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Does Exploring Non-Linear Models Address High School Students' Misconceptions of Linearity and Rate of Change?

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DOES EXPLORING NON-LINEAR MODELS ADDRESS HIGH SCHOOL STUDENTS' MISCONCEPTIONS OF LINEARITY AND RATE OF CHANGE?

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ABSTRACT

Previous studies have shown that students at all levels—elementary, secondary and post-secondary—tend to apply linear reasoning in mathematical problems where non-linear models are needed. They rely on proportionality and linear equations without paying attention to the problem features needed for an accurate model. In the present study, students were taught using an activity-inspired by modeling curricula. In a hands-on activity, they explored the rate of change of linear and nonlinear functions that arise in describing elastic materials, recording average rates of change and noting key graphical features. This led them to articulate the relationship between nonlinear models, local linearity and the derivative. Pre/post-tests revealed a significant difference in performance between the control and experimental groups with respect to conceptual understanding of linearity and its applications.

CHAPTER 1: INTRODUCTION

Research has shown that high school students often rely on proportions and linear relationships to solve problems, even when these tools are not applicable. This may be because linear functions are the first and most basic function that students encounter. Because linear functions have a constant rate of change, they are the easiest to learn about and apply. The mistaken applications of linearity stem from the strong emphasis on linear equations when functions are first introduced in school.

As a result of linear misconceptions, students struggle conceptually when they are introduced to higher-level mathematics, including differential calculus, because of their preconceived idea that everything is linear. This linear default can be both beneficial and harmful in a calculus setting—beneficial in that their strong foundation of linearity will help them find equations of tangent lines and calculate average rates of change; harmful in that, conceptually, it is difficult for students to recognize these are only local linear representations of slope that vary based on the location of a nonlinear model. In general, high school calculus students have sound algebraic skills. However, conceptually, they often do not understand what a derivative means in terms of change for a nonlinear model. Algebraic skill is important, of course. But Wu reminds us that technical skills and conceptual understanding are completely intertwined (Wu, 1999).

The purpose of this study is to address the misconception of universal linearity through student exploration of local linearity and its relationship to rates of change in nonlinear materials. In the physics education research, open-ended, but carefully structured, student investigations are known to be an effective remedy for misconceptions.

The research questions in this study were as follows:

1. How does student exploration and modeling in a mathematics classroom affect students' understanding of the relationship between local linearity and non-linear models?
2. Is incorporating an experiment in a high school mathematics classroom feasible and beneficial?

In the specific study, students participated in an extended version of a Hooke's Law lab to explore displacement versus force for various materials. The students in a traditional calculus classroom were chosen randomly to participate in either the control or experimental group. The research questions were answered through data collected from pre/post-assessments and open response questions designed to observe students' understandings of linearity, rate of change and differential calculus.

Chapter 2 discusses the literature on student misconceptions of linearity and teaching methods that address their misunderstandings. Chapter 3 elaborates the rationale for the study and describes the participants. In Chapter 4, the design of the study is explained, and the results are examined in Chapter 5. Lastly, Chapter 6 discusses conclusions, limitations and extensions of the experiment.

CHAPTER 2: REVIEW OF LITERATURE

This study examines the effectiveness of addressing student misconceptions of linearity through an extended Hooke's Law lab in a high school calculus class. This chapter will discuss the current research on student understandings of linearity, modeling in the high school classroom and the mathematical theory of elasticity. The topics in this chapter justify the design and purpose of this study.

2.1 Linearity

With the development of Differential Calculus, mathematicians were able to study rate of change for nonlinear models through linear approximations. However, studies have shown that students tend to overgeneralize their understandings of linearity, and therefore, misuse it in nonlinear situations (Dirk De Bock, 2002). This can hinder a calculus student's understanding and distinction between local linearity versus a nonlinear model. Students often result to counting vertical change over horizontal change for a parabola when asked to calculate the rate of change of a quadratic. This technique is not entirely wrong, but they do not understand that this is not the rate of change for the entire function. Students do not relate increasing and decreasing intervals to rate of change nor do they understand the idea of local linearity. These are problems that teachers run into as students are introduced to the derivative of polynomials and equations of tangent lines.

Linear functions and slope are two of the first and most foundational concepts in a student's mathematical career. Ratios & Proportional Relationships are a major cluster for Grade 6 in the Common Core State Standards (CCSS, 2013). The two major objectives of this cluster include: "CCSS.Math.Context.6.RP.A.1-Understand the concept of a ratio and use ratio language

to describe a ratio relationship between two quantities" and "CCSS.Math.Content.6.RP.A.2.- Understand the concept of a unit rate a/b associate with a ratio a:b, and use rate language in the context of a ratio relationship" (CCSS, 2013). This is when students are first exposed to the direct relationship between two variables. It is a foundational mathematics unit, and therefore, is what the rest of their mathematical understanding of functions will build from. It is said that the idea of linear functions is immediately understood by children because of the simplicity of the function (De Bock, 2002).

With the implementation of Common Core State Standards, nonlinear models and relationships appear much earlier than it did in previous curriculum maps. CCSS introduces nonlinear models in grade 8, "CCSS.Math.Content.8.F.B.5- Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear)" (CCSS, 2013). In previous curriculums, nonlinear functions were not introduced to a traditional student until grade 10 with quadratics. CCSS includes several strands that require students to analyze data, algebraic equations and graphs to determine the type of function it represents as early as grade 8. This gives students meaningful representations of linear and non-linear models much earlier in their mathematical career.

Because linearity is reinforced numerous times throughout a student's primary and secondary school career, they often "see and apply the linear model everywhere" (De Bock, 2002). Van Dooren and Dirk De Bock refer to this as the 'illusion of linearity.' Throughout a student's mathematics schooling, they are taught the proportional relationships between time and distance at a constant speed, diameter and circumference of a circle, and linear models and approximations in calculus. Students' continuous exposure to linearity in the classroom and everyday life results in student intuition to solve problems proportionally.

A similar misconception of linearity can be seen in student computation errors. Although this is a different approach to their reliance on a linear world, it still portrays a student's misunderstanding of linear versus non-linear relationships. Parish and Ludwig researched structures in mathematical errors. This study is solely based on computational mistakes. Research shows that two of these major typical errors occurred when students were asked to (1) square a binomial and (2) square root a binomial such that the two terms are both perfect squares. The most common incorrect solutions were as follows (Parish, 1994):

(1) $(x + 2)^2 = x^2 + 4$

(2) $\sqrt{x^2 + 4} = x + 2$

Both of these errors show students' heavy reliance on linearity and direct relationships between operations without connecting the meaning behind them. By simply substituting in numerical values, students would see the two expressions are not equivalent. However, these errors are not often evaluated by the teacher or addressed in the classroom resulting in students' continuous dependence on their linear backbone to solve and simplify problems.

Students' prior knowledge of linearity and proportionality is incompatible when introduced to new ideas such as polynomials and non-proportional word problems. These nonlinear concepts are often a challenge for a teacher to transition to in the classroom because it is difficult to build off of students' prior knowledge in these cases. In these circumstances, there is a call for conceptual change in which classroom learning requires a reorganization of a student's pre-existing knowledge (Mason, 2002). In Van Dooren's study of linear illusions in geometry, he claims that students must have meaningful learning experiences and be actively engaged in activities in order to change their original conceptual structures. These activities

should offer external representations that aid in clarifying concepts that are not as clearly defined through pure mathematical symbols and language.

Van Dooren addresses these geometric misconceptions through the idea of external representations. For example, a teacher can demonstrate the quadratic relationship between the side and area of a square in four different ways: (1) algebraically (A = side x side = side^2), (2) a table of input and outputs for different values of a side, (3) graphically illustrate a non-linear model and (4) a visual drawing of a square being covered with smaller squares (Van Dooren, 2004). The data from Van Dooren's study showed that with various representations, students' illusions of linearity were de-constructed (Van Dooren, 2004).

2.2 Modeling

The Common Core State Standards refer to modeling as a way to incorporate everyday life and decision making into the math classroom. It allows for open-ended discussion and student creativity. Modeling should make the abstract mathematics that is taught during lecture more meaningful and relevant to a high school student. CCSS suggests a modeling cycle for the high school math classroom that is shown in Figure 1 (CCSS, 2013).

Although modeling is a relatively new technique in education, De Bock recalls a very early example of using modeling as an aid in conceptual understanding. This is seen in Plato's *Meno* when a slave is asked to double the area of a square. Before Socrates offers a drawing as a visual model, the slave approaches the area problem with linear proportionality. De Bock Claims that the misuse of linearity in non-linear situations is one of the oldest in the literature of mathematical though. This early example of linear misconception depicts the importance of visualization and modeling in mathematics (De Bock, 2002).

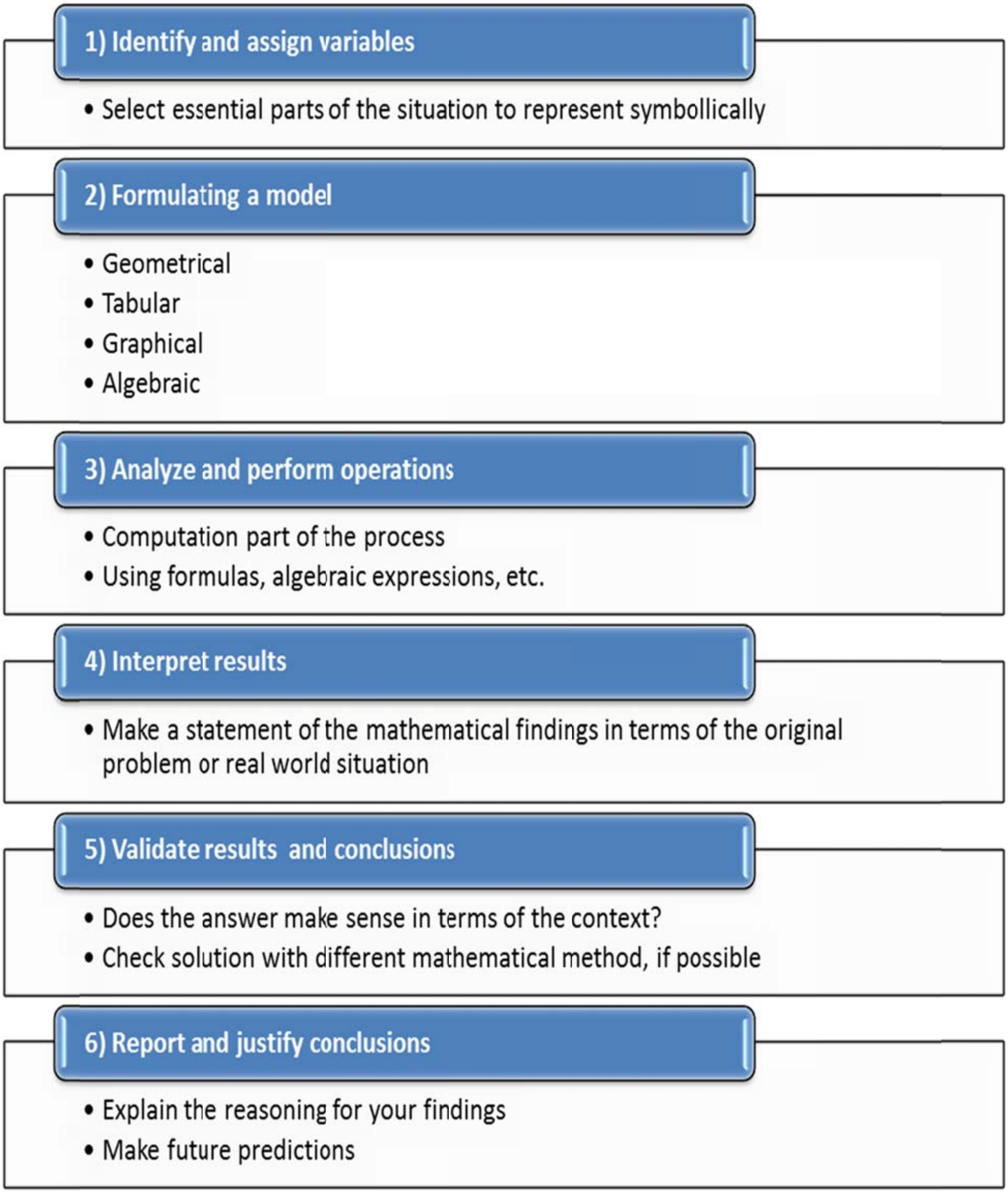

Figure 1: Common Core State Standards Modeling Cycle

Evidence has shown that modeling in the high school physics classroom can produce large gains in student understanding and coherence of the course. Wells refers to a complete solution in physics as being a model instead of just a number. Because the number is not meaningful without a given situation, modeling the problem and solution are imperative to a coherent understanding. Textbook problems are often distorted examples of physics and misguide student understanding. In contrast, solving problems comes easier through modeling with a carefully structured activity or experiment, making the problem meaningful, an aspect of physics that a textbook cannot create. Modeling is an extension of cooperative learning in which students engage in investigation, active participation, laboratories and student-centered learning. According to the physics literature, the structure of modeling depends on addressing the misconceptions expected to arise throughout the process. The teacher serves as a facilitator guiding students in the direction of both problem solving and conceptual understanding. The modeling method of instruction is defined by three stages: exploration, invention and discovery (Wells, 1995).

Stage I, exploration, gives students some type of hands-on activity to investigate. Unexpected tribulations may come with this stage, such as multiple trial failures and equipment malfunctions. This is expected and should be planned for accordingly by the teacher. For example, creating pre-made guiding questions allows teachers to redirect students without giving them the answers. Failure is part of investigation, but teachers can lead students in another direction through guidance. High school students learn from the process and will adjust accordingly to collect the data needed (Wells, 1995).

Stage II, invention, refers to the mathematical tools used to represent the investigation findings from Stage I into quantitative relationships. Students are not expected to derive these

quantitative concepts on their own; the math has already been invented. However, they are expected to use these mathematical tools correctly in order to connect the science and mathematical concepts to the model being explored (Wells, 1995).

Stage III, discovery, is the final stage within the modeling cycle. This stage is devoted to conceptual application. Students are asked to make future predictions and applications of the concept explored within the first two stages. A student is said to fully understand a concept when he/she can put it into their own words or relate it to additional situations. This stage is meant to create a holistic understanding of the concept that was modeled for the student through the activity (Wells, 1995).

Both CCSS and Wells create similar depictions of what modeling should look like in a classroom. Their interpretations share a lot of similarities, Wells' being vaguer than that of CCSS, but the overall process is the same. Throughout these stages, the teacher has a well prepared agenda of specific objectives that include concepts, terminology, expected conclusions and misconceptions to address. The prerequisites for such an activity depend on the concepts being modeled. There may need to be some type of foundational skill taught before allowing the students the freedom to explore something higher-order. Mathematical misconceptions can be addressed with modeling real world applications through scientific representations, blending the two learning processes together. With the integration of science in a math classroom, characteristics and relationships of linear and nonlinear models can be investigated (Frykholm, 2005). The effectiveness of modeling mathematics through science by incorporating a Hooke's Law lab in a high school calculus classroom was conducted for the present study.

2.3 Mathematical Theory of Elasticity

All substances resist distortion when external forces are applied. When these forces are removed, many materials recover from their deformation. Those solids that are able to recover to their original state without much harm are said to be elastic. The analyses of stress and strain, which make up mathematical theory of elasticity, are both dependent of the elastic properties of materials. Instead, stress and strain stem from the concepts of mechanics and geometrical theory on a continuous body. This mathematics is based on differential equations and integration applied to specific elastic problems and provides answers to the relationship between external forces and the deformation of a substance's body (Sokolnikoff, 1956). These mathematical ideas are far too complex for a traditional high school calculus class; however, because students overgeneralize linearity, it is important to note that the rate of change of this continuous body may or may not be linear depending on the material being tested.

2.4 Hooke's Law

"Robert Hooke gave the first rough law of proportionality between the forces and displacements" (Sokolmikoff, 1956). Hooke's Law is a linear relationship between the extension of a material and the force applied. This law is applicable when the tensile force of a substance is not too great. Tensile force is the maximum stress of a substance before its breaking or deformation point. The law can be notated $e = kT$ where e is the extension of the material, T is stress and k represents the constant of the material being displaced. Linear elasticity is useful and provides the equations needed to calculate the deformations and stresses in a structure. It also offers a geometrical interpretation of the structure. However, linear representations do not

provide analysis for limiting these deformations and stresses nor does it give a comparison of elastic behaviors between different materials (Ratner, 2003).

2.5 Non-linear Theory of Elasticity

Non-linear functions become crucial to the study of elasticity when trying to detect the physical limits of a structure. Ratner states the limit of elasticity is characterized by an increase in the rate of change of deformation. This is when calculus comes into play. Linear theory of elasticity only describes a deformation at a constant rate. For example, the derivative of the Hooke's Law function, $e = kT$, is k. This is not an accurate description for materials whose deformation rates of change are not independent of the loading intensity. Therefore, the linear function is not meaningful for structures that exhibit non-linear behavior as they are deformed. A derivative equation that describes the rate of change of the function at each point is needed. This equation will describe the variable rate of change of a structure's elastic behavior at the interval of failure (Ratner, 2003).

2.6 Calculus

Research has shown that implementing engineering and science problems on understanding calculus had significant effects and improvements on student achievement. Students that participated in integrating applied problems showed higher levels of motivation and understanding in their multivariable calculus course. Berman claims that with implementation of applied problems, students were able to "discover the meaning of the procedures' and apply it to solving similar problems. Students in this study expressed the need for implementation of applied problems in earlier mathematics and calculus courses stating, "it is a pity that applied problems were not given in the first Calculus course" (Berman, 2007).

Boas discusses in *Calculus as an Experimental Science* that calculus is used as a facilitator of studying models of observed phenomena (Boas, 1971). He claims that when the opportunity presents itself, teachers should allow students to observe Calculus concepts happening rather than asking students to trust these theorems and proofs of the Calculus are true. For example, we teach students to set the derivative equal to zero to find where horizontal tangent lines occur. This is just a recipe. There is no meaning to setting an equation equal to zero unless conceptually, the student understands what this says about the function. Boas suggests that teachers have students observe this occurrence where the derivative is zero or does not exist. Through observation of a minimum or maximum, which is where the derivative is equal to zero, the second derivative test can be avoided and students have a better understanding from the model graphically than from the seemingly meaningless algebraic representations and calculations (Boas, 1971).

Kajander and Lovric researched calculus textbooks in an effort to understand some of the misunderstandings of tangent lines and derivatives seen in first year colleges. In this study, they found that all of the examined textbooks defined the tangent line as a limit of secant lines but most did not discuss the vertical tangent line within this definition. Most of the textbooks mentioned this concept later with the definition of differentiability. This is said to be because textbooks summarize definitions to make them more clear omitting special cases in an effort to simplify the interpretation for the reader. This oversimplification results in holes and misunderstandings of the reader's knowledge (Kajander, 2009). Project 2061 was founded in 1985 and is a long-term project focusing on resources in science and mathematics. Although there are no major errors or incorrect statements made in the textbooks being investigated, there are also no textbooks that are considered satisfactory in building off of student prior knowledge.

In addition, no textbook investigated has shown to address and/or help overcome student misconceptions. Project 2061 findings show that textbook authors do not examine student understanding or their prerequisite understandings as a guide to writing textbooks (Project 2061, 2003). As a result, student understanding cannot be dependent solely on school textbook resources and examples. Teachers must find other means of delivering a concept and building off of students' prerequisite skills and understandings. With the modeling initiative, instructional research offers an alternative to traditional textbook practice in the classroom.

2.7 Future Research

"The Common Core State Standards focus on core conceptual understandings and procedures starting in the early grades, thus enabling teachers to take the time needed to teach core concepts and procedures well" (CCSS, 2013). H. Wu's concludes in a study of basic skill versus conceptual understanding that there is no "royal road" to conceptual understanding. Wu claims that there is not one without the other in that both skill and concepts must be taught hand-in-hand (Wu, 1999). Conceptual understanding is difficult to build from if misconceptions exist. Dirk De Bock's research has shown that the misconceptions of linearity do exists, but he calls for further research on these misconceptions and how to address them in the classroom (Dirk De Bock, 2002). Modeling mathematics through science has been proven to clarify student misconceptions. The present study uses the modeling approach to incorporate a lab into a high school math classroom in an effort to address and correct student misconceptions of linearity.

CHAPTER 3: THE STUDY

Chapter 3 elaborates on the rationale for incorporating a hands-on experiment into a high school math classroom based on the Standards of Mathematical Practice implemented with the Common Core State Standards. Additionally, the participants and demographics of the high school are described.

3.1 Rationale

This study focused on high school students' misunderstandings of linearity in a differential calculus class in which students had already encountered linear and non-linear functions is previous courses. Misconceptions of linearity and non-linear models are what leads to misunderstandings in higher-level mathematics due to a student's inability to connect his/her prior knowledge when misconceptions exist. Students continue to apply linear and proportional relationships to non-linear situations because of their lack of conceptual understanding of functions. One of the goals of implementing the Common Core State Standards is to create a more holistic mathematical understanding in primary and secondary education. Although there are no current Common Core State Standards for calculus, there are Standards for Mathematical Practice that accompanies the standards which are applicable to any math classroom. These practices suggest eight instructional routines and classroom characteristics that lead to a successful high school mathematics classroom. Although all of these practices are effective and should be observed in any math classroom, several in particular, support the main goals of this study (Table 1).

Table 1: Standards of Mathematical Practice observed during this study

Mathematical Practice	Standard	Descriptor
CCSS.Math. Practice.MP1	Make sense of problems and persevere in solving them.	Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends.
CCSS.Math. Practice.MP4	Model with mathematics	Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.
CCSS.Math. Practice.MP5	Use appropriate tools strategically	Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. They are able to use technological tools to explore and deepen their understanding of concepts.
CCSS.Math. Practice.MP6	Attend to precision	Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other.

Math Practice 1 and Math Practice 4 are observed throughout this study because students are asked to analyze function relationships in many forms. Students created data tables, graphed their results, interpreted the relationship between coordinates and created equations to represent tangent lines at different positions. With these various representations of data, students interpreted their results into words to make sense of the lab and relate the math to the experiment that they performed. Making sense of the mathematics allows them to predict future results and relate their understandings to similar situations, or materials, for this particular lab.

As stated in Math Practice 5 and Math Practice 6, students should be proficient in the use of mathematical tools available to them in the classroom. Activities should be designed so that students are exposed to these tools and learn how to use them correctly. Precise information should be collected for the most accurate and reliable results. These practices were observed throughout the lab as students stretched different materials to measure the displacement in metered distance versus force applied in Newtons. As a pre-lab activity, students familiarized themselves with spring scales, yard sticks, ring stands, units of measurement for force and displacement, spreadsheets, *Wolfram Mathematica* and linear and non-linear fitted regressions. Understanding these parts of the lab created a smoother process for me as the teacher when answering questions and troubleshooting throughout the activity.

These four practices were evident throughout the study. The lab created a variety of experiences for the students allowing them to explore with tools, be precise and accurate with measurement, interpret their results and make sense of the mathematics taking place. Anytime a real world or hands-on activity can be incorporated into a mathematics classroom, the CCSS math practices are easier to observe because of the many stages involved with modeling.

3.2 Participants

Destrehan High School is a public school in Destrehan, Louisiana. The total population is approximately 1,500 students. Although a relatively large population, Destrehan is a small community outside of New Orleans, Louisiana and consists of generations of families that have attended this high school over decades. It is a suburban school that sets high expectations for both students and teachers. Figure 2 represents racial demographics of Destrehan High School's population.

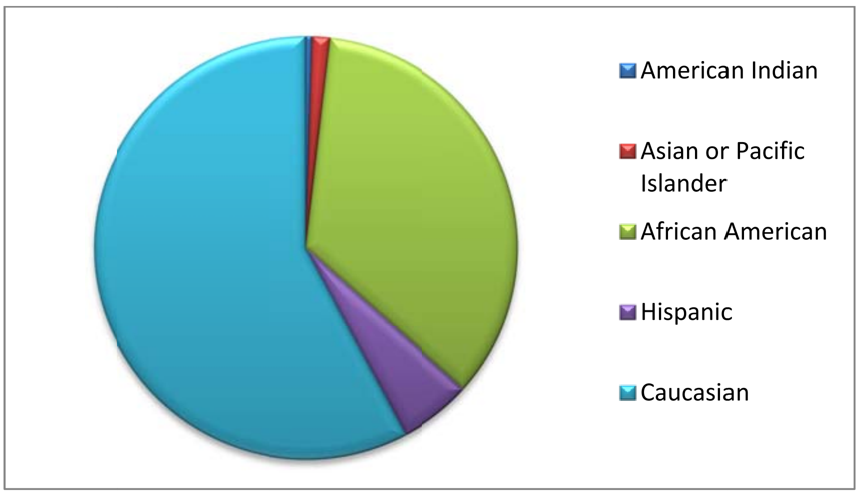

Figure 2: Destrehan High School demographics
Destrehan High School consists of approximately 58% Caucasian, 35% African American, 6% Hispanic, 1% Asian and less than 1% American Indian.

Twenty high school seniors participated in this study ranging in age from 17-18. The study was conducted with a traditional Calculus class. These are all students that, at some point in their high school career, were studying honors mathematics but opted out of Advanced Placement Calculus. Therefore, although in a traditional Calculus class, these are cognitively advanced students. The traditional calculus class population is seen in Figure 3.

All students participated in the study. Destrehan High School runs a 4x4 block schedule in which students take four classes in the fall semester and change schedules for an additional four classes in January for the spring semester. Due to the 4x4 set-up, the traditional calculus class only has time to cover differential calculus including limits (but excluding infinite limits), derivatives and rates of change.

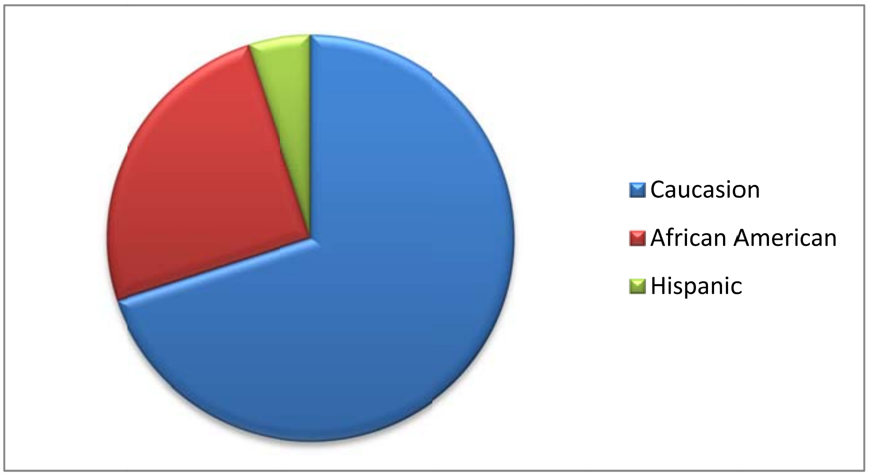

Figure 3: Destrehan High School traditional calculus class population
The calculus class consisted of 70% Caucasian, 25% African American and 5% Asian students.

CHAPTER 4: METHODOLOGY

This chapter describes the design of the study. The experiment was conducted in a differential calculus class at Destrehan High School in Fall 2013. I was the teacher of the class which was offered to 20 students who were considered advanced, but were not enrolled in the honors or Advanced Placement programs. Topics of the course included limits, derivatives and their applications. All students participated in the pre- and post-test. Students were chosen at random to participate in the Hooke's Law lab while the control group continued textbook practice. Seven of the twenty calculus students were randomly selected to participate in the Hooke's Law lab activity. The other thirteen students continued traditional textbook practice. The set-up of the lab was designed using the three stages of modeling based on the physics literature as discussed in Chapter 2.

4.1 Pre-tests

There were two pre-tests given before the course began. I created a test that assessed students' understanding of linearity and rate of change in a multiple choice and free response format called Linearity Pre-Test (Appendix B). The design was based on past research has shown about student misconceptions of linearity. It consisted of both multiple choice and free response portions. The multiple choice section included questions on the effect on area of increasing the dimensions of a rectangle, the slope of a linear function, the graph of a quadratic function and predictions about the function models for the stretch of a spring and elastic string. The free response section required students to analyze critical points on a distance-time graph and explain what was happening in context of the distance of a car traveling over time.

The second pre-test administered was the Calculus Concept Inventory (CCI) written by Jerome Epstein, Professor of Mathematics at Polytechnic University. This is a calculus test that focuses on the conceptual understandings of rate of change, the derivative and applications of differential calculus. The tests consists of twenty-two multiple choice questions, but fourteen of the twenty-two responses were focused on for this study. These fourteen questions were chosen because they assessed student understanding of either rate of change, linearity or the derivative. Due to the block schedule, curve sketching is not taught in the traditional Calculus class so these additional eight questions that appeared on the CCI were omitted.

Both of these pre-tests were given before any instruction began. The first pre-test provided baseline data on student understanding of rate of change, different types of models and linearity misconceptions. The CCI provided additional baseline data for student prior knowledge of rate of change and any pre-existing knowledge of the derivative. Students were given the entire class period of 90 minutes to complete these tests at their own pace. However, all students had completed all three tests within the first hour of class since many of them were merely guessing for the CCI having no prior knowledge of terms such as derivative, differentiate and $f'(x)$.

4.2 The Lab

Most of the course was taught in the traditional manor, student notes, guided and independent practice. The lab activity was conducted after derivatives and rates of change hand been introduced. It was the last activity of the semester before the final exam.

College Board's Advanced Placement program offers a standard Hooke's Law Lab for a high school classroom where students experiment with the displacement of a spring and applied

force leading to the discovery of a linear relationship between displacement and force (Jacobs, 2014). For this study, the traditional Hooke's Law Lab was altered to create both linear and non-linear paths for students to explore. Two professors from the Louisiana State University Mathematics Department advised me in designing the lab. The lab included different materials—springs, rubber bands and elastic string—that would display both linear and nonlinear behavior when force was applied. Experiments with these materials would provide a set of data that could be graphed and analyzed to challenge student misconceptions of universal linearity versus local linearity on a curve.

The materials included different sized springs and elastic string. Our lab was an extension of a traditional Hooke's Law lab. In addition to testing linearly elastic springs, this lab went further by investigating the behavior of other materials outside the domain of linearity. Students measured the force applied in Newton's (dependent variable) as the material was displaced to various selected lengths (independent variable).

Stage I of the modeling method incorporated students participating in a hands-on activity to explore. In the basic lab experiment, a selected material was hung from a ring stand. A spring scale was attached to the loose end. Various forces were applied to the free end of the spring scale. The length of the material was measure and recorded, as well as the reading on the scale. Students were grouped in pairs. The apparatus used for the Hooke's Law lab is seen in Figure 4. One student displaced the material while the other recorded the data. Two different materials were tested, and data was collected for each material.

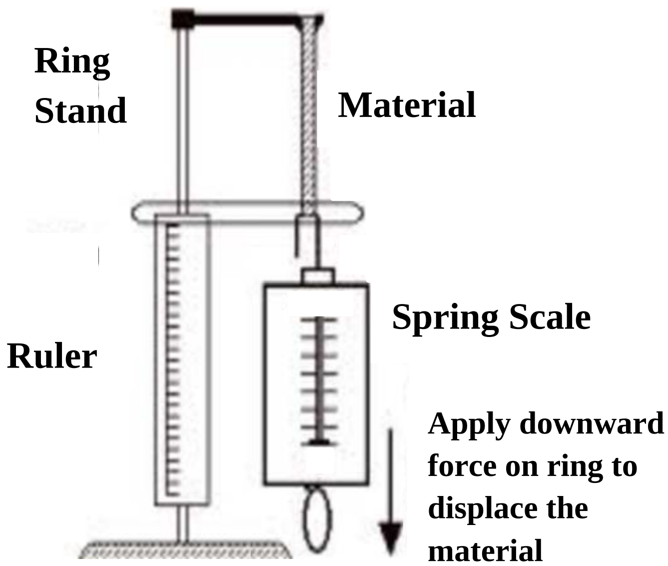

Figure 4: The apparatus used for the Hooke's Law experiment

Stage II of the modeling cycle introduces calculation and quantitative relationships. In this stage, students were given a template for entering their data into *Wolfram Mathematica.* When executed, this template graphed the data collected from the labs. I was able to each group their graphed data and the corresponding fitted curves of degree 1, 2, 3 and 4. Students were then able to see the best fit function for each of their experimented materials.

Because of the time constraint, most groups were only able to collect ten trials for each material. For this purpose, an ideal elastic model was used for the next phase of the lab, the post-lab activity, based off of pre-collected and graphed data of an elastic string by the teacher. Students were then given a lab packet for the post-lab activity. They calculated average rates of change between neighboring points in the data set. They used this to estimate the derivative and write equations of tangent lines near these points. Students were given no teacher guidance on how to perform these calculations, but were able to discuss these problems with one another. Not

that they had already lecture on average rate of change, derivatives and equations of tangent lines.

Stage III, the application stage, was also accomplished through the post-lab packet. After students calculated average rates of change and derived equations of tangent lines, they were asked to explain the meaning of these things in terms of the phenomena they had observed during the lab. This is where the student's conceptual understanding was challenged. Key features of the graph explain important aspects of the material's displacement and rate of change. This part of the post-lab packet allowed students to relate both the calculus and the real world aspect of the lab activity. Interpreting rate of change and applying it to a given situation allowed them to formulate a more complete understanding of the concept and its applications to real life.

4.3 Post-tests

The post-tests were given to all students the day after all parts of the lab were completed for the experimental group and the textbook practice was completed by the control group. The two post-tests mirrored the pretests. The Calculus Concept Inventory was given first, followed by the Free Response and Multiple Choice Linearity tests. Students were given the entire class period of 90 minutes to complete all three tests at their own pace. All students finished within the 90 minutes, but as opposed to the pretests, most students took the entire time to reflect on their answers and really dissect each question of the post-tests.

CHAPTER 5: ANALYSIS OF DATA

In Chapter 5, results of the pre- and post-tests are revealed. Two-sample t-tests and normalized learning gains were calculated to examine student performance between the control and experimental groups. Pre/post-test analysis and student artifacts are also discussed. Student scores on the two linearity tests (multiple choice and free response) and scores on the Calculus Concept Inventory were used as baseline data. A two sample t-test test was calculated in Microsoft Excel which showed no significant difference in the CCI pre-test scores ($P > 0.05$) indicating similar student prior knowledge. In addition, a two sample t-test test was run on both linearity pre-tests indicating similar prior knowledge as well ($P > 0.05$). Both pre-tests were administered on the first day of class prior to any lessons being taught. Pre-test scores showed that student misapplications of linearity existed. This supported the literature on suggested common misconceptions of linearity.

5.1 Results from Pre-test to Post-test

After the post-lab activity, all students were assessed again on rate of change and linearity. The same linear tests were administered. The answers to the pre-tests were never given during the semester. Additionally, other than having completed the lab, the experimental group had no advantage over the control. The lab packets and/or discussions did not include the type of questions that appeared on the pre/post-tests. Pre/post-test questions included very basic ideas of linearity, proportionality and rate of change. Scores on the multiple choice and free response linearity post-tests showed significant difference, $P = 0.000725$ and $P = 0.009$ respectively, between the control and experimental groups. Results showed that there was significant difference on the linear assessments from pre- to post-test performance; however, there was no

significant difference in performance between the two groups on the Calculus Concept Inventory ($P > 0.05$).

Based on individual student normalized learning gains, the experimental group had an overall better understanding of slope and its applications after they had participated in the Hooke's Law activity. Normalized Learning Gains (NLG) were calculated using the formula $< g >= \dfrac{\text{Student's individual gain}}{\text{Possible total gain}}$. Once individual gains were computed, normalized learning gain means were found for each group. The control group's NLG mean on the linear pre-test was 0.03 while the experimental group's NLG mean was 0.91. This is a significant difference in student growth and performance from the pre-test scores. These differences between the control and experiment groups can be seen in Figure 5 and Figure 6, representing the score of each calculus student on the pre- and post-tests.

In addition to the two linear post-tests, the Calculus Concept Inventory was administered again after the lab had been completed but before it was time to review for the final exam coming up. The effect of the lab did not show to be significant on student understanding of the basic differential calculus ideas as it did on slope and linearity. Though there was growth, there were smaller student gains on the CCI. The control group's NLG mean was 0.04, and the experimental group's NLG mean was 0.11. This shows no significant difference ($P = 0.58$). Therefore, the Hooke's Law type lab affected the experimental groups' understandings of rate of change and linear concepts more than it did on differential calculus concepts.

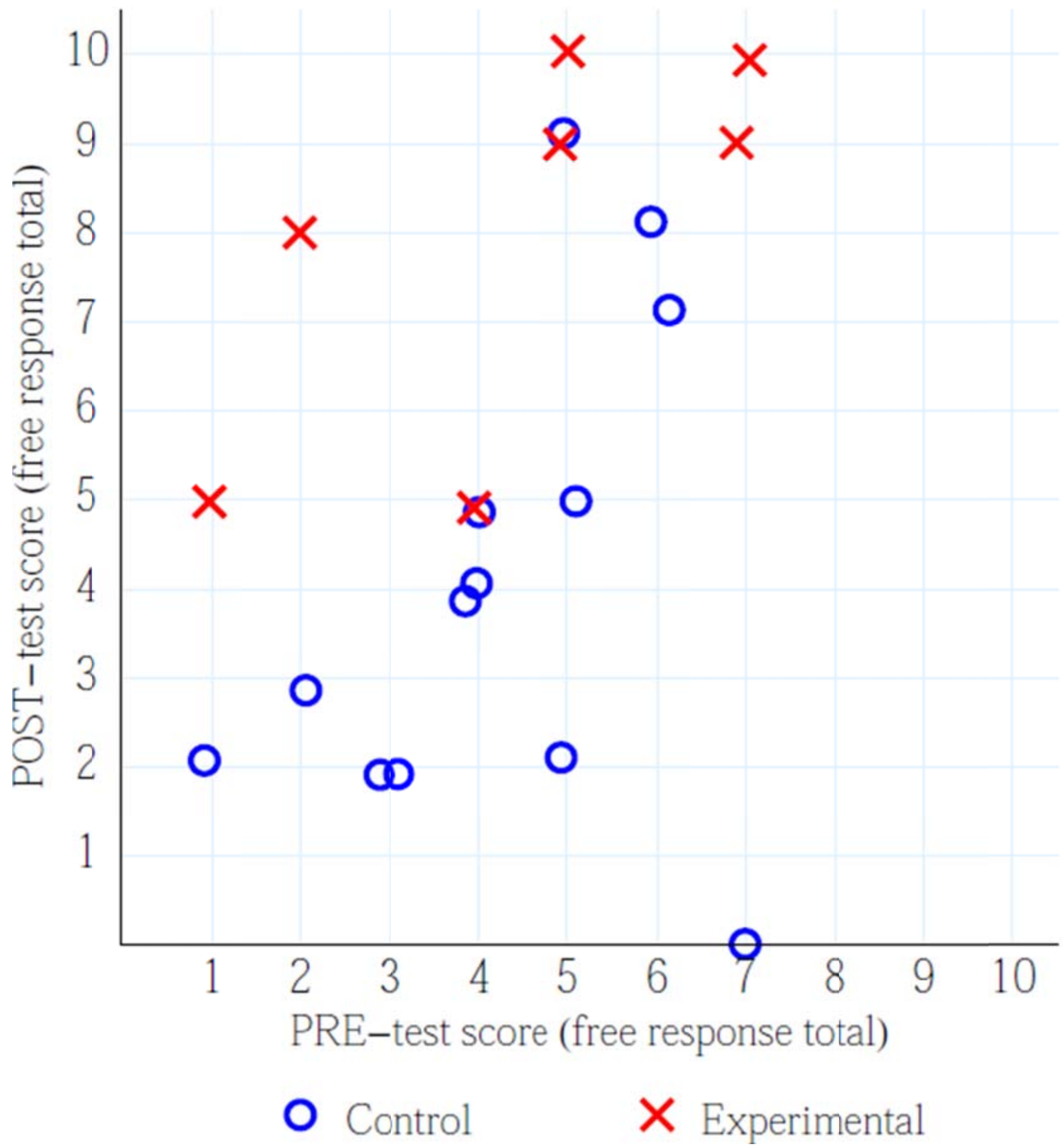

Figure 5: Free Response Linear Pre-Test versus Post-Test Scores
This is a comparison of pre-test (x-axis) to post-test scores (y-axis) of each student on the free response portion of the linearity tests. The red x's represent each student score part of the experimental group. The blue circles represent control group student scores. Questions were scored based on level of correctness of each answer. There were 5 questions for a total of 10 points.

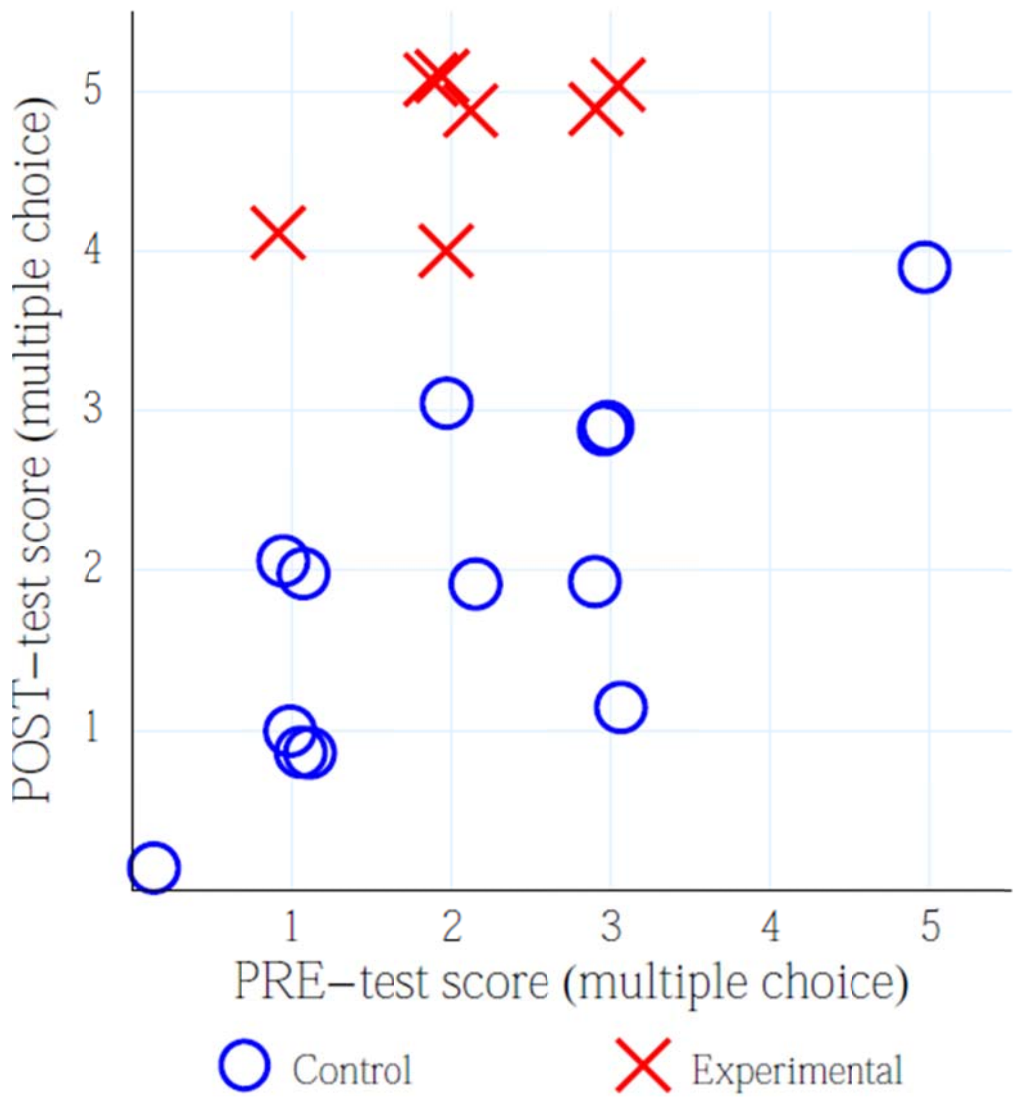

Figure 6: Multiple Choice Linear Pre-Test versus Post-Test Scores
This is a comparison of pre-test (x-axis) to post-test scores (y-axis) of each student on the multiple choice portion of the linearity tests. The red x's represent each student score part of the experimental group. The blue circles represent control group student scores. Scores are based on the number of questions students answered correctly. 1 point was awarded for each correct answer for a possible total of 5 points.

5.2 Post-test Responses

The Linearity Free Response assessment was a more difficult assessment to grade because it was free response. The post-tests were graded by their traditional calculus class teacher. Each question was scored based on whether it was completely correct (2 points), partially correct (1 point) or incorrect (0 points). The distinction between correct and partially correct was the students' use of the situation in their answer. If the student related their interpretation of the graph to the car traveling distance over time, then they received the full 2 points. However, if the student explained the correct graphical behavior but did not reference the context of the problem, he/she received 1 point. Question 2 on this test was focused on because of the nature of this study on linearity. Given a distance/time graph of a traveling car, students were asked to explain the behavior of the car over a specific time interval. Based on the interval that the students were given, the correct answer should be that the car is stopped. There was a major difference in graphical interpretation of this interval between the experimental and control groups as seen in Figures 7-12. Only 15 percent of the students from the control group answered this question correctly, while all of the other students in the control group claimed that the car was moving at a constant speed. From the experimental group, 71 percent of the students answered this question correctly showing a clear gap in conceptual understanding of this constant slope relative to a real world situation. Because the post-lab activity required the experimental group to interpret specific features of the non-linear model and explain these features in the context of the lab, students were able to relate their understandings to another situation. Applications and making future predictions is seen in the final stages of the modeling process.

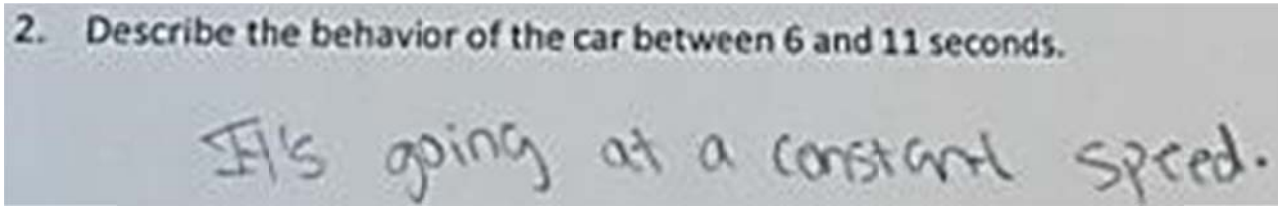

Figure 7: Post-test student 1 response from control group
Student claims the horizontal line in the graph represents a constant speed of the car. This is incorrect.

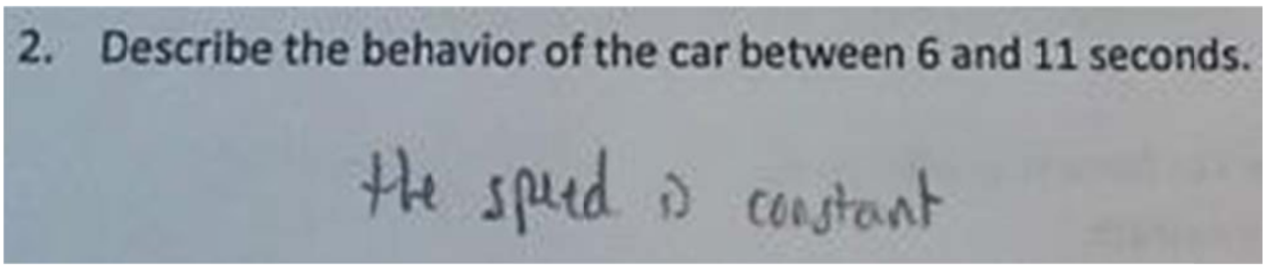

Figure 8: Post-test student 2 response from control group
Student is incorrect stating the horizontal line in the graph represents a constant speed of the car.

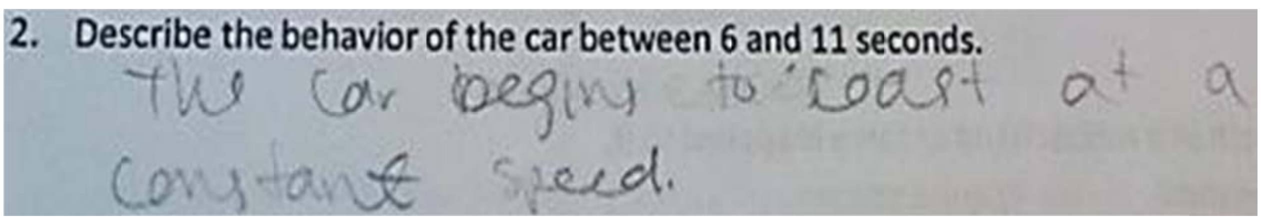

Figure 9: Post-test student 3 response from control group
Student is incorrect in claiming that the horizontal line in the graph means that the car is coasting.

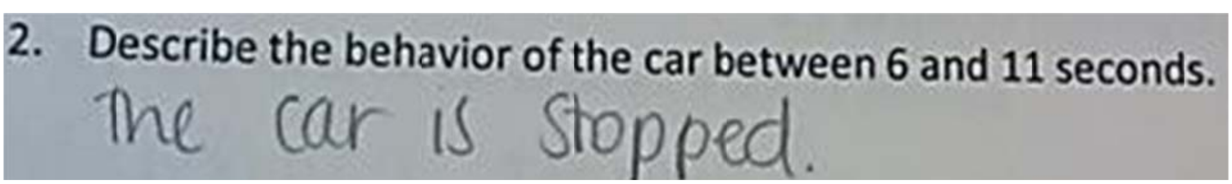

Figure 10: Post-test student 1 response from experimental group
Student is correct in stating that the horizontal line on the graph means that the car has stopped.

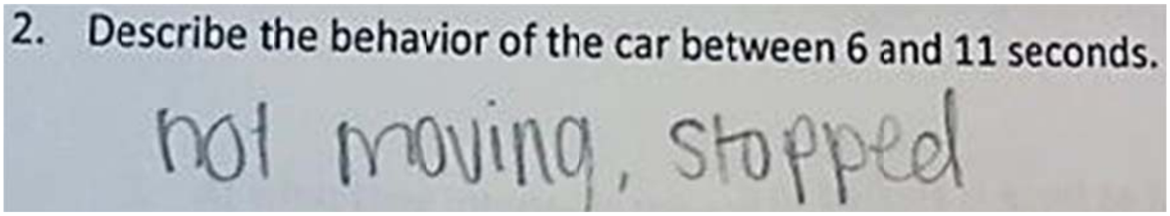

Figure 11: Post-test student 2 response from experimental group
Student is correct in stating that the horizontal line on the graph means that the car is not moving.

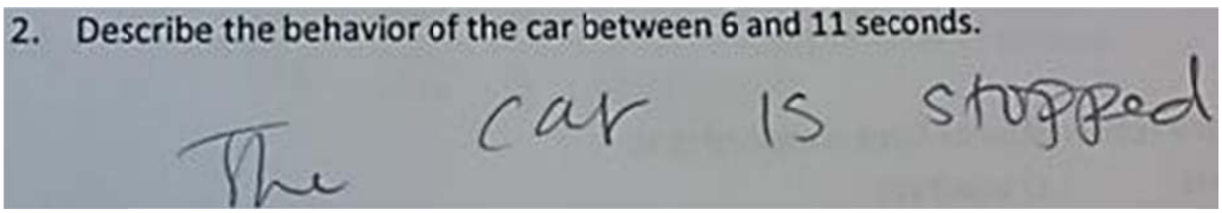

Figure 12: Post-test student 3 response from experimental group
Student is correct in stating that the horizontal line on the graph means that the car has stopped.

An interesting finding that was not originally planned for the study came from student responses on their final exam, as seen in Figures 13, Figure 14, Figure 15 and Figure 16. Although students were asked about the relationship between a scale factor and the area of a figure on the Linearity Multiple Choice pre/post-test, these open ended responses were added to the final to grasp reasoning behind their understandings. This again reinforces the impact of the lab. The question asks how the areas compare from a pre-image to an image after being dilated by a scale factor of 3. 50% of the students from the control group stated there was a direct relationship between the scale factor and area; whereas, 75% of the students from the experimental group claimed the area would be squared.

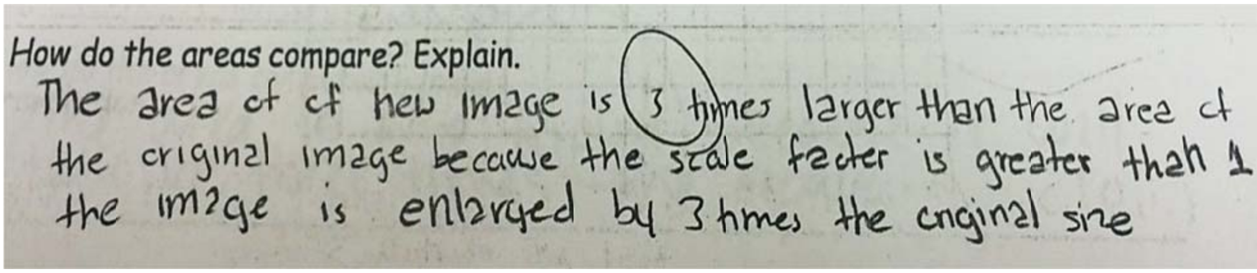

Figure 13: Student response on area from control group
Student claims there is a direct relationship between a scale factor and the dilated area of an image.

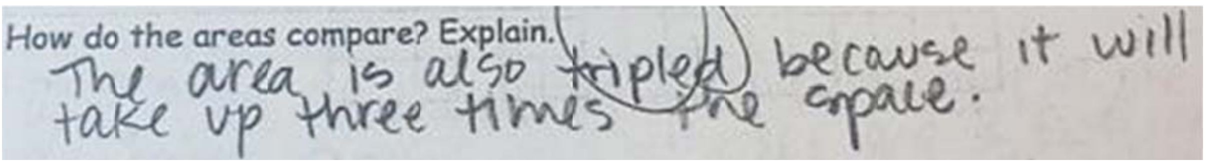

Figure 14: Student response on area from control group
Student claims there is a direct relationship between a scaled factor and the dilated area of an image.

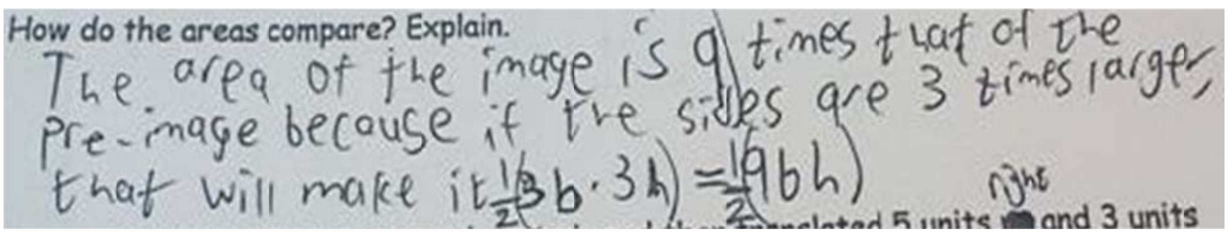

Figure 15: Student response on area from experimental group
Student claims that the dilated image's area will be a result of the squared scale factor.

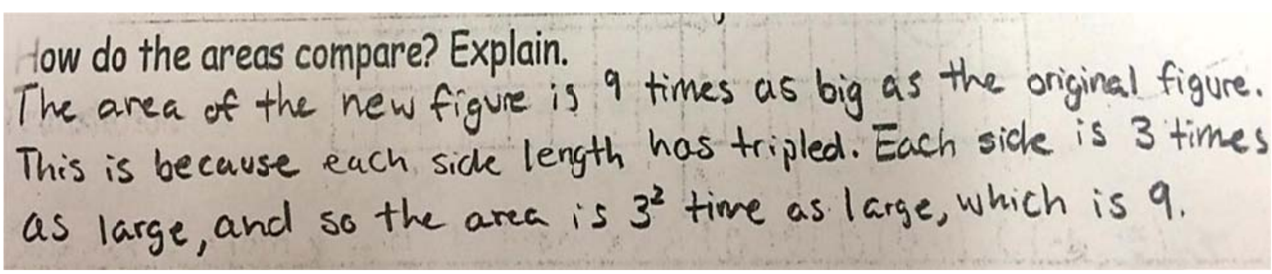

Figure 16: Student response on area from experimental group
Student claims that the dilated image's area will be a result of the squared scale factor.

The students in Figure 15 and Figure 16 are correct in claiming that the relationship between an image and its area is quadratic. In Figure 13 and Figure 14, students claimed this relationship would be linear due to their misconception of a direct relationship between a scale factor and area. This may be because students in the control group had more difficulty visualizing a non-linear model to draw accurate conclusions. It is not determined if the lab was directly related to the difference in responses, but can be assumed by the significant difference in the performance on the linear reasoning pre/post-tests, that there was some impact as a result of the lab on the experimental groups' understanding of linear and non-linear relationships.

5.3 Student Artifacts

The Hooke's Law lab activity was conducted over two 90 minute class periods. On the third day, students were given a template to enter their data into *Wolfram Mathematica* for graphical analysis. Figure 17 shows one group's results from stretching a tightly coiled spring. Students found that a spring's rate of change was constant as it was displaced. Data from the linearity pretest suggest that the students may have already assumed this, but they could not justify their answer choice mathematically prior to the lab activity. On the pre-test, students were asked what model would be produced by a spring's stretch as force was applied—40% of the students chose linear, 25% selected constant and the remaining 35% chose either quadratic or polynomial of degree ≥ 3. These results show that the majority had an intuition of the linear behavior of a spring, and the lab was able to confirm their beliefs through data collection.

```
springdata2(*spring,{length (cm),force (grams)}*) =
  {{6, 60}, {7, 700}, {8, 800}, {9, 1000}, {10, 1100}, {11, 1400}, {12, 1600}, {13, 1700}, {14, 1900}, {15, 2000}, {16, 2200}, {18, 2400}};
f = Fit[Take[springdata2] , {1, x}, x]

-693.459 + 181.162 x

Fit[springdata2, {1, x}, x]

-693.459 + 181.162 x

Graphics[{Map[Point, springdata2], Thick, Red, Line[{{x, f} /. x → 6, {x, f} /. x → 18}]}, AspectRatio → 1, Axes → True]
```

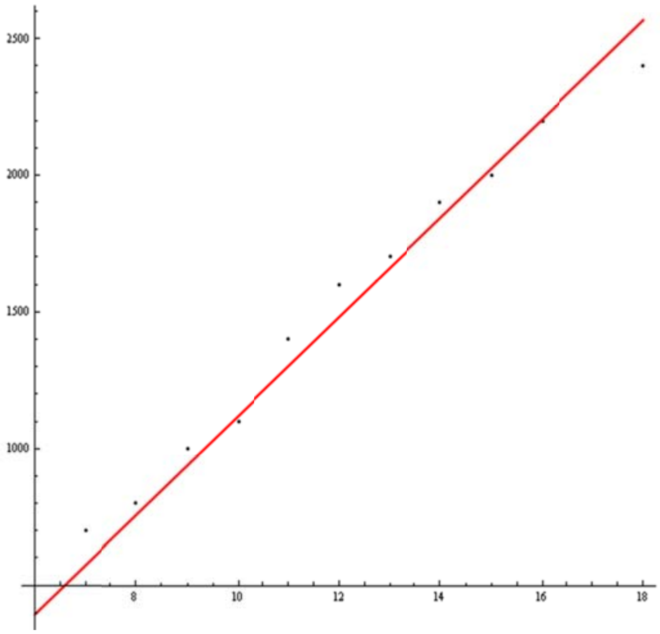

Figure 17: Student graph from data collected during experiment displacing a spring
Linear regression from student data collection of displacing a spring and measuring the applied force as stretched.

During an informal class discussion, students were asked why they chose a linear model for the spring stretch. They claimed that springs are visually stretched in a straight line. This pre-conceived idea is later discredited when they see that although all materials are physically stretched in a straight line, their rates of change can be non-linear. Figure 18 shows a group's findings after displacing a piece of elastic string. This relationship between displacement and force was proven to be non-linear through the students' data and graphical analysis. Both materials chosen by each group were stretched to their distorted points. Students displaced each material until it either broke or permanently became deformed.

Wolfram *Mathematica* | FOR STUDENTS

```
elasticdata(*elastic string,{length (cm),force (tenths of a N)}*) = {{20, .1}, {21, .1}, {22, .2},
   {23, .25}, {24, .3}, {25, .3}, {26, .4}, {27, .45}, {28, .5}, {29, .55}, {30, .6}, {31, .8}, {32, 1.0},
   {33, 1.1}, {34, 1.2}, {35, 1.3}, {40, 2.25}, {41, 2.4}, {42, 2.75}, {43, 3.25}, {44, 4.0}, {45, 4.25},
   {46, 4.9}, {47.5, 5.5}};

ff4 = Fit[elasticdata , {1, x, x^2, x^3, x^4}, x];
ff3 = Fit[elasticdata , {1, x, x^2, x^3}, x];
ff2 = Fit[elasticdata , {1, x, x^2}, x];
ff1 = Fit[elasticdata , {1, x}, x];
```

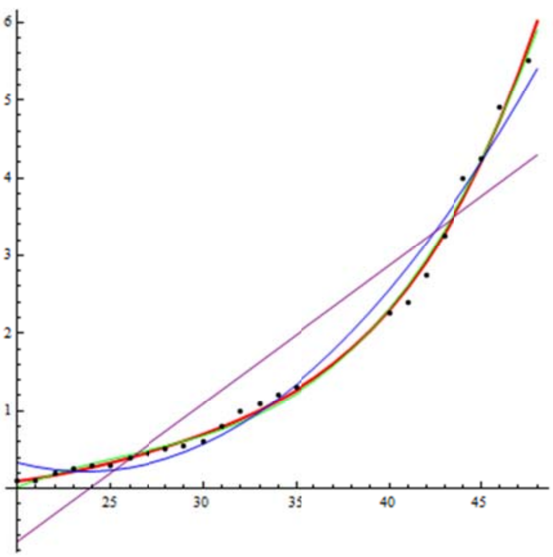

Figure 18: Graph that models student data collected while stretching elastic string
Student group's graphed data from stretching elastic string (x-axis) and force applied (y-axis).
The purple line represents the best linear model, blue quadratic, green cubic and red quartic.

Due to varying results from each group, I conducted the lab with elastic string and collected data that was more precise than that of the high school students' to create the post-lab activity. The post-lab activity asked the students to estimate rate of change at five different points on the graph and to calculate equations of tangent lines at these chosen points. Figure 19 shows the graph that was given to the students in the post-lab activity.

Wolfram *Mathematica* | FOR STUDENTS

```
data(*thin rubber strand from cloth elastic, {length (cm), force (tenths of a gram)}*) =
 {{25, 149}, {26, 153}, {27, 161}, {28, 166}, {29, 170}, {30, 172}, {31, 181}, {32, 183}, {33, 193},
  {34, 216}, {35, 227}, {36, 238}, {37, 251}, {38, 260}, {39, 272}, {40, 284}, {41, 297}, {42, 309}, {43, 319},
  {44, 331}, {45, 344}, {46, 356}, {47, 372}, {48, 385}, {49, 400}, {51, 430}, {53, 461}, {55, 480}, {57, 551},
  {59, 592}, {61, 645}, {63, 705}, {65, 800}, {67, 901}, {69, 1041}, {71, 1161}, {73, 1356}, {75, 1636},
  {77, 1970}, {79, 2296}, {81, 2501}};
```

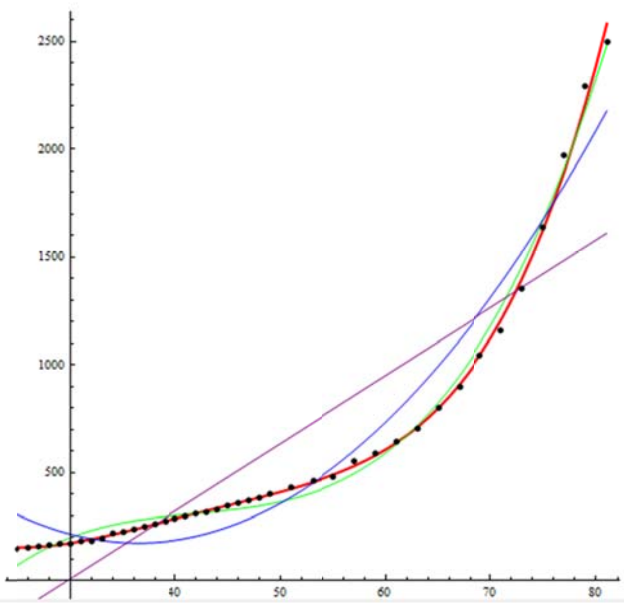

Figure 19: Quartic model graphed from teacher data collected while stretching elastic string
The red line indicates a 4th degree polynomial which is the best fit model for this set of data that models the rate of change of displacement of the material as force is applied. This graph was included in the post-lab activity where students were asked to choose five different points to calculate slopes and tangent line equations. The purple line represents the best linear model, blue quadratic, green cubic and red quartic.

The post-lab activity assessed computational and conceptual understandings of rate of change. Students were prompted to analyze the graph and explain the meaning of specific graphical characteristics in terms of the lab and material. Figure 20 shows one definition of the derivative in a student's own words based on the lab.

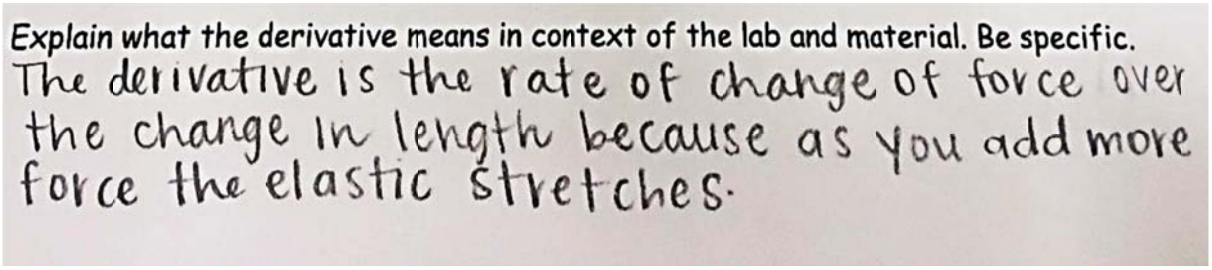
Explain what the derivative means in context of the lab and material. Be specific.
The derivative is the rate of change of force over the change in length because as you add more force the elastic stretches.

Figure 20: The meaning of a derivative in context
This figure shows a student response to the meaning of a derivative in context of the lab conducted. This is based off of stretching elastic at a constant rate and measuring the force applied.

Similar responses were given to the question presented in Figure 20 by all 7 students in the experimental group. Something missing from each of these responses was the reference to a derivative being the rate of change at a particular point in time. However, requiring students to put into words how a derivative, or rate of change, directly relates to a real world situation, allows them to connect the math in a meaningful way.

Another question that compelled students to analyze the graph and bring meaning to the mathematics is seen in Figure 21. This question asks the meaning of a vertical tangent line relative to the lab. Because students were directed to stretch the material until it either broke or deformed, students were able to see the behavior of the rate of change at this critical time interval. Through observation during the lab and post-lab activity, students noticed an increase in the elastic's rate of change as the material began to break which they related to the vertical tangent line, or asymptote, of the graph. This is illustrated in Figure 21 as a student response and in Figure 22 as student's scratch work of tangent line sketches.

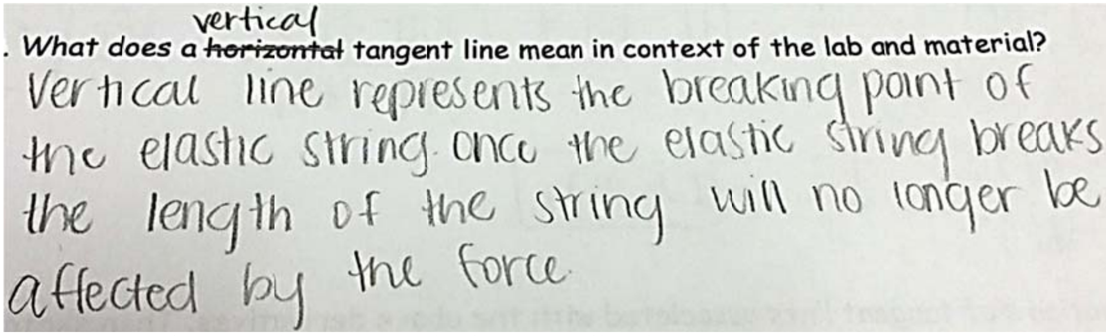

Figure 21: The meaning of a vertical tangent line in context
Student response to the meaning of a vertical tangent line based on the conducted lab. Student claims that the vertical tangent line exists on the graph at the breaking point of the elastic string being stretched.

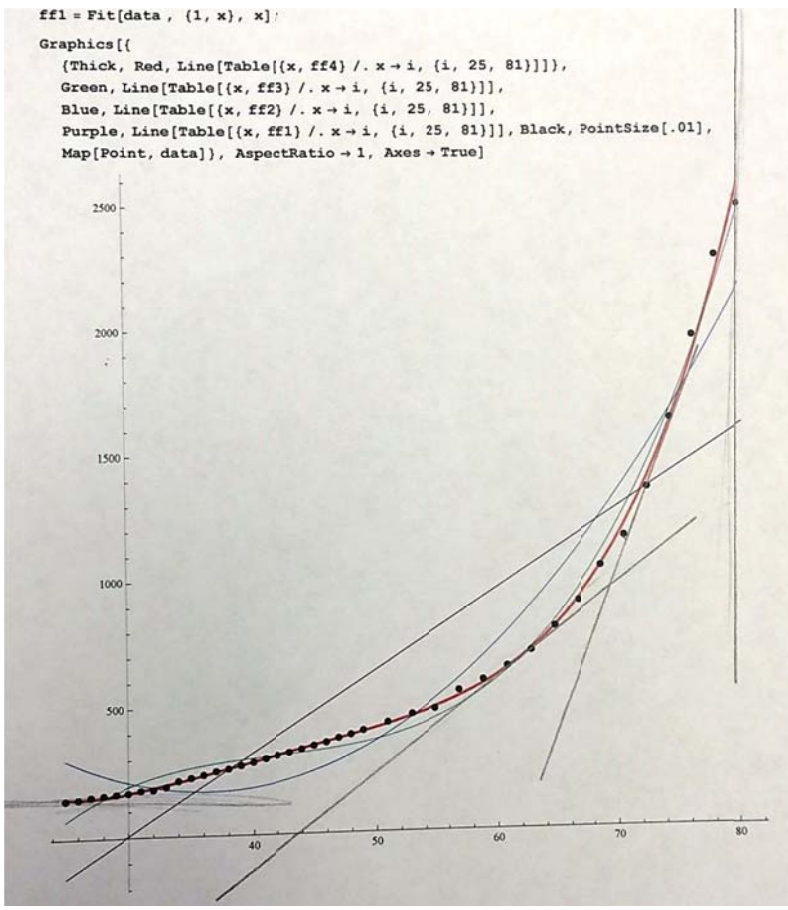

Figure 22: Vertical Tangent Line Student Sketch
Student scratch work on post-lab activity illustrating where he/she believes the vertical tangent line exists.

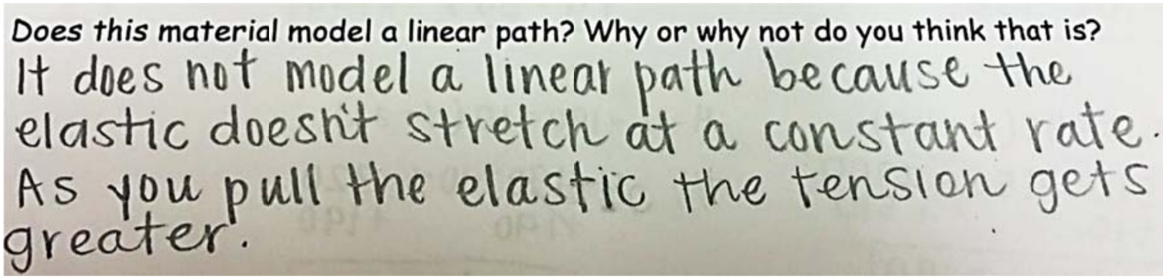
Does this material model a linear path? Why or why not do you think that is?
It does not model a linear path because the elastic doesn't stretch at a constant rate. As you pull the elastic the tension gets greater.

Figure 23: Nonlinear Path
Student response on why the stretch of the elastic is nonlinear. However, this justification excludes a discussion of the composition of the material.

Lastly on the post-lab activity, students were asked to describe whether or not the elastic string modeled of the material was linear and why. Figure 23 gives an example of a student response. All students in the experimental group found the path to be nonlinear for the elastic string and linear for the spring. While monitoring the activity, it seemed that students justified this nonlinear path with the fact that individual strings within the elastic were breaking at different rates as opposed to the material of a spring which is not made of smaller individual pieces of material. This is another interesting misconception that arose which was anticipated as Sokolnikoff claimed in *Mathematical Theory of Elasticity.* This was addressed on an group-by-group basis where I explained to each group the idea of a continuous body being deformed instead of individual pieces of material.

CHAPTER 6: CONCLUSIONS

Data collected comparing pre- to post-tests showed significant difference between the understandings of linearity and nonlinear applications of the control and experimental groups. Calculated learning gains showed that the experimental group's overall knowledge of rate of change, linear and non-linear applications grew significantly after completing the extended Hooke's Law lab that explored displacement versus force for various materials. The three stages of modeling—exploration, invention and discovery—were used to create the structure of the lab. Through teacher observations, the modeling method seemed to be a beneficial progression as students experimented, collect and analyzed data.

Results of this study support past studies proving that modeling can be an effective tool for not only learning, but addressing misconceptions, in the high school classroom. Though this was done on a small scale, the lab created a real world connection to functional relationships allowing students a visual of the mathematics taking place. In addition, they were able to model their findings graphically to analyze critical points of the experiment to relate them mathematically. Previous studies have claimed when students can connect to the material, they have a clearer conceptual understanding and are able to apply a concept to future situations. This study also supports previous studies conducted on student misconceptions of linearity. Pre-test responses illustrated the proportional relationship that students rely on to represent nonlinear situations. This is an existing issue in the high school classroom. It is the most basic function that students encounter in school and continues to be their go-to equation. The lab contradicted many student beliefs of what they thought would happen during the lab, and therefore, gave them reason to change their original ideas of linear models globally versus locally. It is important to note that there was no significant difference between the control and experimental groups'

performance on the Calculus Concept Inventory post-test. This suggests that the lab directly affected student understanding of linearity and rate of change, not calculus, implying that there were no likely outlying factors such as student motivation or enthusiasm within the experimental group.

If I could do this study differently, I would require students to give more precise measurements during the experiment. Some groups' data collections were small and did not produce adequate models to use in the next phase of the lab where rates of change and tangent line equations were calculated. This study could be expanded by incorporating more materials during the experiment for the students to displace. I would also like to have them investigate the interval of failure of the material more intensely to create discussion on the nonlinear behavior of the model at this point.

Limitations of this study include the small population that was tested and the limited classroom resources allowed for measurement error. The spring scales were provided from the physics department at my school and were not in optimal condition. Also, students' lack of proficiency with measurement tools and recording measurements accurately was an obstacle throughout the experimental phase. Though significant difference was found between the two groups, this study was done on a small sample size and cannot be generalized for the general high school population. There are also much more complex mathematical concepts involved when discussing elasticity which we were not able to be explored or discussed considering the prior knowledge, age and population for this study. This allowed for a very basic discussion of linear and non-linear theories of elasticity.

Students were not allowed to discuss the pre-test questions following the administration of the three tests. During my lectures throughout the semester, I also made a point not to address any of the questions that appeared on any of the pre-tests to obtain valid results. The lab was designed to get students to explore linear and non-linear models. The basic ideas of linearity that the pre-tests assessed did not arise during the experiment. Students in the control group continued practice to find derivatives and equations of tangent lines as the experimental group completed the same calculations but was based off of a lab conducted.

Overall, I was pleased with the lab in my classroom. Students were able to see the mathematics at work through the lab. It was a change of pace in the traditional high school math classroom which required students to explore, experiment and create results on their own. I still was heavily involved in monitoring the experiments, answering questions and problem-shooting. The discussion that the lab created during the post-lab activity was exciting to me as a teacher. Students were talking about the graph at different points and explaining how the lab was related to different points on the graph. They were much more involved with the behavior of the graph than I have experienced in the past when asking questions about graphical characteristics such as minimums, maximums and derivatives at specific points during a traditional lecture setting. The experiment gave them something to relate the numbers to and enriched their discussion. Modeling mathematics through other means of visualizations, manipulatives, experiments and/or real world situations is not always easy to plan or design and things can always go wrong. However, this study has proven that experimentation can have a positive effect on student understanding and even improve misunderstandings that once existed. This was not a study to test the effectiveness of the modeling technique, but rather to show the likelihood that teachers will see results when incorporating an experiment to address specific conceptual ideas. I will

encourage my colleagues to implement such methods in their own classroom and strive to create more activities that give students a new way to see math.

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APPENDIX A
DATA TABLES

Table 2: Student scores on Multiple Choice Linearity Pre-test

Student	Question 1		Question 2		Question 3		Question 4		Question 5		Total
1	B	0	A	1	C	1	B	1	A	0	3
2	B	0	A	1	C	1	B	1	B	0	3
3	B	0	C	0	B	0	C	0	D	1	1
4	B	0	B	0	C	1	A	0	B	0	1
5	B	0	B	0	C	1	A	0	A	0	1
6	B	0	B	0	B	0	B	1	A	0	1
7	B	0	D	0	B	0	A	0	A	0	0
8	B	0	B	0	C	1	B	1	B	0	2
9	B	0	A	1	B	0	C	0	D	1	2
10	B	0	D	0	A	0	B	1	C	0	1
11	B	0	A	1	C	1	B	1	A	0	3
12	D	1	A	1	C	1	B	1	D	1	5
13	D	1	C	0	C	1	B	1	B	0	3
14	B	0	A	1	C	1	C	0	B	0	2
15	A	0	A	1	D	0	B	1	B	0	2
16	C	0	A	1	B	0	C	0	D	1	2
17	B	0	D	0	C	1	A	0	A	0	1
18	B	0	A	1	C	1	B	1	A	0	3
19	B	0	A	1	C	1	A	0	C	0	2
20	B	0	A	1	C	1	B	1	B	0	3

0 = Incorrect Response
1 = Correct Response

Table 3: Student scores on Multiple Choice Linearity Post-test

Student	Question 1	Question 2	Question 3	Question 4	Question 5	Total	
1	B	0 A	1 C	1 A	0 C	0	2
2	B	0 A	1 C	1 B	1 B	0	3
3	B	0 C	0 B	0 C	0 D	1	1
4	A	0 A	1 C	1 A	0 B	0	2
5	B	0 A	1 C	1 D	0 B	0	2
6	B	0 D	0 C	1 A	0 B	0	1
7	B	0 D	0 B	0 A	0 A	0	0
8	D	1 A	1 C	1 D	0 D	1	4
9	D	1 A	1 C	1 B	1 D	1	5
10	B	0 D	0 A	0 B	1 C	0	1
11	D	1 A	1 C	1 B	1 D	1	5
12	D	1 A	1 C	1 B	1 B	0	4
13	D	1 A	1 C	1 B	1 D	1	5
14	D	1 A	1 C	1 A	0 A	0	3
15	A	0 A	1 C	1 A	0 B	0	2
16	D	1 A	1 C	1 B	1 D	1	5
17	D	1 A	1 C	1 D	0 D	1	4
18	B	0 A	1 C	1 B	1 A	0	3
19	D	1 A	1 C	1 B	1 D	1	5
20	B	0 B	0 C	1 A	0 A	0	1

0 = Incorrect Response
1 = Correct Response

Table 4: Student scores on Free Response Linearity Pre-test

Student	Question 1	Question 2	Question 3	Question 4	Question 5	Mean
1	2	2	1	0	0	1
2	2	2	0	2	1	1.4
3	2	2	1	0	0	1
4	2	0	1	0	1	0.8
5	2	0	0	0	0	0.4
6	1	2	1	0	0	0.8
7	1	2	1	0	0	0.8
8	2	2	0	0	0	0.8
9	2	2	1	0	0	1
10	2	2	1	0	0	1
11	0	0	0	1	1	0.4
12	2	2	1	1	0	1.2
13	2	2	1	0	0	1
14	0	0	0	0	1	0.2
15	0	2	1	0	0	0.6
16	0	0	1	0	0	0.2
17	2	2	0	2	1	1.4
18	2	2	0	2	0	1.2
19	2	2	0	2	1	1.4
20	2	0	0	1	0	0.6

2 = Correct
1 = Partially correct
0 = Incorrect

Table 5: Student scores on Free Response Linearity Post-test

Student	Question 1	Question 2	Question 3	Question 4	Question 5	Mean
1	0	2	0	0	0	0.4
2	0	0	0	0	0	0
3	2	0	1	0	2	1
4	2	0	0	1	1	0.8
5	2	0	1	0	0	0.6
6	2	0	1	2	0	1
7	1	0	1	0	2	0.8
8	2	0	0	2	1	1
9	2	2	2	2	2	2
10	2	2	2	1	2	1.8
11	2	2	1	2	1	1.6
12	2	2	0	2	1	1.4
13	2	2	2	2	1	1.8
14	1	0	0	0	1	0.4
15	0	0	0	2	0	0.4
16	2	0	0	2	1	1
17	2	2	2	2	2	2
18	2	2	1	2	1	1.6
19	2	2	2	2	1	1.8
20	2	0	0	0	0	0.4

2 = Correct
1 = Partially correct
0 = Incorrect

Table 6: Calculus Concept Inventory Multiple Choice Pre-test

Student	Question 2	Question 3	Question 4	Question 5	Question 7	Question 8	Question 9	Question 10	Question 11	Question 12	Question 15	Question 17	Question 18	Question 22	Total
1	C	0 B	0 E	1 B	0 B	1 D	0 B	0 B	0 E	0 B	0 B	0 E	1 B	0 C	0 3
2	C	0 C	0 E	1 A	0 D	0 E	1 D	0 B	0 E	0 A	1 B	0 B	0 A	0 A	1 4
3	E	1 E	0 C	0 C	1 E	0 A	0 A	0 B	0 B	0 C	0 D	0 E	1 E	0 D	0 3
4	C	0 D	0 D	0 C	1 B	1 D	0 A	0 B	0 C	1 B	0 A	0 E	1 C	1 A	1 6
5	A	0 B	0 C	0 A	0 B	1 B	0 B	0 D	0 D	0 A	1 B	0 B	0 B	0 A	1 3
6	B	0 D	0 C	0 C	1 A	0 E	1 A	0 B	0 D	0 B	0 A	0 B	0 D	0 A	1 3
7	C	0 B	0 C	0 A	0 B	1 A	0 A	0 C	0 A	0 B	0 A	0 B	0 D	0 A	1 2
8	A	0 C	0 C	0 A	0 B	1 B	0 B	0 C	0 C	1 C	0 B	0 B	0 D	0 A	1 3
9	A	0 B	0 C	0 C	1 B	1 C	0 A	0 B	0 B	0 A	1 C	1 C	0 B	0 C	0 4
10	C	0 B	0 C	0 C	1 B	1 A	0 A	0 B	0 D	0 A	1 D	0 B	0 C	1 E	0 4
11	C	0 D	0 C	0 B	0 B	1 A	0 A	0 B	0 D	0 C	0 C	1 E	1 B	0 A	1 4
12	C	0 D	0 C	0 C	1 D	0 B	0 A	0 B	0 C	1 A	1 C	1 D	0 B	0 A	1 5
13	A	0 C	0 C	0 A	0 B	1 A	0 A	0 B	0 C	1 B	0 C	1 B	0 D	0 A	1 4
14	D	0 D	0 E	1	0 B	1 D	0 A	0 C	0 E	0 C	0 A	0 A	0 B	0 A	1 3
15	B	0 B	0 E	1 A	0 E	0 E	1 E	1 B	0 E	0 B	0 B	0 B	0 C	1 A	1 5
16	B	0 C	0 C	0 A	0 B	1 C	0 B	0 E	0 A	0 C	0 A	0 C	0 B	0 C	0 1
17	C	0 D	0 C	0 E	0 B	1 C	0 A	0 B	0 A	0 C	0 C	1 E	1 B	0 A	1 4
18	D	0 D	0 E	1 B	0 B	1 D	0 D	0 B	0 E	0 C	0 C	1 E	1 E	0 D	0 4
19	C	0 B	0 C	0 A	0 A	0 E	1 D	0 B	0 E	0 A	1 C	1 E	1 B	0 A	1 5
20	A	0 D	0 C	0 C	1 B	1 B	0 D	0 E	0 A	0 A	1 C	1 C	0 A	0 C	0 4

Table 7: Calculus Concept Inventory Multiple Choice Post-test

Student	Question 2	Question 3	Question 4	Question 5	Question 7	Question 8	Question 9	Question 10	Question 11	Question 12	Question 15	Question 17	Question 18	Question 22	Total	
1	C	0 B	0 E	1 B	0 B	1 E	1 A	0 C	0 E	0 C	0 C	1 E	1 B	0 A	1	6
2	A	0 B	0 E	1 C	1 B	1 D	0 B	1 D	0 A	0 E	0 B	0 B	0 B	0 A	1	5
3	C	0 B	0 C	0 B	0 B	1 B	0 A	0 B	0 C	1 A	1 A	0 E	1 E	0 C	0	4
4	C	0 C	0 E	1 C	1 B	1 B	0 A	0 B	0 C	1 C	0 A	0 E	1 E	0 B	0	5
5	C	0 C	0 C	0 E	0 B	1 E	1 D	0 B	0 B	0 A	1 C	1 B	0 B	0 A	1	5
6	C	0 C	0 C	0 B	0 B	1 C	0 A	0 B	0 C	1 B	0 B	0 C	0 B	0 B	0	2
7	C	0 C	0 C	0 B	0 C	0 A	0 A	0 B	0 E	0 B	0 C	1 C	0 B	0 A	1	2
8	A	0 C	0 B	0 A	0 B	1 B	0 C	0 B	0 C	1 C	0 B	0 B	0 D	0 A	1	3
9	C	0 C	0 C	0 C	1 C	0 B	0 B	1 B	0 B	0 A	1 B	0 E	1 B	0 A	1	5
10	C	0 B	0 C	0 B	0 D	0 A	0 A	0 B	0 B	0 B	0 A	0 E	1 E	0 A	1	2
11	A	0 C	0 C	0 D	0 B	1 A	0 A	0 B	0 B	0 A	1 B	0 D	0 B	0 A	1	3
12	C	0 C	0 C	0 B	0 B	1 A	0 B	1 B	0 B	0 A	1 D	0 D	0 A	0 A	1	4
13	C	0 C	0 C	0 E	0 A	0 C	0 C	0 B	0 B	0 B	0 D	0 E	1 B	0 A	1	2
14	C	0 A	1 E	1 C	1 B	1 A	0 A	0 E	0 D	0 A	1 C	1 B	0 A	0 C	0	6
15	E	1 C	0 E	1 A	0 E	0 A	0 D	0 B	0 E	0 B	0 C	1 B	0 D	0 A	1	4
16	E	1 A	1 E	1 E	0 B	1 D	0 A	0 C	0 E	0 B	0 D	0 C	0 A	0 C	0	4
17	C	0 C	0 E	1 C	1 B	1 C	0 A	0 A	1 B	0 A	1 C	1 D	0 B	0 A	1	7
18	E	1 C	0 E	1 A	0 A	0 B	0 A	0 C	0 D	0 B	0 A	0 E	1 C	1 A	1	5
19	D	0 B	0 E	1 C	1 B	1 B	0 B	1 B	0 E	0 A	1 C	1 E	1 C	1 A	1	9
20	A	0 B	0 E	1 C	1 B	1 D	0 B	1 B	0 B	0 A	1 D	0 B	0 B	0 B	0	5

APPENDIX B
MULTIPLE CHOICE LINEAR PRE/POST-TEST

1. If all of a rectangle's dimensions are increased by 2 units, then the rectangle's area is…
   a) unchanged c) tripled
   b) doubled d) quadrupled

2. Which statement best describes the slope of a linear function?
   a) The slope is constant.
   b) The slope strictly increases or decreases.
   c) The slope increases and decreases.
   d) The slope cannot be zero.

Choose the best type of model for the following situations:

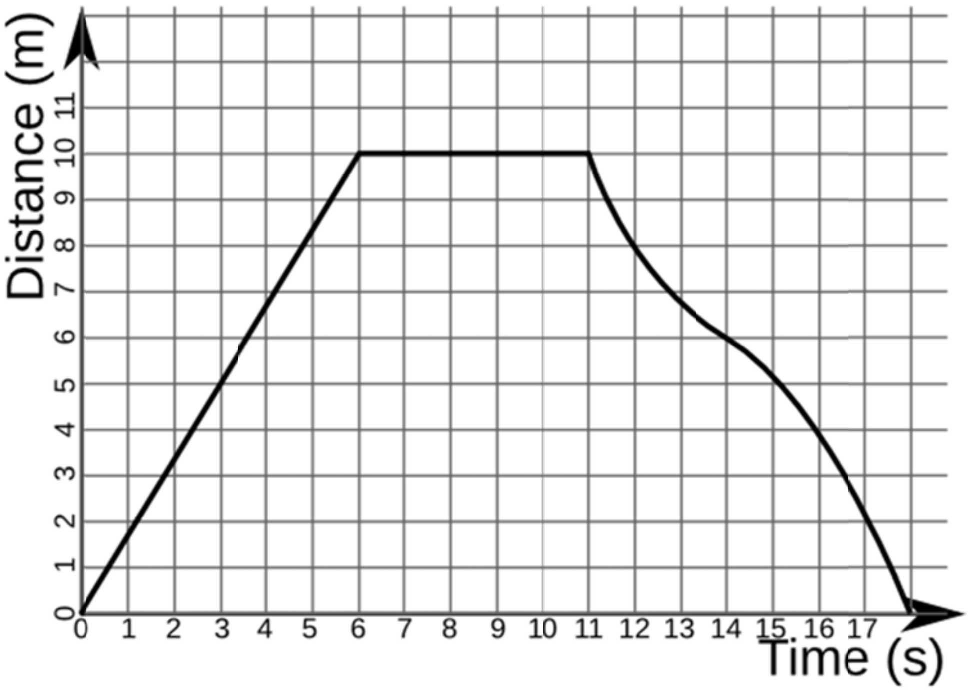
3. The height in feet of a punted football over distance in yards.
   a) constant c) quadratic
   b) linear d) polynomial of degree ≥ 3

4. The cost of a cab ride that is \$2.50 initially and \$1.75 for each additional mile traveled.
   a) constant c) quadratic
   b) linear d) polynomial of degree ≥ 3

5. The area of a rectangle after being dilated by a scale factor.
   a) unchanged c) linear
   b) constant d) quadratic

APPENDIX C
FREE RESPONSE LINEAR PRE/POST-TEST

The graph below represents the distance that a pink Cadillac traveled in meters after time in seconds.



1. What is the average rate of change of the car for $0 \leq t \leq 6$?

2. Describe the behavior of the car between 6 and 11 seconds.

3. At what time interval is the car moving the fastest? Why?

4. There is a change in behavior of the graph at t = 14. Explain what this graphical behavior means in terms of the car's rate of change.

5. Estimate the slope of the graph at t =16. List the steps you took to arrive at your estimation.

APPENDIX D
POST-LAB ACTIVITY

Group Member Names:

____

Complete the Post-lab activity for both materials you chose to experiment. Answer the following questions according to the material used during the lab. Attach your Mathematica graph and data to this handout.

Material: ____

1. Estimate five rates of change at five different points on the graph.

2. Find the equation s of tangent lines associated with the estimated derivatives above. Then, sketch these lines on your graph.

3. Explain what the derivative means in context of the lab and material. Be specific.

4. Does the graph have a horizontal tangent line? If so, where? Explain why. If not, explain why.

5. What does a vertical tangent line mean in context of the lab and material?

6. Does this material model a linear path? Why or why not do you think that is?

APPENDIX E
IRB APPROVAL, PARENTAL AND CHILD CONSENT FORMS

Application for Exemption from Institutional Oversight

Institutional Review Board
Dr. Robert Mathews, Chair
130 David Boyd Hall
Baton Rouge, LA 70803
P: 225.578.8692
F: 225.578.5983
irb@lsu.edu
lsu.edu/irb

Unless qualified as meeting the specific criteria for exemption from Institutional Review Board (IRB) oversight, ALL LSU research/projects using living humans as subjects, or samples, or data obtained from humans, directly or indirectly, with or without their consent, must be approved or exempted in advance by the LSU IRB. This Form helps the PI determine if a project may be exempted, and is used to request an exemption.

– Applicant, Please fill out the application in its entirety and include the completed application as well as parts A-F, listed below, when submitting to the IRB. Once the application is completed, please the completed application to the IRB Office or to a member of the Human Subjects Screening Committee. Members of this committee can be found at http://sites01.lsu.edu/wp/ored/human-subjects-screening-committee-members/

– A Complete Application Includes All of the Following:
  (A) A copy of this completed form and a copy of parts B thru F.
  (B) A brief project description (adequate to evaluate risks to subjects and to explain your responses to Parts 1&2)
  (C) Copies of all instruments to be used.
  *If this proposal is part of a grant proposal, include a copy of the proposal and all recruitment material.
  (D) The consent form that you will use in the study (see part 3 for more information.)
  (E) Certificate of Completion of Human Subjects Protection Training for all personnel involved in the project, including students who are involved with testing or handling data, unless already on file with the IRB. Training link: (http://phrp.nihtraining.com/users/login.php)
  (F) IRB Security of Data Agreement: (https://sites01.lsu.edu/wp/ored/files/2013/07/Security-of-Data-Agreement.pdf)

1) Principal Investigator: Kailyn Brabham **Rank:** Graduate Student

Dept: Natural Sciences **Ph:** 504-401-3426 **E-mail:** kbrabh1@tigers.lsu.edu

2) Co Investigator(s): please include department, rank, phone and e-mail for each
*If student, please identify and name supervising professor in this space

Dr. Blaise Bourdin, Mathematics Department, Professor, (225)578-1612, bourdin@math.lsu.edu

Dr. James Madden, Mathematics Department, Professor, (225)578-7988, jamesjmadden@gmail.com

IRB# E8389 LSU Proposal #
Complete Application
Human Subjects Training
IRB Security of Data Agreement

3) Project Title: Does exploring non-linear models address high school students' misconceptions of linearity and rate of change?

Study Exempted By:
Dr. Robert C. Mathews, Chairman
Institutional Review Board
Louisiana State University
203 B-1 David Boyd Hall
225-578-8692 | www.lsu.edu/irb
Exemption Expires: 8/28/2016

4) Proposal? (yes or no) **If Yes, LSU Proposal Number**

Also, if YES, either
○ **This application completely matches the scope of work in the grant**
OR
○ **More IRB Applications will be filed later**

5) Subject pool (e.g. Psychology students) 30 high school senior calculus students

*Circle any **"vulnerable populations" to be used**: (children <18; the mentally impaired, pregnant women, the ages, other). Projects with incarcerated persons cannot be exempted.

6) PI Signature [signature] **Date** 8/26/13 (no per signatures)

**** I certify my responses are accurate and complete.** If the project scope or design is later changes, I will resubmit for review. I will obtain written approval from the Authorized Representative of all non-LSU institutions in which the study is conducted. I also understand that it is my responsibility to maintain copies of all consent forms at LSU for three years after completion of the study. If I leave LSU before that time the consent forms should be preserved in the Departmental Office.

Screening Committee Action: Exempted ✓ Not Exempted ____ Category/Paragraph 1
Signed Consent Waived?: Yes (No)
Reviewer Mathews **Signature** [signature] **Date** 8/29/13

Calculus Consent Form

1. Study Title: Linearity: Student misconceptions addressed with hands-on exploratory labs

2. Performance Site: Destrehan High School - St. Charles Parish, Louisiana

3. Purpose of the Study: The purpose of this research project is to identify students' misconceptions of linearity and address these misconceptions through exploring local linearity and finding derivatives as a way to correct their misunderstandings.

4. Subject inclusion: Individuals between 16 and 18 who are currently enrolled in Calculus.

5. Number of subjects: 20

6. Study Procedures: Data for this test will be collected over a 4 week period during the differential introduction of Calculus. Subjects will complete a pretest consisting of questions from the Calculus Concept Inventory that the lab is designed to focus on. Subjects will also complete a survey on their current understandings of linearity. In class labs will then be conducted by the subjects exploring the elasticity of various materials (rubber band, extension spring and elastic string). During the labs, subjects will collect data on the extended length of the material and the mass applied to it. Subjects will then graph their data to calculate derivatives and explain the use local linearity for nonlinear models. Subjects will then be given a posttest mirroring the questions from the pretest for data analysis. A correlation study will be conducted to note the possible relationship between linear misconceptions and differential calculus performance as well as growth from pretest to posttest after the lab has been completed. Subjects will then be interviewed by their Calculus teacher to address what they liked and/or disliked about the lab.

7. Benefits: Subjects will have the opportunity to explore the application of math in the real world through the integration of math and science. This study may help correct misunderstandings of linearity and improve student performance on specific differential calculus concepts such as rate of change and slopes of tangent lines.

8. Risks: There are no known risks.

9. Right to Refuse: Participation is voluntary, and a child will become part of the study only if both student and parent agree to the student's participation. At any time, either the subject may withdraw from the study or the subject's parent may withdraw the subject from the study without penalty or loss of any benefit to which they might otherwise be entitled. If a student is does not participate in the study, the subject

will still be required to complete the lab as a part of the classroom lesson, but data will not be collected on said subject.

10. Privacy: The school records of participants in this study may be reviewed by investigators. Results of the study may be published, but no names or identifying information will be included for publication. Subject identity will remain confidential unless disclosure is required by law.

11. Financial Information: There is no cost for participation in the study, nor is there any compensation
to the subjects for participation.

Signatures:

The study has been discussed with me and all my questions have been answered. I may direct additional questions regarding study specifics to the investigator. If I have questions about subjects' rights or other concerns, I can contact Robert C. Mathews, Chairman, Institutional Review Board, (225) 578-8692, irb@lsu.edu, www.lsu.edu/irb. I will allow my child to participate in the study described above and acknowledge the investigator's obligation to provide me with a signed copy of this consent form.

Parent/Guardian Signature: ____________________________________ Date: __________

The parent/guardian has indicated to me that he/she is unable to read. I certify that I have read this consent form to the parent/guardian and explained that by completing the signature line above he/she has given permission for the child to participate in the study.

Signature of Reader: ____ Date: _____________

VITA

Kailyn Brabham was born to Jimmy and April Brabham in 1988 in New Orleans, Louisiana. She attended Louisiana State University where she majored in Mathematics with a concentration in secondary education. She received her Bachelor of Science in May 2010 graduating Summa Cum Laude. Kailyn began her teaching career in August 2010 at Destrehan High School teaching Algebra I and Honors Algebra II. Throughout her career, she has been awarded the 2012 Outstanding New Teacher award for Louisiana Teachers of Math (LATM) and Region 1 LACUE Teacher of the Year for incorporating technology into the classroom. She entered Louisiana State University Graduate School in June 2012 and is a candidate for the Master of Natural Sciences degree. She is currently teaching Honors Geometry, Honors Algebra II and calculus at Destrehan High School in Destrehan, Louisiana.