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Hamilton-Jacobi theory for optimal control problems on stratified domains

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HAMILTON-JACOBI THEORY FOR OPTIMAL CONTROL PROBLEMS ON STRATIFIED DOMAINS

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Abstract

This thesis studies optimal control problems on stratified domains. We first establish a known proximal Hamilton-Jacobi characterization of the value function for problems with Lipschitz dynamics. This background gives the motivation for our results for systems over stratified domains, which is a system with non-Lipschitz dynamics that were introduced by Bressan and Hong. We provide an example that shows their attempt to derive a Hamilton-Jacobi characterization of the value function is incorrect, and discuss the nature of their error. A new construction of a multifunction is introduced that possesses properties similar to those of a Lipschitz multifunction, and is used to establish Hamiltonian criteria for weak and strong invariance. Finally, we use these characterizations to show that the minimal time function and the value function for a Mayer problem, both over stratified domains, satisfy and are the unique solutions to a proximal Hamilton-Jacobi equation.

Chapter 1

Introduction

This thesis studies a nonstandard control system in which the dynamics have a particular structure that allows for the development of some known results under a different set of hypotheses. We shall first review the mathematical description of (standard) optimal control theory, and then with this background in place, briefly describe the system of our study and our contribution to the literature.

Control systems can generally be described as dynamical systems that are being influenced by parameters. Our interest here is in continuous dynamical systems in which the state space is finite-dimensional Euclidean space and the dynamics are given by an ordinary differential equation that are influenced by additional variables. First, we recall the initial value problem from ordinary differential equations:

$$\text{(ODE)} \qquad \begin{cases} \dot{x}(t) &= \tilde{f}(t, x(t)) \quad \text{a.e. } t \in [a, b], \\ x(a) &= x_0. \end{cases}$$

The velocity data is the function $\tilde{f} : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$, in which it is typically assumed to be at least continuous (usually Lipschitz) in the state variable x, but can be merely measurable in t. Here, $\dot{x}(t)$ is the derivative of $x(\cdot)$ with respect to time (the variable t), which changes according to the right hand function; the state of the system at initial time a is prescribed as x_0. The solution $x(t)$ is absolutely continuous and describes the state evolution of the system as it changes continuously as a solution to the differential equation. It is immediate that $x(t)$ solves (ODE) if and only if it satisfies the integral equation

$$x(t) = x_0 + \int_a^t \tilde{f}\big(s, x(s)\big)\, ds.$$

One can see from this latter formulation why assumptions on the t-dependence in $\tilde{f}$ can be weaker than those in the state, but nonetheless, one needs at least continuous dependence in x for a satisfactory existence theory.

With control systems, the right hand side depends not only on time and the current state of the system but also an additional term (called the *control* variable). Control Dynamics are given by

$$\text{(CD)} \qquad \begin{cases} \dot{x}(t) &= f(t, x(t), u(t)) \quad \text{a.e. } t \in [a,b], \\ u(t) &\in U \quad \text{a.e. } t \in [a,b], \\ x(a) &= x_0. \end{cases}$$

The velocity data now is a function $f : [a,b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ which, in addition to the assumptions of (t,x) mentioned above, is assumed to be continuous in u. The control function $u(\cdot)$ is always taken to be measurable, and is called *admissible* if it only takes values in the given control set U, which is nonempty, convex, and compact. Given such an admissible control function $u(\cdot)$, by setting $\tilde{f}(t,x) = f(t,x,u)$, the formulation (CD) reduces to (ODE). Of course different choices of $u(\cdot)$ in general give different trajectories.

The control variable u can be viewed in a myriad of ways. It is usually thought of as an input that an external user can determine to influence the behavior of the trajectory, but it could also be thought of as an external forcing term acting as a disturbance, or as a term describing uncertainty in the system. It really doesn't matter how it is interpreted as far as the theory is concerned.

Here are a variety of issues that arise in the study of control systems:

(1) Given a closed set E and an initial state $x_0 \in E$, how can one characterize the property that there is a trajectory of (CD) that remains in E for all time $t \geq a$? Or the property

that all trajectories remain in E? These questions refer to the (weak and strong) *invariance* properties of the system.

(2) To what states can the system be driven by the selection of different controls? This fundamental question is one on the *controllability* of the system.

(3) Is it possible for the system to be sent to a particular state and then kept there or nearby for all subsequent times? This is a question of *stability* of the system at this given state.

(4) Is there a best way to control the system? More specifically, if there is a function which describes some benefit or cost associated with the state at a prescribed time, can we drive the system in such a way as to optimize this function?

The results of this thesis will answer only the first question (1) in the case of *stratified domains* (to be defined below). The second and third questions lay out a research program for future work. The last question (4) motivates the subject of *optimal control theory*, which is the background framework for this thesis. We discuss aspects of this theory next, which also provides the opportunity to introduce some notation.

An optimal control problem is formulated as minimizing a cost function over admissible controls and their associated trajectories that satisfy (CD). For problems involving maximization, an equivalent minimization problem can be readily found by simply taking the negative of the cost function. Thus, without loss of generality, we need only concern ourselves with minimization. The cost function may take several forms. For instance, it could be the integral of some running cost of the system:

$$J\big(x(t), u(t)\big) = \int_a^b \ell\big(t, x(t), u(t)\big)dt; \tag{1.1}$$

or a function of the system at the terminal time b :

$$J\big(x(t), u(t)\big) = g\big(x(b)\big); \tag{1.2}$$

or a combination of the two:

$$J\big(x(t),u(t)\big)=\int_a^b \ell\big(t,x(t),u(t)\big)dt+g\big(x(b)\big). \tag{1.3}$$

Here, depending on the type of problem being studied, the (not necessarily finite) end time $b>a$ could be a prescribed time or a parameter to be selected as part of the optimization. Whichever cost function is being used, the optimal control problem has the form:

$$\min\ J\big(x(t),u(t)\big) \tag{1.4}$$

where the minimization is over the pairs $\big(x(\cdot),u(\cdot)\big)$ that satisfy (CD). The respective problems that involve the costs given by (1.1), (1.2), and (1.3) are called Lagrange, Mayer, and Bolza problems, respectively.

If the control dynamics (CD) are essentially trivial, that is if $f(t,x,u)=u$, and so the dynamics reduce to $\dot{x}(t)=u(t)$, the optimal control problem reduces to one of the calculus of variations. Actually, this is not quite true since there is also the constraint $u(t)\in U$ which cannot be adequately treated by clasical calculus of variations methods. See Clarke [Cla90] for a complete discussion of the relationships between different problem formulations. The point is that one can view optimal control problems as path-constrained problems in the calculus of variations, and various results and techniques from the latter play a role in the theory of optimal control.

We briefly describe some immediate issues that arise in optimal control. The first, naturally, is whether a control function $u(\cdot)$ can be chosen so as to minimize the cost function J. Just as in the calculus of variations, this is a nontrivial problem. But nonetheless, it has a satisfactory answer by requiring the velocity set $f(t,x,U):=\{f(t,x,u):u\in U\}$ to be convex (along with some standard ones: regularity assumptions on f as mentioned above, a linear growth condition, and lower semicontinuity of the cost data).

The next question is what are the necessary conditions for a given control to be optimal? The main necessary condition is the well-known Pontryagin Maximum Principle (PMP) (cf.

[BGP61]), which we briefly describe here since we can use this context to introduce the Hamiltonian, a function which is in some sense dual to the dynamics and cost function and which plays a major role. We henceforth only consider autonomous problems, where the data has no explicit time dependence. The Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$H(x,\zeta) = \max_{u \in U} \big\{ \langle \zeta, f(x,u) \rangle - \ell(x,u) \big\}, \tag{1.5}$$

where we are considering the Bolza cost function (1.3). (The other two problems have a similarly defined Hamiltonian). Suppose $\bar{u}(\cdot)$ is optimal with associated state trajectory $\bar{x}(\cdot)$. The adjoint arc $\bar{p}(\cdot)$ is the solution of the (nonautonomous) linear ODE (where we have assumed the data is smooth)

$$\begin{aligned} -\dot{p}(t) &= D_x f\big(\bar{x}(t), \bar{u}(t)\big)^{\mathrm{tr}} p(t) - D_x \ell\big(\bar{x}(t), \bar{u}(t)\big) \quad \text{a.e. } t \in [a,b] \\ -p(b) &= \nabla g\big(\bar{x}(b)\big). \end{aligned}$$

The (PMP) is the necessary condition that $\bar{u}(\cdot)$ satisfies

$$H\big(\bar{x}(t), \bar{p}(t)\big) = \big\langle \bar{p}(t), f\big(\bar{x}(t), \bar{u}(t)\big) \big\rangle - \ell\big(\bar{x}(t), \bar{u}(t)\big) \quad \text{a.e. } t \in [a,b].$$

That is, the Hamiltonian given in (1.5) and evaluated at $\big(\bar{x}(t), \bar{p}(t)\big)$ achieves its maximum over $u \in U$ at $u = \bar{u}(t)$.

One may additionally inquire into sufficient conditions for the optimality of a particular control. A particular method of addressing this issue is called *dynamic programming*, and centers around the study of the so-called *value function* $V : [-\infty, T] \times \mathbb{R}^n \to \mathcal{R}$ (T is a fixed terminal time). We define it here only for the Mayer problem, although it could equally be defined for Bolza problems:

$$V(t,x) := \min_{u(\cdot)} \big\{ g\big(\big(x(T)\big),\ \big(x(\cdot), u(\cdot)\big) \text{ satisfies (CD) with initial condition } x(t) = x \big\}$$

. The value function is the function describing the best cost that can be acheved given an initial state and time of the system. In the last 20 years, a major research area has developed around the study of the value function and to what extent it solves the Hamilton-Jacobi equation

$$\phi_t(t,x) + \min_{u \in U}\{\phi_x(t,x) \cdot f(x,u)\} = 0$$

with

$$\phi(T,x) = g(x).$$

Generally, the Hamilton-Jacobi equation is not classically solvable (that is, there are no smooth solutions), however the value function can be shown to be unique the *viscosity solution* of the Hamilton-Jacobi equation. For instance, see [BCD97]. This solution concept is in essence a nonsmooth version of the equation. After determining the value function as the solution to the Hamilton-Jacobi equation, we can then look for a control which provides this cost and know that it is optimal. This final step, known as synthesis, is not a simple process and can be quite involved, even in low dimensions (for instance, see [BP04]).

In the standard theory for optimal control theory, the dynamics are assumed to be Lipschitz continuous with respect to the state variable. This is because techniques similar to those used in the study of ordinary differential equations can be applied to Lipschitz continuous control systems. Moreover, the Hamilton-Jacobi equations associated with optimal control problems involving Lipschitz systems and continuous cost functions have continuous coefficients, which simplifies the establishment of sufficient conditions for optimality. However, there has been a growing interest in systems which are not Lipschitz; the main topic in this work is such a system. We study systems whose given state space is partitioned into regions, where each region has a different velocity set. Mathematically, these regions are manifolds with boundary. This means that as a trajectory stays in one of the regions, its available velocity set resembles a classical system, but as it moves into a different region, the admissable velocity set may make a non-Lipschitz (or typically noncontinuous) switch. This type of model has proven to be of great use in several modeling applications.

Our work has been largely motivated by Bressan and Hong [BH07], which introduced a class of optimal control problems, called stratified domain control problems, where the dynamics may change instantly. In this class of problems, there is a prescribed collection of embedded submanifolds where the dynamics switch discontinuously. That is, when the state reaches a predescribed subset of the state space, the dynamics immediately change. In these problems, the dynamics are not locally Lipschitz, which means that the coefficients of the associated Hamilton-Jacobi equation are generally discontinuous, leading to theoretical problems which are of interest here.

The remaining chapters are ordered in the following manner. In Chapter 2, we introduce differential inclusions, which is a generalization of differential equations that subsumes control systems. A standard form of Hamilton-Jacobi theory for Lipschitz continuous systems is also presented in order to contextualize the extensions in later chapters. Optimal control problems on stratified domains in Chapter 3 are introduced, where the dynamics are no longer assumed to be locally Lipschitzean. We also discuss a recent attempt in [BH07] at establishing the value function as the solution of the Hamilton-Jacobi equation for such a problem and show it is incorrect by providing a counterexample. Chapter 4 contains the main new work where a differential inclusion problem related to the stratified domain problem is studied. Characterizations of weak and strong invariance are established. Finally, in Chapter 5 we show that a Hamilton-Jacobi equation characterizes the value function for problems on stratified domains. We show this first for the minimal time problem, and then for Mayer problem. We conclude with some remarks in Chapter 6 on remaining questions and possible work in the study of optimal control problems on stratified domains.

We conclude this introduction with a few remarks regarding notation. The notation $y \to_v x$, which should be read as "y converging to x along the vector v" means that

$$y \to x, \text{ and } \frac{y-x}{||y-x||} \to \frac{v}{||v||}.$$

The open unit ball centered at the origin is denoted by B and $\overline{B}$ denotes its closure. The open ball of radius δ centered at x is denoted $B(x,\delta)$ and its closure is denoted $\overline{B}(x,\delta)$. As mentioned earlier, for a function of time $x : \mathbb{R} \to \mathbb{R}^n$, the notation $\dot{x}(t)$ denotes the derivative with respect to time. We use X to denote a general Banach space. We use the notation coF defined as

$$coF := \{\sum_{i=1}^{k} \theta_i x_i : k < \infty, \theta_i \geq 0, \sum_{i=1}^{k} \theta_i = 1, x_i \in F\}$$

to denote the convex hull of the set F. The notation $\overline{co}F$ denotes the closure of the convex hull of the set F. Finally, the abbreviation "a.e." means almost everywhere in terms of Lebesgue measure on the real numbers.

Chapter 2

Differential Inclusions and the Mayer Problem

This chapter introduces differential inclusions, which generalize differential equations and control systems. Section 2.1 introduces differential inclusions and discusses the standard hypotheses invoked when dealing with optimal control theory in order to provide the structure needed to develop an existence theory and conditions for optimality. In Section 2.2, we show how these hypotheses lead to criteria for characterizing invariance properties with respect to a given set. Finally, in Section 2.3, the value function for the Mayer problem is shown to uniquely solve the associated proximal Hamilton Jacobi equation.

2.1 Autonomous Differential Inclusions

Autonomous standard differential inclusions describe a problem where we look for a function $x : [0, T] \to \mathbb{R}^n$ whose derivative is an element of a prescribed velocity set. Differential inclusions with an initial condition take the form

$$\begin{cases} \dot{x}(t) & \in & F(x(t)) \quad \text{a.e. } t \in [0, T], \\ x(0) & = & x_0. \end{cases} \tag{2.1}$$

Here $F(x) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a multifunction; in other words, for each $x \in \mathbb{R}^n$, $F(x)$ is a subset (possibly a singleton) of $\mathbb{R}^n$. We say the differential inclusion is *autonomous* because F is not directly dependent on time. An *arc* is an absolutely continuous function $x(\cdot) : [a, b] \to \mathbb{R}^n$

for $a, b \in \mathbb{R},\ a < b$. A *trajectory* of (2.1) is an arc $x(\cdot)$ on $[0, T]$, for some positive time T, such that $x(0) = x_0$ and $\dot{x}(t)$ lies in the set $F(x(t))$ for almost all $t \in [0, T]$.

Recall that classical autonomous ordinary differential equations take the form,

$$\begin{cases} \dot{x}(t) &= \tilde{f}(x(t)) \quad \text{a.e. } t \in [0, T], \\ x(0) &= x_0. \end{cases} \tag{2.2}$$

If we define the singleton-valued multifunction $F(x) = \{f(x)\}$, clearly (2.2) is a special case of (2.1). The existence of solutions to an ordinary differential equation require additional hypotheses. The following is a well-known result.

Proposition 2.1. Assume that f is continuous and satisfies a linear growth hypothesis; that is, there exist $\alpha,\ c \geq 0$ such that

$$||f(x)|| \leq \alpha||x|| + c,\ \forall x.$$

Then there exists a solution $x(\cdot) : (-\infty, \infty) \to \mathbb{R}^n$ to (2.2). If we further assume that f is Lipschitz continuous, then the solution is unique.

Let us consider the existence of solutions to (2.1). A natural question is whether analagous properties exist that guarantee the existence of selections that generate trajectories. That is, can we construct a function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(x) \in F(x)$, for all x and there is an arc $x(\cdot)$ such that $\dot{x}(t) = f(x(t))$ for almost all times t? We will require throughout this work that

(H-1) For each $x \in \mathbb{R}^n$ the set $F(x)$ is nonempty, compact, and convex.

(H-2) There exist positive $\alpha,\ c$ such that for any x,

$$||v|| \leq \alpha||x|| + c,\ \forall v \in F(x).$$

We also need an appropriate analogue to the above assumption of the continuity of the right hand side for ordinary differential equations. However, for our purposes, an assumption of full continuity on the multifunction F is not always necessary. Rather, requiring that the function is *upper semicontinuous* at every $x \in \mathbb{R}^n$ will be sufficient.

Definition 2.2. A multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *upper semicontinuous* at x if for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$F(y) \subset F(x) + \epsilon B, \ \forall y \in \{x\} + \delta B.$$

We recall the standard definition of the distance between a point x and a nonempty set Y in $\mathbb{R}^n$:

$$d(x, Y) := \inf\{d(x, y) : y \in Y\}.$$

Upper semicontinuity in this setting can be characterized in the following manner.

Proposition 2.3. Assume F satisfies (H-1) and (H-2). Then F is upper semicontinuous if and only if its graph

$$grF := \{(x, v) : v \in F(x)\}$$

is closed.

Proof. ($\Rightarrow$) Let $(x_n, v_n) \in grF$ be such that $(x_n, v_n) \to (x, v)$. We need to show that $v \in F(x)$. We note by upper semicontinuity that there is a sequence of $\epsilon_n \downarrow 0$ such that $d(v_m, F(x)) < \epsilon_n$ for all $m > n$. This means that the distance $d(v, F(x)) = 0$ and, as $F(x)$ is compact, we see that $v \in F(x)$ which means that the graph of F is closed. ($\Leftarrow$) Let $x \in \mathbb{R}^n$ and $\epsilon > 0$. Suppose that $x_n \to x$ and $v_n \in F(x_n)$. We assume that $d(v_n, F(x)) \geq \epsilon$ for all n. From (H-2), we know that $\{v_n\}$ is bounded and so, passing to a subsequence if necessary, we may assume that $v_n \to v$. But then $d(v, F(x)) \geq \epsilon$ which contradicts the closure of the graph of F. □

For more on the uppersemicontinuity property of multifunctions, see Chapter 1 of [Dei92]. In the introduction, we discussed questions involving a control system, which takes the general form (in the autonomous case)

$$\begin{cases} \dot{x}(t) &=& f(x(t), u(t)) \quad \text{a.e. } t \in [a,b], \\ u(t) &\in& U \quad \text{a.e. } t \in [a,b], \\ x(a) &=& x_0. \end{cases} \tag{2.3}$$

for some $T > 0$ where U is a prescribed compact, nonempty set. Define $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ as

$$F(x) := \{v : \exists u \in U \text{ s.t. } v = f(x,u)\} =: f(x,U).$$

Clearly, if we define $F(x)$ as $f(x,U)$ then any trajectory of (2.3) is a trajectory of (2.1). But is the converse true? That is, is a trajectory of the differential inclusion (2.1) a trajectory of the control system (2.3)? Filippov's Lemma (see [AC84] and [Vin00]) shows, among other things, that if f in (2.3) is a continuous function of both x and u and U is compact and nonempty, then the trajectories of (2.1) are also trajectories of (2.3). That is, if $x(\cdot)$ is an arc on $[0,T]$ with $\dot{x}(t) \in F(x(t))$ *a.e.* in time then there exists a measurable control $u : [0,T] \to U$ with

$$\dot{x}(t) = f(x(t), u(t))$$

Therefore, differential inclusions subsume the class of control systems with continuous state dependence. With this in mind, we will focus our attention on the differential inclusion formulation of optimal control problems.

However, in so doing, we must take care in moving between control system formulations and differential inclusion formulations. Even if U is a convex set, the multifunction $F(x) := f(x,U)$ is not necessarily convex. However, it can be shown (for instance, in [AC84]) that if F satisfies (H-2) and Lipschitz continuity (see below), the trajectories of coF are arbitrarily close to trajectories of F in the sup norm. Therefore, we can always approximate a differential inclusions involving a nonconvex-valued multifunction with a differential inclusion involving a multifunction that is convex. Because of this, the requirement that F be convex-valued is generally not a very restrictive requirement. However, if we have to regularize a differential inclusion in order to work with an uppersemicontinuous multifunction which satisfies (H-1) and (H-2), a trajectory of this new multifunction may not have an associated control. That is, if $F(x) = f(x, U)$ and G is a regularization of F–i.e. $F(x) \subset G(x)$ everywhere and G satisfies the above hypotheses–a trajectory $x(\cdot)$ of G may be arbitrarily close to a trajectory of F; however, there may not be a control function $u(t)$ such that $\dot{x}(t) = f(x(t), u(t))$ almost everywhere. As we shall see in Section 3.2, significant errors may arise when we assume that a trajectory of a regularized differential inclusion associated with a non-Lipschitz control system has a control which generates this selection as described above.

As in the case of ordinary differential equations, the role of the linear growth hypothesis (H-2) arises most notably in applying Gronwall's Lemma to provide bounds on the growth of trajectories of (2.1).

Proposition 2.4. Suppose that x is an arc on $[0, T]$ such that

$$||\dot{x}(t)|| \leq \alpha ||x(t)|| + c$$

almost everywhere on $[0, T]$ for some $\alpha, c \geq 0$. Then for any $t \in [0, T]$ we have the following bound

$$||x(t) - x(0)|| \leq (e^{\alpha t} - 1)||x(0)|| + \int_0^t ce^{\alpha(t-s)} ds.$$

Proof. See [CLSW98]. □

One of the key motivations for assuming the upper semicontinuity of $F(x)$ is the following result, known as the compactness of trajectories. It is in fact more powerful than this name might suggest. While it implies that bounded sets of trajectories are relatively compact, it also shows that sequences of arcs which are "almost" trajectories are relatively compact and, by passing to a subsequence if necessary, converge to a trajectory of (2.1). This is proved in Theorem 3.1.7 of [Cla90] and in Theorem 2.5.3 of [Vin00].

Lemma 2.5. Suppose that F satisfies (H-1), (H-2), is uppersemicontinuous, and that there is a sequence of arcs $\{x_i(\cdot)\}$ on $[0,T]$ such that $\{x_i(0)\}$ is bounded and

$$\dot{x}_i(t) \in F(x_i(t) + y_i(t)) + r_i(t)B$$

almost everywhere. Here $\{y_i(\cdot)\}, \{r_i(\cdot)\}$ are sequences of measurable functions converging in L^2 to 0. Then there exists a subsequence of $\{x_i(\cdot)\}$ converging uniformly to an arc $x(\cdot)$ which is a trajectory of F, and the derivatives $\dot{x}_i(\cdot)$ weakly converge to $\dot{x}$.

2.2 Invariance

As in differential equations, the question of invariance is of significant interest in the study of differential inclusions. Given a closed set $E \subset \mathbb{R}^n$, the invariance property, in the case of differential equations, states that a trajectory with initial point $x_0 \in E$ will remain in E for all positive times–i.e. $x(t) \in E$, for every $t \geq 0$. However, because trajectories of (2.1) are not unique, there are two different notions of invariance to be investigated.

Definition 2.6. Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a multifunction and $E \subset \mathbb{R}^n$ a closed set. Then

- The pair (F, E) is *weakly invariant* if for each $T > 0, x_0 \in E$, there exists a trajectory $x(\cdot)$ of (2.1) such that $x(t) \in E,\ t \in [0,T]$.
- The pair (F, E) is *strongly invariant* if foreach $T > 0, x_0 \in E$, any trajectory $x(\cdot)$ of (2.1) is such that $x(t) \in E,\ t \in [0,T]$.

Of course, these definitions coincide when trajectories are unique (as in the case of Lipschitz differential equations). It is trivial to find examples where weak invariance holds but not strong invariance (if $F(x) \equiv \overline{B}$, any nontrivial compact subset $S \subset \mathbb{R}^n$ will suffice for an example). We next address the criteria that guarantee weak(strong) invariance. We will use the *lower Hamiltonian*, associated with the multifunction F, $h_F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$h_F(x,\zeta) := \inf_{v \in F(x)} \langle v, \zeta \rangle. \tag{2.4}$$

Though we will not make direct use of it, we note the definition of the *upper Hamiltonian*

$$H_F(x,\zeta) := \sup_{v \in F(x)} \langle v, \zeta \rangle. \tag{2.5}$$

Note that for $F(x) := f(x, U)$, where f is a control dynamics, the upper Hamiltonian here conincides with the Hamiltonian in Chapter 1. Of course, $H_F(x,\zeta) = -h_F(x,-\zeta)$ and so either the lower or upper Hamiltonian would suffice for our purposes. The other main element in our criteria will be the proximal normal, a fundamental concept in nonsmooth analysis. We let $\mathrm{proj}_E(y)$ denote the projection of y onto E, that is the set of points in E whose distance to y is minimal.

Definition 2.7. Let $E \subset \mathbb{R}^n$ and $x \in E$. Then if $x \in \mathrm{proj}_E(y)$ for some $y \notin E$, we say that $\zeta = t(y - x)$ for any $t \geq 0$ is a *proximal normal direction* to E at x.

Note that it is possible, particularly if $x \in \mathrm{int}E$, that the only such ζ is 0. The set of all such vectors, the *proximal normal cone*, will be denoted $N_E^P(x)$. Examples of proximal normal directions are seen in Figure 2.2. Note that at x_1, the proximal normal cone has one vector of unit length, whereas $N_E^P(x_2)$ contains infinitely many such vectors. In fact, because $N_E^P(x)$ is convex, for any $x \in E$ such that $N_E^P(x)$ contains two or more vectors of unit length, $N_E^P(x)$ will contain infinitely many such vectors. Finally, $N_E^P(x_3)$ is trivial, containing only 0, because no point outside of E has x_3 as its closest point in E. It is not hard to show the following

Proposition 2.8. The following are equivalent

- $\zeta \in N_E^P(x)$
- $d_E(x + t\zeta) = t||\zeta||$ for sufficiently small $t > 0$.
- There exists $\sigma_{\zeta,x} \geq 0$ such that

$$\langle \zeta, y - x \rangle \leq \sigma_{\zeta,x} ||y - x||^2, \ \forall y \in E.$$

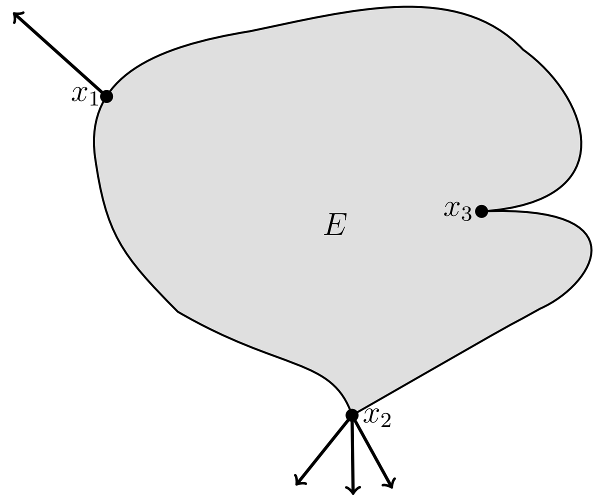


Figure 2.1: Some proximal normal directions. Note that $N_E^P(x_3)$ is trivial, while $N_E^P(x_2)$ contains multiple vectors of unit length

The second statement provides a different viewpoint on the proximal normal cone. It says that a vector ζ is proximally normal at a point if, for any sufficiently small $t > 0$,

$$B(x + t\zeta, t) \cap E = \emptyset,\ x \in \overline{B}(x + t\zeta, t) \cap E.$$

This is shown in Figure 2.2. The last statement in the proposition is known as the *proximal normal inequality.* We note that $N_E^P(x)$ is indeed a cone (i.e. it is closed under nonnegative multiplication), may be trivial (equal to $\{0\}$) even if E is closed and x is in the boundary, and is convex but not generally open or closed.

Along with the proximal normal cone we can construct the proximal subgradient (for more, see [CLSW95],[CLSW98]). Letting X be any real Hilbert space, and $f : X \to (-\infty, +\infty]$, we say that f is *lower semicontinuous* at a point x if

$$\liminf_{y \to x} f(y) \geq f(x).$$

An equivalent definition is related to the *epigraph* of the function, defined as

$$\text{epi} f := \{(x, a) \in \mathbb{R}^n \times \mathbb{R} : a \geq f(x)\}.$$

Proposition 2.9. Let $f : X \to (-\infty, \infty]$. Then f is lower semicontinuous if and only if epi$f$ is closed in $X \times \mathbb{R}$.

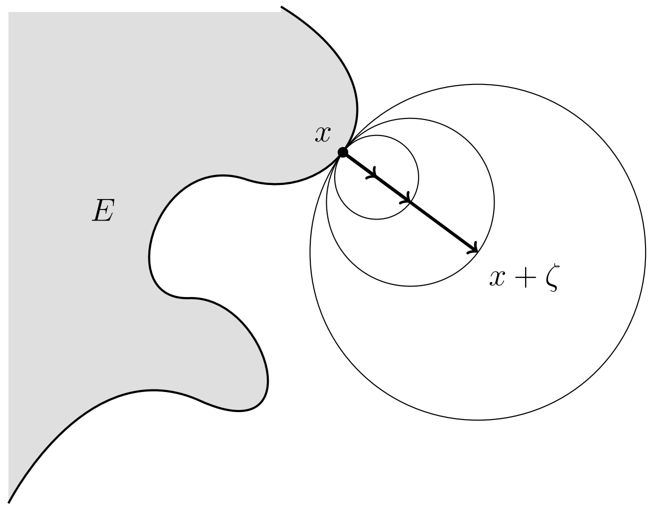


Figure 2.2: An alternative perspective on $N_E^P(x)$

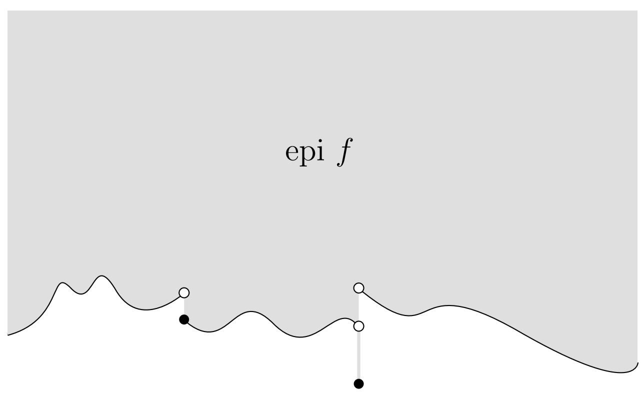


Figure 2.3: A lower semicontinuous function, f, with its epigraph

Proof. Assume that epi$f$ is closed. Then for any $y_i \to x$ with $f(y_i) < \infty$, we have that $(y, \liminf_{y_i \to y} f(y_i)) \in \text{epi}f$ and so, immediately, we have that $\liminf_{y_i \to y_i} f(y_i) \geq f(x)$. Now suppose that f is lower semicontinuous on X, and $y_i \to y$, and $w_i \to v$ where $w_i \geq f(y_i)$. Then, by definition,

$$v \geq \liminf_{y_i \to y} f(y_i)$$

which implies that $v \geq f(x)$ by assumption. Thus, $(y, v) \in \text{epi}f$ and so the epigraph is closed. □

We now may define the proximal subgradient of f.

Definition 2.10. Let $f : x \to (-\infty, +\infty]$ be lower semicontinuous on X. Then $\xi \in X$ is a *proximal subgradient* of f at x if

$$(\xi, -1) \in N^P_{\text{epi}f}(x, f(x)).$$

All such ξ form the *proximal subdifferential*, denoted as $\partial_P f(x)$.

Like the proximal normal cone, the proximal subdifferential is convex. Similar to the proximal normal inequality is the *proximal subgradient inequality*:

Proposition 2.11. Let f be lower semicontinuous. Then $\xi \in \partial_P f(x)$ if and only if there exist $\sigma, \eta > 0$ such that

$$f(y) \geq f(x) + \langle \xi, y - x \rangle - \sigma ||y - x||^2, \ \forall y \in B_\eta(x).$$

Proof. See Theorem 2.5, pages 33-4 in [CLSW98]. □

Intuitively, the classical derivative requires the existence of a linear function which is locally bounded by f from above. The proximal subgradient inequality says that instead of a linear function, we look for quadratic functions locally bounded from above by f. This intuition may seem familiar to the reader who has studied viscosity solutions to partial differential equations (for instance in [Eva10], [BCD97]); in this context, at points where a function is not classically differentiable, the behavior of a quadratic majorized or minorized locally by the function is studied. The proximal subdifferential does in fact coincide with the classical derivative when f is C^2, a fairly immediate consequence of Proposition 2.11.

Corollary 2.12. Let f be as in Proposition 2.11. Then if $f \in C^2(U)$ for some U open, then for all $x \in U$,

$$\partial_P f(x) = \{f'(x)\}.$$

Finally, we note the following simple fact. The proof follows immediately from the proximal subgradient inequality.

Proposition 2.13. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous with nontrivial epigraph. Suppose that $x^* \in X$ is a local minimizer of f. Then

$$\{0\} \subset \partial_P f(x^*).$$

The proximal normal and subgradient naturally complement results involving differential inclusions. As the differential inclusion, informally, takes the question of constructing appropriate controls in a control system and transforms this into a question in set-valued analysis, the proximal subgradient takes questions regarding differentiability and transforms them into the property of a set, namely the epigraph. Of course, this is a bit reductive, but it hints at why the proximal objects will fit nicely into the analysis of differential inclusions. When discussing optimality conditions for the Mayer problem in Section 2.3, we will make use of the proximal subgradient. The proximal normal coupled with the lower Hamiltonian allow us to establish the following necessary and sufficient condition for weak invariance.

Proposition 2.14. Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfy (H-1) and (H-2) and the set $E \subset \mathbb{R}^n$ be closed. Then (F, E) is weakly invariant if and only if, for every $x \in E$,

$$h_F(x, \zeta) \leq 0$$

holds for all $\zeta \in N_E^P(x)$.

Proof. ($\Rightarrow$) Let $e(x)$ be an arbitrary selection from $\text{proj}_E(x)$, which will be unique if E is convex. Immediately, we have that for each x, there is a $v \in F(e(x))$ such that $\langle v, x - e(x)\rangle \leq 0$. Again choosing arbitrarily when needed, we define a selection $f(x) \in F(e(x))$. We now proceed to prove that there is a trajectory which solves, on $[0, T]$, the differential equation

$$\dot{x}(t) = f(x(t)),\ x(0) = x_0$$

for any $x_0 \in E$. To do this, we assume that E is compact, with M_E the bound for F on E (by (H-2)). Let $\pi_n = \{t_0 = 0, t_1, \ldots, t_n = T\}$ be a partition of $[0, T]$ into n segments maximum length δ_n. We define $x_i = f(x_{i-1})x_{i-1}$ and the piecewise-linear function $x_{\pi_n}(t)$ as such that $\dot{x}(t) = f(x_i(t))$ on (t_{i-1}, t_i). Then, for any specific π_n,

$$d_E(x_1) \leq M_E(t_1 - t_0)$$

and, similarly,

$$\begin{aligned} d_E^2(x_2) &\leq |x_2 - s(x_1)|^2 \\ &= |x_2 - x_1|^2 + |x_1 - s(x_1)|^2 + 2\langle x_2 - x_1, x_1 - s(x_1)\rangle \\ &\leq M^2[(t_2 - t_1)^2 + (t_1 - t_0)^2] + 2\int \langle f(x_1), x_1 - s(x_1)\rangle dt \\ &\leq M^2[(t_2 - t_1)^2 + (t_1 - t_0)^2] \end{aligned}$$

which leads to concluding that

$$d_E^2(x_k) \leq M^2 T\delta_n.$$

Letting $\delta_n \to 0$, we define the resulting uniform limit of x_{π_n} as $x(\cdot)$. By Lemma 2.5, we know that $x(\cdot)$ is a trajectory and, by the above, remains in E. Finally, if we remove the assumption that E be compact, we obtain the result by establishing the result, for fixed T and x_0, on all compact subsets of E containing x_0. ($\Leftarrow$) See the proof of Theorem 2.2 of [CLSW95]. □

Weak invariance is, then, equivalent to finding at every point in the set E a direction in F which does not point outside of E. Intuitively, then, we would expect that strong invariance is equivalent to every $v \in F(x)$ not pointing outisde of E for each x in E. This can be expressed at each $x \in E$, for $\zeta \in N_E^P(x)$ in terms of the lower hamiltonian:

$$h_F(x, -\zeta) \geq 0.$$

However, the regularity from our assumptions (H-1), (H-2), and upper semicontinuity are not enough for this criterion to be equivalent to strong invariance.

Example 2.15. Let $E = [-1, 1]$ and

$$F(x) = \begin{cases} [-\sqrt{1-|x|^2}, \sqrt{1-|x|^2}], & x \in [-1,1] \\ [-\sqrt{|x|^2-1}, \sqrt{|x|^2-1}], & x \notin [-1,1]. \end{cases}$$

Then, clearly, for $x \in E$, and any proximal normal direction ζ,

$$h_F(x, \zeta) = 0.$$

However, for instance, at $x_0 = 1$, there exists a trajectory originating at x_0 with derivative $\sqrt{|x|^2 - 1}$ for all $t \in [0, T]$. This clearly will not remain in E.

Note that, in the example, if $x_0 = 1$, the set $F(x_0) = \{0\}$ does not encompass all possible directions that a trajectory can travel in. Any condition for strong invariance will need to capture every direction a trajectory can travel. Often, Lipschitz continuity is invoked in order to provide the requisite regularity for such a statement.

Definition 2.16. A multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *Lipschitz continuous* on $\Omega \subset \mathbb{R}^n$ if there exists a $k > 0$ such that for all $x, y \in \Omega$

$$F(y) \subset F(x) + k|x - y|B.$$

We should note here that for the results in the remainder of this chapter, the dynamics can be assumed to be locally Lipschitz. That is, for every compact set Ω, there exists k_Ω such that

$$F(y) \subset F(x) + k_\Omega ||x - y|| \overline{B}, \ \forall x, y \in \Omega.$$

If, rather than global Lipschitz continuity, this is assumed then the proofs follow in a similar manner. However, for simplicity, we assume global (i.e. $\Omega = \mathbb{R}^n$), rather than local, Lipschitz continuity. The Lipschitz continuity of F often is used in order to make use of the following density result:

Proposition 2.17. Suppose that $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfies (H-1), (H-2), and is Lipschitz continuous. Let $x(\cdot)$ be a trajectory of (2.1) on $[0, T]$. For all $\epsilon > 0$, there exists a C^1 trajectory $x_\epsilon(\cdot)$ of (2.1) on $[0, T]$, such that

$$||x(t) - x_\epsilon(t)|| < \epsilon, \quad \forall t \in [0, T].$$

Proof. See Theorem 3.1 of [Wol90a]. □

However, perhaps the most important property of Lipschitz differential inclusions is the following:

Proposition 2.18. Suppose that $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfies (H-1), (H-2), and is Lipschitz continuous. Then for every x_0 and $v \in F(x_0)$, there exists a C^1 trajectory for (2.1), $x(\cdot)$, such that $\dot{x}(0) = v$.

Proof. See pages 115-117 of [AC84]. □

This says, informally, that Lipschitz multifunctions contain only elements which can "generate" a trajectory. Intuitively, Proposition 2.14 says that weak invariance requires that at every point $x \in E$, there is a direction $v \in F(x)$ pointing inside E. We may then use this to construct a trajectory which remains in E. For a Lipschitz multifunction F, a similar condition may be established, as a Lipschitz multifunction contains exactly the possible velocities a trajectory may have.

Lemma 2.19. Suppose that F satisfies (H-1), (H-2), and is Lipschitz continuous with constant K. Then, for a closed E, the pair (F, E) is strongly invariant if and only if, for every $x \in E$,

$$h_F(x, -\xi) \geq 0, \ \forall \xi \in N_E^P(x).$$

Note that the condition on the lower hamiltonian above is equivalent to requiring that

$$H_F(x, \zeta) \leq 0, \ \forall \zeta \in N_E^P(x).$$

The following proof is similar to proofs in [CLSW98] (page 199) and in [WZ98] (Theorem 3.1); also see Theorem 2.2 of [CLSW95].

Proof. ($\Rightarrow$) Let $x_0 \in E,\ v \in F(x),\ \xi \in N_E^P(x)$. Then let $x(\cdot)$ be a C^1 trajectory originating at x_0 with $\dot{x}(0) = v$. The assumption of strong invariance implies that $x(t) \in E$ for all positive times t; this in turn implies by the proximal normal inequality that there is a $\sigma > 0$ such that

$$\langle \xi, x(t) - x \rangle \leq \sigma ||x(t) - x||^2,\ \forall t \geq 0.$$

Dividing both sides by t, and letting $t \to 0$, gives that $\langle \xi, v \rangle \leq 0$. Which means that

$$h_F(x, -\xi) \geq 0.$$

($\Leftarrow$) Let $x_0 \in E, x(\cdot)$ be a trajectory originating at x_0. By Proposition 2.17, for any $\epsilon > 0$, there is a C^1 trajectory $x_\epsilon(\cdot)$ originating at x_0 such that

$$||x(t) - x_\epsilon(t)|| \leq \epsilon,\ \forall t \in [0.T].$$

Clearly, if $x_\epsilon(\cdot)$ remains in E for each $\epsilon > 0$, so must $x(\cdot)$ (by the closure of E). Therefore, we assume that $x(\cdot)$ is C^1. Let

$$\tilde{F}(t, y) = \{1\} \times \{v \in F(y) : ||v - \dot{x}(t)|| \leq K||y - x(t)||\}.$$

Noticing that

$$N^P_{\mathbb{R}\times E}(t, y) = \{0\} \times N_E^P(y)$$

and

$$0 \leq h_F(y, -\xi) \leq -h_{F_\epsilon}((t, y), (0, \xi))$$

for any $\xi \in N_E^P(y)$, we conclude that the pair $(\mathbb{R} \times E, \tilde{F})$ is weakly invariant by Proposition 2.14. Which means that there exists a trajectory of $\tilde{F}$ which originates at $(0, x_0)$ which has the form $(t, y(t))$. We note that

$$\frac{d}{dt}||x(t) - y(t)|| \leq K||x(t) - y(t)||$$

And, as $x(0) = y(0)$, we conclude, by Gronwall's inequality, that $x(t) = y(t)$ for all $t > 0$. $\square$

2.3 The Mayer Problem

The Mayer problem associated with an autonomous differential inclusion takes the following form

Problem 2.20. Minimize $g(x(T))$ such that $\dot{x}(t) \in F(x(t))$ *a.e.* $t \in [0, T]$ and $x(0) = x_0$.

Here $T > 0$ and a continuous $g : \mathbb{R}^n \to \mathbb{R}$ are given. As mentioned in Chapter 1, this recalls the Mayer problem in the classical calculus of variations. In the previous chapter, some of the different forms a cost functional might take in an optimal control problem were presented. Specifically, we looked at the cost function for the Lagrange problem, which takes the form for a differential inclusion formulation:

$$\int_0^T \ell(t, x(t), \dot{x}(t))dt.$$

This form of the functional corresponds to a differential inclusion form of the optimal control problem; specifically, the Lagrangian ℓ does not depend directly on a control variable u. In order to see that the Mayer framework covers the Lagrange problem, we simply augment our dynamics by appending to our state variable $x_{n+1} = r$ which is governed by the dynamics

$$\dot{x}_{n+1}(t) = \ell(t, x(t), \dot{x}(t)).$$

Then we can formulate the Laplace problem as minimizing the functional

$$g(x(t)) := r(t)$$

for $t = T$. In a similar manner, we can see that the Bolza problem, with cost functional written as

$$\int_0^T \ell(t, x(t), u(t))dt + g_B(x(T))$$

also can be written as a Mayer problem. Thus, we will restrict our attention to the Mayer problem. We next define the *value function* as the function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that $V(t, x)$

is the minimal value of $g(x(T))$ where $x(\cdot)$ satisfies the differential inclusion in Problem 2.20 and $x(t) = x$. In other words, $V(t,x)$ is the optimal cost attainable with initial time and state (t,x). The goal of the remainder of this chapter is to characterize $V(t,x_0)$. Specifically, we wish to prove the following result.

Theorem 2.21. There is a unique continuous function $\phi : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ such that for every $(t,x) \in (-\infty,T) \times \mathbb{R}^n$, and $(\theta,\zeta) \in \partial_P \phi(t,x)$

$$\theta + h_F(x,\zeta) = 0, g(x) = \phi(T,x),\ x \in \mathbb{R}^n.$$

This function is the value function V associate with the Mayer problem.

Before doing so, we prove a few preliminary results. First, we note the continuity of the value function. Suppose that the value function is finite at (t_0,x_0). This means that there is an optimal trajectory $x^*(\cdot)$ with $x^*(t_0) = x_0$. We know by the continuity of g that there is some neighborhood around $x^*(T)$ such that g on this neighborhood stays close to $g(x^*(T))$. We also know by the continuity of F and the compactness of trajectories that there is some neighborhood of (t_0,x_0) such that there is a trajectory starting at every point in this neighborhood which is near $x^*(\cdot)$ in the sup norm. Therefore, we know that there is some neighborhood, O around (t_0,x_0) such that for any (τ,y) there is some trajectory $x(\cdot)$ with $x(\tau) = y$, and $|g(x(T)) - g(x^*(T))| < \epsilon$ for some prescribed $\epsilon > 0$. This in turn implies that $V(\tau,y) \le V(t_0,y_0) + \epsilon$. A similar argument will give us that there is a (possibly smaller) neighbhorhood around (t_0,y_0) such that $V(t_0,y_0) < V(\tau,y) + \epsilon$ which establishes:

Proposition 2.22. Let $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be the value function for the Mayer problem with continuous cost function g and Lipschitz dynamics F. Then V is continuous.

The next two results we prove require the following definitions.

Definition 2.23. Let F be a multifunction and $\phi : \Omega \to \mathbb{R}$ be lower semicontinuous. Then (ϕ, F) is *weakly decreasing* on Ω if for any $\alpha \subset \Omega$ with $\phi(\alpha) < \infty$, there exists a trajectory

$x(\cdot)$ of F with $x(0) = \alpha$ such that for any interval $[0, T]$ where $x([0, T]) \subset \Omega$,

$$\phi(x(t)) \leq u(\alpha), \ \forall t \in [0, T].$$

Definition 2.24. Let F be a multifunction and $\phi : \Omega \to \mathbb{R}$ be lower semicontinuous. Then (ϕ, F) is *strongly increasing* on $(t_0, t_1) \times \Omega$ if for any trajectory on any interval $[a, b]$ where $x([a, b]) \subset \Omega$,

$$\phi(b, x(b)) \geq \phi(t, x(t)), \ \forall t \in [a, b].$$

Strongly increasing and weakly decreasing systems can be characterized in terms of the lower hamiltonian as follows.

Lemma 2.25. Let $\phi : (t_0, t_1) \times \Omega \to \mathbb{R}$ be lower semicontinuous and F satisfy (H-1), (H-2), and be upper semicontinuous. Then (ϕ, F) is weakly decreasing on $(t_0, t_1) \times \Omega$, if and only if

$$\theta + h_F(x, \zeta) \leq 0, \ \forall (\theta, \zeta) \in \partial_P \phi(t, x), \ \forall (t, x) \in (t_0, t_1) \times \Omega$$

Proof. ($\Rightarrow$) Assume that (ϕ, F) is weakly decreasing and let $x_0 \in \Omega$. Then define, for $\delta > 0$ such that $\overline{B}(x_0, \delta) \subset \Omega$,

$$S := \{(t, x, r) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} : (t, x) \in (t_0, t_1) \times \overline{B}(x_0, \delta), \phi(t, x) \leq r\}.$$

then if $(\theta, \zeta) \in \partial_P \phi(\tau, \alpha)$, for $(\tau, \alpha) \in (t_0, t_1) \times \Omega$, we can see that $(\theta, \zeta, -1) \in N_S^P(\tau, \alpha, \phi(\tau, \alpha))$. We define

$$\overline{F}(1, x, r) := \begin{cases} \{1\} \times F(x) \times \{0\}, & x \in B(x_0, \delta) \\ \{1\} \times \overline{\text{co}}\{\bigcup_{||y-\alpha||=\delta} F(y) \cup \{0\}\} \times \{0\}, & x \notin B(x_0, \delta). \end{cases}$$

Then $\overline{F}$ satisfies (H-1), (H-2), and is upper semicontinuous. Furthermore, $(S, \overline{F})$ is weakly invariant. Thus there is some $(1, v, 0) \in \overline{F}(t, \alpha, u(\alpha))$ such that

$$\langle (1, v, 0), (\theta, \zeta, -1) = \theta + \langle (v, 0), (\zeta, -1) \rangle \leq 0.$$

This implies that $h_{\overline{F}}(x, (\theta, \zeta)) \leq 0$ for all $(\theta, \zeta) \in \partial_P \phi(t, x)$ on $(t_0, t_1) \times \Omega$ which in turn means that $\theta + h_F(x, \zeta) \leq 0$ as claimed. ($\Leftarrow$) See the proof of Theorem 6.2 of [CLSW98]. $\square$

As we might expect, a Hamiltonian characterization of the strongly increasing property is available.

Lemma 2.26. Suppose that ϕ and F are as in Proposition 2.25. Also, let F be Lipschitz. Then (ϕ, F) is strongly increasing on $(t_0, t_1) \times \Omega$ if and only if

$$\theta + h(x, \zeta) \geq 0,\ \forall(\theta, \zeta) \in \partial_P \phi(t, x),\ \forall(t, x) \in (t_0, t_1) \times \Omega$$

Proof. See Proposition 6.5 of Chapter 4 in [CLSW98]. □

With these two preliminary results, we can now prove Theorem 2.21.

Proof. (of Theorem 2.21) Suppose that $V(t, x)$ is finite for some (t, x). This implies that there is an optimal arc $x^*(\cdot)$ originating at (t, x). Along $x^*(\cdot)$, V is constant, as the same optimal path will be used. In other words, $V(\tau, x^*(\tau))$ is constant on the interval $[t, T]$. This implies that (V, F) is weakly decreasing and so with the notation

$$\overline{h}_F(x, \theta, \zeta) := \theta + h_F(x, \zeta)$$

for the augmented Hamiltonian,

$$\overline{h}_F(x, \partial_P V(t, x)) \leq 0,\ \forall(t, x) \in (-\infty, T) \times \mathbb{R}^n.$$

Suppose that $t' > \tau$ and $x(\cdot)$ is a trajectory. Then, using the reasoning associated with *the principle of optimality*, $V(t', x(t')) \geq V(\tau, x(\tau))$ because at $(\tau, x(\tau))$ we are "more free" to select a different path with a better terminal cost. Therefore, along any trajectory, $V(t, x(t))$ is increasing as a function of time, meaning that (V, F) is strongly increasing. By Lemma 2.26, we have that

$$\theta + h(x, \zeta) \geq 0, \forall(\theta, \zeta) \in \partial_P V(t, x), \forall(t, x) \in (-\infty, T) \times \mathbb{R}^n$$

Also, we note that it is clear that V is such that $g(x) = V(T, x)$. We now turn to the question of uniqueness.

Let $\phi(t, x)$ be another continuous function satisfying the proximal Hamilton Jacobi equation. We first show that $V \geq \phi$. Let (t, y) be a point with $t < T$, then there exists an optimal trajectory $\overline{x}(\cdot)$ for the Mayer problem with initial time t and state y. Because $t \mapsto \phi(t, x(t))$ increases for every trajectory $x(t)$ (see above), we know that $\phi(T, \overline{x}(T)) \geq \phi(t, y)$ and since

$$\phi(T, \overline{x}(T)) = g(x(T) = V(t, y),$$

we conclude that $V \geq \phi$.

To show that $\phi \geq V$, we again let (t, y) be a point with $t < T$ and note that there is a trajectory $x(t)$ with $x(t) = y$ such that

$$\phi(\tau, x(\tau)0 \leq \phi(t, y), \ \tau \in [t, T).$$

If $\tau \uparrow T$, we see that $g(x(T)) = \phi(T, x(T)) \leq \phi(t, y)$ and so, as $V(t, y) \leq g(x(T))$, we conclude that $\phi \geq V$. This completes the proof. $\square$

Chapter 3

Problems on Stratified Domains

In this chapter, we will introduce problems on stratified domains. In Chapter 2, we looked at problems where the dynamical systems are sufficiently regular. Here, the same regularity properties will not hold. Specifically, the dynamics are not globally, or even locally, Lipschitz. Throughout, we shall consider autonomous systems. In Section 3.1, we introduce the structure of dynamics over stratified domains. In [BH07], this problem was introduced and an attempt was made to characterize the value function for such problems as the solution to the Hamilton-Jacobi equation in a modifiication of the viscosity sense. In Section 3.2, we present the work of Bressan and Hong and provide a counterexample to show that their main result is incorrect.

3.1 Control on Stratified Domains

3.1.1 Control System Formulation

We will begin with a *stratification* of $\mathbb{R}^n$, $\{\Gamma_1, \ldots, \Gamma_N\}$, which is a collection of disjoint embedded submanifolds such that

1.
$$\mathbb{R}^n = \bigcup_{i=1}^{N} \Gamma_i,$$

2. if $\Gamma_i \cap \overline{\Gamma_j} \neq \emptyset$ then $\Gamma_i \subset \overline{\Gamma}_j$,

3. each $\overline{\Gamma}_i$ is proximally smooth of radius δ_i; that is, the distance function

$$d_{\overline{\Gamma}_i}(x) = \inf_{y \in \overline{\Gamma}_i} ||x - y||$$

is differentiable on the set $\{y : 0 < d_{\Gamma_i}(y) < \delta_i\}$,

4. for each i, the interior of $\overline{\Gamma_i}$ is equal to Γ_i.

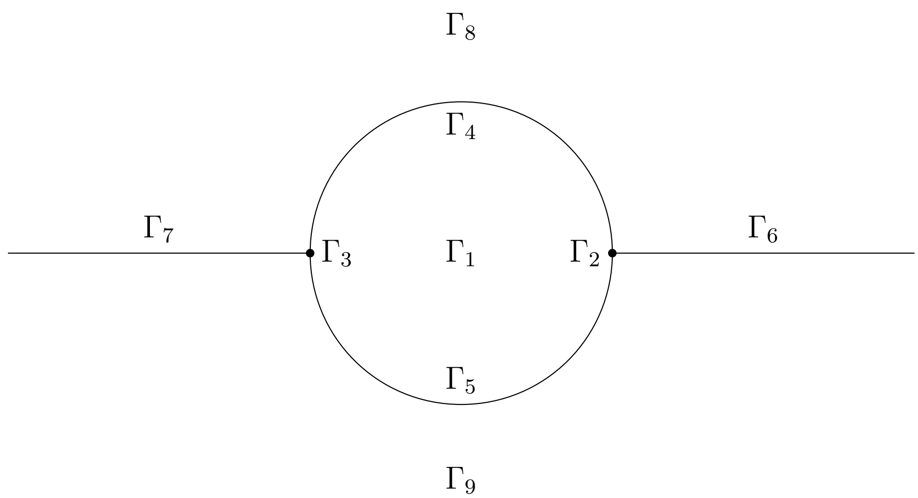


Figure 3.1: An allowable stratification

Such a stratification can be seen in Figure 3.1. There are alternative ways of viewing the proximal smoothness requirement. Geometrically, this requirement is equivalent to what might be called an *external sphere condition*; that is, for every Γ_i, there exists a $\delta_i > 0$ such that for every $x \in \partial\Gamma_i$, there is a ball of radius δ_i which intersects $\overline{\Gamma}_i$ at only x. A consequence of requiring that each Γ_i be proximally smooth, which we shall make use of later, is that the proximal normal cone $N^P_{\overline{\Gamma}_i}(x)$ at any point x on the boundary of Γ_i will be nontrivial. In [CSW95], there is a much more extensive discussion of proximal smoothness, including proofs that the proximal normal cone will be nontrivial. In Figure 3.2, we see a stratification

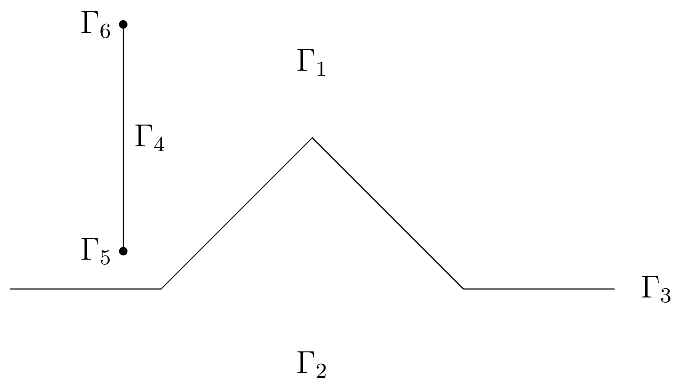


Figure 3.2: A stratification which is not allowable

which violates our above requirements: Γ_4 violates our requirement on the interior of $\overline{\Gamma}_1$ and $\overline{\Gamma}_1$ is not proximally smooth because it has a reentrant corner. We should note that the formulation of this problem in [BH07] did not assume the last two hypotheses on the stratification.

We first present the control system version of this problem. Associated with this stratification, we have a collection of convex, compact sets U_i and functions $f_i : \Gamma_i \times U_i \to \mathbb{R}^n$. We will assume that for each i, f_i is Lipschitz with respect to x with constant k_i and that for any $u \in U_i$ and $x \in \Gamma_i$, the vector $f_i(x,u) \in \mathcal{T}_{\Gamma_i}(x)$ where $\mathcal{T}_{\Gamma_i}(x)$ is the tangent space of Γ_i at x. Finally, we assume for each f_i, that there is some $M_i, c_i \geq 0$ such that at each $x \in \Gamma_i$

$$|f_i(x,u)| \leq M_i|x| + c_i, \ \forall u \in U_i.$$

As in [BH07], we will use the notation that $i(x)$ is the index i such that $x \in \Gamma_i$. For a given trajectory $x(\cdot)$ of $f(x,u)$ on $[0,T]$, we will define its *switching times* $\{t_\alpha\} \subset [0,T]$ as times for which there is no i such that for some $\delta > 0$, we have $x(t) \in \Gamma_i$ for all $t \in [t_\alpha - \delta, t_\alpha + \delta]$. We note that, as mentioned before, f is not locally Lipschitz.

3.1.2 The Differential Inclusion Formulation

As in the previous chapter, we will use differential inclusions for our analysis. Specifically, we assume there is a finite collection of multifunctions

$$F_i : \Gamma_i \rightrightarrows \mathbb{R}^n$$

such that $F_i(x) \subset \mathcal{T}_{\Gamma_i}(x)$. We assume that each $F_i(x)$ is nonempty, compact, and convex for each x and that F_i satisfies a linear growth condition and that on each Γ_i, F_i is Lipschitz continuous.We let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be such that

$$F|_{\Gamma_i} \equiv F_i.$$

We note that the definition of switching times for trajectories of the control system can also be used for trajectories of the differential inclusion. Because F is not in general Lipschitz continuous or even upper-semicontinuous, we will in later sections use multifunctions related to F in our analysis. Before we do, we consider a few motivating examples for this type of dynamics.

The first is a rather general example introduced, in a slightly different form, in [BH07]. The second is similar to the bouncing ball problem often used in the study of hybrid systems (see [CTG08].) The third problem is similar to an optics problem in [CV89]. The examples give different possible viewpoints to the intuition behind stratified domains. The most immediate interpretations of stratifications are viewing the lower dimension manifolds as highways where the possible velocities are much greater or much smaller in magnitude than nearby regions, or that they are interfaces between media with very different dynamics, or some combination of these two notions.

Example 3.1. Consider a stratification as seen in 3.3. We can think of $\Gamma_5, \Gamma_6, \Gamma_7, \Gamma_8, \Gamma_9$, and Γ_{10} as "highways" where $F_i(x) = \{v \in \mathcal{T}_{\Gamma_i}(x) : |v| \leq c_i\}$ for c_i is significantly higher for $i = 5, 6, 7, 8, 9, 10$ than the c_i for the 2-dimensional submanifolds.

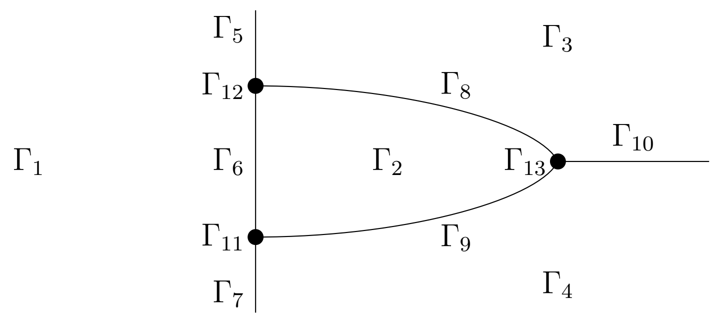


Figure 3.3: Boundary submanifolds as "highways"

Example 3.2. Consider a homogeneous ball of radius r with mass m. Let x_1 be the position of the center of the ball. Then we can use the following dynamical system to describe the ball falling from one fluid into another fluid. Define

$$\Gamma_1 = \{x \in \mathbb{R}^2 | x_1 > r\}$$

$$\Gamma_2 = \{x \in \mathbb{R}^2 | x_1 = r\}$$

$$\Gamma_3 = \{x \in \mathbb{R}^2 | x_1 < r\}$$

$$F_1(x) = (x_2, -mg + m\rho_1 \frac{4\pi r^3}{3})$$

$$F_2(x) = (0, -mg)$$

$$F_3(x) = (x_2, m(-g + \rho_3 \Upsilon(x_1)))$$

where ρ_1 is the density of the top fluid, ρ_3 is the density of the bottom fluid and $\Upsilon(y) : \mathbb{R} \to \mathbb{R}$ is the volume of the ball contained in Γ_3 when the center is at height y; that is,

$$\Upsilon(y) = \begin{cases} \pi(r-y)^2(r - \frac{(r-y)}{3}) & y \in [0, r) \\ \frac{4\pi r^3}{3} - \pi(r+y)^2(r - \frac{(r+y)}{3}) & y \in [-r, 0) \\ \frac{4\pi r^3}{3} & y \in (-\infty, r). \end{cases}$$

Clearly, this system is not locally Lipschitz on the x_1 axis, but is Lipschitz when restricted to the individual submanifolds.

Example 3.3. It is well established that Fermat's principle–that light travels between two points x_0, x_1 in a path in minimal time–can be used to derive Snell's law of refraction/reflection. In [CV89], Clarke and Vinter use optimal multiprocesses and a maximum principle, to give a such a derivation. To do this, they make two assumptions. First, they require that the light passes through two media in a prescribed order; this is of course trivial when only two media are involved, but their formulation can easily be extended to multiple media as long as it is known the order of the media the light will travel through. Second, they require only that the surface between the two media is closed. Here, the dependent variable upon which trajectories depended upon was not time t, but arclength τ, and the relevant differential inclusion and cost function were

$$\dot{x}(\tau) \in \overline{B}, \; x(0) = x_0 \tag{3.1}$$

$$\int_0^{\tau^*} n_1 d\tau + \int_{\tau^*}^{T} n_2 d\tau$$

where τ^* is the arclength where the path passes through the interface and n_1, n_2 are the two refractive indices.

Using stratified domains, we can describe a related situation. Here, we will assume that there is a collection of 2-dimensional media Ω_i and boundary interfaces which satisfy the hypotheses in 3.1.1 with non-identical optical densities/refractive indices n_i. We define $F(x) = \frac{c}{n_{i(x)}}\overline{B}$ for $x \in \Omega_{i(x)}$ where c is the speed of light in a vacuum. For $x \notin \bigcup_i \Omega_i$, we define $F(x) \equiv 0$. Then Fermat's principle can be seen as stating that the path of light is the optimal trajectory solving the minimal time problem (to be discussed in detail in Section 5.1) for the target set $S = \{x_1\}$. This problem does not allow for the same types of boundaries allowed in the optics problem of [CV89], but it makes no prior assumption on the light's path, in regards to which media are entered and in what order.

3.2 A Previous Attempt at Hamilton-Jacobi Theory

In [BH07], Bressan and Hong studied the infinite horizon optimal control problem over stratified domains; specifically, these take the following form using a control formulation.

Problem 3.4. Let $f(x)$ be as in Section 3.1.1. We define

$$J(x_0, u(\cdot)) = \int_0^\infty e^{-\beta t}\ell(x(t), u(t))dt$$

where $\ell(x, u) = \ell_i(x, u)$ for $x \in \Gamma_i$; we assume $\ell_i(x, u)$ is continuous and nonnegative for each i. The problem, then, is to minimize $J(x_0, u(\cdot))$ for all functions $u(t)$ where $x(t)$ satisfies

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0.$$

They then use a related differential inclusion for their analysis. As noted above, $F := f(x, U)$ is not sufficiently regular for standard optimality conditions to hold, and so they instead work with the following, related, multifunction.

$$G(x) = \bigcap_{\epsilon>0} \overline{\text{co}}\{v \in F(\tilde{x}) : \ |x - \tilde{x}| < \epsilon\} \tag{3.2}$$

This is the the smallest upper-semicontinuous, convex-valued multifunction containing $F(x)$ (see [AC84]). We will assume that

$$G(x) \cap T_{\Gamma_i}(x) = F_i(x). \tag{H-3}$$

With this hypothesis, trajectories of G and F coincide (see Theorem 1 of [BH07] for an explanation of this). Bressan and Hong proved the following result.

Proposition 3.5. Consider the minimization problem 3.4 with the dynamics given by F and the assumption (H-3). If there exists an $u(\cdot)$ such that $J(x_0, u(\cdot)) < \infty$, then there exists an optimal solution to the problem.

They then introduced a modification to the classical viscosity solution sense and attempted to prove that the value function

$$V(x) := \inf_{\{u(\cdot):u(t)\in U\}} J(x,u(\cdot))$$

is a viscosity solution–in a new sense–of the associated Hamilton-Jacobi equation. In particular, this new sense of viscosity solutions involves a modification of the definition of a lower viscosity solution. Their definition of a lower solution is as follows.

Definition 3.6. A continuous function W is a *lower viscosity solution* if the following holds. For $\overline{x} \in \Gamma_i$, if a C^1 function ϕ is such that $(W-\phi)|_{\Gamma_i}$ has a local maximum at $\overline{x}$, then the following holds:

$$\beta W(\overline{x}) + \sup_{(y,\eta)\in\tilde{G}(\overline{x})} \{-y\cdot D\phi(\overline{x}) - \eta\} \leq 0.$$

Here $\tilde{G}$, defined by $\tilde{G}(x) = \overline{\text{co}}\{(w,\eta)|w \in F_{i(x)}(x),\ \eta \geq \ell_{i(x)}(x,u), w = f_{i(x)}(x,u)\}$, is the standard augmented differential inclusion associated with G.

The motivation for this seems natural in light of the following lemma.

Lemma 3.7. For each submanifold Γ_i, the function $G(x)|_{\Gamma_i}$ is Lipschitz with constant $k = \frac{\tilde{k}}{3(N+1)}$ where k_i is the Lipschitz constant for F_i, $\tilde{k} = max\{k_i\}$, and N is the number of submanifolds.

Proof. Let $x, \tilde{x} \in \Gamma_i$ and $v \in G(x)$. For a given $\epsilon > 0$, we define

$$G_\epsilon(x) := \overline{\text{co}}\{F_i(z) : z \in x + \epsilon B\}.$$

Clearly, $v \in G_\epsilon(x)$. Suppose that $\epsilon \leq |x - \tilde{x}|$ and that ϵ is sufficiently small enough that $x + \epsilon B$ and $\tilde{x} + \epsilon B$ intersect the same set of manifolds $\{\Gamma_j\}$. The latter is possible due to the assumption that if $x \in \overline{\Gamma}_k$, then $\Gamma_i \subset \overline{\Gamma}_k$. Then, by Carathéodory's Theorem (Section 17 of [Roc97]), we have $x_j \in x + \epsilon B$ and $\lambda_j > 0$ such that $\sum_{j=1}^{n+1} \lambda_j = 1$ and $v = \sum_{j=1}^{n+1} \lambda_j v_j$ where

$v_j \in F(x_j)$. Then let $w_j \in F_j(\tilde{x}_j)$ where $\tilde{x}_j \in (\tilde{x} + \epsilon B) \cap \Gamma_{i(x_j)}$ and let $w = \sum_{j=1}^{n+1} \lambda_j w_j$. Then $w \in G_\epsilon(\tilde{x})$. We then have the following inequality

$$\begin{aligned}
||v - w|| &= ||\sum_{j=1}^{N+1} \lambda_j (v_j - w_j)|| \\
&\leq \sum_{j=1}^{N+1} \lambda_j ||v_j - w_j|| \leq \sum_{j=1}^{N+1} \lambda_j k_j ||x_j - \tilde{x}_j|| \\
&\leq \sum_{j=1}^{N+1} \lambda_j k_j (||x_j - x|| + ||x - \tilde{x}|| + ||\tilde{x} - \tilde{x}_j||) \\
&\leq \tilde{k} \sum_{j=1}^{N+1} \lambda_j (2\epsilon + ||x - \tilde{x}||). \\
&\leq 3\tilde{k}(N+1)||x - \tilde{x}||
\end{aligned}$$

Note that ϵ can be arbitrary as long as it is sufficiently small, and $G(\tilde{x})$ is nonempty. Thus, we have the desired result. □

Because the regularization is Lipschitz when restricted to any of the Γ_i's, it seems reasonable that criteria should only depend on the behavior of G restricted to the individual submanifolds. With this new definition, Bressan and Hong attempted to show that the value function is a lower viscosity solution. There is an error, however, in the proof of Proposition 1 in [BH07] which claims that the value function is a viscosity solution as described above. Specifically, it is assumed there that for an optimal trajectory of G, there is at every point, a control associated with this trajectory. As we noted in Section 2.1, when taking regularizations of multifunctions obtained from the control formulation of a system, optimal trajectories may not have, at a given time, a corresponding control that provides the derivative of the trajectory. Here, even though the trajectories of F and G coincide, there generally exist $v \in G(x) \setminus F(x)$ which may provide the velocity of a trajectory. Clearly, for such a v there is no control which will give this velocity at x. But does this mean that we can still establish that the value function is a solution in this new sense?

As seen below, G does not have the standard property of Lipschitz multifunctions that

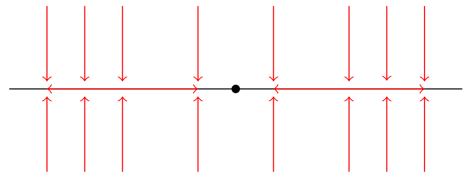


Figure 3.4: The dynamics in the counterexample

every $v \in G(y)$ is such that a trajectory exists originating at y with initial velocity v. This is partially because G is not sufficiently regular on arbitrary compact sets. Thus, $G(x)$ is the incorrect multifunction to consider as it does not contain all of the possible velocities of trajectories of $F(x)$. As mentioned before, Lipschitz continuity on compact sets of the multifunction is often assumed in order to obtain results such as Proposition 2.3 in [WZ98]; that is, such multifunctions have the property that every element may be used to generate a trajectory and that C^1 trajectories are dense. But G does not have this property. To illustrate this, we turn to the following counterexample to Proposition 1 of [BH07]. Specifically, it shows that the value function is not a lower solution in the sense described above.

Example 3.8. Let $X = \mathbb{R}^2$ and

$$\begin{aligned} \Gamma_1 &= \{(x_1, x_2) : x_2 > 0\} \\ \Gamma_2 &= \{(x_1, x_2) : x_2 = 0\} \\ \Gamma_3 &= \{(x_1, x_2) : x_2 < 0\} \end{aligned}$$

with dynamics

$$\begin{aligned} f_1(x, \alpha) &= (0, -\alpha) \\ f_2(x, \alpha) &= (\alpha, 0) \\ f_3(x, \alpha) &= (0, \alpha) \end{aligned}$$

where the control spaces are $A_1 = A_3 = \{u \in [0, 1]\}$ and $A_2 = \{u \in [-1, 1]\}$. This is shown in Figure 3.8.

We will consider the weighted minimal time problem, as in [BH07] where our target set is the origin. That is,

$$J(x,u) := \int_0^\infty e^{-\beta t}\ell(x(t),u(t))dt$$

where

$$\ell(x,\cdot) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

where $\beta > 0$. Then the value function takes the form

$$V(\overline{x}) = \frac{1 - e^{-\beta(|\overline{x}_1| + |\overline{x}_2|)}}{\beta}.$$

Using the definitions in [BH07], we define the set-valued function

$$E(x) = \begin{cases} [1,\infty) & \text{if } x \neq 0 \\ [0,\infty) & \text{if } x = 0 \end{cases}$$

We can then get the following

$$\widehat{F}(x) = \{(v,\eta) : v \in F_{i(x)} \text{ and } \eta \in E(x)\}$$

and

$$\tilde{G}(x) = \begin{cases} \widehat{F}(x) & \text{if } x \notin \Gamma_2 \\ \overline{B}_{\ell^1} \times E(x) & \text{if } x \in \Gamma_2 \end{cases}$$

where $\overline{B}_{\ell^1}$ is the closed unit ℓ^1 ball centered at the origin. Clearly, $\tilde{G}(x)$ satisfies the hypotheses in [BH07]. If we define

$$\phi(x) = (x_1 - (1 - \frac{e^{-2}}{2}))^2 + 10x_2$$

and set $\beta = 2$ then $V - \phi|_{\Gamma_2}$ attains a local maximum at $\overline{x} = (1,0)$. If we let $(y,\eta) = ((0,-1),1)$, then

$$\begin{aligned} y \cdot D\phi(\overline{x}) - \eta &= -[(0,-1)\cdot(-2(x_1+1),10)] - 1 \\ &= -1 + 10 = 9 \end{aligned}$$

while $-\beta V(\overline{x}) = e^{-2} - 1$. This means that

$$\sup_{(y,\eta)\in\tilde{G}(\overline{x})} \{y \cdot D\phi(\overline{x}) - \eta\} \geq \beta V(\overline{x}).$$

Thus, the value function fails to be a lower solution.

Therefore, the error in [BH07] is not simply one of a mistake in the proof; the desired result–that the value function is a solution of the Hamilton-Jacobi equation associated with the dynamics given by G– is false. We note that the counterexample relies on an element of G which cannot be the velocity of a trajectory: no trajectory can move from Γ_2 into Γ_1 or Γ_3 and so at $(1,0)$, no trajectory can use $(0,1)$ as a velocity. In fact, although (Γ_2, G) is strongly invariant, as just described, at any point $x \in \Gamma_2$, we have $(0,1) \in N^P_{\Gamma_2}(x)$, and so the Hamiltonian condition for strong invariance (that is, Lemma 2.19 of Chapter 2) is not met. The connection between Hamiltonian conditions for invariance and the value function as a solution to the Hamilton-Jacobi equation suggests that G is not the appropriate multifunction for our purposes.

Chapter 4

Invariance on Stratified Domains

In light of the error discussed in Chapter 3, in this chapter we turn to the question of invariance for problems on stratified domains, recalling that obtaining a Hamiltonian characterization of invariance led to the relevant Hamilton-Jacobi characterization of the value function in Chapter 2. In order to do this, we introduce a new multifunction in Section 4.1 which we show possesses several useful properties similar to the properties of Lipschitz multifunctions. After this, we show that the Hamiltonian of this new multifunction provides the proper criteria for weak and strong invariance. Throughout, we assume that there is a stratification $\{\Gamma_i\}$ with associated dynamics F_i as in Chapter 3. Also, F and G are defined as in that chapter.

4.1 An Alternative Multifunction

As in [BH07], we first introduce a multifunction related to F and G. In this section, we construct this multifunction, which we call $G^\sharp$. While not having in general any of the standard properties assumed in Chapter 2, $G^\sharp$ does possess certain useful properties similar to those of a Lipschitz multifunction. First, we define $F_i^\sharp$ on each Γ_i. as an extension of F_i to the boundary of Γ_i which is restricted in the following way: Let $\tilde{F}_i(x)$ be the continuous extension of $F_i(x)$ to $\partial\Gamma_i$ for each i (we let $\tilde{F}_i(x) \equiv \emptyset$ for $x \notin \overline{\Gamma}_i$); this is well-defined because of our requirement that the stratification has the property that the interior of the closure of

Γ_i is equal to Γ_i. Then define

$$F_i^\sharp(x) = \begin{cases} F_i(x) & \text{if } x \in \Gamma_i \\ \{v \in \tilde{F}_i(x) | \langle v, \zeta \rangle \leq 0, \ \forall \zeta \in N^P_{\overline{\Gamma}_i}(x)\} & \text{if } x \in \partial\Gamma_i \end{cases}, \tag{4.1}$$

which is a well defined and nontrivial restriction because, by proximal smoothness (as proved in [CLSW95]), $N^P_{\overline{\Gamma}_i}(x) \neq \{0\}$ for any $x \in \partial(\overline{\Gamma}_i)$. We then define

$$G^\sharp(x) := \bigcup_{i=1}^n F_i^\sharp(x).$$

Similar to (H-3), we make the following assumption on $G^\sharp$:

$$G^\sharp(x) \cap T_{\Gamma_i}(x) = F_i(x). \tag{H-4}$$

We note that for x such that $\Gamma_{i(x)}$ is n-dimensional, $G^\sharp(x) = G(x) = F(x)$. Also, for any x, $G^\sharp(x) \subset G(x)$; informally, $G^\sharp$ filters out certain elements from G at each point on the boundary manifold. We define $G^\sharp$ without reference to G because for v to be an element of $G^\sharp(x)$, we require it satisfy a criterion dependent on which F_i it is obtained from.

The multifunction $G^\sharp$ is, unlike G, not generally convex-valued or upper semicontinuous. It is compact, nonempty, and at every point the finite union of convex sets. It does not satisfy the standing hypotheses from Chapter 2. However, some useful properties of $G^\sharp$ are seen in the following propositions. The first is similar to Proposition 2.18 in Chapter 2.

Proposition 4.1. For all $\overline{x}$, and $v \in G^\sharp(\overline{x})$ there is an i and a $\delta > 0$ such that there is a C^1 trajectory $x(\cdot)$ originating at $\overline{x}$, $\dot{x}(0) = v$, and $x(t) \in \Gamma_i$ on the interval $(0, \delta)$.

Proof. This is well known if $\overline{x} \in \Gamma_i$ and $v \in F_i^\sharp(\overline{x})$ (see Proposition 2.3(a) of [WZ98]). Suppose that $\overline{x} \in \partial\Gamma_i$ and assume $v \in F_i^\sharp(\overline{x})$. Let $\{x_j\}$ be such that $x_j \to_v \overline{x}$ and $x_j \in \Gamma_i$. Let S be a compact set containing x_j for all j and $\overline{x}$ with $S \subset \overline{\Gamma}_i$ and $S \cap \partial\Gamma_i = \{\overline{x}\}$. Then, as seen in Proposition 2.3 of [WZ98], there is a time δ associated with S such that for each

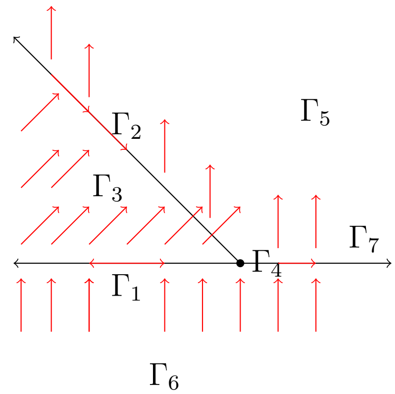


Figure 4.1: The time δ in Proposition 4.1 depends on location of x_0

x_j, we can find a C^1 trajectory $y_j(t)$ on interval $[0, \delta]$ starting at x_j with initial velocity $\dot{y}_j(0) = \text{proj}_{F(\overline{x}+tv)}v$. Then we can define the arc

$$x_j(t) := \begin{cases} \overline{x} + tv & \text{for } t \in [0, \frac{|x_j - \overline{x}|}{j}) \\ y_j(t - \frac{\epsilon}{j}) & \text{for } t \geq \frac{|x_j - \overline{x}|}{j} \end{cases} \tag{4.2}$$

By a result of Fillipov [AC84] for Lipschitz dynamics, $x_j(\cdot)$ converges to a trajectory $x(\cdot)$ with

$$\dot{x}_j(0) \to v.$$

This concludes the proof. □

The time δ in Proposition 4.1 is dependent in some cases on the initial point $\overline{x}$. For instance, suppose we have $\Gamma_1 = \{(x,y)|x < 1,\ y = 0\}$, Γ_2 the ray originating at (1,0) and passing through $(0,1)$ but not including $(1,0)$, Γ_3 the wedge lying between the two, $\Gamma_4 = \{(1,0)\}$, $\Gamma_6 = \{x_1 < 0\}$, $\Gamma_7 = \{(x_1, 0) : x_1 > 1\}$ and Γ_5 the rest of $\mathbb{R}^2$ with $F_1 = [-1,1] \times \{0\}$, $F_2 = [-1,-.5] \times \{1\}$, $F_3 = \{1\} \times \{1\}$, $F_5 = F_6\{(0,1)\}$ and $F_7 = \{(1/2, 0)\}$(see Figure 4.1). Then, for $\overline{x} \in \Gamma_1$, $(1,1) \in G^\sharp(\overline{x})$ but the time δ is clearly dependent on the distance between $\overline{x}$ and $(1,0)$. However, the time δ merely gives a limit on how long the trajectory is C^1; here we can continue the trajectory past time δ.

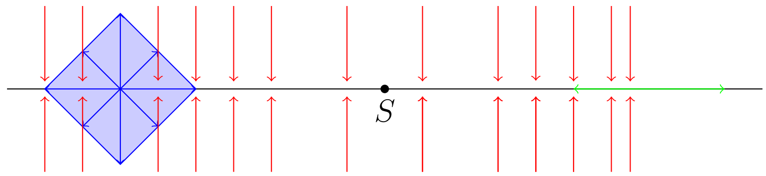


Figure 4.2: A comparison of G (on left in blue) and $G^\sharp$ (on right in green) along Γ_2 in Example 3.8

Proposition 4.2. Let $x(\cdot)$ be a trajectory with $x(0) \in \overline{\Gamma_i}$ such that for $x(t) \in \overline{\Gamma_i}$ for small time t. Then, if it exists, $\dot{x}(0) \in G^\sharp(x(0))$.

Proof. Note that if the trajectory originates in Γ_i then for small time, $x(t) \in \Gamma_i$; then, clearly, $\dot{x}(0) \in F_i(x(0)) \in G^\sharp(x(0))$ if it exists. Now assume $x(t)$ is a trajectory originating at $\overline{x} \in \partial\Gamma_i$ with $\dot{x}(0) = v$ and $x(t) \in \Gamma_i$ for small $t > 0$. Let $\zeta \in N^P_{\overline{\Gamma}_i}(\overline{x})$. Then for any $t \in (0, \delta)$ for sufficiently small δ we have

$$\langle \zeta, x(t) - x(0) \rangle < 0 \tag{4.3}$$

which implies that

$$\langle \zeta, \frac{x(t) - x(0)}{t} \rangle < 0.$$

Taking the limsup as $t \to 0$, we have $\dot{x}(0) \in F_i^\sharp(\overline{x})$ if it exists. Finally, if $x(0) \in \partial\Gamma_i$ and for small time $x(t) \in \partial\Gamma_i$ then, by (H-4), one of the above cases holds for some $\Gamma_j \subset \overline{\Gamma_i}$. □

These two results suggest, in some sense, that $G^\sharp$ contains only elements which can be used as trajectories. Also, anywhere the derivative of a particular trajectory exists, it is an element of $G^\sharp$. This suggests that $G^\sharp$ has removed those elements of G which invalidated Bressan and Hong's result as discussed in Section 3.2.

It can easily be seen that Proposition 4.1 does not hold for G. For example,

Example 4.3. Let Γ_i, F_i be as in Example 3.8 of Chapter 3. Clearly, $G(x)$ is the ℓ^1 unit ball on Γ_2, and there can be no trajectory starting at $(1, 0)$ with initial velocity, $\dot{x}(0) = (0, 1)$. However, $G^\sharp(x) = F_2(x)$ on Γ_2. This is shown in Figure 4.2.

Finally, we note that using $G^\sharp$ as our dynamics does not change the set of trajectories under consideration, as $F(x) \subset G^\sharp(x) \subset G(x)$ everywhere and the trajectories of G and F coincide as already mentioned.

Proposition 4.4. The trajectories of $G^\sharp$ and the trajectories of F coincide.

As mentioned before, $G^\sharp$ is not convex-valued.

Example 4.5. Let $\Gamma_1,\ \Gamma_2,\ \Gamma_3$ and $F_2(x)$ be as in Example 3.8 from Section 3.2, but let $F_1(x) = F_3(x) = \{0\} \times [-1.1]$. Then $G^\sharp(x) = \{\{0\} \times [-1,1]\} \cup \{[-1,1] \times \{0\}\}$ which, of course, is not convex.

We next provide a description of the set of switching times of a trajectory of $G^\sharp$.

Lemma 4.6. Let $x(\cdot)$ be a trajectory of $G^\sharp$ on $[0,T]$. Then the set of switching times is nowhere dense.

Proof. Assume that the result does not hold on some interval $(t_{\alpha_1}, t_{\alpha_2})$. Then for any t_α in this interval assume that there is a sequence of times $t_n \downarrow t_\alpha$ such that $x(t_n) \in \Gamma_i$ where the dimension of Γ_i is greater than the dimension of $\Gamma_{i(x(t_\alpha))}$. Then, because of our requirements on the stratification, there is a $\delta_n = d(x(t_n), \partial\Gamma_i) > 0$. However, Gronwall's inequality implies that there is a minimum time τ_n such that, for any $x_0 \in \mathbb{R}^n$, there exists a trajectory $x(\cdot)$ originating at x_0 where

$$||x(\tau_n) - x_0|| \geq \delta_n.$$

For the moment, we will assume that Γ_i is n-dimensional. There must be a switching time t_β on the interval $(t_\alpha, t_\alpha + \tau_n)$ for each n. This means that $x(t_\beta) \in \partial\Gamma_i$. However, this contradicts the definition of τ_n. For the case where the dimension of Γ_i is less than n, we can embed $\overline{\Gamma_i}$ in $\mathbb{R}^m$ where m is the dimension of Γ_i. The same argument can then be used. This contradicts the assumption that the trajectory can switch into a higher dimensional submanifold.

Note by the requirements on the stratification, we can not have $\Gamma_i,\ \Gamma_j$ of the same dimension be such that $\Gamma_i \cap \overline{\Gamma_j}$ is nonempty. That is, for a trajectory to be such that $x(t_1) \in \Gamma_i$ and $x(t_2) \in \Gamma_j$ for such a pair of submanifolds, there must be a $t_3 \in (t_1, t_2)$ such that $x(t_3) \in \Gamma_\ell$ where the dimension of Γ_ℓ is either greater or smaller than the dimension of Γ_i. This means that the only possible switches that can occur on $(t_{\alpha_1}, t_{\alpha_2})$ are switches into smaller dimensional submanifolds. However, we have assumed that there are finitely many such Γ. Therefore, there cannot be an infinite number of such switching times on $(t_{\alpha_1}, t_{\alpha_2})$. □

Because the dynamics are not globally Lipschitz, we can not approximate trajectories by C^1 trajectories. In fact, this can be seen quite easily using the now-familiar Example 3.8 from the previous chapter. Any trajectory with nonzero derivative for all time originating at at $\overline{x}$ with $\overline{x}_2$ nonzero will have a time $t > 0$ where $x(t) = (0, \overline{x}_2)$ and $x(t)$ has no derivative. We note the following result, which shows we can approximate a trajectory by a function which is piecewise a C^1 trajectory. This is of limited present utility, however, as the number of required discontinuities will increase as the desired closeness of the approximation increases. We note that if the set of switching times is finite, we can naturally remove this caveat: the number of discontinuities in our approximate trajectory is then unrelated to the desired closeness of the approximate trajectory in the sup norm.

Lemma 4.7. Let $x(\cdot)$ be a trajectory of F on $[0, T]$ and $\epsilon > 0$. Then there exists a function $x_\epsilon : [0, T] \to \mathbb{R}^n$ such that there is a finite set of times t_j with $0 = t_0 < \ldots t_j \ldots t_m = T$ where x_ϵ is C^1 on each interval $[t_j, t_j + 1]$ and for any $t \in (t_j, t_{j+1})$, $\dot{x}_\epsilon(t) \in F(x_\epsilon(t))$.

Proof. For each i, if $v \in F_i(x)$, then $|v| \leq c_i|x| + b_i$. This implies, because there are finitely many Γ_i's, that for any $v \in F(x)$, $|v| \leq c|x| + b$ for $c = \sup\{c_i\}$, $b = \sup\{b_i\}$. This implies that there exists a $\tau > 0$ such that for each time $t_a \in [0, T - \tau]$, and any trajectory $y(\cdot)$, $||y(t) - y(t_a)|| < \epsilon/2$ for $|t - t_a| < \tau$ by Gronwall's inequality. Our goal, then, is to construct a piecewise C^1 trajectory on $[n\tau, (n+1)\tau]$ for $0 \leq n < \frac{T}{\tau}$ originating at $x(n\tau)$. We begin with the interval $[0, \tau]$; all other intervals will follow in the same manner.

Let i be such that $x(0) \in \Gamma_i$. We construct x_ϵ as follows. Because F_i is Lipschitz on $\overline{\Gamma}_i \cap \overline{B}_{\frac{\epsilon}{2}}(x(0))$, we know that there exists a C^1 trajectory, $x_{\epsilon,0}$ starting at $x(0)$ which remains in $\overline{\Gamma}_i$ on $[0, t_0]$ for some $t_0 > 0$, which depends only on τ and $x(0)$ (see Lemma 5.3 of [Wol90b]). Assume that $t_0 < \tau$; if $t_0 \geq \tau$, then we have our desired trajectory. If $x_{\epsilon,0} \to y \in \Gamma_j \subset \overline{\Gamma}_i$ as $t \to t_0$ for some j, then we can create another C^1 trajectory of F_j starting at $\lim_{t \to t_0} x_{\epsilon,0}(t)$ on $[t_0, t_1]$ for some $t_1 > t_0$ by the same reasoning, where again t_1 depends only upon τ and $x(0)$. If $x_{\epsilon,0}(t_0) \in \Gamma_i$, then we know that we can repeat this so that we have a C^1 trajectory originating at $x_{\epsilon,0}(t_0)$ on $[t_0, 2t_0]$. Because there are only finitely many Γ_i's and the times t_m are independent of anything except τ and $x(0)$, we can construct a piecewise C^1 trajectory $x_{\epsilon,0}$ on $[0, \tau]$.

We then can construct a trajectory $x_{\epsilon,1} : [\tau, 2\tau) \to \mathbb{R}^n$ of F which is piecewise C^1 and originating at $x(\tau)$ by the above process. In this manner, we can define x_ϵ such that $x_\epsilon|_{[n\tau,(n+1)\tau]} = x_{\epsilon,n}$ for $n = 0, 1, \ldots$ For any $t \in [0, T]$, we have

$$\begin{aligned} ||x_\epsilon(t) - x(t)|| &\leq ||x_\epsilon(t) - x(n\tau)|| + ||x(n\tau) - x(t)|| \\ &\leq ||\frac{\epsilon}{2} + \frac{\epsilon}{2}|| = \epsilon \end{aligned}$$

where n is such that $t \in [n\tau, (n+1)\tau)$.

□

4.2 Hamiltonian Conditions for Invariance

In order to characterize both weak and strong invariance we will use, as in Chapter 2, the lower Hamiltonian. With this function, we can immediately characterize weak invariance for F over the stratification $\{\Gamma_i\}_{i=1}^N$.

Lemma 4.8. For a closed set E, (E, F) is weakly invariant if and only if $h_{G^\sharp}(x, \zeta) \leq 0$ for all $x \in E$ and $\zeta \in N_E^P(x)$.

Proof. ($\Leftarrow$) We will use the proximal aiming construction from [CLSW95]. For this, we first define $s(x)$ as a point in E nearest to x. Then, by assumption, for any x there is a $v \in G^\sharp(s(x))$ such that $\langle v, x - s(x)\rangle \leq 0$ because $(x - s(x)) \in N_E^P(s(x))$. For any x, we can arbitrarily select such a v as $g(x)$. Fixing $x_0 \in S$, the implication is proven if we can solve

$$\dot{x}(t) = g(x(t)),\ x(0) = x_0 \tag{4.4}$$

and show that the resulting solution remains in E. To do this, we use an idea very similar to the Euler polygonal arc construction found in [CLSW95]. We first partition the interval $[0, T]$ into subintervals of equal length with endpoints $0 = t_0 < t_1 < \cdots < t_N = T$ where we have fixed T. On $[t_0, t_1]$, we solve

$$\dot{y}(t) = \phi_0(g(x_0)),\ y(t_0) = \phi_0(x_0)$$

where $\phi_0 \in C(\mathbb{R}^n, \mathbb{R}^n)$ is a locally invertible continuous mapping from Γ_i to $\{x_n = x_{n-1} = \cdots = x_{m+1} = 0\}$ where $m = \dim(\Gamma_i)$ for i such that $g(x_0) \in F_i^\sharp(x_0)$. We denote the resulting $\phi_0^{-1}y(t_1)$ as x_1. We then solve on $[t_1, t_2]$

$$\dot{y}(t) = \phi_1(g(x_1)),\ y(t_1) = \phi_1(x_1).$$

We continue in this fashion for each subinterval. By linear growth, there exists an M such that

$$d_E(x_1) \leq M(t_1 - t_0)$$

and thus

$$\begin{aligned}
d_E^2(x_2) &\leq |x_2 - s(x_1)|^2 \\
&= |x_2 - x_1|^2 + |x_1 - s(x_1)|^2 + 2\langle x_2 - x_1, x_2 - s(x_1)\rangle \\
&\leq M^2(t_2 - t_1)^2 + d_E^2(x_1) + 2\int_{t_1}^{t_2} \langle f(x_1), x_1 - s(x_1)\rangle dt \\
&\leq M^2(t_2 - t_1) + (t_1 - t_0)^2.
\end{aligned}$$

This leads to

$$d_E^2(x_k) \le M^2 T \sup\{t_{j+1} - t_j\}.$$

Let i be such that $g(x_0) \in F_i^\sharp(x_0)$. Then there is a $\delta > 0$, such that if $|t_1 - t_0| < \delta$ then $x(t_1) \in \Gamma_i$ and, as $F_i^\sharp(x)$ is convex-valued and continuous on $\overline{\Gamma_i} \cap B_r(x_0)$ for some $r > 0$, we know that $x(t_1) \in \overline{\Gamma_i}$ for small t_1. Thus as $\sup\{t_{j+1} - t_j\} \to 0$ we know that for small $j > 1$, $g(x(t_j)) \in F_i^\sharp(x(t_j))$ and so, if we let the partition mesh go to 0, the standard compactness of trajectories allows us to state that the resulting limit will satisfy 4.4 and remain in E.

($\Rightarrow$) This is proven in Theorem 2.2, in [CLSW95] which shows that $G^\sharp(x) \cap T_E^D(x) \ne \emptyset$ for each $x \in U$ where $T_E^D(x)$ is the standard *Dini tangent cone or Bouligand cone*. This immediately implies that $h_{G^\sharp}(x,\zeta) \le 0$.

□

Because $G(x)$ is upper semicontinuous, non-empty, and convex-valued everywhere, Proposition 2.14 states that (E, F) is weakly invariant if and only if for every x,

$$h_G(x,\zeta) \le 0, \ \forall \zeta \in N_E^P(x).$$

This is why (using the weak decrease results from Chapter 2) the value function in [BH07] is an upper viscosity solution in the usual sense. This along with Lemma 4.8 means that the $v \in G(x)$ which achieves the infimum for the Hamiltonian is also an element of $G^\sharp(x)$ at each x. This is not surprising, as the proof must construct a trajectory remaining in the set E and we have already seen that the derivative of a trajectory must lie in $G^\sharp$. However, as noted, we can not assume that strong invariance can be characterized by requiring at each $x \in E$, that for every $\zeta \in N_E^P(x)$

$$h_G(x,-\zeta) \ge 0.$$

This would mean that $\sup_{v\in G(x)}\langle v,\zeta\rangle \le 0$ at all $x \in E$ and for each $\zeta \in N_E^P(x)$. As seen in our counterexample, requiring this for $v \in G(x) \setminus G^\sharp(x)$ is not appropriate, as no trajectory can use such a v as a velocity. In the following, we establish that strong invariance is equivalent to a condition on the Hamiltonian associated with $G^\sharp$.

Lemma 4.9. For a closed set E, (E, F) is strongly invariant if and only if for all $x \in E$ and $\zeta \in N_E^P(x)$, we have the following inequality

$$h_{G^\sharp}(x, -\zeta) \geq 0.$$

Proof. $(\Rightarrow)$ Let $\overline{x} \in E$, $v \in F_i^\sharp(\overline{x})$, and $\zeta \in N_E^P(\overline{x})$. We know that there is a trajectory $x(\cdot)$ which is C^1 on an interval $[0, \delta]$ with $x(0) = \overline{x}$ and $\dot{x}(0) = v$. Then by the proximal normal inequality, we know that there exists $\sigma > 0$ such that

$$\langle \zeta, \tilde{x} - \overline{x} \rangle \leq \sigma ||\tilde{x} - \overline{x}||^2, \qquad \forall \tilde{x}.$$

Since for all $t > 0$, we have that $x(t) \in E$ we can use the above in order to get for $t \in [0, \delta)$

$$\langle \zeta, x(t) - \overline{x} \rangle \leq \sigma ||x(t) - \overline{x}||^2$$

Dividing both sides by t and letting $t \to 0$ we have

$$\langle \zeta, v \rangle \leq 0. \tag{4.5}$$

The implication is proven by taking the supremum and multiplying both sides by -1. We now to turn to the converse.

$(\Leftarrow)$ Let $x(\cdot)$ be a trajectory on $[0, T]$ with $x(0) \in E$. Then we know that the multifunction

$$G^*(t, y) := \begin{cases} \{1\} \times \{v \in G^\sharp(y) : ||v - \dot{x}(t)|| \leq \tilde{k}||y - x(t)||\} & y \in \overline{\Gamma}_{i(x(t))} \\ \{(1, 0)\} & y \notin \overline{\Gamma}_{i(x(t))}. \end{cases}$$

The multifunction satisfies the same properties as $G^\sharp$ for each t, specifically, it is the compact union of finitely many convex sets at almost every time t, a restriction of the continuous extension of stratified domain dynamics on $\mathbb{R} \times \mathbb{R}^n$, which satisfies the same hamiltonian condition as $G^\sharp$ along the boundary of Γ_i. So, we claim that, as in the proof of Lemma 2.19, $(\mathbb{R} \times E, G^*)$ is weakly invariant. As in that proof, we note that

$$N^P_{\mathbb{R}\times E}(t,y) = \{0\} \times N^P_E(y)$$

and

$$0 \leq h_{G^\sharp}(y, -\zeta) \leq -h_{G^*}((t,y),(0,\zeta)),$$

for all $\zeta \in N^P_E(y)$. We conclude that $(\mathbb{R} \times E, G^*)$ is weakly invariant by Lemma 4.8. By Gronwall's inequality (as in Lemma 2.19), we conclude that (E, F) is strongly invariant as the resulting trajectory of G^* is $(t, y(t))$ will necessarily be such that $y(t) = x(t)$. $\square$

Chapter 5

Hamilton-Jacobi Equation Theory on Stratified Domains

In this chapter, we present the main results. First, in Section 5.1, we show that the minimal time function for problems on stratified domains is the unique proximal solution for a Hamilton Jacobi equation in a manner similar to that in [WZ98]. We provide a numerical example for this. In Section 5.2, we then look at the Mayer problem on stratified domains with continuous endpoint cost and show that its value function also satisfies a proximal form of the Hamilton-Jacobi equation. Again, we assume that we have a $\{\Gamma_i\}$, F_i, F and G as in Chapter 3; we define $G^\sharp$ as in the previous chapter.

5.1 The Minimal Time Problem on Stratified Domains

The general minimal time problem can be stated in the following way: Given a multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, closed target set S, and initial x_0, find a trajectory of

$$\dot{x}(t) \in F(x(t)),\ x(0) = x_0$$

which reaches S in the shortest time possible. The minimal time function $T_S(x)$ is defined to be minimal time it takes for a trajectory to reach S from x. We set $T_S(x) = +\infty$ if no trajectory originating at x can reach S in finite time. We have already noted that Snell's law provides a necessary condition for a trajectory to be optimal in the minimal time problem with dynamics inherited from the structure of optical media. Another classical problem can be seen as minimal time problem on stratified domains.

Example 5.1. The classical problem of geodesics can be formulated as finding the shortest path between $x_0, x_1 \in \mathcal{M}$ where $\mathcal{M}$ is a Riemannian manifold with boundary such that the path remains in $\mathcal{M}$. We embed $\mathcal{M}$ in $\mathbb{R}^n$ and let this be denoted as Γ_1 and its metric be denoted $\mathfrak{g}$. We then can define

$$F_1(x) = \{(\sqrt{\mathfrak{g}(v,v)})^{-1}v : v \in \mathcal{T}_{\mathcal{M}}(x)\}.$$

We do the same for the embedding of the boundary, denoting it Γ_2. Finally, we define Γ_i for the complement and define $F_i(x) = \{0\}$. Then finding a geodesic from x_0 to x_1 is found by finding a trajectory of F for which $T_{x_1}(x_0)$ is attained.

In Chapter 2, proving that the value function satisfies a proximal Hamilton-Jacobi equation hinged on establishing the relevant invariance of the epigraph of the value function. The proofs for the following propositions can be found in [WZ98] as, respectively, Propositions 3.1 and 3.2.

Proposition 5.2. Let $E := \text{epi}T_S$. The multifunction $G \times \{-1\} : \mathbb{R}^n \times \mathbb{R} \rightrightarrows \mathbb{R}^{n+1}$ is defined as

$$(G^\sharp \times \{-1\})(x,r) := \{v : v \in G(x)\} \times \{-1\}\}. \tag{5.1}$$

The multifunction $G^\sharp \times \{1\}$ is defined similarly. Then:

1. $(G^\sharp \times \{-1\}, E)$ is weakly invariant .
2. $(-G^\sharp \times \{1\}, E)$ is strongly invariant.

Proposition 5.3. For a lower semicontinuous $\theta : \mathbb{R}^n \to (-\infty, \infty]$ such that $\theta|_S \equiv 0$, we have the following for $E = \text{epi}\theta$

1. If $(G^\sharp \times \{-1\}, E)$ is weakly invariant in $S^C \times \mathbb{R}$ and θ is bounded below, then $\theta(x) \geq T_S(x)$.

2. If $(-G^\sharp \times \{1\}, E)$ is strongly invariant in $\mathbb{R}^{n+1}$, then $\theta(x) \leq T_S(x)$ for all $x \in \mathbb{R}^n$.

Though $G^\sharp$ does not satisfy the hypotheses in Propositions 3.1 and 3.2 in [WZ98]– specifically it is not convex-valued– G is convex-valued and uppersemicontinuous; as we have already seen, the trajectories of G, F, and $G^\sharp$ coincide. Thus, $G^\sharp \times \{1\}$ and $G \times \{1\}$ have the same trajectories.

The following result regarding the epigraphs of lower semicontinuous functions uses Lemmas 4.8 and 4.9 in Chapter 4 in the context of augmented dynamics. The proof is similar to that of Proposition 3.3 of [WZ98].

Lemma 5.4. Let $\theta : \mathbb{R}^n \to (-\infty, \infty]$ be lower semicontinuous and $\theta|_S \equiv 0$ and $E = \text{epi}\theta$.

1. $(G^\sharp \times \{-1\}, E)$ is weakly invariant if and only if for all $x \notin S,\ \zeta \in \partial_P \theta(x)$,

$$1 + h_{G^\sharp}(x, \zeta) \leq 0.$$

2. $(-G^\sharp \times \{1\}, E)$ is strongly invariant if and only if for all $x \in \mathbb{R}^n,\ \zeta \in \partial_P \theta(x)$,

$$1 + h_{G^\sharp}(x, \zeta) \geq 0.$$

Proof. We prove the first statement only; the second is proved in an analagous manner. The only difference being that we will rely on Lemma 4.9 instead of Lemma 4.8. Notice that for $(x, \zeta) \in \mathbb{R}^{2n}, r \in \mathbb{R}$ and $\rho < 0$ we have

$$\begin{aligned} h_{(G^\sharp \times \{-1\})}((x, r), (\zeta, \rho)) &= \inf_{v \in G^\sharp(x)} \{\langle v, \zeta \rangle - \rho\} \\ &= -\rho(1 + h_{G^\sharp}(x, -\frac{\zeta}{\rho})). \end{aligned}$$

($\Rightarrow$) Suppose that $x \notin S$ and $\zeta \in \partial_P \theta(x)$; we then know that

$$h_{(G^\sharp \times \{-1\})}((y, r), -\xi) \leq 0 \tag{5.2}$$

for any $(y,r) \in E,\ y \notin S,\ \xi \in N^P_E(x,r)$. Then, for $(y,r) = (x, \theta(x)) \in \text{epi}\theta$ with $\xi = (\zeta, -1)$. Then

$$1 + h_{G^\sharp}(x,\zeta) \leq 0.$$

($\Leftarrow$) Let $x \in E \cap S^c$ and $r \in \mathbb{R}^n$. If $\xi = (\zeta, \rho) \in N^P_E(x,r)$, then $\rho \leq 0$. If $\rho < 0$, then $r = \theta(x)$. This in turn means that $(-\frac{\zeta}{\rho}, -1) \in N^P_E(x, \theta(x))$ because $N^P_E(x,\theta(x))$ is a cone. Therefore, $-\zeta/\rho \in \partial_P\theta(x)$. This means that

$$h_(G^\sharp \times \{-1\})((x,\theta(x)),(\zeta,\rho)) = -\rho(1 + h_{G^\sharp}(x, -\zeta/\rho)) \geq 0.$$

Now we turn to the case where $\rho = 0$. Then $(\zeta, 0) \in N^P_E(x,\theta(x))$ and so, by Rockafellar's horizontality theorem (see [Roc81]), there exist sequences $\{x_i\}, \{\xi_i\}, \{\rho_i\}$ such that $x_i \to x, \theta(x_i) \to \theta(x), \xi_i \to \xi, \rho_i \uparrow 0$ and $-\xi_i/\rho_i \in \partial_P\theta(x_i)$. Then, as seen above, for each i,

$$-\rho_i(1 + h_{G^\sharp}(x_i, -\zeta_i/\rho_i)) \geq 0.$$

Taking a limit, we get $h_{G^\sharp}(x,\xi) \geq 0$. We conclude by noting that $h_{(G^\sharp \times \{-1\})}((x,r),(\xi,0)) = h_{G^\sharp}(x,\xi) \geq 0$. Thus, by 4.8 in Chapter 4, we conclude that $(G^\sharp \times \{-1\}, E)$ is weakly invariant for trajectories remaining in $S^c \times \mathbb{R}$. □

With this we are able to state our main result for this section: that the minimal time function $T_S(\cdot)$ uniquely solves the proximal Hamilton-Jacobi equation on S^c for certain boundary conditions.

Theorem 5.5. Let $S \subset \mathbb{R}^n$ be closed. Then there exists a unique lower semicontinuous function $\theta : \mathbb{R}^n \to (-\infty, \infty]$ bounded below satisfying

- For each $x \notin S$ and $\zeta \in \partial_P\theta(x)$, we have

$$1 + h_{G^\sharp}(x,\zeta) = 0.$$

- Each $x \in S$ is such that $\theta(x) = 0$ and

$$1 + h_{G^\sharp}(x,\zeta) \geq 0$$

for $\zeta \in \partial_P \theta(x)$.

The unique such function is $\theta(\cdot) = T_S(\cdot)$.

Proof. By definition, $T_S(\cdot)$ is bounded below by zero and is equal to zero on S. It is lower semicontinuous by Proposition 2.6 of [WZ98]. Proposition 5.2 and Lemma 5.4 both imply that

$$1 + h_{G^\sharp}(x, \zeta) \leq 0,\ \forall x \notin S,\ \zeta \partial_P T_S(x) \tag{5.3}$$

and

$$1 + h_{G^\sharp}(x, \zeta) \geq 0,\ \forall x \in \mathbb{R}^n,\ \zeta \in \partial_P T_S(x). \tag{5.4}$$

By 5.3 and 5.4, $\theta(\cdot) = T_S(\cdot)$. By Propositions 5.2 and 5.3, we conclude both that $\theta(x) \leq T_S(x)$ and $\theta(x) \geq T_S(x)$ for all $x \in \mathbb{R}^n$. Thus, $\theta(\cdot) = T_S(\cdot)$. □

We conclude this section by revisiting Example 3.3.

Example 5.6. Continuing with the discussion in Example 3.3, we will consider the simple case where, in $\mathbb{R}^2$, s $\Gamma_1 = \{x_1 < 0\}$, $\Gamma_2 = \{x_1 > 0\}$, $\Gamma_3\{x_3 = 0\}$. Again, we define $F_i(x) = \frac{c}{n_i}\overline{B}$ for $i = 1, 2$ and $F_3(x) = \{0\}$. This can be seen in Figure 5.1. In [CV89], Clarke and Vinter derived, from Fermat's principle, the necessary conditions for an optimal trajectory from the relevant maximum principle. With Theorem 5.5, we can determine the time it takes for the light to travel from $\overline{x}$ to the point s by solving for $T_s(\overline{x})$. We assume that $s \in \Gamma_2$. Given a point of incidence $\alpha_{s,\overline{x}}$ we let $V(\overline{x})$ be

$$V(\overline{x}) = \begin{cases} \frac{n_2}{c}||\overline{x} - s||, & \overline{x} \in \Gamma_2 \cup \Gamma_3 \\ \frac{n_1}{c}||\overline{x} - (0, \alpha_{s,\overline{x}})|| + \frac{n_2}{c}||s - (0, \alpha_{s,\overline{x}})||, & \overline{x} \in \Gamma_1 \end{cases}$$

This $\alpha_{s,\overline{x}}$ minimizes

$$\frac{n_1}{c}||\overline{x} - (0, \alpha)|| + \frac{n_2}{c}||s - (0, \alpha)||.$$

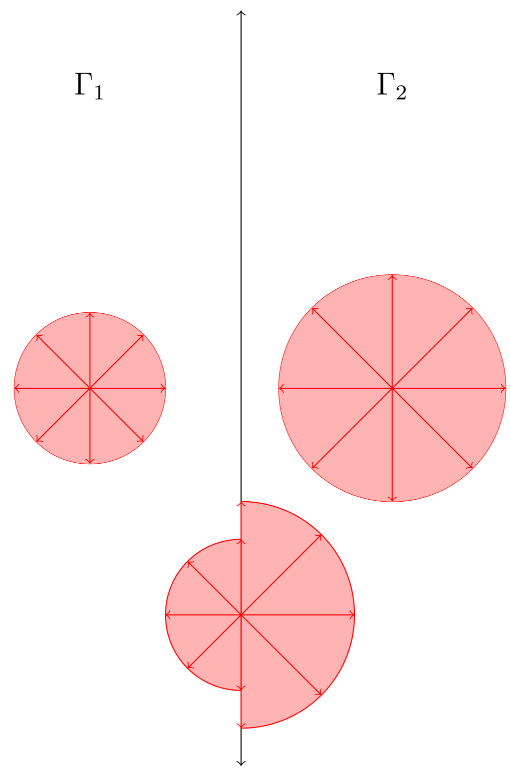


Figure 5.1: The interface between two optical media with velocity sets in red

That is, $\alpha_{s,\overline{x}}$ solves

$$\frac{-n_1(\overline{x}_2-\alpha)}{c||\overline{x}-(0,\alpha)||}-\frac{n_2(s_2-\alpha)}{c||s-(0,\alpha)||}=0 \tag{5.5}$$

Note that on $\Gamma_2 \setminus \{s\}$, the function V is C^2. This means that at every $\overline{x} \in \Gamma_2 \setminus \{s\}$, the proximal subdifferential $\partial_P V(\overline{x}) = \nabla V(\overline{x})$. Therefore, on $\Gamma_2 \setminus \{s\}$,

$$\inf_{v\in\frac{c}{n_2}\overline{B}}\langle v, \nabla V(\overline{x})\rangle = \inf_{v\in\overline{B}}\langle v, \frac{x-\overline{x}s}{||\overline{x}x-s||}\rangle = -1.$$

For $\overline{x} \in \Gamma_1$, in order to determine $V(\overline{x})$, we need to solve for α, which in general involves solving a non-factorizable quartic. With this in mind, we turn to a specific example. Let $s = (1, 0)$, $n_1 = 1.000271374$, $n_2 = 1.33356$, which correspond to approximate refractive indices of light of wavelength 633 nm for air (at 100 kPa and 20° centigrade:[D$^+$04]) and water (at the same pressure and temperature:[HGS98]),respectively. We can then numerically solve for α using Equation 5.5. The solution for this on $[-1,0]\times[-1,1]$ is seen in Figures 5.2(a) and 5.2(b). With this, we can compute $V(\overline{x})$ on the same region. This is shown in Figure 5.2(c). If we take the norm of the gradient of $V(x)$ using finite differences, we see that on the region, the gradient is of norm $\frac{c}{n_1}$. We conclude by this that at each $\overline{x}$, there is a vector $v \in \frac{n_1}{c}\overline{B}$ such that $\langle v, \nabla V\rangle = -1$.

Finally, we look at $\overline{x} \in \Gamma_3$. We note that along this manifold, the proximal subgradient is trivial, as the norm of the gradient on Γ_1 as we approach Γ_3 is less than the norm of the gradient on the other side. As we see in Figure 5.2(d), there is a rentrant edge to the epigraph along Γ_3. Thus the subdifferential is trivial along Γ_3. This in turn implies that V satisfies the Hamilton-Jacobi equation for the minimal time problem and conclude that $V(\overline{x}) = T_{(1,0)}(\overline{x})$. Finally, we note that if we denote θ_1 as the angle between the vector $\overline{x}-(0,\alpha))$ and $(-1,0)$, and θ_2 as the angle between the vector $(1,0)-(0,\alpha)$ and $(1,0)$, that (5.5) is simply the familiar Snell's law:

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{n_2}{n_1}.$$

Therefore, the trajectory satisfying Snell's law is the optimal trajectory with time from $\overline{x}$ to $(1,0)$ equal to $T_{(1,0)}(\overline{x})$.

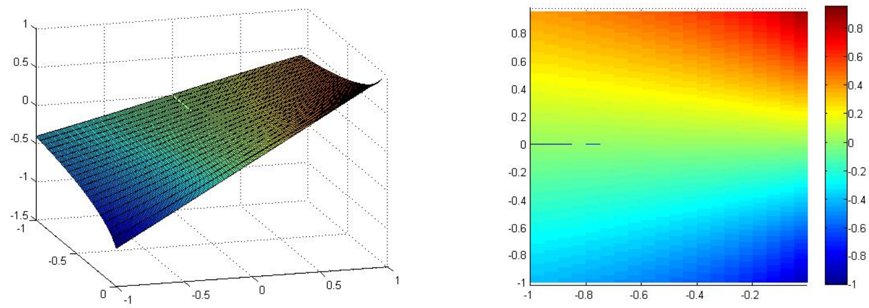
5.2 The Mayer Problem on Stratified Domains

We turn now to the Mayer problem on stratified domains. Recall that this takes the form, for prescribed $T > 0$

$$\text{minimize } g(x(T)), \text{ s.t. } \dot{x}(t) \in F(x(t)) a.e., x(0) = x_0.$$

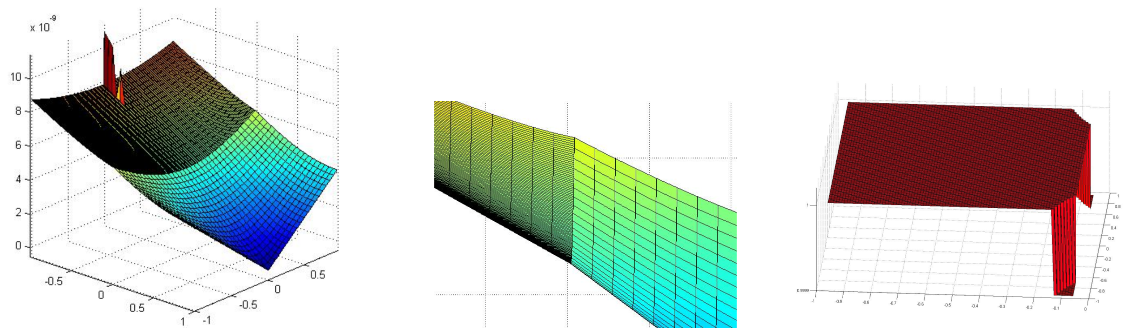
We assume, as in Chapter 2 that the endpoint cost g is continuous.

Proposition 5.7. Let F be the dynamics associated with the stratification $\{\Gamma_i\}$. Then, if the endpoint cost in the Mayer problem, g, is continuous, the associated value function $V(t,x)$ is lower semicontinuous.



(a) The point of incidence as a function of x for target $(1, 0)$.

(b) Level sets of point of incidence



(c) The function V with appropriate α computed

(d) Closeup of V at interface

(e) The norm of the gradient of $\frac{n_1}{c}V$ on Γ_1

Figure 5.2: The point of incidence and the function $V(x)$

Proof. Let $V(t, x_0) < \infty$ and let $(\tau, y) \to (t, x_0)$. We know that there exists an optimal trajectory $x^*_{\tau,y}$ with $x^*_{\tau,y}(\tau) = y$. As $\tau \to t$ and $y \to y_0$, we define $x(\cdot)$ to be the uniform limit of these optimal trajectories. These trajectories are all trajectories of G as well as F. By the uppersemicontinuity of G, we invoke the compactness of trajectories (Lemma 2.5 of Chapter 2) to conclude that $x(\cdot)$ is a trajectory with $x(t) = x_0$. We know that $g(x(T)) \geq V(t, y)$. This impiles that $\liminf_{(\tau,y)\to(t,x_0)} V(\tau, y) \geq V(t, x_0)$. Thus, the value function is lower semicontinuous. □

With that, it is reasonable to predict, as in the case of the minimal time problem, the value function of the Mayer problem is the unique solution to the proximal Hamilton-Jacobi equation.

Theorem 5.8. For a stratification $\{\Gamma_i\}$ and its associated F, as in Chapter 3, if g is continuous, then the value function for the Mayer problem is the unique lower semicontinuous function such that for every $(\theta, \zeta) \in \partial_P V(t, x)$,

$$\theta + h_{G^\sharp}(x, \zeta) = 0, \ (t, x) \in (-\infty, T) \times \mathbb{R}^n$$

$$g(x) = V(T, x), \ x \in \mathbb{R}^n.$$

Proof. We first show that $(V, G^\sharp)$ is weakly decreasing and strongly increasing on $(t, T) \times \mathbb{R}^n$. Assume that $V(t_0, x_0)$ is finite for a particular (t_0, x_0). Then let x^* be the arc with $x^*(t_0) = x_0$ and $V(t_0, x_0) = g(x^*(T))$. Then, as in Theorem 2.21 in Chapter 2, we conclude that $V(\tau, x^*(\tau))$ is constant as a function of τ on the interval (t_0, T). This in turn means that (V, F) is weakly decreasing, and therefore, so is $(V, G^\sharp)$. We note that the principle of optimality argument from Theorem 2.21, Chapter 2 still holds, and so we conclude that along any trajectory $x(\cdot)$, if $t' > \tau$, then $V(t', x(t')) \geq V(\tau, x(\tau))$. So we conclude that $(V, G^\sharp)$ is strongly decreasing.

We now show that because $(V, G^\sharp)$ is weakly decreasing, for each (t, x), if $(\theta, \zeta) \in \partial_P V(t, x)$, then

$$\theta + h_F(t, x, \zeta) \leq 0.$$

We know that $S = \text{epi}V$ is weakly invariant on $G^\sharp$. We also know that the augmented multifunction $\{1\} \times G^\sharp \times \{0\}$ possesses the same relevant properties as $G^\sharp$ with respect to Lemma 4.8 in Section 4.2. By this lemma, we conclude that for $i(\theta, \zeta, -1) \in N_S^P(\tau, \alpha, V(\tau, \alpha))$, we know that there exists some $(1, v, 0) \in \{1\} \times G^\sharp \times \{0\}$ such that

$$\langle (1, v, 0), (\theta, \zeta, -1) \rangle = \theta = \langle (v, 0), (\zeta, -1) \rangle \leq 0.$$

We next show that

$$\theta + h_{G^\sharp}(x, \zeta) \geq 0.$$

Similarly, we know that because $(V, G^\sharp)$ is strongly increasing, the pair $(S, \{-1\} \times -G^\sharp \times \{0\})$

is strongly invariant. Which means that for any $(1, v, 0) \in \{1\} \times -G^\sharp \times \{0\})$, we have

$$\langle(-1, v, 0), (\theta, \zeta, -1)\rangle = \theta + \langle(v, 0), (\zeta, -1)\rangle \leq 0.$$

Which in turn means that

$$\theta + h_{G^\sharp}(t, x, \zeta) \geq 0.$$

Finally, we show uniqueness. Suppose that u is a lower semicontinuous function which satisfies the proximal Hamilton-Jacobi equation. We know that $(u, G^\sharp)$ is weakly decreasing with respect to t on any open interval (τ, T) and so for some pair (τ, y), we know there is a trajectory $x(\cdot)$ with $x(\tau) = y$ and so

$$u(t, x(t)) \leq u(\tau, y),\ \forall t \in [\tau, T).$$

Which in turn means that $g(x(T)) = u(T, x(T)) \leq u(\tau, y)$, and so $V(\tau, y) \leq u(\tau, y)$.

Next, we know that there is an optimal x^* originating at (τ, y) and because $(u, G^\sharp)$ is strongly increasing, we note that $u(T, x^*(T)) \geq u(\tau, y)$. Observing that $u(T, x^*(T)) = g(x^*(T)) = V(\tau, y)$ establishes uniqueness. With this, the proof is concluded. □

Chapter 6

Concluding Remarks and Future Work

In Chapter 1, we introduced several general questions that may arise when studying control problems. These included questions regarding the invariance properties of the control system and those regarding conditions for optimality for optimal control problems. For control problems on stratified domains, we have provided necessary and sufficient conditions for both weak and strong invariance. The associated dynamical system used to provide a Hamiltonian condition for strong invariance is a new construction. It is perhaps surprising that a multifunction which does not possess the majority of regularity properties assumed in the standard theory provides a very similar Hamiltonian characterization of invariance and possesses a structure more advantageous than the usual regularization of a discontinuous dynamical system.

With this characterization of invariance, we are also able to partially answer the question regarding characterizing optimality in optimal control problems. The Hamiltonian characterization of weak and strong invariance immediately provides a sufficient condition for optimality for control problems with continuous cost functions. Indeed, much of the analysis here is very similar to that found in the standard theory; we simply rely on the new Hamiltonian inequalities. However, this is only a partial question answer to the question of whether we can create conditions for optimality. A proper necessary condition, in the form a maximum principle similar to that discussed in Chapter 1, would be an appropriate next avenue of investigation. This is especially true given the connection between the Hamilton-Jacobi equation and the maximum principle for the standard control theory, as described in [CV87] for instance; informally, for Lipschitz control problems the gradient of the value function along an optimal trajectory is the adjoint arc $p(\cdot)$ of the Pontryagin Maximum Principle described in Chapter 1. The other questions introduced in the first chapter–namely, controllability and stability–have not been addressed at all for problems on stratified domains. However, in light of the Hamiltonian criteria for invariance, it is highly likely that any future work in these areas will require further analysis of this new construction.

Bibliography

[AC84] J.P. Aubin and A. Cellina. *Differential Inclusions*. Springer-Verlag, 1984.

[BCD97] M. Bardi and I. Capuzzo-Dolcetta. *Optimal control and Viscosity Solutions to the Hamilton-Jacobi-Bellman Equations*. Birkahuser, 1997.

[BGP61] V. G. Boltjanskiĭ, R. V. Gamkrelidze, and L. S. Pontryagin. The theory of optimal processes. I. The maximum principle. *Amer. Math. Soc. Transl. (2)*, 18:341–382, 1961.

[BH07] A. Bressan and Y. Hong. Optimal control problems on stratified domains. *Networks and Heterogeneous Media*, 2(2):313–331, 2007.

[BP04] U. Boscain and B. Piccoli. *Optimal syntheses for control systems on 2-D manifolds*, volume 43 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 2004.

[Cla90] F.H. Clarke. *Optimization and nonsmooth analysis*. Society for Industrial Mathematics, 1990.

[CLSW95] F.H. Clarke, Yu.S. Ledyaev, R.J. Stern, and P.R. Wolenski. Qualitative properties of trajectories of control systems. *J. of Dynamical and Control Systems*, 1(1):1–48, 1995.

[CLSW98] F.H. Clarke, Yu.S. Ledyaev, R.J. Stern, and P.R. Wolenski. *Nonsmooth Analysis and Control Theory*. Springer, 1998.

[CSW95] F. H. Clarke, R. J. Stern, and P. R. Wolenski. Proximal smoothness and the lower-C^2 property. *J. Convex Anal.*, 2(1-2):117–144, 1995.

[CTG08] C. Cai, A. R. Teel, and R. Goebel. Smooth Lyapunov functions for hybrid systems. II. (Pre)asymptotically stable compact sets. *IEEE Trans. Automat. Control*, 53(3):734–748, 2008.

[CV87] F. H. Clarke and R. B. Vinter. The relationship between the maximum principle and dynamic programming. *SIAM J. Control Optim.*, 25(5):1291–1311, 1987.

[CV89] F.H. Clarke and R. Vinter. Applications of optimal multiprocesses. *SIAM J. Control Optim.*, 27(5):1048–1071, 1989.

[D+04] T. Doiron et al. Engineering metrology toolbox. `http://emtoolbox.nist.gov/Main/Main.asp`, November 2004.

[Dei92] K. Deimling. *Multivalued differential equations*, volume 1 of *de Gruyter Series in Nonlinear Analysis and Applications*. Walter de Gruyter & Co., Berlin, 1992.

[Eva10] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[HGS98] A. H. Harvey, J. S. Gallagher, and J. M.H. Levelt Senger. Revised formulation for the refractive index of water and steam as a function of wavelength, temperature, and density. *J. Phys. Chem. Ref. Data*, 27(4):761–774, 1998.

[Roc81] R.T. Rockafellar. Proximal subgradients, marginal values, and augmented lagrangians in nonconvex optimization. *Math. Oper. Res.*, 6:424–436, 1981.

[Roc97] R.T. Rockafellar. *Convex analysis*. Princeton Univ Pr, 1997.

[Vin00] R. Vinter. *Optimal control (Systems & Control: Foundations & Applications)*. Birkahuser, 2000.

[Wol90a] P.R. Wolenski. The exponential formula for the reachable set of a lipschitz differential inclusion. *SIAM J. Control Optim.*, 28:1148–1161, 1990.

[Wol90b] P.R. Wolenski. A uniqueness theorem for differential inclusions. *J. Differential Equations*, 84:165–82, 1990.

[WZ98] P.R. Wolenski and Y. Zhuang. Proximal analysis and the minimal time function. *SIAM J. Control Optim.*, 35(3):1048–1072, 1998.

Vita

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