Some Results on Quadratic Forms Over Non-Formally Real Fields.

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ABSTRACT

The set of all non-zero elements represented by the binary quadratic form $x^2 + ay^2$ over a field forms a multiplicative group called the value group of that form. These value groups essentially determine what is called the quadratic form structure of the field. The quadratic form structure of three types of non-formally real fields is investigated and determined.

There are three field invariants closely associated with the study of quadratic forms. For any non-formally real field $F$, $q(F) = |F/F^2|$ is called the square class number of $F$, $u(F)$ the $u$-invariant of $F$, and $s(F)$ the level (or Stufe) of $F$. It is known for nonreal fields that $s \leq u \leq q$.

The quadratic form structure of fields with $u = q < \infty$ is known, and if $u \neq q < \infty$, then $u \leq q/2$. The first type of field analyzed is one with $u = q/2 < \infty$. The quadratic form structure of fields with $q \leq 8$ is known, and $q$ is known to be a 2-power. We next categorize fields with $q = 16$ as to quadratic form structure. There are many unanswered questions concerning the quadratic form structure of quadratic extensions of nonreal fields. The third type of field discussed is a quadratic extension of a nonreal field which has exactly two quaternion algebras.
A quadratic form $\varphi$ over a field $F$ is a homogeneous polynomial of degree two over $F$. Such a polynomial in $n$ indeterminants is called an $n$-ary quadratic form over $F$. We will only consider fields which have characteristic different from two. A quadratic form $\varphi$ over such a field $F$ can be put in the form $\varphi = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ with $a_{ij} = a_{ji} \in F$ for $1 \leq i,j \leq n$. The matrix $(a_{ij})$ is called the matrix associated with the form $\varphi$. The determinant of $\varphi$, written $\det \varphi$, is defined to be the determinant of the matrix $(a_{ij})$. $\varphi$ is called regular (or non-singular) if $\det \varphi \neq 0$. We consider only regular forms in this work.

Let $\psi = \sum_{i,j=1}^{n} b_{ij} y_i y_j$ be another $n$-ary quadratic form over $F$. We say that $\varphi$ and $\psi$ are equivalent (written $\varphi \sim \psi$) if there exists an $n \times n$ invertible matrix $C$ over $F$ such that $(b_{ij}) = C^t (a_{ij}) C$ where $C^t$ is the transpose of $C$. It can be shown that $\sim$ is an equivalence relation on the quadratic forms over $F$ and that the determinant of an equivalence class of quadratic forms is unique up to a non-zero square factor.

Let $V$ be a vector space over $F$. Consider any map $Q : V \to F$ such that $Q(fv) = f^2 Q(v)$ for all $f \in F$ and $v \in V$ and let $B : V \times V \to F$ be defined by $B(u,v) = \frac{1}{2} [Q(u+v) - Q(u) - Q(v)]$. If $B$ is a bilinear form, we say that $Q$ is a quadratic map and call the pair $(V,Q)$ a quadratic space. If $(V,Q)$ and $(V',Q')$ are quadratic spaces and there
is a one-to-one, onto linear transformation between them which preserves the quadratic maps, we say that the spaces are isometric and call the transformation an isometry.

Given any n-ary quadratic form $\varphi$ over $F$, let $V$ be any n-dimensional vector space over $F$. Let $\{v_1, \ldots, v_n\}$ be a basis for $V$, and define $Q : V \to F$ by $Q(f_1 v_1 + \ldots + f_n v_n) = \varphi(f_1, \ldots, f_n)$. It can be shown that $Q$ is a quadratic map, so $(V, Q)$ is a quadratic space. Given this quadratic space $(V, Q)$ and basis $\{v_1, \ldots, v_n\}$, the form $\sum_{i,j=1}^{n} B(v_i, v_j) x_i x_j$ turns out to be $\varphi$. This gives a one-to-one correspondence between the equivalence classes of n-ary quadratic forms and the isometry classes of n-dimensional quadratic spaces. For this reason, n-ary forms are sometimes said to have dimension $n$, abbreviated dim $n$. Forms of dimension 2, 3, and 4 are given the special names binary, ternary, and quaternary, respectively.

Every quadratic form is equivalent to a diagonalized form, i.e., one of the form $\sum_{i=1}^{n} a_i x_i^2$. Since we are concerned only with quadratic forms up to equivalence, we will always assume our forms are diagonalized. We will denote the form $\sum_{i=1}^{n} a_i x_i^2$ by $(a_1, \ldots, a_n)$. The form $(a_1, \ldots, a_n)$ is abbreviated $n \times (a)$. We can always select the leading coefficient to be any non-zero element that the form represents, and equivalent forms represent the same elements in $F$. Therefore, a binary form over $F$ which represents 1 can be written as $(1, a)$ for some non-zero $a$ in $F$. 
A form \( \varphi \) is called isotropic over \( F \) if it represents zero non-trivially, otherwise \( \varphi \) is called anisotropic. A form which represents every element in \( F \) is said to be universal over \( F \). It can be shown that every isotropic binary quadratic form has determinant \(-1\) and is equivalent to the form \((1,-1)\). Such a form (and corresponding space) is called a hyperbolic plane. Hyperbolic planes are universal, and this fact can be used to show that isotropic forms are universal.

We will consider only fields \( F \) which are non-formally real. This means \(-1\) is the sum of a finite number of squares of elements of \( F \). The non-zero elements of \( F \) will be denoted by \( \hat{F} \), their squares by \( \hat{F}^2 \). We let \( Q(F) = \hat{F}/\hat{F}^2 \), and \( q(F) = |Q(F)| \) is called the square-class number of \( F \). The level (or Stufe) of \( F \), denoted by \( s(F) \), is the smallest number of squares in \( F \) of which \(-1\) is the sum. Pfister showed that \( s(F) \) is a power of 2 (see [11], [12]). The \( u \)-invariant of \( F \), denoted by \( u(F) \), is defined to be \( \max \{ \dim \varphi \} \) where \( \varphi \) ranges over all anisotropic forms over \( F \). If no such maximum exists, \( u(F) \) is defined to be \( \infty \). The quaternion algebra over \( F \) generated by elements \( x, y \) where \( x^2 = 1 \cdot a, y^2 = 1 \cdot b, xy = -yx \) is denoted by \([a,b]\). The number of quaternion algebras over \( F \) is denoted by \( m(F) \). Let \( h'(F) \), called the reduced height of \( F \), be the smallest positive integer such that any element in \( \hat{F} \) is a sum of at most \( h'(F) \) squares.

It can be easily shown that \( h'(F) = s(F) \) or \( s(F) + 1 \).
If there is no danger of confusion, the above field invariants are abbreviated to $q$, $u$, $s$, $m$, and $h'$. Pfister has shown that $s \leq u \leq q$ for non-formally real fields [13, Satz 19]. Kaplansky conjectured that $u$ is a 2-power like $s$ and $q$ are, but this has neither been shown nor negated by example. A non-formally real field with $q < \infty$ is called a Kneser field, and we will for the most part study quadratic forms over this type of field.

For a quadratic form $\varphi$, $D_\mathbb{F}(\varphi) = \{a \in \mathbb{F} \mid \varphi \text{ represents } a \}$ over $\mathbb{F}$ is called the value set of the form $\varphi$. Quadratic forms represent entire cosets of $\mathbb{F}^2$ in $\mathbb{F}$, so we sometimes also use the symbol $D_\mathbb{F}(\varphi)$ to stand for $D_\mathbb{F}(\varphi)/\mathbb{F}^2 = \{a \in Q(\mathbb{F}) \mid \varphi \text{ represents } a\}$. Which usage is meant will be clear from the context. When there is no danger of confusion, we abbreviate $D_\mathbb{F}(\varphi)$ to $D(\varphi)$. $D(\varphi) = \{a_1, \ldots, a_n\}$ is an abbreviation for $D(\varphi)/\mathbb{F}^2 = \{a_1^\mathbb{F}^2, \ldots, a_n^\mathbb{F}^2\}$, and $D(\varphi) = \langle a_1, \ldots, a_n \rangle$ means $D(\varphi)/\mathbb{F}^2$ is the subgroup of $Q(\mathbb{F})$ generated by the independent elements $a_1^\mathbb{F}^2, \ldots, a_n^\mathbb{F}^2$. Also, $a \in D(\varphi)$ means either of the equivalent statements $a^\mathbb{F}^2 \in D(\varphi)$ or $\varphi$ represents $a$. The fact that $a \in b^\mathbb{F}^2$ will sometimes be written $a \equiv b$. If $A \subseteq \mathbb{F}$, then $\langle A \rangle$ is the subgroup in $\mathbb{F}$ generated by $A$. If $\mathbb{F}^2 \subseteq A \cap B$ and $A$, $B$ are subgroups of $\mathbb{F}$, then $A \otimes B$ denotes that $\langle A, B \rangle/\mathbb{F}^2$ is the direct sum (as a vector subspace of $Q(\mathbb{F})$ over $GF(2)$) of $A/\mathbb{F}^2$ and $B/\mathbb{F}^2$. The symbol $\otimes$ will have another meaning, as described in a subsequent paragraph, but which meaning is intended will always be clear from the context.
If \( \psi = \sum_{i,j=1}^{n} a_{ij} x_i x_j \) and \( \psi = \sum_{i,j=1}^{m} b_{ij} y_i y_j \), then we define

\[
\psi \otimes \psi = \sum_{i,j=1}^{m+n} c_{ij} z_i z_j
\]

where \( c_{ij} = a_{ij} \) for \( 1 \leq i, j \leq n \), \( c_{n+i, n+j} = b_{ij} \) for \( 1 \leq i, j \leq m \), and \( c_{k\ell} = 0 \) otherwise. For example, the sum of the diagonalized forms \((1,a)\) and \((1,b)\) is \((1,a) \oplus (1,b) = (1,a,1,b)\). An important result known as Witt's Theorem states that \( \psi \otimes \psi = \psi \otimes \psi' \) if and only if \( \psi \equiv \psi' \). Moreover, it can be shown that any regular form \( \varphi \) can be written as \( \varphi \equiv \varphi_0 \otimes \varphi_a \), where \( \varphi_0 \) is zero or is a sum of hyperbolic planes and \( \varphi_a \) is either zero or anisotropic. \( \varphi_a \) is unique by Witt's Theorem and is called the anisotropic part of \( \varphi \).

The set of equivalence classes of anisotropic forms over \( F \) can be made into an additive group. The sum of two anisotropic equivalence classes containing \( \varphi \) and \( \psi \) respectively is defined to be the equivalence class of the anisotropic part of \( \varphi \otimes \psi \). This operation does, in fact, make the anisotropic equivalence classes into an abelian group, called the Witt group of \( F \).

Further structure can be added to this group so as to make it a ring. The product of two forms \( \psi = \sum_{i,j=1}^{n} a_{ij} x_i x_j \) and \( \psi = \sum_{k,\ell=1}^{m} b_{k\ell} y_k y_\ell \) is defined to be the form \( \varphi \otimes \psi = \sum_{i,j,k,\ell} c_{ij} b_{k\ell} z_i z_j z_k z_\ell \). For diagonalized forms this becomes

\[
(\sum_{i=1}^{n} a_i x_i^2) \otimes (\sum_{i=1}^{m} b_i y_i^2) = \sum_{i,j} a_i b_j z_{ij}^2.
\]

This product is well-defined up to equivalence. The product of two equivalence classes is defined as for the sum except we consider \( \varphi \otimes \psi \) instead of \( \varphi \otimes \psi' \). These operations then yield a commutative
ring with identity, called the Witt ring of \( F \). The study of the Witt ring of a field \( F \) gives what is called the quadratic form structure of \( F \).

Two fields \( F, K \) are said to be equivalent with respect to quadratic forms if there is an isomorphism \( t : Q(F) \to Q(K) \) (called an equivalence map) such that \( t(-1) = -1 \) and 
\[
t[D_F(a_1, \ldots, a_n)] = D_K[t(a_1), \ldots, t(a_n)] 
\]
for all \( n \) and \( a_i \in F \). It was shown in [1, Proposition 2.2] that \( n = 2 \) suffices. Also pointed out in [1] was that two fields are equivalent with respect to quadratic forms if and only if their Witt rings are isomorphic. Thus equivalent fields have the same quadratic form structures and conversely. Therefore, in order to study the quadratic form structure of \( F \), we need only consider value sets of binary quadratic forms over \( F \).

The scaling of the form \( \varphi = (a_1, \ldots, a_n) \) by \( a \in \hat{F} \) is defined to be the form \( a \varphi = (aa_1, \ldots, aa_n) \). Since \( D_F(a \varphi) = aD_F(\varphi) \), in our study of quadratic form structure we need only consider \( D_F(1, -a) \) for all \( a \in \hat{F} \). These \( D_F(1, -a) \) are always multiplicative subgroups of \( \hat{F} \).

Kaplansky [8] introduced the radical of a field. It is given by \( R(F) = \{ a \in \hat{F} \mid D_F(1, -a) = Q(F) \} \). We will sometimes use the symbol \( R(F) \) to mean \( R(F)/\hat{F}^2 \), since the radical represents entire cosets of \( \hat{F}^2 \) in \( \hat{F} \). The context will clarify which meaning is intended. When no misunderstanding is possible, we abbreviate \( R(F) \) to \( R \). To give an example of these conventions, \( |R| = 1 \) really means \( |R(F)/\hat{F}^2| = 1 \), or \( R(F) = \hat{F}^2 \). Properties of the radical are discussed in [2],
The radical will be very prominent throughout this work since the quadratic form structure strongly depends on it.

It is easily seen that $D(\varphi \oplus \psi) = \bigcup_{b \in D(\psi)} D(\varphi \oplus (b)) = \bigcup_{a \in D(\varphi), b \in D(\psi)} D(a, b)$ for forms $\varphi, \psi$. Clearly $a \in D(1, b)$ if and only if $-b \in D(1, -a)$ for $a, b \in \hat{F}$. If $c \in D(1, a) \cap D(1, b)$, then $c \in D(1, a)$ and $c \in D(1, b)$, which implies $-a, -b \in D(1, -c)$. Thus $ab \in D(1, -c)$, so $c \in D(1, -ab)$. We have then that for all $a, b \in \hat{F}$, $D(1, a) \cap D(1, b) \subseteq D(1, -ab)$. These three results will be used throughout, usually with no mention.

We study the quadratic form structures of three different types of non-formally real fields. These are fields with $u = q/2 < \infty$, fields with $q = 16$, and quadratic extensions of fields with $m = 2$. Although many results are true more generally, we will assume throughout that every field is a Kneser field, i.e., $q < \infty$ and the field is non-formally real. Finally we discuss an important example which relates several of our individual topics.
CHAPTER I. NON-FORMALLY REAL FIELDS WITH $u = q/2 < \infty$

In both [1] and [15], fields with $u = q$ were characterized as to quadratic form structure, and in [5, Theorems 2.7, 2.7'] it was shown for non-formally real fields that if $u < q$, then $u \leq q/2$. In this chapter the equivalence with respect to quadratic forms of non-formally real fields with $u = q/2 < \infty$ is investigated.

All possible non-real fields with $q \leq 8$ have been determined up to equivalence in [1], so it suffices to consider non-real fields with $u = q/2$ and $q \geq 16$. Such fields will be seen to be equivalent to a field whose radical has $|R| = 2$, a power series extension of such a field, or a power series extension of the 2-adic numbers.

The first three propositions listed occur in [4] and are listed here for convenience of reference.

Proposition 1.1. Suppose for some positive integer $r$ that $2^r \times (1)$ is not universal over a nonreal field $K$. If $D(1,x) = \langle x \rangle$ for some $x \in K$, then $D(1,ax) = \langle ax \rangle$ for all $a \in D[2^r \times (1)]$.

Proposition 1.2. If $K$ is a nonreal field, then for $x \in K$, $D(1,x) = \langle x \rangle$ if and only if $D(1,-x) = \langle -x \rangle$.

Proposition 1.3. If $K$ is a nonreal field with $u < \infty$ and $u < q$, then $D(1,1) \neq \langle -1 \rangle$. 

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Elman and Lam showed in [5] that an anisotropic form could be split into binary subforms and a "remainder" subform. They then showed that this splitting, called a $\beta$-decomposition, had several interesting properties which will be exploited here.

**Proposition 1.4.** Let $F$ be a nonreal field with $u = q/2 < \infty$ and $s > \frac{u}{4}$. Then any $\beta$-decomposition of a $u$-dimensional anisotropic form has distinct entries.

**Proof.** Let $\phi \cong \beta_1 \oplus \ldots \oplus \beta_r \oplus \phi_0$ where this is Elman and Lam's $\beta$-decomposition [5]. Since $u = q/2$ is even, either $2\phi_0$ is anisotropic or $\dim \phi_0 = 0$. Now, letting $\beta_i = (x_i, y_i)$ for $1 \leq i \leq r$, we have $\beta_i \oplus \beta_i \cong (1, -1, 1, -1)$. Also, from the proofs of Theorems 2.4 and 2.8 in [5], $x_i \not\in y_i \cdot F^2$, $\pm \{x_i, y_i\} \subseteq D(x_i, y_i)$ for $1 \leq i \leq r$ and $D(\beta_1), \ldots, D(\beta_r), D(\phi_0), -D(\phi_0)$ are pairwise disjoint. So the entries of $\beta_1 \oplus \ldots \oplus \beta_r$ are distinct and $|D(\beta_1 \oplus \ldots \oplus \beta_r)| \geq |D(\beta_1)| + \ldots + |D(\beta_r)| \geq \frac{u}{4}r$, since $|D(\beta_i)| \geq \frac{u}{4}$ for $1 \leq i \leq r$ (see [5]).

If $\dim \phi_0 = 0$, then $\phi \cong \beta_1 \oplus \ldots \oplus \beta_r$, which has distinct entries. So we may assume that $2\phi_0$ is anisotropic. Suppose for such a $\phi_0$ that $\phi_0$ contained two like entries, say $\phi_0 \cong (x, x, w_1, \ldots, w_l)$, where the $w_i$ are not necessarily distinct or disjoint from $D(x, x)$. Suppose $l > 0$. By Kneser's Theorem (see 2.1 of [5]), there exists $s_1 \in D(x, x, w_1)$ such that $s_1 \not\in D(x, x)$, and there exists $s_2 \in D(x, x, w_1, w_2)$ such that $s_2 \not\in D(x, x, w_1)$. Since $s_1 \in D(x, x, w_1)$, $s_2 \not\in D(x, x, w_1)$, $s_1 \not\in s_2 \cdot F^2$. If $s_1 \in -s_2 \cdot F^2$, then $(s_1, s_2)$ is isotropic. This
is a contradiction since \((s_1, s_2)\) can be written as a subform of \(2\varphi_0\). Thus \(s_1 \not\in -s_2\mathbb{F}^2\). Suppose now that \(s_1, \ldots, s_{j-1}\) have been selected as above. By Kneser's Theorem there exists \(s_j \in D(x, x, w_1, \ldots, w_j)\) such that \(s_j \not\in D(x, x, w_1, \ldots, w_{j-1})\). As above, \(s_j \not\in s_i\mathbb{F}^2\) and \(s_j \not\in -s_i\mathbb{F}^2\) for \(1 \leq i \leq j-1\). Let \(S = \{s_1, \ldots, s_j\}\). We have seen that the elements of \(S\) are distinct and \(S \cap -S = \emptyset\). Also, \(S \cap D(x, x) = \emptyset\) by the choice of the \(s_i\). If \(s_i \in -D(x, x)\) for some \(i, 1 \leq i \leq \ell\), then \((s_i, -s_i)\), which is isotropic, can be written as a subform of \(2\varphi_0\), a contradiction. So \(-S \cap D(x, x) = \emptyset\) also. Since \(S\), \(D(x, x) \subseteq D(\varphi_0)\), clearly \(S \cap D(\beta_i) = \emptyset = D(x, x) \cap D(\beta_i)\) for \(1 \leq i \leq r\). By isotropy, \(-S \cap D(\beta_i) = \emptyset = -D(x, x) \cap D(\beta_i)\) for \(1 \leq i \leq r\). Thus the sets \(D(\beta_1), \ldots, D(\beta_r), D(x, x), S, \) and \(-S\) are pairwise disjoint. By Pfister's proof of Satz 18(d) in [13], \(|D(1,1)| \geq s\), so \(|D(x, x)| = |D(1,1)| \geq s > 4\). Also, \(u = \dim \varphi = 2r + 2 + \ell\). So 

\[
|D(\varphi)| \geq |D(\beta_1 \oplus \ldots \oplus \beta_r)| + |D(x, x)| + |S| + |-S| \\
> 4r + 4 + \ell + \ell = 2(2r + 2 + \ell) = 2u = q,
\]

which is a contradiction. If \(\ell = 0\), then \(|D(\varphi)| \geq |D(\beta_1 \oplus \ldots \oplus \beta_r)| + |D(x, x)| > q\) as above. So \(\varphi_0\) must have distinct entries, and since the entries of \(\beta_1 \oplus \ldots \oplus \beta_r\) are distinct and disjoint from the entries of \(\varphi_0\), any \(\beta\)-decomposition of a \(u\)-dimensional anisotropic form consists of distinct entries. \(\square\)

**Proposition 1.5.** Let \(F\) be a nonreal field with \(u = q/2 < \infty\) and \(s \geq \frac{1}{4}\). Then \(s = \frac{1}{4}\) and \(|D(1,1)| = \frac{1}{4}\).
Proof. Let \( \varphi \) be a \( u \)-dimensional anisotropic form over \( F \), and suppose \( \varphi \cong \beta_1 \oplus \ldots \oplus \beta_r \oplus \varphi_0 \) is its \( \beta \)-decomposition. As in the proof of Proposition 1.4, we let \( \beta_i = (x_i, y_i) \), \( 1 \leq i \leq r \), and \( \varphi_0 = (z_1, \ldots, z_k) \). Suppose for some \( i, j \), \( 1 \leq i < j \leq k \), that \( |D(z_i, z_j)| \geq 4 \). Then as in the proof of Proposition 1.4, using \( (z_i, z_j) \) in the place of \( (x, x) \), we would have \( |D(\varphi_0)| \geq k + 2 \). But implicit in that proof is the fact that \( |D(\varphi_0)| = k \), so this is a contradiction. Thus \( |D(z_i, z_j)| = 2 \) for all \( 1 \leq i < j \leq k \). At any rate we have

\[
Q(F) = \pm \{ x_1, y_1, x_2, y_2, \ldots, x_r, y_r, z_1, \ldots, z_k \}.
\]

Also, since \( \beta_i \oplus \beta_i \cong \langle 1, -1, 1, -1 \rangle \), we have \( (x_i, y_i) \cong -(x_i, y_i) \) and \( (x_i, x_i) \cong -(y_i, y_i) \) for \( 1 \leq i \leq r \).

Let \( \psi = \beta_1 \oplus \ldots \oplus \beta_r \). Now \( -z_1 \in D(\varphi) \), so \( -z_1 = a + b \) where \( a \in D(\psi) \), \( b \in D(\varphi_0) \). If \( b \in z_1^F \) for \( i \neq 1 \), then \( -a \in D(z_1, z_1) \) and \( \varphi \) would be isotropic. Thus \( b \in z_1^F \) and it follows that \( D(\psi) \cup \{ \pm z_1 \} \subseteq D[\psi \oplus (z_1)] \). We would like to show \( D(\psi) \cup \{ \pm z_1 \} = D[\psi \oplus (z_1)] \).

We may suppose the above \( a \in D(\psi) \) is \( a = x_1 \), and then obtain \( -x_1 \in D(z_1, z_1) \) from \( -z_1 = a + b \). Consider \( D(x_1, y_1, z_1) \). Clearly this must be a subset of \( D(x_1, y_1) \cup \{ \pm z_1 \} \cup \{ z_2, \ldots, z_k \} \).

Suppose \( z_2 \in D(x_1, y_1, z_1) \). As above, \( -z_2 = a_1 + b_1 \) where \( a_1 \in D(\psi) \) and \( b_1 \in z_2^F \). Moreover, \( a_1 \in D(x_1, y_1) \) for otherwise \( a_1 \in D(x_i, y_i) \), \( i \neq 1 \); and \( \varphi \) would represent \( z_2 - z_2 = 0 \) non-trivially. Also as above, \( -a_1 \in D(z_2, z_2) \). So we have \( (z_1, z_1, z_2, z_2) \cong -(x_1, x_1, a_1, a_1) \). But \( 2\varphi_0 \) is anisotropic and \( 2\beta_1 \) is isotropic, so \( a_1 \notin -x_1^F \), \( y_1^F \). If \( a_1 \in x_1^F \), then \( -z_2 \in D(x_1, z_2) \) and \( -z_1 \in D(x_1, z_1) \) give
-1 ∈ \(D(1, x_1z_2) \cap D(1, x_1z_1) \subseteq D(1, -z_1z_2)\). Thus \(|D(1, -z_1z_2)| > 2\) where \(|D(1, z_1z_2)| = 2\), which is a contradiction to Proposition 1.2. This forces \(a_1 \in -y_1\hat{F}^2\).

Now if some other \(z_1\), say \(z_3\), is also in \(D(x_1, y_1, z_1)\), then \(a_2 \in -y_1\hat{F}^2\) where \(-z_3 = a_2 + b_2\). But then \(-z_2 \in D(-y_1, z_2)\), \(-z_3 \in D(-y_1, z_3)\) give \(-1 \in D(1, -y_1z_2) \cap D(1, -y_1z_3) \subseteq D(1, -z_2z_3)\), which yields the same contradiction as above. So \(D(x_1, y_1, z_1) \subseteq \{+x_1, +y_1, +z_1, z_2\}\) and \(\{+x_1, +y_1, +z_1, z_2\} \subseteq D(x_1, y_1, z_1)\). If \(z_2 \in D(x_1, y_1, z_1)\), \((x_1, y_1, z_1) \equiv (z_2, u, v)\) where \(uv \in x_1y_1z_1\hat{F}^2\) and \(u, v \in \{+x_1, +y_1, +z_1, z_2\}\). The possibilities for \(uv\) are

\[\pm \{1, x_1y_1, x_1z_1, x_1z_2, y_1z_2, z_1z_2\}\]. Clearly \(uv \not\in \hat{F}^2\); and \(uv \not\in \hat{F}^2\) for if so \(x_1y_1 \in z_1z_2\hat{F}^2\) and then \(2 = |D(1, z_1z_2)| = |D(1, x_1y_1)| = |D(x_1, y_1)| = 4\), a contradiction. The other possibilities also all lead to contradictions of the types \(x_1 \in +y_1\hat{F}^2\) and \(z_1 \in +x_1\hat{F}^2\) or \(+y_1\hat{F}^2\), \(i = 1, 2\). So \(z_2 \not\in D(x_1, y_1, z_1)\) and \(D(x_1, y_1, z_1) = \pm \{x_1, y_1, z_1\}\).

Now consider \(D(x_2, y_2, z_1)\). Clearly \((x_2, y_2, z_1)\) can only represent elements from \(D(x_2, y_2) \cup \{z_1\} \cup \{z_2, \ldots, z_k\}\).

Suppose \(z_2 \in D(x_2, y_2, z_1)\). There exists an \(i\) such that \(-z_2 \in D(x_1, y_1, z_2)\). In fact \(i = 2\) or else \(\varphi\) will be isotropic.

This means \(D(x_2, y_2, z_2) = \pm \{x_2, y_2, z_2\}\). Moreover,

\(z_2 \in D(x_2, y_2, z_1)\) if and only if \(-z_1 \in D(x_2, y_2, -z_2) = D(-x_2, -y_2, -z_2) = -D(x_2, y_2, z_2)\). So \(z_1 \in D(x_2, y_2, z_2)\). This cannot happen so we must have \(D(x_2, y_2, z_1) \subseteq \pm \{x_2, y_2, z_1\}\). So \(D(x_1, y_1, z_1) \subseteq \pm \{x_1, y_1, z_1\}\) for \(1 \leq i \leq r\). Of course the same argument applies to any \(z_j\), \(1 \leq j \leq k\).
Next, scale \( \varphi \) so as to put it in the form \( \varphi = (1, y_1, x_2, y_2, \ldots, x_r, y_r, z_1, \ldots, z_k) \). The result of the last paragraph, applied to this scaling of \( \varphi \), yields

\[
D(1, z_j) \not\subseteq D(1, y_1, z_j) \subseteq \pm \{1, y_1, z_j\}, \quad 1 \leq j \leq k. \quad \text{Now} \quad
\]

\[\pm y_1 \not\in D(1, z_j) \quad \text{since} \quad \pm y_1 z_j \not\subseteq \pm \{1, y_1, z_j\}. \quad \text{So we have} \quad
\]

\[D(1, z_j) \subseteq \langle -1, z_j \rangle, \quad 1 \leq j \leq k. \quad \text{Furthermore,} \quad D(1, -z_j) \not\subseteq \langle -1, z_j \rangle. \quad \text{By Proposition 1.2,} \quad |D(1, z_j)| > 2 \quad \text{if and only if} \quad |D(1, -z_j)| > 2. \quad \text{So if} \quad D(1, z_j) = \langle -1, z_j \rangle, \quad \text{then} \quad D(1, -z_j) = \langle -1, z_j \rangle \quad \text{also; thus} -1 \in D(1, z_j) \cap D(1, -z_j) \subseteq D(1, 1), \quad \text{a contradiction to} \quad s \geq \frac{4}{3}. \quad \text{So} \quad D(1, z_j) = \langle z_j \rangle, \quad D(1, -z_j) = \langle -z_j \rangle, \quad 1 \leq j \leq k.
\]

Now, suppose \( |D(1, 1)| \geq 8 \). Clearly \( D(1, x_i) \not\subseteq \)

\[D(1, y_1, x_i, y_i) = \pm \{1, y_1, x_i, y_i\} \quad \text{for} \quad 2 \leq i \leq r. \quad \text{Now} \quad
\]

\( y_1 \not\in D(1, x_i) \) for if so, \( (1, y_1, x_i, y_i) \approx (y_1, y_1, y_i, x_i y_1) \) and

\[|D(1, 1)| \geq 8 \quad \text{yields} \quad |D(1, y_1, x_i, y_i)| \geq 10, \quad \text{a contradiction.} \quad \text{Similarly} \quad y_i \not\in D(1, x_i), \quad \text{and since} \quad \varphi \quad \text{is anisotropic,}
\]

\[-y_1, -y_i \not\in D(1, x_i). \quad \text{Hence} \quad D(1, x_i) \subseteq \langle -1, x_i \rangle, \quad 2 \leq i \leq r. \quad \text{Since} \quad (x_i, y_i) \sim -(x_i, y_i), \quad \text{this same technique leads to}
\]

\[D(1, -x_i) \subseteq \langle -1, x_i \rangle. \quad \text{By Proposition 1.2 again,} \quad D(1, x_i) = \langle -1, x_i \rangle, \quad \text{so} \quad D(1, -x_i) = \langle -1, x_i \rangle \quad \text{also; thus} -1 \in D(1, x_i) \cap D(1, -x_i) \subseteq D(1, 1), \quad \text{a contradiction to} \quad s \geq \frac{4}{3}. \quad \text{Hence} \quad D(1, x_i) = \langle x_i \rangle, \quad D(1, -x_i) = \langle -x_i \rangle, \quad 2 \leq i \leq r. \quad \text{Similarly} \quad D(1, y_i) = \langle y_i \rangle, \quad D(1, -y_i) = \langle -y_i \rangle, \quad 2 \leq i \leq r.
\]

Since \( a \in D(1, 1) \) if and only if \( -1 \in D(1, -a) \), we may conclude \( D(1, 1) \cap \pm \{x_2, y_2, \ldots, x_r, y_r, z_1, \ldots, z_k\} = \emptyset \). Thus \( D(1, 1) \subseteq \langle -1, y_1 \rangle \), and this contradicts \( |D(1, 1)| \geq 8 \). Thus
Proposition 1.5 says that for all non-real fields with $u = q/2 < \infty$, $s$ is 1, 2, or 4. We now consider each of these cases, beginning with $s = 4$.

**Proposition 1.6.** Let $F$ be a non-real field with $u = q/2 < \infty$ and $s = 4$. Then any $u$-dimensional quadratic form $\varphi$ over $F$ can be put in the form $\varphi \cong (x_1, x_1, x_1, x_1, \ldots, x_t, x_t, x_t, x_t, z_1, \ldots, z_k)$, where the $z_i$ are distinct.

**Proof.** Again let $\varphi \cong (x_1, y_1, \ldots, x_r, y_r, z_1, \ldots, z_k)$ be a $\beta$-decomposition of $\varphi$. Notice that all results in the last proposition which were proved before the supposition $|D(1, 1)| \geq 8$ was made can be used here also. The fact that $|D(1, \pm z_1)| = 2$ when $\varphi$ was in scaled form in that proposition yields $|D(1, \pm x_i z_j)| = 2$ for $1 \leq i \leq r, 1 \leq j \leq k$. Since $\beta_1 \oplus \beta_1 = (1, -1, 1, -1)$, $x_1, -y_1 \in D(x_1, x_1)$. But there are two other square classes in $D(x_1, x_1)$ too, say $a F$ and $b F$. Since $|D(1, \pm x_i z_j)| = 2$, $a, b \notin D(z_1, \ldots, z_k)$. Since $s = 4$, we have $a, b \notin \{-x_1, y_1\}$. Hence $a, b \in D(x_2, y_2, \ldots, x_r, y_r)$ and we may assume $a = x_2$. Then $D(x_1, x_1) = \{x_1, -y_1, x_2, b\}$ and since $D(1, 1) = \{1, -x_1 y_1, x_1 x_2, b x_1\}$ is a group, $b x_1 \in -x_2 y_1 F^2$.

Moreover, $x_1 x_2 \in D(1, 1)$ implies $\langle -1, x_1 x_2 \rangle \subseteq D(1, -x_1 x_2)$. So by Proposition 1.2, $|D(1, x_1 x_2)| \geq 4$. Also, $D(1, x_1 x_2) = x_1 D(x_1, x_2) \supseteq x_1 D(x_1, y_1 x_2, y_2) = \pm \{1, x_1 y_1, x_1 x_2, x_1 y_2\}$. However, $-x_1 y_1, -x_1 y_2 \notin D(1, x_1 x_2)$ since $\varphi$ is anisotropic. If $-1 \in D(1, x_1 x_2)$ then $-1 \in D(1, x_1 x_2) \cap D(1, -x_1 x_2) \subseteq D(1, 1)$. 

which contradicts $s = 4$. Thus $D(1, x_1 x_2) = \{1, x_1 y_1, x_1 x_2, x_1 y_2\}$, and in particular, $x_1 x_2 \in y_1 y_2 F^2$ from which follows $(x_1, x_2) \sim (y_1, y_2)$. Finally, we may conclude that $(x_1, y_1, x_2, y_2) \sim (x_1, x_2, x_1, x_2) \sim (x_1, x_1, x_1, x_1)$.

Now consider $(x_3, y_3)$. As above $x_3, -y_3, a_1, b_1 \in D(x_3, x_3)$.

Suppose $a_1 \in D(x_1, y_1, x_2, y_2)$. If $a_1 \in D(x_1, y_1)$, then as before $(x_1, y_1, x_3, y_3) \sim (x_1, x_1, x_1, x_1)$. Hence $(x_1, y_1, x_2, y_2, x_3, y_3) \sim (x_1, x_1, x_1, x_1, x_2, y_2)$ with $D(x_2, y_2) \not\subseteq D(x_1, x_1, x_1, x_1) = -D(x_1, x_1, x_1, x_1)$ and $\varphi$ would be isotropic. So $a_1 \not\in D(x_1, y_1)$ and similarly $a_1 \not\in D(x_2, y_2)$. In fact, then, $a_1 \not\in D(x_1, y_1, x_2, y_2)$.

Therefore $a_1, b_1 \in D(x_1, y_1, \ldots, x_r, y_r)$; we can assume $a_1 = x_1$ and as above $(x_3, y_3, x_1, y_1) \sim (x_3, x_3, x_3, x_3)$. We can continue this process to obtain $\varphi \sim (x_1, x_1, x_1, \ldots, x_t, x_t, x_t, x_t, z_1, \ldots, z_k)$ where $t = r/2$. The $z_i$'s are distinct by the properties of the $\beta$-decomposition. □

**Theorem 1.7.** Every non-real field $F$ with $u = q/2 < \infty$ and $s = 4$ is equivalent with respect to quadratic forms to the 2-adic numbers $\mathbb{Q}_2$ or a formal iterated power series extension of $\mathbb{Q}_2$. $F$ is equivalent to $\mathbb{Q}_2$ if and only if $q = 8$.

**Proof.** If $q \leq 4$, $s \neq 4$ [1, Theorem 6.1]. If $q = 8$ and $s = 4$, then $F$ is equivalent to the 2-adic numbers [1, Theorem 6.11]. Hence it suffices to assume $q \geq 16$. Let $\varphi$ be a $u$-dimensional anisotropic form over $F$. By Proposition 1.6, $\varphi$ can be written after scaling as $\varphi \sim (1, 1, 1, 1, x_2, x_2, x_2, \ldots, x_t, x_t, x_t, x_t, z_1, \ldots, z_k)$ where $t = r/2$ and the $z_i$ are distinct. We know $|D(1, 1, 1, 1)| \geq 8$, and so the properties of the above $\beta$-
decomposition of $\varphi$ yield $|D(1,1,1,1)| = 8$. Recall that $|D(1,1)| = \frac{4}{1}$, and let $D(1,1) = \langle a, b \rangle$. This implies that $D(1,1,1,1) = \langle -1, a, b \rangle$.

Suppose $x \in D(1,1)$. Then $D(1,x) \subseteq D(1,1,1,1)$. If $D(1,x) = D(1,1,1,1)$, then $-1 \in D(1,x)$ means $-1 \in D(1,x) \cap D(1,-x) \subseteq D(1,1)$, a contradiction to $s = \frac{4}{1}$. So $D(1,x) \nsubseteq D(1,1,1,1)$, thus $|D(1,x)| \leq \frac{4}{1}$. If $|D(1,x)| = 2$, then since $(1,1,1,1)$ is not universal over $F$ and $-x \in D(1,1,1,1)$, $|D(1,-1)| = |D(1,(-x)x)| = 2$ by Proposition 1.2. We have reached a contradiction.

Hence, for all $x \in D(1,1)$, $|D(1,x)| = \frac{4}{1}$ and $-1 \notin D(1,x)$.

Now $|D(1,1,1)| \geq 6$ by Elman and Lam [5, Lemma 3.2], and $D(1,1,1) \nsubseteq D(1,1,1,1)$. So $D(1,1,1) \supseteq \{1,a,b,ab\} \cup \{c,d\}$, where $c,d \in D(1,1,1,1) - D(1,1) = -D(1,1)$. Suppose for definiteness that $-a, -b \in D(1,1,1)$, so that $D(1,1,1) \supseteq \{1,a+b,ab\}$.

Since $D(1,a) \subseteq D(1,1,1)$ and $|D(1,a)| = \frac{4}{1}$, $D(1,a) = \langle a,y \rangle$ for some $y \in D(1,1,1)$. Since the only groups that can be formed from the possible elements of $D(1,1,1)$ are $\langle a,b \rangle$, $\langle a,-b \rangle$, $\langle -a,b \rangle$, and $\langle -a,-b \rangle$, then $y$ is $b$ or $-b$. Now consider $D(1,-a)$. Since $-a \in D(1,1,1)$, we have $D(1,-a) \subseteq D(1,1,1,1)$, so $|D(1,-a)| \leq 8$. If $D(1,-a) = D(1,1,1,1)$, then $D(1,\langle a \rangle) \subseteq D(1,-a)$, hence $D(1,\langle a \rangle) = D(1,\langle a \rangle) \cap D(1,-a) \subseteq D(1,a)$, yielding $D(1,a) = D(1,\langle a \rangle)$. This means $D(1,1,1,1) = D(1,a)$, but $-a \in D(1,1,1)$ implies $0 \in D(1,1,1,1) = D(1,1,1,1)$, so $-1 \in D(1,1,1,1)$, a contradiction. Hence $|D(1,-a)| \leq \frac{4}{1}$, and $-1 \in D(1,-a)$, so $D(1,-a) = \langle -1,a \rangle$. Similarly, $D(1,-b) = \langle -1,b \rangle$. 

If \( b \in D(1,a) \), then \(-a \in D(1,-b)\), a contradiction. Thus 
\( y = -b \) and \( D(1,a) = \langle a, -b \rangle \). But \(-b \in D(1,a)\) implies \(-a \in D(1,b)\), so \( D(1,b) = \langle -a, b \rangle \). Also, \(-ab \in D(1,a)\) implies 
\(-a \in D(1,ab)\), so \( D(1,ab) = \langle -a, -b \rangle \).

If \(-ab \not\in D(1,1,1)\), then \((1,1,1,ab)\) is anisotropic. But 
\((1,1,1,ab)\) represents \(0 = a + (-a)\) non-trivially. So \(-ab \in D(1,1,1)\), and a similar technique to that above for \( D(1,-a) \)
shows \( D(1,-ab) = \langle -1, ab \rangle \). Hence \( D(1,1,1) = \{1, +a, +b, +ab\} \); so \( |D(1,1,1)| = 7 \).

We have now shown \( |D(1,x)| = \frac{1}{4} \) and \( D(1,x) \subseteq D(1,1,1,1) \) for all \( x \in D(1,1,1,1) - (-F^2) \). The following argument, due
to Cordes, shows that every quaternary anisotropic form
\( \mu = (w_1, w_2, w_3, w_4) \) with \( w_i \in D(1,1,1,1) \) represents \( D(1,1,1,1) \).
From calculations above and the fact that \( D(\mu) = \bigcup D(\alpha, \beta) \),
\( \alpha \in D(w_1, w_2), \beta \in D(w_3, w_4) \), it follows that \( D(\mu) \subseteq D(1,1,1,1) \).
If three \( w_i \) are in the same square class, then \( |D(1,1,1)| = 7 \)
implies \( |D(\mu)| \geq 8 \) and hence \( D(\mu) = D(1,1,1,1) \). If two \( w_i \)
are in the same square class, then \( \mu \preceq (w, w, w_3, w_4) \). We may
assume \( D(w,w) \cap D(w_3, w_4) = \emptyset \) or else \( \mu \) has three like
entries. But then \( |D(\mu)| \geq |D(w,w)| + |D(w_3, w_4)| = 8 \) and so
\( D(\mu) = D(1,1,1,1) \). Finally, if \( \mu \) contains no like entries,
we may assume \( D(w_1, w_2) \cap D(w_3, w_4) = \emptyset \) and the reasoning is
the same. Since \(-1 \in D(1,1,1,1)\), then any five dimensional
form with coefficients out of \( D(1,1,1,1) \) is isotropic.

Returning to our scaled \( \beta \)-decomposition of \( \varphi \), \( \varphi \simeq 
(1,1,1,1,x_2,x_2,x_2,x_2,\ldots, x_t, x_t, x_t, x_t, x_t, z_1, \ldots, z_k) \), recall
that the \( z_j \) are distinct. Let \( D(1,1,1,1) \) be denoted by \( D \)
for simplicity of notation, and consider the cosets of $D$.
Then we have $Q(F)/D = \{D, x_2D, \ldots, x_tD, b_1D, \ldots, b_mD\}$, where for
all $j$, $1 \leq j \leq k$, $z_j \in b_mD$ for some $m$, $1 \leq m \leq t$. Clearly
$|Q(F)/D| = q/8$. Suppose now that $z_1, \ldots, z_5 \in b_mD$ for some
$m$, $1 \leq m \leq t$. Then, since $b_mD = \langle -1, a, b \rangle$, $z_i \in -z_jF^2$, for
some $i, j$, $1 \leq i < j \leq 5$. This contradicts the anisotropy
of $\varphi$. Similarly, no 5 of the $z_j$, $1 \leq j \leq k$, can belong to
any $b_mD$, $1 \leq m \leq t$. If some $b_mD$ contained fewer than 4 of
the $z_j$, then $\dim (1,1,1,x_2,x_2,x_2,\ldots,x_t,x_t,x_t,x_t, z_1,\ldots,z_k) \leq \dim (1,1,1,x_2,x_2,x_2,\ldots,x_t,x_t,x_t,x_t, b_1,b_1,b_1,\ldots,b_1,b_1, b_1, b_1, b_1, b_1) = |Q(F)/D| \cdot 4 = q/2 = r$, which
contradicts the $u$-dimensionality of $\varphi$. Thus exactly 4 of
the $z_j$ in $\varphi$ are in each $b_mD$, and $\varphi$ can be reordered so that
$\varphi \cong (1,1,1,x_2,x_2,x_2,\ldots,x_t,x_t,x_t,x_t, z_1,\ldots,z_k,\ldots,z_k, z_k)$, where $z_{ij} \in b_mD$ for $1 \leq i \leq 4,$
$1 \leq j \leq t$.

Next, consider some quaternary subform $(z_{1j}, z_{2j}, z_{3j}, z_{4j})$
from the above ordering of $\varphi$, $1 \leq j \leq t$. We have
$(z_{1j}, z_{2j}, z_{3j}, z_{4j}) \subseteq b_jD$, so $b_jz_{ij} \in D(1,1,1,1)$ for
$i = 1,\ldots,4$. Now $(z_{1j}, z_{2j}, z_{3j}, z_{4j})$ is anisotropic since it
is a subform of $\varphi$, hence $b_j(z_{1j}, z_{2j}, z_{3j}, z_{4j}) = (b_jz_{1j}, b_jz_{2j}, b_jz_{3j}, b_jz_{4j})$ is anisotropic. Thus by the result
of Cordes shown in the next to last paragraph,
$b_j(z_{1j}, z_{2j}, z_{3j}, z_{4j}) \cong (1,1,1,1)$, i.e., $(z_{1j}, z_{2j}, z_{3j}, z_{4j}) \cong (b_j,b_j,b_j,b_j), 1 \leq j \leq t$. So $\varphi \cong (1,1,1,x_2,x_2,x_2,\ldots,$
x_t,x_t,x_t,b_1,b_1,b_1,\ldots,b_1,b_1,b_1, b_1,\ldots, b_1, b_1, b_1)$. 
To simplify notation again, let \( \varphi \equiv (1,1,1,1, a_2, a_2, a_2, a_2, \ldots, a_n, a_n, a_n, a_n) \) where \( n = q/8 \) and \( Q(F)/D = \{ D, a_2 D, \ldots, a_n D \} \). Suppose \( |D(1,x)| \geq \frac{q}{4} \) for some \( x \in D(a_2, a_2, a_2, a_2) \). Then \( \varphi \equiv (1,1,1,1, x, x, x, x, a_3, a_3, a_3, a_3, \ldots, a_n, a_n, a_n, a_n) \). If \( z \in D(1,x) \cap D(1,1,1,1) \), then \( z \in \pm F^2, \pm aF^2, \pm bF^2, \) or \( \pm abF^2 \). But if \(-1 \in D(1,x)\), then \(-x \in D(1,1), \) a contradiction. Also, if \( a \in D(1,x) \), then \(-x \in D(1,-a) \subseteq D(1,1,1), \) a contradiction. Similarly, \( b, ab \not\in D(1,x) \). The anisotropy of \( \varphi \) implies that \(-a, -b, -ab \not\in D(1,x); \) hence \( D(1,x) \cap D(1,1,1) = \{1\} \). If \( z \in D(1,x) \cap xD(1,1,1,1) \), then \( zx \in D(1,x) \cap D(1,1,1,1) = \{1\}, \) so \( z \in xF^2 \). Thus \( D(1,x) \cap D(x,x,x,x) = \{x\} \), and so \( |D(1,1,1,1, x, x, x, x, x, x)| \geq |D(1,1,1,1)| + |D(1,x)| + |D(x,x,x,x) - \{x\}| \geq 7 + \frac{q}{4} + 7 = 18. \) Thus \( |D(\varphi)| \geq 18 + 2(u - 8) = q + 2, \) a contradiction. So for all \( x \in D(a_2, a_2, a_2, a_2) \), \( |D(1,x)| = 2, \) and similarly, for all \( x \in D(a_i, a_i, a_i, a_i) \), \( 2 \leq i \leq n, \) \( |D(1,x)| = 2. \)

Summarizing, then, the binary quadratic form structure of \( F \) is given by the following: \( D(1,-1) = Q(F) = \langle -1, a, b \rangle \), \( D(1,1) = \langle a, b \rangle \), \( D(1,a) = \langle a, -b \rangle \), \( D(1,-a) = \langle -1, a \rangle \), \( D(1,b) = \langle -a, b \rangle \), \( D(1,-b) = \langle -1, b \rangle \), \( D(1,ab) = \langle -a, -b \rangle \), \( D(1,-ab) = \langle -1, ab \rangle \), \( D(1,x) = \langle x \rangle \) for all other \( x \) in \( Q(F) \).

The 2-adic numbers \( Q_2 \) have \( D(1,-1) = Q(Q_2) = \langle -1, 2, -3 \rangle \). Moreover, \( D(1,1) = \langle 2, -3 \rangle \), \( D(1,2) = \langle 2, 3 \rangle \), \( D(1,-2) = \langle -1, 2 \rangle \), \( D(1,-3) = \langle -2, -3 \rangle \), \( D(1,3) = \langle -1, -3 \rangle \), \( D(1,-6) = \langle -2, 3 \rangle \), and \( D(1,6) = \langle -1, 6 \rangle \). So \( \sigma : Q(Q_2) \to Q(F) \) defined by \( \sigma(-1) = -1, \sigma(2) = a, \sigma(-3) = b \) and extended homomorphically satisfies
\( \sigma D(\alpha, \beta) = D(\sigma(\alpha), \sigma(\beta)) \) for all \( \alpha, \beta \in Q(Q_2) \). If \( q(F) = 2^{r+3} \geq 16 \) and if \( Q_2((y)) \) denotes the field of formal power series over \( Q_2 \), it is clear that any isomorphic extension of \( \sigma : Q(Q_2) \to F \) to \( Q[Q_2((y_1))((y_2)) \cdots ((y_r))] \) satisfies the hypothesis of Proposition 2.2 of [1]. Hence \( \sigma \) is an equivalence map and the Theorem is proved. □

The above result for \( s = 4 \) is more definite and satisfying than the analagous results for \( s = 1, 2 \). Although the binary quadratic form structure can be determined when \( s = 1, 2 \), the resulting fields are equivalent with respect to quadratic forms to fields whose existence is unknown or to power series extensions of such fields.

From the discussion preceding Theorem 4 in [2] and by the same techniques used for Proposition 5.15 in [1], it is easily seen that there are only three types of possible fields \( F \) (up to equivalence) with \( u(F) = q(F)/2 \), \( R(F) \neq \hat{F}^2 \), and \( s(F) = 1 \) or \( 2 \):

1. A field \( F \) with \( R(F) = \hat{F}^2 \cup -\hat{F}^2 \), \( s(F) = 2 \)
2. A field \( F \) with \( R(F) = \hat{F}^2 \cup a\hat{F}^2 \), \( a \not\in \hat{F}^2 \), \( s(F) = 2 \)
3. A field \( F \) with \( R(F) = a\hat{F}^2 \), \( a \in \hat{F}^2 \), \( s(F) = 1 \).

These types of fields will be referenced as numbered above in the work that follows. Although each type is unique, their existence is not determined for any \( q \geq 8 \).

We also list two fields \( F \) with \( q(F) = 8 \) and "trivial" radical for ease of reference:

4. A field \( F \) with \( q(F) = 8 \), \( u(F) = 4 \), \( R(F) = \hat{F}^2 \), \( m(F) = 4 \) and \( s(F) = 2 \).
(5) a field $F$ with $q(F) = 8$, $u(F) = \frac{1}{4}$, $R(F) = F^2$, $m(F) = \frac{1}{4}$, and $s(F) = 1$.

These are actually power series extensions of fields with $u = 2$ and $q = \frac{1}{4}$, as will be noted later, and their existence is known (see Theorem 6.11(4) and Propositions 6.6 and 6.7 of [1]).

**Proposition 1.8.** Let $F$ be a nonreal field with $u = q/2 < \infty$, $q \geq 16$, and $s = 2$. Suppose there exists an anisotropic $u$-dimensional form over $F$ which has some repeated entries. Then there exists one of the form $\varphi \approx (1, 1, a_1, a_1, \ldots, a_n, a_n)$, $n + 1 = q/4$; $|D(1, 1)| = 4$; and $|D(1, a_i)| = 2$ or $4$, $1 \leq i \leq n$.

**Proof.** Since $D(1, 1, 1) = Q(F)$, the maximal number of times any entry is repeated in an anisotropic $u$-dimensional form is two. Let $\varphi$ be an anisotropic $u$-dimensional form with some repeated entries, and arrange the entries of $\varphi$ so that the pairs of like entries are listed first, i.e., $\varphi \approx (z_1, z_1, z_2, z_2, \ldots, z_k, z_k, x_1, \ldots, x_r)$. By scaling there exists a $u$-dimensional form $\varphi$ where $\varphi \approx (1, 1, a_1, a_1, \ldots, a_k, a_k, x_1, \ldots, x_r)$, with the $x_i$ distinct if they actually are present.

Since $-1 \in D(1, 1)$, $D(1, 1)$, $D(a_j, a_j)$, $D(a_i, a_i)$, $\{x_i\}_{i=1}^r$, $\{-x_i\}_{i=1}^r$ are pairwise disjoint sets (e.g., $z \in D(a_i, a_i) \cap D(a_j, a_j) = a_i z \in D(1, 1) = -a_i z \in D(1, 1)$, hence $-z \in D(a_i, a_i) \Rightarrow 0 = -z + z \in D(a_i, a_i, a_j, a_j) \subseteq D(\varphi)$, a contradiction). By Proposition 1.3, $|D(1, 1)| \geq 4$. Since $u = 2(k+1) + r$, we have the following inequality:
\[ |D(\varphi)| \geq |D(1,1) \cup D(a_1^{i}, a_1^{i}) \cup \ldots \cup D(a_k^{i}, a_k^{i}) \cup \{x_i^{r} \}_{i=1}^{r} \cup \{-x_i^{r} \}_{i=1}^{r} | \]
\[ \geq 4(k+1) + 2r = 2u = q; \]
and this says \( Q(F) = D(1,1) \cup D(a_1^{i}, a_1^{i}) \cup \ldots \cup D(a_k^{i}, a_k^{i}) \cup \{x_i^{r} \}_{i=1}^{r} \).
This then forces \( |D(1,1)| = 4 = |D(a_1^{i}, a_1^{i})|, 1 \leq i \leq k. \)
Let \( D(1,1) = \langle -1, a \rangle \) for some \( a \in Q(F). \)

Suppose at least one \( x_i^{j} \) actually appears in \( \varphi \) and consider the cosets of \( D = D(1,1) \) in \( Q(F) \), where we let \( Q(F)/D = \{ D, a_1 D, \ldots, a_k D, b_1 D, \ldots, b_{\ell} D \} \) for some \( b_j \in Q(F) \), \( 1 \leq j \leq \ell \). Clearly the \( x_i^{j} \) are in the \( b_j D \), and \( |Q(F)/D| = q/4 \). Consider \( x_1, \ldots, x_r \) and suppose that \( x_1, x_2, x_3 \in b_1 D \). Then since \( b_1 D = b_1 \langle -1, a \rangle \), \( -x_m \in x_n F^2 \) for some \( 1 \leq m \neq n \leq 3 \), which contradicts the anisotropy of \( \varphi \). Similarly, at most 2 of the \( x_i^{j} \) can belong to each \( b_j D \). Suppose next that some \( b_j D \) contained only one of the \( x_i^{j} \). Then \( \dim (1,1, a_1^{i}, a_1^{i}, \ldots, a_k^{i}, a_k^{i}, \ldots, x_i^{r}) < \dim (1,1, a_1^{i}, a_1^{i}, \ldots, a_k^{i}, a_k^{i}, b_1, b_1, \ldots, b_{\ell}, b_{\ell}) \)
\[ = |Q(F)/D| \cdot 2 = (q/4) \cdot 2 = u, \]
which contradicts the dimension of \( \varphi \). Thus exactly two of the \( x_i^{j} \) are in each \( b_j D \) and \( \varphi \) may be rearranged so that \( \varphi \cong (1,1, a_1^{i}, a_1^{i}, \ldots, a_k^{i}, a_k^{i}, x_1^{11}, x_2^{11}, x_1^{12}, x_2^{12}, \ldots, x_1^{i}, x_2^{i}), \) where \( x_{ij} \in b_j D \) for \( i = 1, 2; \) for each \( j, 1 \leq j \leq \ell \). Consider a pair \( (x_1^{j}, x_2^{j}) \) where \( x_1^{j}, x_2^{j} \in b_j D \). Now \( b_j (x_1^{j}, x_2^{j}) \cong (b_j x_1^{j}, b_j x_2^{j}) \) must be anisotropic since \( \varphi \) is, and \( b_j x_1^{j}, b_j x_2^{j} \in D \). By inspection, the only distinct binary forms with entries in \( D \) which are anisotropic are \((1,1), (1,a), \) and \((1,-a). \) If \( (b_j x_1^{j}, b_j x_2^{j}) \cong (1,1), \) then \( (x_1^{j}, x_2^{j}) \cong (b_j, b_j) \); and if \( (b_j x_1^{j}, b_j x_2^{j}) \cong (1,a), \) then \( (x_1^{j}, x_2^{j}) \cong (b_j, b_j a). \) Since this pair appears in \( \varphi, |D(b_j, b_j a)| \leq 4, \) so \( |D(1,a)| \leq 4. \) But \( -a \in D, \) so
-1 ∈ D(1,a), hence |D(1,a)| = 4. Thus D(1,a) = <-1,a> = D, and so D(b_j,b_ja) = D(b_j,b_j). Thus (b_j,b_ja) may be replaced in \( \varphi \) by (b_j,b_j). If (b_jx_1j,b_jx_2j) \cong (1,-a), a similar argument shows (x_1j,x_2j) may be replaced in \( \varphi \) by (b_j,b_j). Thus, in all cases (x_1j,x_2j) may be replaced by (b_j,b_j) in \( \varphi \). Hence, \( \varphi \cong (1,1,a_1,a_1,\ldots,a_k,a_k,b_1,b_1,\ldots,b_l,b_l) \), and it may be assumed that no distinct entries \( x_i \) actually appear.

By the above, let \( \varphi \cong (1,1,a_1,a_1,\ldots,a_n,a_n) \) where n + 1 = \( q/4 \). For any \( i \), \( 1 \leq i \leq n \), \( D(a_i,a_i) = \{±a_i,±aa_i\} \), and \( D(1,a_i) \subseteq D(1,1) \cup D(a_i,a_i) = \{±1,±a_i,±a_i,±aa_i\} \). Thus \( |D(1,a_i)| < 8 \), and since \( D(1,a_i) \) is a 2-group, \( |D(1,a_i)| = 2 \) or 4, \( 1 \leq i \leq n \). Since for all \( i \), \( 1 \leq i \leq n \), \( D(-a_i,-a_i) = D(a_i,a_i) \), we have then \( D(1,-a_i) \subseteq D(1,1) \cup D(a_i,a_i) \), so by the same argument, \( |D(1,-a_i)| = 2 \) or 4, \( 1 \leq i \leq n \). This completes the proof of the proposition.

Proposition 1.9. Let \( K \) be a nonreal field with \( u = q/2 < \infty \), \( q \geq 8 \), and \( s = 2 \), and suppose there exists an anisotropic \( u \)-dimensional form over \( K \) which has some repeated entries. Then \( K \) is equivalent with respect to quadratic forms to a field of type (2) or (4) or to an iterated formal power series extension of one of them.

Proof. This is clear for \( q = 8 \) by Theorem 6.11 of [1], so let \( q \geq 16 \). Since an anisotropic \( u \)-dimensional form over \( K \) with some repeated entries exists, by Proposition 1.8 there also exists \( \varphi \cong (1,1,a_1,a_1,\ldots,a_n,a_n) \) anisotropic with
\[ n + 1 = q/4, \quad |D(1,1)| = 4, \quad \text{and} \quad |D(1,\pm a^1)| = 2 \text{ or } 4, \]

\[ 1 \leq i \leq n. \] Suppose for all \( i, 1 \leq i \leq n \), that \( |D(1,a^i)| = 2 \).

Then by Proposition 1.1, \( |D(1,-a^1)| = 2, \quad |D(1,aa^1)| = 2, \) and \( |D(1,-aa^1)| = 2 \). Since \( \mathbb{Q}(K) \) is given by \( \mathbb{Q}(K) = \{ \pm 1, \pm a, \pm a^1, ..., \pm a^n, \pm aa^1, ..., \pm aa_n \} \), \( |D(1,x)| = 2 \) for all \( x \in \mathbb{Q}(K) - D(1,1) \). Now \( <-l, a> \subseteq D(1,+a) \), so suppose there exists \( b \in D(1,+a) - D(1,1) \). Then \( \pm a \in D(1,-b) \), which is a contradiction. So \( D(1,a) = <-l, a> = D(1,-a) \), and the binary quadratic form structure is given by \( D(l,l) = D(l,+a) = <-l,a>, D(l,x) = <x> \) for all \( x \in \mathbb{Q}(K) - D(l,l) \). This field \( K \) is thus clearly equivalent to an iterated formal power series extension of a field of type \((4)\), which is known to exist (see Proposition 6.6 of \([1]\)).

Next suppose there exists some \( i, 1 \leq i \leq n \), such that \( |D(1,a^i)| = 4 \), say \( i = 1 \), so that \( |D(1,a^1)| = 4 \). Now \(-1 \notin 1 \), so \(-a^1 \notin D(1,a^1) \). Thus, since \( D(l,a^1) \subseteq \{ \pm 1, \pm a, \pm a^1, \pm aa^1 \} \), either \( a \) or \(-a \) (but not both) is in \( D(l,a^1) \), so \( D(l,a^1) = <a,a^1> \) or \( D(l,a^1) = <-a,a^1> \). Suppose for definiteness that \( D(l,a^1) = <a,a^1> \). The proof is similar in the other case and will be omitted, since it yields equivalent results. Since \( a \in D(l,a^1) \), then \( D(l,a) \subseteq D(l,l,a^1) \). Now \( |D(l,l,a^1)| \leq |D(l,l,a^1,a^1)| - 1 = 7 \), so \( D(l,1,a^1) \supseteq \{ \pm 1, \pm a, a^1 \} \) and possibly two more of \( \{-a^1, aa^1\} \). Therefore \( D(l,a) \supseteq \{ \pm 1, \pm a, a^1, aa^1 \} \), so \( |D(l,a)| = 2 \) or \( 4 \).

Now \(-a \in D(1,1) \), so \(-1 \in D(1,a) \), thus \(-a \in D(l,a) \), and this gives \( |D(l,a)| = 4 \) and \( D(l,a) = <-1,a> = D(1,1) \). Also, \( |D(l,a^1)| = 4 \) implies \( |D(1,-a^1)| = 4 \) since \( |D(1,-a^1)| \leq 4 \).
by Proposition 1.1. Now $D(1,aa_1) \not\subseteq D(1,1,aa_1,aa_1)$, so $D(1,aa_1) \not\subseteq \{\pm1, \pm a, \pm a_1, \pm aa_1\}$, thus $|D(1,aa_1)| \leq 4$. Since $a \in D(1,a_1) \cap D(1,-a) \subseteq D(1,aa_1)$, we have $D(1,a_1) = \langle a, a_1 \rangle = D(1,aa_1)$. Similarly, we have $D(1,-a) = \langle a, -a_1 \rangle = D(1,-aa_1)$.

Consider $D(1,a_2)$. Now $a \in D(1,a_1)$ means $-a \not\in D(1,a_2)$ by the anisotropy of $\varphi$. Also, $D(1,a_2) \not\subseteq \{\pm1, \pm a, \pm a_2, \pm aa_2\}$, so either $D(1,a_2) = \langle a, a_2 \rangle$, in which case $a_2 \in D(1,-a)$, or $D(1,a_2) = \langle a_2 \rangle$. Similarly, for all $i$, $2 \leq i \leq n$, $D(1,a_i) = \langle a, a_i \rangle$, so that $a_i \in D(1,-a)$, or $D(1,a_i) = \langle a_i \rangle$. By rearranging $\varphi$ then, $\varphi \cong (1,1,b_1,b_1', \ldots, b_k, b_k', c_1, c_1', \ldots, c_m, c_m')$, where $D(1,b_i) = \langle a, b_i \rangle$ for all $i$, $1 \leq i \leq k$, and $D(1,c_i) = \langle c_i \rangle$ for all $i$, $1 \leq i \leq m$. Notice also that $D(1,-a) = \{\pm1, \pm b_1, \pm a, \pm b_2, \pm a_2, \ldots, \pm b_k, \pm a_k \} = D(1,1,b_1,b_1', \ldots, b_k, b_k')$.

Suppose that no pair $(c_i,c_i')$ appears in $\varphi$, i.e., $m = 0$. Then $\varphi \cong (1,1,b_1,b_1', \ldots, b_n, b_n')$, $n + 1 = q/4$, where $D(1,b_i) = \langle a, b_i \rangle = D(1,ab_i)$ for all $i$, $1 \leq i \leq n$. Then $D(1,-b_i) = \langle a, -b_i \rangle = D(1,-ab_i)$ for all $i$, $1 \leq i \leq n$ also. The binary quadratic form structure in this case is then $D(1,1) = \langle -1, a \rangle = D(1,a)$, $D(1,-a) = Q(K) = D(1,-1)$, and $D(1,x) = \langle a, x \rangle = D(1,ax)$ for all $x \in Q(K) - D(1,1)$. So $K$ has $R(K) = K^2 \cup aK^2$ and is thus equivalent to a field of type (2). At present no such field is known to exist.

Suppose now that some pair $(c_i,c_i')$ appears in $\varphi$, i.e., $m \neq 0$, where $c_i \not\in D(1,1,b_1,b_1', \ldots, b_k, b_k') = D(1,-a)$. Thus $|Q(K)/D(1,-a)| \geq 2$, which means $|D(1,-a)| \leq q/2$. Thus there exist at least $q/2$ elements of $Q(K)$ such that $D(1,x) = \langle x \rangle$. 

The binary quadratic form structure here is thus $D(1,1) = \langle -1, a \rangle = D(1,a), D(1,-a) = D(1, -1), D(1,x) = \langle a, x \rangle = D(1,ax)$ for all $x \in D(b_1, b_1, \ldots, b_k, b_k)$, and $D(1,x) = \langle x \rangle$ for all other $x \in Q(K)$. So $K$ is equivalent to an iterated formal power series extension of a field $F$ with $Q(F) = \{ +l_1/a_1/b_1/b_1, \ldots, +b_k/a_k/b_k \}$ where $D_F(1,1) = \langle -1, a \rangle = D_F(1,a), D_F(1,-a) = Q(F)$, and for all $y \in Q(F) - D(1,1), D_F(1,y) = \langle a, y \rangle$. $F$ is clearly a field of type $(2)$. Thus all cases have been explored and the proposition is proved.

Proposition 1.10. Let $K$ be a nonreal field with $u = q/2 < \infty$, $q \geq 16$, and $s = 2$, and suppose every $u$-dimensional anisotropic form over $K$ has distinct entries. Then $K$ is equivalent with respect to quadratic forms either to a field of type $(1)$ or to an iterated formal power series extension of such a field.

Proof. Let $\varphi$ be a $u$-dimensional anisotropic form, say $\varphi = (1, x_1, \ldots, x_{u-1})$ where the $x_i$ are distinct. Consider the groups $D(1, x_i), 1 \leq i \leq u-1$, examining $D(1, x_1)$ in particular. Now for all $i, 2 \leq i \leq u-1$, $+x_i \not\in D(1, x_1)$ by the anisotropy of $\varphi$ and the fact that every $u$-dimensional anisotropic form has distinct entries. We then have $|\{+x_i\}_{i=1}^{u-1}| = 2(u-2) = q-4$, so there exist at least $q-4$ elements of $Q(K)$ that are not in $D(1,x_1)$, hence $|D(1,x_1)| \leq q-4$. Similarly, for all $i, 1 \leq i \leq u-1, |D(1,x_i)| \leq q-4$. If $|D(1,x_i)| = 2$ for all $i, 1 \leq i \leq u-1$, then by Proposition 1.2, $|D(1,-x_i)| = 2$ for all $i, 1 \leq i \leq u-1$, also. But $|\{+x_i\}_{i=1}^{u-1}| = 2(u-1) = q-2$, so for
all \( a \in Q(K) - \{ \pm K^2 \} \). \( |D(1,a)| = 2 \). Also, if \( a \in D(1,1) \), \( a \not\in \pm K^2 \), then \(-1 \in D(1,-a)\), a contradiction. Thus \( |D(1,1)| = 2 \) also, which directly contradicts Proposition 1.3.

Thus there exists some \( i, 1 \leq i \leq u-1 \), such that 
\( |D(1,x^1_i)| = \frac{4}{4}, \) say \( |D(1,x^1)| = 4 \). Since for all \( i \), 
\[ 2 \leq i \leq u-1, \] \[ +x_i \not\in D(1,x^1_1), \] and \( Q(K) = \{ \pm x^1_i \}^{u-1}_i=2 \cup \{ \pm 1, \pm x^1_1 \} \), we have \( D(1,x^1_1) = \langle -1,x^1_1 \rangle \). This in turn implies by Proposition 1.1 that \( |D(1,-x^1_1)| \geq \frac{4}{4} \). Suppose for some \( i \), 
\[ 2 \leq i \leq u-1, \] that \( x^1_i \in D(1,-x^1_1) \). Then \( 1 \in D(x^1_1,x^1_1) \), so \( \varphi \) can be written with two like entries, a contradiction. Next suppose for some \( i, 2 \leq i \leq u-1, \) that \(-x^1_i \in D(1,-x^1_1) \). Then \( x^1_i \in D(1,x^1_1) \), so \( \varphi \) can be written with like entries, a contradiction. Thus for all \( i, 2 \leq i \leq u-1, \) \( +x^1_i \not\in D(1,-x^1_1) \).

Hence \( D(1,-x^1_1) \subseteq \{ \pm 1, \pm x^1_1 \} \), and since \( |D(1,-x^1_1)| \geq \frac{4}{4} \), 
\( D(1,-x^1_1) = \langle -1,x^1_1 \rangle = D(1,x^1_1) \). Similarly, if any \( x^1_i \) has \( |D(1,x^1)| = \frac{4}{4} \), then \( |D(1,-x^1_1)| = \frac{4}{4} \) also; and if any \( x^1_i \) has \( |D(1,x^1_1)| \geq \frac{4}{4} \), then \( |D(1,x^1)| \geq \frac{4}{4} \), so \( |D(1,x^1)| = \frac{4}{4} \), hence \( |D(1,-x^1_1)| = \frac{4}{4} \). In either situation, \( D(1,x^1) = \langle -1,x^1_1 \rangle = D(1,-x^1_1) \) for that \( x^1_i \).

Thus \( \varphi \) can be rearranged so that \( \varphi \sim (1,x^1_1,\ldots,x^1_n, y^1_1,\ldots,y^1_m) \), where for all \( i, 1 \leq i \leq n, \) \( D(1,x^1_i) = \langle -1,x^1_i \rangle = D(1,-x^1_i) \), and for all \( i, 1 \leq i \leq m, \) \( D(1,y^1_i) = \langle y^1_i \rangle = D(1,-y^1_i) \) = \( \langle -y^1_i \rangle \). Clearly \( |D(1,x^1_1,\ldots,x^1_n)| \leq 2(n+1) \), so 
\( D(1,x^1_1,\ldots,x^1_n) = \{ \pm 1, \pm x^1_1,\ldots,\pm x^1_n \} \). If \( a \in D(1,1) \), then \(-1 \in D(1,-a) \). Thus \(-a \in D(1,x^1_1,\ldots,x^1_n) \), and we then have \( a \in D(1,x^1_1,\ldots,x^1_n) \). This says \( D(1,1) \subseteq D(1,x^1_1,\ldots,x^1_n) \). If \( a \in D(1,x^1_1,\ldots,x^1_n) \), then \( a \in \pm x^1_i K^2 \) where \( 1 \leq i \leq n \) or
a ∈ χ^2. If a ∈ ±x_i^K^2, then |D(1,a)| = 4, so -1 ∈ D(1,a), hence -a ∈ D(1,1), so a ∈ D(1,1). Clearly ±1 ∈ D(1,1), so we have D(1,x_1, ..., x_n) ⊆ D(1,1). Hence D(1,1) = D(1,x_1, ..., x_n), which shows D(1,x_1, ..., x_n) is a 2-group and also D(1,1) = {±1, ±x_1, ..., ±x_n}.

Suppose that no y_i appear in ϕ, i.e., ϕ = (1,x_1, ..., x_{u-1}) where D(1,x_i) = <-l,x_i> = D(l,-x_i) for all i, 1 ≤ i ≤ u-1. Then for all a / ∈ χ^2, |D(1,a)| = 4, -1 ∈ D(1,a), hence -a ∈ D(1,1). Thus |D(1,1)| = q, R(K) = K^2 ∪ -K^2 and D(1,x) = <-l,x> for all x ∈ Q(K) - (+K^2). So K is equivalent with respect to quadratic forms to a field of type (1).

If on the other hand, at least one y_i appears in ϕ, say y_1, then y_1 / ∈ D(1,x_1, ..., x_n) = D(1,1), so |Q(K)/D(1,1)| ≥ 2, thus |D(1,1)| ≤ q/2. So there are at least q/2 elements of Q(K) not in D(1,1), i.e., there exist at least q/2 elements a ∈ Q(K) such that |D(1,a)| = 2. In this case the binary quadratic form structure is given by D(1,1) = <-l,x_1, ..., x_n>, D(1,x_i) = <-l,x_i> = D(l,-x_i) for all i, 1 ≤ i ≤ n, and D(1,y) = <y> for all other y ∈ Q(K). Thus K is equivalent with respect to quadratic forms to an iterated formal power series extension of a field of type (1), and the proposition is proved.

The following theorem summarizes the results obtained for the s = 2 case.

**Theorem 1.11.** Every nonreal field K with u(K) = q(K)/2 < ∞, s(K) = 2, and q(K) ≥ 8, is equivalent with respect to
quadratic forms to a field of type (1), (2), or (4), or to an iterated formal power series extension of such a field.

Proof. If \( q(K) = 8 \), see Theorem 6.11 of [1]. If \( q(K) \geq 16 \), Propositions 1.9 and 1.10 complete the proof. □

As previously mentioned, fields of type (4), and thus iterated formal power series extensions of them, are known to exist. The existence of the other fields possible for \( s = 2 \) is not presently known. We now turn to the remaining case to be considered, namely \( s = 1 \).

Proposition 1.12. Let \( F \) be a nonreal field with \( u = q/2 < \infty \), \( q \geq 16 \), and \( s = 1 \). Then there exists an anisotropic \( u \)-dimensional form \( \varphi \) such that \( \varphi \cong (1, x, a_1 x, \ldots, a_n x) \) for some \( x \in \mathbb{Q}(F) \), where \( n + 1 = q/4 \), \( |D(1,x)| = \frac{q}{4} \), and \( \{1, a_1, \ldots, a_n\} \) is a full set of coset representatives of \( D(1,x) \) in \( \mathbb{Q}(F) \). Also, \( D(1,x) = \langle a, x \rangle \) and \( D(1, a_i), D(1, a_i x), D(1, a_ia_i), D(1, a_ia_i x) \) have order 2 or 4 for all \( i, 1 \leq i \leq n \).

Proof. Let \( \varphi \cong (1, x_1, \ldots, x_{u-1}) \) where the \( x_i \) are distinct. Suppose for any such \( \varphi \) that \( |D(1, x_i)| = 2 \) for all \( i, 1 \leq i \leq u-1 \). However, there does exist some \( x \in \mathbb{Q}(F) - F \) such that \( |D(1,x)| \geq \frac{q}{4} \), since \( u \neq q \) (see Theorem 5.13 of [1]). If it is assumed that for all \( i \neq j, 1 \leq i, j \leq u-1 \), \( x_i x_j \not\in \mathbb{F}^2 \), then \( x \not\in D(1, x_i) \) yields \( x_i \not\in D(1,x) \), hence \( xx_i \not\in D(1,x) \) for all \( i, 1 \leq i \leq u-1 \). Since \( xx_i \not\in x_j \mathbb{F}^2 \) for any \( i \neq j, 1 \leq i, j \leq u-1 \), then \([x_i] \cup \{xx_i\}]_{i=1}^{u-1} \) contains \( 2(u-1) = q - 2 \) elements. Thus there are \( q-2 \) elements of \( \mathbb{Q}(F) \) not in
D(1,x), hence |D(1,x)| = 2, a contradiction. So for some \( i \neq j \), \( 1 \leq i,j \leq u-1 \), we have \( x_i x_j \in xF^2 \). Then \( \varphi \) has (\( x_i,x_i x \)) as a subform, so (\( 1,x \)) is a subform of \( x_i \varphi \), which is a \( u \)-dimensional anisotropic form with distinct entries also.

On the other hand, if there exists some \( x_i \) such that \( |D(1,x_i)| \geq \frac{1}{4} \), then \( \varphi \) itself has a subform \((1,x)\) (namely, \((1,x_i)\)) with \(|D(1,x)| \geq \frac{1}{4} \).

Rearrange \( \varphi \) so that if some product \( x_i x_j \in xF^2 \), \((x_i,x_j)\) is placed after \((1,x)\); and continue until \( \varphi \) has the form \( \varphi \cong (1,x,y_1,y_1 x,y_2,y_2 x,\ldots,y_m,y_m x,z_1,\ldots,z_n) \), where \( u = 2(m+1) + n \). Clearly \( z_1,\ldots,z_n \notin D(1,x,\ldots,y_m,y_m x) \). Suppose \( z_1 x \in D(1,x,y_1,y_1 x,\ldots,y_m,y_m x) \). Then \( z_1 \in xD(1,x,y_1,y_1 x,\ldots,y_m,y_m x) = D(1,x,y_1,y_1 x,\ldots,y_m,y_m x) \), a contradiction. Also, \( z_1 x \notin \{z_2,\ldots,z_n\} \) by the above arrangement of \( \varphi \). Thus \(|\{z_i\} \cup \{z_i x\}|_{i=1}^n = 2n \), and none of these elements is in \( D(1,x,y_1,y_1 x,\ldots,y_m,y_m x) \). Hence \(|D(\varphi)| \geq \frac{1}{4}(m+1) + 2n = 2u = q \), so if \(|D(1,x)| > \frac{1}{4}, |D(\varphi)| > q \), a contradiction. Hence \(|D(1,x)| = \frac{1}{4} \), so let \( D(1,x) = <a,x> \) for some \( a \in Q(F) \), and abbreviate \( D(1,x) \) by \( D \). Let \( \{1,y_1,\ldots,y_m,w_1,\ldots,w_l\} \) be a full set of coset representatives of \( D \) in \( Q(F) \). Clearly there are \(|Q(F)/D| = q/4 \) elements in this set, and the \( z_i \) are in the \( w_j D \).

Suppose that \( z_1,z_2,z_3 \in w_1 D = \{w_1,a,w_1 x,w_1 ax\} \). By inspection, some pair out of \( \{z_1,z_2,z_3\} \) will have product \( x \), contradicting the way \( \varphi \) was arranged above. Thus in general at most two of the \( z_i \) are in each \( w_j D \). Suppose some \( w_j D \)
contained only one $z_i$. Then, $\dim \varphi \leq$

$\dim (1, x, y_1, y_1 x, \ldots, y_m, y_m x, w_1, w_1 x, \ldots, w_\ell, w_\ell x) = |Q(F)/D| \cdot 2$

$= (q/4) \cdot 2 = u$, which contradicts the dimension of $\varphi$. Thus exactly two of the $z_i$ are in each $w_j D$, and we again rearrange $\varphi$ so that $\varphi \cong (1, x, y_1, y_1 x, \ldots, y_m, y_m x, z_1, z_1 x, z_2, z_2 x, \ldots, z_\ell, z_\ell x)$, where $z_{ij} \in w_j D$ for $i = 1, 2$; for each $j$, $1 \leq j \leq \ell$.

Consider any pair $(z_{1j}, z_{2j})$, where $z_{1j}, z_{2j} \in w_j D$. The form $w_j(z_{1j}, z_{2j}) \cong (w_j z_{1j}, w_j z_{2j})$ must be anisotropic since $\varphi$ is, and $w_j z_{1j}, w_j z_{2j} \in D$. By inspection, the only possible anisotropic binary forms with entries out of $D$ are $(1, x), (1, ax), (a, x), (a, ax)$, and $(x, ax)$. Suppose $(w_j z_{1j}, w_j z_{2j}) \cong (1, x)$. Then $(z_{1j}, z_{2j}) \cong (w_j, w_j x)$. If $(w_j z_{1j}, w_j z_{2j}) \cong (1, a)$, then $(z_{1j}, z_{2j}) \cong (w_j, w_j a)$. Since this is a subform of $\varphi$, a counting argument on $|D(\varphi)|$ shows $|D(w_j, w_j a)| \leq \frac{1}{4}$, hence $|D(1, a)| \leq \frac{1}{4}$. Since $x \in D(1, a)$, $D(1, a) = D(1, x)$, so $(z_{1j}, z_{2j})$ may be replaced in $\varphi$ by $(w_j, w_j x)$ thus getting a new $u$-dimensional form but without affecting the anisotropy. If $(w_j z_{1j}, w_j z_{2j}) \cong (1, ax)$, then $z_{1j}, z_{2j}) \cong (w_j, w_j ax)$, and as above we may replace $(z_{1j}, z_{2j})$ in $\varphi$ by $(w_j, w_j x)$. If $(w_j z_{1j}, w_j z_{2j}) \cong (a, x)$, then $(z_{1j}, z_{2j}) \cong (w_j a, w_j x)$. Thus $|D(w_j a, w_j x)| \leq \frac{1}{4}$, so $|D(a, x)| \leq \frac{1}{4}$, hence $|D(1, ax)| \leq \frac{1}{4}$, and as before $D(1, ax) = D(1, x)$. This gives $D(a, x) = aD(1, ax) = aD(1, x) = D(1, x)$, so $(z_{1j}, z_{2j})$ may be replaced in $\varphi$ by $(w_j, w_j x)$. The other cases, $(a, ax)$ and $(x, ax)$, are done similarly and yield the same result. So for all $j$, $1 \leq j \leq \ell$, $(z_{1j}, z_{2j})$ can be replaced in $\varphi$ by
\[(w_j, w_jx),\] and there is a \(\varphi\) of the form \(\varphi \cong (l, x, y_1, y_1x, \ldots, y_m, y_mx, w_1, w_1x, \ldots, w_k, w_kx)\). Thus we can write \(\varphi \cong (l, x, a_1, a_1x, \ldots, a_n, a_nx)\) where \(n + 1 = q/4\), and \(\{1, a_1, \ldots, a_n\}\) is a full set of coset representatives of 
\[D(l, x)\] in \(Q(F)\).

Moreover, \(D(l, a_i)\), \(D(l, a_ix)\) are properly contained in 
\[D(l, x, a_i, a_ix)\], for all \(i, 1 \leq i \leq n\); and \(D(l, aa_i)\), \(D(l, aa_ix)\) are properly contained in 
\[D(l, x, aa_i, aa_ix)\], for all \(i, 1 \leq i \leq n\). But \(D(l, x, a_i, a_ix) = D(l, x) \cup D(a_i, a_ix)\) and 
\[D(l, x, aa_i, aa_ix) = D(l, x) \cup D(aa_i, aa_ix)\] by counting arguments on \(|D(\varphi)|\), and both of these sets are equal to 
\[\{1, a, x, ax, a_i, aa_i, a_ix, aa_ix\}\]. Thus \(D(l, a_i)\), \(D(l, a_ix)\), 
\(D(l, aa_i)\), and \(D(l, aa_ix)\) are properly contained in this set of 8 elements. Since all these sets are 2-groups, they must have order 2 or \(4\), for all \(i, 1 \leq i \leq n\). The proposition is proved. \(\square\)

**Theorem 1.13.** Every nonreal field \(K\) with \(u(K) = q(K)/2 < \infty\), \(s(K) = 1\), and \(q(K) \geq 8\) is equivalent with respect to quadratic forms to a field of type (3) or (5), or to an iterated formal power series extension of such a field.

**Proof.** If \(q(K) = 8\), see Theorem 6.11 of [1]. So suppose \(q(K) > 16\), and let \(\varphi \cong (l, x, a_1, a_1x, \ldots, a_n, a_nx)\) be the \(u\)-dimensional anisotropic form whose existence is guaranteed by Proposition 1.12.

Suppose that \(|D(l, a_i)| = 2\) for all \(i, 1 \leq i \leq n\). Consider 
\[D(l, aa_i) \not\subseteq \{l, a, x, ax, a_1, aa_1, a_ix, aa_ix\}\] (see proof of
Proposition 1.12). Now $aa_i \not\in D(1,x)$ implies $x \not\in D(1,aa_i)$. Moreover, $ax \not\in D(1,a_i)$ means $a_i \not\in D(1,ax)$. Since $a \in D(1,a) \cap D(1,x) \subseteq D(1,ax)$ also, we have $aa_i \not\in D(1,ax)$, or $ax \not\in D(1,aa_i)$. So $D(1,aa_i) \subseteq \{1,a,a_i,aa_i\}$. But $a \not\in D(1,a_i)$, so $aa_i \not\in D(1,a_i)$, hence $a_i \not\in D(1,aa_i)$ and $a \not\in D(1,aa_i)$. Hence $D(1,aa_i) = \langle a,a_i \rangle$, so $|D(1,aa_i)| = 2$ also. Consider $D(1,a_i x)$. Since $a_i x \not\in D(1,x)$, then $x \not\in D(1,a_i x)$. Moreover, $a_i x \not\in D(1,aa_i)$ implies $aa_i \not\in D(1,a_i x)$, and so $ax \not\in D(1,a_i x)$. Similarly, $a_i, aa_i \not\in D(1,a_i x)$. If $a \in D(1,a_i x)$, then $a \in D(1,a_i x) \cap D(1,x) \subseteq D(1,a_i)$, a contradiction, so $a \not\in D(1,a_i x)$, thus $aa_i x \not\in D(1,a_i x)$. Hence $D(1,a_i x) = \langle a_i x \rangle$, and $|D(1,a_i x)| = 2$. Similarly $|D(1,aa_i x)| = 2$.

Clearly $\langle a,x \rangle \not\in D(1,a)$, and $\langle a,x \rangle \subseteq D(1,ax)$. If $b \in D(1,a) - D(1,x)$, then $a \in D(1,b)$ but we have just shown that for all $y \in \mathbb{Q}(K) - D(1,x)$, $D(1,y) = \langle y \rangle$. Thus $D(1,a) = \langle a,x \rangle$ and similarly $D(1,ax) = \langle a,x \rangle$. The binary quadratic form structure of $K$ is therefore given by $D(1,x) = D(1,a) = D(1,ax) = \langle a,x \rangle$ and $D(1,y) = \langle y \rangle$ for all $y \in \mathbb{Q}(K) - D(1,x)$. This field is clearly equivalent with respect to quadratic forms to an iterated formal power series extension of a field $F$ of type(5). $F$ (and thus $K$ also) is known to exist (see Proposition 6.7 of [1]).

Now suppose instead that there exists some $i$, $1 \leq i \leq n$, such that $|D(1,a_i)| = \frac{3}{4}$. Assume without loss of generality that $|D(1,a_i)| = \frac{3}{4}$. Now $D(1,a_i) = \langle a,a_i \rangle$ or $\langle ax,a_i \rangle$, so suppose $D(1,a_i) = \langle a,a_i \rangle$. (The other choice yields analogous results). Now we have $a \in D(1,a_i) \cap D(1,x) \subseteq D(1,a_i x)$, so
|D(l,a_1x)| = 4 and D(l,a_1x) = \langle a,a_1, x \rangle. Similarly, D(l,aa_1)
= \langle a,a_1 \rangle = D(l,a_1) and D(l,aa_1x) = \langle a,a_1x \rangle = D(l,a_1x). Now
a \in D(l,a_1), so for all i, 2 \leq i \leq n, ax \not\in D(l,a_i). (If
so, a \in D(x,a_i x), and a \in D(l,a_1), hence \varphi can be written
with 2 like entries.) So for all i, 2 \leq i \leq n, either
D(l,a_1) = \langle a,a_1 \rangle, in which case a \in D(l,a), D(l,a_1x) =
\langle a,a_1x \rangle = D(l,aa_1x), and D(l,aa_1) = \langle a,a_1 \rangle = D(l,a_1) as
above for i = 1; or D(l,a_1) = \langle a^2 \rangle, in which case a \in
D(l,a), D(l,aa_1) = \langle aa_1 \rangle, D(l,a_1x) = \langle a_1x \rangle, and D(l,aa_1x) =
\langle aa_1x \rangle. Thus \varphi can be rearranged so that \varphi \sim
(l,x,a_1,a_1x,\ldots,a_n,a_nx,b_1,b_1x,\ldots,b_m,b_mx) where D(l,y) =
\langle a,y \rangle for all y \in D(l,x,a_1,a_1x,\ldots,a_n,a_nx) and D(l,y) = \langle y \rangle
for all y \in D(b_1,b_1x,\ldots,b_m,b_mx). Notice that D(l,a) =
\{l,a,x,ax,a_1,aa_1,a_1x,aa_1x,\ldots,a_n,aa_n,a_nx,aa_nx\}, so D(l,a) =
D(l,x,a_1,a_1x,\ldots,a_n,a_nx).

Suppose that no pair (b_1,b_1x) appears in \varphi, i.e., m = 0.
Then \varphi \sim (l,x,a_1,a_1x,\ldots,a_n,a_nx), where D(l,a_1) = \langle a,a_1 \rangle =
D(l,aa_1) and D(l,a_1x) = \langle a,a_1x \rangle = D(l,aa_1x). Notice also
that if b \in D(l,ax) - D(l,x), then ax \in D(l,b), a contradic­tion.
So the binary quadratic form structure in this case
is given by D(l,x) = \langle a,x \rangle = D(l,ax), D(l,a) = Q(K) = D(l,l),
and D(l,y) = \langle a,y \rangle for all y \not\in D(l,x). So K is equivalent
with respect to quadratic forms to a field F with R(F) =
F^2 \cup aF^2, u(F) = q(F)/2, and s(F) = 1 (type (3)).

Suppose on the other hand that some pair (b_1,b_1x)
appears in \varphi, b_1 \not\in D(l,x,a_1,a_1x,\ldots,a_n,a_nx) = D(l,a). Thus
\mid Q(K)/D(l,a) \mid \geq 2, so \mid D(l,a) \mid \leq q/2, and hence there exist
at least $q/2$ elements $y$ such that $D(1,y) = \langle y \rangle$. Also, if $b \in D(1,ax) - D(1,x)$, then $ax \in D(1,b)$, a contradiction. So the binary quadratic form structure in this case is given by $D(1,x) = \langle a,x \rangle = D(1,ax)$, $D(1,y) = \langle a,y \rangle$ for all $y \in D(1,a) = D(1,x,a_1^2,x,a_2,a_1^2,x,...,a_n^2,a_n^2)$, and $D(1,y) = \langle y \rangle$ for all other $y \in \mathbb{Q}(K)$. Since there are at least $q/2$ elements $y$ such that $D(1,y) = \langle y \rangle$, $K$ is equivalent with respect to quadratic forms to an iterated formal power series extension of a field $F$ with $Q(F) = \{1,a,x,ax,a_1^2,a_1^2,x,a_2,a_1^2,x,...,a_n^2,a_n^2\}$, where $D_F(1,a) = Q(F) = D_F(1,1)$, so that $R(F) = \hat{F}^2 \cup a\hat{F}^2$, and for all other $y \in Q(F)$, $D(1,y) = \langle a,y \rangle$. Clearly $F$ is a field of type (3). The proposition is proved, and the $s = 1$ case is completed.

The fields of type (4) and (5) are actually power series extensions of fields with $u = 2$ and $q = 4$, and there are no fields with $u = 1$ and $q = 2$ (see [1]). Thus we can generalize the results for the cases $s = 1, 2$ even further, as stated in the following theorem.

**Theorem 1.14.** Every nonreal field $K$ with $u(K) = q(K)/2 < \infty$ and $s(K) = 1$ or 2 is equivalent with respect to quadratic forms to a field of one of the following types:

1. a field $F$ with $R(F) = \hat{F}^2 \cup \hat{F}^2$, $u(F) = q(F)/2$, $s(F) = 2$

2. a field $F$ with $R(F) = \hat{F}^2 \cup a\hat{F}^2$, $a \not\in \hat{F}^2$, $u(F) = q(F)/2$, $s(F) = 1$ or 2

3. a field $F$ with $R(F) = \hat{F}^2$, $u(F) = 2$, $q(F) = 4$, $s(F) = 1$ or 2

4. an iterated formal power series extension of a field $F$ of one of these three types.
CHAPTER II. NON-FORMALLY REAL FIELDS WITH q = 16.

This chapter contains results which enable us to classify with respect to quadratic forms all possible nonreal fields \( K \) with \( q(K) \leq 16 \). Since fields with \( q \leq 8 \) were classified in [1], we restrict our attention to fields with \( q = 16 \). Also, since fields \( K \) with \( q(K) = 16 \) and non-trivial radical have \( |\hat{K}/R(K)| \leq 8 \), they have been classified in [3]. So we may assume all fields \( K \) under consideration here to have \( q(K) = 16 \) and \( R(K) = \hat{K}^2 \).

By the work of Elman and Lam [5, Theorems 2.7, 2.7'], we know that if \( u \neq 16 \) then \( u \leq 8 \). Since it is well known that \( u \neq 3, 5, \) or 7, we only need consider \( u = 2, 4, 6, 8, \) or 16. Fields with \( u = q < \infty \) and \( u = 2 \) were classified in [1], and the results of Chapter I classify fields with \( u = q/2 = 8 \). Finally, by Pfister [13, Satz 18(d)], \( q = 16 \) implies \( s = 1, 2, \) or 4. So the cases left to be considered are \( q = 16, |R| = 1, u = 4 \) or 6, and \( s = 1, 2, \) or 4. It will turn out, with two possible exceptions, that all fields that can exist with these properties are equivalent to formal power series extensions of fields with \( q = 8 \), which are listed in 6 categories in Theorem 6.11 of [1]. At the end of the chapter we will combine our results into a theorem giving the possible nonreal fields with \( q = 16 \).

**Proposition 2.1.** Let \( K \) be a field with \( u \leq 6 \) and \( q \geq 16 \). If there is an \( x \in \hat{K} \) with \( D(1,x) = \langle x \rangle \), then there is no \( a \in \hat{K} \) with \( 4 \leq |D(1,a)| \leq q/4 \).
Proof. Suppose there was such an $a$. Consider $D(l,x,a,ax) = \bigcup D(a,a\beta) = \langle x \rangle D(l,a) \cup \langle x \rangle D(l,ax)$. Since $u \leq 6$, every quaternary form of determinant 1 is universal. Since no group is the union of two proper subgroups and $|\langle x \rangle D(l,a)| \leq q/2$, we must have $\langle x \rangle D(l,ax) = Q(K)$. Hence $|D(l,ax)| \geq q/2$. But $D(l,ax) \neq Q(K)$ for if so, $(l,x,a,ax)$ would be isotropic and this would mean $-a \in D(l,x)$. This is impossible under the hypotheses so $|D(l,ax)| = q/2$. It now follows that $|D(l,a) \cap D(l,ax)| \geq \frac{1}{2} |D(l,a)| \geq 2$. However, $D(l,a) \cap D(l,ax) \subseteq D(l,-x) = \langle -x \rangle$ by Proposition 1.2. So $D(l,a) \supseteq \langle -x \rangle$ and this again contradicts $-a \notin D(l,x)$. Thus $a$ cannot exist. □

We remark that in the course of this proof it was shown that, under the hypothesis of Proposition 2.1, $|D(l,a)| = 2$ implies $|D(l,ax)| \geq q/2$, and in fact $|D(l,ax)| = q/2$. So, in particular, if $s \geq 2$, then $|D(l,1)| = q/2$.

Proposition 2.2. Let $K$ be a field with $u = 4$ and with $m$ quaternion algebras. If $d \in \mathbb{K}$, then $K$ has $m - \frac{q}{|D(l,-d)|}$ anisotropic quaternary forms of determinant $d$.

Proof. Since $u = 4$, every quaternary form is universal. So every such form of determinant $d$ looks like $(d,-a,-b,ab)$. Forms over fields with $u = 4$ are determined by their dimension, determinant, and Hasse algebra [10, Theorem 58:8]. The Hasse algebra for $(d,-a,-b,ab)$ is $[-d,-d][a,b]$. Hence there are $m$ quaternary forms of determinant $d$, but some are isotropic. All of these have the form $(1,1,a,-ad)$. It is
easy to see \((1,-1,a,-ad)\) and \((1,-1,b,-bd)\) are equivalent if and only if \(ab \in D(1,-d)\). So there are \(\frac{q}{|D(1,-d)|}\) isotropic \((1,-1,a,-ad)\).

**Proposition 2.3.** Let \(K\) be a field with \(q(K) = 16, u(K) = 4, R(K) = K^2\), and suppose there exists \(x \in K\) such that \(D(1,x) = \langle x \rangle\). Then \(K\) is equivalent with respect to quadratic forms to a formal power series extension of a field \(F\) with \(q(F) = 8, u(F) = 2, s(F) = s(K),\) and \(R(F) = F^2\).

**Proof.** If \(s(K) = 4\), then \(h = 4 = s\), and we also have \(|D(1,x)| < 2 \cdot \dim(1,x)\). This contradicts a result of Elman and Lam [5, Corollary 3.6]. So \(s(K) = 1\) or \(2\).

If \(s(K) = 1\), then by [1, Theorem 5.13], there must be an element \(a \in K\) with \(|D(1,a)| > 2\). By Proposition 2.1, no element \(y\) satisfies \(|D(1,y)| = \frac{1}{2}\) so \(D(1,a) = \langle a, b, c \rangle\) for \(b, c \in K\). Since \(a \in D(1,a)\), we clearly have that \(|D(1,a)| > \frac{1}{4}\) for \(a \in \{a, b, c\} - \{1\}\). Thus \(|D(1,a)| = 8\) for \(a \in \{a, b, c\} - \{1\}\). Now \(D(1,ax) = \langle ax \rangle\) for \(a \in \{a, b, c\}\) because if not, \(|D(1,ax)| = 8\) and so \(|D(1,a \cap D(1,ax)| > \frac{1}{4}\) but \(D(1,a) \cap D(1,ax) \subseteq D(1,x) = \langle x \rangle\). This shows \(Q(K) = \langle a, b, c, x \rangle\) and \(D(1,a) = \langle a, b, c \rangle\) for all \(a \in \{a, b, c\} - \{1\}\). So \(K\) is equivalent with respect to quadratic forms to a formal power series extension of the field with \(s = 1\) listed in (1) of Theorem 6.11 of [1].

If \(s(K) = 2\), then by Propositions 1.2 and 2.1, \(D(1,1) = \langle -1, a, b \rangle\). By Propositions 1.1, \(D(1,ax) = \langle ax \rangle\) for all \(a \in \{a, b, c\}\). Hence, \(Q(K) = \langle -1, a, b, x \rangle\) and \(D(1,a) = \langle -1, a, b \rangle\).
for all $\alpha \in \langle -1, a, b \rangle - \{-1\}$. But $\langle -1, \alpha \rangle \subseteq D(1, a)$ for $\alpha \in \langle -1, a, b \rangle$. So $|D(1, a)| \geq 4$ implies $D(1, a) = \langle -1, a, b \rangle$ for all $\alpha \in \langle -1, a, b \rangle - \{-1\}$. So $K$ is equivalent with respect to quadratic forms to a formal power series extension of the field with $s = 2$ listed in (1) of Theorem 6.11 of [1]. □

Proposition 2.4. Let $K$ be a field with $q = 16$, $u = 4$, $|D(1, a)| \geq 4$ for all $a \in K$, and $|D(l, a)| = 8$ for $k$ elements out of $Q(K)$. If $k \geq 2$, then $m \leq 4$.

Proof. Let $p$ denote the maximum index of all $D(1, a)$ in $Q(K)$. So $p = 2$ or $\frac{1}{4}$. If $p = 2$, then $m = 2$ by Kaplansky [8, Theorem 2]. Consider $p = \frac{1}{4}$. In the beginning of section 4 of [3], there is a counting argument for the maximum $m$ in terms of $p$ and $q$. Our hypothesis here allows us to change that numerator (the maximum number of anisotropic ternary forms of determinant 1) to $(q-1-k)(p-1) + k(p/2-1)$. Dividing by the same denominator, we see that in this case $m-1 \leq \frac{2p[(q-1-k)(p-1) + k(p/2-1)]}{3q}$. For $p = \frac{1}{4}$, $q = 16$, the right side of this inequality is less than seven if and only if $k \geq 3/2$. It is well known that for $u \leq \frac{1}{4}$, the quaternion algebras form a subgroup of the Brauer group. So $m$ is a power of two. Thus $k \geq 2$ implies $m \leq 4$ in the case $p = \frac{1}{4}$. □

Proposition 2.5. Let $K$ be a nonreal field with $m = 4$.

Suppose for $a \in \hat{K}$ that $D(1, -a)$ has index four in $Q(K)$. Then $D(l, x) \cap yD(1, ax) \neq \emptyset$ for all $x, y \in \hat{K}$.

Proof. By Lam [9, Corollary 4.12], $m = 4$ implies $u = 4$. By
Proposition 2.2 then, the form \((l,x,-y,-axy)\) is isotropic for all \(x,y \in \mathbb{K}\). So \(D(l,x) \cap yD(l,ax) \neq \emptyset\).

Proposition 2.6. Let \(K\) be a field with \(q = 16, u = 4, |R| = 1\), and \(|D(l,x)| \geq \frac{1}{4}\) for all \(x \in \mathbb{K}\). If \(|D(l,a)| = 8\) for some \(a\), then there is a \(b \not\in \mathbb{K}^2\) such that \(|D(l,b)| = 8\).

Proof. Suppose there exists exactly one \(a \mod \mathbb{K}^2\) such that \(|D(l,a)| = 8\), so that for all \(x \not\in \mathbb{K}^2 \cup -\mathbb{K}^2, |D(l,x)| = 4\).

If \(s = 1\), let \(D(l,a) = \langle a,b,c \rangle\) where \(Q(K) = \langle a,b,c,d \rangle\). Now \(|D(l,b)| = \frac{1}{4} = |D(l,ab)|\), so \(D(l,b) = \langle a,b \rangle = D(l,ab)\). Thus \(D(l,c) = \langle a,c \rangle = D(l,ac)\). Now, clearly \(d,da \not\in D(l,a)\) for \(a \not\in \langle a,b,c \rangle - \{1\}\), and this forces \(D(l,d) = \langle d \rangle\), a contradiction.

Next, consider the case \(s = 2\). If \(a \in \mathbb{K}^2\), let \(D(l,l) = \langle -l,f,g \rangle\) where \(Q(K) = \langle -l,f,g,h \rangle\). Thus \(D(l,-a) = \langle -l,a,b,c \rangle\). Since \(-1 \in D(l,l)\), \(a \in D(l,1)\) also. So \(-1 \in D(l,-a)\), hence \(D(l,-a) = \langle -l,-a,-b \rangle\), a contradiction. If \(-a \not\in D(l,1)\), let \(D(l,a) = \langle a,b,c \rangle\) where \(Q(K) = \langle -l,a,b,c \rangle\). Now since \(a \not\in D(l,1)\), \(-l \in D(l,1), \) and \(|D(l,1)| = 4\), we may assume \(D(l,1) = \langle -l,b \rangle\). So \(-l \in D(l,-b)\). But \(b \in D(l,a)\) implies \(-a \in D(l,-b)\), thus \(D(l,-b) = \langle -l,-a,-b \rangle\), a contradiction.
Finally consider the case $s = 4$. If $a \in \mathbb{K}^2$, let $D(1,1) = \langle a, b, c \rangle$ with $Q(K) = \langle -1, a, b, c \rangle$. So $D(1,-a) = \langle -1, a \rangle$ for all $a \in \langle a, b, c \rangle - \{1\}$. Thus $D(1,a) = \langle a, y \rangle$ where $y \in -bk^2, -cK^2$, or $-bcK^2$. These yield similar results so we will assume $y \in -bk^2$. Then $D(1,b) = \langle -a, b \rangle$, $D(1,ab) = \langle -a, -b \rangle$. These, with the above, force $D(1,c) = \langle c \rangle$, a contradiction.

If $-a \in D(1,1) - \{-1\}$, let $D(1,a) = \langle -1, a, b \rangle$ with $Q(K) = \langle -1, a, b, c \rangle$. If $D(1,-a) \subseteq \langle -1, a, b \rangle$, then $D(1,-a) = D(1,a) \cap D(l,a) \subseteq D(1,1)$ implies $D(1,-a) = D(1,1)$. It now follows that $D(1,c) = \langle c \rangle$, a contradiction. So $D(1,-a) \not\subseteq D(1,a)$ and we may assume $D(1,-a) = \langle -a, c \rangle$. The only possibility for $D(1,c)$ then is $\langle -1, c \rangle$. Hence $D(1,1) = \langle -a, -c \rangle$. So $-bc \not\in D(1,a)$ for $a \in \langle -1, a, b \rangle - \{1\}$ then shows $D(1,bc) = \langle bc \rangle$, a contradiction. If $-a \not\in D(1,1)$, let $D(1,a) = \langle a, b, c \rangle$ where $Q(K) = \langle -1, a, b, c \rangle$. Now since $-1, -a \not\in D(1,1)$ and $|D(1,1)| = 4$, we have $D(1,1) = \langle x, y \rangle$ where $x, y \in D(1,a) - \{-a\}$. This forces $D(1,1) \cap D(1,a) \neq \emptyset$. For definiteness assume $b \in D(1,1)$, so that $-1 \in D(1,-b)$. However, $b \in D(1,a)$ implies $-a \in D(1,-b)$, so $D(1,-b) = \langle -1, -a, -b \rangle$, a contradiction. □

Combining the last three propositions, we get the following useful corollary.

**Corollary.** Let $K$ be a field with $q = 16$, $u = \frac{1}{4}$, $|R| = 1$, $|D(l,x)| \geq 4$ for all $x \in \mathbb{K}$, and $|D(l,b)| = 8$ for at least one $b$. If $|D(l,-a)| = 4$, then for $x \in \mathbb{K}$ we have

1. $D(l,x) \cap D(l,ax) = \{1\}$ if $|D(l,x)| = |D(l,ax)| = 4$ and
(2) \( yD(1,ax) \not\subseteq D(1,x) \) for all \( y \in \hat{K} \) if \( |D(1,x)| = 8 \).

The following unpublished result of C. M. Cordes will be helpful in analyzing the \( u = 4 \) case.

**Proposition 2.7 (Cordes).** Let \( K \) be a nonreal field with \( 16 \leq q < \infty \) and \( R(K) = \hat{K}^2 \). Suppose \( D(1,-a) \) has index at most 4 for all \( a \in \hat{K} \). If \( H \) is a subgroup in \( \hat{K} \) which contains \( \hat{K}^2 \), has index 2 in \( \hat{K} \), and satisfies \( |D(1,-x)| \geq q/2 \) for all \( x \in H \), then \( m(K) \leq 2 \).

In the next six propositions we will be assuming that \( |D(1,x)| \geq 4 \) for all \( x \in \hat{K} \) and \( m(K) > 2 \). From Kaplansky [8] we know that \( |D(1,x)| = 8 \) for all \( x \not\in \hat{K}^2 \) implies \( m = 2 \). So if \( m > 2 \) it can be assumed that \( |D(1,x)| = 4 \) for at least one value of \( x \in \hat{K} \).

**Proposition 2.8.** Let \( K \) be a nonreal field with \( q(K) = 16 \), \( R(K) = \hat{K}^2 \), \( u(K) = 4 \), \( s(K) = 1 \), \( m(K) > 2 \), and \( |D(1,x)| \geq 4 \) for all \( x \in \hat{K} \). Then \( m(K) = 4 \) and \( K \) has the following binary form value set structure, where \( \Omega(K) = \langle a,b,c,d \rangle = D(1,1) \):

\[
\begin{align*}
D(1,a) &= \langle a,b,c \rangle & D(1,abc) &= \langle a,bc \rangle & D(1,ac) &= \langle a,c \rangle \\
D(1,b) &= \langle a,b,d \rangle & D(1,ad) &= \langle b,c,ad \rangle & D(1,bc) &= \langle a,d,bc \rangle \\
D(1,ab) &= \langle a,b \rangle & D(1,acd) &= \langle c,ad \rangle & D(1,abd) &= \langle b,ad \rangle \\
D(1,c) &= \langle a,c,d \rangle & D(1,cd) &= \langle c,d \rangle & D(1,bcd) &= \langle d,bc \rangle \\
D(1,d) &= \langle b,c,d \rangle & D(1,bd) &= \langle b,d \rangle & D(1,abcd) &= \langle bc,ad \rangle.
\end{align*}
\]

**Proof.** Let \( \Omega(K) = \langle a,b,c,d \rangle \) and first suppose that \( |D(1,x)| = 4 \) for all \( x \not\in \hat{K}^2 \). For definiteness assume \( D(1,a) = \langle a,b \rangle \). Then \( D(1,b) = \langle a,b \rangle = D(1,ab) \) is clear. Since \( c \not\in D(1,a) \)
for $a \in \langle a, b \rangle - \{1\}$, we may assume $D(1,c) = \langle c, d \rangle$. This forces $D(1,d) = \langle c, d \rangle = D(1,cd)$ also. Since $ac \notin D(1,a)$ for $a \in \langle a,b,d \rangle - \{1,bd,abd\}$, we may assume that $D(1,ac) = \langle ac, bd \rangle$. Thus $D(1,bd) = \langle ac, bd \rangle = D(1,abcd)$ also. Now $D(1,bc) = \langle bc, w \rangle$ where $w \in \langle a,b,d \rangle$. Clearly, if $a,b,d,ab$, or $bd \in D(1,bc)$, we contradict our previous structure. So $w = ad$ or $abd$. Suppose that $w = ad$, so that $D(1,ac) = \langle bc, ad \rangle = D(1,ad)$. These then force $D(1,abcd) = \langle bc, ad \rangle$, a contradiction. The choice $w = abd$ yields the following binary quadratic form value set structure:

$$
D(1,a) = D(1,b) = D(1,ab) = \langle a, b \rangle
$$
$$
D(1,c) = D(1,d) = D(1,cd) = \langle c, d \rangle
$$
$$
D(1,ac) = D(1,bd) = D(1,abcd) = \langle ac, bd \rangle
$$
$$
D(1,bc) = D(1,abd) = D(1,acd) = \langle bc, abd \rangle
$$
$$
D(1,ad) = D(1,bcd) = D(1,abc) = \langle ad, bcd \rangle.
$$

However, this cannot determine a quadratic form structure since $bc \in D(1,a,c) = \bigcup_{a \in D(1,c)} D(a,a)$, but $bc \notin \bigcup_{a \in D(1,a)} D(a,c)$. Next, assume there is an $x$ such that $|D(1,x)| = 8$, say $x \in aK^2$. Let $D(1,a) = \langle a,b,c \rangle$. If $|D(1,b)| = 4 = |D(1,ab)|$, then $D(1,b) = \langle a,b \rangle = D(1,ab)$, which contradicts (2) of the corollary following Proposition 2.6. So $|D(1,b)| = 8$ or $|D(1,ab)| = 8$. These will give equivalent results, so let us suppose $|D(1,b)| = 8$, so that $D(1,b) = \langle a,b,x \rangle$ with $x \in \langle c,d \rangle$. If $D(1,b) = \langle a,b,c \rangle$, then $c \in D(1,a) \cap D(1,b) \subseteq D(1,ab)$, so $D(1,ab) = \langle a,b,c \rangle$ also. This forces $D(1,c) = D(1,ac) = D(1,abc) = D(1,bc) = \langle a,b,c \rangle$, which implies $D(1,d) = \langle d \rangle$, a contradiction. So $D(1,b) = \langle a,b,d \rangle$ or $\langle a,b,cd \rangle.$
If $D(l,b) = \langle a, b, d \rangle$, suppose first that $|D(l,c)| = 4 = |D(l,ac)|$, which implies $D(l,c) = \langle a, c \rangle = D(l,ac)$, a contradiction of (2) of the corollary after Proposition 2.6. Thus $|D(l,c)| = 8$ or $|D(l,ac)| = 8$. These will give equivalent results, so let us suppose $|D(l,c)| = 8$, so that $D(l,c) = \langle a, c, x \rangle$ with $x \in \langle b, d \rangle$. If $D(l,c) = \langle a, b, c \rangle$, then $b \in D(l,a) \cap D(l,c) \subseteq D(l,ac)$, so $ac \in D(l,b)$, a contradiction. So $D(l,c) = \langle a, c, d \rangle$ or $\langle a, c, bd \rangle$. These cases yield analogous results, and here we will analyze only the first case in detail. So let $D(l,c) = \langle a, c, d \rangle$, where $D(l,b) = \langle a, b, d \rangle$. So $d \in D(l,b) \cap D(l,c) \subseteq D(l,bc)$, and $D(l,ac) = \langle a, c, x \rangle$, $x \in \langle b, d \rangle$, or $|D(l,ac)| = 4$. If $b \in D(l,ac)$, then $b \in D(l,a) \cap D(l,ac) \subseteq D(l,c)$, a contradiction. If $d \in D(l,ac)$, then $d \in D(l,c) \cap D(l,ac) \subseteq D(l,a)$, a contradiction. So $D(l,ac) = \langle a, c, bd \rangle$ or $\langle a, c \rangle$. If $D(l,ac) = \langle a, c, bd \rangle$, we have $D(l,ab) = \langle a, b, cd \rangle$, $D(l,abc) = \langle a, bc, bd \rangle$, $D(l,bc) = \langle a, bc, d \rangle$, $D(l,d) = \langle b, d, c \rangle$, $D(l,bd) = \langle b, d, ac \rangle$, $D(l,ad) = \langle b, c, ad \rangle$, $D(l,cd) = \langle c, d, ab \rangle$, $D(l,acd) = \langle c, ad, ab \rangle$, $D(l,bcd) = \langle ab, ac, d \rangle$, $D(l,abd) = \langle b, ac, ad \rangle$, and $D(l,abcd) = \langle ab, cd, bc \rangle$. But we know from the remarks following Proposition 2.7 that there is an $x$ such that $|D(l,x)| = 4$, so this structure cannot exist. Next we will suppose that $D(l,ac) = \langle a, c \rangle$. Now either $D(l,ab) = \langle a, b \rangle$ or $D(l,ab) = \langle a, b, x \rangle$ where $x \in \langle c, d \rangle$. If $D(l,ab) = \langle a, b, x \rangle$ where $x \in \langle c, d \rangle$, we have $D(l,abc) = \langle a, bc \rangle$, $D(l,bc) = \langle a, bc, d \rangle$, $D(l,d) = \langle b, c, d \rangle$, $D(l,bd) = \langle b, d, c \rangle$, and $D(l,ad) = \langle b, c, ad \rangle$. If $c \in D(l,ab)$, $c \in D(l,a) \cap D(l,ab) \subseteq D(l,b)$, a contradiction. If $d \in D(l,ab)$, $d \in D(l,b) \cap
D(l,ab) ⊆ D(l,a), a contradiction. So D(l,ab) = <a,b,cd>,
which forces D(l,cd) = <c,d,ab> and D(l,abd) = <b,ad,cd>.
This scheme yields ac ∈ D(l,abd), which implies abd ∈ D(l,ac),
a contradiction.

If, on the other hand, D(l,ab) = <a,b>, we will have
D(l,d) = <b,c,d>. Clearly D(l,bd) ⊇ <b,d>, and a,c,ac ∉
D(l,bd). Hence D(l,bd) = <b,d>. Similarly then, D(l,abc) ⊇
<a,bc>, and b,d,bd ∉ D(l,abc), hence D(l,abc) = <a,bc>.
Also, it is clear that D(l,ad) = <b,c,ad> and D(l,bc) =
<a,d,bc>. Consider D(l,cd) ⊇ <c,d>. Since a,b,ab ∉ D(l,cd),
D(l,cd) = <c,d>. Continuing in this manner we obtain the
following list.

D(l,a) = <a,b,c> D(l,abc) = <a,bc> D(l,ac) = <a,c>
D(l,b) = <a,b,d> D(l,ad) = <b,c,ad> D(l,bc) = <a,d,bc>
D(l,ab) = <a,b> D(l,acd) = <c,ad> D(l,abd) = <b,ad>
D(l,c) = <a,c,d> D(l,cd) = <c,d> D(l,bcd) = <d,bc>
D(l,d) = <b,c,d> D(l,bd) = <b,d> D(l,abcd) = <bc,ad>

This is clearly a list which has the structure of a field
with m = 4.

Finally, if D(l,b) = <a,b,cd>, then D(l,ab) = <a,b,d>,
so the results in this case will clearly be equivalent to
the results obtained when D(l,b) = <a,b,d>, by consideration
of the homomorphism on Q(K) determined by -1 ↦ -1, a ↦ a,
b ↦ ab, c ↦ c, d ↦ d. □

Proposition 2.9. Let K be a nonreal field with q(K) = 16,
R(K) = K², u(K) = 4, s(K) = 2, m(K) > 2, and |D(l,x)| ≥ 4
for all x ∈ K. Then m(K) = 4 and K has the following binary
form value set structure, where \( Q(K) = \langle -l, a, b, c \rangle = D(1, -l) \):

\[
D(1,1) = \langle -l, a, b \rangle \quad D(1,c) = \langle a, c \rangle \quad D(1,-ac) = \langle a, b, -c \rangle
\]

\[
D(1,a) = \langle -l, a \rangle \quad D(1,-c) = \langle a, -c \rangle \quad D(1,bc) = \langle ac, bc \rangle
\]

\[
D(1,-a) = \langle -l, a, c \rangle \quad D(1,ab) = \langle -l, ab \rangle \quad D(1,-bc) = \langle -ac, -bc \rangle
\]

\[
D(1,b) = \langle -l, b \rangle \quad D(1,-ab) = \langle -l, ab, ac \rangle \quad D(1,abc) = \langle ac, b \rangle
\]

\[
D(1,-b) = \langle -l, b, ac \rangle \quad D(1,ac) = \langle a, b, c \rangle \quad D(1,-abc) = \langle -ac, b \rangle.
\]

**Proof.** First suppose that \( |D(1,x)| = 4 \) for all \( x \not\in -K^2 \).

Then if we assume \( D(1,1) = \langle -l, a \rangle \), \( D(1,a) = \langle -l, a \rangle = D(1,-a) \)
is immediate. Suppose \( Q(K) = \langle -l, a, b, c \rangle \). Then \( D(1,b) \cap D(1,1) = \{1\} \) means we can assume \( D(1,b) = \langle b, c \rangle \) from which

\( D(1,-c) = \langle -b, -c \rangle \) and \( D(1,-bc) = \langle -b, c \rangle \) are clear. \( D(1,-b) = \langle -b, x \rangle \) where \( x \in \langle -l, a, c \rangle \). But \( x \not\in -K^2, \pm ak^2 \), or \( ck^2 \), otherwise \( b \in D(1,-x) \) for those \( x \). If \( x \in -ck^2 \), then \( bc \in D(1,-b) \) and so \( b \in D(1,-bc) \), a contradiction. This leaves \( \pm ac \) as the only choices. Choose \( ac \). It makes no difference since if it were \(-ac\), the homomorphism on \( Q(K) \) determined by

\[
-1 \mapsto -1, \quad a \mapsto -a, \quad b \mapsto b, \quad c \mapsto c
\]

would not alter what has been done and would make the two situations equivalent with respect to quadratic forms. So \( D(1,-b) = \langle -b, ac \rangle \), and the remaining binary form value sets are then easily determined, giving the following list.

\[
D(1,1) = \langle -l, a \rangle \quad D(1,c) = \langle c, ab \rangle \quad D(1,-ac) = \langle b, -ac \rangle
\]

\[
D(1,a) = \langle -l, a \rangle \quad D(1,-c) = \langle -b, -c \rangle \quad D(1,bc) = \langle -ac, bc \rangle
\]

\[
D(1,-a) = \langle -l, a, c \rangle \quad D(1,ab) = \langle ab, -ac \rangle \quad D(1,-bc) = \langle -b, c \rangle
\]

\[
D(1,b) = \langle b, c \rangle \quad D(1,-ab) = \langle -c, -ab \rangle \quad D(1,abc) = \langle b, ac \rangle
\]

\[
D(1,-b) = \langle -b, ac \rangle \quad D(1,ac) = \langle -ab, ac \rangle \quad D(1,-abc) = \langle -c, ab \rangle.
\]

However, this cannot determine a quadratic form structure.
since \( ac \in D(1,1,b) = \bigcup_{a \in D(1,1)} D(a,b) \), but \( ac \notin \bigcup_{a \in D(1,1)} D(1,a) \).

We conclude that this type of field cannot exist.

Next, assume there is an \( x \) such that \( |D(1,x)| = 8 \).

Propositions 2.4, 2.5, 2.6, 2.7 and the corollary following Proposition 2.6 now apply to our situation.

Suppose \( |D(1,1)| = 4 \) and \( D(1,1) = <-l,a> \). It is easy to see that if \( |D(1,x)| = 8 \), then either \( |D(1,a)| \) or \( |D(1,-a)| \) must be 8. Suppose \( D(1,a) = <-l,a,b> \). By Proposition 2.7 there is a \( c \notin <-l,a,b> \) such that \( |D(1,c)| = 4 \). Clearly then we may suppose \( x \in cK^2 \) for definiteness. There are only the two non-equivalent situations \( D(1,c) = <a,c> \) and \( D(1,c) = <b,c> \).

Suppose \( D(1,c) = <a,c> \). Then \( D(1,-a) = <-l,a,c> \), and \( <-a,b> \subset D(1,b) \). Thus \( D(1,b) = <-a,b,c> \) or \( D(1,b) = <-a,b> \).

If \( D(1,b) = <-a,b,c> \), the remaining binary form value sets are readily determined and the structure is given by the following list.

\[
\begin{align*}
D(1,1) &= <-l,a> & D(1,c) &= <a,c> & D(1,-ac) &= <a,-c> \\
D(1,a) &= <-l,a,b> & D(1,-c) &= <a,-b,-c> & D(1,bc) &= <-ab,bc> \\
D(1,-a) &= <-l,a,c> & D(1,ab) &= <-a,-b,c> & D(1,-bc) &= <-b,c> \\
D(1,b) &= <-a,b,c> & D(1,-ab) &= <-a,b> & D(1,abc) &= <-b,-ac> \\
D(1,-b) &= <-a,-b> & D(1,ac) &= <a,-b,c> & D(1,-abc) &= <c,-ab>
\end{align*}
\]

This cannot, however, determine a quadratic form structure since \( b \notin D(1,1,-ac) = \bigcup_{a \in D(1,1)} D(a,-ac) \), but \( b \notin \bigcup_{a \in D(1,1)} D(1,a) \).

If \( D(1,b) = <-a,b> \), then \( D(1,-b) \supset <a,-b> \). If \( D(1,-b) = <a,-b> \), \( D(1,-b) \cap D(1,b) \neq \{1\} \) would contradict the corollary following Proposition 2.6. So \( D(1,-b) = <a,-b,c> \) is forced,
and then $b \mapsto -b$ (other basis elements go to themselves) gives an equivalent binary form value set structure to the above, which does not exist.

Now suppose $D(1,c) = \langle b, c \rangle$. Proceeding as above yields a structure clearly equivalent to the last list by the homomorphism $b \mapsto -b, c \mapsto -bc$. Again we recall that this structure cannot exist.

The remaining case is $|D(1,1)| = 8$. Suppose $D(1,1) = \langle -1, a, b \rangle$. By Proposition 2.7, applied to $H = \langle -1, a, b \rangle$, we may assume $D(1,a) = \langle -1, a \rangle$. Now $D(1,-a) \not\subseteq \langle -1, a \rangle$ for otherwise $D(1,-a) \subseteq D(1,1)$ contradicts the corollary after Proposition 2.6. For the same reason $D(1,-a) \not\subseteq \langle -1, a, b \rangle$. So $D(1,-a) = \langle -1, a, c \rangle$. Next consider $D(1,c) \supseteq \langle a, c \rangle$. If $D(1,c) \not\supseteq \langle a, c \rangle$, then $D(1,c) = \langle a, b, c \rangle$ or $\langle a, -b, c \rangle$. We will consider each of these three cases, beginning with $D(1,c) = \langle a, c \rangle$.

If $D(1,c) = \langle a, c \rangle$, then $D(1,-ac) \supseteq \langle a, -c \rangle$. But $|D(1,-ac)| = 4$ would lead to $D(1,c) \cap D(1,-ac) \neq \{1\}$, which contradicts the corollary after Proposition 2.6. The only other non-equivalent possibility is $D(1,-ac) = \langle a, b, -c \rangle$.

The remaining forms are easily analyzed to give this list.

- $D(1,1) = \langle -1, a, b \rangle$
- $D(1,c) = \langle a, c \rangle$
- $D(1,-ac) = \langle a, b, -c \rangle$
- $D(1,a) = \langle -1, a \rangle$
- $D(1,-c) = \langle a, -c \rangle$
- $D(1,bc) = \langle ac, bc \rangle$
- $D(1,-a) = \langle -1, a, c \rangle$
- $D(1,ab) = \langle -1, ab \rangle$
- $D(1,-bc) = \langle -ac, -bc \rangle$
- $D(1,b) = \langle -1, b \rangle$
- $D(1,-ab) = \langle -1, ab, ac \rangle$
- $D(1,abc) = \langle ac, b \rangle$
- $D(1,-b) = \langle -1, b, ac \rangle$
- $D(1,ac) = \langle a, b, c \rangle$
- $D(1,-abc) = \langle -ac, b \rangle$

If $D(1,c) = \langle a, b, c \rangle$, then $D(1,-b) = \langle -1, b, c \rangle$, $D(1,-c) = \langle a, b, -c \rangle$, and $D(1,ac) \supseteq \langle a, c \rangle$. The only possibility for
D(l,ac) other than \( <a,c> \) is \( <a,c,-b> \). The latter is out, however, since \(-bc \notin D(l,a)\) but \(-bc \in D(l,ac) \cap D(l,-c)\). So \( D(l,ac) = <a,c>\) and \( c \mapsto ac \) yields an equivalence to the last list.

If \( D(l,c) = <a,-b,c>\), proceeding as in the last paragraph yields a structure equivalent to the last list by \( b \mapsto -b \). It is easy to see that the last list gives the binary form value set structure for a field with \( m = 4 \).

**Proposition 2.10.** There are no fields \( K \) with \( q(K) = 16 \), \( R(K) = \mathbb{K}^2 \), \( u(K) = \frac{1}{4} = s(K) \), \( m(K) > 2 \), \( |D(l,x)| \geq \frac{1}{4} \) for all \( x \in K \), and \( |D(l,1)| = \frac{1}{4} \), where there exists \( y \in D(l,1) - \{1\} \) such that \( |D(l,y)| = \frac{1}{4} \).

**Proof.** Suppose \( D(l,1) = <a,b> \) where \( Q(K) = \{-1,a,b,c\} \), and let \( y \in a\mathbb{K}^2 \) for definiteness, so that \( |D(l,a)| = \frac{1}{4} \). Suppose \( |D(l,x)| = \frac{1}{4} \) for all \( x \in K \). This means \( D(l,-a) = <-1,a> \), \( D(l,-b) = <-1,b> \), and \( D(l,-ab) = <-1,ab> \), and these force \( D(l,a) = <a,w> \), where \( w \notin D(l,1) \). So for definiteness choose \( w = c \), so that \( D(l,a) = <a,c> \). Now \( D(l,b) = <b,v> \) where \( v \in <-1,a,c> \). If \(-1, +a, c, ac \in D(l,b)\), we contradict the previous structure. So \( v = -c \) or \( ac \), and we choose \( v = -c \) for definiteness. This means \( D(l,b) = <b,-c> \). Finally, \( D(l,ab) = <ab,u> \) where \( u \in <-1,a,c> \). Clearly, if \(-1, +a, +c, ac \in D(l,ab)\), the previous structure is contradicted. And if \(-ac \in D(l,ab)\), we have \(-bc \equiv (ab)(-ac) \in D(l,ab) \cap D(l,b) \subseteq D(l,-a)\), a contradiction. So there is no choice for \( u \), and we have contradicted \( |D(l,x)| = \frac{1}{4} \) for all \( x \in K \).
Thus we assume there is an $x$ such that $|D(1,x)| = 8$. Propositions 2.4, 2.5, 2.6, 2.7, and the corollary following Proposition 2.6 now apply to our situation. Recall that we have $D(1,1) = \langle a, b \rangle$ where $Q(K) = \langle -1, a, b, c \rangle$, and $|D(1,a)| = \frac{1}{2}$. If $|D(1,-a)| = \frac{1}{2}$, then we have a contradiction of the corollary following Proposition 2.6. So $|D(1,-a)| = 8$, and thus $D(1,-a) = \langle -1, a, w \rangle$ where $w \in \langle b, c \rangle$.

If $D(1,-a) = \langle -1, a, b \rangle$, then $D(1,1) \subseteq D(1,-a)$, which contradicts the corollary following Proposition 2.6. So $D(1,-a) = \langle -1, a, c \rangle$ or $\langle -1, a, bc \rangle$. These choices give similar results, so suppose for definiteness that $D(1,-a) = \langle -1, a, c \rangle$. By Proposition 2.7, there is some $x \in \langle a, b, c \rangle$ such that $|D(1,-x)| = \frac{1}{2}$, and we know $x \not\in -a^2$, $x \not\in -a^2$. If $x \in -b^2$, then $D(1,-b) = \langle -1, b \rangle$. If $|D(1,c)| = 8 = |D(1,-c)|$, then $D(1,c) = \langle a, c, -b \rangle$ and $D(1,-c) = \langle a, -c, -b \rangle$, since $b \not\in D(1,\pm c)$. But this means $-b \in D(1,c) \cap D(1,-c) \subseteq D(1,1)$, a contradiction. So one of $|D(1,c)|$, $|D(1,-c)|$ must be $\frac{1}{2}$, and we may assume for definiteness that $|D(1,c)| = \frac{1}{2}$, which means $D(1,c) = \langle a, c \rangle$. If $|D(1,ac)| = 8 = |D(1,-ac)|$, then $-b \in D(1,ac) \cap D(1,-ac) \subseteq D(1,1)$, a contradiction. If $|D(1,ac)| = \frac{1}{2}$, then $D(1,ac) = \langle a, c \rangle \subseteq D(1,-a)$, which contradicts the corollary again. If $|D(1,-ac)| = \frac{1}{2}$, then $D(1,-ac) = \langle a, -c \rangle$, so $D(1,c) \cap D(1,-ac) = \{a\}$, a contradiction of the corollary.

Since similar results are obtained with the choice of $|D(1,-c)| = \frac{1}{2}$, we now have shown $x \not\in -ab^2$ and thus $|D(1,-b)| = 8$. Similarly we can show $x \not\in -ab^2$ and so $|D(1,-ab)| = 8$. If $x \in -bc^2$, suppose first that $|D(1,c)| = 8 = |D(1,-c)|$. 

Then $D(l,c) = \langle a, c, u \rangle$ and $D(l,-c) = \langle a, -c, v \rangle$ where $u, v \in \langle -l, b \rangle - \{\pm 1\}$. If $D(l,c) = \langle a, c, b \rangle$, then $-b \notin D(l,-c)$ for if so $bc \in D(l,c) \cap D(l,-c) \subseteq D(l,l)$, a contradiction. So $D(l,-c) = \langle a, -c, b \rangle$, which means $c \in D(l,-bc)$. But $bc \in D(l,c)$ implies $c \in D(l,bc)$. Thus $c \in D(l,bc) \cap D(l,-bc) \subseteq D(l,l)$, a contradiction. If $D(l,c) = \langle a, c, -b \rangle$ a similar contradiction is reached. So one of $|D(l,c)|$, $|D(l,-c)|$ must be 4, and we suppose here that $|D(l,c)| = \frac{4}{4}$, which means $D(l,c) = \langle a, c \rangle$. If $|D(l,ac)| = 8 = |D(l,-ac)|$ we reach a contradiction as before in this proof in similar situations, and if either of $|D(l,-ac)|$, $|D(l,ac)|$ is 4 we contradict the corollary as before. If instead we suppose $|D(l,-c)| = \frac{4}{4}$ we get the same result. So $x \notin -bc^2$ and thus $|D(l,-bc)| = 8$. Similarly, $x \notin -abc^2$ and so $|D(l,-abc)| = 8$. If $x \in -ck^2$, then $D(l,-c) = \langle a, -c \rangle$, and since $D(l,-c) \subseteq D(l,-a)$ we must have $|D(l,-ac)| = 8$. So $D(l,-ac) = \langle a, -c, w \rangle$ where $w \in \langle -l, b \rangle - \{\pm 1\}$. Suppose $|D(l,ac)| = \frac{4}{4}$. Then $D(l,ac) = \langle a, c \rangle = D(l,-a)$, which means $|D(l,c)| = 8$. So $D(l,c) = \langle a, c, v \rangle$ where $v \in \langle -l, b \rangle - \{\pm 1\}$. If $b \in D(l,c)$, then $b \in D(l,1) \cap D(l,c) \subseteq D(l,-c)$, a contradiction. This forces $D(l,c) = \langle a, c, -b \rangle$, and so $D(l,-ac) = \langle a, -c, -b \rangle$, since $b \in D(l,-ac)$ would imply $b \in D(l,1) \cap D(l,-ac) \subseteq D(l,ac)$, a contradiction. Thus $D(l,ab) = \langle ab, -c, ac \rangle = \langle -a, -b, -c \rangle$ and $D(l,b) = \langle b, -c, ac \rangle = \langle -a, b, -c \rangle$. Now, as shown previously, $|D(l,-b)| = 8$, so $D(l,-b) = \langle -l, b, w \rangle$ where $w \in \langle a, c \rangle - \{1\}$. Now $b \notin D(l,-c)$ for all $a \in \langle a, c \rangle - \{1\}$, so $a, c, ac \notin D(l,-b)$. So we have reached a contradiction, and we have $|D(l,ac)| = 8$, which
means $D(1,ac) = \langle a, c, u \rangle$ where $u \in \langle -1, b \rangle - \{+1\}$. If $-b \in D(1,ac)$, then $-b \in D(1,-ac)$ also for otherwise, $bc \in D(1,ac) \cap D(1,-ac) \subseteq D(1,1)$, a contradiction. But $-b \in D(1,ac) \cap D(1,-ac) \subseteq D(1,1)$ gives a contradiction also, so we must have instead $D(1,ac) = \langle a, b, c \rangle$ and $D(1,-ac) = \langle a, b, -c \rangle$. If $|D(1,c)| = \frac{1}{4}$, then $D(1,c) = \langle a, c \rangle$, so $D(1,-abc) \supsetneq \langle b, -ac \rangle$. Since it has been shown that $|D(1,-abc)| = 8$, we must have $D(1,-abc) = \langle b, -ac, w \rangle$ where $w \in \langle -1, c \rangle - \{+1\}$. But if $+c \in D(1,-abc)$, we have $abc \in D(1,+c)$, a contradiction. So $|D(1,c)| = 8$, and this forces $D(1,c) = \langle a, -b, c \rangle$, since if $b \in D(1,c)$ we have $b \in D(1,c) \cap D(1,ac) \subseteq D(1,-a)$, a contradiction. Similar arguments as before yield a contradiction, so we know now that $x \not\in -cK^2$. Similarly we can show $x \not\in -acK^2$. Summarizing, we have shown that for all $x \in \langle a, b, c \rangle$, $|D(1,-x)| = 8$, which is a contradiction to Proposition 2.7. So no such field exists.

Proposition 2.11. There are no fields $K$ with $q(K) = 16$, $R(K) = \hat{K}^2$, $u(K) = \frac{1}{4} = s(K)$, $m(K) > 2$, $|D(1,x)| \geq \frac{1}{4}$ for all $x \in \hat{K}$, and $|D(1,1)| = \frac{1}{4}$, where there exists $y \in -D(1,1)$ such that $|D(1,y)| = \frac{1}{4}$.

Proof. Suppose $D(1,1) = \langle a, b \rangle$ where $Q(K) = \langle -1, a, b, c \rangle$, and let $y \in -a\hat{K}^2$ for definiteness, so that $D(1,-a) = \langle -1, a \rangle$. In the last proposition we showed that there exists $x \in \hat{K}$ such that $|D(1,x)| = 8$, and that proof is applicable here also. Propositions 2.4, 2.5, 2.6, 2.7, and the corollary following Proposition 2.6 now apply here. If $|D(1,-b)| = \frac{1}{4} = |D(1,-ab)|$. 
then the corollary forces $|D(1,a)| = |D(1,b)| = |D(1,ab)| = 8$. Since $D(1,c) \neq <c>$, we must have $-c \in D(1,a), D(1,b),$ or $D(1,ab)$. Suppose for definiteness that $-c \in D(1,a)$, so that $D(1,a) = <a,-c,x>$ for $x \in <-l,b> - \{+1\}$. Clearly $b \notin D(1,a)$, so $D(1,a) = <a,-b,-c>$. Thus we have $D(1,b) = <-a,b,u>$ and $D(1,ab) = <-a,-b,v>$, where $u,v \in <-l,c> - \{+1\}$. So we see that at least two of $D(1,a), D(1,b),$ and $D(1,ab)$ contain either $c$ or $-c$, which gives a contradiction (e.g., if $c \in D(1,a)$ and $c \notin D(1,b)$, then $c \notin D(1,-ab)$, which contradicts $D(1,-ab) = <-l,ab>$).

So at least one of $|D(1,-b)|, |D(1,-ab)|$ must be 8. Since these choices yield equivalent results, suppose $|D(1,-b)| = 8$. We know $|D(1,a)| = 8$ by the corollary as in the last paragraph. So here we have $|D(1,-b)| = <-l,b,x>$, where $x \in <a,c> - \{1\}$. If $D(1,-b) = <-l,b,a>$, then $D(1,a) = <a,b,w>$ where $w \in <-l,c> - \{+1\}$. So we have $|D(1,-a)| = 4$, $|D(1,a)| = 8$, and $D(1,1) \subseteq D(1,a)$, which contradicts the corollary. If $D(1,-b) = <-l,b,c>$, then $D(1,a) = <a,u,v>$ where $u,v \in \{+b,+c\}$. We have just shown $b \notin D(1,a)$, so we may let $-b \in D(1,a)$. Then exactly one of $c, -c$ is in $D(1,a)$. These choices yield the same result so suppose $c \in D(1,a)$, giving $D(1,a) = <a,-b,c>$. Thus $-a \in D(1,b), D(1,ab)$, and this gives $c \in D(1,a) \cap D(1,-b) \subseteq D(1,ab)$. So $D(1,ab) = <-a,-b,c>$. If $|D(1,b)| = 4$, then $D(1,-a) \cap D(1,ab) = \{1\}$, a contradiction. If $|D(1,b)| = 8$, then $D(1,b) = <-a,b,u>$ where $u \in <-l,c> - \{+1\}$. But if $+c \in D(1,b)$, we have $+c \in D(1,b) \cap D(1,-b) \subseteq D(1,1)$, a contradiction. So the u
cannot be chosen, which means \( D(l, -b) \neq \langle -l, b, c \rangle \). If \( D(l, -b) = \langle -l, b, ac \rangle \), we proceed as above to reach a contradiction on \( |D(l, b)| \) again. So \( D(l, -b) \neq \langle -l, b, ac \rangle \). Thus the third generator \( x \) of \( D(l, -b) \) cannot be chosen, which gives us our final contradiction. Therefore, no such field can exist. □

**Proposition 2.12.** There are no fields \( K \) with \( q(K) = 16 \), \( R(K) = K^2 \), \( u(K) = 4 = s(K) \), \( m(K) > 2 \), \( |D(l, x)| \geq 4 \) for all \( x \in K \), and \( |D(l, 1)| = 4 \), where there exists \( y \in K - [\pm D(l, 1)] \) such that \( |D(l, y)| = 4 \).

**Proof.** Suppose \( D(l, 1) = \langle a, b \rangle \) where \( q(K) = \langle -1, a, b, c \rangle \), and let \( y \in cK^2 \) for definiteness, so that \( D(l, c) = \langle c, x \rangle \) for some \( x \in K \). If \( x \in D(l, 1) \), let \( x \in aK^2 \) for definiteness, so that \( D(l, c) = \langle c, a \rangle \) and thus \( D(l, -a) = \langle -l, a, c \rangle \). If \( |D(l, a)| = 4 \) we have the same situation as in a case in Proposition 2.10, so we may assume \( |D(l, a)| = 8 \). Thus \( D(l, a) = \langle a, u, v \rangle \), where \( u, v \in \langle -l, b, c \rangle - \{\pm 1\} \). But \( \pm c \notin D(l, a) \) for if so, \( \pm c \in D(l, 1) \), a contradiction. So \( u, v \in \{b, bc\} \), and only one from each of the pairs \( \pm b, \pm bc \) may be in \( D(l, a) \). So there is no way to pick \( u \) and \( v \), which means \( x \notin D(l, 1) \).

If \( x \in D(l, 1) \), let \( x \in -aK^2 \) for definiteness, so that \( D(l, c) = \langle c, -a \rangle \), and thus \( D(l, a) \supseteq \langle a, -c \rangle \). Suppose that \( D(l, a) = \langle a, -c \rangle \). If \( |D(l, -a)| = 8 \), then we have \( D(l, -a) = \langle -l, a, w \rangle \), where \( w \in \langle b, c \rangle - \{1\} \). In each case we reach a contradiction of the corollary, so \( |D(l, -a)| = 4 \) and thus \( D(l, -a) = \langle -l, a \rangle \). This, however, means that \( a \in D(l, a) \cap D(l, -a) \) with \( |D(l, a)| = |D(l, -a)| = |D(l, 1)| = 4 \), which also
contradicts the corollary. Hence, \( D(1,a) \neq \langle a,-c \rangle \), and 
\(|D(1,a)| = 8\). Suppose \(|D(1,-a)| = 8\). If \( D(1,a) = \langle a,-c,-b \rangle \), then \(+b, +c / D(1,-a)\), a contradiction, so \( D(1,a) = \langle a,-c,b \rangle \) and \( D(1,-a) = \langle -l,a,b \rangle \). This means \( D(1,-b) = \langle -l,a,b \rangle = D(1,-ab) \) also. Consider \( D(1,b) \supseteq \langle a,b \rangle \). If \( D(1,b) = \langle a,b \rangle \), then \( D(1,-b) = \langle l,a,b \rangle \) where \(|D(1,-b)| = 8\) and \(|D(1,1)| = 4\), which contradicts the corollary. So \( D(1,b) = \langle a,b,w \rangle \) where \( w \in \langle -l,c \rangle - \{-1\} \). If \(-c \in D(1,b)\), then \(-c \in D(1,a) \cap D(1,b) \subseteq D(1,-ab)\), a contradiction. So \( D(1,b) = \langle a,b,c \rangle \).
Similarly, \( D(1,ab) = \langle a,b,x \rangle \) with \( x \in \langle -l,c \rangle - \{-1\} \), and \(+c / D(1,ab)\). This means \( x \) cannot be chosen, so we must have \(|D(1,-a)| = 4\), thus \( D(1,-a) = \langle -l,a \rangle \). Now \( D(1,a) = \langle a,-c,u \rangle \) where \( u \in \langle -l,b \rangle - \{-1\} \), and as before, \( b \notin D(1,a) \), so \( D(1,a) = \langle a,-c,-b \rangle \). This means \( D(1,c) \supseteq \langle -a,c \rangle \). Consider \( D(1,-c)\), which has at least one generator from \( \langle -l,a,b \rangle - \{+1\} \). Clearly \(+a / D(1,-c)\), and if \( b \) and one of \(+ab\) were in \( D(1,-c)\) then one of \(+a \in D(1,-c)\), a contradiction. Similarly \(-b\) and one of \(+ab\) are not in \( D(1,-c)\). So \(|D(1,-c)| = 4\) and thus \( D(1,-c) = \langle -c,w \rangle \) where \( w \in \{+b,+ab\} \). Suppose \( D(1,-c) = \langle -c,b \rangle \). Then \( D(1,w) = \langle -a,w \rangle \) and \( D(1,ab) = \langle -a,-w \rangle \), so \( -a \in D(1,w) \cap D(1,aw) \) and \(|D(1,-a)| = 4\), which contradicts the corollary. So \( x \notin D(1,1) \).

If \( x \in \hat{K} - [+D(1,1)] \), then let \( x \in wcK^2 \) where \( w \in +D(1,1) \), so that \( D(1,c) = \langle c,wc \rangle \). This means \( w \in D(1,c) \), so we are reduced to one of the two previous cases.
Proposition 2.13. There are no fields $K$ with $q(K) = 16$, $R(K) = \hat{K}^2$, $u(K) = 4 = s(K)$, $m(K) > 2$, $|D(l,x)| \geq 4$ for all $x \in \hat{K}$, and $|D(l,1)| = 8$.

Proof. Suppose $D(l,1) = \langle a, b, c \rangle$ where $Q(K) = \langle -l, a, b, c \rangle$.

By Proposition 2.7 there is an $x \in \langle a, b, c \rangle$ such that $|D(l,-x)| = 4$. Clearly $x \notin \hat{K}^2$, so suppose for definiteness that $x \in a\hat{K}^2$, i.e., $D(l,-a) = \langle -l, a \rangle$. If $|D(l,a)| = 8$, then $|D(l,1) \cap D(l,a)| \geq 4$, and we may for definiteness assume $b \in D(l,a)$. This means $D(l,-b) = \langle -l, a, b \rangle$, which contradicts $D(l,-a) = \langle -l, a \rangle$. Therefore, $|D(l,a)| = 4$. As above, if $x \in D(l,a) - \{1,a\}$, then $x \notin \langle a, b, c \rangle$. So we assume $D(l,a) = \langle a, -b \rangle$. Suppose $|D(l,-b)| = 8$. Then there is an $x \in \langle a, c \rangle \cap D(l,-b)$. Since $a \notin D(l,-b)$, either $c$ or $ac$ is in $D(l,-b)$, thus $D(l,-b) = \langle -l, b, c \rangle$ or $\langle -l, b, ac \rangle$. Since these choices yield equivalent results we will let $D(l,-b) = \langle -l, b, c \rangle$. This forces $D(l,-c) = \langle -l, b, c \rangle = D(l,-bc)$ and $D(l,b) = \langle -a, b, c \rangle$. Consider $D(l,-ac)$, and suppose there is an $x \in \langle a, b \rangle \cap D(l,-ac)$. Clearly $a, b \notin D(l,-ac)$, and if $ab \in D(l,-ac)$, we have $bc = (ab)(ac) \in D(l,-ac)$, which means $ac \in D(l,-bc)$, a contradiction. So $D(l,-ac) = \langle -l, ac \rangle$, and similarly we have $D(l,-abc) = \langle -l, abc \rangle$ and $D(l,-ab) = \langle -l, ab \rangle$. Consider $D(l,ab)$, and suppose there is an $x \in \langle -l, c \rangle \cap D(l,ab)$. Clearly $-l, c \notin D(l,ab)$, and if $-c \in D(l,ab)$ then $-ab \in D(l,c)$. So $-a = (-ab)(b) \in D(l,c)$, which implies $-c \in D(l,a)$, a contradiction. So $D(l,ab) = \langle -a, -b \rangle$, and similarly we have $D(l,ac) = \langle -b, ac \rangle$, $D(l,abc) = \langle -b, -ac \rangle$, $D(l,c) = \langle b, c \rangle$, and $D(l,bc) = \langle b, c \rangle$. But this
cannot determine a quadratic form structure since \(-ab \in D(1,-a, c) = \bigcup_{\alpha \in D(1,c)} D(\alpha, c)\), but \(-ab \not\in \bigcup_{\alpha \in D(1,c)} D(\alpha, -a)\). We conclude that this type field cannot exist.

Now suppose \(|D(1,-b)| = \frac{1}{4}\), which means \(D(1,-b) = \langle -1, b \rangle\). Consider \(D(1,-ab)\). Since \(|D(1,1)| = 8\) and \(|D(1,-a)| = \frac{1}{4} = |D(1,-b)|\), \(|D(1,-ab)| = \frac{1}{4}\) would imply \(-1 \in D(1,-b) \cap D(1,-ab)\) which contradicts the corollary. So \(|D(1,-ab)| = 8\), and thus there is an \(x \in \langle a, c \rangle - \{1\} \cap D(1,-ab)\). Since \(a \not\in D(1,-ab)\), we must have \(c\) or \(ac \in D(1,-ab)\), which means \(D(1,-ab) = \langle -1, ab, c \rangle\) or \(\langle -1, ab, ac \rangle\). In either case we continue as in the last paragraph to get an equivalent form scheme to the one listed there, but this type structure is not consistent.

We summarize the results for the \(u = \frac{1}{4}\) case in the following theorem.

**Theorem 2.14.** Let \(K\) be a nonreal field with \(q(K) = 16\) and \(u(K) = \frac{1}{4}\). Then \(K\) must be equivalent to one of the following types of fields.

1. \(|R| = \frac{1}{4}\), \(m = 2\), \(s = 1\) or \(s = 2\) (with \(-1 \in R\) or \(-1 \not\in R\))
2. \(|R| = 2\), \(m = \frac{1}{4}\), \(s = 1\) or \(s = 2\) (with \(-1 \in R\) or \(-1 \not\in R\))
3. \(|R| = 2\), \(m = 2\), \(s = 4\)
4. \(|R| = 1\), \(m = 2\), \(s = 1\) or \(2\)
5. \(|R| = 1\), \(m = 8\), \(s = 1\) or \(2\)
6. \(|R| = 1\), \(m = \frac{1}{4}\), \(s = 1\) or \(2\)

**Proof.** If \(R(K) \not\subseteq K^2\), then \(|R| = 2, \frac{1}{4}, 8, \) or \(16\). By Kaplansky [8], \(R(K)\) can never have index two in \(K\) for nonreal \(K\), so
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\(|R| \neq 8\). By Theorem 2 of [3] then, \(K\) must be of type (1), (2), or (3) above.

If \(R(K) = K^2\) the results of this chapter will be applied. If there exists \(x \in K\) such that \(D(1, x) = \langle x \rangle\), then by Proposition 2.3 we have that \(K\) must be of type (5) above. So suppose \(|D(1, x)| \geq \frac{4}{2}\) for all \(x \in K\). If \(m(K) = 2\) then \(|D(1, x)| = 8\) for all \(x \in K\) and \(K\) must be of type (4) above. So let \(m(K) > 2\). Suppose \(s(K) = 4\) and \(|D(1,1)| = 4\), where \(Q(K) = \langle -1, d, e, f \rangle\). By Proposition 2.7 there is a \(y \in -\langle d, e, f \rangle\) such that \(|D(1, y)| = 4\). So \(y \not\in K^2\), and \(K = +D(1,1) \cup \{K - [+D(1,1)]\}\). Thus \(y \in D(1,1) - \{1\}, -D(1,1),\) or \(K - \{+D(1,1)\}\), and Propositions 2.10, 2.11, and 2.12 respectively tell us no such field can exist. Proposition 2.13 negates the existence of a field \(K\) with \(s(K) = \frac{4}{2}\) and \(|D(1,1)| = 8\). So \(s(K) = 1\) or 2. Propositions 2.8 and 2.9 then tell us \(K\) must be of type (6) above.

Techniques very similar to those used in the \(u = 4\) case show that \(u \neq 6\) when \(q = 16\). Because of this similarity, and also since B. Blaszczyk has recently shown that \(u\) is a 2-power when \(q \leq 32\), we omit these arguments here.

Theorem 2.15. Let \(K\) be a non-formally real field with \(q(K) = 16\). Then \(K\) must be equivalent to one of the following 26 types of fields.

I. \(|R| \neq 1\)

a. \(|R| = 2:\)

1. \(u = 4\), \(m = 4\), \(s = 1\) or \(s = 2\)
   (with \(-1 \in R\) or \(-1 \not\in R\))
2. \( u = \frac{1}{4}, m = 2, s = 4 \)
3. \( u = 8, s = 1 \) or \( s = 2 \) (with \(-1 \in R\) or \(-1 \not\in R\))
   a. \(|R| = \frac{1}{4}, u = \frac{1}{4}, m = 2, s = 1\) or \( s = 2 \)
   b. \(|R| = \frac{1}{4}, u = \frac{1}{4}, m = 2, s = 1\) or \( s = 2 \)
   c. \(|R| = 16, u = 2, s = 1\) or \( s = 2 \)

II. \( |R| = 1 \)
   a. \( u = \frac{1}{4}, m = 2, s = 1\) or \( s = 2 \)
   b. \( u = \frac{1}{4}, m = 8, s = 1\) or \( s = 2 \)
   c. \( u = \frac{1}{4}, m = 4, s = 1\) or \( s = 2 \)
   d. \( u = 8, m = 32, s = 1\) or \( s = 2 \)
   e. \( u = 8, m = 16, s = 1 \)
   f. \( u = 8, m = 16, s = 2, |D(1,1)| = \frac{1}{4}\) or \(8 \)
   g. \( u = 8, m = 16, s = \frac{1}{4} \)
   h. \( u = 16, s = 1\) or \( s = 2 \)

**Proof.** If \( u = 2 \), then \( K \) is of type \( Ic \) by [1, Theorem 4.1]. If \( u = \frac{1}{4} \), then \( K \) is of type \( Ia1, 2, Ib, Ila,b, \) or \( c \) by Theorem 2.14. If \( u = 8 \) and \( s = 1\) or \( 2 \), then \( K \) is of type \( Ia3, IId,e, \) or \( f \) by Theorem 1.14. If \( u = 8 \) and \( s = \frac{1}{4} \) then \( K \) is of type \( IIg \) by Theorem 1.7. If \( u = 16 \), then \( K \) is of type \( IIh \) by [1, Theorem 5.13].

It should be pointed out that examples have been found illustrating only the fields listed in the above theorem under \( Ic, IIa,b,d,g,h \). All of the remaining ones except \( IIc \) appear to rest on the existence of a field with non-trivial radical.
CHAPTER III. QUADRATIC EXTENSIONS OF
FIELDS WITH TWO QUATERNION ALGEBRAS

This chapter is concerned with the behavior of quadratic forms under quadratic extensions of non-formally real fields with two quaternion algebras. Let $F$ be any non-formally real field with $q(F) < \infty$ and $\text{char}(F) \neq 2$. The extension fields $K = F(\sqrt{a}), a \in \hat{F} \setminus \hat{F}^2$, are studied. Let $q(F) = 2^n$, $n \geq 1$, and suppose $D_F(1,-a) = \langle a_1, \ldots, a_j \rangle$ and $Q(F) = \langle f_1, \ldots, f_{n-1}, a \rangle$, $j \leq n$. Also, for all $i = 1, \ldots, j$, let $a_i^{1/2} = a_i^{1/2} - a a_2, a_i^{1/2} \in D_F(1,-a)$ and $x_i = a_i^{1/2} + a_i^{1/2} \sqrt{a} \in K$. Gross and Fischer have shown that $q(K) = 2^{n-1+j}$ and that $Q(K) = \langle f_1, \ldots, f_{n-1}, x_1, \ldots, x_j \rangle$ (see [7, pp. 298-299]). Notice also that, for all $i = 1, \ldots, j$, $N_{K/F}(x_i) = a_i$. We begin with some general results, and then we consider the situation when $F$ has exactly two quaternion algebras.

One result that will be used is the fact that $R(F) \subseteq R(K)$, which is shown by the corollary to Proposition 3 in [2]. The norm $N_{K/F}$ will also play an important role, and we abbreviate $N_{K/F}$ to $N$. Elman and Lam's Norm Principle [6, Proposition 2.13] is very useful and is restated as Lemma 3.1 here for binary forms representing one. Recall that throughout this chapter $F$ and $K$ are related as stated above; i.e., $K = F(\sqrt{a})$ where $a \in \hat{F} \setminus \hat{F}^2$. Also, although many results are also true for arbitrary fields $F$, we will remain under the assumption that $F$ is not formally real with $q(F) < \infty$. Excepting theorems, these last assumptions will not be stated in the results of this chapter.
Lemma 3.1 (Norm Principle). Let $x \in \hat{K}$ and $c \in \hat{F}$. Then $N(x) \in D_F(1,-c)$ if and only if $x \in \hat{F} \cdot D_K(1,-c)$.

Lemma 3.2. Let $x \in \hat{K}$ and $c \in \hat{F}$. Then $c \in D_F(1,-N(x))$ if and only if there exists $d \in \hat{F}$ such that $c \in D_F(1,-dx)$.

Proof. $c \in D_F(1,-N(x)) \iff N(x) \in D_F(1,-c)$ (Lemma 3.1) 
\[ \iff x \in \hat{F} \cdot D_K(1,-c) \]
\[ \iff \text{there exists } d \in \hat{F} \text{ such that } dx \in D_K(1,-c) \]
\[ \iff \text{there exists } d \in \hat{F} \text{ such that } c \in D_K(1,-dx). \]

Lemma 3.3. If $x \in \hat{K}$ and $N(x) \in \hat{F}^2$, then there is a $c \in \hat{F}$ and a $y \in \hat{K}$ such that $x = cy^2$.

Proof. This is an immediate consequence of Theorem 2.11 of [6] with $\varphi = (1)$.

If $m(F) = 2$, then it is well-known that $D_F(1,-b)$ has index 1 or 2 for all $b \in \hat{F}$. From Theorems 1,2 in [8], it follows that if $D_F(1,-b)$ has index 1 or 2 for all $b \in \hat{F}$ (and is 2 for at least one $b$), then $m(F) = 2$. This is the criterion we will use in analyzing $m(K)$.

Proposition 3.4. If $m(F) = 2$, then $\hat{F} \cap R(K) = R(F) \cup aR(F)$.

Proof. $R(F) \cup aR(F) \subseteq \hat{F} \cap R(K)$ follows from the corollary to Proposition 3 in [2]. On the other hand, if $c \in \hat{F} \cap R(K)$, then $x \in D_K(1,-c)$ for all $x \in \hat{K}$. Hence the Norm Principle implies $N(x) \in D_F(1,-c)$ and so $D_F(1,-a) \subseteq D_F(1,-c)$. If
equality holds, then \( ac \in R(F) \) by Lemma 1 of [3], so \( c \in aR(F) \). If \( D_F(1,-a) \nsubseteq D_F(1,-c) \), \( |D_F(1,-a)| = q(F)/2 \) forces \( c \in R(F) \). Therefore \( c \in R(F) \cup aR(F) \).

**Corollary 3.5.** If \( m(F) = 2 \) and \( R(F) = \hat{F}^2 \), then \( R(K) = \hat{K}^2 \).

**Proof.** If \( x \in R(K) \), then the Norm Principle gives \( N(x) \notin R(F) = \hat{F}^2 \). Thus, by Lemma 3.3, \( x \in c\hat{K}^2 \) for some \( c \in \hat{F} \). But \( c \in F \cap R(K) = R(F) \cup aR(F) \subseteq \hat{K}^2 \).

The next three results are true for any field of characteristic different from 2. For any \( y = m + n/a \in \hat{K} \), consider the \( F \)-linear functional \( s_y \) on \( K \) determined by \( s_y(1) = n, s_y(\sqrt{a}) = -m \). Thus \( s_y : K \to F \), and if \((V,Q)\) is a quadratic space over \( K \), then the quadratic space \((V,s_yQ)\) is called the transfer of \( V \) and is denoted by \( s_y^*V \).

**Lemma 3.6.** Let \( \varphi \) be a quadratic form over \( K \), and \( x,y \in \hat{K} \). Then:

1. If \( x \in D_K(\varphi) \), then \( s_y(x) = 0 \) or \( s_y x \in D_F(s_y^*\varphi) \).
2. \( s_y(x) = 0 \Rightarrow y = fx \) for some \( f \in \hat{F} \).
3. \( s_y^*\varphi \) is \( F \)-isotropic \( \Rightarrow \) there exists \( f \in \hat{F} \) such that \( fy \in D_K(\varphi) \).
4. \( s_y^*(x) \) is \( F \)-isotropic \( \Rightarrow fy \in x\hat{K}^2 \) for some \( f \in \hat{F} \).

**Proof.** Let \((V,Q)\) be the quadratic space over \( K \) associated with the quadratic form \( \varphi \).

1. If \( x \in D_K(\varphi) \) \( \Rightarrow \) there exists \( v \in V \) such that \( Q(v) = x \). Therefore, \( x \in D_K(\varphi) \) implies there exists \( v \in V \) such that \( s_y(Q(v)) = s_y(x) \), which implies that \( s_y(x) = 0 \) or \( s_y(x) \in D_F(s_y^*(\varphi)) \).
(2) Since the range of $s_y$ is $F$ and $[K:F] = 2$, then, by a well-known result from linear algebra, $\dim \{ x | s_y(x) = 0 \} = 1$.

(3) $s_y \circ \varphi$ is $F$-isotropic $\Rightarrow$ there exists a non-zero $v$ in $V$ such that $s_y(Q(v)) = 0$ $\Rightarrow$ there exists $x \in \hat{K}$ such that $s_y(x) = 0$ and $x = Q(v)$ for some $v \in V$ (by (2)) $\Rightarrow$ there exists $x \in \hat{K}$ such that $x = fy$ for some $f \in \hat{F}$ and $x \in D_K(\varphi)$ $\Rightarrow$ there exists $f \in \hat{F}$ such that $fy \in D_K(\varphi)$.

(4) $s_y \ast (x)$ is $F$-isotropic $\Rightarrow$ there exists $f \in \hat{F}$ such that $fy \in D_K((x))$ $\Rightarrow$ there exists $f \in \hat{F}$ such that $fy \in xK^2$.

Lemma 3.7. Let $x, y \in \hat{K}$, with $y \notin \hat{F} \hat{K}^2 \cup x \hat{F} \hat{K}^2$. Then $s_y(l)$ and $s_y(-x)$ are non-zero and $s_y \ast (l,-x) = s_y(l)(1,-N(y)) \oplus s_y(-x)(1,-N(xy))$.

Proof. $s_y(l)$ and $s_y(-x)$ are non-zero by Lemma 3.6(1), (2). The transfer $s_y \ast (-x)$ has $F$-bilinear form $(u,v) \mapsto s(-xuv)$. With respect to the $F$-basis $\{1, \sqrt{a}\}$ on $K$, this bilinear form has the associated matrix $\begin{pmatrix} s_y(-x) & s_y(-\sqrt{ax}) \\ s_y(-\sqrt{ax}) & s_y(-ax) \end{pmatrix}$. If $-x = b + c/\sqrt{a}$, this matrix has determinant $\begin{vmatrix} bn-cm & acn-bm \\ acn-bm & abn-acm \end{vmatrix} = -(m^2-an^2)(b^2-ac^2) = -N(xy)$. By a determinant argument, $s_y \ast (-x) \simeq (s_y(-x), -s_y(-x) N(xy))$.
\[ \cong s_y (-x)(1,-N(xy)). \] In particular, \( s_y*(l) \cong s_y(l)(1,-N(y)). \)

Thus \( s_y*(l,-x) \cong s_y*(l) \otimes s_y*(-x) \)
\[ \cong s_y(l)(1,-N(y)) \otimes s_y(-x)(1,-N(xy)). \]

To facilitate notation and to avoid awkward subscripting, we henceforth let \( s_y(l) = \alpha_y \) and \( s_y(-x) = \alpha_{-x} \).

**Proposition 3.8.** Let \( x,y \in \hat{K} \) with \( y \not\in F \hat{K}^2 \cup x\hat{K}^2 \). Then \( \alpha_y(l,-N(y)) \otimes \alpha_{-x}(y)(1,-N(xy)) \) is F-isotropic if and only if there exists \( f \in \hat{F} \) such that \( fy \in D_K(l,-x) \).

**Proof.** It was shown in Lemma 3.7 that \( \alpha_y(l,-N(y)) \otimes \alpha_{-x}(y)(1,-N(xy)) \) is F-isotropic if and only if \( s_y*(l,-x) \) is F-isotropic. By Lemma 3.6(3), this is true if and only if there exists \( f \in \hat{F} \) such that \( fy \in D_K(l,-x) \).

We now analyze the structure when \( m(F) = 2 \) and \( a \not\in R(F) \). In this case we will be able to determine \( |D_K(l,-x)| \) for all \( x \in \hat{K} \), and show that \( m(K) = 2 \) and \( |R(K)/\hat{K}^2| = |R(F)/\hat{F}^2| \) as well.

**Proposition 3.9.** If \( m(F) = 2 \) and \( a \not\in R(F) \), then \( \hat{F} \subseteq D_K(l,-c) \) for all \( c \in \hat{F} \).

**Proof.** Suppose \( b,c \in \hat{F} \). Since \( a \not\in R(F) \), \([a,a]\) takes on both possible values as \( a \) runs through \( \hat{F} \). Choose \( a \) such that \([b,c] \cong [a,a]\). Then over \( F \), \( (1,-b) \otimes (1,-c) = (1,-b,-c,bc) \cong (1,-a) \otimes (1,-a) \). Hence \( (1,-b) \otimes (1,-c) \) is hyperbolic over \( K \) and \( b \in D_K(l,-c) \). Thus \( \hat{F} \subseteq D_K(l,-c) \).
The Norm Principle now gives the next corollary, of which the second corollary is an immediate consequence.

**Corollary 3.10.** Suppose \( m(F) = 2 \) and \( a \notin R(F) \). Then for \( x \in K \) and \( c \in F \), \( x \in D_K(1,-c) \) if and only if \( N(x) \in D_F(1,-c) \).

**Corollary 3.11.** Suppose \( m(F) = 2 \) and \( a \notin R(F) \). Then for \( x \in K \), \( F \subseteq D_K(1,-x) \) if and only if \( N(x) \in R(F) \).

**Proposition 3.12.** Suppose \( m(F) = 2 \) and \( a \notin R(F) \). Then for \( c \in F \), \( |D_K(1,-c)| = q(K) \) or \( q(K)/2 \).

**Proof.** Suppose \( x, y \notin D_K(1,-c) \). By Corollary 3.10, \( N(x) \), \( N(y) \) do not belong to \( D_F(1,-c) \). But \( m(F) = 2 \) then implies \( N(xy) \in D_F(1,-c) \) and so \( xy \in D_K(1,-c) \).

**Proposition 3.13.** If \( m(F) = 2 \) and \( a \notin R(F) \), then for \( x \in K \), \( F \cap D_K(1,-x) = D_F(1,-N(x)) \).

**Proof.** Let \( f \in D_F(1,-N(x)) \). Then \( f \in D_K(1,-dx) \) for some \( d \in F \), by Lemma 3.2. Since \( f \in D_K(1,-d) \) also, we have \( f \in D_K(1,-x) \). The other containment is clear by Lemma 3.2.

This last proposition will be used many times in the next two proofs, sometimes without mention. Also used often is the fact that for any \( x \in R(K) \), \( N(x) \in R(F) \) [2, Proposition 5].

**Proposition 3.14.** Suppose \( m(F) = 2 \) and \( a \notin R(F) \). If \( x \in K \) and \( N(x) \notin R(F) \), then \( |D_K(1,-x)| = q(K)/2 \).
Proof. Let \( D_F(1,-N(x)) = \langle d_1, \ldots, d_{n-1}, h \rangle \) where \( Q(F) = \langle d_1, \ldots, d_{n-1}, h \rangle \), \( h \not\in D_F(1,-N(x)) \). Since \( N(x) \in D_F(1,-a) \), \( a \in D_F(1,-N(x)) \), so we may assume without loss of generality that \( a = d_{n-1} \). Then \( D_K(1,-x) \supseteq \langle d_1, \ldots, d_{n-2} \rangle \) and there exist \( y_1, \ldots, y_{n-1} \in \hat{K} - \hat{F} \) such that \( Q(K) = \langle d_1, \ldots, d_{n-2}, h \rangle \). 

For each \( i = 1, \ldots, n-1 \), clearly \( y_i \not\in \hat{FK} \). If \( y_i \in x\hat{FK} \), then \( y_i \in \hat{FD}_K(1,-x) \), so there is an \( f_i \in \hat{F} \) such that \( f_i y_i \in D_K(1,-x) \). 

If \( y_i \not\in x\hat{FK} \), the quadratic form \( \rho_i = \alpha_1(y_i)(1,-N(y_i)) \otimes \alpha_x(y_i)(1,-N(xy_i)) \) has determinant \( N(x) \). Since \( N(x) \not\in R(F) \) and \( m(F) = 2 \), then \( |D_F(1,-N(x))| = \frac{q(F)}{2} \). By Proposition 2.2, since \( m(F) = 2 \) implies \( u(F) = 4 \), \( F \) has no anisotropic quaternary forms of determinant \( N(x) \). 

Hence the form \( \rho_i \) is \( F \)-isotropic, so by Proposition 3.8 there is an \( f_i \in \hat{F} \) such that \( f_i y_i \in D_K(1,-x) \). Thus for each \( i = 1, \ldots, n-1 \) there is an \( f_i \in \hat{F} \) such that \( f_i y_i \in D_K(1,-x) \). 

So \( D_K(1,-x) \supseteq \langle d_1, \ldots, d_{n-2}, f_1 y_1, \ldots, f_{n-1} y_{n-1} \rangle \). Since \( Q(K) = \langle d_1, \ldots, d_{n-2}, h, f_1 y_1, \ldots, f_{n-1} y_{n-1} \rangle \) and \( h \not\in D_K(1,-x) \) by Proposition 3.13, then equality holds and \( |D_K(1,-x)| = q(K)/2 \).

Using the notation of the last proposition, notice that for any \( w \in \hat{F} \), exactly one of \( w, hw \) is in \( D_K(1,-x) \), and in fact \( D_K(1,-x) = \langle d_1, \ldots, d_{n-2}, f_1 y_1, \ldots, f_{n-1} y_{n-1} \rangle \), where \( f_i = 1 \) or \( h \) for all \( i = 1, \ldots, n-1 \).

Proposition 3.15. Let \( m(F) = 2 \) and \( a \not\in R(F) \). If \( x \in \hat{K} \) and \( N(x) \in R(F) \), then \( |D_K(1,-x)| = q(K) \) or \( q(K)/2 \).

Proof. If \( x \in R(K) \), then \( |D_K(1,-x)| = q(K) \). If \( x \not\in R(K) \), then \( |D_K(1,-x)| \leq q(K)/2 \). Let \( y_1, y_2 \not\in D_K(1,-x) \), \( y_1 \neq y_2 \).
To show $|D_K(1,-x)| = q(K)/2$, it suffices to show $Y_1Y_2 \in D_K(1,-x)$. By Corollary 3.11, $D_K(1,-x) \supset \hat{F}$, so $D_K(1,-x) \supset \langle f_1, \ldots, f_{n-1} \rangle$ where $Q(\hat{F}) = \langle f_1, \ldots, f_{n-1}, a \rangle$. So $Y_1, Y_2 \not\in \hat{F}$.

If $y_i \in f x \hat{K}^2$ for some $f \in \hat{F}$, then since $-f \in D_K(1,-x)$, $-x = (fx)(-f) \not\in D_K(1,-x)$, a contradiction. So $y_i \not\in x \hat{F} \hat{K}^2$ for $i = 1$ or 2. By Proposition 3.8, for each $i$ the form $\alpha_1(y_i)(1,-N(y_i)) \oplus \alpha_x(y_i)(1,-N(x)N(y_i))$ is anisotropic over $F$. Since $N(x) \in R(\hat{F})$, these are anisotropic if and only if the forms $\alpha_1(y_i)(1,-N(y_i)) \oplus \alpha_x(y_i)(1,-N(y_i))$ are anisotropic for $i = 1$ and 2, by the corollary to Proposition 1 of [2]. Clearly then $(1,-N(y_1))(1,-N(y_2))$ are not $F$-universal, i.e., $N(y_1), N(y_2) \not\in R(\hat{F})$. Thus $Y_1, Y_2 \not\in R(K)$.

If $y_2 \in f y_1 \hat{K}^2$ for some $f \in \hat{F}$, then $y_1 y_2 \in \hat{F} \subset D_K(1,-x)$, and we are done. So in the following, suppose that $y_1 y_2 \not\in \hat{F} \hat{K}^2$.

Similarly, $y_1 y_2 \not\in x \hat{F} \hat{K}^2$. If $D_F(1,-N(y_1)) = D_F(1,-N(y_2))$, then, since $m(F) = 2$, $N(y_1 y_2) \in R(F)$, hence $(1,-N(y_1 y_2))$ is $F$-universal. So the form $\alpha_1(y_1 y_2)(1,-N(y_1 y_2)) \oplus \alpha_x(y_1 y_2)(1,-N(x)N(y_1 y_2))$ is isotropic, since $N(x) \in R(F)$.

So by Proposition 3.8, there exists $f \in \hat{F}$ such that $f y_1 y_2 \in D_K(1,-x)$. But $\hat{F} \subset D_K(1,-x)$ shows $y_1 y_2 \in D_K(1,-x)$, and we are done.

So assume in the following that $D_F(1,-N(y_1)) \not\subset D_F(1,-N(y_2))$. We now have $N(y_1 y_2) \not\in R(F)$, hence $y_1 y_2 \not\in R(K)$.

Thus $|D_F(1,-N(y_1))| = q(F)/2 = |D_F(1,-N(y_2))|$ and so the intersection of these subgroups of $Q(\hat{F})$ forms a subgroup with exactly $q(F)/4$ elements. Let $D_F(1,-N(y_1)) \cap D_F(1,-N(y_2)) = \langle c_1, \ldots, c_{n-2} \rangle$. Clearly, $a \in D_F(1,-N(y_1)) \cap D_F(1,-N(y_2))$, with $a \not\in Q(\hat{F})$. So
so without loss of generality let $a = c_{n-2}$. Thus there exist $r, s \in \mathbb{F}$ such that $D_F(1,-N(y_1)) = \langle c_1, \ldots, c_{n-2}, r \rangle$ and $D_F(1,-N(y_2)) = \langle c_1, \ldots, c_{n-2}, s \rangle$, where $Q(F) = \langle c_1, \ldots, c_{n-2}, r, s \rangle$. Since $N(y_1), N(y_2) \not\in R(F)$, the proof of Proposition 3.14 yields $D_F(1,N(1,-N(y_1))) = \langle c_1, \ldots, c_{n-2}, r \rangle$ and $D_F(1,-N(y_2)) = \langle c_1, \ldots, c_{n-2}, s \rangle$.

Now $D_F(1,-N(y_1)) = \langle c_1, \ldots, c_{n-2}, r \rangle$, $D_F(1,-N(y_2)) = \langle c_1, \ldots, c_{n-2}, s \rangle$, and $Q(F) = \langle c_1, \ldots, c_{n-2}, r, s \rangle$ forces $D_F(1,-N(1,-N(y_1 ))) = \langle c_1, \ldots, c_{n-2}, r \rangle$. Notice that now $r \not\in D_K(1,-y_2)$ and $s \not\in D_K(1,-y_2)$. Suppose $y_1 y_2 \not\in D_K(1,-x)$. Then $x \not\in D_K(1,-y_1 y_2)$, so $sx \in D_K(1,-y_1 y_2)$. But $y_1 \not\in D_K(1,-x)$, so $x \not\in D_K(1,-y_1)$, hence $sx \in D_K(1,-y_1)$. Thus $sx \in D_K(1,-y_1) \cap D_K(1,-y_1 y_2) \subseteq D_K(1,-y_2)$. But $y_2 \not\in D_K(1,-x)$, so $x \not\in D_K(1,-y_2)$, hence $rx \in D_K(1,-y_2)$. Therefore, $rs = (rx)(sx) \in D_K(1,-y_2)$, which implies that $rs \in D_F(1,-N(y_2))$ by Proposition 3.13, a contradiction. So $y_1 y_2 \in D_K(1,-x)$, and we are done.

Now that we know the value set structure for $K$ in relation to that of $F$, we consider the values of $m(K)$ and $|R(K)|$.

**Theorem 3.16.** Let $F$ be a non-formally real field with $q(F) < \infty$ and $m(F) = 2$. Let $K = F(\sqrt{a})$, $a \not\in R(F)$. Then $m(K) = 2$. 
Proof. Let \( x \in K \). If \( x \in \hat{F} \), then \( |D_K(1,-x)| = q(K) \) or \( q(K)/2 \), by Corollary 3.12. If \( x \in \hat{K} \) and \( N(x) \not\in R(F) \), then \( |D_K(1,-x)| = q(K)/2 \) by Proposition 3.14. If \( x \in \hat{K} \) and \( N(x) \in R(F) \), then \( |D_K(1,-x)| = q(K) \) or \( q(K)/2 \) by Proposition 3.15. So for all \( x \in \hat{K} \), \( |D_K(1,-x)| = q(K) \) or \( q(K)/2 \). Since \( m(F) = 2 \) implies \( u(F) = 4 \), we will have \( u(K) = 4 \) or 6 [5, Theorems 4.3, 4.11(2)]. Thus \( \hat{K} \not\in R(K) \), hence there is some \( x \in \hat{K} \) such that \( |D_K(1,-x)| \neq q(K) \). It follows from the remarks in the second paragraph of this chapter that \( m(K) = 2 \). □

Proposition 3.17. If \( m(F) = 2 \), \( a \not\in R(F) \), and \( S(K) = \{ x \in \hat{K} | N(x) \in R(F) \} \), then \( S(K) = \langle R(K) \cup \hat{F}K^2 \rangle \).

Proof. Clearly \( \langle R(K) \cup \hat{F}K^2 \rangle \subseteq S(K) \), so suppose \( y \in S(K) \).

Then \( N(y) \in R(F) \). If \( y \in R(K) \) we are done, so suppose \( y \not\in R(K) \). Then by Proposition 3.15, \( D_K(1,-y) = \langle f_1, \ldots, f_{n-1}, w_1, \ldots, w_{n-2} \rangle \) where \( Q(K) = \langle f_1, \ldots, f_{n-1}, w_1, \ldots, w_{n-1} \rangle \), \( D_F(1,-a) = \langle N(w_1), \ldots, N(w_{n-1}) \rangle \), and \( Q(F) = \langle f_1, \ldots, f_{n-1}, a \rangle \) = \( \langle N(w_1), \ldots, N(w_{n-1}), b \rangle \) for some \( b \in \hat{F} \setminus D_F(1,-a) \). Let \( A = \langle N(w_1), \ldots, N(w_{n-2}), b \rangle \). Suppose there exists \( g \in \hat{F} \) such that \( g \in R(F) \setminus A \). Then \( g \in R(F) \subseteq D_F(1,-a) \) implies \( g = N(z) \) for some \( z \in \hat{K} \). Since \( N(z) \not\in A \), we have \( z \not\in D_K(1,-y) \) by the construction of \( A \). Clearly \( z \not\in \hat{F}K^2 \). If \( z \in y\hat{F}K^2 \), then \( z \in D_K(1,-fy) \) for some \( f \in \hat{F} \). Since \( z \not\in D_K(1,-y) \), we have \( z \not\in D_K(1,-f) \). By Corollary 3.10, this implies \( N(z) \not\in D_F(1,-f) \). Hence \( g = N(z) \not\in R(F) \), a contradiction. Thus we may suppose \( z \not\in y\hat{F}K^2 \). The form \( \rho = a_1(y)(1,-N(y)) + \)

\( \alpha_z(y)(1,-N(z)N(y)) \) is clearly \( F \)-isotropic. Proposition 3.8 then implies that there exists \( h \in \hat{F} \) such that \( z \in D_K(1,-hy) \). Since \( z \notin D_K(1,-y) \), we have \( z \notin D_K(1,-h) \). Corollary 3.10 then gives \( N(z) \notin D_F(1,-h) \), which means \( g = N(z) \notin R(F) \), a contradiction. Therefore, \( R(F) \subseteq A \). By the corollary to Lemma 2 of [3], there exists \( c \in \hat{F} \) such that \( D_F(1,-c) = A \).

By Proposition 3.9 and Corollary 3.10, \( D_K(1,-c) = \langle f_1, \ldots, f_{n-1}, w_1, \ldots, w_{n-2} \rangle = D_K(1,-y) \). Since \( m(K) = 2 \) by Theorem 3.16, we have \( cy \in R(K) \) by Lemma 1 of [3]. Thus \( y \in \langle R(K) \cup \hat{F}^2 \rangle \) and we are done.

### Theorem 3.18.

Let \( F \) be a non-formally real field with \( q(F) < \infty \) and \( m(F) = 2 \), \( K = F(\sqrt{a}) \), \( a \notin R(F) \). Then

\[
|R(K)/K^2| = |R(F)/\hat{F}^2|^2.
\]

**Proof.** Let \( R(F) = \langle b_1, \ldots, b_\ell \rangle \). If \( d \in \hat{F} \cap R(K) \) then by Proposition 3.4, \( d \in R(F) \) or \( ad \in R(F) \), but not both. Thus we may assume \( R(K) = \langle b_1, \ldots, b_\ell, y_1, \ldots, y_k \rangle \) where for all \( i = 1, \ldots, k \), \( y_i \notin \hat{F} \). Let \( Q(F) = \langle b_1, \ldots, b_\ell, a \rangle \).

\[
c_1 \cdots c_{n-(\ell+1)}.
\]

Clearly \( \langle R(K) \cup \hat{F}^2 \rangle = \langle b_1, \ldots, b_\ell, c_1 \cdots c_{n-(\ell+1)}, y_1, \ldots, y_k \rangle \). Since \( R(F) \subseteq D_F(1,-a) \), let \( b_i = N(z_i) \) for all \( i = 1, \ldots, \ell \). Then \( S(K) = \langle b_1, \ldots, b_\ell, c_1 \cdots c_{n-(\ell+1)}, z_1, \ldots, z_\ell \rangle \). Thus \( |S(K)| = 2^{n-1+\ell} \) and \( |\langle R(K) \cup \hat{F}^2 \rangle| = 2^{n-1+k} \). By Proposition 3.17 we have \( 2^{n-1+\ell} = 2^{n-1+k} \), hence \( \ell = k \). Thus \( R(K) = \langle b_1, \ldots, b_\ell, y_1, \ldots, y_\ell \rangle \) and \( |R(K)/K^2| = (2^\ell)^2 = |R(F)/\hat{F}^2|^2 \). □

We now turn to the case when \( m(F) = 2 \) and \( a \in R(F) \). Here we will be able to determine \( |D_K(1,-x)| \) for all \( x \in K \).
and show that \( 4 \leq m(K) \leq 8 \). The results in this case are not as easily attained as for \( a \notin R(F) \). The primary reason for this is that \( \hat{F} \subseteq D_K(1,-c) \) is no longer true for all \( c \in \hat{F} \). The next proposition, which does not depend on \( m(F) = 2 \), shows what is true.

**Proposition 3.19.** Suppose \( a \in R(F) \). For every \( c \in \hat{K} \), \( \hat{F} \cap D_K(1,-c) = D_F(1,-c) \).

**Proof.** Clearly \( D_F(1,-c) \subseteq \hat{F} \cap D_K(1,-c) \). So suppose \( b \in \hat{F} \cap D_K(1,-c) \). Then \( (1,-c) \otimes (1,-b) \) is isotropic over \( K \) and hence hyperbolic [11, Theorem 2]. Therefore, by [14, Remark 2.29] there are \( e,f \in \hat{F} \) such that \( (1,-c) \otimes (1,-b) \sim (1,-a) \otimes (e,f) \) over \( F \). But \( a \in R(F) \) implies \( (1,-a) \otimes (e,f) \) is isotropic over \( F \) and so \( b \in D_F(1,-c) \).

**Corollary 3.20.** Suppose \( m(F) = 2 \) and \( a \in R(F) \). Then for \( c \in \hat{F} \), \( \hat{F} \subseteq D_K(1,-c) \) if and only if \( c \in R(F) \). Moreover, \( c \in R(F) \) if and only if \( c \in R(K) \).

**Proposition 3.21.** Let \( m(F) = 2 \) and \( a \in R(F) \). If \( c \in \hat{F} - R(F) \), then \( |D_K(1,-c)| = q(K)/4 \).

**Proof.** Let \( q(F) = 2^n \). \( \hat{F} \cap D_K(1,-c) = D_F(1,-c) \) contains \( 2^{n-2} \) distinct representatives of \( K^2 \) in \( \hat{K} \), so choose \( b \in \hat{F} - D_F(1,-c) \) and then select \( y \in \hat{K} \) so that \( N(y) = b \). By the Norm Principle \( y \), by \( \not \in D_K(1,-c) \). Hence \( D_K(1,-c) \) has index at least \( 4 \) in \( K \). But for \( x \in \hat{K} \), either \( N(x) \) or \( N(xy) \) \( \in D_F(1,-c) \). Thus again by the Norm Principle, either \( x \) or
xy ∈ \hat{F} \cdot D_K(1,-c) = D_K(1,-c) \cup bD_K(1,-c). So x is in one of the cosets of \( D_K(1,-c) \) represented by 1, b, y, or by. Therefore \( |D_K(1,-c)| = q(K)/4. \)

Recall that over \( K \) a does not continue to be a basis element. To make the results of the last proposition more explicit, let \( q(F) = 2^n \) again. Then \( \hat{F} \cap D_K(1,-c) = D_F(1,-c) \) contains \( n-2 \) representatives of a basis of \( K^2 \) in \( \hat{K} \), and \( n-1 \) representatives from \( D_K(1,-c) - D_F(1,-c) \) may be added to form a basis for \( D_K(1,-c) \). Therefore, \( |D_K(1,-c)| = 2^{n-2}2^{n-1} = 2^{2n-3} \) where \( q(K) = 2^{n-1}2^n = 2^{2n-1} \). So a basis of \( D_K(1,-c) \) which "misses" one basis element in \( \hat{F}K^2 \) and one basis element not in \( \hat{F}K^2 \) can be found.

Notice that Proposition 3.21 also implies that \( m(K) \geq \frac{4}{3} \), since \((x,y) \cong (u,v)\) if and only if \( xy \in uvK^2 \) and \([x,y] = [u,v]\).

**Proposition 3.22.** Let \( m(F) = 2 \) and \( a \in R(F) \). If \( x \in \hat{K} \), then the index of \( \hat{F} \cap D_K(1,-x) \) in \( D_F(1,-N(x)) \) is 1 or 2.

**Proof.** By the Norm Principle, \( \hat{F} \cap D_K(1,-x) \subseteq D_F(1,-N(x)) \).
Suppose there exist \( c_1, c_2 \in \hat{F} \) such that \( c_1, c_2 \in D_F(1,-N(x)) - D_K(1,-x) \). Thus \( c_1, c_2 \not\in D_K(1,-x) \), so \( x \not\in D_K(1,-c_1) \cup D_K(1,-c_2) \). But \( N(x) \in D_F(1,-c_1) \) and \( N(x) \in D_F(1,-c_2) \), so \( x \in \hat{F} \cdot D_K(1,-c_1) \) and \( x \in \hat{F} \cdot D_K(1,-c_2) \), by the Norm Principle. By Proposition 3.19, \( \hat{F} \cap D_K(1,-c_i) = D_F(1,-c_i) \) for \( i = 1, 2 \). So \( \hat{F} \cdot D_K(1,-c_i) = D_K(1,-c_i) \cup b_i D_K(1,-c_i) \), for any \( b_i \in \hat{F} - D_F(1,-c_i) \), \( i = 1, 2 \), since \( m(F) = 2 \). Thus \( x \in \hat{F} \)
b_1D_K(1,-c_i) for i = 1,2. Choose b = b_1 = b_2 ∈ \mathbb{F} - [D_F(1,-c_1) ∪ D_F(1,-c_2)]. This is possible since 
|D_F(1,-c_1) ∩ D_F(1,-c_2)| = q(F)/4 whenever D_F(1,-c_1) ≠ D_F(1,-c_2). Then bx ∈ D_K(1,-c_1) ∩ D_K(1,-c_2) ⊆ D_K(1,-c_1c_2).
But b ∉ D_F(1,-c_i), i = 1,2, so c_1, c_2 ∉ D_F(1,-b). Since 
m(F) = 2, this implies c_1c_2 ∈ D_F(1,-b) ⊆ D_K(1,-b). So b ∈ D_K(1,-c_1c_2), bx ∈ D_K(1,-c_1c_2), thus x ∈ D_K(1,-c_1c_2), hence 
c_1c_2 ∈ D_K(1,-x). This shows that whenever the index of 
\hat{F} ∩ D_K(1,-x) in D_F(1,-N(x)) is not 1, it is 2.

This last result together with the realization that, over K, a does not continue to be a basis element, give the following results when q(F) = 2^n. We have q(K) = 2^{n-1}2^n = 2^{2n-1} by the remarks of Gross and Fischer [7]. If N(x) ∉ R(F), then \hat{F} ∩ D_K(1,-x) contains 2^{n-2} or 2^{n-3} representatives of \hat{K}^2 in \hat{K}. If N(x) ∈ R(F), then \hat{F} ∩ D_K(1,-x) contains 2^{n-1} or 2^{n-2} representatives of \hat{K}^2 in \hat{K}.

Proposition 3.23. Suppose m(F) = 2 and a ∈ R(F). Let x ∈ \hat{K} with N(x) ∉ R(F). Then |D_K(1,-x)| = q(K)/2 or q(K)/4.

Proof. Since N(x) ∉ R(F) and |D_F(1,-N(x))| = q(F)/2 = 2^{n-1}, we have by the above remarks that \hat{F} ∩ D_K(1,-x) contains 2^{n-2} or 2^{n-3} representatives of \hat{K}^2 in \hat{K}, where q(K) = 2^{2n-1}.

Suppose Q(K) = <N(w_2),...,N(w_n),w_1,...,w_n>, where a = N(w_1).
For each i = 1,...,n, we know w_i ∉ \hat{F}K^2, and we suppose w_i ∉ \hat{F}K^2. (Otherwise, we have w_i ∈ \hat{F} ∩ D_K(1,-x) which is the desired conclusion.) Consider the form \rho_i defined by
\[ \rho_i = \alpha_1(w_i)(1,-N(w_i)) \oplus \alpha_x(w_i)(1,-N(xw_i)) \] of determinant \( N(x) \). By Proposition 2.2, since \( m(F) = 2 \) implies \( u(F) = 4 \), \( F \) has no anisotropic quaternary forms of determinant \( N(x) \). So for \( i = 1, \ldots, n \) the form \( \rho_i \) is \( F \)-isotropic, which by Proposition 3.8 implies that, for each \( i = 1, \ldots, n \), there is an \( f_i \in \hat{F} \) such that \( f_iw_i \in D_K(1,-x) \). So \( D_K(1,-x) \) contains \( 2^{n-2} \) or \( 2^{n-3} \) representatives of \( \mathbb{K}^2 \) in \( \mathbb{K} \) that are in \( \hat{F} \) as well as \( \langle f_1w_1, \ldots, f_nw_n \rangle \). Hence \( |D_K(1,-x)| = 2^{n-2}2^n = q(K)/4 \) or \( 2^{n-3}2^n = q(K)/2 \). 

Proposition 3.24. Suppose \( m(F) = 2 \) and \( a \in R(F) \). Let \( x \in \hat{K} \) with \( N(x) \not\in R(F) \). Then there exists \( f \in \hat{F} \) such that 
\[ \hat{F} \cap D_K(1,-fx) = D_F(1,-N(x)). \]

**Proof.** Suppose \( D_F(1,-N(x)) = \langle a_1, a_2, \ldots, a_{n-1} \rangle \) where \( Q(F) = \langle a_1, \ldots, a_{n-1}, a_n \rangle \) and suppose \( \hat{F} \cap D_K(1,-x) \neq D_F(1,-N(x)) \).

Then, by Proposition 3.22, the index of \( \hat{F} \cap D_K(1,-x) \) in \( D_F(1,-N(x)) \) is 2. Thus we may choose the \( a_i \) in such a manner that \( \hat{F} \cap D_K(1,-x) = \langle a_2, \ldots, a_{n-2} \rangle \), where \( a = a_1 \).

Suppose also that \( R(F) = \langle a_1, \ldots, a_k \rangle, k \leq n-2, \) for definiteness. By the corollary to Lemma 2 in [3], there exists some \( b \in \hat{F} \) such that \( D_F(1,-b) = \langle a_1, a_2, \ldots, a_{n-2}, a_n \rangle \). Thus \( a_{n-1} \not\in D_F(1,-b) \), and \( D_K(1,-b) \supseteq \langle a_2, \ldots, a_{n-2}, a_n \rangle \). So \( D_K(1,-bx) \supseteq D_K(1,-b) \cap D_K(1,-x) \supseteq \langle a_2, \ldots, a_{n-2} \rangle \). Now, \( a_{n-1} \in D_F(1,-N(x)) \), so \( N(x) \in D_F(1,-a_{n-1}) \). By the Norm Principle, \( x \in \hat{F} \). 

\[ D_K(1,-a_{n-1}) = D_K(1,-a_{n-1}) \cup \hat{d}D_K(1,-a_{n-1}) \] for any \( d \in \hat{F} - D_K(1,-a_{n-1}) \) since \( D_K(1,-a_{n-1}) \cap \hat{F} = D_F(1,-a_{n-1}) \) has order \( q(F)/2 \). Letting \( d = b \) yields \( x \in D_K(1,-a_{n-1}) \) or
bx ∈ D_k(1,-a_{n-1}). Since a_{n-1} \not\in D_k(1,-x), this forces bx ∈ D_k(1,-a_{n-1}), or a_{n-1} ∈ D_k(1,-bx). Thus D_k(1,-bx) ⊇ \langle a_2, \ldots, a_{n-2}, a_{n-1} \rangle, so \hat{F} \cap D_k(1,-bx) = D_F(1,-N(x)).

**Proposition 3.25.** Suppose m(F) = 2 and a ∈ R(F). Let x ∈ \hat{F} with N(x) ∈ R(F). Then |D_k(1,-x)| = q(K), q(K)/2, or q(K)/4.

**Proof.** By the remarks following Proposition 3.22, \hat{F} \cap D_k(1,-x) contains 2^{n-1} or 2^{n-2} representatives of k^2 in \hat{k}. Let Q(K) = \hat{k}^2 \otimes \langle z_1, \ldots, z_n \rangle. If for any z_i ∈ \{z_1, \ldots, z_n\}, there exists an f_i ∈ \hat{F} such that f_i z_i ∈ D_k(1,-x), then clearly |D_k(1,-x)| = 2^{n-1} or 2^{n-2} = q(K) or q(K)/2, and we are done. So assume there exists z_i ∈ \{z_1, \ldots, z_n\}, say z_1, such that z_1 \not\in \hat{F} \cdot D_k(1,-x). Consider another z_i ∈ \{z_2, \ldots, z_n\}, say z_2. We will show that z_2 ∈ \hat{F} \cdot D_k(1,-x) or z_1 z_2 ∈ \hat{F} \cdot D_k(1,-x). This will show |D_k(1,-x)| = 2^{n-1} = q(K)/2 or 2^{n-2} = q(K)/4. Assume z_2 \not\in \hat{F} \cdot D_k(1,-x). Since z_1 \not\in z_2, and z_1, z_2 \not\in \hat{F} \cdot D_k(1,-x), we have z_1 z_2 \not\in D_k(1,-x). Since z_1, z_2 \not\in \hat{F} \cdot D_k(1,-x), we know z_1 z_2 \not\in xFk^2. By Proposition 3.8 the forms \alpha_1(z_i)(1,-N(z_i)) ⊕ \alpha_{-x}(z_i)(1,-N(xz_i)) are anisotropic over F for i = 1, 2, and this is true if and only if the forms \alpha_1(z_i)(1,-N(z_i)) ⊕ \alpha_{-x}(z_i)(1,-N(z_i)) are anisotropic since N(x) ∈ R(F). This last statement follows from the corollary to Proposition 1 of [2]. So N(z_1), N(z_2) \not\in R(F), hence z_1 z_2 \not\in R(K). Suppose D_F(1,-N(z_1)) = D_F(1,-N(z_2)). Then since m(F) = 2, N(z_1 z_2) ∈ R(F), hence (1,-N(z_1 z_2)) is
clearly \( z_{1z_2} \notin \tilde{F}K^2 \). If \( z_{1z_2} \in x\tilde{F}K^2 \in \hat{\mathcal{F}} \cdot D_K(1,-x) \), we are done. So suppose \( z_{1z_2} \notin \tilde{F}K^2 \). Then the
form \( \rho = a_1(z_{1z_2})(1,-N(z_{1z_2})) \oplus a(-x(z_{1z_2})(1,-N(z_{1z_2})) \) is 
\( F \)-isotropic by construction. But \( \rho \) is \( F \)-isotropic if and
only if the form \( a_1(z_{1z_2})(1,-N(z_{1z_2})) \oplus a(-x(z_{1z_2})(1,-N(z_{1z_2})) \)

is \( F \)-isotropic. Thus, by Proposition 3.8, there exists \( f \in 
\hat{\mathcal{F}} \) such that \( fz_{1z_2} \in D_K(1,-x) \), hence \( z_{1z_2} \in \hat{\mathcal{F}} \cdot D_K(1,-x) \),
and we are done. So we may suppose \( D_F(1,-N(z_1)) \neq 
D_F(1,-N(z_2)) \), thus \( N(z_{1z_2}) \notin R(F) \) and so \( z_{1z_2} \notin R(K) \).
Therefore \( |D_F(1,-N(z_1))| = q(F)/2 = |D_F(1,-N(z_2))| \) and so 
\( |D_F(1,-N(z_1)) \cap D_F(1,-N(z_2))| = q(F)/4 \). Let \( D_F(1,-N(z_1)) \cap 
D_F(1,-N(z_2)) = <a_1, \ldots, a_{n-2}> \) where \( D_F(1,-N(z_1)) = 
<\ldots, a_{n-2}, a_{n-1}, a_n> \), and \( Q(F) = <a_1, \ldots, a_{n-2}, a_n> \). If \( a = a_1 \), then \( Q(K) =
<\ldots, a_{n-2}, a_{n-1}, a_n> \) where \( a_i = N(w_i) \) for all \( i = 1, \ldots, n \).
Since \( N(z_{1z_2}) \notin R(F) \), we are forced to have \( D_F(1,-N(z_{1z_2})) =
<\ldots, a_{n-2}, a_{n-1}, a_n> \). If \( r = a_{n-1} \) and \( s = a_n \), then we have 
\( D_F(1,-N(z_1)) = <a_1, \ldots, a_{n-2}, r> \), \( D_F(1,-N(z_2)) = <a_1, \ldots, a_{n-2}, s> \),
\( D_F(1,-N(z_{1z_2})) = <a_1, \ldots, a_{n-2}, r, s> \), and \( Q(F) = <a_1, \ldots, a_{n-2},
<\ldots, a_{n-2}, r, s> \). By Propositions 3.21 and 3.24, there exist \( d_1, d_2, d_3 \in 
\hat{\mathcal{F}} \) such that \( D_K(1,-d_{1z_1}) = <\ldots, a_{n-2}, r, f, w_1, \ldots, f w_n> \),
\( D_K(1,-d_{2z_2}) = <\ldots, a_{n-2}, s, g_1 w_1, \ldots, g_n w_n> \), and
\( D_K(1,-d_{3z_1z_2}) = <\ldots, a_{n-2}, rs, h_1 w_1, \ldots, h_n w_n> \) for appropriate 
\( f_i, g_i, h_i \in \hat{\mathcal{F}} \), where \( Q(K) = <\ldots, a_{n-2}, r, s, w_1, \ldots, w_n> \). Now,
for all \( f \in \hat{\mathcal{F}}, s, rs \notin D_K(1,-fz_1), r, rs \notin D_K(1,-fz_2), \)
and 
\( r, s \notin D_K(1,-fz_1z_2) \), by Proposition 3.22.
Assume that \( x \not\in D_K(1,-d_1 z_1 z_2) \) and \( x \not\in D_K(1,-d_2 z_1 z_2) \). Then \( s,x \not\in D_K(1,-d_1 z_1) \), so \( sx \in D_K(1,-d_1 z_1) \) by choice of \( d_1 \) and the proof of Proposition 3.24. Also, \( s \in D_K(1,-d_2 z_2) \) implies \( sx \not\in D_K(1,-d_2 z_2) \). Combining these gives \( sx \not\in D_K(1,-d_2 z_2) \). Since \( s,x \not\in D_K(1,-d_2 z_1 z_2) \), \( sx \in D_K(1,-d_2 z_1 z_2) \) and this means \( sx \not\in D_K(1,-d_1 d_2 d_3) \). In a similar manner we can show \( rx \not\in D_K(1,-d_1 d_2 d_3) \). Hence \( d_1 d_2 d_3 \not\in R(F) \) and so by the remarks following Proposition 3.21,
\[
|D_K(1,-d_1 d_2 d_3) \cap \hat{F}| = 2^{n-2}.
\]
Also, \( \langle a_2, \ldots, a_{n-2} \rangle \subseteq D_K(1,-d_1 d_2 d_3) \cap D_K(1,-d_2 z_2) \cap D_K(1,-d_3 z_1 z_2) \subseteq D_K(1,-d_1 d_2 d_3) \).

So one of \( r,s,rs \) is in \( D_K(1,-d_1 d_2 d_3) \). Also, since \( N(x) \in R(K) \), there exists \( f \in \hat{F} \) such that \( fx \in D_K(1,-d_1 d_2 d_3) \). Clearly, \( f \) is in \( \langle a_2, \ldots, a_{n-2}, r, s \rangle \).

If \( f \in r\langle a_2, \ldots, a_{n-2} \rangle \) or \( s\langle a_2, \ldots, a_{n-2} \rangle \), we have \( rx \) or \( sx \), respectively, in \( D_K(1,-d_1 d_2 d_3) \), which is a contradiction.

So \( f \in \langle a_2, \ldots, a_{n-2}, rs \rangle \). If \( f \in \langle a_2, \ldots, a_{n-2} \rangle \), we have
\[
x \in D_K(1,-d_1 d_2 d_3),
\]
so that \( r,s \not\in D_K(1,-d_1 d_2 d_3) \). Thus \( rs \in D_K(1,-d_1 d_2 d_3) \). If \( f \in \langle a_2, \ldots, a_{n-2}, rs \rangle - \langle a_2, \ldots, a_{n-2} \rangle \), then \( rsx \in D_K(1,-d_1 d_2 d_3) \). Since \( rx,sx \not\in D_K(1,-d_1 d_2 d_3) \), we have \( r,s \not\in D_K(1,-d_1 d_2 d_3) \), which means \( rs \in D_K(1,-d_1 d_2 d_3) \).

In any case, then, \( rs \in D_K(1,-d_1 d_2 d_3) \). So \( rs \in D_K(1,-d_1 d_2 d_3) \cap D_K(1,-d_2 z_1 z_2) \subseteq D_K(1,-d_1 d_2 z_1 z_2) \). So
\[
D_K(1,-d_1 d_2 z_1 z_2) = \langle a_2, \ldots, a_{n-2}, rs, j_1 w_1, \ldots, j_n w_n \rangle
\]
for some \( j_i \in \hat{F} \) and thus has index 2 in \( K \). Now \( s,x \not\in D_K(1,-d_1 d_2 z_1 z_2) \), so \( sx \in D_K(1,-d_1 d_2 z_1 z_2) \). Moreover, \( s,x \not\in D_K(1,-d_1 z_1) \), so \( sx \in D_K(1,-d_1 z_1) \). This means \( sx \not\in D_K(1,-d_1 z_1) \cap D_K(1,-d_1 z_1 z_2) \subseteq D_K(1,-d_2 z_2) \). But \( x \not\in D_K(1,-d_1 z_1) \), so
\[ s = (sx) \cdot (x) \notin D_K(1, -d_2 z_2), \] which is a contradiction. So the assumption is false, and either \( x \in D_K(1, -d_1 z_1 z_2) \) or \( x \in D_K(1, -d_1 d_2 z_1 z_2) \). In either case, \( z_1 z_2 \in F \cdot D_K(1, -x) \), which was to be shown.

**Theorem 3.26.** Let \( F \) be a non-formally real field with \( q(F) < \infty \) and \( m(F) = 2 \). Let \( K = F(\sqrt{a}) \), \( a \in \mathbb{R}(F) \). Then \( \frac{4}{2} \leq m(K) \leq 8 \).

**Proof.** The remarks after Proposition 3.21 show \( m(K) \geq \frac{3}{2} \). By Propositions 3.21, 3.23, and 3.25, the maximum index of all \( D_K(1, -x) \) in \( K \) is \( 4 \). By Theorem 3 of [3] we have \( m(K) - 1 \leq 8(q-1)/q < 8 \), so \( m(K) \leq 8 \).

We can summarize some of our findings about \( m(K) \) in the following theorem, which is clear by Theorems 3.16 and 3.26.

**Theorem 3.27.** Let \( F \) be a non-formally real field with \( q(F) < \infty \) and \( m(F) = 2 \). Let \( K = F(\sqrt{a}) \), \( a \notin \mathbb{F}^2 \). Then \( m(K) = 2 \) if and only if \( a \notin \mathbb{R}(F) \).

Using similar techniques, Cordes has shown further that, when \( a \in \mathbb{R}(F) \) and \( m(F) = 2 \), \( m(K) = 4 \) and \( |\mathbb{R}(K)/K^2| = \frac{1}{2} |\mathbb{R}(F)/F^2|^2 \). He also answered these questions for real fields and eliminated finiteness conditions from some results.

In this chapter we have been concerned with quadratic extensions of nonreal fields with two quaternion algebras. In order to show how to apply techniques similar to those
developed here to extensions of other types of fields, we conclude by considering an example. Recall the possible form scheme with \( q = 16, u = 4, s = 2, m = 4, \) and \(|R| = 1\) that is listed in Proposition 2.9. Suppose there exists some field \( F \) which has this form scheme. We will exhibit a quadratic extension of \( F \) which has a non-trivial radical.

Let \( K = F(\sqrt{c}) \). By [7, p. 299], \( q(K) = \frac{1}{2}q(F) \cdot |D_F(1,-c)| = 32 \). If \(-1 \in K^2\), then \(-1 = (\alpha + \beta/c)^2 = \alpha^2 + c\beta^2 + 2\alpha\beta/c\) for some \( \alpha, \beta \in \hat{F} \). Thus \( 2\alpha\beta = 0 \), hence \( \alpha = 0 \) or \( \beta = 0 \). This means \(-1 = \alpha^2 \) or \( c\beta^2 \). So \(-1 \notin \hat{K}^2 \) or \(-c \notin \hat{F}^2 \), a contradiction. So \(-1 \notin K^2 \). Also, \(-1 \notin D_F(1,1) \subseteq D_K(1,1)\), so \( s(K) = 2 \). Although we will not show this here, \( K \) has exactly four quaternion algebras. Then \( m(K) = 4 \) implies \( u(K) = 4 \) by [9, p. 323, Corollary 4.12(proof)]. Thus \( q(K) = 32, u(K) = 4 = m(K), \) and \( s(K) = 2 \).

We can now construct the binary quadratic form scheme for \( K \) by using the techniques of Gross and Fischer. Here, however, we will not exhibit this structure; we instead concentrate on \( R(K) \). Clearly \( D_F(1,-c) = \{ N(x) \in \hat{F} | x \in \hat{K} \}, \) and \( D_F(1,-c) \subseteq D_F(1,-a) \) by inspection of the form scheme for \( F \) as listed in Proposition 2.9. Also, since \( D_F(1,-c) \) has index 4 in \( \hat{F} \), there exist 4 distinct quaternion algebras over \( F \) of the form \([c,f]\) for \( f \in \hat{F} \). Since \( m(F) = 4 \), every quaternion algebra over \( F \) must be of this form.

Let \( e \in \hat{F} \). Then \([a,e] = [c,f]\) for some \( f \in \hat{F} \). So \((1,-a,-e,ae) \tilde{=} (1,-c,-f,cf)\) over \( F \) and hence over \( K \). Since \((1,-c,-f,cf)\) is \( K \)-isotropic, \((1,-a,-e,ae)\) is \( K \)-isotropic.
also, which implies \( e \in D_K(1,-a) \). So \( F \subseteq D_K(1,-a) \). Thus the Norm Principle of Lam and Elman [6, Proposition 2.13] says in particular here that for all \( x \in K \), \( x \in D_K(1,-a) \) if and only if \( N(x) \in D_F(1,-a) \). But \( D_F(1,-c) \subseteq D_F(1,-a) \), so \( D_K(1,-a) = Q(K) \), i.e., \( a \in R(K) \). Therefore \( K \) has a non-trivial radical.

In this example, we have started with a field with trivial radical and extended it quadratically to obtain a field with non-trivial radical. Moreover, the "new" member of the radical is a square class representative of the original field. A similar construction could have been done with the scheme listed in Proposition 2.8, where \( s = 1 \).

A Kneser field is a nonreal field with \( q < \infty \). At present there are no known Kneser fields with non-trivial radical. If a field with \( q = 16 \) is discovered satisfying the form structure listed in either Proposition 2.8 or Proposition 2.9, the above construction (or one similar to it) will yield a Kneser field with non-trivial radical.
BIBLIOGRAPHY


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