independent of the choice of basis, as well as the Witt representative of the given Witt equivalence class.

Let \([M,B] \in H_1(E)\). Let \(B_1\) and \(B_2\) denote two different matrices of \(B\). By 2.7 we can write \(B_1 = CB_2C^t\), for a non-singular matrix \(C\). Let \(\det B_1\) denote the determinant of \(B_1\). Then \(\det B_1 = \det C \det B_2 \det C^t = \det C \det B_2 (\det C)\).

Thus, we can read the determinant of \(B\) in \(F^*/NE^*\), since when \(B_1\) is diagonalized, the diagonal elements must be in \(F^*\) as \(B\) is Hermitian. Unfortunately, this is not a Witt invariant. For example, \(\det B\) need not be in \(NE^*\) even when \([V,B] = 0\). Let \(B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\). \(\det B = -1\), which may not be a norm.

We are thus led to define a corresponding notion:

**Definition 2.8** Let \(B_1\) be a matrix with coefficients in \(E\), corresponding to a Hermitian form \(B_1\). Then \(\text{dis } B = (-1)^{n(n - 1)/2} \det B_1\), where \(n\) is the dimension of \(M\), is called the discriminant of the inner product space \((M,B)\).

**Lemma 2.9** If \([M,B] = 0\) then \(\text{dis } B \in NE^*\).
**Proof:** Let $N$ be a metabolizer for $M$. Let $\{n_1, \ldots, n_t\}$ be a basis for $N$. Extend this to a basis for $M$, say $\{n_1, \ldots, n_t, n_t + 1, \ldots, n_{2t}\}$. With respect to this basis, $B$ has matrix \[
abla \begin{pmatrix} 0 & C \\ \bar{C} & X \end{pmatrix}.
\]
Interchanging the first $t$ columns with the last $t$ columns, we obtain a matrix \[
abla \begin{pmatrix} C & 0 \\ X & \bar{C} \end{pmatrix}.
\]
This requires interchanging $t^2$ columns. Hence $B$ has
\[
\det B = (-1)^{t^2} \det C \cdot \det \bar{C} = (-1)^t \det C \cdot \det \bar{C}.
\]
Thus,
\[
\text{dis } B = (-1)^{2t(2t - 1)/2} (-1)^t \det C \cdot \det \bar{C}
\]
\[= (-1)^t + t(2t - 1) \det C \cdot \det \bar{C}
\]
\[= (-1)^t + t \det C \cdot \det \bar{C}
\]
\[= \det C \cdot \det \bar{C} \in \text{NE}^* \text{ as claimed.} \quad \square
\]

It follows that dis is exactly the kind of invariant we seek. There is still a problem, namely dis is not additive. Hence, we do not obtain a group homomorphism:
\[
H_1(E) \to \text{F}^*/\text{NE}^*.
\]
To remedy this problem, we invent the group
\[
Q(E) = \text{F}^*/\text{NE}^* \times F_2 \quad \text{F}_2 = \{0, 1\}
\]
[LM 38]. The binary operation in $Q(E)$ is given by:
\[
(d_1, e_1) \cdot (d_2, e_2) = ((-1)^{e_1e_2} d_1 d_2, e_1 + e_2)
\]
This is an associative, commutative operation with $(1, 0)$ the identity. The inverse of $(d, e)$ is $((-1)^e d, e)$. 

Proposition 2.10  The map \( \varphi: H_1(E) \to \mathbb{Q}(E) \) defined by \([M,B] \mapsto \det B \cdot \text{rk} M\) is a group homomorphism.

Proof: Consider \([M,B]\) and \([W,B']\) in \(H_1(E)\). Suppose that: \(\text{rank } M = n\) and \(\text{rank } W = m\).

We have then:

\[
\varphi([M,B]) = ((-1)^n(n - 1)/2 \det B, n)
\]
\[
\varphi([W,B']) = ((-1)^m(m - 1)/2 \det B', m)
\]
\[
\varphi([M,B]) \cdot \varphi([W,B']) = ((-1)^{nm}(-1)^n(n - 1)/2(-1)^{m(m - 1)/2} \det B \det B', n + m)
\]
\[
\varphi([V,B] \oplus [W,B']) = ((-1)^{(n + m)(n + m - 1)/2} \det B \det B', n + m)
\]

But

\[
(-1)^{nm} + n(n - 1)/2 + m(m - 1)/2 = (-1)^{(n^2 - n + m^2 - m + 2nm)/2}
\]
\[
= (-1)^{(n+m)^2 - (n+m)/2}
\]
\[
= (-1)^{(n+m)(n+m-1)/2}
\]

Thus \(\varphi\) indeed gives a well-defined group homomorphism. \(\square\)

Now consider the kernel of \(\text{rk}\), call this \(J\). So \(J\) is the subgroup of \(H_1(E)\) generated by the even dimensional forms.

Proposition 2.11  \(H_1(E)/J^2\) is isomorphic to \(\mathbb{Q}(E)\).

Proof: \(J\) is additively generated by 2-dimensional forms, \(\langle 1, a \rangle\). To see this, write \(\langle a, b \rangle \sim \langle 1, a \rangle - \langle 1, -b \rangle\).

Thus \(J^2\) is additively generated by the
forms:

\[ \langle 1, a \rangle \otimes \langle 1, b \rangle = \langle 1, a, b, ab \rangle. \]

Applying \( \varphi \) to a generator, we obtain:

\[ \varphi \langle 1, a, b, ab \rangle = ((-1)^0(ab)^2, 0) = (1, 0). \]

Thus \( \varphi \) induces a map \( \varphi : H_1(E)/J^2 \to Q(E) \). We now construct an inverse of \( \varphi \), \( \Upsilon \).

Define \( \Upsilon : Q(E) \to H_1(E)/J^2 \) by

\[
\begin{align*}
(a, 0) & \mapsto \langle 1, -a \rangle \mod J^2 \\
(a, 1) & \mapsto \langle a \rangle \mod J^2.
\end{align*}
\]

It is easy to check that \( \Upsilon \) is a homomorphism, and \( \Upsilon \circ \varphi = \text{id} \), \( \varphi \circ \Upsilon = \text{id} \), where \( \text{id} \) is the appropriate identity map. Proposition 2.11 implies that \( \varphi \) is 1-1. Hence,

**Corollary 2.12** \( J^2 \) consists of even dimensional forms \([M, B]\), with \( \text{dis } B = 1 \in F*/N^{*}, \) ie. \( \det B = (-1)^{n(n-1)/2} \)

where \( n = \text{rank } M \). □

**Corollary 2.13** Restricting \( \varphi \) to \( J/J^2 \), we have \( J/J^2 \approx F*/N^{*} \)

In fact, we may think of this as follows:

\( F*/N^{*} \) is embedded into \( Q(E) \) by: \( d \mapsto (d, 0) \). This is a subgroup of index 2. We may represent the non-identity coset by \( (1, 1) \). \( (1, 1)^2 = (-1, 0) \). Thus, we have the exact sequence:

\[
q_2
1 \to F*/N^{*} \to Q(E) \to F_2 \to 0
\]

\( q_2 \) is projection onto the second factor. By the above
Corollary 2.14 This sequence splits if and only if
\[(1,0) = (-1,0) \text{ in } Q(E) \text{ if and only if } -1 \text{ is a norm in } F*/N(E)^*.
\]

Remark 2.15 This defines the discriminant for \(H_1(E)\).

In order to define discriminant for \(H_u(E)\), we fix an isomorphism:

\[f_{x_1} : H_u(E) \rightarrow H_1(E), \text{ by Proposition 2.2.}\]

\[B \rightarrow B_1 \quad B_1(x,y) = (1/x_1)B(x,y).\]

Then define \(\text{dis } B = \text{dis } f_{x_1} B = \text{dis } B_1\). We must note that this depends on the isomorphism chosen, i.e. this depends on \(x_1\), where \(x_1x_1^{-1} = u\).

Remark 2.16 The discriminant inner product space, Chapter I 2.5, yields the information crucial to the discriminant invariant above, namely the determinant of \(B\). Its advantage is that it generalizes the notion to \(H(D)\), for \(D\) the Dedekind ring of integers.
3. **Signatures**

The real infinite primes, \( P_\infty \), give rise to the signature invariant which we now discuss. Suppose then that \( E = F(\sqrt{\sigma}) \), and \( \sigma < 0 \) with respect to \( P_\infty \). Thus \( P_\infty \) is an infinite ramified prime.

**Lemma 3.1** If \( x \in E \), then \( N_{E/F}(x) > 0 \) with respect to \( P_\infty \).

**Proof:** \( N_{E/F} \) denotes the norm. Write \( x = a + b/\sigma \).

Then \( N(x) = a^2 - b^2/\sigma > 0 \) since \( \sigma < 0 \).

Let \([M,B] \in H_1(E)\). By 2.3, we can find a basis \( \{e_i\} \) of \( M \) in which \( B \) is diagonalized. We can thus write \( M = X^+ \oplus X^- \), where \( B(e_i,e_i) > 0 \) for \( e_i \in X^+ \), \( B(e_i,e_i) < 0 \) for \( e_i \in X^- \).

Now, let \( v \in X^+ \), so \( v = \sum a_i e_i \). We compute

\[
B(v,v) = B(\sum a_i e_i, \sum a_i e_i) = \sum_i B(a_i e_i, a_i e_i) = \sum_i a_i \bar{a}_i B(e_i, e_i) = \sum_i N(a_i) B(e_i, e_i) > 0
\]

by Lemma 3.1.

Similarly, for all \( v \in X^- \), \( B(v,v) < 0 \). We now define:

\[
\text{sgn}[M,B] = \dim X^+ - \dim X^-.
\]

\( \text{sgn}[M,B] \) is called the signature of \([M,B]\). In order to show \( \text{sgn} \) is well-defined, we first need:
Lemma 3.2 \textit{sgn}[M,B] is independent of the basis chosen for M.}

\textbf{Proof:} Suppose M has two bases, \(\{e_i\}, \{f_i\}\) which make B diagonal. Let \(B_1, B_2\) be the matrices of B with respect to \(\{e_i\}, \{f_i\}\).

Consider \([M,B_1] - [M,B_2]\) which is Witt equivalent to 0. With respect to the basis \(\{e_i,f_i\}\) of \(M \oplus M\), \(B_1 \oplus -B_2\) has matrix
\[
\begin{pmatrix}
B_1 & 0 \\
0 & -B_2
\end{pmatrix}
\]
It follows that
\[
\text{sgn} [M \oplus M,B_1 \oplus -B_2] = \text{sgn} B_1 - \text{sgn} B_2.
\]
Thus in order to show \(\text{sgn} B_1 = \text{sgn} B_2\), it clearly is sufficient to show:
Any metabolic space \([V,h]\) has \(\text{sgn}[V,h] = 0\) with respect to an arbitrary basis.

So suppose \(V = X^+ \oplus X^-\). Let N be a metabolizer for V, \(N = N^+\). We note that \(n \in N\) implies \(h(n,n) = 0\). Thus, by the remarks preceding this theorem, \(X^+ \cap N = 0 = X^- \cap N\).

However, \(X^+ \cap N = 0\) implies \(\dim N \leq \dim V - \dim X^+ = \dim X^-\). Similarly, \(\dim N \leq \dim X^+\). Now \(2\dim N = \dim V\), so that \(\dim X^-\) and \(\dim X^+\) are both \(\geq (1/2)\dim V\). However,
\[
\dim X^+ + \dim X^- = \dim V.
\]
Thus, \(\dim X^+ = \dim X^- = (1/2)\dim V\), and \(\text{sgn}[V,h] = 0\) with respect to any basis. \(\Box\)

In the process of this proof, we have shown:
Corollary 3.3  If \([V,h]\) is metabolic, \(\text{sgn}[V,h] = 0\). □

It is thus clear that sgn gives a well-defined Witt-invariant, which is a group homomorphism:

\[
\text{sgn}: H_1(E) \rightarrow \mathbb{Z}.
\]

It is clear that if \([M,B]\) has finite order, \(\text{sgn}[M,B] = 0\), since every element in \(\mathbb{Z}\) has infinite order. Thus sgn is non-trivial only on the non-torsion elements in \(H_1(E)\).

We finally recall Landherr's Theorem which explicitly computes \(H_1(E)\), for \(E\) an algebraic number field, \([Lh]\).

\text{Landherr's Theorem 3.4}  There is an exact sequence:

\[
0 \rightarrow (\frac{1}{4} \mathbb{Z})^r \rightarrow H(E) \rightarrow \mathbb{Q}(E) \rightarrow 0
\]

\(\varphi[M,B] = (\text{dis } B, \text{rk } V)\).

The kernel of \(\varphi\), \(\ker \varphi\), is determined by the real infinite ramified primes and the corresponding signatures, each of which is divisible by \(\frac{1}{4}\). Here \(r\) is the number of real infinite primes. When \(r = 0\), \(\varphi\) is an isomorphism.

This theorem is important in the boundary computation in Chapter VIII.

By Proposition 2.10, \(H(E)/J^2 \approx \mathbb{Q}(\mathbb{Z})\), so we can state:

Corollary 3.5  \(J^2 \approx \frac{1}{4}(\mathbb{Z} \oplus \mathbb{Z} \cdots \oplus \mathbb{Z})\) \(r\) times

Remark 3.6  As with the discriminant, we can define a
signature invariant for $H_u(E)$. This is done by picking an isomorphism $f_{x_1} : H_u(E) \to H_1(E)$ as in 2.15. We then define $\text{sgn}(M,B) = \text{sgn}(f_{x_1}[M,B])$. 
Chapter VII  AN EXACT HEXAGON

We have been examining $H_u(K)$, where $u$ is a unit in $O(E)$ with $uu = 1$, and $K \subset E = F(\sqrt{a})$ is a fractional $O(E)$-ideal with $K = \overline{K}$. We begin this chapter by studying $H_u(K)$ for different $u$ and $K$. Thus an equivalence relation is defined on pairs $(u,K)$ as above, and the collection of equivalence classes is collected into a group, $\text{Iso}(E/F)$. The reason for the name is Theorem 1.5, which shows that if $(u,K) \sim (u_1,K_1)$, then there is a canonical isomorphism $H_u(K) \sim H_{u_1}(K_1)$.

The next major idea in this chapter is the study of the cohomology groups $H^*(C_2;O(E)^*)$ and $H^*(C_2;C(E))$, where $O(E)^*$ is the multiplicative group of units in $O(E)$, and $C(E)$ is the ideal class group. The action of $C_2$ is from the $-$ involution on $E$. These groups are defined and examined in Section 2.

In Section 3, a group of pairs $(y,A)$, where $y \in F^*$, $A \subset E$ is a fractional $O(E)$ ideal, and $yA\overline{A} = O(E)$ is invented. This group arises from the discriminant. The name Gen is given because of the manner in which such pairs give rise to a system of additive generators for $H(O(E))$.

All of these ideas are important to the study of the Witt ring. Our reason for including all of them in
this chapter is their interrelationship as given in the following Exact Hexagon.

\[
\begin{array}{ccc}
H^1(C_2; C(E)) & \xrightarrow{d_1} & H^1(C_2; O(E^*) \\
\downarrow j & & \downarrow \mu \\
\text{Gen} & & \text{Iso}(E/F) \\
i & \leftarrow & \eta \\
H^2(C_2; O(E^*)) & \xleftarrow{d_2} & H^2(C_2; C(E)) 
\end{array}
\]

The maps in this hexagon are given in Section 4, where exactness is proven.
1. \textit{Iso} (E/F)

We begin by studying the group \textit{Iso}(E/F). In order to define this group we consider pairs \((u, K)\), where \(u \in \mathcal{O}(E)^*\), the units in \(\mathcal{O}(E)\), with \(uu = 1\), \(K \subset E\) is a fractional \(\mathcal{O}(E)\)-ideal with \(K = K\).

\textbf{Definition 1.1} \textit{Iso}(E/F) consists of equivalence classes of such pairs \((u, K)\) under the relation:

\[(u, K) \sim (u_1, K_1) \iff \text{there exists } x \in E^* \text{ and a fractional } \mathcal{O}(E)\text{-ideal } A \text{ satisfying:}
\]

\begin{enumerate}
  \item \(xA\bar{\bar{A}}K = K_1\)
  \item \(x^{-1}u = u_1\)
\end{enumerate}

\textbf{Lemma 1.2} \(\sim\) \textit{is an equivalence relation}. □

We denote the equivalence class of \((u, K)\) by \(\vert u, K \vert\). We make \(\textit{Iso}(E/F)\) into a group by defining

\[\vert u, K \vert \cdot \vert u_1, K_1 \vert = \vert uu_1, KK_1 \vert.\]

With this multiplication, \(\vert 1, \mathcal{O}(E) \vert\) is the identity, and \(\vert u, K \vert^{-1} = \vert u^{-1}, K^{-1} \vert\). The following lemma is then clear:

\textbf{Lemma 1.3} The multiplication in \(\textit{Iso}\) is well-defined, and makes \(\textit{Iso}\) into an abelian group. □

The reason for the name \(\textit{Iso}\) becomes evident with:
Theorem 1.4: If \((u, K) \sim (u_1, K_1)\), then \(H_u(K)\) is canonically isomorphic to \(H_{u_1}(K_1)\).

**Proof:** Suppose \(x \AA K = K_1\) and \(xx^{-1}u = u_1\). Let 
\([M, B] \in H_u(K)\). Then define 
\([M_1, B_1] \in H_{u_1}(K_1)\) by:

\[M_1 = M \otimes_D A\]

\[B_1(n_0 \otimes a_0, n_1 \otimes a_1) = xa_0 \overline{a_1}B(n_0, n_1).\]

In other words, we first scale \(B_1\) by \(x\), then apply \(\otimes_D A\).

By I. 1.2, \(B_1\) is an inner product. \(B_1\) is \(u_1\) Hermitian since:

\[B_1[n_0 \otimes a_0, n_1 \otimes a_1] = a_0 \overline{a_1}xB[n_0, n_1] = a_0 \overline{a_1}xuB[n_1, n_0]\]

\[= a_0 \overline{a_1}xuB[n_1, n_0] = u_1x \overline{a_0}a_1B[n_1, n_0]\]

\[= u_1 \overline{B_1}[n_1 \otimes a_1, n_0 \otimes a_0]\]

We have thus constructed a homomorphism:

\[f : H_u(K) \rightarrow H_{u_1}(K_1)\] given by

\([M, B] \mapsto [M_1, B_1]\]

In constructing \(f\), we used \(x \AA K = K_1\), \(xx^{-1}u = u_1\).

However, we also have that

\[x^{-1}A^{-1}A^{-1}K_1 = K\] and \(x^{-1}x u_1 = u\)

so that we can similarly write down an inverse of \(f\).

Hence \(f\) is an isomorphism. \(\square\)

Next we compute the group \(\text{Iso}(E/F)\). Let \(\mathcal{O}(E)\) be the group of fractional \(O(E)\)-ideals, under multiplication. Let \(\mathcal{R}\) be the subgroup of \(\mathcal{O}(E)\) generated by prime ideals \(\mathfrak{p}\) over (finite) ramified primes. We form \(\mathcal{R}/\mathcal{R}^2\).
Theorem 1.5 If at least one prime, finite or infinite is ramified, \( \text{Iso}(E/F) \) is isomorphic to \( R/R^2 \).

Comment: If no finite prime ramifies, \( R = 0 \) by definition.

Proof: Define the map \( f: \text{Iso}(E/F) \to R/R^2 \) by

\[
|u,K| \rightarrow \prod_{\varphi = \overline{\varphi}} v_\varphi(z) + v_\varphi(K),
\]

where \( z \) is given by \( \varphi = \overline{\varphi} \) over ramified Hilbert 90 satisfying \( \overline{zz^{-1}} = u \), since \( uu = 1 \). \( v_\varphi \) arises from the \( \varphi \)-adic valuation on \( E \).

We must check that \( f \) is well-defined. First we show \( f \) preserves \( \sim \). ie. Let \( (u,K) \sim (u_1,K_1) \). Then we will show \( f(u,K) = f(u_1,K_1) \).

Since \( (u,K) \sim (u_1,K_1) \), we can find \( x, A \) with \( xA\overline{A}K = K_1 \) and \( xx^{-1}u = u_1 \). Suppose \( \overline{zz^{-1}} = u \). Then \( (xz)(xz)^{-1} = u_1 \). Also, \( xA\overline{A}K = K_1 \) implies

\[
v_\varphi(x) + v_\varphi(K) \equiv v_\varphi(K_1) \pmod{2} \quad \text{over ramified,}
\]

since \( v_\varphi(A) = v_\varphi(\overline{A}) = v_\varphi(\overline{A}) \) as \( \varphi = \overline{\varphi} \) is over ramified. ie. \( v_\varphi(x) + v_\varphi(K_1) \equiv v_\varphi(K) \pmod{2} \).

Thus \( v_\varphi(xz) + v_\varphi(K_1) \equiv v_\varphi(x) + v_\varphi(z) + v_\varphi(K_1) \pmod{R^2} \)

\[
= v_\varphi(z) + v_\varphi(K) \pmod{R^2}
\]

as desired.
Second, we claim \( f \) is independent of the choice of \( z \) with \( z^{-1} z = u \). So suppose \( z^{-1} y^{-1} = u \). Then \( z z = y z \), so that \( z y \) is fixed, hence in \( F \). Thus \( v_\varphi(z y) \equiv 0 \pmod{2} \) over ramified. So \( v_\varphi(z) \equiv v_\varphi(y) \equiv v_\varphi(y) \equiv v_\varphi(y) \pmod{2} \) for \( \varphi = \overline{\varphi} \) over ramified. It follows that \( f \) is well-defined.

In order to show \( f \) is an isomorphism, we must show:

(a) \( f \) is onto: Clearly \( \mathcal{O}/\mathcal{O}_E^2 \) is generated by ramified primes \( \varphi \). Such a prime \( \varphi \) is in the image of \( f \), since \( f|1,\varphi| = \varphi \). (b) \( f \) is 1-1. Suppose \( |u, K| \) is in the kernel of \( f \). Then \( v_\varphi(z) + v_\varphi(K) \equiv 0 \pmod{2} \) at all \( \varphi = \overline{\varphi} \) over ramified. By realization of Hilbert symbols, V 2.9, there exists \( y \in F^* \) with

\[
(y, \sigma)_p = (-1)^v_\varphi(z) + v_\varphi(K) = (-1)^v_\varphi(y)
\]

which are inert. We take care of realization by using \( (y, \sigma)_p = -1 \) at \( P \) ramified. Since \( y \in F^* \), \( y y^{-1} = y y^{-1} = 1 \).

Our object is to show that \( (1, O(E)) \sim (u, K) \). Thus, we need to produce \( x \in E^* \), \( A \) a fractional \( O(E) \)-ideal, with:

\[
xA = K \quad \text{and} \quad xx^{-1} = u. \quad \text{We let} \quad x = yz. \quad \text{Then} \quad xx^{-1} = yz y^{-1} z^{-1} = u.
\]

For \( \varphi \) over \( P \) inert,

\[
v_\varphi(x^{-1} K) = v_\varphi(x K) \equiv v_\varphi(y) + v_\varphi(z) + v_\varphi(K) \equiv 0 \pmod{2} \]

by our choice of \( y \).

For \( \varphi \) over \( P \) ramified,
\[ v(\phi^{-1}K) = v(\phi xK) = v(\phi y) + v(\phi z) + v(\phi K) = v(\phi z) + v(\phi K) \]

(since \( y \in F^* \), so that \( v(\phi y) = 0 \) when \( \phi \) is over ramified)

\[ \equiv 0 \pmod{2} \text{ since } (u,K) \in \text{ker } f. \]

If \( \phi \neq \tilde{\phi} \), \( \phi \) is over split, observe \( xx^{-1} = u \), and \( u \in O(E)^* \). Thus \( v(\phi xx^{-1}) = 0 \), so that

\[ v(\phi x) + v(\phi x) = 0, \text{ hence } v(\phi x) = v(\phi x) \pmod{2}. \]

Hence, for \( \phi \) over split,

\[ v(\phi xK) = v(\phi x) + v(\phi K) \]

\[ = v(\phi x) + v(\phi K) \]

\[ = v(\phi x) + v(\phi K) \]

\[ = v(\phi xK) \pmod{2}. \]

It follows that \( xK \), and \( x^{-1}K \) is \( \phi \)-invariant. The first two computations showed \( x^{-1}K \) has even order at inert and ramified primes. Thus, \( x^{-1}K \) factors as

\[ x^{-1}K = AA, \text{ or } xAA = K \text{ as desired. Hence} \]

\[ (1,0(E)) \sim (u,K). \]

Theorem 1.5 describes the situation when there is ramification. Next, we examine \( \text{Iso}(E/F) \) when there is no ramification at all, finite or infinite.

**Theorem 1.6**  If no prime finite or infinite is ramified then \( \text{Iso}(E/F) \cong C_2 \)

**Proof:** Let \( (u,K) \in \text{Iso}(E/F) \).

Claim: \( (u,K) \sim (1,0(E)) \) or \( (1,\phi_1) \) where \( \phi_1 \) is over inert.
Proof: Since $u \bar{u} = 1$, by Hilbert 90, there exists $x \in E$ such that $x\bar{x}^{-1} = u$.

Choose $y \in F$ by realization of Hilbert symbols as follows:

\[
(1) \quad (-1)^{v_\varphi(y)} \equiv (y, \sigma)_p \equiv (-1)^{v_\varphi(x) + v_\varphi(K)} \quad \text{(mod 2)}
\]

for $\varphi \neq \varphi_1$ over inert.

\[
(2) \quad (y, \sigma)_p = (-1) \text{ if needed, where } \varphi_1 \cap F = p_1.
\]

Now let $z = xy$. $(xy)(\bar{xy})^{-1} = u$.

Consider $xyK$. $v_\varphi(x) + v_\varphi(y) + v_\varphi(K) \equiv 0$ for $\varphi \neq \varphi_1$ by 1.

\[
v_{\varphi_1}(xyK) = v_\varphi(y) + v_{\varphi_1}(x) + v_{\varphi_1}(K)
\]

\[
\equiv \begin{cases} 
0 & \text{(a)} \\
1 & \text{(b)}
\end{cases}
\]

In case (a), $xyK$ has even valuation at all inerts.

$xy = u(\bar{xy})$, so $xyK = \bar{xy}K$, and $xyK$ is - invariant. Thus, since there are no ramified primes $(u, K) \sim (1, 0(E))$. In case (b), it is likewise evident that $(u, K) \sim (1, \varphi_1)$.

Finally, we must show that $(1, 0(E)) \not\sim (1, \varphi_1)$. But this would imply that there exists $x \in E^*$ with $x\bar{x}^{-1} = 1$, ie. $x = \bar{x}$ and $x \in F^*$; and $A$ a fractional $O(E)$-ideal with $xA\bar{A} = \varphi_1$.

But consider $x^{-1}\varphi_1$. Since $x^{-1}\varphi_1 = A\bar{A}$, $v_{\varphi_1}(x) \equiv 1 \text{ (mod 2)}$
and \( v_\varphi(x) = 0 \) for \( \varphi \neq \varphi_1 \) over inerts. But \( (x, \sigma)_p = (-1)^{v_\varphi(x)} \) at inerts. Thus \( \prod(-1)^{v_\varphi(x)} = \prod(x, \sigma)_p = -1 \). However, \( (x, \sigma)_p \) is non-trivial only over inerts, as there are no ramified primes. This contradicts \( \prod_{p \text{ inert}} (x, \sigma)_p = +1 \). Hence \( (1, \mathcal{O}(E)) \neq (1, \varphi_1) \) and \( \text{Iso}(E/F) = C_2 \) as claimed. \( \square \)

This completes the description of \( \text{Iso}(E/F) \). Next we examine the cohomology groups which arise in the hexagon.
2. **Cohomology groups**

E has an involution - . This gives rise to several cohomology groups induced by the \( C_2 \) action from - . We now examine these.

Let \( \mathcal{O} \) be a - invariant prime ideal in \( \mathcal{O}(E) \). Then the - involution makes the local units in \( \mathcal{O}_E(\mathcal{O}) \), denoted \( \mathcal{O}_E(\mathcal{O})^* \), into a \( C_2 \)-module. We examine the resulting cohomology groups, \( H^i(C_2;\mathcal{O}_E(\mathcal{O})) \), \( i = 1,2 \).

**Lemma 2.1** If \( \mathcal{O} = \bar{\mathcal{O}} \) is over inert, then
\[
H^1(C_2;\mathcal{O}_E(\mathcal{O})^*) = 1.
\]
If \( \mathcal{O} = \bar{\mathcal{O}} \) is over ramified,
\[
H^1(C_2;\mathcal{O}_E(\mathcal{O})^*) = C_2.
\]

**Proof:** Recall that \( H^1(C_2;\mathcal{O}_E(\mathcal{O})^*) = \{ x \in \mathcal{O}_E(\mathcal{O})^* : x\bar{x} = 1 \} \) modulo \( \{ v/\bar{v} : v \in \mathcal{O}_E(\mathcal{O})^* \} \).

If \( x \) is a local unit of norm 1, then by Hilbert 90, there exists \( z \in E^* \) with \( z\bar{z}^{-1} = x \).

Write \( z = \pi \varphi(z) \bar{v} \), where \( \pi \) is a local uniformizer for \( \mathcal{O} \), and \( v \in \mathcal{O}_E(\mathcal{O})^* \). If \( \mathcal{O} \) is over inert, we may choose \( \pi \in F^* \), so that \( \pi = \bar{\pi} \). Thus \( z\bar{z}^{-1} = v\bar{v}^{-1} = x \), and \( H^1 \) is trivial in this case.

If \( \mathcal{O} \) is over ramified, \( (\pi\bar{\pi}^{-1}) \) is a local unit, and \( z\bar{z}^{-1} = (\pi\bar{\pi}^{-1}) \varphi(z) \bar{v} \bar{v}^{-1} = x \). Thus \( H^1 \) is generated by the class of \( \pi\bar{\pi}^{-1} \), \( \text{cl}(\pi\bar{\pi}^{-1}) \). Of course \( (\pi\bar{\pi}^{-1})^2 \) is trivial in \( H^1 \), since \( (\pi\bar{\pi}^{-1})^2 = (\pi\bar{\pi}^{-1})(\pi\bar{\pi}^{-1})^{-1} \), a quotient of local
units. Thus, to complete the proof we need only show that 
\( cl(\pi^{-1}) \) is non-trivial in \( H^1 \).

Suppose to the contrary that there is a local unit
\( v \) with \( vv^{-1} = \pi^{-1} \). Then \( \pi v^{-1} = \pi v^{-1} \), so that \( \pi v^{-1} \) is
a local uniformizer of \( \mathcal{O}_E(\mathfrak{p}) \) which lies in \( F \). This is
impossible as we are in the ramified case. Hence,
\( cl(\pi v^{-1}) \) is non-trivial as claimed. \( \square \)

In the case of the localized completion, \( \mathcal{O}_E(\mathfrak{p}) \),
of \( E \) at \( \mathfrak{p} \), we can continue.

**Lemma 2.2**
\[
H^2(C_2; \mathcal{O}_E(\mathfrak{p})^*) = \{1\} \text{ if } \mathfrak{p} \text{ is over inert.}
\]
\[
H^2(C_2; \mathcal{O}_E(\mathfrak{p})^*) = C_2 \text{ if } \mathfrak{p} \text{ is over ramified.}
\]

**Proof:** Recall \( H^2(C_2; \mathcal{O}_E(\mathfrak{p})^*) = \{x \in \mathcal{O}_E(\mathfrak{p})^*: x = \overline{x}\} \)
modulo \( \{w: w = \gamma \overline{y}, \text{ for some } y \in \mathcal{O}_E(\mathfrak{p})^*\} \)
If \( \mathfrak{p} \) is inert, by [O'M 169], any local unit \( x \) in \( F \) is the norm of a unit in \( \mathcal{O}_E(\mathfrak{p})^* \)
For \( \mathfrak{p} \) ramified, [O'M 176], we have \([\overline{F}(\mathfrak{p}): \mathcal{O}_E(\mathfrak{p})]\) = 2.
However, the uniformizer \( \pi \) of \( \overline{E}(\mathfrak{p}) \) has norm \( \overline{\pi} \) which is
a uniformizer for \( \overline{F}(\mathfrak{p}) \). It follows that if \( x \in \mathcal{O}_F(\mathfrak{p})^* \)
is a norm, it must be the norm of a unit. Hence,
\[
[\mathcal{O}_F(\mathfrak{p})^*: \mathcal{O}_E(\mathfrak{p})^*] = [\overline{F}(\mathfrak{p}): \mathcal{O}_E(\mathfrak{p})] = 2, \text{ as claimed. } \square
\]
The ramified primes \( \mathfrak{p} \subset \mathcal{O}(F) \) are divided into two
types.
Definition 2.3 \( P \) is of type 1 provided \( \text{cl}(-1) \neq 1 \in H^1(C_2; \mathcal{O}_\mathcal{E}(\varphi)^*) \). \( P \) is of type 2 provided \( \text{cl}(-1) = 1 \in H^1(C_2; \mathcal{O}_\mathcal{E}(\varphi)^*) \).

We can classify type by:

Lemma 2.4 \( P \) is of type 1 if and only if there exists a local uniformizer \( \pi \in \mathcal{O}_\mathcal{E}(\varphi) \) with \( \overline{\pi} = -\pi \). \( P \) is of type 2 if and only if there is a skew unit, \( u = -\overline{u} \in \mathcal{O}_\mathcal{E}(\varphi)^* \).

Proof: \( \text{cl}(-1) \neq 1 \) if and only if for any local uniformizer \( \pi \), \( \text{cl}(-1) = \text{cl}(\overline{\pi}^{-1}) \). Hence, for some \( v \in \mathcal{O}_\mathcal{E}(\varphi)^* \), \( -v\overline{v}^{-1} = \pi^{-1} \). Replacing \( \pi \) by \( \overline{\pi} \), we obtain a skew uniformizer. \( \text{cl}(-1) = 1 \) if and only if there is a local unit \( v \) with \( v\overline{v}^{-1} = -1 \). \( \square \)

Lemma 2.5 If \( P \) is non-dyadic ramified, meaning the characteristic of the residue field \( \mathcal{O}(\mathcal{F})/\mathcal{P} \neq 2 \), then \( P \) is of type 1.

Proof: Since \( P \) is ramified, \( e = 2 \) and \( f = 1 \). Thus \( \mathcal{O}(\mathcal{E})/\varphi = \mathcal{O}(\mathcal{F})/\mathcal{P} \), and the induced involution on \( \mathcal{O}(\mathcal{E})/\varphi \) is trivial.

Suppose \( P \) is of type 2. Then by 2.4, there is a skew unit \( v \), with \( v = -\overline{v} \). However, viewing \( v \) in the residue field, \( v = \overline{v} \). Thus \( 1 = -1 \). This is a contradiction unless the characteristic is 2. \( \square \)
Finally, we wish to examine type in terms of the local different. Let \( P \) ramify, say \( \mathcal{O}(E) = \varphi^2 \). By [A 83], we can find \( \alpha \in \mathcal{O}_E(\varphi) \) such that \( 1, \alpha \) forms a basis of \( \mathcal{O}_E(\varphi) \), as an \( \mathcal{O}_P(P) \)-module. \( \alpha \) satisfies 
\[ t^2 - (\alpha + \bar{\alpha})t + \alpha\bar{\alpha} = 0. \]
By [A 92], \( 2\alpha - (\alpha + \bar{\alpha}) = \alpha - \bar{\alpha} \) generates the different of the extension.

Factor \( \alpha - \bar{\alpha} = \pi^d \nu \), where \( d_p \) is called the local differential exponent, \( \pi \) is a local uniformizer for \( \varphi \), \( \nu \in \mathcal{O}_E(\varphi)^* \).

Now, \( (\alpha - \bar{\alpha}) = -(\alpha - \bar{\alpha}) = \frac{d_p}{\pi} \nu = -\pi \frac{d_p}{\pi} \nu \). Thus
\[ -\nu \frac{1}{\nu} = (\pi^{-1}) \frac{d_p}{\pi} \quad \text{and} \quad \text{cl}(-1) = \text{cl}(\pi^{-1}) \frac{d_p}{\pi}. \]
Thus, by definition,

**Lemma 2.6** Let \( P \) be dyadic ramified. Then \( P \) is of type 1 if and only if the local differential exponent \( d_p \) is odd. \( \square \)

The cohomology groups \( H^*(C_2; \mathcal{O}_E(\varphi)^*) \) form part of the exact hexagon. We have explicitly calculated these in the case of the completion, \( \mathcal{O}_E(\varphi)^* \).

Let \( \mathcal{O}(E) \) be the group of all fractional \( \mathcal{O}(E) \)-ideals. This is likewise a \( C_2 \)-module by \( A \to \bar{A} \), for \( A \in \mathcal{O}(E) \).

We may embed \( E^*/\mathcal{O}(E)^* \to \mathcal{O}(E) \) by \( x \mapsto x\mathcal{O}(E) \). This yields a short exact sequence of \( C_2 \)-modules:
1 \to \mathbb{E}^*/O(E)^* \to \mathfrak{J}(E) \to C(E) \to 1

where \( C(E) \) = ideal class group.

The groups which arise in the hexagon are \( H^*(C_2;C(E)) \).

For the purposes of the next section, we now describe \( H^*(C_2;\mathfrak{J}(E)) \).

To begin with, \( \mathfrak{J}(E) \) is the free abelian group generated by the prime ideals in \( O(E) \). If \( \wp \) is such a prime, then either

(a) \( \wp = \wp \) is over inert or ramified

or (b) \( \wp \neq \wp \) is over split.

Let \( \mathfrak{S} \) be the subgroup of \( \mathfrak{J}(E) \) generated by (a), those \( \wp \) with \( \wp = \wp \).

Lemma 2.7 \( H^1(C_2;\mathfrak{J}(E)) = \{1\} \) and \( H^2(C_2;\mathfrak{J}(E)) = \mathfrak{S}/\mathfrak{S}^2 \).

Proof: \( H^1 \) is generated by \([A:A \mathfrak{A} = O(E)]\), modulo \([A/A:A \in \mathfrak{J}(E)]\). Write \( A = \prod \wp_i^{a_i} \). By unique factorization, it follows that if \( \wp_i^{a_i} \) occurs in this factorization, then so does \( \wp_i^{-a_i} \), since \( AA = O(E) \). Thus, \( H^1 \) is trivial, since these quotients are divided out.

\( H^2(C_2;\mathfrak{J}(E)) \) is generated by the collection of fractional ideals \([A:A = \mathfrak{A}]\) modulo \([A:A = BB]\). Again write \( A = \prod \wp_i^{a_i} \). If \( \wp_i^{a_i} \) occurs in this factorization, then so does \( \wp_i^{-a_i} \), since \( A = \mathfrak{A} \). However, for \( \wp_i \neq \wp_i \).
in the factorization, the product \( \phi_{i}^{a_{i}} \) occurs in the factorization.

Thus, it is clear that \( \gamma \) generates \( H^{2} \). Again, \( \gamma^{2} \) is trivial in \( H^{2} \), so that \( H^{2}(C_{2}; \mathcal{H}(E)) \cong \gamma / \gamma^{2} \). □

We shall refer to this later. Now, we continue the discussion of the groups in the hexagon with Gen, the group of Generators.
3. Generators

In this section we define a group which gives rise to additive generators of $\mathbb{H}(O(E))$. In order to define this group, which we call Gen, it is necessary to consider pairs $(y,A)$, where $y \in F^*$, and $A \subseteq E$ is a fractional $O(E)$-ideal with $yA \overline{A} = O(E)$. We make this into a group by component-wise multiplication.

$$(y,A) \cdot (y_1,A_1) = (yy_1,AA_1).$$

The identity is $(1,O(E))$. The inverse of $(y,A)$ is $(y^{-1},A^{-1})$.

We then factor out the subgroup consisting of pairs of the form $(z\overline{z},z^{-1}\overline{B}B^{-1})$, where $z \in E^*$, $B$ is a fractional $O(E)$-ideal. We denote the resulting group by Gen and an element in Gen by $\langle y,A \rangle$. Of course, $\langle y,A \rangle$ is a coset.

The study of Gen begins with:

**Lemma 3.1** If $y \in F^*$, then there exists a fractional $O(E)$-ideal $A$ with $yA \overline{A} = O(E)$ if and only if $v_p y \equiv 0 \pmod 2$ at all inert primes $P$.

**Proof**: Necessity is clear.

Conversely, suppose $v_p y \equiv 0 \pmod 2$ at all inert primes. Then $v_p y^{-1} = v_p y^{-1} = v_p y^{-1} = 0 \pmod 2$ at $P = \emptyset \cap F$ inert. If $P$ splits, $v_p y^{-1} = v_p y^{-1} = v_p y^{-1}$.
since \( y \in F^* \). If \( P \) ramifies, \( v_p y^{-1} = 2v_p y^{-1} \equiv 0 \).

Hence, \( y^{-1} O(E) \) factors as \( A A \), i.e. \( y A A = O(E) \) as claimed. 

Lemma 3.1 states that the elements in Gen arise precisely from \( y \in F^* \) satisfying \( v_p y \equiv 0 \pmod{2} \) at all inerts.

Lemma 3.2 Suppose \( y A A = O(E) \). Then \( \langle y, A \rangle = \langle 1, O(E) \rangle \) if and only if \( y \) is a norm from \( E^* \).

Proof: Necessity is clear, since pairs of the form 
\((z z, z^{-1} B B^{-1})\), \( z \in E^* \) are trivial in Gen.

Conversely, suppose \( y = z z \). Then \( (z A)(z A) = O(E) \).

Since \( H^1(C_2, \mathfrak{o}(E)) = \{1\} \), there exists a fractional \( O(E) \)-ideal \( B \) with \( z A = B B^{-1} \).

Thus, \( \langle y, A \rangle = \langle z z, z^{-1} B B^{-1} \rangle = \langle 1, O(E) \rangle \), the identity in Gen by definition of the equivalence relation. 

We are now ready to describe Gen.

Theorem 3.3 If no primes ramify, \( \text{Gen} = \{1\} \). If at least one prime ramifies, \( \text{Gen is a 2-group, with 2 rank } (N - 1) \) where \( N \) is the total number of ramified primes, finite and infinite.

Proof: Suppose first that no prime ramifies. Let
\langle y, A \rangle \in \text{Gen}. By Lemma 3.1, \( v_p y = 0 \) at all inerts.

However, \( (y, \sigma)_p = (-1)^{v_p(y)} \), so \( (y, \sigma)_p = +1 \) at all inerts.

Since no primes ramify, \( (y, \sigma)_p = +1 \) everywhere. Thus, by the Hasse's Theorem V 2.1, \( y \) is a norm. By Lemma 3.2, \( \langle y, A \rangle = \langle 1, 0(E) \rangle \).

Next, suppose there are ramified primes. Let \( \langle y, A \rangle \in \text{Gen} \). By Lemmas 3.1, 3.2, the class of \( \langle y, A \rangle \) in \( \text{Gen} \) is determined by the norm class of \( y \), i.e. by the Hilbert symbols \( (y, \sigma)_p \). At inerts, \( (y, \sigma)_p = (-1)^{v_p(y)} = +1 \), by 3.1. By Hilbert reciprocity, \( \prod_{p \text{ ramified}} (y, \sigma)_p = +1 = \prod (y, \sigma)_p \).

Thus to finish the proof, we must show:

To each prime \( P \) which ramifies, assign a number \( \epsilon(P) = +1 \), satisfying \( \prod_{P \text{ ramified}} \epsilon(P) = +1 \). We now need to produce a pair \( \langle y, A \rangle \in \text{Gen} \) with \( (y, \sigma)_p = \epsilon(P) \) for all \( P \) ramified.

By realization of Hilbert symbols, there exists \( y \in F^* \) with \( (y, \sigma)_p = 1 \) if \( P \) is split or inert \( (y, \sigma)_p = \epsilon(P) \) if \( P \) is ramified.

Consider \( y^{-1}0(E) \). \( (y, \sigma)_p = (-1)^{v_p(y)} = (-1)^{v_p(y^{-1})} = +1 \) at inerts. Thus \( v_p(y) \equiv 0 \pmod{2} \) at inerts.

\[ v_{\sigma}(y^{-1}) = 2v_p(y^{-1}) \equiv 0 \text{ at } P \text{ ramified.} \]

\[ v_{\sigma}(y^{-1}) = v_{\sigma}(y^{-1}) = v_{\sigma}(y^{-1}) \text{ at } \sigma \text{ split.} \]

Hence we may factor \( y^{-1}0(E) = A\tilde{A} \), yielding the pair \( \langle y, A \rangle \in \text{Gen} \) as desired. \( \Box \)

We still must discuss where the name Gen comes from.
Thus, recall our study of discriminants. Each inner product space \((M, B)\) gives rise to the discriminant inner product space \((y, A)\), where \(A = \bigwedge^n M\), \(\langle y \rangle = \bigwedge^n B\). This inner product space satisfies \(yA\bar{A} = O(E)\), thus giving rise to an element in \(Gen\).

By Lemma 3.2, the equivalence relation in \(Gen\) is the same as the Witt equivalence of the two one-dimensional forms.

**Lemma 3.4** If \(\langle z, A \rangle = \langle z_1, A_1 \rangle\) in \(Gen\), then the one dimensional inner product spaces \((A, \langle z \rangle)\), \((A_1, \langle z_1 \rangle)\), where \(\langle z \rangle\) is multiplication by \(z \in F^*\), are Witt equivalent.

**Proof:** Since \(\langle z, A \rangle = \langle z_1, A_1 \rangle\), we may write \(zz^{-1}_1 = \overline{ww}\), \(AA^{-1}_1 = w^{-1}CC^{-1}\). However, \([w^{-1}CC^{-1}, \langle \overline{ww} \rangle]\) is the multiplicative identity in \(H(E)\) and hence in \(H(O(E))\). Thus

\[(A_1, \langle z_1 \rangle) \sim (A_1, \langle z_1 \rangle) \otimes (w^{-1}CC^{-1}, \langle \overline{ww} \rangle)\]

\(\sim (A, \langle z \rangle)\) as desired. \(\square\)

Now consider \(H(O(E))\). Let \([M, B] \in H(O(E))\). By Landherr's Theorem VI 3.4, \([M, B]\) is determined by rank mod 2, discriminant, and signatures. Forming the discriminant inner product space, \((y, A)\), the discriminant is \(ty\).

At this point, we wish to show how \(Gen\) gives rise to additive generators for \(H(O(E))\). There are two cases.
Case 1. If there exist finite ramified primes in $E/F$.

Case 2. If there are only infinite ramified primes.

**Case 1:** Number the ramified primes $P_1, \ldots, P_t, P_{t+1}, \ldots, P_N$, where $P_1, \ldots, P_t$ are the infinite ramified primes, with $N \geq t + 1$. Let $y_i \in F^*$ satisfy:

\[(y_i, \sigma)_p = +1\] at all inert primes.
\[(y_i, \sigma)_p = -1\] for all infinite primes except $P_i$.
\[(y_i, \sigma)_\infty = +1\] at $P_i$.

This is possible since $N \geq t + 1$, by realization. The collection $\{y_i\}_{i=1}^t$ above is then extended arbitrarily to a basis for $\text{Gen}$, $\{y_i\}_{i=1}^{N-1}$.

**Lemma 3.5** The one-dimensional forms $[y_i, A_i]$, corresponding to $i=1, \ldots, t$ above, together with $[1, \Omega(E)]$ additively generate $J^2$.

**Proof:** By construction and Lemma 3.1, each $y_i$, $i = 1, 2, \ldots, t$, determines a corresponding pair $\langle y_i, A_i \rangle$ in $\text{Gen}$. In fact, as observed, we may view $\text{Gen}$ as given by elements $y \in F^*$, with $v_p y \equiv 0 \pmod{2}$ at inert. The elements $\{y_i\}_{i=1}^t$ are clearly part of a basis for $\text{Gen}$ by Theorem 3.3.

In order to see that these 1-dimensional forms generate $J^2$, we need to see that we can obtain an
arbitrary signature in \( \frac{1}{4}(\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}) \cong J^2 \). This is clear, since \( \langle 1, y_i \rangle \otimes \langle 1, y_i \rangle = \langle 1, y_i, y_i, y_i^2 \rangle \sim \langle 1, y_i, y_i, 1 \rangle \) has signature +4 only at the \( i \)th prime by construction. \( \square \)

**Theorem 3.6** In Case 1, the elements \( \{y_i\}_{i=1}^{N-1} \) above, together with \([1, 0(\text{E})]\) determine an additive generating set for \( H(0(\text{E})) \).

**Proof:** Let \([M, B] \in H(0(\text{E}))\). By Landherr, \([M, B]\) is determined by rank, discriminant, and signatures. By 3.5, it suffices to show that the generators given yield an element with the same discriminant and rank as \([M, B]\). Let \( \pm y \) be the discriminant. Then \( y = \prod y_i \), since the \( \{y_i\} \) generate \( \text{Gen} \). It thus is clear that we need only show:

If \( z_1 \) and \( z_2 \) are in the set generated additively by \( \{y_i, 1\}_{i=1}^{N-1} \), then so is their product \( z_1 z_2 \). But consider \( \langle 1, z \rangle \otimes \langle 1, z_2 \rangle = \langle 1, z_1, z_2, z_1 z_2 \rangle \). This is in \( J^2 \), hence \( \langle 1 \rangle \otimes \langle z_1 \rangle \otimes \langle z_2 \rangle \otimes \langle z_1 z_2 \rangle \) is in the set generated by \( \{y_i, 1\} \). However, so is \( \langle 1 \rangle, \langle z_1 \rangle, \langle z_2 \rangle \). It follows that \( \langle z_1 z_2 \rangle \) is also in the span of \( \{y_i, 1\} \) as desired. \( \square \)

**Case 2:** If there are no finite ramified primes, we label the infinite ramified primes \( p_1, \ldots, p_N \). \( N \) is even since \( \prod (-1, \sigma)_p = \prod (-1, \sigma)_{p_i} = +1 \) by Hilbert reciprocity.
In this case, consider the collection of elements \( \{y_i\}_{i=1}^{N-1} \), where the \( y_i \in F^* \) satisfy:

\[
(y_i, \sigma)_P = +1 \text{ at all inert } P \\
(y_i, \sigma)_P = -1 \text{ at all infinite ramified primes except } P_i \text{ and } P_N.
\]

A \( y_i \) satisfying these conditions exists by realization of Hilbert symbols, since \( N \) is even. Clearly too this set \( \{y_i\}_{i=1}^{N-1} \) determines a basis for \( \text{Gen} \).

**Lemma 3.7** The one-dimensional forms \([y_i, A] \in H(O(E))\) corresponding to \( \{y_i\} \) above, for \( i = 1, \ldots N-1 \), together with \([1, 0(E)]\), and a four-dimensional form in \( J^2 \) with signature +4 at the \( N^{th} \) prime only, additively generate \( J^2 \).

**Proof:** \( J^2 \) is the square of the fundamental ideal in \( H(O(E)) \). To prove the lemma, it suffices to produce a form in \( J^2 \) which is in the span of the given set of additive generators, with signature +4 at only one given infinite prime. Note that \( \langle 1, y_i \rangle \otimes \langle 1, y_i \rangle = \langle 1, y_i, y_i, 1 \rangle \) has signature +4 at only \( P_i \) and \( P_N \). We then add the given 4-dimensional form to this to produce a form with signature +4 at only \( P_i \). □

**Remark 3.8** It is possible to choose only 1-dimensional generators for \( H(O(E)) \) provided \( N > 2 \). For when \( N > 2 \),
let $y_1, y_2$ be two of the 1-dimensional forms from Lemma 3.7. Add to the collection $\{y_i\}$, the forms $[1, O(E)]$ and the form corresponding to $y_1 y_2$. Then $[1, y_1, y_2, y_1 y_2]$ will be in the span of the 1-dimensional forms given, with signature $+4$ at only $p N$, and Lemma 3.7 is still true.

For $N = 2$, this is not possible. Let $[M, B]$ be a 4-dimensional form in $J^2$ with signature $+4$ at only one of the infinite primes. A discriminant argument using Hilbert reciprocity implies any 1-dimensional form in $H(O(E))$ has equal signatures at both infinite primes. Thus, $[M, B]$ gives rise to an element in $H(O(E)) \cap J^2$ which is not a sum of 1-dimensional forms in $H(O(E))$.

**Example:** Let $p, q$ be two distinct positive primes with $-p \equiv 1 \pmod{4}$. Let $F = \mathbb{Q}(\sqrt{pq})$, $E = F(\sqrt{-p})$. Then $N = 2$ as above.

**Theorem 3.9** In Case 2, when there are no finite ramified primes, the elements in $H(O(E))$ given in 3.7 additively generate $H(O(E))$.

**Proof:** Same as Theorem 3.6. □

Thus Gen, together with $[1, O(E)]$, and possibly one additional form, determines a collection of forms which additively generate $H(O(E))$.

This completes the introduction to the groups in the hexagon. We are now ready for the main theorem of this chapter.
4. The exact hexagon

In this section we describe the relation of the groups we have been discussing. This is given by the exact hexagon which appears below. The maps in the hexagon are defined immediately below it for reference in reading this section.

Exactness at each vertex of the hexagon is proven beginning at $H^2(C_2; O(E)^*)$, and moving in a clockwise manner around the hexagon.

Theorem 4.0 There is an exact hexagon with maps as described below.

\[
\begin{array}{cccccc}
H^1(C_2; C(E)) & \xrightarrow{d_1} & H^1(C_2; O(E)^*) & \xleftarrow{\mu} & I_0(E/F) \\
\langle y, A \rangle & \xrightarrow{j} & \langle v, 0(E) \rangle \\
H^2(C_2; O(E)^*) & \xrightarrow{d_2} & H^2(C_2; C(E)) \\
i: cl(v) & \rightarrow & \langle v, 0(E) \rangle \\
j: \langle y, A \rangle & \rightarrow & cl |A| \\
d_1: cl |A| & \rightarrow & cl(x\bar{x}^{-1}) \text{ where } x\bar{x}A \bar{A} = 0(E) \\
u: cl(u) & \rightarrow & |u, 0(E)| \\
\eta: |u, k| & \rightarrow & clk \\
d_2: clA & \rightarrow & cl(x\bar{x}) \\
\bar{A}A^{-1} = x0(E)
\end{array}
\]
We begin by considering $H^2(C_2; O(E)^\ast)$. This is generated by \{\(v \in O(E)^\ast : v = \tilde{v}\}\) modulo \{\(v \in O(E)^\ast : v = \tilde{w}\) for some \(w \in O(E)^\ast\)\}. Define \(i : H^2(C_2; O(E)^\ast) \rightarrow \text{Gen by}
\cl(v) \rightarrow \langle v, O(E) \rangle\). If \(\cl v = \cl w\) in \(H^2\), \(v = \tilde{wx}\tilde{x}\), so that \(|v, O(E)| = |w, O(E)|\) in Gen by 3.2. Thus \(i\) is well-defined.

In order to study the kernel of \(i\), we consider the \(C_2\) module structure on \(C(E)\), the ideal class group of \(E\). If \(A\) is an \(O(E)\)-ideal, let \(|A|\) denote the class of \(A\) in \(C(E)\). Now consider \(H^2(C_2; C(E))\). An element in this cohomology group is represented by an ideal class \(|A|\) which is equivalent to \(|\tilde{A}|\). This means that \(A = x\tilde{A}\) for some \(x \in E^\ast\). Hence, \(AA^{-1} = x0(E)\), so \(AA^{-1} = \tilde{x}0(E)\).

Thus \(O(E) = xx0(E)\). It follows that \(xx \in O(F)^\ast\), ie. \(xx\) is a unit. We now define
\[d_2 : H^2(C_2; C(E)) \rightarrow H^2(C_2; O(E)^\ast)\] by
\[d_2 : \cl |A| \rightarrow \cl(xx) \in H^2(C_2; O(E)^\ast)\].

We must show \(d_2\) is well-defined. Suppose \(AA^{-1} = x_10(E)\). Then \(x_1 = xv\) for some \(v \in O(E)^\ast\). Thus \(x_1\tilde{x_1} = xxv\tilde{v}\), so that \(\cl(x_1\tilde{x_1}) = \cl(xx)\).

Next suppose \(\cl |A| = \cl |C| \in H^2(C_2; C(E))\). Then we may write \(C = yABB\). So \(CC^{-1} = yy^{-1}AA^{-1}BB^{-1}BB^{-1}B^{-1}\)
\(=yy^{-1}AA^{-1}\). Suppose \(CC^{-1} = z0(E)\), so that \(AA^{-1} = y^{-1}yz0(E)\).

However, \(zz = (y^{-1}yz)(\tilde{y}^{-1}y\tilde{z})\), so that
\[d_2\cl |A| = (y^{-1}yz)(\tilde{y}^{-1}y\tilde{z}) = zz = d_2\cl |C|\].
Thus $d_2$ is well-defined. This brings us to:

**Lemma 4.1** $H^2(C_2;O(E)) \xrightarrow{d_2} H^2(C_2;O(E)^*) \xrightarrow{i} \text{Gen}$ is exact.

**Proof:** $i \circ d_2 = 0$ since $xx \in O(F)^*$ is a norm, so that $\langle xx, O(E) \rangle = \langle 1, O(E) \rangle$ by 3.2.

Conversely, suppose $i(cl(v)) = \langle v, O(E) \rangle = \langle 1, O(E) \rangle$ in Gen. Then $v = xx$ and $x^{-1}BB^{-1} = O(E)$ by definition of Gen. Hence $d_2 \mid |B| = cl \mid |v|$, so that $\ker i \subseteq \im d_2$. □

Next we consider the cokernel of $i$. When $yA\tilde{A} = O(E)$, we may form $cl \mid |A| \in H^1(C_2;C(E))$. Thus define a homomorphism $j: \text{Gen} \rightarrow H^1(C_2;C(E))$ by $\langle y, A \rangle \mapsto cl \mid |A|$. If $\langle y, A \rangle = \langle z\tilde{z}, z^{-1}BB^{-1} \rangle$, clearly

$$cl\mid z^{-1}BB^{-1} = 1 \in H^1(C_2;C(E)),$$

so that $j$ is well-defined.

**Lemma 4.2** The sequence

$$H^2(C_2;O(E)^*) \xrightarrow{i} \text{Gen} \xrightarrow{j} H^1(C_2;C(E))$$

is exact.

**Proof:** $j \circ i(cl(v)) = j(\langle v, O(E) \rangle) = cl \mid O(E) \mid$ is clearly trivial. Thus $\im i \subseteq \ker j$.

Conversely, let $\langle y, A \rangle \in \ker j$. Then $cl \mid |A| = 1 \in H^1(C_2;C(E))$. Thus $A = z^{-1}BB^{-1}$ for some $z \in E^*$ and fractional ideal $B$. Hence $z\tilde{z}A\tilde{A} = O(E) = yA\tilde{A}$ and we can
find $u \in O(F)^*$ with $uzz = y$. Consider $\text{cl}(u) \in H^2(C_2; O(E)^*)$. $i(\text{cl}(u)) = \langle u, O(E) \rangle = \langle y, A \rangle$ in Gen since $y/u = z\bar{z}$ is a norm. So $\ker j \subseteq \text{im } i$ as desired. □

We continue by defining the map $d_1 : H^1(C_2; C(E)^*) \to H^1(C_2; 0(E)^*)$ as follows. Suppose $x\bar{A} = 0(E)$, so that $\text{cl} |A| \in H^1(C_2; C(E)^*)$. Then $x\bar{A} = x\bar{A}$, and $xx^{-1} = v \in O(E)^*$. Clearly $v\bar{v} = 1$; so we define $d_1$ by: $\text{cl} |A| \rightarrow \text{cl}(xx^{-1})$.

We must show that $d_1$ is well-defined. Suppose first that $x$ is replaced by $x^\prime$. Then $x^\prime = xw$ for some $w \in O(E)^*$. Thus $x^\prime x_{1}^{-1} = xx^{-1}ww^{-1}$, and $\text{cl}(x^\prime x_{1}^{-1}) = \text{cl}(xx^{-1})$ in $H^1(C_2; O(E)^*)$. Continuing, suppose $A$ is trivial in $H^1(C_2; C(E)^*)$, so that $A = z\bar{B}B^{-1}$. Then $x\bar{A} = xz\bar{z}0(E) = 0(E)$. Thus $xz\bar{z} = u \in O(E)^*$. Hence $xx^{-1} = uu^{-1}$ is trivial in $H^1(C_2; O(E)^*)$, and $d_1$ is well-defined.

Lemma 4.3 The sequence

$$Gen \xrightarrow{j} H^1(C_2; C(E)^*) \xrightarrow{d_1} H^1(C_2; O(E)^*)$$

is exact.

Proof: Let $\langle y, A \rangle \in \text{Gen}$, with $y\bar{A} = O(E)$, $y \in F^*$. $d_1 \circ j \langle y, A \rangle = \text{cl}(yy^{-1}) = \text{cl}(1)$ in $H^1(C_2; O(E)^*)$. Hence $d_1 \circ j$ is trivial, and $\text{im } j \subseteq \ker d_1$.

Conversely, suppose $d_1(\text{cl} |A|) = \text{cl}(xx^{-1}) = \text{cl}(1)$. Then $xx^{-1} = vv^{-1}$, for $v \in O(E)^*$. Replace $x$ by $x\bar{v} = y$. 

since $yA\overline{A} = 0(E)$. However $y = \overline{y}$, so that $\langle y, A \rangle \in \text{Gen.}$ j$\langle y, A \rangle = \text{cl } |A|$, and $\ker d_1 \subseteq \text{im } j$. □

The next map is $\mu: H^1(C_2;O(E)^*) \to \text{Iso}(E/F)$.

Let $\text{cl}(u) \in H^1(C_2;O(E)^*)$, so $uu = 1$. Thus we can form $|u, O(E)| \in \text{Iso}(E/F)$. Define $\mu$ by: $\text{cl}(u) \to |u, O(E)|$.

We must show $\mu$ is well-defined. Suppose $u = vv^{-1}$, with $v \in O(E)^*$. Then $vO(E) = O(E)$ since $v$ is a unit. Thus $|vv^{-1}, O(E)| = |1, O(E)|$ in $\text{Iso}(E/F)$, by Definition 1.1, with $x = v$ and $A = O(E)$, and $\mu$ is well-defined.

The sequence

$$H^1(C_2;C(E)) \xrightarrow{d_1} H^1(C_2;O(E)^*) \xrightarrow{\mu} \text{Iso}(E/F)$$

is exact.

Proof: Let $\text{cl } |A| \in H^1(C_2;C(E))$, with $xA\overline{A} = 0(E)$. Then $\mu \circ d_1(\text{cl } |A|) = |xx^{-1}, O(E)|$ which is clearly trivial in $\text{Iso}(E/F)$ since $xA\overline{A}O(E) = O(E)$. Hence $\text{im } d_1 \subseteq \ker \mu$.

Conversely, let $\text{cl}(u) \in \ker \mu$. So $|u, O(E)| = |1, O(E)|$. Thus there exists $x \in E^*$ and a fractional $O(E)$-ideal $A$ with $xx^{-1} = u$, $xA\overline{A} = O(E)$. Hence we may form $\text{cl } |A| \in H^1(C_2;C(E))$. $d_1(\text{cl } |A|) = \text{cl}(xx^{-1}) = \text{cl}(u)$. □

Next, we define $\eta: \text{Iso}(E/F) \to H^2(C_2;C(E))$ by $|u, K| \to \text{cl}(K)$. In order to show that $\eta$ is well-defined, suppose $|u, K| = |u_1, K_1|$. Then there exists $x \in E^*$ and a fractional $O(E)$-ideal $A$ satisfying:
(1) $xx^{-1}u = u_1$ and (2) $xAAK = K_1$. By (2), $cl(K) = cl(K_1)$ in $H^2(C_2;\mathcal{C}(E))$, so that $\eta$ is well-defined.

**Lemma 4.5** The sequence

$$H^1(C_2;O(E)^*) \xrightarrow{\mu} Iso(E/F) \xrightarrow{\eta} H^2(C_2;\mathcal{C}(E))$$

is exact.

**Proof:** Let $cl(u) \in H^1(C_2;O(E)^*)$. Then $\eta \circ \mu(cl(u)) = \eta|u,O(E)| = cl(O(E))$, and $\eta \circ \mu$ is trivial. Hence $im \mu \subseteq ker \eta$.

Conversely, suppose $|u,K| \in ker \eta$. Then $cl(K)$ is trivial in $H^2(C_2;\mathcal{C}(E))$. This means that we can write $K = xAA$, for $x \in E^*$, $A$ a fractional $O(E)$-ideal. Now $K = \overline{K} = xAA = x\overline{A}$, so that $x^{-1}x$ is a unit in $O(E)^*$. Consider $w = ux^{-1}x$. Clearly, $w\overline{w} = uux^{-1}xx^{-1} = 1$, so we can form $cl(w) \in H^1(C_2;O(E)^*)$. Applying $\mu$ we obtain $|w,O(E)|$.

Notice, however, that $xx^{-1}w = u$ and $xAAO(E) = K$. This implies $|w,O(E)| = |u,K|$ in $Iso(E/F)$, so that $|u,K| \in im \mu$. □

Finally, we conclude the proof of exactness in the hexagon by:
Lemma 4.6  The sequence

$$\eta: \text{Iso}(E/F) \xrightarrow{d_2} H^2(C_2;C(E)) \xrightarrow{d_2} H^2(C_2;O(E)^*)$$

is exact.

Proof: Let $|u,K| \in \text{Iso}(E/F)$. $d_2 \circ \eta|u,K| = \text{cl}(1)$ since $K = \bar{K}$. So $\text{im} \eta \subseteq \ker d_2$.

Conversely, suppose $\text{cl}(K) \in \ker d_2$. Write $\bar{K}K^{-1} = x0(E)$, so that $d_2 \text{cl}(K) = \text{cl}(x\bar{x})$. Since $\text{cl}(x\bar{x})$ is trivial in $H^2(C_2;O(E)^*)$, $x\bar{x} = vv$ for $v \in O(E)^*$. Consider $w = xv^{-1}$, $w\bar{w} = xv^{-1}xv^{-1} = 1$ by above. By Hilbert 90, there exists $z \in E^*$ with $z\bar{z}^{-1} = w$. Now, $z\bar{z}^{-1}0(E) = w0(E) = xv^{-1}0(E) = x0(E) = K\bar{K}^{-1}$. Hence $z\bar{K} = \bar{z}K$, so that we may consider the element $|1,z\bar{K}| \in \text{Iso}(E/F)$. Applying $\eta$, we obtain $\text{cl}(z\bar{K}) = \text{cl}(K) \in H^2(C_2;C(E))$, and $\ker d_2 \subseteq \text{im} \eta$. □

This completes the proof of the exactness in the hexagon.
5. Further results

To begin with, we assume that at least one prime, finite or infinite, ramifies in $E/F$.

Recall that if $\mathfrak{p} = \mathfrak{p}_E$ is over ramified, then

$$H^1(C_2; O_E(\mathfrak{p})^*) = C_2,$$

generated by $\text{cl}(\pi^{n-1})$ where $\pi$ is a local uniformizer of $O_E(\mathfrak{p})$.

**Lemma 5.1** Let $S = \tilde{S} \subset E$ be a fractional $O(E)$-ideal. Then there exists $z \in E^*$, and a fractional ideal $A$ with $zA\bar{A} = S$ if and only if there is a unit $u \in O(E)^*$ satisfying (a) $uu = 1$ and (b) $\text{cl}(u) = \text{cl}(\pi^{n-1})^\mathfrak{p}(S)$

$\in H^1(C_2; O_E(\mathfrak{p})^*)$ for all $\mathfrak{p} = \mathfrak{p}_E$ over ramified. ($zz^{-1} = u$).

**Proof:** Necessity. Suppose $zA\bar{A} = S$. Then $zA\bar{A} = z\bar{A}A$, so that $zz^{-1} = u$ a unit in $O(E)^*$, and $uu = 1$.

Write $z = \pi^v(\mathfrak{p})(z)^v$, where $v \in O(E)^*$. Then $\text{cl}(zz^{-1})$

$= \text{cl}(u) = \text{cl}(\pi^{n-1})^\mathfrak{p}(z)$

$\in H^1(C_2; O_E(\mathfrak{p})^*)$. However, $zA\bar{A} = S$, so that at each $\mathfrak{p}$ over ramified, $v(\mathfrak{p})(z) \equiv v(\mathfrak{p})(S) \pmod{2}$. This completes necessity.

Conversely, suppose $uu = 1$ and $\text{cl}(u) = \text{cl}(\pi^{n-1})^\mathfrak{p}(S)$

for all $\mathfrak{p} = \mathfrak{p}_E$ over ramified. By Hilbert 90, there exists $z_1 \in E^*$ with $z_1z^{-1} = u$. Since at least one prime ramifies by assumption, there exists $y \in F^*$ such that for all $\mathfrak{p} = \mathfrak{p}_E$ over $\mathfrak{P}$ inert,
This $y$ exists by realization of Hilbert symbols. Let $z = z_1 y$. Note that we still have $z^{-1} z_1 = u$ since $y \in F^*$. By (A), we also have $v_{\bar{\varphi}}(z) \equiv v_{\varphi}(S)$ for all $\varphi = \bar{\varphi}$ over inert.

By assumption, $\text{cl}(u) = \text{cl}(\pi^{-1}) v_{\varphi}(S)$ for $\varphi = \bar{\varphi}$ over ramified. Writing $z = \pi^\varphi w$, where $w \in O_E(\varphi)^*$, for $\varphi$ ramified, we apply the relation $z^{-1} z_1 = u$ to conclude:

$$v_{\varphi}(z)^{\varphi}(w^{-1}) = (\pi^{-1}) v_{\varphi}(S)$$

since $\text{cl}(u) = \text{cl}(\pi^{-1}) v_{\varphi}(S)$. This implies $v_{\varphi}(z) \equiv v_{\varphi}(S) \pmod{2}$ for $\varphi$ over ramified.

Finally, if $\varphi \neq \bar{\varphi}$ is over split, then $z = \bar{z} u$, so that $v_{\varphi}(z) = v_{\varphi}(\bar{z}) = v_{\bar{\varphi}}(z)$. Also, $v_{\bar{\varphi}}(S) = v_{\varphi}(S)$ since $S = \bar{S}$.

It thus follows that $z^{-1} S$ factors as $A \bar{A}$, i.e. $z A \bar{A} = S$ as claimed. □

As an immediate application of Lemma 5.1,

**Theorem 5.2** Let $\mathcal{D}(E/F) \subset O(E)$ be the different. Then there exists $z \in E^*$, $z = -\bar{z}$, and a fractional ideal $A$ with $z A \bar{A} = \mathcal{D}(E/F)$.

**Proof:** If $\varphi = \bar{\varphi}$ is over ramified of type 1, we have $v_{\varphi}(\mathcal{D}(E/F)) \equiv 1 \pmod{2}$, and $\text{cl}(-1) \equiv \text{cl}(\pi^{-1})$ in $H^1(C_2; O_E(\varphi)^*)$.

On the other hand, if $\varphi$ is over type 2 ramified, $v_{\varphi}(\mathcal{D}(E/F)) \equiv 0 \pmod{2}$, and $\text{cl}(-1) = 1 \in H^1(C_2; O_E(\varphi)^*)$. 

It follows that \( \text{cl}(-1) = \text{cl}(\pi^{-1})^\varphi(E/F) \).

Applying Lemma 5.1, there exists \( z \in E^* \) with \( zz^{-1} = -1 \), and \( zA^* = B(E/F) \). □

We next consider the homomorphism: \( C(F) \to C(E) \) which sends an \( \mathcal{O}(F) \)-ideal \( A \to \mathcal{O}(E) \). Let \( \mathcal{X}(E/F) \) be the kernel.

There is an embedding \( i: \mathcal{X}(E/F) \to H^1(C_2;\mathcal{O}(E)^*) \). In order to define \( i \), suppose \( |A| \in C(F) \), and \( A_0(E) = z_0 \mathcal{O}(E) = \tilde{z}_0 \mathcal{O}(E) \). Then \( \tilde{z}_0^{-1} \in \mathcal{O}(E)^* \). \( i|A| \) is defined to be \( \text{cl}(\tilde{z}_0^{-1}) \in H^1(C_2;\mathcal{O}(E)^*) \).

To see that \( i \) is well-defined, suppose \( A \) is replaced by \( yA \), where \( y \in F^* \). Then \( yA_0(E) = zy \mathcal{O}(E) \), and \( (zy)(\tilde{z}_0y)^{-1} = \tilde{z}_0^{-1} \). This is independent of the choice of \( z \), since if \( z_0 \mathcal{O}(E) = z_1 \mathcal{O}(E) \), \( z = z_1u \), for \( u \in \mathcal{O}(E)^* \). Hence \( \tilde{z}_0^{-1} = z_1 \tilde{z}_1^{-1} uu^{-1} \), and \( \text{cl}(\tilde{z}_0^{-1}) = \text{cl}(z_1 \tilde{z}_1^{-1}) \in H^1(C_2;\mathcal{O}(E)^*) \).

In order to show that \( i \) is 1-1, we need the norm homomorphism: \( N: \mathcal{O}(E) \to \mathcal{O}(F) \) given by \( N(\mathcal{O}) = P \) if \( P \) is split or ramified, \( N(\mathcal{O}) = P^2 \) if \( P \) is inert.

Lemma 5.3 \( i: \mathcal{X}(E/F) \to H^1(C_2;\mathcal{O}(E)^*) \) is 1-1.

Proof: Let \( |A| \in \mathcal{X}(E/F) \). Then \( A_0(E) = z \mathcal{O}(E) \). Suppose \( |A| \in \ker i \). Then there exists \( v \in \mathcal{O}(E)^* \) with \( \tilde{z}^{-1} = vv^{-1} \). Set \( y = z \bar{v} \), so that \( y = \bar{y} \in F^* \). Then
\( A_0(E) = z_0(E) = y_0(E) \). Applying the norm homomorphism, 
\( A^2 = y^2_0(F) \). By unique factorization, \( A = y_0(F) \), so 
that \( A \) was principal in \( F \). Hence \( i \) is 1-1 as claimed. \( \square \)

**Lemma 5.4** If no finite primes ramify, \( i \) is onto, 
so that \( \chi(E/F) \) is isomorphic to \( H^1(C_2; O(E)^*). \)

**Proof:** Let \( \text{cl}(v) \in H^1(C_2; O(E)^*) \), so that \( vv = 1 \).
By Hilbert 90, there exists \( z \in E^* \) with \( zz^{-1} = v \). We 
wish to write \( z_0(E) = A_0(E) \) for some ideal \( A \) in \( O(F) \), 
so that \( i|A| = \text{cl}(v) \). We define \( A \) to have 
\[ v_p A = v_\varphi(z) \] for \( P \) inert.
\[ v_p A = v_\varphi(z) = v_\varphi(z) \] since \( z = vz \) if \( \varphi \) is split.
Since there are no ramified primes, the result follows. \( \square \)

**Lemma 5.5** There is a commutative diagram:

\[
\begin{array}{ccc}
\chi(E/F) & \xrightarrow{\phi} & \text{Iso}(E/F) \\
\downarrow i & & \downarrow \text{identity} \\
H^1(C_2; O(E)^*) & \xrightarrow{\mu} & \text{Iso}(E/F)
\end{array}
\]

\( \phi \) is defined by \( \phi|A| = |1, A_0(E)|. \) \( \phi \) is well-defined, 
since \( |1, A_0(E)| = |1, yA_0(E)| \), for \( y \in F^* \). This follows 
by Definition 1.1 of \( \text{Iso}(E/F) \), with \( x = y \).
Proof of Lemma 5.5: Let $|A| \in \chi(E/F)$. Then

$AO(E) = zO(E) = \overline{z}O(E)$, so that $zz^{-1} = u \in O(E)^*$. Hence

$zAO(E) = z^2O(E) = (zO(E))(\overline{z}O(E))$.

$\mu \circ i|A| = \mu(cl(zz^{-1})) = |u, O(E)|$.

$\varphi|A| = |1, AO(E)|$.

Let $C = (zO(E))^{-1}$, $zz^{-1} = u$, so that $zCCA = O(E)$ by above. Hence, $|u, O(E)| = |1, AO(E)|$ and $\mu \circ i = \varphi$ as claimed. □

Lemma 5.6 If at least one prime ramifies in $E/F$, the sequence

\[ 0 \to \chi(E/F) \to H^1(C_2; O(E)^*) \to \text{Iso}(E/F) \to \] is exact.

Proof: $i$ is 1-1 by 5.3. We shall show the sequence is exact at $H^1(C_2; O(E)^*)$ by showing that $i(\chi(E/F)) = d_1(H^1(C_2; C(E))) = \ker \mu$ by exactness of the hexagon.

We begin by showing that $\text{im} d_1 \subseteq \text{im} i$. Let $cl|A| \in H^1(C_2; C(E))$, with $x\overline{A}A = O(E)$. Then $\overline{A}A = x^{-1}O(E)$, so that applying the norm map, $N(A) \cdot O(E) = \overline{A}A = x^{-1}O(E)$. We compute,

$\overline{d_1}cl|A| = cl(x\overline{A}^{-1}) = cl(x^{-1}\overline{A}) = i(N(A))$. Thus $\text{im} d_1 \subseteq \text{im} i$.

Conversely, consider $|A| \in \chi(E/F)$, where $AO(E) = zO(E) = \overline{z}O(E)$. Let $u = zz^{-1}$, so $u \in O(E)^*$, and $i(|A|) = cl u$. 
Since at least one prime ramifies, we can find \( y \in F^* \) with 
\[
(y, \sigma)_P = (-1)^{v_p(y)} = (-1)^{v_p(A)} \quad \text{at } P \text{ inert.}
\]
Thus \( yA \) has even order at \( P \) inert.

Consider \( yA0(E) \). We can write 
\[
yA0(E) = BB, \quad \text{for some fractional } O(E)\text{-ideal } B.
\]
Since \( A0(E) = z0(E) \), we have \( yA0(E) = yz0(E) = BB \).

Thus, 
\[
d_1\text{cl}|B| = \text{cl}(yz)^{-1}(\overline{yz}) = \text{cl}(z^{-1}z) = \text{cl}(u^{-1}).
\]
However, \( \text{cl}(u) = \text{cl}(u^{-1}) \) since \( uu = 1 \). Thus 
\[
i(|A|) = \text{cl}(u) = \text{cl}(u^{-1}) = d_1\text{cl}|B|, \quad \text{and } \text{im } i \subseteq \text{im } d_1 \text{ as was to be shown. } \Box
\]

We may now state:

**Theorem 5.7** If \( E/F \) has no ramified primes, there is an exact sequence:
\[
1 \to H^1(C_2; \mathcal{C}(E)) \xrightarrow{d_1} H^1(C_2; O(E)^*) \xrightarrow{\mu} \text{Iso}(E/F)
\]
\[
\eta H^2(C_2; \mathcal{C}(E)) \xrightarrow{d_2} H^2(C_2; O(E)^*) \to 1.
\]
This follows since Gen is trivial in this case. \( \Box \)

We may interpret this sequence by Lemma 5.5. Since there are no ramified primes, \( \eta(E/F) \) is isomorphic to \( H^1(C_2; O(E)^*) \) by 5.4. Thus the image of \( \mu \) is the same as the image of \( \phi \). In other words, the image of \( \mu \) consists of pairs \(|1, A0(E)|\). If \( A \) splits, then clearly 
\[
|1, A0(E)| = |1, BB| = |1, O(E)|.
\]
Hence, in order for \( \mu \)
to be non-trivial, it is necessary and sufficient that some inert prime $p$ in $O(F)$ become principal in $O(E)$. 
Chapter VIII  THE BOUNDARY

Our ultimate goal is to study the octagon over Z, when \( k = \pm 1 \). In order to do this, we relate \( W(k, Z) \) to \( W(k, Q) \) by means of an exact sequence. In Section 1, the boundary homomorphism, \( \partial \), is defined. This enables us to establish an exact sequence:

\[
0 \to W(k, Z) \to \mathcal{W}(k, Q) \to W(k, Q/Z) \to 0.
\]

Next let \( S = \mathbb{Z}[t, t^{-1}]/(f(t)) \), where \( f(t) \) is a \( T_k \)-fixed monic, integral, irreducible polynomial. We have the decomposition \( \mathcal{W}(k, Q) \cong \bigoplus_{f} W(k, Q; f) = \bigoplus_{f} W(k, Q; S) \). It is thus natural to consider the restriction of \( \partial \) to \( W(k, Q; S) \). We denote this restriction \( \partial(S) \). We wish to compute \( \partial(S) \). This will eventually allow us to analyze \( \partial \).

The first step is the reduction of the study of \( \partial(S) \) to the study of \( \partial(D) : W(k, Q; D) \to W(k, Q/Z; D) \) where \( D \) is the Dedekind ring of integers in \( E = \mathbb{Q}[t, t^{-1}]/(f(t)) \). This is done in Section 2.

This reduction involves the trace homomorphism and the set \( T(\mathfrak{m}) \). \( T(\mathfrak{m}) = \{ \text{maximal ideals } \mathfrak{p} \text{ in } D : \mathfrak{p} \cap S = \mathfrak{m} \} \).
maximal in $S$, and $\mathcal{F} = \overline{\mathcal{F}}$. We thus discuss $\text{tr}_*$ and $T(\mathfrak{m})$ in Section 3.

In Section 4, we begin the computation of $\mathfrak{a}$ by studying the local case, $\mathfrak{a}(D,\mathcal{F})$. Here $\mathfrak{a}(D,\mathcal{F})$ is the localization of the $\mathfrak{a}(D)$ sequence at an invariant maximal ideal, $\mathcal{F}$, in $D$. This is used in Section 5 to compute the cokernel of $\mathfrak{a}(D)$.

As $D$ varies, $\mathfrak{a}(D)$ will have image in $W(k,\mathbb{Q}/\mathbb{Z};D)$. However, different $\mathfrak{a}(D)$ may have the same image in $W(k,\mathbb{Q}/\mathbb{Z})$. This coupling between different $\mathfrak{a}(D)$ is discussed in Section 6.

These results then enable us to study $\mathfrak{a} : \mathcal{W}(k,\mathbb{Q}) \to W(k,\mathbb{Q}/\mathbb{Z})$. For $k = \pm 1$, we show $\mathfrak{a}$ is onto in Section 7. This will be the key to analyzing the octagon over $\mathbb{Z}$ in Chapter 10.
1. The Boundary Homomorphism

We shall construct an exact sequence

\[ 0 \rightarrow W(k, \mathbb{Z}) \xrightarrow{i} \mathcal{V}(k, \mathbb{Q}) \xrightarrow{\mathcal{A}} W(k, \mathbb{Q}/\mathbb{Z}). \]

The script \( \mathcal{V} \) indicates that we have placed a restriction on the minimal polynomial of \( \ell \) in the degree \( k \) mapping structure \([M, B, \ell] \). This restriction is that the minimal polynomial of \( \ell \) be a monic integral polynomial.

We also should observe that this construction works equally well for \( \mathbb{Z} \) replaced by \( D \) a Dedekind Domain, \( \mathbb{Q} \) replaced by \( E \) the quotient field of \( D \), and \( \mathbb{Q}/\mathbb{Z} \) replaced by \( E/D \). The sequence then reads:

\[ 0 \rightarrow W(k, D) \xrightarrow{i} \mathcal{V}(k, E) \xrightarrow{\mathcal{A}} W(k, E/D). \]

Note that this construction also applies to the asymmetric case. We similarly will obtain an exact sequence:

\[ 0 \rightarrow A(\mathbb{Z}) \rightarrow A(\mathbb{Q}) \rightarrow A(\mathbb{Q}/\mathbb{Z}). \]

If \([M, B] \in A(\mathbb{Q})\), with symmetry operator \( s \) satisfying \( B(x, y) = B(y, sx) \), we then require that the minimal polynomial of \( s \) be a monic integral polynomial in order for
Proposition 1.1  A degree $k$ mapping structure $(M, B, \ell)$ over $\mathbb{Q}$ contains an $\ell$ invariant integral lattice $A$ if and only if the minimal polynomial of $\ell$ is a monic, integral polynomial.

Comments: (1) $\ell$ is replaced by the symmetry operator $s$ for the asymmetric case, $\mathcal{G}(\mathbb{Q})$.

(2) All lattices are assumed to be full.

[BS99] This means that $A$ is a finitely generated $\mathbb{Z}$-submodule of $M$ with rank $A = \text{rank } M$.

(3) An integral lattice $A$ is one on which the inner product $B$ is integrally valued.

Proof: Necessity. We assume $M$ contains an $\ell$ invariant integral lattice $A$. Since $B|_A$ is integral the characteristic polynomial of $\ell$ is integral. Since the characteristic polynomial of $\ell$ is integral, so is the minimal polynomial $[L - 1 | 402]$.

Sufficiency. Conversely, suppose that the minimal polynomial, $f(x)$, of $\ell$ is integral. Write
\[ f(x) = a_0 + a_1x + \ldots + a_{m-1}x^{m-1} + x^m. \]

By Lemma II. 1.1, \( f(\ell) = f(\ell^*) = 0 \). Thus, we obtain the identities
\[
\ell^m = -\sum_{i=0}^{m-1} a_i \ell^i, \quad (\ell^*)^m = -\sum_{i=0}^{m-1} a_i (\ell^*)^i.
\]

We now construct an \( \ell, \ell^* \) invariant integral lattice. Let \( \{e_1, \ldots, e_n\} \) be a basis for \( M \) over \( \mathbb{Q} \).

Since \( B(x, y) \in \mathbb{Q} \) for \( x, y \in M \), we can find integers \( r_{ijk} \) and \( s_{ijk} \) so that \( r_{ijk} B(\ell^k e_i, e_j) \in \mathbb{Z} \) and \( s_{ijk} B((\ell^*)^k e_i, e_j) \in \mathbb{Z} \) for \( i, j \leq n \) and \( k < m \). Let \( d = \prod r_{ijk} s_{ijk} \). Then clearly \( \{de_1, \ldots, de_n\} \) generates a free \( \mathbb{Z} \)-module on which \( B \) is integrally valued. Let \( f_i = de_i \). We then define \( A \) to be the \( \mathbb{Z} \)-lattice spanned by

\[
\{f_1, \ldots, f_n, \ell f_1, \ell f_2, \ldots, \ell f_n, \ell^2 f_1, \ldots, \ell^2 f_n, \ldots
\]
\[
\ell^{m-1} f_1, \ldots, \ell^{m-1} f_n, \ell^* f_1, \ldots, \ell^* f_n, \ldots, \ell^{m-1} f_1, \ldots
\]
\[
\ell^{m-1} f_n \}.
\]

Here \( m = \text{degree } f(x) \), the minimal polynomial of \( \ell \) and \( \ell^* \). The identities for \( \ell^m \), (\( \ell^* \))^m, together with \( \ell \ell^* = k \) show that \( A \) is \( \ell \) and \( \ell^* \) invariant. Since \( B \) is integral on \( \{f_1, \ldots, f_n\} \), it follows that \( B \) is integral on \( A \) because \( \ell \) has degree \( k \). Thus \( A \) is an \( \ell, \ell^* \) invariant integral lattice as desired. \( \square \)

Definition 1.2 \( \mathcal{Y}(k, \mathbb{Q}) \) (respectively \( \mathcal{G}(\mathbb{Q}) \)) denotes
Witt equivalence classes \([M, B, \ell]\) in \(W(k, Q)\) for which the minimal polynomial of \(\ell\) (respectively \(s\)) is integral.

We now define the maps in the boundary sequence:

\[
0 \rightarrow W(k, Z) \rightarrow \frac{k}{(k,0)} \rightarrow W(k, Q/Z) \rightarrow 0
\]

If \([M, B, \ell] \in W(k, Z)\), \(i[M, B, \ell] = [M, B, \ell] \otimes_{Z} Q\).

The boundary homomorphism \(\partial\) is more involved. Let \([M, B, \ell] \in \frac{k}{(k, Q)}\). By Proposition 8.1, we can find an \(\ell\) invariant integral lattice \(L \subset M\). We define the dual lattice of \(L\) to be

\[
L^\# = \{x \in M : B(x, L) \subset Z\}.
\]

We observe that \(L^\# \supset L\), with rank \(L^\# = \text{rank} L\).

Thus \(L^\# / L\) is a finitely generated torsion \(Z\)-module.

If \(\bar{x}, \bar{y} \in L^\# / L\), we let \(x\) and \(y\) be preimages of these equivalence classes in \(L^\#\). Let \(q : Q \rightarrow Q/Z\) be the quotient map. We define \(B' : L^\# / L \times L^\# / L \rightarrow Q/Z\) by:

\[
B'(\bar{x}, \bar{y}) = q \circ B(x, y).
\]

We define \(\ell' : L^\# / L \rightarrow L^\# / L\) by:
\[ \ell' (\bar{x}) = \overline{\ell(x)} \]

We now need to show that \([L^\# / L, B', \ell']\) is an inner product space in \(W(k, \mathbb{Q}/\mathbb{Z})\). Once this is done, we define \(\mathfrak{d}\) by:

\[ \mathfrak{d} : [M, B, \ell] \to [L^\# / L, B', \ell'] \]

To begin with, since \(L\) is \(\ell\) invariant so is \(L^\#\). Thus \(\ell'\) is well-defined.

The fact that \(B\) is integral on \(L\) implies that \(B'\) is well-defined on \(L^\# / L\), with values in \(\mathbb{Q}/\mathbb{Z}\).

We must show that \(B'\) is non-singular. In order to do this we define

\[ h : L^\# \to \text{Hom}_\mathbb{Z}(L^\# / L, \mathbb{Q}/\mathbb{Z}) \]

by

\[ x \mapsto B' (\bar{x}, -) \]

We need to show that \(h\) is epic, with kernel \(L\). We shall need:

Lemma 1.3 For a lattice \(L\), \((L^\#)^\# = L\)

Proof: If \([v_1, \ldots, v_n]\) is a basis for \(L\), then a basis for \(L^\#\) is given by \([v_1^\#, \ldots, v_n^\#]\), where
\( B(v_i, v_j) = \delta_{ij} \). It follows that a basis for \((L^\#)\) is \(\{v_1, \ldots, v_n\} \). 

We now show \( h \) is onto.

Let \( g : L^\# / L \to \mathbb{Q}/\mathbb{Z} \) be given. Let \( \{w_1, \ldots, w_n\} \) be a basis for \( L^\# \). Then define \( \hat{g} : L^\# / L \to \mathbb{Q} \) as follows. \( \hat{g}(w_i) \) is chosen so that:

\[
q \circ \hat{g}(w_i) = g(w_i) \quad \text{for} \quad w_i \in L^\# / L
\]

Then extend \( \hat{g} \) linearly to \( L^\# \).

In fact, tensoring with \( \mathbb{Q} \), we may assume \( \hat{g} \in \text{Hom}_L(M, \mathbb{Q}) \). Thus, since \( B \) is an inner product we may write

\[
B(v_1, -) = \hat{g}(-).
\]

However \( B(v_1, -) \) is integer valued on \( L \) since \( \hat{g}(-) = B(v_1, -) \) and \( \hat{g} \) is integral on \( L \). Thus \( v_1 \in L^\# \). It is now clear that

\[
g(-) = q \circ \hat{g}(-) = q \circ B(v_1, -) = B'(v_1, -),
\]

and \( h \) is onto.

In order to show that the kernel of \( h \) is \( L \), suppose \( h(w) = 0 \). So \( B'(\bar{w}, -) = 0 \). Thus, for all \( v \in L^\# \), we have \( B(w, v) \in \mathbb{Z} \). Hence, \( w \in (L^\#)^\# = L \) by Lemma 8.3, and the kernel of \( h \) is \( L \).
Thus, given \([M, B, \ell] \in \mathcal{W}(k, Q)\), we have a method for obtaining an element in \(W(k, Q/\mathbb{Z})\). We define:

\[ \alpha[M, B, \ell] = [L^\# / L, B', \ell'] \]

**Lemma 1.4** \(\alpha\) is well-defined.

In order to show that \(\alpha\) is well-defined, we must show that this construction is independent of the lattice chosen; and that \(\alpha\) is trivial on metabolic forms. If we do this, then \(\alpha\) preserves Witt-equality and gives a homomorphism of Witt groups.

(a) Independence of lattice.

Let \(L_0\) be another choice of \(\ell\) invariant integral lattice. Without loss of generality, \(L_0 \subseteq L\), for otherwise, we can show that both \(L_0\) and \(L\) give the same result as the \(\ell\) invariant integral lattice \(L_0 \cap L\).

We have then \(L_0 \subseteq L \subseteq L^\# \subseteq L_0^\#\). Consider \(L/L_0 \subseteq L_0^\# / L_0\). The annihilator of \(L/L_0\) in \(L_0^\# / L_0\) is \((L/L_0)^\perp = L^\# / L_0\). Thus, by I.5.4, \([L_0^\# / L_0, B', \ell']\) is Witt-equivalent to \([(L/L_0)^\perp / (L/L_0), B_0'', \ell_0'']\) where \(B_0', \ell_0''\) indicate the appropriate induced forms. However,
(L/L_0)^(-1)/(L/L_0) = (L'/L_0)/(L/L_0) \cong L'/L.

Hence \([L_0'/L_0, B_0', \ell_0'] = [L'/L, B', \ell']\) as was to be shown.

(b) We now show \(\hat{\alpha}\) (metabolic) \(\sim 0\). So let 
\((M, B, \ell) \sim 0\) with metabolizer \(N\). Let \(L\) be an \(\ell\) invariant lattice. Define \(N_1 = N \cap L'\).

Clearly \(N_1\) is \(\ell\) invariant. We also have the exact sequence 
\[0 \to N_1 \to L' \to L'/N_1 \to 0.\]

Claim: This sequence splits so that \(N_1\) is a summand of \(L'\).

To show this, it clearly suffices to show \(L'/N_1\) is torsion free and hence projective. Suppose to the contrary that there is 
\[x \in L', x \not\in N_1, d \neq 0\] with \(dx \in N_1\).

Now 
\[dx \in N_1 \subset N,\]
so

\[B(dx, y) = dB(x, y) = 0\] for all \(y \in N_1\).

Tensoring with \(Q\),

\[(N_1 \cap L') \otimes Q \cong N\]
Therefore
\[ dB(x,y) = 0 \quad \text{for all} \quad y \in N, \quad \text{and} \]
\[ B(x,y) = 0 \quad \text{for} \quad y \in N. \]

Hence
\[ x \in N^\perp = N, \quad x \in L^\# \]
so that
\[ x \in N \cap L^\# = N_1. \quad \text{Contradiction.} \]

It follows that \( L^\#/N_1 \) is torsion free, and \( N_1 \) is a summand of \( L^\# \).

Now let \( H = (L^\# \cap N) / L \subset L^\#/L \). Clearly \( H \subset H^\perp \).

Conversely, suppose \( k \in H^\perp \). This means \( B'(k,h) \in Z \) for all \( h \in H \). Let \( k \) be a lift of \( k \) to \( L^\# \). Consider

\[ B(k,\cdot) : L^\# \cap N \rightarrow Z \]

\[ L^\# \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow 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\rightarrow \rightarrow \rightarrow \rightarrow \right Fraction Handling in Large Sums, Continued}
Thus $k-w \in (L^\# \cap N)^\perp$. Since $L^\#$ is a lattice, $k-w \in N^\perp = N$.
Clearly $k-w \in L^\#$. Thus $k-w \in L^\# \cap N = N_1$.

Now $(\overline{k-w}) \in H$. However, $w \in L$, so $\overline{w} = 0 \in H$. Thus $\overline{k} \in H$ as desired. So $H^\perp = H$ and $\mathfrak{a}$ (metabolic) \sim 0. □

Since we have now shown that $\mathfrak{a}$ is well-defined and independent of the choices made, we are ready to prove:

**Theorem 1.5**

The sequence

\[
0 \twoheadrightarrow W(k,Z) \xrightarrow{i} \gamma(k,Q) \xrightarrow{\mathfrak{a}} W(k,Q/Z)
\]

is exact.

**Proof:** $i$ is 1-1 by Lemma I 4.4.

$\text{im } i \subseteq \ker \mathfrak{a}$.

Let $[M,B,\mathfrak{k}] \rightarrow [M,B,\mathfrak{k}] \otimes Z Q$. Choose the lattice $M \otimes I = L$ in $M \otimes Z Q$. For this choice of $L$, we have $L \approx \text{Hom}_Z(L,Z)$ since $B$ is non-singular on $M$. Thus $L^\# = L$ and $\mathfrak{a} \circ i = 0$.

$\ker \mathfrak{a} \subseteq \text{im } i$.

Suppose $\mathfrak{a}([M,B,\mathfrak{k}] = 0$. Let $H \subseteq L^\#/L$ be a metabolizer for $(L^\#/L, B', \mathfrak{k}')$.

Let $L_0 = \text{inverse image of } H \in L^\# \subseteq M$ under the projection $L^\# \twoheadrightarrow L^\#/L$. Then $B|_{L_0}$ has values in $Z$ since $H$ is a self-annihilating subspace of $L^\#/L$, meaning that $B' = 0 \in Q/Z$ on $H$. 
\[ L^\#_0 = \{ x \in M : B(x, L_0) \in \mathbb{Z} \}. \text{ If } x \in L^\#_0, \text{ projecting to } L^\#_0/L, \quad \overline{x} \in H^1 = H. \text{ Thus } x \in L_0. \text{ Obviously } L_0 \subset L^\#_0, \]

so \( L_0 = L^\#_0 \cong \text{Hom}_\mathbb{Z}(L_0, \mathbb{Z}) \), and the adjoint is an isomorphism on \( L_0 \).

\( L_0 \) is \( \ell \) invariant since \( H \) is \( \ell' \) invariant. Thus consider \( [L_0, B|_{L_0}, \ell|_{L_0}] \in W(k, \mathbb{Z}) \). Applying \( i \), we obtain \( [M, B, \ell] \), since \( L \) is a full lattice. \( \square \)

**Corollary 1.6**  The sequence

\[
0 \to A(\mathbb{Z}) \to A(\mathbb{Q}) \to A(\mathbb{Q}/\mathbb{Z}) \text{ is exact.} \]
2. **Reducing to the maximal order**

We continue our study of the boundary homomorphism by recalling the computation of \( \mathcal{H}(k, Q) \) and \( \mathcal{G}(Q) \). This was done as follows. Let

\[
S = \mathbb{Z}(\theta) = \mathbb{Z}[t, t^{-1}]/(f(t)),
\]

where \( f(t) \) is a monic, integral, irreducible \( T_k \) fixed polynomial. Let

\[
E = \mathbb{Q}[t, t^{-1}]/(f(t)).
\]

Then, as we have seen, the field \( E \) has an involution induced from \( T_k \). The fixed field of \( - \) is denoted \( F \). \( S \) is an order in \( \mathcal{O}(E) \), the ring of integers in \( E \). To simplify our notation, we write \( \mathcal{O}(E) = D \).

We recall Theorem 1.8, which computes

\[
\mathcal{H}(k, Q) \approx \bigoplus_{f \in \mathcal{F}} \mathcal{W}(k, Q; f) \approx \bigoplus \mathcal{W}(k, Q; S) - \bigoplus_{f \in \mathcal{F}} \mathcal{W}(k, Q; f).
\]

**Note:** Here we are using the symbol \( \mathcal{F} \) to denote the collection of \( T_k \) fixed, monic, integral, irreducible polynomials. This should not be confused with III 1.8 where the integral requirement is omitted, for \( \mathcal{W}(k, Q) \).

We denote the restriction of \( \mathcal{O} \) to \( \mathcal{W}(k, Q; S) \) by \( \mathcal{O}(S) \).

**Lemma 2.1** There are exact sequences:

\[
0 \rightarrow \mathcal{W}(k, \mathbb{Z}; S) \overset{i}{\rightarrow} \mathcal{W}(k, Q; S) \overset{\mathcal{O}(S)}{\rightarrow} \mathcal{W}(k, Q/\mathbb{Z}; S).
\]
0 \rightarrow A(Z;S) \xrightarrow{\delta} A(Q;S) \xrightarrow{\delta(S)} A(Q/Z;S)

These follow from Theorem 1.5, since the $S$-module structure is preserved by $i$ and $\delta(S)$. □

Remark: $W(k,Q;S) \approx W(k,Q;S)$ since the $S$ given determines that the minimal polynomial of the degree $k$ map is integral.

We identified each of the above groups with their Hermitian counterparts in III 3.2 and 3.3. Letting $I = \text{inverse different of } S \text{ over } Z = \Delta^{-1}(S/Z)$, we obtained:

\begin{align*}
W(k,Z;S) & \approx H(I) \\
W(k,Q;S) & \approx H(E) \\
W(k,Q/Z;S) & \approx H(E/I) \approx \bigoplus_{\mathfrak{m} \text{ maximal in } S} H(E/I(\mathfrak{m}))
\end{align*}

In the asymmetric case, we had:

\begin{align*}
A(Z;S) & \approx H_{\phi}(\Delta^{-1}(S/Z)) \\
A(Q;S) & \approx H_{\phi}(E) \\
A(Q/Z;S) & \approx H_{\phi}(E/\Delta^{-1}(S/Z))
\end{align*}

Thus, these isomorphisms together with Lemma 2.1 lead us to the exact sequence:

\begin{align*}
0 \rightarrow H_u(I) \xrightarrow{\delta} H_u(E) \xrightarrow{\delta(S)} H_u(E/I)
\end{align*}

We need to discuss how this boundary map induced
from Lemma 2.1, which we continue to call \( \partial(S) \), is computed. \( \partial(S) \) is defined as follows.

Let \([M,B] \in H_u(E)\). We first construct an \( S \)-lattice \( L \) in \( M \) so that \( B|_L \) takes values in \( I \). To do this, we follow Proposition 1.1. Let \( \{e_1, \ldots, e_n\} \) be a basis for \( M \). Then we can find non-zero integers \( r_{ijk}, s_{ijk} \) so that \( r_{ijk} B(\theta^k e_i, e_j) \in I \) and \( s_{ijk} B(\overline{e}_i, e_j) \in I \) for \( i,j \leq n \) and \( k < m \), where \( m \) is the degree of the minimal polynomial of \( \theta \). Let \( d = \prod r_{ijk} s_{ijk} \). Then

\[ L = \langle d e_1, \ldots, d e_n, \theta d e_1, \ldots, \theta^m d e_1, \ldots, \theta^m d e_n \rangle \]
generates an \( S \)-lattice on which \( B \) is \( I \)-valued, as in 1.1.

Let \( L^\# = \{ v \in M : B(v, L) \in I \} \). \( L^\# \) is also an \( S \)-lattice. We form \( L^\#/L \), with induced \( E/I \)-valued inner product \( B' \) as before. Then define \( \partial(S) \)

\[ \partial(S) : [M,B] \to [L^\#/L, B'] \] . With \( \partial(S) \) so defined, the following is clear.

\textbf{Proposition 2.2} \( \partial(S) \) is well-defined for Hermitian and the following diagram commutes.

\[
\begin{array}{c}
0 \to \mathbb{W}(k, Z; S) \xrightarrow{i} \mathbb{W}(k, Q; S) \xrightarrow{\partial(S)} \mathbb{W}(k, Q/Z; S) \\
\uparrow \cong \quad \uparrow \cong \quad \uparrow \cong \\
0 \to H(\Delta^{-1}(S/Z)) \to H(E) \xrightarrow{\partial(S)} H(E/\Delta^{-1}(S/Z))
\end{array}
\]
In order to compute \( \alpha(S) \), we begin by comparing this with the exact sequence for \( \alpha(D) \), where \( D \) is the maximal order. We have the exact sequence of "forgetful" maps.

**Lemma 2.3** Let \( i_1, i_2, i_3 \) be the maps which forget the \( D \)-module structure and remember only the \( S \)-module structure. Then the diagram below commutes:

\[
\begin{array}{ccc}
0 & \rightarrow & W(k, \mathbb{Z}; D) \\
\downarrow i_1 & & \downarrow i_2 \\
0 & \rightarrow & W(k, \mathbb{Z}/\mathbb{Z}; D) \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & W(k, \mathbb{Z}; S) \\
\downarrow i_3 & & \downarrow i_4 \\
0 & \rightarrow & W(k, \mathbb{Z}/\mathbb{Z}; S) \\
\end{array}
\]

**Proof:** We begin by remarking that the notation \( W(k, \mathbb{Q}; D) \) or \( W(k, \mathbb{Q}; S) \) is somewhat redundant. Indeed, \( W(k, \mathbb{Q}; D) \approx W(k, \mathbb{Q}; S) \approx H(E) \).

In order to see commutativity, recall that \( \alpha(D) \) is defined by choosing an integral \( D \)-lattice, examining \( L^\# / L \), etc. However, "by forgetting," the \( D \)-lattice \( L \) is also an \( S \)-lattice, \( L^\# \) is the same, and so consequently is \( L^\# / L \).

Combining Lemmas 2.2 and 2.3, we have

**Corollary 2.4** The diagram below commutes:
0 \to H(\Delta^{-1}(D/Z)) \to H(E) \overset{\partial(D)}{\to} H(E/\Delta^{-1}(D/Z)) \\
\downarrow i_1' \quad \downarrow i_2' \quad \downarrow i_3'  \\
0 \to H(\Delta^{-1}(S/Z)) \to H(E) \overset{\partial(S)}{\to} H(E/\Delta^{-1}(S/Z)) .

i_1', i_2', i_3' are the composition of the isomorphisms in Lemma 2.2, with i_1, i_2 and i_3 . □

We next recall the computation of \( W(k, Q/Z; D) \),

\( W(k, Q/Z; D) \cong H(E/\Delta^{-1}(D/Z)) \) given in Chapter III 3.5,3.8. We had:

\[
W(k, Q/Z; D) \cong \bigoplus_{\varphi=\overline{\varphi}} W(k, F_p; D/\varphi) \cong \bigoplus_{\varphi=\overline{\varphi}} H(D/\varphi)
\]

Here we sum over \( - \)-invariant maximal ideals in \( D \). Similar computations apply to \( W(k, Q/Z; S) \). We denote by \( \{ \mathfrak{m} \} \) the collection of \( - \)-invariant maximal ideals in \( S \).

**Proposition 2.5** The diagram below commutes.

\[
\begin{array}{ccc}
W(k, Q/Z; D) \cong \bigoplus_{\varphi=\overline{\varphi}} W(k, F_p; D/\varphi) & \xrightarrow{\text{tr}_*} & \bigoplus_{\varphi=\overline{\varphi}} H(D/\varphi) \\
\downarrow i_3 & & \downarrow \oplus \text{tr}_* \\
W(k, Q/Z; S) \cong \bigoplus_{\mathfrak{m}=\overline{\mathfrak{m}}} W(k, F_p; S/\mathfrak{m}) & \xrightarrow{\text{tr}_*} & \bigoplus_{\mathfrak{m}=\overline{\mathfrak{m}}} H(S/\mathfrak{m})
\end{array}
\]
The map \( \text{tr}_*: H(D/\mathfrak{m}) \to H(S/\mathfrak{m}) \) where \( \mathfrak{m} = \mathfrak{m} \cap S \)

is given by trace on finite fields.

Proof: Let \([M,B] \in H(D/\mathfrak{m})\). Then applying \( \text{tr}_* \)
we obtain a corresponding element in \( W(k,F_p;D/\mathfrak{m}) \).
We apply \( i_3 \) and "forget" to obtain an element in \( W(k,Q/Z;S) \).

However \( \text{tr}_*: D/\mathfrak{m} \to F_p \) is the same as the com-
position \( \text{tr}_*: D/\mathfrak{m} \to S/\mathfrak{m} \cap S \to F_p \). Thus
\( i_3 \circ \text{tr}_* = \text{tr}_* \circ \text{tr}_* \), ie. the above diagram commutes. \( \square \)

We now wish to collect those ideals in \( D \) which lie
over a given ideal \( \mathfrak{m} \) in \( S \). So let \( T(\mathfrak{m}) = \{ \mathfrak{m} : \mathfrak{m} \)
is a \( \mathfrak{m} \)-invariant maximal ideal in \( D \) with \( \mathfrak{m} \cap S = \mathfrak{m} \} \),
where \( \mathfrak{m} \) is a \( \mathfrak{m} \)-invariant maximal ideal in \( S \). \( T \) may
or may not be empty. In Section 3, we shall discuss
this further.

We now define the local boundary

\( \partial(D,\mathfrak{m}) : H(E) \to H(D/\mathfrak{m}) \) by :

\( \partial(D,\mathfrak{m}) = q(\mathfrak{m}) \circ \partial(D) \)

where \( q(\mathfrak{m}) \) is projection to the \( \mathfrak{m}^{th} \) coordinate.

\[ H(E/\Delta^{-1}(D/\mathfrak{m})) \cong \bigoplus_{\mathfrak{m} = \mathfrak{m}} H(D/\mathfrak{m}) \]
The local boundary $\partial(S, \mathfrak{m})$ is similarly defined. These are related by:

**Theorem 2.6** \[ \partial(S, \mathfrak{m}) = \bigoplus_{\mathfrak{q} \in T(\mathfrak{m})} \text{tr}_* \circ \partial(D, \mathfrak{q}). \]

**Proof:** Since $i_3' \circ \partial(D) = \partial(S)$, we need only determine which ideals $\mathfrak{q}$ in $D$ lie over $\mathfrak{m}$ in $S$ by Proposition 2.5. These are given by $T(\mathfrak{m})$ by definition. The commutative diagram in 2.5 then yields:

\[ \partial(S, \mathfrak{m}) = \bigoplus_{\mathfrak{q} \in T(\mathfrak{m})} \text{tr}_* \circ \partial(D) \]
\[ = \bigoplus_{\mathfrak{q} \in T(\mathfrak{m})} \text{tr}_* \circ \partial(D, \mathfrak{q}) \]

In order to tabulate the progress we have made, we record the following diagram.

\[
\begin{array}{ccccccccc}
0 & \rightarrow & H(\Delta^{-1}(D/Z)) & \rightarrow & H(E) & \rightarrow & H(E/\Delta^{-1}(D/Z)) \approx \oplus H(D/\mathfrak{q}) \\
& & \downarrow \approx & & \downarrow \approx & & \downarrow \approx \\
0 & \rightarrow & W(k, Z; D) & \overset{i_1}{\rightarrow} & W(k, Q; D) & \overset{\partial(D)}{\rightarrow} & W(k, Q/Z; D) \\
& & \downarrow i_1 & & \downarrow i_2 & & \downarrow i_3 \\
0 & \rightarrow & W(k, Z; S) & \overset{i}{\rightarrow} & W(k, Q; S) & \overset{\partial(S)}{\rightarrow} & W(k, Q/Z; S) \\
& & \approx \uparrow & & \approx \uparrow & & \approx \uparrow \\
0 & \rightarrow & H(\Delta^{-1}(S/Z)) & \rightarrow & H(E) & \overset{\partial(S)}{\rightarrow} & H(E/\Delta^{-1}(S/Z)) \approx \bigoplus_{\mathfrak{m} = \mathfrak{m}} H(S/\mathfrak{m})
\end{array}
\]
We wish to compute the group \( W(k, \mathbb{Z}; S) \approx H(\Delta^{-1}(S/\mathbb{Z})). \) The idea is the following. We compute the group 
\( W(k, \mathbb{Z}; D) \), for the maximal order \( D \) by using the exact sequence with \( \partial(D) \). Then we compute the map \( i_1 \).

Lemma 2.7 \( i_1 \) is 1-1

Proof: Clear by chasing the diagram, since \( i \) is 1-1 and \( i_2 \) is an isomorphism. □

In order to compute the coker of \( i_1 \) we use homological algebra. From \( \text{[M 50]} \) \( \text{im} \partial(D) \cap \ker i_3 = \text{coker} i_1 \). Thus we need to compute \( \partial(D) \), and \( i_3 \).

By Proposition 2.5, \( i_3 \) is determined by two things:

1. The map \( \text{tr}^* : H(D/\mathfrak{p}) \to H(S/\mathfrak{p} \cap S) \)
2. \( T(\mathfrak{m}) = \{ \text{maximal ideals } \mathfrak{p} \text{ in } D: \mathfrak{p} \cap S = \mathfrak{m}, \text{ with } \mathfrak{p} = \overline{\mathfrak{p}} \} \)

These will be discussed in Section 3.

In analyzing the group \( W(k, \mathbb{Z}; D) \approx H(\Delta^{-1}(D/\mathbb{Z})), \) we use the exact sequence from Lemma 2.2.

\[
0 \to H(\Delta^{-1}(D/\mathbb{Z})) \to H(E) \to H(E/\Delta^{-1} D/\mathbb{Z})
\]

Recall that \( H(E) \), for \( E \) an algebraic number field, and \( H(E/\Delta^{-1}) \approx \oplus H(D/\mathfrak{p}) \) are known. (Here we abbreviate \( \Delta^{-1}(D/\mathbb{Z}) \) by \( \Delta^{-1}. \)) Thus we need to
analyze the boundary map $\partial(D)$. We reduce to the local case using:

**Proposition 2.8** Localization at a $-\infty$ invariant, maximal ideal $\mathfrak{p}$ in $D$ induces the commutative diagram:

$$
0 \to H(\Delta^{-1}(D/\mathbb{Z})) \to H(E) \xrightarrow{\partial(D)} H(E/\Delta^{-1}) \cong \bigoplus_{\mathfrak{p}} H(D/\mathfrak{p}) \\
\downarrow \quad \downarrow \\
0 \to H(\Delta^{-1}(\mathfrak{p})) \to H(E) \xrightarrow{\partial(D,\mathfrak{p})} H(E/\Delta^{-1}(\mathfrak{p})) \cong H(D/\mathfrak{p})
$$

This diagram also commutes for an order $S$.

**Proof:** $\partial(D,\mathfrak{p}) = q(\mathfrak{p}) \circ \partial(D)$ by definition. What we show here is that these maps are induced by localization.

As long as $S$ is an order in $D$, maximal or not, $S$ is a finitely generated $\mathbb{Z}$-algebra, and hence Noetherian by $[A,M 81]$ . Thus by $[B-2 20]$ , every finitely generated $S$-module $M$ is finitely presented. Hence $[B-2 76]$ the adjoint isomorphism $M \to \text{Hom} (M,E/\Delta^{-1})$ localizes to $M(\mathfrak{m}) \to \text{Hom}_S(\mathfrak{m})(M(\mathfrak{m}), (E/\Delta^{-1})(\mathfrak{m}))$, where $\mathfrak{m}$ is a maximal ideal in $S$. Thus localization preserves the non-singularity of the forms, and indeed defines a map.
In order to identify \((E/\Delta^{-1})(\varphi)\) with \(E/\Delta^{-1}(\varphi)\), use the exact sequence

\[ 0 \to \Delta^{-1} \to E \to E/\Delta^{-1} \to 0, \]
and [A,M 39]. □

Comment: This re-emphasizes the remarks before Theorem III 1.9.

Corollary 2.9 \(H(\Delta^{-1}(D/Z)) = \bigcap_{\varphi=\varphi} H(\Delta^{-1}(D/Z)(\varphi))\)

Proof: \(H(\Delta^{-1}) = \ker \partial(D)\)

\[ = \bigcap_{\varphi=\varphi} \ker q(\varphi) \circ \partial(D) \]
\[ = \bigcap_{\varphi=\varphi} \ker \partial(D, \varphi) \]
\[ = \bigcap_{\varphi=\varphi} H(\Delta^{-1}(\varphi)) \] □

The computation of \(\partial(D, \varphi)\) will be made in Section 4.
3. Computing $\text{tr}_*$ and describing $T(\mathfrak{m})$

The computation of $\text{tr}_*$ is given by:

**Theorem 3.1** Suppose $[E : K] < \infty$ is an extension of finite fields. $E$ has an involution - which is possibly trivial. Then the homomorphism $\text{tr}_*$ is given as follows:

1. If $-$ is non-trivial on both $E$ and $K$,
   $\text{tr}_* : H(E) \to H(K)$ is an isomorphism.
2. If $-$ is trivial on both $E$ and $K$,
   $\text{tr}_* : W(E) \to W(K)$ is an isomorphism if $[E : K]$ is odd. If $[E : K]$ is even,
   $\text{tr}_*$ is an isomorphism on the fundamental ideal, and has kernel $C_2$.
3. If $-$ is non-trivial on $E$, trivial on $K$,
   then $\text{tr}_* : H(E) \to W(K)$ is \[ 1 - 1 \text{ if } p \neq 2 \]
   \[ 0 \text{ if } p = 2 \]
   where $p$ is the characteristic of the finite field $E$.

**Proof:** We should observe that since $E$ is finite, $K$ is automatically - invariant, and thus has an involu-
tion induced on it. Thus the statements make sense.

(1) If \([E : K]\) is even, then \(K\) is contained in the fixed field of \(-\) by Galois theory, since the fields are finite. Hence in case (1) we may assume \([E : K]\) is odd.

We recall that the Hermitian group of a finite field is determined by rank mod 2 \([M,H\ 117]\). So let \([M,B] \in H(E)\). Then \(\text{tr}^*[M,B]\) has rank equal to \([\text{rank } M] \cdot [E : K]\). Hence rank modulo 2 is preserved when \([E : K]\) is odd, so that \(\text{tr}^*\) is an isomorphism in this case as claimed.

(2) Let \([E : K]\) be odd, with \(-\) trivial on \(E\).

By \([Lm\ 193]\), the composition

\[
W(K) \xrightarrow{\otimes E} W(E) \xrightarrow{\text{tr}^*} W(K)
\]

is multiplication by \(\text{tr}^*<1>\), which is of odd rank, hence a unit in \(W(K)\). Thus the composition is an isomorphism. Since all groups have order 4, \(\text{tr}^*\) must be an isomorphism too.

Now consider the case \([E : K]\) is even, and \(-\) is trivial on \(E\). By Galois theory, \(E/K\) factors into a maximal odd order extension and successive extensions
of degree 2.

\[ E/K : K \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq E \]

\([K_1 : K] = \text{odd} \quad [K_i : K_{i-1}] = 2 \ \text{for } i>1\]

\(\text{tr}_* : W(E) \rightarrow W(K)\) is the composition of

the trace maps:

\[ W(E) \xrightarrow{\text{tr}_*} W(K_i) \rightarrow W(K_{i-1}) \rightarrow \cdots \rightarrow W(K_2) \rightarrow W(K_1) \xrightarrow{\text{tr}_*} W(K) \]

To finish the proof of (2), we need to examine separately

each \(\text{tr}_* : W(K_i) \rightarrow W(K_{i-1})\) and show that each

\(\text{tr}_*\) is an isomorphism on the fundamental ideal, with

kernel \(C_2\) when \([K_i : K_{i-1}] = 2\).

\(\text{tr}_* : W(K_1) \rightarrow W(K)\) is an isomorphism by the first part

of the theorem for \([K_1 : K] = \text{odd}\).

Note: We assume now that the characteristic of \(E\) is not 2, for in that case, rank mod 2 determines everything, and (2) is clearly true as stated.

We thus consider \(\text{tr}_* : W(F(\sqrt{w})) \rightarrow W(F)\). Since the characteristic of \(E\) is not 2, any quadratic extension of \(F\) is given by adjoining \(\sqrt{w}\), where \(w \not\in F^{**}\).

In order to describe a basis for \(G_F^*(\sqrt{w}) = F(\sqrt{w})^*/F(\sqrt{w})^{**}\) as an \(F_2\)-vector space, we recall the
general theory from \([G,F]\). Let \(G_F = F^*/F^{**}\) have a
basis \(\{w, b_1, \ldots, b_n\}\) as an \(F_2\)-vector space. Then
\(G_{F(\sqrt{w})}\) has a basis given by \(\{b_1, \ldots, b_n\}\) together with
\(\{x_i + y_iw\}\) as \(x_i, y_i\) run thru distinct square classes
represented by \(x_i^2 - y_i^2w\) in \(F^*/F^{**}\).

Thus for the case of \(G_{F(\sqrt{w})}\) where \(F\) is a finite
field we have two cases:

(1) \((-1)\) is a square in \(F\). Then, letting
\(x_i = 0, y_i = 1\) \(x_i^2 - y_i^2w = -w = w\) in \(F^*/F^{**}\), so that
we may choose \(\sqrt{w} = g\) the non-square class in \(F(\sqrt{w})\).

(2) \((-1)\) is not a square in \(F\). In order to find \(g\)
the non-square class in \(F(\sqrt{w}) = F(\sqrt{-1})\), we must solve
the equation \(x^2 + y^2 = -1\). Then \(g = x + y\sqrt{-1}\) is the
new square class. We could as well choose \(g = 1 + x\sqrt{-1}\),
since \(1^2 - x^2(-1) = 1 + x^2 = -y^2\) = non-square class in
\(F^*\).

Now the Witt group of any finite field is generated by
the 1-dimensional forms \(\langle 1 \rangle, \langle g \rangle\), where \(g\) is a non-
square. We have just computed \(g\) for a quadratic exten-
sion. We now compute \(\text{tr}_\ast\) in terms of the \(g\) given in
cases 1,2 above.
(1) (-1) is a square in \( F \). \( g = \sqrt{w} \). \( W(F(\sqrt{w})) \) is generated by the 1-dimensional forms \( \langle 1 \rangle \) and \( \langle \sqrt{w} \rangle \).

We compute.

\[
\text{tr} \_1 <\langle 1 \rangle> = \begin{pmatrix} 1 & \sqrt{w} \\ \sqrt{w} & 0 \end{pmatrix}
\]

Here the matrix of \( \text{tr} \_1 <\langle 1 \rangle> \) is with respect to the basis \( 1, \sqrt{w} \) as indicated. It has discriminant \(-4w\) which is non-trivial.

\[
\text{tr} \_1 <\langle \sqrt{w} \rangle> = \begin{pmatrix} 1 & \sqrt{w} \\ \sqrt{w} & 0 \end{pmatrix}
\]

in this case, the discriminant \( 4w^2 \) is trivial.

By additivity, \( \text{tr} \_1 <\langle 1 \rangle> \odot <\langle \sqrt{w} \rangle> \neq 0 \), and case (2) follows.

(2) (-1) is not a square. \( g = 1 + x/\sqrt{-1} \).

\[
\text{tr} \_1 <\langle 1 \rangle> = \begin{pmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}
\]

The discriminant \( 4 \) is trivial.

\[
\text{tr} \_1 <\langle 1 + x/\sqrt{-1} \rangle> = \begin{pmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}
\]

The discriminant \( \frac{1}{4} (1 + x^2) = \frac{1}{4} (-y^2) \) = non-square = non-trivial.
Again, by additivity
\[ \text{tr}_* <l> \oplus <1+x/-1> \neq 0 \], and \( \text{tr}_* \) is an isomorphism on the fundamental ideal, with kernel \( C_2 \) as claimed.

(3) Finally, let \( \sigma \) be non-trivial on \( E \), and trivial on \( K \). Let \( F \) be the fixed field of \( \sigma \). Then \( E \supset F \supset K \).
\[ \text{tr}_* : H(E) \to W(K) \] is the composition
\[ H(E) \xrightarrow{\text{tr}_*} W(F) \xrightarrow{\text{tr}_*} W(K) \]

By Jacobson's Theorem [M,H 115], we have the exact sequence \( 0 \to H(E) \xrightarrow{\text{tr}_*} W(F) \), and \( \text{tr}_* \) is injective into the fundamental ideal in \( W(F) \). By part (2) of this theorem, \( W(F) \xrightarrow{\text{tr}_*} W(K) \), \( \text{tr}_* \) is an isomorphism on the fundamental ideal in \( W(F) \). Thus, \( \text{tr}_* \) is 1-1, \( \text{tr}_* : H(E) \to W(K) \).

For \( p = 2 \), the fundamental ideal is trivial, so that \( \text{tr}_* = 0 \) as claimed. \( \Box \)

In order to finish describing \( i_3 \), the last task then is to describe which \( \sigma \)-invariant maximal ideals in \( S \) have \( \sigma \)-invariant maximal ideals in \( D \) lying over them.
**Definition 3.2** The conductor of $D$ over $S$ is the largest set $C$ which is an ideal in both $D$ and $S$.

We shall need the following theorem:

**Theorem 3.3** If $A$ factors as $A = \prod_{i=1}^{n} \mathfrak{p}_i$ where $A$ is prime to the conductor $C$, then

$$A \cap S = \prod_{i=1}^{n} (\mathfrak{p}_i \cap S).$$

**Proof:** See [G 38]. □

We shall also need a few results from ideal theory.

**Lemma 3.4** Let $D$ be a domain, $A, B, C$ ideals in $D$. Let $C$ be generated by $k$ elements, $C = \langle c_1, \ldots, c_k \rangle$. Then $AC = BC$ implies $A^k \subseteq B$, and similarly $B^k \subseteq A$.

**Proof:** It clearly suffices to show an arbitrary product $(a_1 \cdots a_k)$ of $k$ elements in $A$ is contained in $B$. $A^k$ is generated by finite sums of such products. Since $AC = BC$ we may write

$$a_1 c_1 = \sum_j b_{j1} c_j, \ldots, a_k c_k = \sum_j b_{jk} c_j$$

This system of equations can then be written as
Solving for $c_i$, using Cramer's rule we obtain

$$\Delta c_i = 0 \quad \text{where} \quad \Delta = \text{determinant of coefficient matrix}.$$ 

$c_i \neq 0$ yields $\Delta = 0$. However, the determinant $\Delta$
can be written as $\Delta(a_1 \cdots a_k) + b$ where $b \in \mathbb{B}$.
Thus 

$(a_1 \cdots a_k) \in \mathbb{B}$ as desired. \(\square\)

**Lemma 3.5** If a prime ideal $\mathfrak{P}$ in $\mathbb{D}$ factors as

$$\mathfrak{P} = A \cdot B,$$

then $A = \mathbb{D}$ or $B = \mathbb{D}$ ($\mathbb{D}$ a Dedekind Domain)

(A Dedekind Domain has unique factorization)

Proof: $\mathfrak{P} = AB$ clearly implies $\mathfrak{P} \supset A$ or $\mathfrak{P} \supset B$.

Say $\mathfrak{P} \supset A$. Then we may write $A = \mathfrak{P}W$, $W$ an ideal
in $\mathbb{D}$. Hence $\mathfrak{P} = \mathfrak{P}(WB) = \mathfrak{P}Q$. So $\mathfrak{P} \cdot \mathbb{D} = \mathfrak{P} \cdot \mathbb{Q}$.

Since $\mathbb{D}$ is Dedekind, $\mathfrak{P}$ is generated by 2 elements,

$[0'M]$, and we may apply Lemma 3.4.

$$D^2 = D \subset \mathbb{Q}.$$ Hence $\mathbb{Q} = D = WB$, so that

$W = B = D$, and $\mathfrak{P} = A$. \(\square\)

**Note:** This Lemma is also clearly true for an order $S$.

We also recall the following:

**Theorem 3.6** Let $S \subseteq \mathbb{D}$ be rings, with $\mathbb{D}$ integral
over $S$. Let $\mathfrak{m}$ be a prime ideal in $S$. Then there
exists a prime ideal $\mathfrak{p}$ in $\mathbb{D}$ with $\mathfrak{p} \cap S = \mathfrak{m}$.
Proof: See [A,M 62]

With these preliminaries, we are in a position to give a sufficient condition for a \( \sigma \)-invariant maximal ideal in \( S \) to have a unique \( \sigma \)-invariant maximal ideal in \( D \) lying over it.

**Theorem 3.7** Let \( \mathfrak{m} \) be a \( \sigma \)-invariant maximal ideal in \( S \). If \( \mathfrak{m} \) is prime to the conductor \( C \), then \( D \) has a unique \( \sigma \)-invariant maximal ideal \( \mathfrak{p} \) with \( \mathfrak{p} \cap S = \mathfrak{m} \).

**Proof:** Let \( \mathfrak{m} \) be maximal in \( S \). By Theorem 3.6, we can find \( \mathfrak{p} \) maximal in \( D \) with \( \mathfrak{p} \cap S = \mathfrak{m} \). We claim that \( \mathfrak{p} \) is unique, and hence \( \sigma \)-invariant, for clearly \( \mathfrak{p} \cap S = \mathfrak{m} = \mathfrak{m} \). So suppose \( \mathfrak{p}_i \neq \mathfrak{p} \) with \( \mathfrak{p}_i \cap S = \mathfrak{m} \). \( \mathfrak{p}_i \) may be \( \mathfrak{p} \), or some other maximal ideal in \( D \). Each such \( \mathfrak{p}_i \) will clearly appear in the factorization of \( D\mathfrak{m} = \prod_{i=1}^{w} \mathfrak{p}_i \). We claim that each \( \mathfrak{p}_i \cap S = \mathfrak{m} \). Clearly, \( \mathfrak{p}_i \cap S \supseteq \mathfrak{m} \), since \( \mathfrak{p}_i \supseteq D\mathfrak{m} \). However, \( \mathfrak{p}_i \cap S \neq \mathfrak{m} \) implies that \( \mathfrak{p}_i \cap S = S \), so that \( 1 \in \mathfrak{p}_i \), contradiction. Thus \( \mathfrak{p}_i \cap S = \mathfrak{m} \).

Now note that \( D\mathfrak{m} \cap S = \mathfrak{m} \). This follows since \( D\mathfrak{m} \supseteq \mathfrak{m} ; D\mathfrak{m} \subseteq \mathfrak{p}_i \) and \( \mathfrak{p}_i \cap S = \mathfrak{m} \) so \( D\mathfrak{m} \cap S \subseteq \mathfrak{p}_i \cap S = \mathfrak{m} \).
We now apply Theorem 3.3. Since \( \mathfrak{D} \) is prime to \( \mathfrak{C} \), \( \mathfrak{D} \cap S = \mathfrak{m} = \prod_{i=1}^{w} (\mathfrak{q}_i \cap S) = \prod_{i=1}^{w} \mathfrak{m}_i \). However, for \( \mathfrak{m} \) a prime ideal, by Lemma 3.5, \( w = 1 \), and consequently \( \varphi \) is unique. \( \Box \)

For an order \( S \subseteq D \), the conductor \( \mathfrak{C} \neq 0 \). For \( m \neq 0 \in \mathbb{Z} \) satisfy \( mD \subseteq S \). Then \( (m) \) generates an ideal in \( S \) which is also an ideal in \( D \).

Thus, we see that the cardinality of the set

\[
T(\mathfrak{m}) = \{ \varphi : \varphi \cap S = \mathfrak{m}, \varphi = \mathfrak{e} \}
\]

is one except possibly for those ideals \( \mathfrak{m} \) which are not prime to \( \mathfrak{C} \). Since \( \mathfrak{C} \neq 0 \), there are only a finite number of maximal ideals \( \mathfrak{m} \) in \( S \) which are not prime to \( \mathfrak{C} \). Theorems 3.1 and 3.7 then enable us to compute the map \( i_3 \). The set \( T(\mathfrak{m}) \) must be computed explicitly only at the finite set of ideals not prime to \( \mathfrak{C} \).

Given \( \mathfrak{e} \) in \( D \), we should like to relate \( D/\mathfrak{e} \) to \( S/\mathfrak{e} \cap S \).

**Definition 3.8** Let \( \mathfrak{e} \) be a maximal ideal in \( D \), so that \( \mathfrak{e} \cap S = \mathfrak{m} \) is a maximal ideal in \( S \). [A,Mc 61] We will say \( S \) is integrally closed at \( \mathfrak{m} \) if \( S(\mathfrak{m}) \), the localization of \( S \) at \( \mathfrak{m} \), is integrally closed.
D is integrally closed, hence so also is \( D(\varphi) \).

[A, Mc 62] Thus, \( S \) is integrally closed at \( \mathfrak{m} \) if and only if \( D(\varphi) = S(\mathfrak{m}) \).

**Proposition 3.9** If \( D(\varphi) = S(\mathfrak{m}) \), then \( D/\varphi = S/\mathfrak{m} \).

**Proof:** \( D/\varphi = D(\varphi)/\mathfrak{m}(\varphi) = S(\mathfrak{m})/\mathfrak{m}(\mathfrak{m}) = S/\mathfrak{m} \). □

Note that \( D \) and \( S \) have the same quotient field, \( E \), and \( D(\varphi) = S(\mathfrak{m}) \) except at finitely many primes.

It follows by 3.9 then that \( D/\varphi = S/\mathfrak{m} \) with only finitely many exceptions also.

Since we have computed \( i_3 \), we next must face the task of computing the local boundary, \( \partial(D, \varphi) \).

This comes in the next section.
4. Computing the local boundary \( \partial(D, \varnothing) \)

We consider the general case of \( \partial(D) : H_u(E) \to H_u(E/I) \) where \( I = \bar{I} \) is a - invariant fractional ideal. Of course, we have in mind \( I = \Delta^{-1}(D/Z) \).

Following Proposition 2.8, we wish to compute the localization, \( \partial(D, \varnothing) \) of \( \partial(D) \) at a - invariant maximal ideal \( \varnothing = \bar{\varnothing} \) in \( D \). Since \( D \) will be fixed throughout this section, we simplify our notation of \( \partial(D, \varnothing) \) to \( \partial(\varnothing) \).

From 2.8, \( (E/I)(\varnothing) = E/I(\varnothing) \). We now consider \( E/I(\varnothing) \). (See III 3.8). We embed the residue field \( D/\varnothing \) into \( E/I(\varnothing) \) as follows. Let \( \varnothing \in E \) satisfy \( v_{\varnothing}(\varnothing) = v_{\varnothing}(I) - 1 \). (V 1.5). Then define

\[
f : D/\varnothing \to E/I(\varnothing)
\]

by

\[
r + \varnothing \to \varnothing r + I(\varnothing).
\]

\( f \) is 1 - 1 since \( \varnothing r \in I(\varnothing) \) implies

\[
v_{\varnothing}(\varnothing r) \geq v_{\varnothing}(I(\varnothing)) = v_{\varnothing}(I)
\]

so

\[
v_{\varnothing}(I) - 1 + v_{\varnothing}(r) \geq v_{\varnothing}(I) \]

\[
v_{\varnothing}(r) \geq 1
\]
Hence $r \in \mathcal{O}$.

For the case when $S$ is an order in $E$, with $I = \Delta^{-1}(S/Z)$, we claim that there is a commutative diagram

$$
\begin{array}{ccc}
0 & \rightarrow & S/\mathfrak{m} & \rightarrow & \tilde{w} & \rightarrow & E/I(\mathfrak{m}) \\
& & \downarrow \text{tr} & & \downarrow t & & \\
& & 0 & \rightarrow & F_p & \rightarrow & Q/Z(p) \\
\end{array}
$$

Here $\text{tr}$ denotes trace on the finite field level. $t$ is induced by trace of $E/Q$. $\tilde{w}$ is given by the canonical choice of uniformizer in $Q/Z(p)$ annihilated by $p$, namely

$$
\tilde{w}: 1 \rightarrow \left(\frac{1}{p}\right).
$$

In order to see that $\tilde{w}$ exists we proceed as follows. Let $A$ be a finitely generated $S$-module. Then $S$ is the image of a free $S$-module, $F_1$, and we have the exact sequence $0 \rightarrow \ker f \rightarrow F_1 \xrightarrow{f} A \rightarrow 0$. Of course $F_1/\ker f \approx A$. This leads to the diagram below, with $F_2 = \ker f$. 
Given $h \in \text{Hom}_Z(A, \mathbb{Q}/\mathbb{Z})$, $h$ lifts to $h_1 : F_1 \to Q$ since $F_1$ is projective. By commutativity, 
$h_1|_{F_2} = h_2 \in \text{Hom}_Z(F_2, \mathbb{Z})$.

For finitely generated projective $S$-modules $A$, we assert that there is a trace $E/\mathbb{Q} = t$ induced isomorphism:

$$\text{Hom}_S(A, E) \xrightarrow{\hat{t}} \text{Hom}_Z(A, \mathbb{Q}). \quad \hat{t}(g) = t \circ g$$

$\hat{t}$ is clearly onto since $A$ is $S$-projective. In order to see $\hat{t}$ is $1-1$, suppose $g \in \text{Hom}_S(A, E)$. Let $a \in A$ satisfy $g(a) \neq 0$. Then clearly there exists $e \in E$ with $t(eg(a)) \neq 0$. However $S$ is an order in $E$ so we can write $m \cdot e \in S$ for some $m \in \mathbb{Z}$. Thus, since $t$ is $\mathbb{Z}$-linear,

$$m \cdot t(eg(a)) = t(me g(a)) = t(g(me a)) \neq 0.$$
It follows that \( t \circ g \neq 0 \), and \( \hat{\epsilon} \) is an isomorphism.

Hence \( h_1 \in \text{Hom}_S(F_1, \mathbb{Q}) \) may be written uniquely as 
\( t \circ k_1 \) where \( k_1 \in \text{Hom}_S(F_1, E) \). Further, since 
\( t \circ k_1 \mid_{F_2} = h_2 \), we observe that 
\( k_1 \mid_{F_2} = k_2 \in \text{Hom}_S(F_2, \Delta^{-1}(S/\mathbb{Z})) \). Thus \( k_1 \) induces an 
\( S \)-module homomorphism \( k \in \text{Hom}_S(F_1/F_2, E/\Delta^{-1}(S/\mathbb{Z})) \).

Clearly \( t \circ k = h \).

We claim that this \( k \) is unique. For suppose 
\( t \circ j = t \circ k = h \). Then \( t \circ (j - k) = 0 \) in \( \mathbb{Q}/\mathbb{Z} \). Now 
consider the diagram:

\[
\begin{array}{ccc}
F_1 & \rightarrow & E \\
\downarrow & & \downarrow \\
F_1/F_2 & \rightarrow & E/\Delta^{-1}
\end{array}
\]

\( (j - k) \) exists since \( F_1 \) is \( S \)-projective.

However, we also have the commutative diagram:
Thus $t \circ (\hat{j} - k) \subseteq \mathbb{Z}$, from which it follows that we have $\text{im}(\hat{j} - k) \subseteq \Delta^{-1}(\mathbb{S}/\mathbb{Z})$. Hence $j - k \equiv 0$ as maps in $\text{Hom}_{\mathbb{S}}(E, E/\Delta^{-1}(\mathbb{S}/\mathbb{Z}))$ and $j = k$ is unique.

We apply this to the finitely generated $\mathbb{S}$-module $\mathbb{S}/\mathfrak{m}$, where $h : \mathbb{S}/\mathfrak{m} \to \mathbb{Q}/\mathbb{Z}(p)$ is the $\mathbb{Z}(p)$-module homomorphism $h = w \circ \text{tr}$.

By the above, there exists a unique $k = \tilde{w}$ with $t \circ \tilde{w} = w \circ \text{tr}$ as claimed.

$\tilde{w}$ is evidently determined by where $\tilde{1} \in \mathbb{S}/\mathfrak{m}$ is taken. Hence $\tilde{w}$ is in fact determined by $\mathfrak{m}$ for a suitable choice of localizer $\rho_{\mathfrak{m}}$.

However, in our computation of $\partial(\varphi)$, we shall find it convenient to specify the localizers $\rho_{\varphi}$ in a different manner. The manner in which we pick these is dictated by our desire to have the boundary computation read by Hilbert symbols.

**Theorem 4.1** Let $[M, B] \in H_u(E)$. We diagonalize $B$ as $B = \langle a_1 x_1 \rangle \oplus \cdots \oplus \langle a_n x_1 \rangle$ where $a_i \in \mathbb{F}/\mathbb{N}$, and $x_1 x_1^{-1} = u$ is chosen in a prescribed manner to be described
in the proof. Having fixed our choice of $x_1$, there is
a choice of localizers $\rho_\varphi$ so that the following holds:

I. If $\varphi$ is over inert, so that

$$\delta(\varphi) : H_u(E) \to H_u(E/I(\varphi)) \cong H(D/\varphi) \cong \{0, 1\}$$

we have the following formulas.

(a) If $B = \langle ax_1 \rangle$ is of rank 1,

$$(a, \sigma)_P = (-1)^\delta(\varphi) \langle ax_1 \rangle + v_\varphi(I).$$

(b) If $B$ has even rank and discriminant $d$ relative
to $x_1$, then:

$$(d, \sigma)_P = (-1)^\delta(\varphi)(B).$$

(c) If there are no ramified primes, and $x_1$ has
odd valuation $v_\varphi(x_1)$ at an even number of primes
formulas (a) and (b) above hold.

(d) If there are no ramified primes and $x_1$ has
odd valuation at an odd number of primes, formulas
(a) and (b) are valid at all inert primes except one
specified prime $\varphi_1$ over $P_1$. At $P_1$,

$$v_{\varphi_1}(ax_1) = v_\varphi(a) + 1$$

and we then have

$$(a, \sigma)_{P_1} = (-1)^\delta(\varphi) \langle ax_1 \rangle + v_\varphi(I) + 1.$$
(b) For $B$ of even rank, formula (b) still holds.

II. Ramified primes are divided into two classes:

(a) $\text{cl}(\varphi) = 0$ if $v_\varphi(I) \equiv v_\varphi(x_1)$ (2).

(b) $\text{cl}(\varphi) = 1$ if $v_\varphi(I) \equiv v_\varphi(x_1) + 1$ (2).

$\delta(\varphi) \equiv 0$ at ramified primes of class 0. $\delta(\varphi)$ preserves rank at ramified primes of class 1.

Note: This determines $\delta(\varphi)$ at dyadic ramified primes.

Further, under the choice of localizers $\varphi$ made, $H_u(E/I(\varphi)) \approx W(D/\varphi)$ reads $\delta(\varphi)$ as follows:

$\delta(\varphi)(\langle ax_1 \rangle)$ is a non-square if and only if $(a, \varphi)_p = -1$.

Further, if $B$ has even rank, and discriminant $d$, then $\delta(\varphi)(B)$ has a non-trivial discriminant if and only if $(d, \varphi)_p = -1$.

Proof: We begin by considering $u$, with $uu = 1$.

By Hilbert's Theorem 90, there exists $x \in E^*$ with $xx^{-1} = u$. Our first task as described in the theorem is to rechoose $x$ appropriately.

Thus, we consider $v_\varphi(x)$, for $\varphi$ over inert. If at least one prime ramifies, finite or infinite, we can find $y \in F^*$ with $(y, \varphi)_p = (-1)^{v_\varphi(x)}$ for all $p = \varphi \cap F$. 
which are inert, by realization of Hilbert symbols.

If there are no ramified primes, finite or infinite, there are two possibilities.

1. If \( v_\mathcal{O}(x) \) is odd at an even number of inert primes, by realization it is still possible to choose \( y \in F^* \) with \( (y, \sigma)_p = (-1)^{v_\mathcal{O}(x)} = (-1)^{v_\mathcal{O}(y)} \) at all inerts.

2. If \( v_\mathcal{O}(x) \) is odd at an odd number of inert primes, we may find \( y \in F^* \) with

\[
(y, \sigma)_p = (-1)^{v_\mathcal{O}(x)}
\]

at all inert primes except one specific inert prime, say \( p_1 \), at which

\[
(y, \sigma)_{p_1} = (-1)^{v_\mathcal{O}_1(x)} + 1.
\]

We now rechoose \( x_1 = xy \). We still have \( x_1^{-1} = u \) since \( y \in F^* \) has \( y^{-1} = yy^{-1} = 1 \). Note, however, that \( x_1 \) now has even valuation at all inert primes, with at most one exception as described above.

We next describe how to choose the localizers \( \rho_\mathcal{O} \).

First, at \( \mathcal{O} \) inert, we choose \( \rho_\mathcal{O} = x_1w \), where \( w \in F^* \) satisfies \( v_\mathcal{O}(w) = v_\mathcal{O}(I) - v_\mathcal{O}(x) - 1 \), so

\[
v_\mathcal{O}(\rho_\mathcal{O}) = v_\mathcal{O}(I) - 1.
\]

This is possible at inert \( \mathcal{O} \), since a local uniform-
mizer for $\varphi \cap F$ is also a local uniformizer for $\varphi$.

In order to describe the ramified primes $\varphi$, we begin as follows. Consider $cl(u) \in H^1(C_2; \Omega_E(\varphi)^*)$, where $\varphi = \overline{\varphi}$ is over ramified. By VII 2.1 $H^1(C_2; \Omega_E(\varphi)^*) \neq 0$ and we may write

$$cl(u) = cl(\pi^{n-1})v_{\varphi}(I) - \varepsilon$$

where $\varepsilon = 0$ or $1$.

**Definition 4.2** If $\varphi = \overline{\varphi}$ is over ramified, $\varphi$ is of class $0$ if $\varepsilon = 0$. $\varphi$ is of class $1$ if $\varepsilon = 1$.

We observe the following:

**Lemma 4.3** We may rephrase class as follows:

(a) $cl(\varphi) = 0$ if and only if $v_{\varphi}(I) = v_{\varphi}(x_1)$ (2)

(b) $cl(\varphi) = 1$ if and only if $v_{\varphi}(I) - 1 = v_{\varphi}(x_1)$ (2).

**Proof:** Here $x_1^{x_1^{-1}} = u$. Write $x_1 = \pi^iw$, $w \in \Omega_E(\varphi)^*$. Then $cl(u) = cl(x_1^{x_1^{-1}}) = cl(\pi^{n-1})$. However, $cl(\pi^{n-1})$ generates $H^1(C_2; \Omega_E(\varphi)^*) \neq 0$, from which the result follows. □

When $\varphi$ is tamely ramified, we choose $\pi = -\pi$, a
skew-uniformizer. When the ramification is wild, any uniformizer will do. Note that $\pi^\pi$ is a uniformizer for $\theta \cap F = P$, since $\theta$ is over ramified. Also, $v_{\phi}(\pi^\pi) = 2$. We now choose $\rho_{\phi}$ at ramified primes as:

$$\rho_{\phi} = x_1(\pi^\pi)^t \pi \text{ with } t \text{ suitably chosen so that }$$

$$v_{\phi}(\rho_{\phi}) = v_{\phi}(I) - 1 \text{ if } \text{cl}(\phi) = 0.$$  

$$\rho_{\phi} = x_1(\pi^\pi)^t \text{ with } t \text{ suitably chosen so that }$$

$$v_{\phi}(\rho_{\phi}) = v_{\phi}(I) - 1 \text{ if } \text{cl}(\phi) = 1.$$  

With these choices of localizers made, we now identify the image groups of $\gamma(\phi)$, $\text{H}_u(E/I(\phi))$. Let $[V,B] \in \text{H}_u(E/I(\phi))$. By assuming that $(V,B)$ is anisotropic, it follows that the annihilator of the finitely generated $0_E(\phi)$-module $V(\phi)$ is the maximal ideal $m(\phi)$ in $0_E(\phi)$. Thus $V$ is an $0_E(\phi)/m(\phi)$-module. This is equivalently phrased by saying $V$ is a vector space over the residue field $0_E(\phi)/m(\phi) = D/\theta$.

Let $x,y \in V$. Suppose $B(x,y) = [a] \in E/I(\phi)$. Letting $\pi$ be a uniformizer for $\theta$ as above, $\pi x = 0$ since $V$ is an $0_E(\phi)/m(\phi)$-module. Thus $\pi[a] \in I(\phi)$.

Let $a_1$ be a lift of $a$ to $E$. Since
\( a_1 \in I(\phi) \), it follows that \( v_\phi(a_1) \geq v_\phi(I) - 1 \).

Also \( B \) is \( \mu \) Hermitian, so that \([a] = \mu[a]\) in \( E/I(\phi) \).

We may thus write \( a_1 - u\overline{a}_1 \in I(\phi) \).

We now consider the \( D/\varphi \)-valued form on \( V \) given by \( B^1 = \rho_\phi^{-1} \cdot B \), where the choice of \( \rho_\phi \) has been previously specified.

\( (1) \). \( \varphi \) inert. With \( a_1 - u\overline{a}_1 = i \in I(\phi) \), we show \( B^1 \) is \(+1\) Hermitian. Here \( \rho_\phi = x_1 w \) where \( w \in F^* \).

\[
\begin{align*}
a_1^{-1} &= \frac{(a_1 - i)}{u} \overline{x}_1 w^{-1} \\
&= \frac{(a_1 - i)}{\mu} \overline{x}_1^{-1} w^{-1} \\
&= \frac{a_1 w^{-1}}{x_1} - i \frac{w^{-1}}{x_1} = a_1 \rho_\phi^{-1} - i \rho_\phi^{-1} \\
&= a_1 \rho_\phi^{-1} \text{ in } D/\varphi.
\end{align*}
\]

This last follows because

\[
\begin{align*}
v(i \rho_\phi^{-1}) &= v(i) - v(I) + 1 \geq 1, \text{ so that } i \rho_\phi^{-1} \in \varphi.
\end{align*}
\]

This shows there is an isomorphism between
\[ H_u(E/I(\Theta)) \text{ and } H_{+1}(O_{E}(\Theta)/m(\Theta)) \]
given by scaling with \( \rho_\Theta \). Since \( \Theta \) is inert, the involution induced on \( O_{E}(\Theta)/m(\Theta) \) is non-trivial, and we have true +1 Hermitian.

(2) The tamely ramified case.

As before, we have the form \( B^1 = \rho_\Theta \beta_1 B \).

(a) \( cl(\Theta) = 0 \), so \( \rho_\Theta = x_1(\tau) \pi \) where \( \pi = -\pi \).

We now compute as before:

\[
\tilde{a}_1 \rho_\Theta^{-1} = \left( \frac{a_1 - i}{\mu_1} \right) (x_1^{-1}) (\pi \bar{\tau})^{-t} (-\pi)^{-1} \\
= \left( \frac{a_1 - i}{\mu_1} \right) (x_1^{-1}) (\pi \tau)^{-t} (-\pi)^{-1} \\
= (a_1 - i) (-\rho_\Theta^{-1}) = -a_1 \rho_\Theta^{-1} \text{ in } O_E(\Theta/m(\Theta)).
\]

Since \( \Theta \) is ramified, we obtain this time an isomorphism between \( H_u(E/I(\Theta)) \approx W^{-1}(D/\Theta) = 0 \).

(b) \( cl(\Theta) = +1 \), so \( \rho_\Theta = x_1(\tau) \pi \). The same computation shows \( \tilde{a}_1 \rho_\Theta^{-1} = a_1 \rho_\Theta^{-1} \) in \( D/\Theta \), and we have \( H_u(E/I(\Theta)) \approx W^+(D/\Theta) \).

(3) The case for wild ramification follows as above.
In either case, \( H_u(\mathbb{E}/I(\mathfrak{p})) \approx W(D/\mathfrak{p}) \).

With these preliminaries, we are ready to compute \( \mathfrak{e}(\mathfrak{p}) \). To begin with, consider a 1-dimensional form in \( H_u(\mathbb{E}) \). By VI 2.15 we may write this as \( \langle ax_1 \rangle \), for \( x_1 \) fixed as described, and a uniquely determined in \( F/N_E/F \).

I. We first compute \( \mathfrak{e}(\mathfrak{p}) \) for \( \mathfrak{p} \) over inert. We begin by considering \( \mathfrak{e}(\mathfrak{p})(\langle ax_1 \rangle) \). Observe the Witt equivalence

\[ \langle ax_1 \rangle \sim \langle ax_1 (\pi^{-t}) \rangle, \]

for \( \pi \) a uniformizer for \( \mathfrak{p} \). It follows that without loss of generality, we may assume either:

(a) \( v_{\mathfrak{p}}(ax_1) = v_{\mathfrak{p}}(I) - 2 \)

or

(b) \( v_{\mathfrak{p}}(ax_1) = v_{\mathfrak{p}}(I) - 1 \).

This is done by rechoosing \( a \) as \( a(\pi^{-t}) \). Here, recall we may choose \( \pi \in \mathfrak{p} \cap F = \mathfrak{p} \) since this is the inert case. In any case, \( v_{\mathfrak{p}}(\pi^{-t}) = 2 \) and \( \pi^{-t} \) is not a uniformizer for \( \mathfrak{p} \). Thus case (a) or (b) only depends on \( v_{\mathfrak{p}}(a) \) compared to \( v_{\mathfrak{p}}(I) \), since by choice \( v_{\mathfrak{p}}(x_1) \equiv 0 \pmod{2} \) with at most one exceptional prime \( \mathfrak{p}_1 \).
Now consider the lattice \( L = \varnothing \). Since \( \langle ax_1 \rangle \) has 
\[ v_\varnothing(ax_1) = v_\varnothing(I) - 1 \text{ or } v_\varnothing(I) - 2, \] 
\( \langle ax_1 \rangle \) is \( I \)-valued.

We consider the dual lattice:

\[
\hat{L} = \{ x : B(x, L) \subseteq I \} \\
= \{ x : x_1 \varnothing \subseteq I \} \\
= \{ x : x \in I \cdot \varnothing^{-1}(ax_1)^{-1} \} \\
= \varnothing \text{ in case (a)} \\
= \mathcal{O}_E(\varnothing) \text{ in case (b)}
\]

Thus viewed, in case (a), we clearly get \( \varnothing(\varnothing)\langle ax_1 \rangle = 0 \)

In case (b), \( \varnothing(\varnothing)\langle ax_1 \rangle = \mathcal{O}_E(\mathcal{O} / \mathcal{M}(\varnothing), \mathcal{B}^{1}) \),

where \( \mathcal{B}^{1} \) is defined on the torsion \( \mathcal{O}_E(\varnothing) \)-module

\[
\hat{L} / L = \mathcal{O}_E(\mathcal{O} / \mathcal{M}(\varnothing)) \text{ with values in } E / I(\varnothing) \text{ by } \langle ax_1 \rangle,
\]

with \( v_\varnothing(ax_1) = v_\varnothing(I) - 1 \). As we have mentioned, we then identify this with the \( D/\varnothing \)-valued form on \( D/\varnothing \) given by \( \langle ax_1 \rho^{-1}_\varnothing \rangle \), where \( \rho_\varnothing = x_1w \), ie.

\[
\langle ax_1 \rho^{-1}_\varnothing \rangle = \langle ax_1 \cdot x_1^{-1}w \rangle = \langle aw \rangle, \ w \in F^*.
\]

Again, \( \varnothing \) is inert so that \( [\mathcal{O}_E(\mathcal{O})/\mathcal{M}(\varnothing) : \mathcal{O}_F(P)/\mathcal{M}(P)] = 2 \) and the induced involution on the finite field \( \mathcal{O}_E(\varnothing)/\mathcal{M}(\varnothing) \) is non-trivial. Hermitian of a finite field is determined by rank modulo 2. Thus, we have completed our computation.
of the local boundary on a 1-dimensional form when $\theta$ is inert.

Identifying $H_\nu(E/I(\theta)) \cong H_{+1}(D/\theta) \cong F_2 = \{0,1\}$, we may summarize this as:

$$\partial(\theta) <ax_1> = (-1)^{v(\theta)(ax_1) - v(\theta)(I)}$$

or

$$\partial(\theta) <ax_1> + v(\theta)(I) = (-1)^{v(\theta)(ax_1)}$$

$$= (-1)^{v(\theta)(a)}$$

$$= (a, \sigma)_p .$$

Continuing in the inert case, let $B$ be a form of even rank. Since $E$ is a field, we may diagonalize $B$ as before,

$$B = <a_1 x_1> \oplus <a_2 x_1> \oplus \ldots \oplus <a_n x_1> ,$$

where $a_i \in F/NE$

As in VI 2.15, we define the discriminant of $B$ to be

$$\frac{n(n-1)}{2} \prod a_i.$$  

Note that this depends on the fixed choice of $x_1 ,$. Adding a hyperbolic form $<x_1> \oplus <-x_1>$ if necessary we may, without loss of generality take $n$ to be a multiple of 4. This does not effect $\partial(\theta)$ or $d$, but it does enable us to write $d = \prod a_i$. We
now use that $\mathfrak{d}(\varnothing)$ is additive, to compute:

\[
(d, \sigma)_p = (a_1 \ldots a_n, \sigma)_p \\
= \prod_i (a_i, \sigma)_p \\
= \prod_i \mathfrak{d}(\varnothing) \langle a_i x_i \rangle + v(\varnothing)(I) \\
= \prod_i \mathfrak{d}(\varnothing) \langle a_i x_i \rangle \\
= \prod_i \mathfrak{d}(\varnothing) [B] 
\]

This completes the inert case.
II. The ramified case.

As in the inert case, we begin by considering a 1-dimensional form \(<ax_1>\) in \(H_u(E)\). Here \(a\) is unique in \(F/\mathfrak{P}\), so \(v_\mathfrak{P}(a) = 0 \pmod{2}\), since \(\mathfrak{P}\) is ramified. Scaling \(a\) by the norm \((\pi^t_\mathfrak{P})\) from \(E\), we may assume \(v_\mathfrak{P}(a) = 0\).

Note: This does not affect \((a,\sigma)_p\), nor \(\vartheta(\mathfrak{P})<ax_1>\).

We now scale the resulting form \(<ax_1>\) and obtain the Witt-equivalent form \(<ax_1(\pi^t_\mathfrak{P})>\), with

\[
v_\mathfrak{P}(ax_1(\pi^t_\mathfrak{P})) = v_\mathfrak{P}(x_1(\pi^t_\mathfrak{P})) = \begin{cases} v_\mathfrak{P}(I) - 2 \\ v_\mathfrak{P}(I) - 1 \end{cases} ,
\]

depending on \(\text{cl}(\mathfrak{P})\).

As in the inert case, we let \(L = \vartheta\), and compute

\[
L^# = \vartheta \quad \text{if} \quad \text{cl}(\vartheta) = 0 \\
= \mathcal{O}_E(\vartheta) \quad \text{if} \quad \text{cl}(\vartheta) = 1
\]

Thus, if \(\text{cl}(\vartheta) = 0\), \(\vartheta(\vartheta) = 0\). If \(\text{cl}(\vartheta) = 1\), we obtain the \(E/\mathcal{I}(\vartheta)\)-valued form \(<\mathcal{O}_E(\vartheta)/\mathfrak{m}(\vartheta), B'>\), where \(B' = <ax_1(\pi^t_\mathfrak{P})>\). We identify this with the \(\mathcal{O}_E(\vartheta)/\mathfrak{m}(\vartheta)\)-valued form on \(\mathcal{O}_E(\vartheta)/\mathfrak{m}(\vartheta)\) given by viewing \(a\) in \(\mathcal{O}_E(\vartheta)/\mathfrak{m}(\vartheta)\):
\[ \langle ax_1(x_1^t \rho^{-1}) \rangle = \langle a \rangle . \]

Note, that this is a Witt inner product since \( \rho \) ramifies, so that \([O_E(\varphi)/m(\varphi) : O_F(\varphi)/m(\varphi)] = 1\), and the induced involution on the finite field \( O_E(\varphi)/m(\varphi) \) is trivial.

(1) If the characteristic of \( O_E(\varphi)/m(\varphi) = 2 \), rank is the only invariant, and we are done.

(2) If the characteristic of \( O_E(\varphi)/m(\varphi) \neq 2 \), we must determine if \( a \) is a square in the residue field.

By \( V 2.4 \), \( a \) is a square in \( O_E(\varphi)/m(\varphi) \) if and only if \( (a, \sigma)_p = +1 \).

We continue by letting \( B \) be a form of even rank. As before, we diagonalize \( B \), \( B = \langle a_1 x_1 \rangle \otimes \ldots \otimes \langle a_n x_n \rangle \).

Again, without loss of generality, \( n \equiv 0 \) \((\text{iv})\). By additivity of the boundary, \( \delta(\varphi)(B) = \langle a_1 \rangle \otimes \ldots \otimes \langle a_n \rangle \), which has discriminant \( \prod_{i=1}^{n} a_i \), since \( n \equiv 0 \) \((\text{iv})\).

Again, \( \prod_{i=1}^{n} a_i \) is a square in \( D/\varphi \) if and only if \( \prod_{i=1}^{n} a_i = d \) is a local norm, if and only if \( (d, \sigma)_{\varphi} = +1 \).

This completes the computation of \( \delta(\varphi) \). \( \square \)
5. **Computing the cokernel of \( \delta(D) \)**

In this section, we use the computation of the local boundary, \( \delta(\Phi) \), to compute

\[
\delta : H_u(E) \rightarrow H_u(E/I) .
\]

We also show how to compute \( H_u(I) \), where \( I = \Delta^{-1}(D/Z) \).

Of course, \( H(\Delta^{-1}(D/Z)) \approx W(k,\mathbb{Z};D) \), and \( H_u(\Delta^{-1}(D/Z)) \approx A(\mathbb{Z};D) \). Further, the computation of the boundary on the Hermitian level will subsequently be used in the computation of the global boundary

\[
\delta : W(k,\mathbb{O}) \rightarrow W(k,\mathbb{O}/\mathbb{Z}) ,
\]

in Section 7.

In order to describe the boundary homomorphism, the complicated case is described in the next Lemma.

**Lemma 5.1** Suppose \( E \) has involution \(-\), fixed field \( F \) as usual. Suppose \( E/F \) has no signatures, no dyadic ramified primes, and all ramified primes are of class 1. Then we may write the collection of ramified primes as \( \varphi_1, \ldots, \varphi_{2t}, \varphi'_1, \ldots, \varphi'_r \) where the \( \varphi_i \), \( i = 1, \ldots, 2t \) have residue fields \( O_E(\varphi_i)/m(\varphi_i) = F_q \) with \( q \equiv 3 \) (4) and the \( \varphi'_i \), \( i = 1, \ldots, r \) have residue fields \( F_q \).
with \( q = 1 \ (4) \) (assuming \( 2t + r \neq 0 \)).

**Proof:** We wish to show that the number of ramified primes whose residue fields \( F_q \) have \( q = 3 \) \( (4) \) is even. Let \( \varphi_i \) be such a prime. Then \(-1\) is not a square in each \( O_E(\varphi_i)/m(\varphi_i) = F_q \) \( \text{[Lm 43]} \). However, at ramified primes the square class in the residue field determines the Hilbert symbol. Thus \((-1, \sigma)_{P_i} = -1\) at each such \( P_i = \varphi_i \cap F \) over ramified. Notice that at the other ramified primes, \( \varphi'_i \), whose residue fields are \( O_E(\varphi'_i)/m(\varphi'_i) = F_q \) with \( q \equiv 1 \ (\text{mod 4}) \), \(-1\) is a square in the residue field, so that \((-1, \sigma)_{P'_i} = +1\). Also, \(-1\) is a local unit at inerts, so that \((-1, \sigma)_P = (-1)^v(-1) = +1\) at all inerts.

We now apply Hilbert reciprocity, \( \prod_{i}(-1, \sigma)_{P_i} = +1 \). This shows that the number of primes with residue fields \( F_q \) with \( q = 3 \) \( (4) \) elements must be even, since \((-1, \sigma)_{P_i} = -1\) only at those primes. \( \square \)

Next we form the group \( G \) given by
\[
G = \mathbb{Z}^* \times (F_2 \times \cdots \times F_2)^{2t} \times (F_2 \times \cdots \times F_2)^r,
\]
where \( 2t = \text{number of ramified primes at which } (-1, \sigma)_P = -1 \), \( r = \text{number of ramified primes at which } (-1, \sigma)_P = +1 \),
as given by Lemma 5.1. We write $Z^* = \{1, -1\}$, $F_2 = \{0, 1\}$. $Z^*$ is designed to keep track of the discriminant and reciprocity; $F_2$ will take care of ranks.

On $G$ we define a multiplication as follows:

$$(c, a_1, \ldots, a_2t, b_1, \ldots, b_r) \times (c', a_1', \ldots, a_2t', b_1', \ldots, b_r')$$

$$= \left( (-1)^{a_1} + \ldots + a_2t a_2t' \right) \left( c' a_1 + a_1' a_2t + a_2t' b_1 + b_1' \ldots \right)$$

Note that $(1, 0, \ldots, 0)$ is the identity in $G$, and that the order of every element divides $4$.

The purpose of this group $G$ is to describe the cokernel of $\bar{\gamma}: H_u(E) \to H_u(E/I)$ in the special case that there are no signatures, no dyadic ramified primes, and there are ramified primes, all of which are of class $1$ in the extension $E/F$. We assume now that we are in this case.

To begin with recall that

$$H_u(E/I) \approx \bigoplus_{\varphi \in \mathcal{G}} H(D/\varphi) \oplus W(D/\varphi).$$

Thus any element in $H_u(E/I)$ can be expressed as a direct sum of elements $\bigoplus [M_i B_i]$, where $[M_i B_i]$ is either in $H(D/\varphi)$ or $W(D/\varphi)$. We define a map $h : H_u(E/I) \to G$.
by:

Let \([M_i, B_i] \in H(D/\mathcal{O})\). This Hermitian element depends only on the rank modulo 2 of \(M_i\) over \(D/\mathcal{O}\).

We define:

\[
h([M_i, B_i]) = ((-1)^{\text{rank } M_i}, 0, 0, \ldots, 0) \in G
\]

for these Hermitian summands.

Continuing, suppose \(\bigotimes [M_i, B_i] \in \bigotimes W(D/\mathcal{O})\).

Let \(d_i = \text{discriminant of } B_i\)

\[
c = \prod_{p_i \text{ ramified}} (d_i^{\mathcal{O}})_{p_i}
\]

\(a_i = \text{local rank of } M_i \text{ over } D/\mathcal{O}, \quad i = 1, \ldots, 2t\)

\(b_i = \text{local rank of } M_i \text{ over } D/\mathcal{O}', \quad i = 1, \ldots, r\).

Define \(h: \bigotimes [M_i, B_i] \rightarrow (c, a_1, \ldots, a_t, b_1, \ldots, b_r)\).

Clearly, by additivity, this defines \(h\) as a map

\[h : H_u(E/I) \rightarrow G\]

\underline{Lemma 5.2} \quad \(h\) is a homomorphism.
Proof: This follows using the product formula for discriminants, VI 2.10. The local discriminants satisfy
\[ \text{rank } M_1 \text{ rank } M_2 \text{ dis } B_1 \cdot \text{dis } B_2. \]

We now use the following abbreviations:
\[ \text{dis } B_1 = d_1 \quad \text{dis } B_2 = d'_1 \quad \text{rank } M_1 = a_1 \quad \text{rank } M_2 = a'_1. \]

At each of the first \(2t\) ramified primes,
\[ (-1)^{a_1 a'_1} \left( d_1 d'_1, \sigma \right)_p = (-1)^{a_1 a'_1} \left( dd'_1, \sigma \right)_p \]
\[ \quad = (-1)^{a_1 a'_1} \left( d, \sigma \right)_p \left( d'_1, \sigma \right)_p \]
since \((-1)\) is not a square in \(D/\mathcal{O}\). At each of the next \(r\) ramified primes,
\[ (-1)^{b_1 b'_1} d_1 d'_1, \sigma \right)_p = \left( d_1 d'_1, \sigma \right)_p = \left( d, \sigma \right)_p \left( d'_1, \sigma \right)_p \]
since \((-1)\) is a square in \(D/\mathcal{O}\).

From these formulas, it clearly follows that \(h\) is a homomorphism. \(\Box\)

Lemma 5.3 The image of \(\delta\) in \(H_u(E/\mathcal{I})\) is mapped under \(h\) to the subgroup \(W_2\) of \(G\) whose elements are \((1,0,...,0)\) and \((\epsilon,1,1,...,1)\) where \(\epsilon\) is given by \(\epsilon = \prod_{\mathfrak{p} \text{ inert}} (-1)^{v_{\mathfrak{p}(I)}}\). (Same hypotheses as 5.1)
Proof: Let \([M_1, B_1] \oplus [M_1, B_1] \oplus [M_1, B_1]\) be in the image of \(\mathcal{A}\), say \(\mathcal{A}[M, B]\) equals this element in \(H_u(E/I)\). In other words \(\mathcal{A}(\mathcal{G})[M, B] = [M_1, B_1]\), where \([M_1, B_1] \in H(D/\mathcal{G})\) or \(W(D/\mathcal{G})\) depending on whether \(\mathcal{G}\) is over inert or ramified.

Case (1) \(M\) has even rank, and discriminant \(d\). By Theorem 4.1, \(\mathcal{A}(\mathcal{G})\) is read by the Hilbert symbol, \((d, \mathcal{G})_p = (-1)^{\mathcal{A}(\mathcal{G})[M, B]}\). Rank is preserved at class 1 ramified primes. Hence by Hilbert reciprocity,

\[
h \circ \mathcal{A} [M, B] = (1, 0, \ldots, 0) \in \mathcal{G}.
\]

Case (2) \(M\) has odd rank and discriminant \(d\). At inert primes:

\[
(d, \mathcal{G})_p = (-1)^{\mathcal{A}(\mathcal{G})[M, B]+v\mathcal{G}(I)}.
\]

At ramified primes, the square class of the discriminant of \(\mathcal{A}(\mathcal{G})[M, B]\) is determined by the Hilbert symbol \((d, \mathcal{G})_p\). Rank mod 2 is preserved, since by hypothesis all ramified primes are of class 1. Hence,

\[
h \circ \mathcal{A} [M, B] = \prod_{\mathcal{A}(\mathcal{G})[M, B]} (-1) \prod_{(d, \mathcal{G})_p \neq 1, 1, \ldots, 1} (d, \mathcal{G})_p.
\]

However, by Hilbert reciprocity,
\[ \prod (d, \sigma)_p = +1. \] Thus, all \( P \)

\[ \begin{align*}
\prod (-1)^{\nu(\phi)} [M, E] \cdot \prod (d, \sigma)_p &= \prod (-1)^{\phi(\text{inert})} \text{P ramified} \\
&= \prod (-1)^{\phi(\text{inert})} \text{as claimed.} \\
\end{align*} \]

Let \( \tilde{h} : H_u(E/I) \to G/W_2 \) by the composition of \( h : H_u(E/I) \to G \) with projection \( G \to G/W_2 \).

**Corollary 5.4** Again with the hypotheses of Lemma 5.1, there is an exact sequence

\[ 0 \to \text{image } \delta \to H_u(E/I) \xrightarrow{\tilde{h}} G/W_2 \to 0. \]

**Proof:** By Lemma 5.3, image \( \delta \) is contained in the kernel of \( \tilde{h} \). It follows by realization of Hilbert symbols and Theorem 4.1 that image \( \delta = \text{kernel } \tilde{h} \).

\( h \) and consequently \( \tilde{h} \) maps onto \( G \) by realization of Hilbert symbols.

**Corollary 5.4** in fact is the proof of part (2) of the following theorem.

**Theorem 5.5** The cokernel of \( \delta : H_u(E) \to H_u(E/I) \) is given as follows.

(1) If there are no ramified primes, finite or in-
finite, the cokernel of \( \partial \) depends on the number of primes at which \( v_\varphi(x_1) + v_\varphi(I) \) is odd. If this number is even, the cokernel of \( \partial \) is \( \mathbb{C}_2 \). If this number is odd, \( \partial \) is onto.

(2) If there are no signatures, no dyadic ramified primes, and there are ramified primes, all ramified primes being class 1, then there is an isomorphism induced by \( \tilde{h} \):

\[
H(E(I)/im \partial = \text{cokernel } \partial \approx G/W_2
\]

Thus the cokernel has order \( 2^{2t+r} \). There is an element of order 4 in cokernel \( \partial \) if and only if \( t \neq 0 \).

(3) If there are signatures, dyadic ramified primes, or ramified primes of class 0 the cokernel of \( \partial \) is a product of \( \mathbb{C}_2 \)'s. Its order is determined as follows.

Let \( a \) be the number of ramified primes of class 1.

Let \( b \) be the number of dyadic ramified primes of class 0. Then the cokernel has order \( 2^{\max(a-1,0)} + b \).

Proof: (1) If there are no ramified primes, we must consider two cases.

(a) \( v_\varphi(x_1) \) is odd at an even number of primes.

(b) \( v_\varphi(x_1) \) is odd at an odd number of primes.
In case (a), we can read

\[ \varphi : H_u(E) \rightarrow H_u(E/I) \cong \frac{\mathcal{O}H(D)}{\mathcal{O}} \]

by Theorem 4.1. The formulas for \( \varphi(\varphi) \) are:

\[ (a, \varphi)_p = (-1)^{\varphi(a)e_1 + \varphi(\varphi)} \quad \text{on rank 1 forms } \langle ax_1 \rangle. \]

\[ (d, \varphi)_p = (-1)^{\varphi(\varphi)}(B) \quad d = \text{discriminant of } B, \text{ on even rank forms}. \]

Suppose \( v_{\varphi}(I) \) is odd at an even number of primes. Then clearly \( v_{\varphi}(x_1) + v_{\varphi}(I) \) is odd at an even number of primes, since we are in case (a).

By realization of Hilbert symbols we may pick \( a \in F \) with \( (a, \varphi)_p \) arbitrarily specified subject only to reciprocity. Since \( v_{\varphi}(I) \) is odd at an even number of primes, the formulas given determine that \( \varphi(\varphi) \) is non-trivial at an even number of primes. Thus the cokernel has order 2 by realization; and \( \varphi(\varphi) \) is subject only to the stated restriction.

Continuing, if \( v_{\varphi}(I) \) is odd at an odd number of primes, \( v_{\varphi}(x_1) + v_{\varphi}(I) \) is odd at an odd number of primes also. Thus by the formula \( (a, \varphi)_p = (-1)^{\varphi(a)e_1 + \varphi(\varphi)} \), we may make \( \varphi(\varphi) \) non-trivial at any one specified prime, and \( \varphi(\varphi) \) is onto.
(b) \( v_\phi(x_1) \) is odd at an odd number of primes. In this case, by Theorem 4.1, we have the previously stated formulas, with one exception \( p_1 \) at which the Hilbert symbol is read "backwards".

Again, suppose \( v_\phi(I) \) is odd at an odd number of primes. This means \( v_\phi(x_1) + v_\phi(I) \) is odd at an even number of primes. However, reciprocity now reads

\[
\prod_{p \mid (\phi)} \frac{3}{\phi} < ax_1 > = \prod_{p \notmid (\phi)} (-1)^{v_\phi(I) + 1} \quad \text{from which it becomes clear that} \quad \phi(\phi) \text{ must be non-trivial at an even number of primes as before. The last case is also similar, and the boundary is onto when} \quad v_\phi(x_1) + v_\phi(I) \text{ is odd at an odd number of primes.}
\]

**Comment:** In this instance we are penalized for our choice of \( x_1 \). Had we chosen \( x_1 \) with \( v_\phi(x_1) \equiv v_\phi(I) \pmod{2} \) almost everywhere, (meaning with at most one exception) the local boundary \( \phi(\phi) \) would have given the formula \( (a, c)_p = (-1)^{\phi(\phi)} < ax_1 > \) almost everywhere, depending on the number of primes at which \( v_\phi(x_1) + v_\phi(I) \) is odd. Part (1) of this theorem would have required no special analysis. Indeed, since the cokernel of \( \phi \) does not
depend on \( x_1 \), this provides an alternate proof.

However, we find it more natural to give \( x_1 \) with even valuation at inert primes. Thus, forms in \( H_{+1}(E) \) do not require a special "\( x_1 \)" in their diagonalization.

(2) As remarked earlier, part (2) follows from Corollary 5.4.

(3) If there are signatures, dyadic ramified primes or tamely ramified primes of class 0, Hilbert symbols determining boundary may be arbitrarily specified, with corrections made at the infinite ramified, dyadic or class 0 primes. Thus, by realization there is a form in \( H(E) \) with non-trivial discriminant at prescribed inert and ramified primes of class 1. Applying \( \partial \), we obtain a form in \( H(E/I) \) with prescribed rank at inert primes, and prescribed discriminant at class 1 ramified primes. However, as required by Theorem 4.1, rank is preserved at class 1 ramified primes, with \( \gamma(\partial) = 0 \) at dyadic ramified primes of class 0.

We may thus write

\[
H(E/I)/\text{im} \partial \cong (F_2 \times \ldots \times F_2)^{a_x} \times (F_2 \times \ldots \times F_2)^{b/W} \text{ via } \gamma
\]

with \( \gamma \) given by:

\[
\gamma : \oplus [M_i, B_i] \rightarrow (r_1, \ldots, r_a, s_1, \ldots, s_b) \quad \text{where}
\]
\( r_i = \text{rank } [M_i, B_i] \) at class 1 ramified primes, dyadic or not.

\( s_i = \text{rank } [M_i, B_i] \) at class 0 dyadic ramified primes.

\( W = C_2 \) subgroup of the product \( \prod_{i=1}^{a+b} F_2 \) generated by

\[
(1, \ldots, 1, 0, 0, \ldots, 0) \\
(0, \ldots, 0, 0, \ldots, 0)
\]

This completes the proof. \( \square \)

**Comment:** \( \hat{\beta}(\varphi) = 0 \) at tamely ramified class 0 primes and also \( H_u(D/\varphi) = 0 \). Thus, there is no contribution to the cokernel.

By applying Theorem 4.1 we are also able to calculate \( H_u(I) \).

We let \( J \subseteq H(E) \) be the subgroup of even rank forms. On \( J \cap H_u(I) \) we may define a local discriminant homomorphism at each dyadic ramified prime:

\[
d_i : J \cap H_u(I) \to H(E) \xrightarrow{(d, \sigma)} P_i \mathbb{Z}^* 
\]

where \( d \) is the discriminant of the form in \( H(E) \), and \( (d, \sigma)_{P_i} \) is the Hilbert symbol at \( P_i \).

Since we have restricted the domain of \( d_i \) to the even
rank forms, the discriminant is multiplicative, as is the Hilbert symbol, so that \( d_i \) is indeed a homomorphism.

Let \( t \) be the number of dyadic ramified primes. We then define the total discriminant homomorphism \( \tilde{d} \) to be the product of the \( d_i \).

\[
\tilde{d} = \prod (d_i) : J \cap H_u(I) \to (\mathbb{Z}^*)^t
\]

There are also the infinite ramified primes to take care of. By Chapter VI 3, we may define a signature \( \text{sgn}_i : H_u(E) \to \mathbb{Z} \) at each infinite ramified prime \( p_i \). Combining, we obtain a total signature homomorphism \( \text{sgn} : J \cap H(I) \supseteq (2\mathbb{Z})^r \), where \( r \) is the number of infinite ramified primes.

Recall how the Hilbert symbol was read at an infinite ramified prime, Chapter V 2.5. Let \( d \) be the discriminant, \( p_\infty \) an infinite ramified prime. Then \( (d, \sigma)_\infty = \pm 1 \) depending on the sign of \( d \) in \( p_\infty \). \( d = (-1)^{\frac{w(w-1)}{2}} \text{det} \) where \( w = \text{rank } B \), \( \text{det} = \text{determinant } B \). Hence \( d \) has signature

\[
(-1)^{\frac{w(w-1)}{2}} (-1)^{\frac{w-2w-\text{sgn}B}{2}} = (-1)^{\frac{w^2-2w-\text{sgn}B}{2}}
\]

For \( [M,B] \) in \( J \cap H_u(I) \), \( w \equiv 0 \mod 2 \), so that

\[
(d, \sigma)_\infty = (-1)^{-\text{sgn}B/2} = (-1)^{\text{sgn}B/2}
\]
We thus can state:

**Theorem 5.6** \( H_u(I) \) is determined as follows. For the even rank forms in \( H_u(I) \), we have an exact sequence:

\[
0 \rightarrow J \cap H_u(I) \rightarrow \mathbb{J}_n \rightarrow \text{sgn}^e \rightarrow H_u(I) \oplus (\mathbb{Z}^*)_t \rightarrow \mathbb{Z}^* \rightarrow 0
\]

where \( H = \Pi(-1) \) and \( d_i \) is the Hilbert reciprocity map.

In order for a rank 1 form to exist in \( H_u(I) \) there must be no class 1 ramified primes. If there are any signatures or class 0 ramified primes, a rank 1 form exists. If there are no ramified primes, a rank 1 form exists if and only if \( v_\varphi(I) + v_\varphi\varphi(x_1) = 1 \) (2) at an even number of primes.

Further, if a rank 1 form exists, there is an exact sequence:

\[
0 \rightarrow J \cap H_u(I) \rightarrow H(I) \rightarrow F_2 \rightarrow 0
\]

This sequence splits if and only if \(-1\) is a norm.

**Proof:** \( J \cap H(I) \) embeds into \( J \cap H(E) \), which by Landherr's Theorem VI 3.4 is determined by the discriminant and multisignatures. However, \( H_u(I) = \bigcap_{\varphi=\varphi} \ker \Delta(\varphi) \).
by Corollary 2.9. Now, \( \delta = \otimes \delta (\theta) \) is read by the discriminant at all inert primes of even rank, and at all class 1 tamely ramified primes. Hence \((d, \sigma)_p \) must be trivial at all these primes in order for a form to be in the kernel of \( \delta = \otimes \delta (\phi) \).

Thus, by Landherr we obtain

\[
\text{sgn } \otimes \tilde{\delta} \\
0 \to J \cap H_u(I) \to (\mathbb{Z})^r \oplus (\mathbb{Z}^\times)^t \to \mathbb{Z}^\times \to 0
\]

is exact.

To complete the discussion, we must examine whether a rank 1 form can exist in \( H_u(I) \). \( \langle ax_1 \rangle \) is such a form if and only if \( \delta(\theta) \langle ax_1 \rangle = 0 \). By theorem 4.1, this is possible only if there are no class 1 ramified primes. Assuming this necessary condition, if there is a signature or a class 0 ramified prime, we apply Realization of Hilbert symbols. Thus, in this case, there exists \( a \in F/E/F \) with \( \delta(\phi) \langle ax_1 \rangle = 0 \) at all inert primes. Further, all ramified primes are of class 2, so that \( \delta \langle ax_1 \rangle = 0 \), and we obtain the desired rank 1 form.

We apply Theorem 4.1 if there are no ramified primes, finite or infinite.

In order to have \( \delta(\phi) \langle ax_1 \rangle = 0 \) we must satisfy:

\[
(a, \mathcal{C})_p = (-1)^{\delta(\phi) \langle ax_1 \rangle + \nu_{\phi}(I)} = (-1)^{\nu_{\phi}(I)}
\]
when \( v_\varphi(x_1) \) is odd at an even number of primes. This is possible if and only if \( v_\varphi(I) \) is odd at an even number of primes, so that \( v_\varphi(I) + v_\varphi(x_1) \) is odd at an even number of primes.

Similarly, if \( v_\varphi(x_1) \) is odd at an odd number of primes, \( \langle ax_1 \rangle \) must satisfy:

\[
(a, \sigma)_p = (-1)^{\frac{3}{2} \langle ax_1 \rangle + v_\varphi(I)} = (-1)^{v_\varphi(I)} \quad \sigma \neq \varphi
\]

\[
(a, \sigma)_p = (-1)^{v_\varphi(I)+1}
\]

in order to have \( 3\langle ax_1 \rangle = 0 \). Again Hilbert reciprocity implies \( v_\varphi(I) \) is odd at an odd number of primes, so that \( v_\varphi(I) + v_\varphi(x_1) \equiv 1 \pmod{2} \) at an even number of primes.

Finally, we must discuss the extension

\[
0 \rightarrow J \cap H_{\mu}(I) \rightarrow H(I) \overset{rk}{\rightarrow} P_2 \rightarrow 0
\]

when a rank 1 form exists. By Landherr, and Corollary VI 2.14, this sequence splits if and only if \(-1\) is a norm in \( F/NE \).

---

**Corollary 5.7** If there are no ramified primes, the boundary sequence depends on the number of primes at which \( v_\varphi(x_1) + v_\varphi(I) \) is odd.

(a) If this number is even, we have:
0 \to H_u(I) \to H_u(E) \to H_u(E/I) \to F_2 \to 0

(b) If this number is odd, $H_u(I) = 0$, and

0 \to H_u(E) \to H_u(E/I) \to 0. \quad \Box

Thus, we have shown how to compute $H_u(I)$, and the boundary homomorphism $\partial : H_u(E) \to H_u(E/I)$. To apply this to $I = \Delta^{-1}(D/Z)$, we clearly need only to know $v_\Phi(\Delta^{-1}(D/Z))$. This is exactly determined by type, see Chapter VII 2.6, for dyadic ramified primes.

Now $\partial : H_u(E) \to H_u(E/I)$ splits as a direct sum of $\oplus \partial(\Phi)$. However, each $H_u(E)$ is only one piece of $\mathcal{H}(k,Q)$. The fact is that the total boundary $\partial : \mathcal{H}(k,Q) \to W(k,Q/Z)$ may have coupling between the boundary pieces $\partial(D,\Phi)$, as both $D$ and $\Phi$ vary.

In the next section then, we examine this coupling.

We should make two remarks concerning the computation of the group $W(k,Z:S) \approx H(\Delta^{-1}(S/Z))$ for $S = Z[t,t^{-1}]/(f(t))$.

(1) In order to complete the computation one must compute $i_3$. This was done in Section 3, except for the remark that the set $T(\mathfrak{m}) = \{\Phi: \Phi \cap S = \mathfrak{m}, \Phi = \overline{\mathfrak{m}}\}$ may not have cardinality one for those ideals $\mathfrak{m}$ not prime to the conductor $C$. In order to complete the computation,
one must examine $T(\mathfrak{m})$ individually at the finite set of maximal ideals $\mathfrak{m}$ not prime to $C$.

(2) As indicated in Section 4, our computation of $\beta(\varphi)$ uses a particular choice of localizers to embed the residue field $D/\varnothing$ into $E/I(\varnothing)$. This set of localizers may not correspond to the canonical embedding: $W : F_p \to \mathbb{Q}/\mathbb{Z}(p)$ given by $1 \to \frac{1}{p}$. Thus, in using the diagram preceding Lemma 2.7, and in particular for Proposition 2.5, the isomorphisms

$$W(k, \mathbb{Q}/\mathbb{Z}; D) \simeq \bigoplus_{\varnothing = \mathfrak{m}} W(k, F_p; D/\mathfrak{m})$$

$$W(k, \mathbb{Q}/\mathbb{Z}; S) \simeq \bigoplus_{\mathfrak{m} = \mathfrak{n}} W(k, F_p; S/\mathfrak{n})$$

are given by the choice of localizers used in the boundary computation in order that the diagram below commute.

$$\begin{array}{ccc}
H(D/\varnothing) & \cong & H(E/I(\varnothing)) \\
\downarrow \text{tr} & & \downarrow t \\
W(k, F_p; D/\varnothing) & \to & W(k, \mathbb{Q}/\mathbb{Z}(p))
\end{array}$$
6. The coupling invariants

We recall our notation: \( W(k,K;f) \) denotes Witt equivalence classes of triples \([M,B,I]\) where the characteristic polynomial of \( I \) is a power of the \( T_k \) fixed irreducible polynomial \( f(x) \). By taking anisotropic representatives, Proposition III 1.11 identified this as:

\[ W(k,K;f) \cong W(k,K;S) \quad \text{where} \quad S = \mathbb{Z}[t]/(f(t)). \]

We used the notation \( \partial(S) \) to denote the restriction of the boundary map to \( W(k,K;S) \). In this section we wish to emphasize the polynomial \( f \) rather than the module structure. Thus we use the notation \( \partial(f) \) to denote the restriction of the boundary map to \( W(k,K;f) \). Of course \( \partial(f) \) is really \( \partial(S) \).

We have the commutative diagram

\[
\begin{array}{ccc}
W(k,Z) & \xrightarrow{i} & W(k,Q) \\
\downarrow & & \downarrow \\
\oplus_{f \in \mathcal{B}} W(k,Q;f) & \xrightarrow{q_f} & W(k,Q;f) \\
\downarrow \epsilon(f) & & \downarrow \partial(f) \\
& & W(k,Q/Z;f)
\end{array}
\]

We label the composition \( \partial(f) \circ q_f \circ i \equiv \epsilon(f) \).

It follows that there is the commutative diagram:
\[ W(k,Z) \xrightarrow{\oplus \varepsilon(f)} \bigoplus_{f} W(k,Q/Z;f) \]
\[ \downarrow i \quad \quad \downarrow \alpha_{1} \]
\[ W(k,Q) \xrightarrow{\beta} W(k,Q/Z) \]

The map \( \alpha_{1} \) just adds up all the terms in \( W(k,Q/Z) \).
We wish to measure how the various groups \( W(k,Q/Z;f) \) couple together.

**Theorem 6.1** There is an exact sequence

\[ 0 \rightarrow \bigoplus_{f \in \mathcal{B}} W(k,Z;f) \xrightarrow{i} W(k,Z) \oplus \varepsilon(f) \bigoplus_{f \in \mathcal{B}} W(k,Q/Z;f) \]
\[ \cong W(k,Q/Z) \oplus \text{coker } \beta(f). \]

**Comment:** \( \alpha \) is onto provided \( \beta : \mathcal{W}(k,Q) \rightarrow W(k,Q/Z) \) is onto. This is the topic of Section 7.

**Proof:** The map \( \alpha \) is \( \alpha_{1} \) on the first factor, and the appropriate projection into the cokernel on the other factor. \( i \) just adds up terms in \( W(k,Z) \), as does \( \alpha_{1} \).

We begin by considering the commutative diagram:
\[ \begin{array}{cccc}
0 & 0 \\
\downarrow & \downarrow \\
\oplus W(k, Z; f) & \overset{i}{\rightarrow} & W(k, Z) \\
\downarrow & \downarrow \\
0 & \overset{i_1}{\rightarrow} & \mathcal{W}(k, Q)
\end{array} \]

\(i\) is then clearly 1-1 since all the other maps are.

To check exactness at \(W(k, Z)\), suppose

\[ i \langle \oplus L_f, B_f, t_f \rangle = \langle L, B, t \rangle . \]

Then \(q_f(L \otimes Q)\) contains the self dual lattice \(L_f = L_f^\#\), hence \(\epsilon(f)(L) = 0\). Thus \((\epsilon(f)) \circ i = 0\).

Conversely, suppose \((\oplus \epsilon(f)) \langle L, B, t \rangle = 0\). Let

\[ M = L \otimes Q = \oplus M(f) \].

Each \(M(f)\) has \(\epsilon(f)(M(f)) \sim 0\), so let \(N(f)\) be a metabolizer for \(\epsilon(f)(M(f))\). Exactly as in the proof of exactness for \(\bar{\alpha}\), this yields a self-dual lattice and hence an element in \(W(k, Z; f)\); which under \(i\) is mapped to \(M(f)\). Thus, \(\ker (\oplus \epsilon(f)) \subseteq \text{im } i\).

Next we show \(\alpha_1(\oplus \epsilon(f)) = 0\). For the first factor, this follows by the commutative diagram \(\alpha_1 \circ (\oplus \epsilon(f)) = \bar{\alpha} \circ i = 0\).

The other components are also 0 since \(\epsilon(f) = \alpha(f) \circ q_f \circ i\) so that elements in the image of \((\oplus \epsilon(f))\) are in the image of \(\alpha(f)\), thus 0 in \(\text{coker } \alpha(f)\).
Conversely, suppose we are given a collection
\[ \oplus_{f} \left[ M(f), B(f), t(f) \right] \]
of torsion forms in \[ \oplus_{f} W(k, \mathbb{Q}/\mathbb{Z}; f) \]
which are in the kernel of \( \alpha \). Write these as
\[ \oplus_{f} [M_f, B_f, t_f] \]. Then each \( [M_f, B_f, t_f] \) is the trivial element
in cokernel of \( \delta(f) \), hence \( [M_f, B_f, t_f] \) is in the image
of \( \delta(f) \).

So let \( \alpha(f)[M_f', B_f', t_f'] = [M_f, B_f, t_f] \). Then
\[ \oplus_{f} [M_f, B_f, t_f] = \oplus_{f} \alpha(f)[M_f', B_f', t_f'] \]

Applying \( \alpha_1 \), i.e. adding up in \( W(k, \mathbb{Q}/\mathbb{Z}) \), we obtain.
\[ \alpha_1(\oplus_{f} [M_f, B_f, t_f]) = \alpha(\oplus_{f} [M_f', B_f', t_f']) \]

However, we are assuming \( \alpha_1(\oplus_{f} [M_f, B_f, t_f]) = 0 \).

Thus \( \alpha(\oplus_{f} [M_f', B_f', t_f']) = 0 \). By the exactness of the boundary sequence, Theorem 1.5, this implies there exists
\( [M, B, t] \in W(k, \mathbb{Z}) \) with \( i[M, B, t] = \oplus_{f} [M_f', B_f', t_f'] \). Hence
\[ \bigoplus_{f} \alpha(f) q_f i([M, B, t]) \]
\[ = \bigoplus_{f} \delta(f) [M_f', B_f', t_f'] \]
\[ = \oplus_{f} [M_f, B_f, t_f] = \bigoplus_{f} \varepsilon(f) [M, B, t] \]

so that \( \ker \alpha \subseteq \text{im}(\bigoplus_{f} \varepsilon(f)) \). \( \square \)
7. **The boundary is onto**

In this Section, we derive the results needed to study the octagon over \( \mathbb{Z} \). In particular, we will show that \( \mathcal{a} : \mathcal{H}(k, \mathbb{Q}) \rightarrow W(k, \mathbb{Q}/\mathbb{Z}) \) is onto when \( k = \pm 1 \), or \( k = \) positive prime, \( k = 2, 3(4) \), or \( k = 1(8) \).

We also show that \( \mathcal{a} : \mathcal{O}(\mathbb{Q}) \rightarrow A(\mathbb{Q}/\mathbb{Z}) \) is onto, and compute the cokernel of \( \mathcal{a} : \mathcal{H}^{-1}(k, \mathbb{Q}) \rightarrow W^{-1}(k, \mathbb{Q}/\mathbb{Z}) \) when \( k = \pm 1 \).

To begin with, we recall our computations:

\[
\mathcal{H}(k, \mathbb{Q}) \approx \bigoplus_{f \in S} H(\mathbb{Q}(\theta)) \otimes W(\mathbb{Q}(\sqrt[k]{f})) \quad k \neq 1
\]

\[
\text{of type } l
\]

\[
\bigoplus_{f \in S} H(\mathbb{Q}(\sqrt[k]{f})) \otimes W(\mathbb{Q}) \otimes W(\mathbb{Q}) \quad k = 1.
\]

Also, we had:

\[
W(k, \mathbb{Q}/\mathbb{Z}) \approx \bigoplus_{p \nmid k} W(k, \mathbb{F}_p) \oplus W(\mathbb{F}_p)_{p|k}
\]

\[
\approx [\bigoplus_{p \nmid k} \bigoplus_{f \in S} \bigoplus_{k \nmid \mathbb{F}_p^*} H(\mathbb{F}_p(\theta)) \otimes W(\mathbb{F}_p(\sqrt[k]{f}))]
\]

\[
\text{of type } l \quad p
\]

\[
\bigoplus_{k \in \mathbb{F}_p^*} [\bigoplus_{p|k} W(\mathbb{F}_p) \oplus W(\mathbb{F}_p)_{p|k} W(\mathbb{F}_p)_{p \neq 2}]
\]

In this decomposition, we sum over all \( T_k \) fixed irreducible polynomials \( f(x) \) of type 1, where \( f \) has coefficients in \( \mathbb{F}_p \). We observe that any such \( f \) can
be lifted (not uniquely) to a $T_k$ fixed integral polynomial.

To see this write $f(x) = \sum_{i=0}^{2n} a_i x^i$, where $a_i \in \mathbb{F}_p$.

Lift the first $n$ coefficients $\bar{a}_i$ to $a_i \in \mathbb{Z}$ with $a_i \equiv \bar{a}_i \pmod{p}$. By Lemma II.1.4, a polynomial $g(x)$ is $T_k$ fixed if and only if its coefficients satisfy $a_i k^i = a_0 a_{2n-i}$. We define $g(x)$ by

$$g(x) = \sum_{i=0}^{2n} c_i x^i \quad \text{where} \quad c_i = a_i \quad i = 0, \ldots, n$$

$$c_i = a_0 a_{2n-i} k^{-i} \quad i = n+1, \ldots, 2n$$

By Lemma II.1.4, $g(x)$ is $T_k$ fixed, and clearly the mod $p$ reduction of $g(x)$ is $f(x)$, since $f(x)$ is $T_k$ fixed also.

In fact, we should observe that if $Q(\theta)$ has a non-trivial involution $\bar{\theta} = k \theta^{-1}$, then the irreducible polynomial of $\theta$ over $Q$ is $T_k$ fixed. For if

$$p(x) = \sum_{i=0}^{2n} a_i x^i$$

is the monic irreducible polynomial of $\theta$, then $p(\theta) = p(\bar{\theta}) = 0$. Thus

$$p(k \theta^{-1}) = \sum_{i=0}^{2n} a_i (k \theta^{-1})^i = 0$$

Multiplying by $\theta^{2n}$, $\sum_{i=0}^{2n} a_i (k)_{i} \theta^{2n-i} = 0$. Thus

$$\theta^{2n} = - \sum_{i=1}^{2n} \frac{a_i k^i \theta^{2n-i}}{a_0}.$$
However
\[ \Theta^{2n} = - \sum_{i=0}^{2n-1} a_i \Theta^i \] since \( p(\Theta) = 0 \).

Using the fact that \( 1, \Theta, \ldots, \Theta^{2n-1} \) is a basis for \( Q(\Theta) \) over \( Q \) since \( p(x) \) is irreducible, we may equate coefficients of the two sums for \( \Theta^{2n} \), and obtain
\[ \frac{a_i k^i}{a_0} = a_{2n-i}, \]
so that \( p(x) \) is \( T_k \) fixed by Lemma II 1.4.

This remark is important when we show that the Hermitian elements in \( W(k, Q/\mathbb{Z}) \) are in the image of boundary. Given \( H(F_p(\Theta_1)) \), we construct an extension \( Q(\Theta) \) of \( Q \), with a non-trivial involution \( \overline{\Theta} = k\Theta^{-1} \), with the property that the mod \( p \) reduction of \( \overline{\Theta} \) is \( \Theta_1 \).

By the above it follows that the irreducible polynomial of \( \Theta \) is \( T_k \) fixed. Hence \( H(Q(\Theta)) \) occurs in the decomposition of \( \mathcal{H}(k, Q) \). Applying \( \delta \) to this \( H(Q(\Theta)) \) we show that \( H(F_p(\Theta_1)) \) is in the image of \( \delta \), possibly together with some Witt contribution. However, all Witt elements are first shown to be in the image of \( \delta \), so that \( \delta \) is onto.

We begin our study of \( \delta : \mathcal{H}(k, Q) \to W(k, Q/\mathbb{Z}) \) by studying \( \delta \mid W(Q(\sqrt{k})) \). Letting \( S = \mathbb{Z}[t]/(t^2 - k) \), we previously used the notation \( \delta(S) \) for \( \delta \mid W(Q(\sqrt{k})) \).
Let $D$ denote the ring of integers in $\mathbb{Q}(\sqrt{k})$. By [S 35] $D = S$ for $k = \pm 1$ or $p$, where $p$ is a prime $p \equiv 2$ or $p \equiv 3 \pmod{4}$. For $p \equiv 1 \pmod{4}$, $D$ = \{all elements of the form $\frac{1}{2}(u+\sqrt{k})$\}, where $u, v \in \mathbb{Z}$ with $u \equiv v \pmod{2}$.

We now recall the computation for $\mathfrak{d}(D)$ given in [M.H 94]. There is an exact sequence:

$$W(\mathbb{Q}(\sqrt{k})) \overset{\delta}{\to} W(D/\mathfrak{d}) \overset{\phi}{\to} C/C^2 \to 0.$$ 

where $C$ = ideal class group, and $\phi$ is defined on each generator $\langle \tilde{u} \rangle$ of $W(D/\mathfrak{d})$ by $\langle \tilde{u} \rangle \mapsto$ ideal class of $\mathfrak{d}$ modulo $C^2$.

We must be careful. This boundary sequence is for $I = D$. We are interested in the case $I = \Delta^{-1}(D/\mathfrak{d})$ of the inverse different. Fortunately, in our case, $\Delta^{-1}$ and $D$ are principal orders, and we may write

$$\Delta^{-1} = D/\alpha \quad \text{for some} \quad \alpha \in D.$$ 

We are of course in the special case of a quadratic extension of $\mathbb{Q}$.

We denote $\mathfrak{d}'(D) =$ boundary for $I = \Delta^{-1}$; $\mathfrak{d}(D) =$ boundary for $I = D$ in the next Lemma.

Lemma 7.1 Scaling by $\alpha$ induces a commutative
\[ W(\sqrt[k]{k}) \xrightarrow{\partial(D)} W(E/D) \]
\[ \cong \downarrow \cdot \frac{1}{\alpha} \cong \downarrow \cdot \frac{1}{\alpha} \]
\[ W(\sqrt[k]{k}) \xrightarrow{\partial'(D)} W(E/I) \]

**Proof:** \(\alpha \Delta^{-1} = D\), so that commutativity follows by definition of \(\alpha\). \(\square\)

Thus, once we have computed \(\partial(D)\), we will also have a computation for \(\partial'(D)\).

To begin with we will show \(C/C^2\) is trivial in the stated cases. This requires some number theory; we refer to [B,S]. We shall show that for \(k = \pm 1\), \(p\) prime, \(p \equiv 3 \pmod{4}\), \(\partial'(D) = \partial(D) = \partial(S)\) is onto.

We also show \(\partial(S)\) is onto when \(k = 1\) (8), and compute the cokernel, \(C_2\) arising from \(W(F_2)\), when \(k = 5\) (8). Caution: This Witt piece, \(W(F_2)\), arising in \(W(k, \mathbb{Q}/\mathbb{Z})\), \(k = 5\) (8), is thus not in the image of the Witt piece \(W(\sqrt[k]{k})\) in \(\mathcal{W}(k, \mathbb{Q})\). However, we have not shown that this \(W(F_2)\) is not the image of a Hermitian piece in \(\mathcal{W}(k, \mathbb{Q})\). This question is still open.

We now aim to prove:

**Theorem 7.2** \(C/C^2\) is trivial for \(C\) the ideal class
group in \( \mathbb{Q}(\sqrt{k}) \), provided \( k = \pm 1 \), or \( k \) a positive prime.

To set our notation, there are three cases:

1. \( k \not\equiv p^\ast \pmod{F} \) (p) remains prime in \( D \). \( e = 1 \quad f = 2 \)
   \( \frac{D}{\mathfrak{p}} = F_p(\sqrt{k}) \) where \( \mathfrak{p} \cap \mathbb{Z} = (p) \).

2. \( k \equiv p^\ast \pmod{F} \) (p) splits in \( D \). \( e = 1 \quad f = 1 \)
   \( \frac{D}{\mathfrak{p}_1} = \frac{D}{\mathfrak{p}_2} \) where \( pD = \mathfrak{p}_1 \mathfrak{p}_2 \).

3. \( p \) divides \( k \), written \( p \mid k \)
   (p) ramifies \( e = 2 \quad f = 1 \)
   \( \frac{D}{\mathfrak{p}} = F_p \) where \( \mathfrak{p} \cap \mathbb{Z} = (p) \).

We follow Borevich-Shafarevich [B,S] in defining:

**Definition 7.3** Two ideals \( A \) and \( B \) of \( D \) are strictly equivalent if there exists \( \alpha \neq 0 \) in \( \mathbb{Q}(\sqrt{k}) \) satisfying

\[
N_{\mathbb{Q}(\sqrt{k})/\mathbb{Q}}(\alpha) > 0 \quad \text{and} \quad A = B(\alpha).
\]

For \( k < 0 \), \( N_{\mathbb{Q}(\sqrt{k})/\mathbb{Q}}(\alpha) > 0 \) always, so that this is the usual definition of equivalence in the ideal class group \( \mathcal{C} \).

However, if \( k < 0 \) and \( N_{\mathbb{Q}(\sqrt{k})/\mathbb{Q}}(\varepsilon) = +1 \) for all units \( \varepsilon \), then each ideal class in \( \mathcal{C} \) will split into two classes equivalent in the strict sense. [B,S 239].

[B,S] calls \( A,B \) divisors. For the case of the maximal order \( D \), [B,S 215], divisors correspond in a 1-1 fashion with ideals in \( D \).

**Notation** If \( \mathfrak{p} \) is an ideal, let \([\mathfrak{p}]\) denote its equivalence class, \( <\mathfrak{p}> \) denotes its strict equivalence class.

When all units \( \varepsilon \) have positive norm and \( k > 0 \) we can write

\[
[\mathfrak{p}] = <\mathfrak{p}> \cup <\sqrt{k}\mathfrak{p}>.
\]
Lemma 7.4 \([\varnothing] is a square in \mathbb{C} if and only if there exists an ideal \(Q \in [\varnothing] with \langle Q \rangle strictly a square.\)

Proof: Sufficiency is clear, since if \(\langle Q \rangle\) is strictly equivalent to \(\langle R^2 \rangle\), then \([Q] = [R^2]\) also.

Conversely, let \([\varnothing] \in \mathbb{C}^2\). Then there exists an ideal \(Q\) with \(\varnothing = \alpha Q^2\). If \(N_Q(\sqrt{k})/Q(\alpha) > 0\), \(\langle \varnothing \rangle \sim \langle Q^2 \rangle\) and we are done. Otherwise, suppose \(N(\alpha) = \alpha \bar{\alpha} < 0\). Consider \(\bar{\varnothing} \in [\varnothing]\). \(\bar{\varnothing} = \bar{\alpha} Q^2\), and \(N(\alpha \bar{\alpha}) > 0\). Thus \(\langle \bar{\varnothing} \rangle \sim \langle Q^2 \rangle\). □

Thus to check if \([\varnothing]\) is a square, we need only check if either of its strict equivalence classes is a square.

By [B,S 246], Theorem 7, \(\langle A \rangle \sim \langle B^2 \rangle\) if and only if \(\left( \frac{N'(A), D}{p} \right) = +1\) for all \(p \mid D\). Here \(N'(A)\) is the norm of \(A\) [B,S 124,219], \(\bar{D}\) is the discriminant of \(\mathbb{Q}(\sqrt{k})\) over \(\mathbb{Q}\), and \(\left( \frac{N'(A), D}{p} \right) = (N'(A), D)_p\) is the Hilbert symbol.

Note: We shall use our usual notation in this section for the Hilbert symbol rather than following [B,S]. \(N'(A)\) is a positive integer, see [B,S 124].

Remark: \(\left( \frac{N'(A), D}{p} \right)_p = +1\) automatically for \(p \nmid \bar{D}\), and \(p = \infty, [B,S 242]\).

We are now ready to examine \(\mathbb{C}/\mathbb{C}^2\) for \(k = \pm 1\), \(k\) a positive prime.

Case I. For \(k = \pm 1\), \(D\) is a principal ideal domain, so that \(\mathbb{C} = 0 = \mathbb{C}/\mathbb{C}^2\).

Case II. Let \(p > 0\) be a prime \(p \equiv 1 \pmod{4}\).

Claim: For \(\mathbb{Q}(\sqrt{p})\), \(\mathbb{C}/\mathbb{C}^2\) is trivial.
Proof: In this case, equivalence coincides with strict equivalence. If \( \mathcal{O} \) is a prime ideal, we shall show \( \langle \mathcal{O} \rangle \sim \langle B^2 \rangle \) by computing \( (N'(\mathcal{O}), \mathfrak{D})_p \), as \( p \) divides \( \mathfrak{D} = \text{discriminant} = p \).

Let \( \mathcal{O} \cap \mathbb{Z} = (q) \).

Case 1. \( (q) \) is inert. Then \( N'(\mathcal{O}) = q^2 \). \( (q^2, p)_p = +1 \).

Case 2. \( (q) \) splits, \( q \) odd, so \( N'(\mathcal{O}) = q \) and \( (\mathfrak{P}_q) = +1 \).

Note: \( (\mathfrak{P}_q) \) is the Legendre symbol.

\[
\begin{align*}
(N'(\mathcal{O}), p)_p &= (q, p)_p = (\frac{q}{p})_p \\
&= (\frac{q}{p})(-1)\frac{p-1}{2} \cdot \frac{q-1}{2} \\
&= +1
\end{align*}
\]

Case 3. \( (q) \) ramifies. Again \( N'(\mathcal{O}) = q = p \).

\[
\begin{align*}
(p, p)_p &= (p, -p)_p(p, -1)_p = (\frac{-1}{p}) = (-1)\frac{p-1}{2} = +1
\end{align*}
\]

Case 4. \( (q) = (2) \)

(a) \( p \equiv 5 \pmod{8} \). \( (2) \) is inert. \( N'(\mathcal{O}) = 2^2 \), and we are done as in Case 1.

(b) \( p \equiv 1 \pmod{8} \). \( (2) \) splits. \( (2, p)_p = (\frac{2}{p}) = (-1)\frac{p^2-1}{8} = +1 \).

Thus, by Theorem 7 from [B, S], all prime ideals \( \mathcal{O} \) in \( \mathcal{C} \) are squares and \( \mathcal{C} = \mathcal{C}^2 \). \( \square \)

Case III. Let \( p > 0 \) be prime, \( p \equiv 3 \pmod{4} \).

Claim: \( \mathcal{C}/\mathcal{C}^2 \) is trivial for \( \mathbb{Q}(\sqrt{p}) \).

In this case, each ideal class \([\mathcal{O}]\) in \( \mathcal{C} \) splits into two strict equivalence classes. We may represent these as
< p > and < √pθ >, since N (√p) = - p < 0.

Let θ be a prime ideal in Q(√p). θ ∩ Z = (q). In this case the discriminant D = 4q. Again, we have 4 cases.

Case 1. (q) is inert. N'(θ) = q^2, and (q^2,4p)_p = + 1 as before.

Case 2. q odd. (q) splits. So (q) = + 1. N(θ) = q.

We compute (N'(θ),4p)_{p_i} for p_i = 2 or p.

(q,4p)_p = (q) = (q) (-1)(p-1)/2 (q-1)/2 = (-1)(q-1)/2

(q,4p)_2 = (q,p)_2 = (-1)(p-1)/2 (q-1)/2 = (-1)(q-1)/2

(see [O'M 206])

If (-1)(q-1)/2 = - 1, θ is a strict square.

If (-1)(q-1)/2 = - 1, consider √pθ ∈ [θ].

In this case, namely (-1)(q-1)/2 = - 1, we will show < √pθ > is a strict square. Hence, [θ] is a square in C.

To begin with, N'(√pθ) = p . q. We compute,

(N'(√pθ),4p)_p = (pq,p)_p = (p,p)_p (q,p)_p

= (p,-p)_p (p,-p)_p (-1)(q-1)/2

= (-1)(p-1)/2 (-1)(q-1)/2 = (-1)(-1) = + 1

(N(√pθ),4p)_2 = (pq,p)_2 = (p,p)_2 (q,p)_2

= (-1)(p-1)/2 (p-1)/2 (-1)(q-1)/2 (p-1)/2

= (-1)(-1) = + 1

Thus, again by Theorem 7, we conclude < √pθ > is a strict square.
Case 3. $\varnothing \cap \mathbb{Z} = (q) = (p)$, so that $(q)$ ramifies. $[\varnothing]$ contains $\sqrt{p}\varnothing$. $N'(\sqrt{p}\varnothing) = pp = p^2$. Hence $[\varnothing]$ contains a strict square class, namely $<\sqrt{p}\varnothing>$.

Case 4. $\varnothing \cap \mathbb{Z} = (2)$. $N'(\varnothing) = 2$. If $p \equiv 3 \pmod{8}$, consider $\sqrt{p}\varnothing \in [\varnothing]$. $N'(\sqrt{p}\varnothing) = 2p$.

\[
\begin{align*}
(2p, 4p)_p &= (2, p)_p (p, p)_p \\
&= (-1)(p^2-1)/8 (-1) \\
&= (-1)(-1) = +1
\end{align*}
\]

Thus $[\varnothing] \in C^2$.

If $p \equiv 7 \pmod{8}$, $N'(\varnothing) = 2$.

\[
\begin{align*}
(2, 4p)_p &= (2, p)_p = (2) = (-1)(p^2-1)/8 = +1 \\
(2, 4p)_2 &= (2, p)_2 = (-1)p^2-1/8 = +1
\end{align*}
\]

Again $[\varnothing] \in C^2$. We have thus completed the proof of Theorem 7.2. □

Remark: For $k$ a negative prime congruent to 1 modulo 4, $Q(\sqrt{k})$ also has $C/C^2$ trivial. The argument is just like the above. It is also possible for one to ana-
lyze \( Q(k) \), for \( k = p \equiv 3 \ (4) \), in which case the above argument fails.

\textbf{Corollary 7.5} \ For \( k = +1 \), \( p \) with \( p \equiv 2, 3 \ (4) \), 
\( \delta'(D) = \delta(D) = \delta(S) \) \ is onto.

\textbf{Proof:} Immediate from the boundary sequence and Theorem 7.1, since \( S = D \) in this case. Recall \( S = \mathbb{Z}[t]/(t^2-k) \).

For \( p = 1 \ (4) \), we apply Proposition 3.9.

It follows that \( D/\mathcal{Q} = S/\mathcal{Q} \cap S \) for \( \mathcal{Q} \cap \mathbb{Z} \neq (2) \).

At \( (2) \) however, when \( \mathcal{Q} \cap \mathbb{Z} = (2), D/\mathcal{Q} = \) has \( \frac{1}{4} \) elements when \( p \equiv 5 \ (8) \), for then \( (2) \) is inert and \( f = 2 \).

For \( p = 1 \ (8) \), \( (2) \) splits as \( \mathcal{Q}_1 \mathcal{Q}_2 \), \( f = 1 \), and \( D/\mathcal{Q}_1 = D/\mathcal{Q}_2 = F_2 \).

Thus, \( \delta(S) = tr_* \circ \delta(\mathcal{Q}) \) cannot possibly be onto \( W(F_2) \) when \( p \equiv 5 \ (8) \), and is onto when \( p \equiv 1 \ (8) \). We summarize,

\textbf{Corollary 7.6} \( \delta(S) \) \ is onto when \( p \equiv 1 \ (8) \), and has cokernel \( C_2 = W(F_2) \) when \( p \equiv 5 \ (8) \).

Thus, in order to show \( \delta : \mathcal{V}(k, \mathbb{Q}) \to W(k, \mathbb{Q}/\mathbb{Z}) \) is onto all of the Witt pieces in \( W(k, \mathbb{Q}/\mathbb{Z}) \), it remains to hit this one last Witt piece, \( W(F_2) \), when \( k = p \equiv 5 \ (8) \).
We thus need to show how to find a Hermitian element in $\mathcal{W}(k, \mathbb{Q})$ which under $\alpha$ hits $W(F_2) \in W(k, \mathbb{Q}/\mathbb{Z})$ whenever $k = p \equiv 5 \pmod{8}$ is a positive prime. This question remains open.

**Corollary 7.7** For $k = \pm 1, 2, 3 \pmod{4}$ or $k \equiv 1 \pmod{8}$, $k$ a positive prime, $\alpha: \mathcal{W}(k, \mathbb{Q}) \rightarrow W(k, \mathbb{Q}/\mathbb{Z})$ is onto all Witt pieces in the decomposition of $W(k, \mathbb{Q}/\mathbb{Z})$.

**Proof.** We observe that all Witt pieces in $W(k, \mathbb{Q}/\mathbb{Z})$ occur in $W(D/H) = W(E/D)$, $E = \mathbb{Q}(\sqrt{k})$.

By Corollaries 7.5, and 7.6 $\alpha(S)$ is onto these Witt pieces. Hence, so is $\alpha$ all the more so. □

**Corollary 7.8** $\alpha: \mathcal{S}(\mathbb{Q}) \rightarrow A(\mathbb{Q}/\mathbb{Z})$ is onto all Witt pieces in $A(\mathbb{Q}/\mathbb{Z})$.

**Proof:** Same as above, since $W(\mathbb{Q})$ occurs in $A(\mathbb{Q})$. □

Thus $\alpha$ restricted to $W(\mathbb{Q}((k))/k)$; $\alpha(D) = \alpha(S)$, for $S = \mathbb{Z}[t]/(t^2 - k)$, is onto when $k$ is a positive prime, $k \equiv 1, 2, 3 \pmod{4}$ or $k \equiv 1 \pmod{8}$, or $k = \pm 1$. 
By onto, we mean all Witt pieces in \( W(k, \mathbb{Q}/\mathbb{Z}) \) will be in the image of \( \phi(S) \), and hence in the image of \( \phi \).

For these \( k \), it remains to show \( \phi \) is onto. To do this, we must show that all Hermitian pieces in \( W(k, \mathbb{Q}/\mathbb{Z}) \) are in the image of \( \phi \).

In this inert case, we show \( \phi \) is onto by hitting each \( H(F_p(\theta_1)) \) separately by \( \phi \), where \( \theta_1 \) satisfies a \( T_k \) fixed polynomial of type 1.

First, Assume \( p \neq 2 \); we will do the case \( p = 2 \) last.

Let \( q \) be a prime, with \( (q, p) = 1 = (2, p) \).

Suppose \( \theta \) satisfies \( x^2 - a_1 x + k = 0 \) over the fixed field of the involution \( -: \theta_1 \rightarrow \tilde{\theta}_1 = k \theta_1^{-1} \). Here \( a_1 \in F_p(\theta_1 + k \theta_1^{-1}) \), the fixed field of \( - \). We write \( a_1 = 2b_1 \), which is possible since 2 and \( p \) are relatively prime.

Let \( F_p(\theta_1) = F_p^{2n} = \text{finite field with } p^{2n} \text{ elements.} \)

The fixed field of \( - \) is \( F_p^n \). We shall now construct an extension \( Q(\theta) \) of \( Q \), together with an involution \( -: \theta \rightarrow k \theta^{-1} \) which is non-trivial. Further, we shall arrange that the monic irreducible polynomial of \( \theta \) when read mod \( p \) is the monic irreducible polynomial of \( \theta_1 \).
Let $F$ denote the fixed field of $-\cdot$ on $\mathbb{Q}(\theta)$. We shall arrange for at least one prime ideal, $p$ in $O(F)$, with $p \cap \mathcal{Z} = (q)$, to ramify in $\mathbb{Q}(\theta)$ over $F$. We then consider the boundary map $\partial$ restricted to $H(Q(\theta))$, with $H(Q(\theta))$ a direct summand of $W(k,\mathbb{Q})$. Since there are ramified primes, the cokernel of $\partial$ will be in terms of these. Hence, $\partial | H(Q(\theta))$ will be onto the Hermitian piece $H(F_p(\theta_1))$, modulo the Witt pieces in $W(k,\mathbb{Q}/\mathbb{Z})$. But by 7.7, these Witt pieces have already been shown to be in the image of $\partial$. Thus, $H(F_p(\theta_1))$ is in the image of $\partial : W(k,\mathbb{Q}) \to W(k,\mathbb{Q}/\mathbb{Z})$ as desired.

Again, $\theta_1$ satisfies $x^2 - 2b_1x + k = 0$.

We begin our construction by defining $b_2$ by the equation: $b_2^2 = q^2 b_2 + q + k$. Suppose now that $b_2$ satisfies $x^m + c_{m-1}x^{m-1} + \ldots + c_0 = f_2(x)$ over $F_p$. We have the following field extensions:

$$F_p \subseteq F_p(b_2) \subseteq F_p(b_1) \subseteq F_p(\theta_1).$$

We now choose $g_2(x) = x^m + d_{m-1}x^{m-1} + \ldots + d_0$, a monic integral irreducible polynomial, with mod $p$ reduction $f_2(x)$. We also arrange for the mod $q$ reduction of $g_2(x)$ to be irreducible. This is possible by the Chinese Remainder Theorem. Thus, both ideals
(p), and (q) remain prime in $\mathbb{Q}(\beta_2)$, where

$\beta_2$ is a root of $g_2(x)$.

Next consider the extension of $\mathbb{Q}(\beta_2)$ given by

adjoining a root of $x^2 - (q^2\beta_2 + q + k) = 0$.

Call this extension $\mathbb{Q}(\beta_1)$. The extension

$\mathbb{Q}(\beta_2) \subset \mathbb{Q}(\beta_1)$ may or may not be proper. In any case,

Lemma 7.9 (q) does not ramify in $\mathbb{Q}(\beta_2) \subset \mathbb{Q}(\beta_1)$.

Proof: The different $\mathfrak{P}$ of this extension is the

greatest common divisor of the element differentials,

$(f'(a))$, where $a$ generates $\mathbb{Q}(\beta_1)$ over $\mathbb{Q}(\beta_2)$.

Hence $\mathfrak{P}$ divides $2\sqrt{q^2\beta_2 + q + k}$. If (q) ramifies, $q$ divides $4(q^2\beta_2 + q + k)$, so that $q$ divides $k$.

This is impossible since $(q,k) = 1$.

Finally, we let $E = \mathbb{Q}(\theta)$, where $\theta$ satisfies

$x^2 - 2\beta_1 x + k = 0$ over $\mathbb{Q}(\beta_1)$. Notice that the mod p

reduction of $\theta$ is $\theta_1$. Thus we have an extension of

degree $2n$, $[F_p(\theta_1) : F_p]$, over $Q$.

There is the fixed field of the involution $\theta \rightarrow k\theta^{-1}$
given by $\mathbb{Q}(\beta_1)$. We are adjoining

$\sqrt{4\beta_1^2 - 4k} = 2\sqrt{\beta_1^2 - k}$ to $\mathbb{Q}(\beta_1)$.

By construction, $\beta_1^2 - k = q^2\beta_2 + q$

$= q(q^2\beta_2 + 1)$
This has q-adic valuation 1, i.e. \( (q) \) ramifies in \( \mathbb{Q}(\theta) \) over \( \mathbb{Q} \). However, also by construction, \( (q) \) does not ramify in \( \mathbb{Q}(\alpha_1) \). Hence, some prime lying over \( (q) \) must ramify in \( \mathbb{Q}(\theta) \) over \( \mathbb{Q}(\mathbb{F}_1) \).

Now consider \( \hat{\alpha}(D) : H(\mathbb{Q}(\theta)) \to H(\mathbb{E}/I) \).

Since there are ramified primes in \( \mathbb{Q}(\theta) \) over the fixed field, the cokernel of \( \hat{\alpha}(D) \) is given in terms of ramified primes. In other words, the term \( H(\mathbb{F}_p(\theta_1)) \) in \( H(\mathbb{E}/I) \) is in the image of \( \hat{\alpha}(D) \), modulo ramified primes. Since all the ramified primes have already been shown to be in the image of \( \hat{\alpha} \), so also is \( H(\mathbb{F}_p(\theta_1)) \) in the image of \( \hat{\alpha} \) as desired.

The final construction is to show that \( H(\mathbb{F}_2(\theta_1)) \) is in the image of boundary. Suppose \( \mathbb{F}_2(\theta_1) = \mathbb{F}_2^{2n} \), with fixed field \( \mathbb{F}_2^{2n} \). Suppose \( \theta_1 \) satisfies \( x^2 - a_1 x + k = 0 \) over \( \mathbb{F}_2^{2n} \). Suppose \( a_1 \) satisfies \( f_1(x) \) over \( \mathbb{F}_2 \). Lift each of these polynomials to \( \mathbb{Q} \) to obtain

\[
\mathbb{Q} \subseteq \mathbb{Q}(a) \subseteq \mathbb{Q}(\theta).
\]

Now consider \( \hat{\alpha}(D) : H(\mathbb{Q}(\theta)) \to H(\mathbb{E}/I) \). This time the cokernel will be in terms of ramified primes, or possibly \( C_2 \) if there is no ramification. This does not matter,
since all cokernel elements are already in the image of \( \varphi \) by previous work. So modulo these pieces, \( \varphi(D) \) hits \( H(F_2(\theta_1)) \) as desired.

We have thus shown how to hit with \( \varphi \) a typical Hermitian term \( H(F_p(\theta_1)) \) in \( W(k,\mathbb{Q}/\mathbb{Z}) \). This of course works equally well for \( \varphi : \mathbb{Q} \to A(\mathbb{Q}(\mathbb{Z})) \). We summarize.

**Theorem 7.10** \( \varphi : \mathbb{A}^1(k,\mathbb{Q}) \to W^1(k,\mathbb{Q}/\mathbb{Z}) \) is onto when \( k = \pm 1 \), or \( k \) a positive prime \( k = 2, 3 \) (4) or \( k = 1 \) (8).

**Theorem 7.11** \( \varphi : \mathbb{A}(k,\mathbb{Q}) \to A(k,\mathbb{Q}/\mathbb{Z}) \) is onto.

In the skew case \( W^{-1}(k,\mathbb{Q}/\mathbb{Z}) \), we need a slight modification. The above argument does show that \( \varphi \) is onto all inert primes modulo ramified primes. There is in fact only one Witt piece in \( W^{-1}(k,\mathbb{Q}/\mathbb{Z}) \), namely \( W(F_2) \).

**Lemma 7.12** \( W(F_2) \subseteq W(k,\mathbb{Q}/\mathbb{Z}) \) is not in the image of \( \varphi \).

**Proof:** Consider the commutative diagram of forgetful maps.
\[ \nabla^{-1}(k, \mathcal{O}) \xrightarrow{\delta} W^{-1}(k, \mathcal{O}/\mathcal{Z}) \]

\[ \downarrow f_1 \quad \downarrow f_2 \]

\[ \nabla^{-1}(\mathcal{O}) \xrightarrow{\delta} W^{-1}(\mathcal{O}/\mathcal{Z}) = W(F_2) \]

\( f_1 \) and \( f_2 \) are the maps which forget the degree \( k \) map in the data of a degree \( k \) mapping structure.

\( f_2 \) is the identity: \( W(F_2) \rightarrow W(F_2) \). However \( W^{-1}(\mathcal{O}) = 0 \), so that \( W(F_2) \) is not in the image of \( \delta \).

As a consequence,

\[ \text{Theorem 7.13} \quad \delta: \nabla^{-1}(k, \mathcal{O}) \rightarrow W^{-1}(k, \mathcal{O}/\mathcal{Z}) \], for \( k = \pm 1 \), or \( k = \text{prime has cokernel} \). \( C_2 \) given by the Witt element \( W(F_2) \) in \( W^{-1}(k, \mathcal{O}/\mathcal{Z}) \) which is not in the image of \( \delta \). □

In order to understand the octagon, and apply the boundary sequences above, it is first necessary to analyze the individual maps in the octagon. We do this next, in terms of the \( T_k \) fixed polynomials determining the Hermitian pieces.
Chapter IX A DETAILED ANALYSIS OF THE OCTAGON

We have an exact octagon over a field with typical term \( W^e(k, F) \). We also have analyzed each term

\[
W^e(k, F) \cong \oplus H^e(F(\theta)) \oplus W^e(F(\sqrt{k})) \quad k \not\in F^{**}
\]

\[
\cong \oplus H^e(F(\theta)) \oplus W^e(F) \oplus W^e(F) \quad k \in F^{**}
\]

In this chapter, we analyze the maps in the octagon using this direct sum decomposition. Each of these Hermitian and Witt summands is determined by a \( T_k \) fixed irreducible polynomial. In Section 1, we classify these polynomials, and discuss several cases that may arise.

In Sections 2, 3, 4, 5 we examine the various maps in the octagon. This analysis involves determining the effect of the homomorphisms on rank mod 2, signature, and discriminant. For \( F \) an algebraic number, these invariants determine \( H^e(F) \) by Landherr's Theorem.
1. The involutions

Recall $K(F) = \{\text{monic Polynomials, non-zero constant term, coefficients in} \ F\}$. On $K(F)$, we have several automorphisms defined.

(1) $T_k p(t) = \frac{t^n}{a_0} p(kt^{-1})$ Denote $\overline{p}(t) = T_k(p(t))$

(2) $T_{-k} p(t) = \frac{t^n}{a_0} p(-kt^{-1})$ $p^*(t) = T_{-k}(p(t))$

(3) $T_0 p(t) = (-1)^n p(-t)$ $p^0(t) = T_0(p(t))$

These arise from the corresponding involutions on $F[t, t^{-1}]$ given by, respectively:

(1) $t \rightarrow kt^{-1}$ $\overline{t} = kt^{-1}$

(2) $t \rightarrow -kt^{-1}$ $t^* = -kt^{-1}$

(3) $t \rightarrow -t$ $t^0 = -t$

These involutions, together with the identity, thus determine an action of the Klein 4 group, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ on $F[t, t^{-1}]$.

We should recall the origin of these involutions. Let $[M, B, \ell] \in W(k, F)$, with $(M, B, \ell)$ anisotropic. Let $p(t)$ be the minimal polynomial of $\ell$. As we have seen in II 1.3, we may assume $p(t)$ is a $T_k$ fixed irreducible polynomial. By II 1.7, the ideal $(p(t))$ in $F[t, t^{-1}]$ is then $-$ invariant.

We then consider $F[t, t^{-1}]/(p(t)) \approx F(\theta)$. $F(\theta)$ has
an involution - induced by $\bar{\theta} = k\theta^{-1}$, since $(p(t))$
was - invariant. Under this involution, $[M, B, \ell]$ is
identified with the Hermitian inner product space $[M, B']$
over $F(\theta)$, with $t_{*} \circ B' = B$, where $t_{*} = \text{trace } F(\theta)/F$.
We will describe the effect of the maps in the octagon
on these Hermitian inner product spaces, and the
associated polynomials.

To begin with, we need a criteria to determine
whether $F(\theta) = F(\theta^{2})$, when $F(\theta)$ has an involution induced
by $T_{k}$ for some $k$. The answer is given by considering
the following cases. We will assume that the characteristic
of $F$ is not 2, since $F(\theta) = F(\theta^{2})$ when $F$ has characteristic
2.

Notation: For the remainder of this chapter, we let
$p(t) = \text{irreducible polynomial of } \theta \text{ over } F$
$q(t) = \text{irreducible polynomial of } \theta^{2} \text{ over } F$.

Case 1: $F(\theta) \neq F(\theta^{2})$ when $\overline{p}(t) = p(t)$ and $p^{*}(t) = p(t)$.

Proof: In this case, the ideal $(p(t))$ is both - and
* invariant. Hence $F[t, t^{-1}]/(p(t)) \simeq F(\theta)$ has the induced
involutions: $\bar{\theta} = k\theta^{-1}, \theta^{*} = -k\theta^{-1}, \theta^{o} = (\bar{\theta})^{*} = -\theta$. These
are Galois automorphisms of $F(\theta)$. The fixed field of
$\circ$ is $F(\theta^{2})$. As long as $\circ$ is non-trivial, which happens
provided the characteristic of $F$ is not 2, $F(\theta^{2}) \neq F(\theta)$
by Galois theory. $\Box$
**Remark:** Since $F(\theta^2) \neq F(\theta)$ in this case, we have degree $p(t) = 2$ degree $q(t)$, and consequently $p(t) = q(t^2)$. 

Case 2: $F(\theta) = F(\theta^2)$ when $\bar{p}(t) = p(t)$ and $p^*(t) \neq p(t)$. 

In this case, the $\sigma$ involution is not present. 

Recall our notation; $q(t)$ is the minimal polynomial satisfied by $\theta^2$ over $F$. $q(t^2)$ has $\theta$ and $-\theta$ as roots. Note that $q(t^2)$ has only even degree terms. The hypothesis $\bar{p}(t) = p(t)$ and $p^*(t) \neq p(t)$ implies $p(t)$ has odd degree terms. Thus, $q(t^2)$ is not irreducible. Hence, we may write

$$q(t^2) = p(t)p(-t)w(t).$$

This follows since $p(t) \neq p(-t)$, else $p^*(t) = (\bar{p}(-t)) = \bar{p}(t) = p(t)$, contradiction. Further, degree $q(t^2) = 2$ degree $q(t) \leq 2$ degree $p(t)$. However, by (a), degree $q(t^2) \geq 2$ degree $p(t)$. Thus, degree $q(t^2) = 2$ degree $p(t)$, and $w(t) = 1$, so $q(t^2) = p(t)p(-t)$. Hence, degree $q(t) = degree p(t)$, and $F(\theta) = F(\theta^2)$. 

Case 3: $F(\theta) = F(\theta^2)$ when $p^*(t) = p(t)$ and $\bar{p}(t) \neq p(t)$. 

Proof: Exactly as above.
2. The map $I_\epsilon : W^\epsilon(k^2,F) \to W^\epsilon(-k,F)$

We begin with the induction map $I_\epsilon$ defined by

$I_\epsilon : [M,B,\ell] \to [M \oplus M,B \oplus -kB,\tilde{\ell}]$ where $\tilde{\ell}(x,y) = (\ell y, x)$.

Consider $[M,B'] = [M,[,]] \in H^\epsilon(F(\sigma))$, where $H^\epsilon(F(\sigma))$ embeds into $W^\epsilon(k^2,F)$ via $t \ast$. To keep our notation consistent with Section 1, we write $\sigma = \theta^2$, and $H^\epsilon(F(\sigma)) = H^\epsilon(F(\sigma^2))$. Here we have $F[t,t^{-1}]/(q(t)) = F(\sigma^2)$.

**Case 1: $F(\sigma) \neq F(\sigma^2)$**

$F(\sigma^2)$ has involution given from $T_{k^2} \theta^2 \mapsto k^2 \theta^{-2}$. This extends in two ways to $F(\sigma)$, namely $\theta \mapsto k \theta^{-1} = \bar{\theta}$, and $\theta \mapsto -k \theta^{-1} = \sigma^x$. In fact, this is the way this case is recognized, namely both the $-$ and $*$ involutions are present on $F(\sigma)$.

We identify $[M,B']$ with $[M,B,\ell] \in W^\epsilon(k^2,F)$.

$B(x,y) = t \ast \circ B'(x,y)$, and $\ell$ is multiplication by $\theta^2$.

Here $B' = [,,]$, so that $B = t \ast \circ [,,]$. Mapping over with $I_\epsilon$ we obtain $[M \oplus M,B \oplus -kB,\tilde{\ell}]$.

We wish now to identify the (Hermitian) form we obtain in $W^\epsilon(-k,F)$. Recall that $\tilde{\ell}^2(x,y) = (\ell x, \ell y)$.

Hence, $\tilde{\ell}$ acts as a square root of $\ell$. Thus, we wish to identify $[M \oplus M,B \oplus -kB,\tilde{\ell}]$ with an Hermitian form over $F(\sqrt{\sigma}) = F(\sigma)$. This is a question of how to define an $F(\sigma)$-vector space structure on $M \oplus M$ compatible with the $F(\sigma^2)$-vector space structure. The following can best
be understood by considering $M \cong F(\theta^2)$, the one dimensional case. The identifications work equally well for $M$ arbitrary.

We now write $F(\theta) \cong F(\theta^2) \oplus F(\theta^2)$. We are naturally thinking of $\theta = \sqrt{\sigma}$ as the ordered pair $(0,1)$. Addition of ordered pairs is componentwise. Multiplication of pairs is given by $(a,b) \cdot (c,d) = (ac + bd\theta^2, ad + bc)$.

If the dimension $[M:F(\theta^2)] = n$, we now view $M \oplus M$ as a vector space over $F(\theta)$, with dimension $[M \oplus M,F(\theta)] = n$ also. A basis for $(M \oplus M,F(\theta))$ is given by $\{(v_i,0)\}$ where $\{v_i\}$ is a basis of $M/F(\theta^2)$. Scalar multiplication by $F(\theta) = F(\theta^2) \oplus F(\theta^2)$ is given by $(a,b) \cdot (v_i,0) = (av_i,bv_i)$ on basis elements by following the above. These operations extend linearly to make $M \oplus M$ into a vector space over $F(\theta)$.

We obtain the form $\langle , \rangle$ in $H^\epsilon(F(\theta))$ given by:

$$\langle (x,y),(z,w) \rangle = 1/2([x,z] + k[y,w] - k\theta^{-1}[x,w] + \theta[y,z])$$

where $[ , ] = B': M \times M \rightarrow F(\theta^2)$.

One easily checks that this respects the vector space operations given; by the identification $(x,y) \approx x + y\theta$, with involution on $F(\theta)$ given by $\theta^* = -k\theta^{-1}$. We compute $\text{tr}_{\theta^*} \circ \langle , \rangle$ by using

$$\text{tr}_{F(\theta)/F} = \text{tr}_{F(\theta^2)/F} \circ \text{tr}_{F(\theta)/F(\theta^2)}.$$ 

Here $\text{tr}_{\theta^*}$ is the map induced by the appropriate trace, denoted $\text{tr}$. 
Note, that \( \text{tr}_{F(\theta)}/F(\theta^2) (r) = 2r \) for \( r \in F(\theta^2) \) and
\[ \text{tr}_{F(\theta)}/F(\theta^2)(\theta) = \text{tr} \theta^{-1} = 0. \]
It follows that \( \text{tr} \circ \langle , , \rangle = B \otimes -kB \) as desired.

We see in this case, that \( I_\epsilon \) is identified with a map \( I_\epsilon : H^e(F(\theta^2)) \to H^e(F(\theta)) \).

1. \( I_\epsilon \) preserves rank.

This is clear, since \( [M:F(\theta^2)] = [M \otimes M,F(\theta)] \), ie.
\[ \dim_{F(\theta^2)}[M] = \dim_{F(\theta)}(M \otimes M). \]

2. Signatures.

If \( k<0 \), \( W(k,F) = \ker I_\epsilon \) is all torsion. Hence, in order that \( [M,[,]] \in \ker I_\epsilon \) there must be no signatures in \( H^e(F(\theta^2)) \).

If \( k>0 \), \( I_\epsilon [M,[,]] \in H^e(F(\theta)) \) is in \( W(-k,F) \); again this group is all torsion. So there is no signature in the image in this case.

3. Discriminant.

Here, we must be careful because the discriminant of \( [M,[,]] \) is read in \( F(\theta^2 + k^2\theta^{-2})/N_{F(\theta^2)/F(\theta^2 + k^2\theta^{-2})} \)
where \( N_{F(\theta^2)/F(\theta^2 + k^2\theta^{-2})} \) denotes elements in \( F(\theta^2 + k^2\theta^{-2}) \)
which are the norms of elements from \( F(\theta^2) \), whereas the discriminant of the image, \( [M \otimes M,\langle ,\rangle] \) is read
\( F(\theta - k\theta^{-1})/N_{F(\theta)/F(\theta - k\theta^{-1})} \). These may be different groups as the example which follows will show.

To summarize, when \( F(\theta) \) is an algebraic number field,
Theorem 2.1 Let $[M, [, ]] \in H^e(F(\theta^2))$, and assume $F(\theta^2) \neq F(\theta)$. Then $[M, [, ]] \in \ker I$ if and only if

(a) $M$ has even rank.

(b) $M$ has signature 0 if $k < 0$.

(c) The discriminant of $M$ when read in $F(\theta - k\theta^{-1})/N_{F(\theta)/F(\theta - k\theta^{-1})}$ must be trivial.

Proof: (a) and (b) have already been discussed. To verify (c), we need to calculate the discriminant of $<[ , ]>$. If $\dim M = n$, this is exactly given by $(1/2)^n \text{dis}([ , ])$, by the formula for $< , >$ applied to a 1-dimensional form, and induction. Hence, (a), (b), (c) follow by Landherr's Theorem. □

An Example.

Let $\sigma = \theta^2 = \sqrt{-1} = i \quad \theta = \sqrt{i}$

We now have the extensions:

\[
\begin{array}{c}
Q(\sqrt{i}) \\
Q(i) \searrow \downarrow \downarrow \\
\searrow \downarrow \\
Q(\sqrt{i} + 1/\sqrt{i}) \\
Q
\end{array}
\]

This example is for $k = -1$.

The involution on $Q(i)$ is $i \rightarrow k^2/i = 1/i = -i$

The involution on $Q(\sqrt{i})$ is $\sqrt{i} \rightarrow 1/\sqrt{i} = -i\sqrt{i}$

By elementary number theory, 3 is not a norm in $Q(i)/Q$. 
since 3 is not the sum of 2 squares.

However, consider \((i - 1) + \sqrt{i}\) in \(\mathbb{Q}(\sqrt{i})\). We compute its norm in \(\mathbb{Q}(\sqrt{i})/\mathbb{Q}(\sqrt{i} + 1/\sqrt{i})\),
\[
((i - 1) + \sqrt{i})((-i - 1) + 1/\sqrt{i}) = 3.
\]
Thus 3 becomes a norm. This leads us to consider the following example.

Let \(k = -1\), and let \(M\) be a 2-dimensional vector space over \(\mathbb{Q}(i)\), with basis \(\vec{e}_1, \vec{e}_2\). With respect to this basis, consider the Hermitian form over \(\mathbb{Q}(i)\) given by
\[
[\ , ] = \begin{pmatrix}
1 & 0 \\
0 & -3
\end{pmatrix}
\]
This 2-dimensional form has signature 0, and discriminant +3, which is not a norm.

If \(\vec{e}_1, \vec{e}_2\) is a basis for \(M\) over \(\mathbb{Q}(i)\), \(\vec{e}_1, i\vec{e}_1, \vec{e}_2, i\vec{e}_2\) is a basis for \(M\) over \(\mathbb{Q}\). We thus identify the Hermitian form \([M,[,]]\) with the Witt class in \(W(+1,\mathbb{Q})\) given by:
\[
[M, t_\ast \circ [,], \ell] \text{ where } \ell \vec{x} = i\vec{x},
\]
so that with respect to the basis given for \(M/\mathbb{Q}, \ell\) has matrix
\[
\begin{pmatrix}
\vec{e}_1 & i\vec{e}_1 & \vec{e}_2 & i\vec{e}_2 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]
We write \(B = t_\ast \circ [,]\).

Next, apply \(I_\varepsilon\) to \([M, B, \ell]\), to obtain \([M \oplus M, B \oplus -kB, \ell]\). This in turn is identified with an Hermitian form over \(\mathbb{Q}(\sqrt{i})\).
With respect to the basis $(e_1, 0), (e_2, 0)$ for $M \oplus M = V$ over $Q(\sqrt{i})$, we obtain the form \[
abla = \begin{pmatrix} 1/2 & 0 \\ 0 & -3/2 \end{pmatrix}
\]
Since this form has discriminant $3/4$, which is a norm, it follows that \([V, <, >] = 0\) in $H(Q(\sqrt{i}))$. In this manner, we see how the norm groups $Q(\theta^2 + k^2 \theta^{-2})/N_{Q(\theta^2)/Q(\theta^2 + k^2 \theta^2)}$ and $Q(\theta - k \theta^{-1})/N_{Q(\theta)/Q(\theta - k \theta^{-1})}$ give rise to kernel elements for $I$.

**Case II:** $F(\theta) = F(\theta^2)$

To begin with, $F(\theta^2)$ has the involution given from $T_k^2$, namely $\theta^2 \sim k^2 \theta^{-2}$. Call this involution $\sim$, so $\theta^2 = k^2 \theta^{-2}$. Hence $\theta^2 \theta^{-2} = k^2$, and $(\theta \theta)^2 = k^2$. It follows that $\theta \theta = \pm k$. This gives us two subcases.

(a) $\theta = + k \theta^{-1}$
(b) $\theta = - k \theta^{-1}$

In either case, we begin with $[M, [\cdot, \cdot]] \in H(F(\theta^2))$. We embed this into $W(k^2, F)$ via $\tau^*$ to obtain $[M, B, \ell]$, where $B = \tau^* \circ [\cdot, \cdot]$ and $\ell(x) = \theta^2 x$. Applying $I_\epsilon$ we obtain $[M \oplus M, B \oplus -k B, \tilde{\ell}]$ where $\tilde{\ell}(x,y) = (\theta^2 y, x)$.

(a) In case (a), consider $N \subset M \oplus M$ given by $N = \{(\theta x, x) : x \in M\}$. $N$ is clearly $\tilde{\ell}$ invariant, with rank $N = 1/2 \text{rank}(M \oplus M)$. Further,

\[
(B \oplus -k B)((\theta x, x), (\theta y, y)) = B(\theta x, \theta y) - k B(x, y)
\]

\[
= \theta \theta B(x, y) - k B(x, y)
\]

\[
= k B(x, y) - k B(x, y) = 0
\]
Thus \( N \subset N^1 \), from which it follows that \( I^e_{\varepsilon}[M, B, \ell] = 0 \). This completes case (a).

(b) In case (b), we consider the rank 1 case, \( M = F(\theta^2) \), the general case follows by diagonalizing.

Let \( \hat{e}_1 = (1,0) \) and \( \hat{e}_2 = (0,1) \) be a basis for \( M \oplus M \) over \( F(\theta^2) \).

\[
\begin{align*}
\hat{\gamma}(\hat{e}_1) &= 0 \cdot \hat{e}_1 + \hat{e}_2 \\
\hat{\gamma}(\hat{e}_2) &= \theta^2 \hat{e}_1 + 0 \cdot \hat{e}_2
\end{align*}
\]

Thus viewed, the matrix of \( \gamma \) relative to \( \hat{e}_1, \hat{e}_2 \) is

\[
\begin{pmatrix}
\hat{e}_1 & \hat{e}_2 \\
0 & \theta^2
\end{pmatrix}
\]

The characteristic polynomial of \( \gamma \) is

\[
\det\begin{pmatrix}
x & -\theta^2 \\
-1 & x
\end{pmatrix} = x^2 - \theta^2 = (x + \theta)(x - \theta).
\]

The eigenvalues are thus \( \theta, -\theta \). We next compute the corresponding eigenvectors.

\[
\begin{pmatrix}
0 & \theta^2 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= 
\begin{pmatrix}
\theta x_1 \\
\theta x_2
\end{pmatrix}
\]

\[
\theta^2 x_2 - \theta x_1 = 0
\]

\[
x_1 - \theta x_2 = 0
\]

Solving, the eigenspace corresponding to eigenvalue \( \theta \) is generated by \( \hat{f}_1 = (\theta,1) \), namely \( \{ t\hat{f}_1 : t \in F(\theta^2) \} \).

Similarly, the eigenspace corresponding to \( -\theta \) is \( \{ t\hat{f}_2 = t(-\theta,1) : t \in F(\theta^2) \} \).

<table>
<thead>
<tr>
<th>Eigenvalue</th>
<th>Eigenvectors</th>
<th>dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta )</td>
<td>( t(\theta,1) )</td>
<td>( t \neq 0 )</td>
</tr>
<tr>
<td>( -\theta )</td>
<td>( t(-\theta,1) )</td>
<td>( t \neq 0 )</td>
</tr>
</tbody>
</table>
Expressing each of $f_i$ in terms of the $e_i$, we have

\[ f_1 = \theta e_1 + e_2 \]
\[ f_2 = -\theta e_1 + e_2 \]

This leads to the change of basis matrix

\[
\begin{pmatrix}
\theta & -\theta \\
1 & 1
\end{pmatrix} = L,
\]

so $L^{-1}\lambda_{\text{orig}}L = \lambda_{\text{new}}$, where $\lambda_{\text{new}}$ is diagonalized with respect to the basis $f_1, f_2$. $\lambda_{\text{orig}}$ denotes $\lambda$ with respect to the basis $e_1, e_2$.

We obtain the diagram:

\[
\begin{array}{ccc}
M \oplus M & \xrightarrow{\lambda} & M \oplus M \\
\downarrow \lambda_{\text{new}} & & \downarrow \lambda \\
M \oplus M & \xrightarrow{L} & M \oplus M \\
\end{array}
\]

$L: (x, y) \mapsto (\theta(x - y), x + y)$. The question arises:

what inner product on $M \oplus M$ makes $L$ into an isometry.

$L: (M \oplus M, B, \lambda_{\text{new}}) \mapsto (M \oplus M, B \oplus -kB, \lambda_{\text{orig}})$. We compute,

\[
b((x, y), (z, w)) = (B \oplus -kB)(L(x, y), L(z, w))
\]

\[
= (B \oplus -kB)((\theta(x - y), x + y), (\theta(z - w), z + w))
\]

\[
= B(\theta(x - y), \theta(z - w)) + -kB(x + y, z + w)
\]

\[
= \theta B(x - y, z - w) - kB(x + y, z + w)
\]

\[
= -kB(x - y, z - w) - kB(x + y, z + w)
\]

\[
= -k[B(x, z) + B(y, w) - B(y, z) - B(x, w)]
\]

\[
+ B(x, z) + B(y, w) + B(y, z) + B(x, w)
\]

\[
= -2k[B(x, z) + B(y, w)]
\].
Thus under the isometry $L$, we may view
\[ (M \oplus M, B \oplus -kB, \mathcal{I}) \cong (M \oplus M, b, \mathcal{I}_{\text{new}}) \], where
\[ b((x, y), (z, w)) = -2k(B(x, z) + B(y, w)) \]
\[ \mathcal{I}(x, y) = (\partial x, -\partial y) \]

Thus, the image of $I_\epsilon$ is in two distinct pieces, $H(F(\theta)) \oplus H(F(-\theta))$ in this case. $I_\epsilon$ is given by:

\[ [M, [,]] \xrightarrow{I_\epsilon} [M, b_1] \oplus [M, b_2], \]
where $b_1(x, y) = b_2(x, y) = -2k[x, y]$. It follows that $I_\epsilon [M, [,]]$ is non-zero if $[M, [,]] \neq 0$. We summarize with:

**Theorem 2.2** Let $[M, [,]] \in H^\epsilon F(\theta^2)$, and assume $F(\theta^2) = F(\theta)$. There is an induced involution $\sim$ on $F(\theta^2)$ given by $\mathcal{I}^2 = k\mathcal{I}^{-2}$. This gives rise to two cases.

(a) $\mathcal{I} = k\mathcal{I}^{-1}$

(b) $\mathcal{I} = -k\mathcal{I}^{-1}$

In case (a), $[M, [,]] \in \ker I_\epsilon$

In case (b), $I_\epsilon : H^\epsilon(F(\theta^2)) \to H^\epsilon(F(\theta)) \oplus H^\epsilon(F(-\theta))$

is 1-1. □

Finally, we should observe that this analysis works equally well for $F$ a finite field. The only difference is that $H^\epsilon(F(\theta))$ is determined by rank mod 2 only.

In particular, when $F$ has characteristic 2, and $k$ and 2 are relatively prime, $W^\epsilon(k, F_2) \cong \oplus H^\epsilon(k, F_2(\theta)) \oplus W^\epsilon(F_2)$.

We examine the map $I_\epsilon$. For all Hermitian summands $H^\epsilon(k, F_2(\epsilon))$, we are in case (a) of Theorem 2.2. Thus
$I_\varepsilon(H^\varepsilon(k, F_2(\theta))) = 0$. We must also check what happens to $W^\varepsilon(F_2)$. So consider $[M, B, \ell]$, where $\ell x = x$. Applying $I_\varepsilon$, we obtain $[M \oplus M, B \oplus -kB, \ell']$. A metabolizer is given by $N = \{(x, x) : x \in M\}$. Thus, when $F$ has characteristic 2, $I_\varepsilon$ is identically 0.
3. The map $d_\epsilon : \mathcal{W}^\epsilon(-k,F) \to A(F)$

Recall that $d_\epsilon$ is defined by: $[M,B,\ell] \to [M,\overline{B}]$
where $\overline{B}(x,y) = k^{-1}B(x,\ell y)$. From IV 2.4, the symmetry operator for $\overline{B}$, satisfying $\overline{B}(x,y) = \overline{B}(y,sx)$ is $s = -\epsilon k\ell^{-2}$.

We consider $[M,B'] = [M,[]] \in \mathcal{H}^\epsilon(F(\Theta))$, where
$F(\Theta) = F[t,t^{-1}]/(p(t))$ has induced involution $\Theta^* = -k\Theta^{-1}$.
This embeds into $\mathcal{W}^\epsilon(-k,F)$ via $t^*$ and we identify $[M,B']$
with $[M,B,\ell] \in \mathcal{W}^\epsilon(-k,F)$ where $B = t^* \circ B'$, $\Delta x = \Theta x$.

Apply $d_\epsilon$ and obtain $[M,\overline{B}]$. We must now identify the Hermitian form we obtain.

For $A(F)$ this is done by using a scaled trace $t_1$

$\text{III 2.5. } t_1$ depends on the scaling factor $u$ chosen

where $uu^{-1} = s = -\epsilon k\Theta^{-2}$. $t_1 : F(\Theta^2) \to F$. Recall
that $u$ may be chosen as $u = s/\ell + s) = -\epsilon k\Theta^{-2}/\ell + -\epsilon k\Theta^{-2})$
$= -\epsilon k/\ell^2 - \epsilon k$ and $t_1(x) = t(xu^{-1})$ where $t$ is the usual trace.
Since $s = -\epsilon k\ell^{-2} = -\epsilon k\Theta^{-2}$, we obtain a Hermitian form with values in $\mathcal{H}^\epsilon(F(\Theta^2))$.

**Case 1.** $F(\Theta) \neq F(\Theta^2)$

In this case, $[M,\overline{B}]$ may be identified with the Hermitian form in $\mathcal{H}^\epsilon(F(-\epsilon k\Theta^{-2})) = \mathcal{H}^\epsilon(F(\ell^2))$ given by:

$\langle x,y \rangle = \text{trace } F(\Theta)/F(\Theta^2) k^{-1}[x,u\Theta y] [ , ] = B'$

We then have $t_1 \langle x,y \rangle = (t \circ h)(k^{-1}[x,u\Theta y])$

$= t \circ k^{-1}[x,\Theta y]$

$= k^{-1}B(x,\Theta y)$

$= B(x,y)$ as desired.

($h$ is defined in III 2.6)
We now examine the Witt invariants. We obtain a form in $H^c(F(\theta^2))$. In this case, $F(\theta) \neq F(\theta^2)$, so that the rank of $M$ as a vector space over $F(\theta^2)$ is twice the rank of $M$ over $F(\theta)$.

We describe the method for obtaining the other invariants by examining the one-dimensional case. So assume $M = F(\theta)$, and $[1,1] = d \in F(\theta - k\theta^{-1})$.

Then $\hat{e}_1 = 1$, $\hat{e}_2 = \theta$ is a basis for $M$ over $F(\theta^2)$. With respect to $\hat{e}_1$, $\hat{e}_2$ we examine the matrix of the Hermitian form $< , >$. We assume $u = -\epsilon k / \theta^2 - \epsilon k \cdot \bar{u}$ is given by the involution $\theta^* = -k\theta^{-1}$; the $*$ involution extends the $-$ involution. Thus the matrix of $< , >$ is:

$$
\begin{pmatrix}
1 & \theta \\
\theta^* & \text{tr}(dk^{-1}u\theta^*) & \text{tr}(dk^{-1}u\theta^*^2)
\end{pmatrix}
$$

Using this, one may determine the signature and discriminant of $< , >$.

Case 2. $F(\theta) = F(\theta^2)$

In this case, $[M, B'] = [M, [\cdot, \cdot]]$ has the same rank as $[M, <, >]$. Again we examine the 1-dimensional case, $M = F(\theta)$, where $[1,1] = d$. Then $< , >$ is given by:

$$
<1,1> = dk^{-1}u\theta^* = dk^{-1}[-\epsilon k \theta^2 \lambda^2 - \epsilon k \theta^2]( -k\theta^{-1})
= d[+\epsilon k \theta \lambda^2 - \epsilon k \theta^2].
$$
Thus, the value of the discriminant depends on this factor 
\( e k \theta / k^2 - e k \theta^2 \).

When \( k > 0 \), \( W^\epsilon(-k, F) \) is all torsion, and there are no signatures. When \( k < 0 \), we must check that the resulting form \([M, \langle, \rangle]\) has 0 signature in order that \([M, [\cdot, \cdot]]\) be in the kernel of \( d_\epsilon \).

Again, we examine \( W^\epsilon(-k, F_2) \). We must be in case (2). Rank is the only invariant, so that \( d_\epsilon \) is 1-1. It is also clear that the Hermitian forms, \( H^\epsilon(F_2(\theta)) \), in \( W^\epsilon(-k, F_2) \) are mapped under \( d_\epsilon \) to Hermitian forms. The form \([M, B, \ell]\), with \( M = F_2, B = \langle 1 \rangle, \ell = \text{identity} \), corresponding to \( W^\epsilon(F_2) \), likewise maps under \( d_\epsilon \) to \( W(F_2) \). These remarks are needed for the computation of the exact octagon over \( Z \) to be made later.
4. The map $S_\varepsilon : \mathbb{W}(k,F) \to \mathbb{W}(k^2,F)$

Recall that $S_\varepsilon$ is defined by $[M,B,\ell] \to [M,B,\ell^2]$.

Let $[M,B'] = [M,\{\cdot\}] \in \mathbb{H}(F(\theta))$, where $F(\theta) = F[t,t^{-1}]/(p(t))$ has induced involution $\overline{\theta} = k\theta^{-1}$. Embed $[M,B']$ into $\mathbb{W}(k,F)$ via $t^*$ and identify $[M,B']$ with $[M,B,\ell]$, where $B = t^* \circ B'$, $\ell x = \ell x$.

We apply $S_\varepsilon$, and obtain $[M,B,\ell^2]$. We wish to identify the Hermitian form we obtain.

**Case 1.** $F(\theta) \neq F(\theta^2)$

We clearly obtain the Hermitian form in $\mathbb{H}\varepsilon(F(\theta^2))$ given by $[M,B_1]$, where $B_1 = \operatorname{tr}_{F(\theta)/F(\theta^2)} B'$. The rank of $M$ over $F(\theta^2)$ is twice the rank of $M$ over $F(\theta)$.

In order to examine the other invariants, consider the 1-dimensional case, $M = F(\theta)$. A basis for $M$ over $F(\theta)$ is $\hat{e}_1 = 1$, $\hat{e}_2 = \theta$. Suppose $B' = \left[ \begin{array}{cc} \ell & 0 \\ 0 & \ell^{-1} \end{array} \right]$. Then with respect to the basis $1$, $\theta$ $B_1$ has matrix:

\[
\begin{pmatrix}
1 & \theta \\
\theta & \operatorname{tr}(\theta \delta) & \operatorname{tr}(\delta)
\end{pmatrix}
\]

This is with $\operatorname{tr}$ denoting $\operatorname{tr}_{F(\theta)/F(\theta^2)}$. Again, this matrix enables one to compute the signature and discriminant invariants.
Case 2. \( F(\ell) = F(\ell^2) \)

In this case \( F(\ell) \) has involution \( \ell \to \bar{\ell} = k\ell^{-1} \), so that \( \ell^2 \to \bar{\ell}^2 = k^2\ell^{-2} \).

It follows that \( S \in [M, B, \ell] \) may be identified with the Hermitian form in \( H^\ell(F(\ell^2)) \) given by \([M, B']\). In this case \( S \in \) is then clearly 1-1.

We remark that when the characteristic of \( F \) is 2, \( S \in \) is 1-1 by case (2). In particular, \( S \in : W(F_2) \cong W(F_2) \); where the non-trivial form \( \langle 1 \rangle \) in \( W(F_2) \) is identified with the form \([M, B, \ell]\) in \( W(k, F_2) \) given by : \( M = F_2 \), \( B = \langle 1 \rangle \), \( \ell = \) identity.
5. The map $m_\varepsilon: A(F) \to W^\varepsilon(k,F)$

$m_\varepsilon$ is defined by: $[M,B] \to [M \oplus M, B_\varepsilon, \ell_\varepsilon]$, where

$B_\varepsilon((x,y),(z,w)) = B(x,w) + \varepsilon B(z,y)$

$\ell_\varepsilon(x,y) = (\varepsilon k s^{-1} y, x)$.

Let $[M,B'] = [M, [\cdot, \cdot]] \in H^e(F(\theta))$, where $F(\theta) = F[t, t^{-1}]/(p(t))$ has the involution $-\cdot$ induced by $\theta \to \bar{\theta} = \theta^{-1}$. We identify $[M,B']$ with $[M,B] \in A(F)$ using a scaled trace, $t_1$. Here the symmetry operator $s$ acts as $\theta$, and $t_1(x) = t(x u^{-1})$, where $u^{-1} = \theta$.

As observed before III 2.6, we may choose $u = \theta \cdot [1 + \theta]$, so that $u^{-1} = 1 + \theta$. Then $B = t_1 \circ B'$.

The analysis of $m_\varepsilon$ is then similar to $I_\varepsilon$. The image of $[M, [\cdot, \cdot]]$ under $I_\varepsilon$ can be viewed in $H^e(F(\sqrt{\varepsilon k \theta^{-1}}))$, this is because $k_\varepsilon^2 = \varepsilon k \theta^{-1}$.

There are two cases.

Case 1: $F(\theta) \neq F(\sqrt{\varepsilon k \theta^{-1}})$

Case 2: $F(\theta) = F(\sqrt{\varepsilon k \theta^{-1}})$

Note: Let $\alpha = \sqrt{\varepsilon k \theta^{-1}}$. The involution on $F(\alpha)$ is then $\alpha \to k \alpha^{-1} = \bar{\alpha}$. So $(\varepsilon k \theta^{-1}) \to k^2 (\varepsilon k \theta^{-1})^{-1}$, and $\varepsilon k \theta^{-1} \to \varepsilon k \theta$, $\theta \to \theta^{-1}$. In other words, the involution on $F(\alpha)$ extends the $-\cdot$ involution on $F(\theta)$.

Case 1: $F(\theta) \neq F(\sqrt{\varepsilon k \theta^{-1}})$

In this case, the dimension $M \oplus M$ over $F(\sqrt{\varepsilon k \theta^{-1}})$ is the same as the dimension of $M$ over $F(\theta)$. As observed
previously, we can make $M \oplus M$ into a vector space over $F(\sqrt{e\theta^{-1}}) = F(\alpha)$. The idea again is to view $F(\alpha) = F(\theta) \oplus F(\theta)$. We view $\alpha$ as the ordered pair $(0,1)$.

Multiplication of ordered pairs is given by identifying

\[(a, b) \rightarrow a + b\alpha\]
\[(c, d) \rightarrow c + d\alpha\]
\[(a, b) \cdot (c, d) \rightarrow ac + bd(\alpha^2) + (ad + bc)\alpha\]

Thus, $m_{\epsilon}: H(F(\theta)) \rightarrow H(F(\alpha))$ in this case. If $M$ has basis {$v_i$} over $F(\theta)$, $M \oplus M$ has basis {$v_i, 0$} over $F(\alpha)$.

Consider the form $\langle , \rangle: M \oplus M \rightarrow F(\alpha)$ given by

$$\langle (x, y), (z, w) \rangle = (1/2k)u^{-1}\alpha([x, z] + (k/\alpha)[x, w] + \alpha[y, z] + k[y, w]).$$

Here $\alpha^2 = e\theta^{-1}$ and $[ , ]$ is the Hermitian form we began with.

We must check that $\langle , \rangle$ is $\epsilon$ Hermitian. There are the identities:

$$\overline{u^{-1}} = 1 + \theta$$
$$\overline{u^{-1}}\alpha = (1 + \theta)\sqrt{e\theta^{-1}}$$
$$u^{-1}\alpha = (1 + \theta/\theta)k\sqrt{e\theta^{-1}/(e\theta^{-1})} = \epsilon\overline{u^{-1}}\alpha.$$

Hence:

$$\langle (z, w), (x, y) \rangle = (1/2k)u^{-1}\alpha([z, x] + k\alpha^{-1}[z, y] + a[w, x] + k[w, y]).$$
\[= (\epsilon/2k)\bar{u}^{-1}\alpha([x,z] + (k/\alpha)[x,w] +
\alpha[y,z] + k[y,w])
= \epsilon\langle(x,y), (z,w)\rangle.\]

- denotes the involution. \([a,b] = [b,a]\) since \([ , ]\)
is Hermitian.

Next we compute \(\text{tr}_* \circ \langle , \rangle\), where \(\text{tr}_*\) is \(\text{trace}_F(a)/F\).

\[
\text{tr}_F(a)/F = \text{tr}_F(\theta)/F \circ \text{tr}_F(a)/F(\theta).
\]

\[
\text{tr}_F(a)/F(\theta) \langle(x,y), (z,w)\rangle =
(1/k)\bar{u}^{-1}k[x,w] + (1/k)\bar{u}^{-1}(\epsilon k\theta^{-1})[y,z].
\]

Hence,

\[
\text{tr}_F(a)/F \langle(x,y), (z,w)\rangle = \text{tr}_F(\theta)/F(\bar{u}^{-1}[x,w] +
\bar{u}^{-1}\epsilon\theta^{-1}[y,z])
= t_1([x,w] + \epsilon[\theta^{-1}y, z])
= B(x,w) + \epsilon B(\theta^{-1}y, z)
= B(x,w) + \epsilon B(z, y).
\]

Hence \(\text{tr}_* \circ \langle , \rangle = B_{\epsilon}\) as desired.

We have thus identified the Hermitian form we obtain in the image of \(m_{\epsilon}\) in this case. Rank mod 2 is clearly preserved, and we read the discriminant and signatures from the extension \(F(a)/F(a + ka^{-1})\) in order to determine if \([M, [, ]]\) is in the kernel of \(m_{\epsilon}\).

**Case 2:** \(F(\theta) = F(\sqrt{\epsilon k}\theta^{-1})\).

\(F(\theta)\) has the involution \(\theta \rightarrow \bar{\theta} = \theta^{-1}\). Under this involution, \((\bar{\alpha}^2) = \epsilon k\theta\). Also \(\alpha^2 = \epsilon k\theta^{-1}\). Thus,

\[
(\alpha^2)(\bar{\alpha}^2) = (\alpha\bar{\alpha})^2 = (\epsilon k)^2 = k^2.
\]
Hence, $\alpha a = \pm k$. This gives two cases:

(a) $\alpha = -k\alpha^{-1}$

(b) $\alpha = k\alpha^{-1}$

Case (a) $\alpha = -k\alpha^{-1}$

Let $N = \{(\alpha x, x) : x \in M\}$. $N$ is clearly $\ell_\epsilon$-invariant, since $\ell_\epsilon(x, y) = (\alpha^2 y, x)$. Further, $N$ is self-annihilating since:

$B_\epsilon((\alpha x, x), (\alpha y, y)) = B(\alpha x, x) + \epsilon B(\alpha y, x)$

$= B(\alpha x, y) + B(\epsilon x, \alpha \theta y)$

$= B(\alpha x, y) + B(\epsilon \alpha \theta x, y)$. However, $\alpha^2 = \epsilon k\alpha^{-1}$, so $\alpha = (\epsilon k \theta^{-1})(\alpha^{-1})$. $\epsilon \alpha \theta = \epsilon (-k\alpha^{-1})(\theta^{-1}) = -\alpha$. Thus the above equals 0 and $N$ is a metabolizer for $[M \oplus M, B_\epsilon, \ell_\epsilon]$. Thus $[M, [\cdot, \cdot]]$ is in the kernel of $m_\epsilon$ in this case.

Case (b) $\alpha = k\alpha^{-1}$

As with $I_\epsilon$, we consider the 1-dimensional case, $M = F(\theta)$. Let $\hat{e}_1 = (1, 0), \hat{e}_2 = (0, 1)$ be a basis for $M \oplus M$ over $F(\theta) = F(\alpha)$. $\ell_\epsilon(\hat{e}_1) = 0 \cdot \hat{e}_1 + 1 \cdot \hat{e}_2$ and $\ell_\epsilon(\hat{e}_2) = \alpha^2 \hat{e}_1 + \hat{e}_2$, so that $\ell_\epsilon$ has matrix

$$
\begin{pmatrix}
0 & \alpha^2 \\
1 & 0
\end{pmatrix}
$$

with respect to $\hat{e}_1, \hat{e}_2$.

As with $I_\epsilon$, we now diagonalize this matrix.
We obtain the diagram:

\[
\begin{array}{c}
M \oplus M & \xrightarrow{L} & M \oplus M \\
\downarrow (\alpha & 0) & \downarrow (0 & \alpha^2) \\
M \oplus M & \xrightarrow{\alpha L} & M \oplus M
\end{array}
\]

where \( L = \begin{pmatrix} \alpha & -\alpha \\ 1 & 1 \end{pmatrix} \) is the change of basis matrix.

We compute:

\[
b((x,y), (z,w)) = B(\varepsilon(L(x,y), L(z,w))
\]

\[
= B(\varepsilon(\alpha(x - y), x + y), (\alpha(z - w), z + w))
\]

\[
= B(\alpha(x - y), (z + w)) + \varepsilon B(\alpha(z - w), (x + y))
\]

\[
= B(\alpha x - \alpha y, z + w) + \varepsilon B(x + y, \theta(\alpha)(z - w))
\]

\[
= B(\alpha x, z) + B(\alpha x, w) - B(\alpha y, z) - B(\alpha y, w)
\]

\[
+ \varepsilon B(x, \alpha \theta z) - \varepsilon B(x, \alpha \theta w) + \varepsilon B(y, \alpha z \theta) - \varepsilon B(y, \alpha w \theta).
\]

However, \( \alpha = (\varepsilon k \theta^{-1}) \alpha^{-1} = \alpha^{-1} \) in this case. Thus, the above becomes

\[
= 2\alpha[B(x, z) - B(y, w)].
\]

We may thus view the image of \( m_\varepsilon \) as in \( H^\varepsilon(F(\alpha)) \oplus H^\varepsilon(F(-\alpha)) \) in this case, namely

\[
[M, [\cdot, \cdot]] \rightarrow [M, b_1, \ell_1] \oplus [M, b_2, \ell_2], \text{ where}
\]
\[ b_1(x, y) = 2B(ax, y) \quad \ell_1(x) = ax \]
\[ b_2(x, y) = -2B(ax, y) \quad \ell_2(x) = -ax \]

B is an F-valued asymmetric form. We must apply the trace lemma to identify \( b_1, b_2 \) with \( \langle , \rangle_1, \langle , \rangle_2 \) where \( \langle , \rangle_1 \) and \( \langle , \rangle_2 \) correspond with \( b_1, b_2 \) above, by \( \text{tr}_* \circ \langle , \rangle_1 = b_1, \; \text{tr}_* \circ \langle , \rangle_2 = b_2 \). Then

\[ m_\epsilon: [M, [\cdot, \cdot]] \rightarrow [M, \langle , \rangle_1] \otimes [M, \langle , \rangle_2]. \]

\( \langle , \rangle_1: M \times M \rightarrow F(\alpha) \) is defined by

\[ \langle x, y \rangle_1 = 2u^{-1}[ax, y]. \] Similarly \( \langle x, y \rangle_2 = -2u^{-1}[ax, y]. \)

\[ \langle , \rangle_1 \] is \( \epsilon \) Hermitian since

\[ \langle y, x \rangle_1 = 2u^{-1}[ay, x] \]
\[ = 2u^{-1}a[y, x] \]
\[ = 2\epsilon u^{-1}a[x, y] \]
\[ = 2\epsilon u^{-1}[ax, y] \]
\[ = \epsilon \langle x, y \rangle_1 \]

\[ \text{tr}_{F(\alpha)}/F \langle x, y \rangle_1 = \text{tr}_{F(\alpha)}/F (2u^{-1}[ax, y]) \]
\[ = t_1(2[ax, y]) \]
\[ = 2B(ax, y) \]
\[ = b_1(x, y) \]

Hence, \([M, \langle , \rangle_1]\) is merely \([M, [\cdot, \cdot]]\) with the scaling factor \( 2u^{-1}a \), so that \( m_\epsilon \) is 1-1 in this case.
Chapter X  THE OCTAGON OVER \(Z\)

We recall the decompositions

\[ \mathcal{W}^c(k,\mathbb{Q}/\mathbb{Z}) \cong \bigoplus_{p\mid k} \mathcal{W}(k,F_p) \oplus \mathcal{W}(k,F) , \quad \mathcal{A}(\mathbb{Q}/\mathbb{Z}) \cong \bigoplus_{p} \mathcal{A}(F_p). \]

For \(p\mid k\), the maps in the octagon for \(\mathcal{W}(k,F)\) do not make sense, as these terms \(\mathcal{W}(k,F)\) have \(k = 0\). Therefore, in this section, we assume \(k = \pm 1\). Hence, by the results for a field, we restate:

**Lemma 1.1** There is an exact octagon where \(k = \pm 1\),

\[
\begin{align*}
\mathcal{W}(k,F_p) & \xrightarrow{S_1} \mathcal{W}(k^2,F_p) \xrightarrow{I_1} \mathcal{W}(-k,F_p) \\
\mathcal{W}(-k,F_p) & \xleftarrow{d_{-1}} \mathcal{W}(k,F_p) \xleftarrow{m_{-1}} \mathcal{W}(k^2,F_p) \xleftarrow{I_{-1}} \mathcal{W}(k,F_p)
\end{align*}
\]

**Proof:** This is the octagon over the field \(F_p\).

Taking the direct sum over all \(p\), we obtain

**Theorem 1.2** For \(k = \pm 1\), there is an exact octagon:

\[
\begin{align*}
\mathcal{W}^1(k,\mathbb{Q}/\mathbb{Z}) & \xrightarrow{S_1} \mathcal{W}^1(k^2,\mathbb{Q}/\mathbb{Z}) \xrightarrow{I_1} \mathcal{W}^1(-k,\mathbb{Q}/\mathbb{Z}) \\
\mathcal{A}(\mathbb{Q}/\mathbb{Z}) & \xleftarrow{d_{-1}} \mathcal{A}(\mathbb{Q}/\mathbb{Z}) \xleftarrow{m_{-1}} \mathcal{W}^1(-k,\mathbb{Q}/\mathbb{Z}) \xleftarrow{I_{-1}} \mathcal{W}^1(k^2,\mathbb{Q}/\mathbb{Z}) \xleftarrow{S_{-1}} \mathcal{W}^1(k,\mathbb{Q}/\mathbb{Z})
\end{align*}
\]
Although we have yet to prove exactness of the octagon over \( \mathbb{Z} \), the homomorphisms nonetheless are defined over \( \mathbb{Z} \). It is easy then to check that we have the commutative diagram which follows. \( i \) denotes the map \( \otimes_{\mathbb{Z}} \), and \( \partial \) denotes the appropriate boundary homomorphism.

\[
\begin{align*}
0 \to & \quad A(\mathbb{Z}) \xrightarrow{i} \mathcal{A}(\mathbb{Q}) \xrightarrow{\partial} A(\mathbb{Q}/\mathbb{Z}) \to 0 \\
\downarrow & \quad \downarrow \quad \downarrow m_1 \\
0 \to & \quad W^1(k, \mathbb{Z}) \xrightarrow{i} \mathcal{W}^1(k, \mathbb{Q}) \xrightarrow{\partial} W^1(k, \mathbb{Q}/\mathbb{Z}) \to 0 \\
\downarrow & \quad \downarrow \quad \downarrow S_1 \\
0 \to & \quad W^1(k^2, \mathbb{Z}) \xrightarrow{i} \mathcal{W}^1(k^2, \mathbb{Q}) \xrightarrow{\partial} W^1(k^2, \mathbb{Q}/\mathbb{Z}) \to 0 \\
\downarrow & \quad \downarrow \quad \downarrow I_1 \\
0 \to & \quad W^1(-k, \mathbb{Z}) \xrightarrow{i} \mathcal{W}^1(-k, \mathbb{Q}) \xrightarrow{\partial} W^1(-k, \mathbb{Q}/\mathbb{Z}) \to 0 \\
\downarrow & \quad \downarrow \quad \downarrow d_1 \\
0 \to & \quad A(\mathbb{Z}) \xrightarrow{i} \mathcal{A}(\mathbb{Q}) \xrightarrow{\partial} A(\mathbb{Q}/\mathbb{Z}) \to 0 \\
\downarrow & \quad \downarrow \quad \downarrow m_{-1} \\
0 \to & \quad W^{-1}(k, \mathbb{Z}) \xrightarrow{i} \mathcal{W}^{-1}(k, \mathbb{Q}) \xrightarrow{\partial} W^{-1}(k, \mathbb{Q}/\mathbb{Z}) \\
\downarrow & \quad \downarrow \quad \downarrow S_{-1} \\
0 \to & \quad W^{-1}(k^2, \mathbb{Z}) \xrightarrow{i} \mathcal{W}^{-1}(k^2, \mathbb{Q}) \xrightarrow{\partial} W^{-1}(k^2, \mathbb{Q}/\mathbb{Z}) \\
\downarrow & \quad \downarrow \quad \downarrow I_{-1} \\
0 \to & \quad W^{-1}(-k, \mathbb{Z}) \xrightarrow{i} \mathcal{W}^{-1}(-k, \mathbb{Q}) \xrightarrow{\partial} W^{-1}(-k, \mathbb{Q}/\mathbb{Z}) \\
\downarrow & \quad \downarrow \quad \downarrow d_{-1}
\end{align*}
\]
The last two columns are exact, as are all rows. The problem is that the last three rows are not short exact, as there is the term $W(F_2)$ in $W^{-1}(k, Q/Z)$, not in the image of $\partial$.

We recall now the diagram chase that would prove exactness of the first column if all the rows were short exact. For simplicity, we label Witt equivalence classes now with symbols $x$, $y$, $z$, $u$, $v$, $w$.

To prove: $\ker S_1 = \text{im } m_1$

Let $x \in W^1(k, Z)$ have $x \in \text{im } m_1$. So we can find $y \in A(Z)$ with $m_1(y) = x$. Now $S_1 \circ m_1 \circ i(y) = 0$ by exactness of the 2nd column. Hence $i \circ S_1 \circ m_1(y) = 0$ by commutativity. So $i(S_1 \circ m_1(y)) = 0$. But $i$ is 1-1, so $S_1 \circ m(y) = S_1(x) = 0$. Thus $\text{im } m_1 \subseteq \ker S_1$.

Pictorially:

```
i
\downarrow m_1 \downarrow m_1
y \rightarrow i(y)
```

$x \quad m_1(i(y))$

```
\downarrow S_1 \downarrow S_1
0 \rightarrow S_1(x) \rightarrow 0
```

Conversely, let $x \in W^1(k, Z)$ have $x \in \ker S_1$.

The picture below will facilitate reading the proof.
The middle column is exact, so we can find \( y \in \mathcal{A}(Q) \) with \( m_1(y) = i(x) \). Now \( m_1(\partial y) = (\partial \circ m_1)(y) = (\partial \circ i)(x) = 0 \).

Thus, by exactness of the last column, we can find \( z \in W^{-1}(-k, Q/Z) \) with \( d_{-1}(z) = \partial y \).

This is the point that we need

\[ \partial : W^{-1}(-k, Q) \to W^{-1}(-k, Q/Z) \text{ is onto.} \]

If \( \partial \) is onto \( z \), we can find \( w \in W^{-1}(-k, Q) \) with \( \partial w = z \).

Then consider \( (y - d_{-1}w) \). \( \partial (y - d_{-1}w) = \partial y - \partial y = 0 \).

Thus, by exactness of the row, we can find \( v \in \mathcal{A}(Z) \) with \( i(v) = y - d_{-1}w \). However, \( (m_1 \circ i)(v) = m_1(y) - (m_1 \circ d_{-1})(w) = m_1y - ix \). Hence, \( (i \circ m_1)(v) = i(x) \). Since \( i \) is 1-1, \( m_1v = x \). \( \square \)

We see that the problem arises from \( z \not\in \text{im} \partial \) going under \( d_{-1} \) to \( \partial y \), which is in the image of \( \partial \). However, we have explicitly calculated what such \( z \) must be. Namely, \( z \) must arise from \( W^{-1}(F_2) \).

We recall the computations given in the last chapter.
We may thus conclude that the octagon is exact over \( \mathbb{Z} \) with one possible exception, the term
\[
A(\mathbb{Z}) \xrightarrow{d} W^{+1}(k, \mathbb{Z}) \xrightarrow{S} W^{+1}(k^2, \mathbb{Z}).
\]

We must carefully analyze exactness at \( W^{+1}(k, \mathbb{Z}) \). To begin with, consider \( W(F_2) \in A(\mathbb{Q}/\mathbb{Z}) \). This is the source of the problem.

Consider \([V, B] \in A(\mathbb{Q})\), where \( V = \langle \hat{e}_1 \rangle \), and \( B = [ , ]\), with \([\hat{e}_1, \hat{e}_1] = 2\).

We apply \( \delta \) to \([V, B]\). So let \( L = \langle \hat{e}_1 \rangle \) be a \( \mathbb{Z} \)-lattice. Let \( L^\# = \langle (1/2)\hat{e}_1 \rangle \) and \( \langle (1/2)\hat{e}_1, (1/2)\hat{e}_1 \rangle = 1/2 \). It follows that \( \delta [V, B] = W(F_2) \) (meaning \( \delta([V, B]) \neq 0 \) in \( W(F_2) \)).

Next, apply \( m_1 \) to \([V, B]\). Since \( s = \) identity is the symmetry operator, we obtain: \([V \oplus V, B_\epsilon, \ell_\epsilon]\).

\( V \oplus V \) has basis \((1, 0) = (\hat{e}_1, 0) = \hat{f}_1 \) and \((0, 1) = (0, \hat{e}_1) = \hat{f}_2\).

With respect of \( \hat{f}_1, \hat{f}_2 \), \( B_\epsilon \) has matrix
\[
\begin{pmatrix}
\hat{f}_1 & \hat{f}_2 \\
\hat{f}_1 & \hat{f}_2
\end{pmatrix}
\begin{pmatrix}
0 & 2 \\
2 \epsilon & 0
\end{pmatrix}
\]

since \( B_\epsilon((x, y), (z, w)) = B(x, w) + \epsilon B(z, y) \).

\( \ell_\epsilon \) has matrix
\[
\begin{pmatrix}
0 & \epsilon k \\
1 & 0
\end{pmatrix}
\]
since $\ell_\epsilon(x,y) = (\epsilon x s^{-1} y, x)$. Of course, $\epsilon = +1$ in this case.

Now $\delta[V \oplus V, B_\epsilon, \ell_\epsilon] = \delta \circ m_1 [V, B] = m_1 \circ \delta [V, B] = m_1 \circ d_1(W^{-1}(F_2)) = 0$. In fact, we may apply $\delta$ by:

Let $L$ be the $\mathbb{Z}$-lattice $\langle f_1, f_2 \rangle$. Then $L^\# = \langle (1/2)\hat{f}_2, (1/2)\hat{f}_2 \rangle$.

A metabolizer for $L^\# / L$ is $N = \langle (1/2)\hat{f}_1 + (1/2)\hat{f}_2 \rangle$.

There is the projection $L^\# \rightarrow L^\# / L$. Then $q^{-1}(N)$ has basis $\{(1/2)\hat{f}_1 + (1/2)\hat{f}_2, (1/2)\hat{f}_1 - (1/2)\hat{f}_2\}$, which we write as $\{\hat{g}_1, \hat{g}_2\}$. This enables us to construct an element in $W^{+1}(k, \mathbb{Z})$, which when tensored with $\mathbb{Q}$ yields $[V \oplus V, B_\epsilon, \ell_\epsilon]$. The element is $W = \langle \hat{g}_1, \hat{g}_2 \rangle$ as a $\mathbb{Z}$-module, with inner product

$$
\begin{pmatrix}
\hat{g}_1 & \hat{g}_2
\end{pmatrix}
\begin{pmatrix}
\hat{g}_1 \\
\hat{g}_2
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
$$

with respect to $\hat{g}_1, \hat{g}_2$, and degree $k$ map

$$
\begin{pmatrix}
(k + 1)/2 & (-k + 1)/2 \\
(-1+k)/2 & (-k - 1)/2
\end{pmatrix}
$$

with respect to $\hat{g}_1, \hat{g}_2$, where $\hat{g}_1 = (1/2)\hat{f}_1 + (1/2)\hat{f}_2$ and $\hat{g}_2 = (1/2)\hat{f}_1 - (1/2)\hat{f}_2$. This follows since $\ell_\epsilon$ is

$$
\begin{pmatrix}
0 & k \\
1 & 0
\end{pmatrix}
$$

with respect to $\hat{f}_1, \hat{f}_2$. We label this element $[W, b_1, t_1] = x$.
where \( W = \langle \vec{g}_1, \vec{g}_2 \rangle \), \( b_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), and

\[
t_1 = \begin{pmatrix} (k + 1)/2 & (-k + 1)/2 \\ (-1 + k)/2 & (-k - 1)/2 \end{pmatrix}.
\]

We observe that \([W, b_1, t_1] = x\) has order 2. When \( k = -1 \), \( x \) has order two since every element in \( W(-1, Z) \) has order two. For if \([W, B, \ell] \in W(-1, Z)\), \( \{(x, \ell x)\} \) will be an \( \ell \otimes \ell \) invariant self-annihilating subspace of \([W \otimes W, B \otimes B, \ell \otimes \ell]\), hence \( W \otimes W \sim 0 \). When \( k = +1 \), we consider \([V \otimes V, B, \ell] = y = ix\). Since \( i \) is injective, \( 2x = 0 \) if and only if \( 2y = 0 \). With the matrices given,

\[
B_\ell = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad \ell_\ell = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Clearly \( \{(x, y, x, -y); x \in V\} \) is a metabolizer for \((V \otimes V \otimes V \otimes V, B \otimes B, \ell \otimes \ell) = 2y\), so that \( 2y = 0 = 2x \).

**Lemma 1.3** \( x \) above is not in the image of \( m_1 \), but \( x \) is in the kernel of \( S_1 \).

**Proof:** By construction, \( x \) is in the kernel of \( S_1 \). This follows since \( i \circ S_1(x) = S_1 \circ i(x) = S_1 \circ m_1[V, B] = 0 \). \( i \) is 1-1, so \( S_1(x) = 0 \).

The picture below explains the proof that \( x \) is not in the image of \( m_1 \).
Suppose m^1(z) = x. Then m^1 ° i(z) = i ° m^1(z) = i(x).

However, m^1(y) = iy also. Thus, m^1(y - i(z)) = 0. By exactness of the middle column, there exists v with d_1(v) = y - i(z). Now consider δ v.

δ v = δ ° d_1v = δ(y - i(z)) = δy.

However, by construction, δy ≠ 0 in W(F_2).

The question then is: Can δv have d_1(δv) = u ≠ 0 in W^{-1}(F_2)? Clearly δ(v) ≠ u as u is not in the image of δ.

However, Hermitian summands are mapped under d_1 to Hermitian summands by the results of the last chapter.

Thus, no such v can exist, and hence x is not in the image of m^1. □

Lemma 1.4 If x_1 ∈ ker S, then either x ∈ im m_1 or x_1 - x ∈ im m_1, where x = [W, b_1, t_1] as described before Lemma 1.3.

Proof: The picture below may be useful.
Since $S_1 x_1 = 0$, $S_1 \circ i x_1 = 0$, and there exists $y_1$ with $m_1 y_1 = i x_1$ by exactness of the middle column.

Now consider $\partial y_1$. By commutativity, $m_1 \circ \partial y_1 = 0$, so that by exactness of the last column we can find $z_1$ with $d_{-1}(z_1) = \partial y_1$. The question is: Is $z_1$ in the image of $\partial$? If the answer is yes, we proceed as in the general case and conclude $x_1$ is in the image of $m_1$. If the answer is no, consider $z_1 + u$. This clearly must be in the image of $\partial$ say $\partial w = z_1 + u$ where $u \neq 0$ in $W^{-1}(kZ)$. Again we proceed as before and conclude $x_1 - x$ is in the image of $m_1$. □

We may now state the theorem we have been aiming for:

**Theorem 1.5** The following octagon is exact.

\[
\begin{array}{cccc}
W^1(kZ) & S^1 W^1(k^2Z) & T^1 W^1(-kZ) \\
\downarrow m_1 \circ & \downarrow i x_1 \circ & \downarrow d_{-1} \circ \\
A(Z) \oplus C_2 & A(Z) & A(Z)
\end{array}
\]

\[
\begin{array}{cccc}
W^{-1}(-kZ) & \epsilon^{-1} W^{-1}(k^2Z) & \epsilon^{-1} W^{-1}(kZ) \\
\downarrow d_{-1} \circ & \downarrow m_{-1} \circ & \downarrow m_{-1} \circ
\end{array}
\]
Proof: Here $C_2$ denotes the element $[W, B_1, t_1] = x$ constructed prior to Lemma 1.3. As we have seen the only question is exactness at $W^{+1}(k, \mathcal{Z})$.

Let $x_1 \in \text{im}(m_1 \oplus i)$, where $i$ is the identity on $x$; so $x_1 = m_1 y$ or $x_1 = m_1 y + x$. Applying $S_1$, we obtain

$$S_1(m_1 y) + S_1 x = S_1 m_1 y \quad \text{by 1.3}$$

$$= 0$$

Conversely, let $x_1 \in \ker S$. By 1.4, either $x_1 \in \text{im} m_1$ or $x_1 - x \in \text{im} m_1$. In either case, $x_1$ is in the image of $m_1 \oplus i$ as desired.

Finally, we should remark that adding the term $x$ to $A(Z)$ does not create new kernel elements for $m_1$. This is because $x \in W^{+1}(k, \mathcal{Z})$ is not in the image of $m_1$ by 1.3. □

Remark: The reason no problem occurred with $S_{-1}: W^{-1}(F_2) \rightarrow W^{-1}(F_2)$ is that neither term is in the image of $\partial$. 
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NOTATION

This is a list of commonly used symbols and abbreviations. A complete definition and description of each symbol is generally given in the text. This list is intended as an index of symbols.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>(Z)</td>
<td>The ring of integers</td>
</tr>
<tr>
<td>(Q)</td>
<td>The rational numbers</td>
</tr>
<tr>
<td>(D)</td>
<td>A Dedekind domain</td>
</tr>
<tr>
<td>(E)</td>
<td>The quotient field of (D)</td>
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<tr>
<td>(-)</td>
<td>An involution on (E)</td>
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<tr>
<td>(F)</td>
<td>The fixed field of (-)</td>
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<tr>
<td>(E^*)</td>
<td>Units in (E)</td>
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<tr>
<td>(E^{**})</td>
<td>Squares in (E)</td>
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<tr>
<td>(N_E^*)</td>
<td>Norms from (E)</td>
</tr>
<tr>
<td>(F^<em>/N_E^</em>)</td>
<td>Group of (-)-fixed elements modulo norms</td>
</tr>
<tr>
<td>(O(E))</td>
<td>Dedekind ring of integers in (E), namely (D)</td>
</tr>
<tr>
<td>(O(F))</td>
<td>Dedekind ring of integers in (F)</td>
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<tr>
<td>(O(E)^*)</td>
<td>Units in (O(E))</td>
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<tr>
<td>(S)</td>
<td>An order in (D)</td>
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<td>(\mathfrak{m})</td>
<td>A prime ideal in (O(E))</td>
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<td>(\mathfrak{P})</td>
<td>A prime ideal in (O(F))</td>
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<tr>
<td>Symbol</td>
<td>Description</td>
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<tr>
<td>----------------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>$O_E(\theta)$</td>
<td>Local ring of integers (at $\theta$)</td>
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<tr>
<td>$\hat{O}_E(\theta)$</td>
<td>Completion of $O(E)$ at $\theta$</td>
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<td>$I$</td>
<td>Fractional ideal in $O(E)$</td>
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<td>$I(\theta)$</td>
<td>$I$ localized at $\theta$</td>
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<td>$M(\theta)$</td>
<td>$M$ localized at $\theta$</td>
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<tr>
<td>$m(\theta)$</td>
<td>The localization of $\theta$ in $O_E(\theta)$</td>
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<tr>
<td>$D/\theta$</td>
<td>The residue field $O(E)/\theta$, also isomorphic to $O_E(\theta)/m(\theta)$</td>
</tr>
<tr>
<td>$\pi_\theta$</td>
<td>Uniformizer for $\theta$</td>
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<tr>
<td>$\pi_P$</td>
<td>Uniformizer for $P$</td>
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<tr>
<td>$v_\theta$</td>
<td>Additive version of $</td>
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<tr>
<td>$v_P$</td>
<td>Additive version of $</td>
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<tr>
<td>$(M,B)$</td>
<td>Inner product space</td>
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<td>$[M,B]$</td>
<td>Witt equivalence class of $(M,B)$</td>
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<tr>
<td>$(M,B,\ell)$</td>
<td>Degree $k$ mapping structure</td>
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<td>$\ell$</td>
<td>Degree $k$ map</td>
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<td>$[M,B,\ell]$</td>
<td>Witt equivalence class of $(M,B,\ell)$</td>
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<td>$\ell^*$</td>
<td>Adjoint operator of $\ell$</td>
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<td>$\text{Ad}_R B$</td>
<td>Right adjoint map of $B$</td>
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<td>$\text{Ad}_L B$</td>
<td>Left adjoint map of $B$</td>
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<td>$N_R$</td>
<td>Right orthogonal complement of $N$</td>
</tr>
<tr>
<td>$N_L$</td>
<td>Left orthogonal complement of $N</td>
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<tr>
<td>$N$</td>
<td>Orthogonal complement of $N</td>
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<td><strong>Symbol</strong></td>
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<td>---------------------</td>
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<tr>
<td>$W^+ (k)$</td>
<td>Witt equivalence classes of degree $k$ mapping structures with $B$ symmetric</td>
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<tr>
<td>$W^{1+} (k,K)$</td>
<td>Witt equivalence classes of inner product spaces, $(M,B)$ with $B$ symmetric</td>
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<td>$A(K)$</td>
<td>Witt equivalence classes of inner product spaces, $(M,B)$ with no symmetry requirements</td>
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<td>$s$</td>
<td>The symmetry operator</td>
</tr>
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<td>$\mathcal{H}(k,K)$</td>
<td>Degree $k$ mapping structures $(M,B,\lambda)$ under Witt equivalence, with the characteristic polynomial of $\lambda$ integral</td>
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<tr>
<td>$\sigma(K)$</td>
<td>The characteristic polynomial of $s$ is integral</td>
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<tr>
<td>$\text{Ann} M$</td>
<td>Annihilator of $M$</td>
</tr>
<tr>
<td>$\text{Ext}$</td>
<td>Cokernel of Hom functor</td>
</tr>
<tr>
<td>$K(F)$</td>
<td>Monic polynomials, coefficients in $F$, non-zero constant term</td>
</tr>
<tr>
<td>$GK(F)$</td>
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VITA

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Major Field: Mathematics

Title of Thesis: Witt Classification of Inner Product Spaces

Approved:

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Major Professor and Chairman

[Signature]
Dean of the Graduate School

EXAMINING COMMITTEE:

[Signatures]

Date of Examination:

April 19, 1979