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Extra structures on three-dimensional cobordisms

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EXTRA STRUCTURES ON THREE-DIMENSIONAL COBORDISMS

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Abstract

A Topological Quantum Field Theory (TQFT) is a functor from a cobordism category to the category of vector spaces, satisfying certain properties. An important property is that the vector spaces should be finite dimensional. For the WRT TQFT, the relevant $2 + 1$-cobordism category is built from manifolds which are equipped with an extra structure such as a p_1-structure, or an extended manifold structure. In chapter 1, we perform the universal construction of [3] on a cobordism category without this extra structure and show that the resulting quantization functor assigns an infinite dimensional vector space to the torus. In chapter 2, we enhance the extended manifold structure through introducing oriented lagrangians. We apply a machinery introduced by Guillemin and Sternberg in [7] to transport oriented lagrangians. Using Lion and Vergne's s map in [12, p66], we defined a modulo 4 invariant for cobordisms equipped with such an enhanced structure. This invariant can be viewed as a generalization of Gilmer and Masbaum's $n_\lambda(f)$ in [6, Theorem 6.6], which is defined on the extended mapping class group. The techniques used here might be useful in finding a index 4 subcategory of the extended cobordism category.

Chapter 1
Introduction

Axioms for a topological quantum field theory (TQFT) were given by Atiyah in [1]. An important TQFT was first described by Witten [19]. The first mathematical rigorous construction of this TQFT was due to Reshetikhin and Turaev in [14] using quantum groups.

A $2+1$ TQFT is a covariant functor (V, Z) from a $2+1$ cobordism category $\mathcal{C}$ to the category of complex vector spaces, satisfying certain properties. It assigns a vector space $V(\Sigma)$ to each closed oriented surface Σ, and a linear map $Z_C : V(\Sigma_1) \to V(\Sigma_2)$ to each cobordism $C : \Sigma_1 \to \Sigma_2$. In particular, such a TQFT assigns to the empty space ground field $\mathbb{C}$, and gives rise to a quantum invariant when applied to closed cobordisms, i.e. closed 3-manifold.

In [3], Blanchet, Habegger, Masbaum and Vogel described an universal construction which can be used to build a quantization functor from many invariants of closed 3-manifolds. When applying this universal construction to the Witten-Reshetikhin-Turaev invariants (constructed in [2] using skein theory), the resulting quantization functor is a TQFT. Their cobordisms are equipped with p_1-structure, see [3, Appendix B]. In chapter 2, we studied the quantization functor V'_p resulted from the universal construction applied to the cobordism cateogry without this p_1-structure. We found V'_p will assign an infinite dimensional vector space to the torus. This contradicts one important property that a $2+1$ TQFT should always assign a finite dimensional vector space to a surface. This chapter is substantially the same as a joint paper [8] with Gilmer .

In chapter 2, we consider the cobordism category endowed with a different extra structure. This so-called extended manifold structure was first introduced by Walker in [18] and further developed by Turaev in [16]. Here we follow Gilmer and Masbaum's description in [6]. An object in extended cobordism category is a closed oriented surface Σ with a lagrangian subspace λ on it. Here and in the future, when we refer to a lagrangian on an oriented surface Σ, we mean a lagrangian in the $H_1(\Sigma, \mathbb{Q})$ with the symplectic form given by the intersection form induced by the surface's orientation. An extended morphism $M : \Sigma_1 \to \Sigma_2$ can be roughly viewed as a 3-dimensional cobordism between Σ_1 and Σ_2 with an integer-valued weight. The weight of the composition of two extended cobordisms is given by the extended gluing formula, see formula 3.3 and [6, eq2.1]. Such extended structures are used in [6] to describe a central extension of mapping class group $\Gamma(\Sigma)$ associated to surface Σ. Furthermore, Gilmer and Masbaum define an integer $n_\lambda(f)$ for each mapping class f in $\Gamma(\Sigma)$. This $n_\lambda(f) \mod 4$ is then used to describe an index 4 subgroup of $\Gamma(\Sigma)$.

We approach this invariant in a different way and generalize it to cobordisms which are not necessarily mapping cylinders. First we enhance the extended structure by putting an orientation on the lagrangians. Besides, as Turaev pointed out in [16, p188], for an extended cobordism $M : \Sigma_1 \to \Sigma_2$, the kernel of the inclusion $H_1(\Sigma_1 \sqcup \Sigma_2) \to H_1(M)$ is a lagrangian subspace on $\Sigma_1 \sqcup \Sigma_2$. This kernel, following Turaev, is called a lagrangian relation from $H_1(\Sigma_1, \mathbb{Q})$ to $H_1(\Sigma_2, \mathbb{Q})$. The enhanced extra structure assigns an orientation on the lagrangian relation as well. Using the machinery of Guillemin and Sternberg in [7, Chap3], we develop a way to transport an oriented lagrangian on one boundary surface through the cobordism to the other boundary surface. Then we consider a heegaard splitting of an enhanced cobordism and transport oriented lagrangians from both ends to the middle splitting surface.

Viewing heegaard splitting from a cobordism viewpoint, we endow the splitting surface with an orientation in a natural way. With two oriented lagrangians on one oriented surface, we can define a modulo 4 invariant j using the s map introduced by Lion and Vergne in [12, p66]. This j invariant agrees with Gilmer and Masbaum's $n_\lambda(f)$ when restricted to the enhanced mapping class group. We also remark that the techniques in chapter 2 might be useful in constructing an index four subcategory of the extended cobordism category.

Chapter 2
Extra Structure and the Universal Construction for the WRT TQFT

2.1 Introduction

A TQFT in dimension $2+1$ is a covariant functor (V, Z) from some $(2+1)$-cobordism category $\mathcal{C}$ to the category of finite dimensional complex vector spaces which assigns to the empty object the vector space $\mathbb{C}$. Other properties are usually required for a TQFT, and other ground rings are sometimes allowed. But for our purposes, this will do.

Recall an object Σ in $\mathcal{C}$ is a closed oriented surface with possibly some specified extra structure, and a morphism C from Σ_1 to Σ_2 is an equivalence class of cobordisms from Σ_1 to Σ_2. Such a cobordism can be loosely viewed as a compact oriented 3-manifold (again possibly with some appropriate extra structure) with a boundary decomposed into an incoming surface $-\Sigma_1$ and an outgoing surface Σ_2. Two cobordisms are considered equivalent if there is a orientation-preserving (extra structure preserving) differeomorphism between them which restricts to the identity on the boundary. Then (V, Z) assigns a vector space $V(\Sigma)$ to an object Σ, and a linear map $Z_C : V(\Sigma_1) \to V(\Sigma_2)$ to a morphism $C : \Sigma_1 \to \Sigma_2$.

The WRT-invariant is a 3-manifold invariant which was first described by Witten in [19] and then rigorously defined by Reshetikhin and Turaev with quantum groups in [14]. The approach to this invariant that we will use was developed by Blanchet, Habegger, Masbaum, and Vogel in [2] with skein theory and then used by them to construct [3] a TQFT on a 2+1 cobordism category where the objects and morphisms are equipped with p_1-structures. The question that we consider in this paper is whether this construction based on the WRT-invariant still yields a TQFT when the extra structure is removed from the cobordism category. The answer is no, as the resulting vector space associated to the torus has infinite dimension. See Theorem 2.1. To be more precise: we follow this construction after assigning to each closed 3-manifold the invariant of this 3-manifold equipped with a certain choice of extra structure. Our choice, which seems to us to be the most natural, is described in the next paragraph.

For each integer $p \geq 5$, consider the complex valued invariant, $\langle \ \rangle_p$ of closed oriented 3-manifolds equipped with a p_1-structure defined in [3]. Here we must choose a primitive $2p$th root of unity $A \in \mathbb{C}$, and scalar $\kappa \in \mathbb{C}$ with $\kappa^6 = A^{-6-p(p+1)/2}$. One may remove the dependence on this extra structure by defining $\langle M \rangle'_p = \langle \check{M} \rangle_p$ where $\check{M}$ is M equipped with a p_1-structure with σ-invariant zero. See [3, Appendix B] for the definition of the σ-invariant. If one uses extended manifold structures in lieu of p_1-structures as in [18, 16, 6], one would instead choose $\check{M}$ to have weight zero.

If M is obtained by surgery to S^3 along a framed link L, then

$$\langle M \rangle'_p = \eta \mu^{-\sigma(L)} L(\omega_p).$$

Here we let $\mu = \kappa^3$ and $\sigma(L)$ stands for the signature of the linking matrix of framed link L. Also ω_p is the skein specified in [3, p.898], η is the scalar as given in [3, p.897] and $L(\omega_p)$ is the Kauffman bracket of the cabling of L by ω_p. One can easily extend this definition to the disconnected case by letting $\langle M_1 \sqcup M_2 \rangle'_p = \langle M_1 \rangle'_p \langle M_2 \rangle'_p$.

A quantization functor is a covariant functor (V, Z) from $\mathcal{C}$ to a category of (not necessarily finite dimensional) complex vector spaces. Like a TQFT, it should assign to the empty object the vector space $\mathbb{C}$. A certain naturally defined Hermitian form on $V(\Sigma)$ must also be non-degenerate. One has that $\langle \ \rangle'_p$ is multiplicative and involutive. So we can perform the universal construction described in [3, Prop.

1.1] to construct a quantization functor from the ordinary $(2+1)$-cobordism category (without any extra structure), which we will denote by $\mathcal{C}'$, to the category of complex vector spaces

This is how the universal construction goes (when applied to $\mathcal{C}'$ and $\langle\ \rangle'_p$): Given an object Σ in $\mathcal{C}'$, denote $\mathcal{V}'_p(\Sigma)$ as the vector space spanned by all compact oriented 3-manifolds with boundary Σ (or equivalently all cobordisms $\{M : \emptyset \to \Sigma\}$). There is a hermitian form $\langle\ ,\ \rangle'_\Sigma$ on $\mathcal{V}'_p(\Sigma)$; This is specified on generators by

$$\langle M, N\rangle'_\Sigma = \langle M \cup_\Sigma -N\rangle'_p$$

and then extended sesquilinearly. Let $\mathrm{rad}\langle\ ,\ \rangle'_\Sigma$ denote the radical of the hermitian form $\langle\ ,\ \rangle'_\Sigma$. Define $V'_p(\Sigma)$ to be $\mathcal{V}'_p(\Sigma)/\,\mathrm{rad}\langle\ ,\ \rangle'_\Sigma$. Given a morphism $C : \Sigma_1 \to \Sigma_2$, define $\mathcal{Z}'_{p,C} : \mathcal{V}'_p(\Sigma_1) \to \mathcal{V}'_p(\Sigma_2)$ by assigning $C \cup_{\Sigma_1} N$ to any $N \in \mathcal{V}'_p(\Sigma_1)$ and extending linearly. Note that $\mathcal{Z}'_{p,C}$ send $\mathrm{rad}\langle\ ,\ \rangle'_{\Sigma_1}$ into $\mathrm{rad}\langle\ ,\ \rangle'_{\Sigma_2}$. So it induces a linear map $Z'_{p,C} : V'_p(\Sigma_1) \to V'_p(\Sigma_2)$. Then the quantization functor (V'_p, Z'_p) is the rule assigning $V'_p(\Sigma)$ to Σ and $Z'_{p,C}$ to C.

Let $\mathcal{S}$ be a standard unknotted solid torus in 3-space, and let T^2 denote the boundary of $\mathcal{S}$. Let w_i denote the 3-manifold obtained by doing surgery to $\mathcal{S}$ along i parallel copies of the core of $\mathcal{S}$ with framing $+1$. Let z_j denote the 3-manifold obtained by doing surgery along the core of $\mathcal{S}$ with framing j. We have the following theorem which will be proved in the next section.

Theorem 2.1. *For all $p \geq 5$, $V'_p(T^2)$ is infinite-dimensional. An infinite set of linearly independent elements in $V'_p(T^2)$ can be given by either $\{w_{2pk}\}$ or $\{z_{2pl}\}$, where k and l vary through the positive integers. Hence V'_p is not a TQFT.*

If (V, Z) is a quantization functor resulting from the universal construction, then [3, p.886] there is a natural map

$$t^V_{\Sigma_1,\Sigma_2} : V(\Sigma_1) \otimes V(\Sigma_2) \to V(\Sigma_1 \sqcup \Sigma_2).$$

It is easy to see that this map must be injective. Quinn [13, Prop. 7.2] gave an argument that shows that the finite-dimensionality of $V(\Sigma)$ is implied by the functoriality of V applied to a "snake-shaped" composition of cobordisms built from copies of $\Sigma \times I$ and the assumption that $t^{V}_{\Sigma,\Sigma}$ is an isomorphism. See also Kock [10, Corollary 1.2.28]. This argument shows:

Corollary 2.2. *The natural map $t^{V'_p}_{T^2,T^2}$ is not surjective.*

2.2 Proof of Theorem

We first construct a bilinear form $B_p(\ ,\ )$ on $V'_p(T^2)$, then write out the $n \times n$ truncated matrix associated to $B_p(\ ,\ )$ with respect to $\{w_{2pk}\}$ and $\{z_{2pl}\}$ as k, and l range from 1 to n. We will show there are infinitely many integers n such that the truncated matrix of size $n \times n$ is non-singular. Hence $\{w_{2pk}\}$ and $\{z_{2pl}\}$ are linearly independent and $V'_p(T^2)$ is infinite-dimensional.

2.2.1 Bilinear form on $V'_p(T^2)$

Every closed orientable connected 3-manifold can be obtained by doing Dehn surgery in S^3, see [11]. This result can be used to show the following well-known related fact: every connected orientable 3-manifold with boundary T^2 can be obtained by doing surgery along some framed link L in the solid torus $\mathcal{S}$. We denote the result of this surgery by $\mathcal{S}(L)$.

According to the universal construction, elements in $V'_p(T^2)$ are represented by linear combinations of connected manifolds with boundary T^2. Given two elements w and z in $V'_p(T^2)$ represented by $\mathcal{S}(L_w)$ and $\mathcal{S}(L_z)$, we glue together $\mathcal{S}(L_w)$ to $\mathcal{S}(L_z)$ by the map on their boundary tori which switches the meridian and the longtitude (note this map is orientation reversing). The resulting manifold is obtained by performing surgery on the 3-sphere along a framed link $(L_w, L_z)_{\text{Hopf}}$ obtained by cabling the Hopf link with L_w on one component and L_z on the other

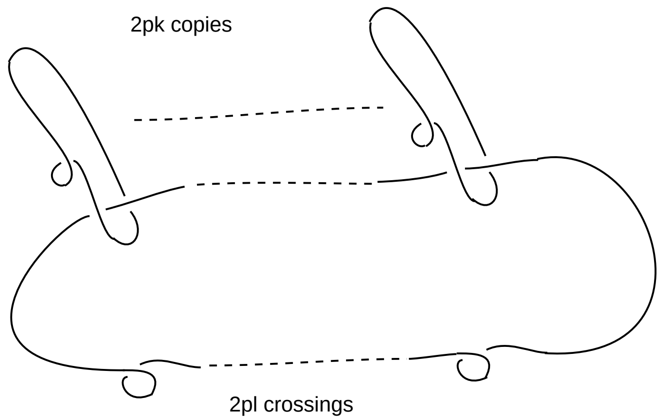

FIGURE 2.1. Framed link $(L_{w_{2pk}}, L_{z_{2pl}})_{\text{Hopf}}$

component. We define

$$B_p(w, z) = \langle S^3((L_w, L_z)_{\text{Hopf}})\rangle'_p \ .$$

This extends to a well-defined symmetric bilinear form $B_p : V'_p(T^2) \times V'_p(T^2) \to \mathbb{C}$, as elements in $\text{rad}\langle\ ,\ \rangle'_{T^2}$ will pair with any other element to give zero.

2.2.2 Truncated square matrices

Note that $B_p(w_{2pk}, z_{2pl})$ is Kauffman bracket of the 3-manifold obtained by doing surgery to S^3 along the framed link pictured in Figure 1. Blowing down [9] the $2pk$ unknotted components with framing $+1$, one by one, and reducing the framing of the single component linked with these, we get an unknot L' with framing $2p(l-k)$. Doing this surgery gives us the lens space $L(2p(l-k), 1)$.

Thus $B(w_{2pk}, z_{2pl}) = \langle L(2p(l-k), 1)\rangle'_p$. We let s abbreviate $l-k$ to simplify our expressions. To compute $\langle L(2ps, 1)\rangle'_p$, we need to compute $U(t^{2ps}\omega_p)$. Here t is the linear map from Kauffman skein module $K(S^1 \times D^2)$ to itself induced by a positive twist and U stands for an unknot with framing zero. According to [2], $t^{2ps}e_i = u_k^{2ps}e_i$, where $u_k = (-A)^{k(k+2)}$ and the e_i are certain generators for $K(S^1 \times D^2)$.

As A is a 2p-th root of unity, and $u_k^{2ps} = 1$, it follows that t^{2ps} is the identity on $K(S^1 \times D^2)$. We obtain

$$\langle L(2ps, 1)\rangle'_p = \eta U(t^{2ps}\omega_p)\mu^{-\sigma(L')} = \eta U(\omega_p)\mu^{-\sigma(L')} = \mu^{-\sigma(L')} \ .$$

This last equation holds as $\eta U(\omega_p) = 1$ [3, p.897]. Alternatively,

$$\eta U(\omega_p) = \langle S^1 \times S^2\rangle'_p = \dim(V_p(S^2)) = 1.$$

Consider the matrix with entries $B(w_{2pk}, z_{2pl})$ as k, l range over the integers from 1 to n (actually any set of n integers in increasing order will do). We will call this a truncated matrix of size n. Then each entry in a truncated matrix just depends on the signature of the linking matrix of L'. Hence every truncated matrix has the form:

$$\begin{bmatrix} 1 & \mu & \mu & \dots & \mu \\ \mu^{-1} & 1 & \mu & \dots & \mu \\ \mu^{-1} & \mu^{-1} & 1 & \dots & \mu \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu^{-1} & \mu^{-1} & \mu^{-1} & \dots & 1 \end{bmatrix} . \tag{*}$$

2.2.3 Determinants of the truncated matrices

We will not try to show that the truncated matrix of every size is non-singular. In fact, the truncated 2×2 matrix has determinant zero. Instead, we show that two truncated matrices of consecutive sizes can not be both singular. Therefore infinitely many non-singular truncated matrices exist.

Consider the $m \times m$ matrix $\mathcal{B}(a,m)$:

$$\begin{bmatrix} a & a-(1-\mu) & a-(1-\mu) & \dots & a-(1-\mu) \\ \mu^{-1} & 1 & \mu & \dots & \mu \\ \mu^{-1} & \mu^{-1} & 1 & \dots & \mu \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu^{-1} & \mu^{-1} & \mu^{-1} & \dots & 1 \end{bmatrix}.$$

Note that $\mu \neq 1$ for $p \geq 5$. We can apply the following elementary row and column operations to $\mathcal{B}(a,m)$: subtract the second column from the first column, add $-\mu^{-1}$ times the first row to the second row, then clearing all but the first entry in the first row, and obtain $(1-\mu) \oplus \mathcal{B}(f(a), m-1)$. Here $f(a) = \mu^{-1}(1-a)$. Applying same operations to the $\mathcal{B}(f(a), m-1)$ part of $(1-\mu) \oplus \mathcal{B}(f(a), m-1)$, we obtain $(1-\mu)I_2 \oplus \mathcal{B}(f^2(a), m-2)$. Repeating this q times for some $q \leq m-1$, we see that $(1-\mu)I_q \oplus \mathcal{B}(f^q(a), m-q)$ is equivalent to $\mathcal{B}(a,m)$. Here we say two matrices are equivalent if they are related by a sequence of determinant preserving elementary row and column operations.

Note that $\mathcal{B}(1,n)$ is exactly the matrix (*) of size n. Following from the above argument, it is clear that $\mathcal{B}(1,n)$ is equivalent to $(1-\mu)I_{n-1} \oplus \mathcal{B}(f^{n-1}(1), 1)$ and that $\mathcal{B}(1,n+1)$ is equivalent to $(1-\mu)I_{n-1} \oplus \mathcal{B}(f^{n-1}(1), 2)$. So both $\det \mathcal{B}(1,n)$ and $\det \mathcal{B}(1,n+1)$ can be written in terms of $f^{n-1}(1)$ as $\det \mathcal{B}(1,n) = (1-\mu)^{n-1} f^{n-1}(1)$ and $\det \mathcal{B}(1,n+1) = (1-\mu)^{n-1}(f^{n-1}(1)(1-\mu^{-1}) + (\mu^{-1}-1))$. Therefore we have

$$\det \mathcal{B}(1,n+1) = \det \mathcal{B}(1,n)(1-\mu^{-1}) + (1-\mu)^{n-1}(\mu^{-1}-1) \ .$$

As a result, as long as $\mu \neq 1$, we can not have two consecutive singular truncated matrices since the term $(1-\mu)^{n-1}(\mu^{-1}-1)$ is non-zero. This completes the proof of Theorem 2.1.

Chapter 3
A Modulo 4 Invariant for Enhanced Cobordisms

3.1 Enhanced Structure on Extended Cobordisms

In this section, we consider a specific extra structure on cobordisms which was first introduced by Walker in [18] and then further developed by Turaev in [16]. The version we are following here is used by Gilmer and Masbaum in [6]. Such an extra structure is called extended manifold structure, and a cobordism with such structure is called an extended cobordism. At the end of this section, we enhance the extended structure via adding orientations on the lagrangians. All the cobordisms mentioned are $2+1$ cobordisms and all the homology considered is with rational coefficient.

A (rational) non-singular symplectic vector space (V, B) is a (rational) vector space with a non-singular skew-symmetric form $B : V \times V \to \mathbb{Q}$. A subspace $\lambda \subset V$ is lagrangian if

$$\lambda = \{x | B(x, y) = 0, \forall y \in \lambda\}.$$

Given an ordered triple of lagrangians $\lambda_1, \lambda_2, \lambda_3$ in (V, B), the associated maslov index $\mu(\lambda_1, \lambda_2, \lambda_3)$ is defined as the signature of the bilinear symmetric form $\langle \cdot, \cdot \rangle$ on $(\lambda_1 + \lambda_2) \cap \lambda_3$ given by $\langle a_1 + a_2, b_1 + b_2 \rangle = B(a_2, b_1)$. (Here $a_i, b_i \in \lambda_i$ for i=1,2, and $a_1 + a_2, b_1 + b_2 \in \lambda_3$).

Given a closed oriented surface Σ, recall that the orientation of Σ induces a non-sigular skew-symmetric intersection form B on $H_1(\Sigma, \mathbb{Q})$. Equipped with this intersection form, $(H_1(\Sigma, \mathbb{Q}), B)$ can be viewed as a (rational) symplectic vector space. An extended surface is a closed oriented surface Σ with a lagrangian $\lambda \subset$

$H_1(\Sigma, \mathbb{Q})$, denoted as a pair (Σ, λ). We will sometimes refer to a lagrangian λ on an oriented surface Σ. We actually mean a lagrangian λ of $H_1(\Sigma, \mathbb{Q})$ with respect to the intersection form.

An extended 3-manifold is a compact oriented manifold M, whose oriented boundary $(\partial M, \lambda(\partial M))$ is an extended surface, together with an integer-valued weight $w(M)$. If ∂M is partitioned into more than one component, the restriction of $\lambda(\partial M)$ on a certain component Σ may or may not be a lagrangian on it. If it is, such component Σ, equipped with the restricted lagrangian $\lambda(\partial M) \cap H_1(\Sigma)$ is called a boundary surface of this extended 3-manifold.

An extended cobordism from extended surface (Σ_1, λ_1) to (Σ_2, λ_2) can be viewed as an extended 3-manifold M whose boundary is partitioned into two components, one is identified with Σ_2 via an orientation preserving differeomorphism, the other is identified with Σ_1 via an orientation reversing differeomorphism. There differeomorphisms are called boundary identifications. Here Σ_1 is referred to as source surface, and Σ_2 is referred to as target surface. We will denote such an extended cobordism as $(M, w(M)) : (\Sigma_1, \lambda_1) \to (\Sigma_2, \lambda_2)$. Two extended cobordisms are equivalent if there is an orientation-preserving differeomorphism which is compatible with their boundary identifications.

Two extended cobordisms can be composed using "extended gluing". To define such an extended gluing, we need to transport the lagrangian on one boundary surface to the other through the cobordism. We transport lagrangians using lagrangian relations as defined by Turaev in [16, p181].

For extended cobordism $(M, w(M)) : (\Sigma_1, \lambda_1) \to (\Sigma_2, \lambda_2)$, there is a lagrangian space $\mathfrak{M}$ in $H_1(\Sigma_1)^- \oplus H_1(\Sigma_2)$ naturally arising from the geometric structure, which is the kernel of the homomorphism $H_1(\Sigma_1)^- \oplus H_1(\Sigma_2) \to H_1(M)$ induced by the inclusion map $\Sigma_1 \sqcup \Sigma_2 \to M$. Here $H_1(\Sigma_1)^-$ denotes symplectic vector space $H_1(\Sigma_1)$

with an opposite symplectic form. We will follow Turaev's notion in [16] and call $\mathfrak{M}$ a lagrangian relation from $H_1(\Sigma_1)$ to $H_1(\Sigma_2)$, denoted as $\mathfrak{M} : H_1(\Sigma_1) \Rightarrow H_1(\Sigma_2)$. We will write an element in $\mathfrak{M}$ as a pair (a, b), where $a \in H_1(\Sigma_1), b \in H_1(\Sigma_2)$. We can see that saying $(a, b) \in \mathfrak{M}$ is equivalent to saying that a and b are homologous in M.

We define

$$\mathfrak{M}_*\lambda_1 = \{y \in H_1(\Sigma_2) | \exists x \in \lambda_1 \ s.t. \ (x, y) \in \mathfrak{M}\}. \tag{3.1}$$

In other words, $\mathfrak{M}_*\lambda_1$ consists of elements y which are homologous to some element $x \in \lambda_1$. From this viewpoint, it is like we transport λ_1 through M to Σ_2. In [16, p181],Turaev proved that $\mathfrak{M}_*\lambda_1$ is a lagrangian on Σ_2. A different proof due to Guillemin and Sternberg can be found in [7, chapter 3]. We can also transport λ_2 on the target surface Σ_2 backwards onto the source surface Σ_1. To be specific, we define it as

$$\mathfrak{M}^*\lambda_2 = \{x \in H_1(\Sigma_1) | \exists y \in \lambda_2 \ s.t. \ (x, y) \in \mathfrak{M}\} \tag{3.2}$$

Now we can define the extended gluing and composition of two extended cobordisms. Let

$$(M_1, \mathfrak{M}_1, w(M_1)) : (\Sigma_1, \lambda_1) \to (\Sigma_2, \lambda_2),$$

and

$$(M_2, \mathfrak{M}_2, w(M_2)) : (\Sigma_2, \lambda_2) \to (\Sigma_3, \lambda_3)$$

be two extended cobordisms. Their composition is an extended cobordism from (Σ_1, λ_1) to (Σ_3, λ_3), whose underlying manifold is obtained by gluing M_1 and M_2, and whose weight is defined as

$$w(M_2 \circ M_1) = w(M_1) + w(M_2) - \mu_{\Sigma_2}(\mathfrak{M}_{1*}\lambda_1, \lambda_2, \mathfrak{M}_2^*\lambda_3). \tag{3.3}$$

Here the subscript Σ_2 means that the maslov index is computed with respect to the intersection form on Σ_2 rather than $-\Sigma_2$.

We remark that the above composition is well-defined on equivalence classes of extended cobordisms, and is associative. Therefore, with the collection of extended surfaces as objects and equivalence classes of extended cobordisms as morphisms, we obtain a category named the extended cobordism category on which WRT TQFT can be defined.

Now we enhance the extended structure via adding orientations to lagrangians. To be specific, an enhanced cobordism $(M, \mathfrak{M}, w(M)) : (\Sigma_1, \lambda_1) \to (\Sigma_2, \lambda_2)$ is an extended cobordism, where $\mathfrak{M}, \lambda_1, \lambda_2$ are endowed with orientations. In following paragraphs, we will sometime abuse our notation and write M for either the underlying cobordism or the whole enhanced cobordism $(M, \mathfrak{M}, w(M))$. We will not specially distinguish oriented lagrangian and unoriented lagrangian in notion. Unless otherwise stated the lagrangians in following sections are oriented. A minus sign in front of an oriented vector space denotes the changing of orientation, that is, V and $-V$ are the vector space V oriented oppositely.

3.2 Guillemin and Sternberg's Machinery

Turaev studied the transportation of unoriented lagrangians induced by unoriented lagrangian relation in [16]. For our purpose, the transportation of oriented lagrangians via oriented lagrangian relations needs to be considered. In this section, we will briefly introduce Guillemin and Sternberg's study of a linear symplectic category in [7, Chapter 3]. In the next section, we apply their machinery to transport oriented lagrangians.

Let V_1 and V_2 be symplectic vector spaces with symplectic forms ω_1 and ω_2, a lagrangian subspace $\Gamma \subset V_1^- \oplus V_2$ is called a linear canonical relation from V_1 to

V_2. Here V_1^- is used to denote the symplectic space with opposite symplectic form $-\omega_1$. We can see that a lagrangian relation $\mathfrak{M} : H_1(\Sigma_1) \Rightarrow H_1(\Sigma_2)$ as defined in the previous section is a linear canonical relation.

Let $\Gamma_1 \subset V_1^- \oplus V_2$ and $\Gamma_2 \subset V_2^- \oplus V_3$ be two linear canonical relations. The composition $\Gamma_2 \circ \Gamma_1$ is a subspace of $V_1^- \oplus V_3$ defined by

$$(x, y) \in \Gamma_2 \circ \Gamma_1 \iff \exists z \in V_2 \ s.t. \ (x, z) \in \Gamma_1, (z, y) \in \Gamma_2.$$

Guillemin and Sternberg proved that $\Gamma_2 \circ \Gamma_1$ is actually a linear canonical relation from V_1 to V_3. Turaev also proved an essentially the same result in studying composition of lagrangian relations. Therefore, with the collection of symplectic vector spaces as objects and linear canonical relations as morphisms, one obtains a category named linear symplectic category.

Guillemin and Sternberg further considered oriented linear canonical relation. Let $\Gamma_1 \subset V_1^- \oplus V_2$ and $\Gamma_2 \subset V_2^- \oplus V_3$ be two linear canonical relations but with orientation. Define

$$\Gamma_2 * \Gamma_1 = \{(x, y, y, z) | (x, y) \in \Gamma_1, (y, z) \in \Gamma_2\}.$$

Let

$$\tau : \Gamma_1 \oplus \Gamma_2 \quad \to \quad V_2$$

be defined by

$$\tau(x_1, y_1, y_2, z_2) \quad = \quad y_1 - y_2.$$

and

$$\alpha : \Gamma_2 * \Gamma_1 \to \Gamma_2 \circ \Gamma_1 \tag{3.4}$$

be defined by

$$\alpha(x, y, y, z) = (x, z). \tag{3.5}$$

One observes that $\Gamma_2 * \Gamma_1$ can be viewed as the kernel of τ and that $\ker\alpha = \{(0,y,y,0)|(0,y) \in \Gamma_1, (y,0) \in \Gamma_2\}$. By abuse of notation, $\ker\alpha$ is also used to denote its isomorphic space in V_2, that is, we will write

$$\ker\alpha = \{y|(0,y) \in \Gamma_1, (y,0) \in \Gamma_2\}.$$

With this notation, Guillemin and Sternberg proved following lemma.

Lemma 3.1 ([7] Eq 3.16). *We have* $\ker\alpha$ *the perpendicular subspace to* $\operatorname{Im}\tau$ *in symplectic space* V_2*.*

We observe that $\ker\alpha \subset \operatorname{Im}\tau$. Thus, lemma 3.1 implies that $\ker\alpha$ is isotropic and $\operatorname{Im}\tau$ is coisotropic. Hence $\operatorname{Im}\tau/\ker\alpha$ can be made a symplectic vector space with symplectic form derived from that on V_2. We have short exact sequence

$$0 \to \ker\alpha \to \operatorname{Im}\tau \to \operatorname{Im}\tau/\ker\alpha \to 0 \tag{3.6}$$

with $\operatorname{Im}\tau/\ker\alpha$ being a symplectic vector space. For any symplectic vector space (V,ω), we associate it with canonical orientation defined by ordered basis

$$\{p_1, q_1, p_2, q_2 \ldots, p_n, q_n\},$$

where

$$\{p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n\}$$

is one of its symplectic basis with $\omega(p_i, q_j) = \delta_{ij}$. This canonical orientation is independent from the choice of symplectic basis. Had we started with a different symplectic basis $\{p'_1, p'_2, \ldots, p'_n, q'_1, q'_2, \ldots, q'_n\}$ with $\omega(p'_i, q'_j) = \delta_{ij}$. It is well-known that there exists a symplectic matrix M serving as the matrix of changing basis, see e.g. [12, 1.1.10]. But any symplectic matrix has determinant $+1$.

Furthermore, we make the following convention which determines the orientation of any vector space in a short exact sequence when the other two vector spaces are oriented. To be specific, for any short exact sequence of vector spaces,

$$0 \to U \xrightarrow{i} V \xrightarrow{p} W \to 0$$

with two of them oriented, we require the orientation of the third vector space to satisfy

$$U \oplus \{lift\ of\ W\} = V. \tag{3.7}$$

Thus, an oriented $\ker \alpha$, together with canonically oriented $\operatorname{Im} \tau / \ker \alpha$, determines an orientation of $\operatorname{Im} \tau$ via short exact sequence 3.6. It is possible that $\ker \alpha = 0$. Then oriented $\operatorname{Im} \tau$ is taken as canonically oriented V_2. It is also possible that $\ker \alpha = \operatorname{Im} \tau$. Then oriented $\operatorname{Im} \tau$ is taken as oriented $\ker \alpha$. In particular, applying convention 3.7 to short exact sequence

$$0 \to \Gamma_1 \to \Gamma_1 \oplus \Gamma_2 \to \Gamma_2 \to 0, \tag{3.8}$$

we obtain a method to oriented $\Gamma_1 \oplus \Gamma_2$ from orientations of Γ_1 and Γ_2.

Guillemin and Sternberg considered following three short exact sequences [7, eq 3.19].

$$\begin{aligned} 0 \to \Gamma_2 * \Gamma_1 \to \Gamma_1 \oplus \Gamma_2 \xrightarrow{\tau} \operatorname{Im} \tau \to 0 \\ 0 \to \ker \alpha \to \Gamma_2 * \Gamma_1 \xrightarrow{\alpha} \Gamma_2 \circ \Gamma_1 \to 0 \\ 0 \to \ker \alpha \to \operatorname{Im} \tau \to \operatorname{Im} \tau / \ker \alpha \to 0 \end{aligned} \tag{3.9}$$

As stated above, an orientation of $\ker \alpha$ together with the canonical orientation of $\operatorname{Im} \tau / \ker \alpha$ determines an orientation of $\operatorname{Im} \tau$. Then the orientation of $\operatorname{Im} \tau$ together with orientation of $\Gamma_1 \oplus \Gamma_2$ determines an orientation of $\Gamma_2 * \Gamma_1$. Then the orientation of $\Gamma_2 * \Gamma_1$ together with that of $\ker \alpha$ determines an orientation of $\Gamma_2 \circ \Gamma_1$. Had one started with the opposite orientation of $\ker \alpha$, then $\operatorname{Im} \tau$ and $\Gamma_2 * \Gamma_1$'s orientation are determined oppositely, but the orientation of $\Gamma_2 \circ \Gamma_1$ remains the same. Thus, this machinery determines the orientation of $\Gamma_2 \circ \Gamma_1$ if Γ_1 and Γ_2 are both oriented.

The case that $\ker \alpha = 0$ is called transverse case. In the transverse case, one just need to consider one short exact sequence

$$0 \to \Gamma_2 * \Gamma_1 \to \Gamma_1 \oplus \Gamma_2 \to V_2 \to 0. \tag{3.10}$$

An orientation of $\Gamma_2 \circ \Gamma_1$ is then determined by $\Gamma_2 \circ \Gamma_1 = \alpha(\Gamma_2 * \Gamma_1)$.

3.3 Transportation of Oriented Lagrangians

In this section we apply Guillemin and Sternberg's machinery to transport oriented lagrangians through oriented lagrangian relations. Then we do three examples to show how to compute such transportations.

To simplify notations in later computation, we introduce following notations of ordered sets. Let

$$\{a_i\}_{i=1,n} = \{a_1, a_2, \ldots, a_{n-1}, a_n\}. \tag{3.11}$$

When we need to deal with cases where more than one subscript are needed, we write

$$\{a_i, b_j\}_{i=1,n;j=1,m} = \{a_1, \ldots, a_n, b_1, \ldots, b_m\}. \tag{3.12}$$

Those abbreviated notations are mainly used to denote ordered bases of a symplectic space. In particular, to represent an ordered symplectic basis of a canonically oriented symplectic space (V, ω), we write

$$\{p_i, q_i\}_{i_{alt}=1,n} = \{p_1, q_1, \ldots, p_n, q_n\}, \tag{3.13}$$

where $\omega(p_i, q_j) = \delta_{ij}$.

Let $(M, \mathfrak{M}, w(M)) : (\Sigma_1, \lambda_1) \to (\Sigma_2, \lambda_2)$ be an arbitrary enhanced cobordism. We identify λ_1 with $0 \oplus \lambda_1$, and then view it as an oriented lagrangian relation from $\{0\}$ to $H_1(\Sigma_1)$. Following Guillemin and Sternberg's machinery, $\mathfrak{M} \circ (0 \oplus \lambda_1)$ is an oriented lagrangian relation from $\{0\}$ to $H_1(\Sigma_2)$, which naturally yields an oriented lagrangian in $H_1(\Sigma_2)$, denoted as $\mathfrak{M}_* \lambda_1$. Without consideration of orientation, we have

$$\mathfrak{M}_* \lambda_1 = \{y \in H_1(\Sigma_2) | \exists x \in \lambda_1 s.t. (x, y) \in \mathfrak{M}\},$$

which agrees with Turaev's definition of transporting lagrangians in equation 3.1.

We can also pull back λ_2 using $\mathfrak{M}$. Let $\{(y_i, x_i)\}_{i=1,\dim \mathfrak{M}}$ be an ordered basis for oriented $\mathfrak{M}$. We define $\mathfrak{M}^s = \{(x_i, y_i)\}_{i=1,\dim \mathfrak{M}}$, then $\mathfrak{M}^s$ is oriented lagrangian relation from $H_1(\Sigma_2)$ to $H_1(\Sigma_1)$. Then we can define the pull back $\mathfrak{M}^* \lambda_2 = \mathfrak{M}^s_* \lambda_2$. Similarly, this definition agrees with Turaev's definition in 3.2.

We conclude this section with three representative examples showing how to compute a transported oriented lagrangian. This kind of computation will be core to our work.

Lemma 3.2. *Let $(M, \mathfrak{M}, w(M)) : (\Sigma_1, \lambda_1) \to (\Sigma_1 \# T^2, \lambda_2)$ be an enhanced cobordism whose underlying cobordism is formed by gluing one 1-handle to thickened surface $\Sigma_1 \times I$ along two 2-disks in $\Sigma_1 \times \{1\}$. Let $\{p_i\}_{i=1,n}$ be an ordered basis for λ_1, we can extend $\{p_i\}$ to an ordered symplectic basis $\{p_i, q_i\}_{i_{alt}=1,n}$, which gives canonical orientation of $H_1(\Sigma_1)$. From the geometric structure of the underlying cobordism, we can see that the lagrangian relation $\mathfrak{M}$, without orientation, can be written as*

$$\mathfrak{M} = \{(p_i, p_i), (q_i, q_i), (0, m)\}_{i_{alt}=1,n}. \tag{3.14}$$

where m is the element in $H_1(\Sigma_1 \# T^2)$ which can be represented by meridian of newly glued 1-handle. With properly choice of m, or rather, with properly choice of *orientation of the meridian representing* m*, we can make equation* 3.14 *hold with orientation. We claim that*

$$\mathfrak{M}_*\lambda_1 = \lambda_1 \oplus m \tag{3.15}$$

as oriented spaces, where the m *is the element which make equation* 3.14 *holds as oriented space.*

Proof. Using Guillemin and Sternberg's construction, we consider short exact sequences 3.9. We can see that $\operatorname{Im}\tau = H_1(\Sigma_1)$, so it suffices to consider the transverse case:

$$0 \to \mathfrak{M} * (0 \oplus \lambda_1) \to (0 \oplus \lambda_1) \oplus \mathfrak{M} \xrightarrow{\tau} H_1(\Sigma_1) \to 0. \tag{3.16}$$

With notations introduced at the beginning of this section, we have that

$$(0 \oplus \lambda_1) \oplus \mathfrak{M} = \{(0, p_i, 0, 0), (0, 0, p_j, p_j), (0, 0, q_j, q_j), (0, 0, 0, m)\}_{i=1,n; j_{alt}=1,n},$$

and that

$$H_1(\Sigma_1) = \{p_j, q_j\}_{j_{alt}=1,n}.$$

Therefore, we can use equation 3.7 to determine the orientation of $\mathfrak{M} * (0 \oplus \lambda_1)$. We first lift $H_1(\Sigma_1)$ through τ up to $(0 \oplus \lambda_1) \oplus \mathfrak{M}$, and get

$$\{A \ lift \ of \ H_1(\Sigma_1)\} = \{(0, 0, -p_i, -p_i), (0, 0, -q_i, -q_i)\}_{i_{alt}=1,n} \tag{3.17}$$

We claim that oriented $\mathfrak{M} * (0 \oplus \lambda_1)$ is determined by

$$\mathfrak{M} * (0 \oplus \lambda_1) = \{(0, p_i, p_i, p_i), (0, 0, 0, m)\}_{i=1,n}. \tag{3.18}$$

To prove this claim, we need to check that

$$\mathfrak{M} * (0 \oplus \lambda_1) \oplus \{A \ lift \ of \ H_1(\Sigma_1)\} = (0 \oplus \lambda_1) \oplus \mathfrak{M}, \tag{3.19}$$

that is to check

$$\begin{aligned}
& \{(0,p_i,p_i,p_i),(0,0,0,m),(0,0,-p_j,-p_j),(0,0,-q_j,-q_j)\}_{i=1,n;j_alt=1,n} \\
= \quad & \{(0,p_i,0,0),(0,0,p_j,p_j),(0,0,q_j,q_j),(0,0,0,m)\}_{i=1,n;j_alt=1,n}. \qquad (3.20)
\end{aligned}$$

Some computation will show that the change of basis matrix has positive determinant. Now recall

$$\alpha : \mathfrak{M} * (0 \oplus \lambda_1) \to \mathfrak{M} \circ (0 \oplus \lambda_1)$$

is defined by

$$\alpha(x, y, y, z) = (x, z).$$

We have that

$$\mathfrak{M} \circ (0 \oplus \lambda_1) = \{(0, p_i), (0, m)\}_{i=1,n} = \lambda_1 \oplus m \qquad (3.21)$$

□

To summarize, in this case, properly orienting meridian of glued 1-handle give an explicit way to describe oriented lagrangian relations and transportation of oriented lagrangian.

The following lemma is very similar to lemma 3.3. We write it out for later citation.

Lemma 3.3. *Let $(N, \mathfrak{N}, w(N)) : (\Sigma_1, \lambda_1) \to (\Sigma_2, \lambda_2)$ be enhanced cobordism whose underlying manifold N is obtained via gluing one 2-handle to a thickened surface $\Sigma_2 \times I$ along a curve l on $\Sigma_1 \times \{1\}$. Let $\{p_i\}_{i=1,n}$ be an ordered basis for λ_2, we can extend $\{p_i\}$ to an ordered symplectic basis $\{p_i, q_i\}_{i_alt=1,n}$, which gives the canonical orientation of $H_1(\Sigma_2)$. From the geometric structure of the underlying cobordism, we can see that the lagrangian relation $\mathfrak{N}$, without orientation, can be written as*

$$\mathfrak{N} = \{(p_i, p_i), (q_i, q_i), (l, 0)\} \qquad (3.22)$$

We can always properly orient l such that equation 3.22 *holds as oriented vector space as well. We claim*

$$\mathfrak{N}^*\lambda_2 = \lambda_2 \oplus (l, 0) \tag{3.23}$$

where l is oriented such that 3.22 *holds as oriented spaces.*

Let $(C(f), \mathfrak{f}, w) : (\Sigma, \lambda) \to (\Sigma, \lambda')$ be an enhanced cobordism, whose underlying manifold $C(f)$ is the mapping cylinder of a mapping class $f : \Sigma \to \Sigma$. Let $\{p_i\}_{i=1,n}$ be an ordered basis of λ_1. Then $\mathfrak{f}_*\lambda$ is defined. Besides, in this case, there is a more direct way to transport oriented lagrangians, using the induced isomorphism $f_* : H_1(\Sigma) \to H_1(\Sigma)$. We define

$$f_*\lambda = \{f(p_i)\}_{i=1,n}. \tag{3.24}$$

In other words, we just transport every element in $\{p_i\}_{i=1,n}$ in order using f_*. The following lemma says $f_*\lambda = \mathfrak{f}_*\lambda$ if $\mathfrak{f}$ is endowed a "natural" orientation.

Lemma 3.4. *Let $(C(f), \mathfrak{f}, w) : (\Sigma, \lambda) \to (\Sigma, \lambda')$ and $\{p_i\}_{i=1,n}$ be as defined above. We can extend $\{p_i\}_{i=1,n}$ to an ordered basis $\{p_i, q_i\}_{i_{alt}=1,n}$, which gives the canonical orientation of $H_1(\Sigma)$. If $\mathfrak{f}$ is oriented as*

$$\mathfrak{f} = \{(p_i, f(p_i)), (q_i, f(q_i))\}_{i_{alt}=1,n}, \tag{3.25}$$

then $\mathfrak{f}_*\lambda = f_*\lambda$.

Proof. We need to check that $\mathfrak{f}_*\lambda = \{f(p_i)\}_{i=1,n}$. This is a transverse case, so we consider short exact sequence

$$0 \to \mathfrak{f} * (0 \oplus \lambda) \to (0 \oplus \lambda) \oplus \mathfrak{f} \to H_1(\Sigma) \to 0.$$

Oriented $H_1(\Sigma)$ is given by $\{p_i, q_i\}_{i_{alt}=1,n}$, and oriented $(0 \oplus \lambda) \oplus \mathfrak{f}$ is given by

$$(0 \oplus \lambda) \oplus \mathfrak{f} = \{(0, p_j, 0, 0), (0, 0, p_i, f(p_i)), (0, 0, q_i, f(q_i))\}_{j=1,n; i_{alt}=1,n}. \tag{3.26}$$

We check that following equation is true.

$$\{A\ lift\ of\ H_1(\Sigma)\} = \{(0,0,-p_i,-f(p_i)),(0,0,-q_i,-f(q_i))\}_{i_{alt}=1,n}.$$

We claim the orientation of $\mathfrak{f} * (0 \oplus \lambda)$ determined by short exact sequence 3.26 is

$$\mathfrak{f} * (0 \oplus \lambda) = \{(0, p_i, p_i, f(p_i))\}_{i=1,n}.$$

One can check this claim by checking the following equation holds with above ordered basis.

$$\mathfrak{f} * (0 \oplus \lambda) \oplus \{A\ lift\ of\ H_1(\Sigma)\} = (0 \oplus \lambda) \oplus \mathfrak{f}.$$

Thus, $\mathfrak{f}_*\lambda = \mathfrak{f} \circ (0 \oplus \lambda) = \alpha(\mathfrak{f} * (0 \oplus \lambda)) = \{(0, f(p_i))\}_{i=1,n} = f_*\lambda.$ □

3.4 Lion and Vergne's s Map

In this section, we briefly introduce several results of Lion and Vergne in [12, section 1.7]. In next section we will use these results to define a modulo 4 invariant for enhanced cobordism.

Let (V, ω) be a symplectic vector space. For an ordered pair of oriented lagrangians (λ_1, λ_2) in V, Lion and Vergne associated with it $\epsilon(\lambda_1, \lambda_2) \in \{-1, 1\}$, which captures the relative orientation between them.

This $\epsilon(\lambda_1, \lambda_2)$ is defined as follows. If λ_1, λ_2 are the same lagrangian with same orientation, $\epsilon(\lambda_1, \lambda_2) = 1$. If λ_1, λ_2 are the same lagrangian with opposite orientation, $\epsilon(\lambda_1, \lambda_2) = -1$. If $\lambda_1 \cap \lambda_2 = 0$, let

$$g_{\lambda_1,\lambda_2} : \lambda_1 \to \lambda_2^* \tag{3.27}$$

be defined by

$$g_{\lambda_1,\lambda_2}(x)(y) = \omega(x, y). \tag{3.28}$$

One can check the kernel of this map is $\lambda_1 \cap \lambda_2$. So $\lambda_1 \cap \lambda_2 = \{0\}$ implies g_{λ_1,λ_2} is invertible. Let $\{a_i\}_{i=1,n}$ be an ordered basis of λ_1, $\{b_i\}_{i=1,n}$ be the ordered basis of λ_2. We define an ordered basis of oriented λ_2^* as $\{b_i^*\}_{i=1,n}$, where $b_i^*(b_j) = \delta_{ij}$. The ij-th entry of the associated matrix of g_{λ_1,λ_2} with respect to these ordered bases is $\omega(a_j, b_i)$. Then we can talk about the determinant of g_{λ_1,λ_2}. One defines $\epsilon(\lambda_1, \lambda_2)$ to be the sign of the determinant of g_{λ_1,λ_2}.

In general, $\lambda_1 \cap \lambda_2 \neq \{0\}$. First denote $\lambda_1 \cap \lambda_2$ by ρ to abbreviate the notation. We can see that ρ is isotropic, and hence $\rho^\perp/\rho$ can be made a symplectic vector space with λ_1/ρ, λ_2/ρ being lagrangians. Choosing an orientation of ρ, we can make use of the short exact sequence:

$$0 \to \rho \to \lambda_i \to \lambda_i/\rho \to 0. \tag{3.29}$$

to determine an orientation of $\lambda_i/\rho, i = 1, 2$. We check that λ_1/ρ and λ_2/ρ intersect trivially. So $\epsilon(\lambda_1/\rho, \lambda_2/\rho)$ is defined. And we define $\epsilon(\lambda_1, \lambda_2) = \epsilon(\lambda_1/\rho, \lambda_2/\rho)$. Here the choice of orientation of ρ is not important. Had we started with opposite orientation of ρ, then both orientations on λ_i/ρ determined by short exact sequence 3.29 would be reversed. Then $\epsilon(\lambda_1, \lambda_2)$ will be defined as $\epsilon(-\lambda_1/\rho, -\lambda_2/\rho)$, but $\epsilon(-\lambda_1/\rho, -\lambda_2/\rho) = \epsilon(\lambda_1/\rho, \lambda_2/\rho)$.

We list the following useful lemmas.

Lemma 3.5 ([12],1.7.3)**.** *Denote n as half of the dimension of the symplectic vector space V, then $\epsilon(\lambda_1, \lambda_2) = (-1)^{n-\dim(\lambda_1\cap\lambda_2)}\epsilon(\lambda_2, \lambda_1)$.*

Lemma 3.6. $\epsilon(\lambda_1, -\lambda_2) = \epsilon(-\lambda_1, \lambda_2) = -\epsilon(\lambda_2, \lambda_1)$.

Lemma 3.7. *Let $(V, \omega), (V', \omega')$ be two symplectic vector spaces, $\lambda_i \subset V$ $(i = 1, 2)$ and $\lambda_i' \subset V'$ $(i = 1, 2)$ being oriented lagrangians, then*

$$\epsilon(\lambda_1 \oplus \lambda_1', \lambda_2 \oplus \lambda_2') = \epsilon(\lambda_1, \lambda_2)\epsilon(\lambda_1', \lambda_2')$$

Proof. Let $\rho = \lambda_1 \cap \lambda_2$ and $\rho' = \lambda_1' \cap \lambda_2'$. We first consider the special case where $\rho = \{0\}$ and $\rho' = \{0\}$. In this special case, we have $(\lambda_1 \oplus \lambda_1') \cap (\lambda_2 \oplus \lambda_2') = \{0\}$ as well. Let $\{a_i\}_{i=1,n}$,$\{b_i\}_{i=1,n}$ be ordered bases of λ_1 and λ_2, and $\{a_i\}_{i=n+1,n+n'}$,$\{b_i\}_{i=n+1,n+n'}$ be ordered bases of λ_1' and λ_2'. Let G be the matrix of $g_{\lambda_1\oplus\lambda_2,\lambda'\oplus\lambda_2'}$, G_1 be the matrix of g_{λ_1,λ_2} and G_2 be the matrix of $g_{\lambda_1',\lambda_2'}$ with respect to above bases. The ij-th entry of G is given by $G_{ij} = (\omega + \omega')(a_j, b_i) = \omega(a_j, b_i)$ if $i, j \leq n$, and $G_{ij} = \omega'(a_j, b_i)$ if $i, j > n$. Hence

$$\det G = \det G_1 \det G_2,$$

and our conclusion follows immediately.

Now we turn to prove the general case. Arbitrarily choosing orientations of ρ, ρ', considering short exact sequences 3.29, we obtain oriented lagrangians $\lambda_1/\rho, \lambda_2/\rho$ in $\rho^\perp/\rho$ and oriented lagrangians $\lambda_1'/\rho', \lambda_2'/\rho'$ in $\rho'^\perp/\rho'$. By definition, we have

$$\epsilon(\lambda_1, \lambda_2)\epsilon(\lambda_1', \lambda_2') = \epsilon(\lambda_1/\rho, \lambda_2/\rho)\epsilon(\lambda_1'/\rho', \lambda_2'/\rho')$$

Here $(\lambda_1/\rho) \cap (\lambda_2/\rho)$ and $(\lambda_1'/\rho') \cap (\lambda_2'/\rho')$ are both trivial, so we can apply the conclusion of the special case and obtain

$$\epsilon(\lambda_1/\rho, \lambda_2/\rho)\epsilon(\lambda_1'/\rho', \lambda_2'/\rho') = \epsilon(\lambda_1/\rho \oplus \lambda_1'/\rho', \lambda_2/\rho \oplus \lambda_2'/\rho').$$

On the other hand, we have $(\lambda_1 \oplus \lambda_1') \cap (\lambda_2 \oplus \lambda_2') = \rho \oplus \rho'$. Orientations of ρ and ρ' determine an orientation of $\rho \oplus \rho'$. Considering short exact sequence 3.29, we obtain oriented lagrangians $(\lambda_1 \oplus \lambda_1')/(\rho \oplus \rho')$ and $(\lambda_2 \oplus \lambda_2')/(\rho \oplus \rho')$ in symplectic space $(\rho \oplus \rho')^\perp/(\rho \oplus \rho')$. So

$$\begin{aligned}
\epsilon(\lambda_1 \oplus \lambda_1', \lambda_2 \oplus \lambda_2') &= \epsilon((\lambda_1 \oplus \lambda_1')/(\rho \oplus \rho'), (\lambda_2 \oplus \lambda_2')/(\rho \oplus \rho')) \\
&= \epsilon(\lambda_1/\rho \oplus \lambda_1'/\rho', \lambda_2/\rho \oplus \lambda_2'/\rho') \\
&= \epsilon(\lambda_1, \lambda_2)\epsilon(\lambda_1', \lambda_2').
\end{aligned}$$

The proof is complete. ∎

Based on the definition of ϵ map, Lion and Vergne defined the s map for an ordered pair of oriented lagrangians (λ_1, λ_2) in symplectic vector space V as

$$s(\lambda_1, \lambda_2) = i^{n-\dim(\lambda_1 \cap \lambda_2)} \epsilon(\lambda_1, \lambda_2). \tag{3.30}$$

Here n is half of the dimension of V.

It is immediate that we have following lemma.

Lemma 3.8 ([12],1.7.4)**.** $s(\lambda_1, \lambda_2)s(\lambda_2, \lambda_1) = 1$.

The following lemma is due to Lion and Vergne as well.

Lemma 3.9 ([12],1.7.5)**.** *Let $f : V \to V$ be a symplectic isomorphism, then*

$$s(f(\lambda_1), f(\lambda_2)) = s(\lambda_1, \lambda_2).$$

Here $f(\lambda_i), i = 1, 2$ is defined in the sense of equation 3.24.

3.5 Enhanced Heegaard Decompositions

In this section, we are going to make use of Heegaard splittings of compact 3-manifolds to define so-called enhanced Heegaard decompositions of an enhanced cobordisms. We will use such decomposition to derive the definition of our modulo 4 invariant of enhanced cobordism.

We briefly recall the definition of a Heegaard splittings of a compact manifolds to set up our notations. A compression body H is obtained from a closed surface Σ by gluing 1-handles to $\Sigma \times I$ along two 2-disks on $\Sigma \times \{1\}$. The negative boundary of H, denoted as $\partial_- H$, is defined as $\Sigma \times \{0\}$. The positive boundary of H, denoted as $\partial_+ H$, is defined as $\partial H - \partial_- H$. We also consider a handlebody to be a compression body whose negative boundary is empty. A trivial compression body is just a thickened surface $\Sigma \times I$. It is well-known that every compact 3-manifold M has a Heegaard splitting with two compression bodies. To be precise, there are two compression

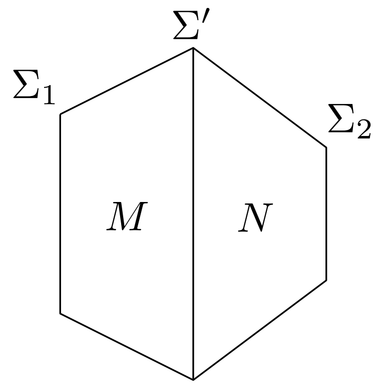

FIGURE 3.1. Enhanced Heegaard Decomposition of C

bodies H_1, H_2, one closed surface $S \in M$, two homeomorphisms $f_1 : \partial_+ H_1 \to S$ and $f_2 : S \to \partial_+ H_2$ such that gluing H_i to S via f_i yields M. We denote this Heegaard splitting as $M = H_1 \cup_S H_2$ and call S as the Heegaard splitting surface.

Given enhanced cobordism $(C, \mathfrak{C}, w(C)) : (\Sigma_1, \lambda_1) \to (\Sigma_2, \lambda_2)$, its underlying compact manifold C has a Heegaard splitting $M \cup_{\Sigma'} N$, shown as in figure 1. To complete the definition of enhanced Heegaard decomposition, we want to properly decompose the oriented lagrangian relation $\mathfrak{C}$ as well.

We can view M, N as cobordisms. As cobordisms, M and N naturally carry unoriented lagrangian relations $\mathfrak{M}$ and $\mathfrak{N}$. According to Turaev [16, p181], $\mathfrak{N} \circ \mathfrak{M}$ is a lagrangian relation from $H_1(\Sigma_1)$ to $H_1(\Sigma_2)$. We can verify that $\mathfrak{N} \circ \mathfrak{M} \subset \mathfrak{C}$. Counting their dimensions, we can see they are actually the same as unoriented vector spaces.

A natural question at this point is, could we always select proper orientations on $\mathfrak{M}, \mathfrak{N}$ such that $\mathfrak{N} \circ \mathfrak{M} = \mathfrak{C}$ as oriented lagrangian relations? This is answered by following lemma.

Lemma 3.10. *Given two $\mathfrak{M}, \mathfrak{N}$ oriented lagrangian relations, reversing orientation of one of them will reverse the orientation of the composition, that is, $\mathfrak{M} \circ (-\mathfrak{N}) = (-\mathfrak{M}) \circ \mathfrak{N} = -(\mathfrak{M} \circ \mathfrak{N})$.*

Proof. Consider Guillemin and Sternberg's three short exact sequences,

$$0 \to \mathfrak{N} * \mathfrak{M} \to \mathfrak{M} \oplus \mathfrak{N} \to \operatorname{Im} \tau \to 0$$

$$0 \to \ker \alpha \to \mathfrak{N} * \mathfrak{M} \to \mathfrak{N} \circ \mathfrak{M} \to 0$$

$$0 \to \ker \alpha \to \operatorname{Im} \tau \to \operatorname{Im} \tau / \ker \alpha \to 0$$

Without loss of generality, we consider the reversion of orientation of $\mathfrak{N}$. This change doesn't have effect on orientation of $\ker \alpha$, $\operatorname{Im} \tau / \ker \alpha$ and hence on that of $\operatorname{Im} \tau$. But it does reverse the orientation of the direct sum, that is, $\mathfrak{M} \oplus (-\mathfrak{N}) = -(\mathfrak{M} \oplus \mathfrak{N})$. Hence the orientation of the $*$ part is reversed, $(-\mathfrak{N}) * \mathfrak{M} = -(\mathfrak{N} * \mathfrak{M})$. Therefore, the orientation of the composition is reversed. □

An immediate consequence of lemma 3.10 is that $\mathfrak{N} \circ \mathfrak{M} = \mathfrak{C}$ as oriented lagrangian relations can always be done with properly selected oriented $\mathfrak{M}, \mathfrak{N}$.

Now we define an enhanced Heegaard decomposition. For any enhanced cobordism

$$(C, \mathfrak{C}, w(C)) : (\Sigma_1, \lambda_1) \to (\Sigma_2, \lambda_2),$$

an enhanced Heegaard decomposition of it is given by two enhanced cobordisms

$$(M, \mathfrak{M}, w(M)) : (\Sigma_1, \lambda_1) \to (\Sigma', \lambda') \tag{3.31}$$

and

$$(N, \mathfrak{N}, w(N)) : (\Sigma', \lambda') \to (\Sigma_2, \lambda_2) \tag{3.32}$$

satisfying: (1) $M \cup_{\Sigma'} N$ is Heegaard splitting of C; (2) $\mathfrak{C} = \mathfrak{N} \circ \mathfrak{M}$ as oriented lagrangian relations; (3) $w(C) = w(M_1) + w(M_2) - \mu_{\Sigma_2}(\mathfrak{M}_{1*}\lambda_1, \lambda_2, \mathfrak{M}^* \lambda_3)$.

The key property of such decomposition is that it allows us to transport oriented lagrangians of boundary surfaces to a middle splitting surface through compression bodies. In such a transportation, no information gets lost. For future convenience, we introduce following terminology. Enhanced cobordism M, as described in 3.31, is called increasing cobordism, since from Σ_1 to Σ', the genus increases. Similarly, enhanced cobordism N, as described in 3.32, is called decreasing cobordism. Readers are warned that not every cobordism M with the property that its target surface has a larger genus than its source surface is increasing cobordism. The underlying manifold of M must be a compression body. Similar warning for decreasing cobordism exists. We conclude this section by pointing out that enhanced Heegaard decomposition is far from being unique. In fact, we can easily found two different enhanced Heegaard decomposition with the same underlying ordinary Heegaard splitting.

3.6 Definition of a Modulo 4 Invariant j(C)

Given an enhanced cobordism $(C, \mathfrak{C}, w(C)) : (\Sigma_1, \lambda_1) \to (\Sigma_3, \lambda_3)$, we associate it with $j(C) \in \mathbb{Z}_4$ as following. Let $(M, \mathfrak{M}, w(M)) : (\Sigma_1, \lambda_1) \to (\Sigma_2, \lambda_2)$, $(N, \mathfrak{N}, w(N)) : (\Sigma_2, \lambda_2) \to (\Sigma_3, \lambda_3)$, be an enhanced Heegaard decomposition of C, we define $j(C)$ by asking it to satisfy

$$i^{j(C)} = i^{k^2} s_{\Sigma_2}(\mathfrak{M}_* \lambda_1, \mathfrak{N}^* \lambda_3). \tag{3.33}$$

Here k is the number of 2-handles involved in the decomposition. The subscript Σ_2 says that s map is defined on symplectic space $H_1(\Sigma_2)$ with the symplectic (intersection) form induced by oriented surface Σ_2. We recall that Σ_2, as the target surface of M and source surface of N, is endowed with a natural orientation. One observes that $w(M), w(N), w(C), \lambda_2$ don't play a direct role in this definition. We will frequently drop them out when refer to enhanced Heegaard decomposition in this section. We will also often drop out the subscript Σ_2 when there is no confusion.

Since an enhanced Heegaard decomposition is not unique, we need to check that $j(C)$ is well-defined, that is, if some $j(C)$ make equation 3.33 hold for one enhanced Heegaard decomposition, it also works for all other enhanced Heegaard decompositions.

We start from the case where orientations of lagrangian relations are reversed. To be specific, let $(C, \mathfrak{C})$ be an enhanced cobordism, and $(M, \mathfrak{M}), (N, \mathfrak{N})$ an enhanced Heegaard decomposition of C. From lemma 3.10, one know that $(M, -\mathfrak{M}), (N, -\mathfrak{N})$ is also an enhanced Heegaard decomposition. Let $j(C)$ is defined using $(M, \mathfrak{M}), (N, \mathfrak{N})$. We verify that

$$i^{j(C)} = i^{k^2} s(-\mathfrak{M}_* \lambda_1, -\mathfrak{N}^* \lambda_3)$$

by observing that

$$i^{k^2} s(-\mathfrak{M}_* \lambda_1, -\mathfrak{N}^* \lambda_3) = i^{k^2} s(\mathfrak{M}_* \lambda_1, \mathfrak{N}^* \lambda_3) = i^{j(C)} \tag{3.34}$$

Here first equation follows from lemma 3.10 and the definition of transportation of oriented lagrangian.

Therefore, if two enhanced Heegaard decompositions share the same underlying Heegaard splitting, then $j(C)$ calculated from one decomposition is the same as the $j(C)$ calculated from the other. From now on, we need to focus on cases where the Heegaard splittings of the underlying manifold are different. We start from following elementary case.

Given a Heegaard splitting $H_1 \cup_S H_2$ of a 3-manifold M, it is well-known that we can obtain a different Heegaard splitting by "inserting one cancelling pair". We take following description of "inserting one cancelling pair" from Martin Scharlemann's survey paper [15]. First, let α be a nicely embedded arc in H_2. By "nicely embedded" we mean there exists an embedded disk D whose boundary is the union of α and an arc in Heegaard splitting surface S. Then we add a neighbourhood

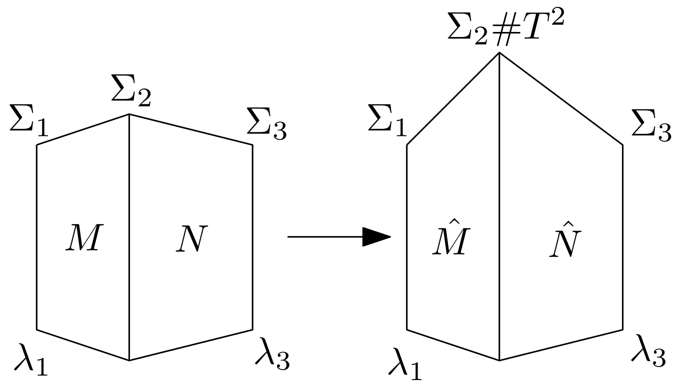

FIGURE 3.2. Stabilization

of α to H_1 and delete it from H_2. This adds a 1-handle to H_1, whose core is α, and adds a 1-handle to H_2, whose cocore is part of D. Thus we obtain a different Heegaard splitting of M, denoted as $\hat{H}_1 \cup_{S\#T^2} \hat{H}_2$. This procedure of "inserting one cancelling pair" is more formally referred to as a "stabilization".

Similarly, we can generalize "stabilization" to enhanced Heegaard decomposition. Let $(M, \mathfrak{M})$,$(N, \mathfrak{N})$ be an enhanced Heegaard decomposition of $(C, \mathfrak{C})$, which means that $M \cup_{\Sigma_2} N$ is a Heegaard splitting of C and that $\mathfrak{N} \circ \mathfrak{M} = \mathfrak{C}$. Now we can stabilize $M \cup_{\Sigma_2} N$ into $\hat{M} \cup_{\Sigma_2\#T^2} \hat{N}$. Viewed as cobordisms, $\hat{M}$ and $\hat{N}$ naturally carry unoriented lagrangian relations $\hat{\mathfrak{M}}$ and $\hat{\mathfrak{N}}$. We need to carefully choose orientations of $\hat{\mathfrak{M}}$ and $\hat{\mathfrak{N}}$ so that $\hat{\mathfrak{N}} \circ \hat{\mathfrak{M}} = \mathfrak{N} \circ \mathfrak{M} = \mathfrak{C}$. Then $(\hat{M}, \hat{\mathfrak{M}}), (\hat{N}, \hat{\mathfrak{N}})$ is an enhanced Heegaard decomposition of $(C, \mathfrak{C})$. We call such procedure "enhanced stabilization", which actually means obtaining an enhanced Heegaard decomposition whose underlying Heegaard splitting is obtained by stabilization.

As stated above, oriented $\hat{\mathfrak{M}}, \hat{\mathfrak{N}}$ must be carefully chosen to guarantee that

$$\hat{\mathfrak{N}} \circ \hat{\mathfrak{M}} = \mathfrak{N} \circ \mathfrak{M} = \mathfrak{C} \tag{3.35}$$

In next few paragraphs we will explore how to choose proper orientations of $\hat{\mathfrak{M}}$ and $\hat{\mathfrak{N}}$.

Recall $\hat{M}$ is obtained from M via gluing a 1-handle. Without consideration of orientation, we can see that all the elements in $\mathfrak{M}$ are still in $\hat{\mathfrak{M}}$. (Recall that $(a, b) \in \mathfrak{M}$ means a and b are homologous in M. Gluing a 1-handle will not make them non-homologous). Let m be a meridian of this newly glued 1-handle, we can see it is homologous to $0 \in H_1(\Sigma_1)$. Hence $(0, m) \in \hat{\mathfrak{M}}$. Thus, we have that $\mathfrak{M} \oplus (0, m) \subset \hat{\mathfrak{M}}$. Counting the dimension, we can see that they are actually the same as unoriented vector spaces, that is,

$$\mathfrak{M} \oplus (0, m) = \hat{\mathfrak{M}}. \tag{3.36}$$

Therefore, an orientation of m, together with the orientation of $\mathfrak{M}$, determines an orientation of $\hat{\mathfrak{M}}$. From the description of stabilization, $\hat{N}$ is obtained via gluing one 2-handle along the longitude l of newly glued 1-handle of $\hat{M}$ to cancel it. Similarly we have that

$$\hat{\mathfrak{N}} = \mathfrak{N} \oplus (l, 0) \tag{3.37}$$

as unoriented vector spaces. An orientation of l, together with the orientation of $\mathfrak{N}$, determines an orientation of $\hat{\mathfrak{N}}$. The following lemma 3.11 states that, in order to make 3.35 hold, m and l must be oriented such that $m \cdot l = (-1)^{\dim \hat{N}}$. From now on, we use $\cdot$ to denote the intersection form of an oriented surface.

For convenience of computation, in the statement and proof of lemma 3.11, we will orient m and l such that $m \cdot l = 1$ and let $\hat{\mathfrak{N}} = \mathfrak{N} \oplus (\varepsilon l, 0)$ as oriented lagrangian relations. The final conclusion will be about ε.

Lemma 3.11. *Let* $\hat{\mathfrak{M}} = \mathfrak{M} \oplus (0, m)$*,* $\hat{\mathfrak{N}} = \mathfrak{N} \oplus (\varepsilon l, 0)$ *and* $m \cdot l = 1$*, then* $\mathfrak{N} \circ \mathfrak{M} = \hat{\mathfrak{N}} \circ \hat{\mathfrak{M}}$ *is equivalent to* $\varepsilon = (-1)^{\dim \hat{N}}$.

Proof. Using Guillemin and Sternberg's construction, to determine the orientation of $\mathfrak{N} \circ \mathfrak{M}$, we need to consider short exact sequences:

$$0 \to \mathfrak{N} * \mathfrak{M} \to \mathfrak{M} \oplus \mathfrak{N} \to \operatorname{Im} \tau \to 0 \tag{3.38}$$

$$0 \to \ker \alpha \to \mathfrak{N} * \mathfrak{M} \to \mathfrak{N} \circ \mathfrak{M} \to 0 \tag{3.39}$$

$$0 \to \ker \alpha \to \operatorname{Im} \tau \to \operatorname{Im} \tau / \ker \alpha \to 0 \tag{3.40}$$

Similarly, to determine the orientation of $\hat{\mathfrak{N}} \circ \hat{\mathfrak{M}}$, we need to consider short exact sequences:

$$0 \to \hat{\mathfrak{N}} * \hat{\mathfrak{M}} \to \hat{\mathfrak{M}} \oplus \hat{\mathfrak{N}} \to \operatorname{Im} \hat{\tau} \to 0 \tag{3.41}$$

$$0 \to \ker \hat{\alpha} \to \hat{\mathfrak{N}} * \hat{\mathfrak{M}} \to \hat{\mathfrak{N}} \circ \hat{\mathfrak{M}} \to 0 \tag{3.42}$$

$$0 \to \ker \hat{\alpha} \to \operatorname{Im} \hat{\tau} \to \operatorname{Im} \hat{\tau} / \ker \hat{\alpha} \to 0 \tag{3.43}$$

Here we introduce $\hat{\alpha}$ and $\hat{\tau}$ just to make it clear that they are not exactly the same maps as α and τ.

We observe that $\mathfrak{N} * \mathfrak{M} = \hat{\mathfrak{N}} * \hat{\mathfrak{M}}$ as unoriented vector spaces. This is because, $(x, y, y, z) \in \mathfrak{N} * \mathfrak{M}$ if and only if $(x, y) \in \mathfrak{M}$ and $(y, z) \in \mathfrak{N}$. But $\hat{\mathfrak{M}} = \mathfrak{M} \oplus (0, m)$ and $\hat{\mathfrak{N}} = \mathfrak{N} \oplus (\varepsilon l, 0)$, so $(x, y) \in \hat{\mathfrak{M}}$ and $(y, z) \in \hat{\mathfrak{N}}$ as well. Thus $(x, y, y, z) \in \hat{\mathfrak{N}} * \hat{\mathfrak{M}}$. If $(x, y, y, z) \in \hat{\mathfrak{N}} * \hat{\mathfrak{M}}$, then we can break $y \in H_1(\Sigma_2 \# T^2)$ into $y_1 + y_2$ where $y_1 \in H_1(\Sigma_2), y_2 \in \operatorname{span}\{m, l\}$. However, $(x, y) \in \hat{\mathfrak{M}}$ implies that $y_2 \in \operatorname{span}\{m\}$, while $(y, z) \in \hat{\mathfrak{N}}$ implies that $y_2 \in \operatorname{span}\{l\}$. So $y_2 = 0$ and hence $(x, y, y, z) \in \mathfrak{N} * \mathfrak{M}$. So we can conclude that $\mathfrak{N} * \mathfrak{M} = \hat{\mathfrak{N}} * \hat{\mathfrak{M}}$.

Second, we observe that $\ker \alpha = \ker \hat{\alpha}$ as unoriented space. Knowing that $\mathfrak{N} * \mathfrak{M} = \hat{\mathfrak{N}} * \hat{\mathfrak{M}}$ and $\mathfrak{N} \circ \mathfrak{M} = \hat{\mathfrak{N}} \circ \hat{\mathfrak{M}}$ as unoriented vector spaces, considering short exact sequences 3.39 and 3.42, we can conclude that $\alpha = \hat{\alpha}$ and hence $\ker \alpha = \ker \hat{\alpha}$.

Third, from lemma 3.1 we have that $(\ker\alpha)^{\perp} = \operatorname{Im}\tau$ in $H_1(\Sigma_2)$ and that $(\ker\hat{\alpha})^{\perp} = \operatorname{Im}\hat{\tau}$ in $H_1(\Sigma_2 \# T^2) = H_1(\Sigma_2) \oplus m \oplus l$. Since $\ker\alpha = \ker\hat{\alpha}$, we conclude that $\operatorname{Im}\hat{\tau} = \operatorname{Im}\tau \oplus m \oplus l$ as unoriented vector space.

Recall that Guillemin and Sternberg's construction allows orientations of $\ker\alpha$ and $\ker\hat{\alpha}$ to be freely chosen. We choose them to be of the same orientation for convenience in computation. Let $[m], [l]$ be the image of m, l of the quotient map from $\operatorname{Im}\hat{\tau}$ to $\operatorname{Im}\hat{\tau}/\ker\hat{\alpha}$. We have $[m]\cdot[l] = m \cdot l = 1$. Therefore, if $\operatorname{Im}\tau/\ker\alpha$ is canonically oriented, then

$$\operatorname{Im}\hat{\tau}/\ker\hat{\alpha} = \operatorname{Im}\tau/\ker\alpha \oplus [m] \oplus [l] \tag{3.44}$$

is also canonically oriented.

Let $\operatorname{Im}\tau/\ker\alpha$ and $\operatorname{Im}\hat{\tau}/\ker\hat{\alpha}$ be canonically oriented, then short exact sequences 3.40 and 3.43 determine orientations on $\operatorname{Im}\tau$ and $\operatorname{Im}\hat{\tau}$ as

$$\operatorname{Im}\tau \quad = \quad \ker\alpha \oplus \{A\ lift\ of\ \operatorname{Im}\tau/\ker\alpha\}, \tag{3.45}$$

and

$$\operatorname{Im}\hat{\tau} \quad = \quad \ker\hat{\alpha} \oplus \{A\ lift\ of\ \operatorname{Im}\hat{\tau}/\ker\hat{\alpha}\}. \tag{3.46}$$

Actually, equation 3.44 says we can choose

$$\{A\ lift\ of\ \operatorname{Im}\hat{\tau}/\ker\hat{\alpha}\} = \{A\ lift\ of\ \operatorname{Im}\tau/\ker\alpha\} \oplus m \oplus l \tag{3.47}$$

because $[m], [l]$ can be lifted to m, l. Combine equations 3.44,3.45 and 3.47, we have that

$$\operatorname{Im}\hat{\tau} = \operatorname{Im}\tau \oplus m \oplus l. \tag{3.48}$$

as oriented vector spaces.

By the short exact sequences 3.38 and 3.43, orientations on $\mathfrak{N} * \mathfrak{M}$ and $\hat{\mathfrak{N}} * \hat{\mathfrak{M}}$ are determined by following two equations.

$$\begin{aligned} \mathfrak{M} \oplus \mathfrak{N} &= \mathfrak{N} * \mathfrak{M} \oplus \{A\ lift\ of\ \ \mathrm{Im}\,\tau\}, & (3.49)\\ \hat{\mathfrak{M}} \oplus \hat{\mathfrak{N}} &= \hat{\mathfrak{N}} * \hat{\mathfrak{M}} \oplus \{A\ lift\ of\ \ \mathrm{Im}\,\hat{\tau}\}. & (3.50)\end{aligned}$$

From equation 3.48, we can choose $\{A\ lift\ of\ \ \mathrm{Im}\,\hat{\tau}\}$ to satisfy

$$\{A\ lift\ of\ \ \mathrm{Im}\,\hat{\tau}\} = \{A\ lift\ of\ \ \mathrm{Im}\,\tau\} \oplus (0, m, 0, 0) \oplus (0, 0, -l, 0) \tag{3.51}$$

because m, l can be lifted to $(0, m, 0, 0), (0, 0, -l, 0)$.

Recall that $\hat{\mathfrak{M}} = \mathfrak{M} \oplus (0, m)$ and $\hat{\mathfrak{N}} = \mathfrak{N} \oplus (\varepsilon l, 0)$. We have

$$\begin{aligned} \hat{\mathfrak{M}} \oplus \hat{\mathfrak{N}} &= \mathfrak{M} \oplus (0, m, 0, 0) \oplus \mathfrak{N} \oplus (0, 0, \varepsilon l, 0)\\ &= (-1)^{\dim \mathfrak{N}} \mathfrak{M} \oplus \mathfrak{N} \oplus (0, m, 0, 0) \oplus (0, 0, \varepsilon l, 0). & (3.52)\end{aligned}$$

as oriented vector spaces. Combining above four equations, we have

$$\begin{aligned} & \hat{\mathfrak{N}} * \hat{\mathfrak{M}} \oplus \{A\ lift\ of\ \ \mathrm{Im}\,\tau\} \oplus (0, m, 0, 0) \oplus (0, 0, -l, 0)\\ =\ & \hat{\mathfrak{N}} * \hat{\mathfrak{M}} \oplus \{A\ lift\ of\ \ \mathrm{Im}\,\hat{\tau}\}\\ =\ & \hat{\mathfrak{M}} \oplus \hat{\mathfrak{N}}\\ =\ & (-1)^{\dim \mathfrak{N}} \mathfrak{M} \oplus \mathfrak{N} \oplus (0, m, 0, 0) \oplus (0, 0, \varepsilon l, 0)\\ =\ & (-1)^{\dim \mathfrak{N}} \mathfrak{N} * \mathfrak{M} \oplus \{A\ lift\ of\ \ \mathrm{Im}\,\tau\} \oplus (0, m, 0, 0) \oplus (0, 0, \varepsilon l, 0). & (3.53)\end{aligned}$$

Comparing the beginning and the end of equation 3.53, we have

$$\hat{\mathfrak{N}} * \hat{\mathfrak{M}} = ((-1)^{\dim \mathfrak{N}+1} \varepsilon) \mathfrak{N} * \mathfrak{M} \tag{3.54}$$

as oriented vector spaces.

Now we use equation 3.39 and 3.42 to determine orientations on $\mathfrak{N} \circ \mathfrak{M}$ and $\hat{\mathfrak{N}} \circ \hat{\mathfrak{M}}$. We have following two equations.

$$\begin{aligned} \mathfrak{N} * \mathfrak{M} &= \ker \alpha \oplus \{A\ lift\ of\ \mathfrak{N} \circ \mathfrak{M}\}, & (3.55)\\ \hat{\mathfrak{N}} * \hat{\mathfrak{M}} &= \ker \hat{\alpha} \oplus \{A\ lift\ of\ \hat{\mathfrak{N}} \circ \hat{\mathfrak{M}}\}. & (3.56)\end{aligned}$$

Recall that we have chosen $\ker\alpha = \ker\hat{\alpha}$ as oriented vector spaces. From equation 3.54, we have

$$\{A\ lift\ of\ \hat{\mathfrak{N}} \circ \hat{\mathfrak{M}}\} = (-1)^{\dim \mathfrak{N}+1}\varepsilon\{A\ lift\ of\ \mathfrak{N} \circ \mathfrak{M}\}. \tag{3.57}$$

Thus

$$\hat{\mathfrak{N}} \circ \hat{\mathfrak{M}} = (-1)^{\dim \mathfrak{N}+1}\varepsilon(\mathfrak{N} \circ \mathfrak{M}). \tag{3.58}$$

Observing that $\dim \hat{\mathfrak{N}} = \dim \mathfrak{N} + 1$, the conclusion follows immediately. □

Now we can prove the following theorem.

Theorem 3.12. *Let $(C, \mathfrak{C}, w(C)) : (\Sigma_1, \lambda_1) \to (\Sigma_3, \lambda_3)$ be an enhanced cobordism. Let $(M, \mathfrak{M}), (N, \mathfrak{N})$ be an enhanced Heegaard decomposition of C. Let $(\hat{M}, \hat{\mathfrak{M}}), (\hat{N}, \hat{\mathfrak{N}})$ be another enhanced Heegaard decomposition obtained from $(M, \mathfrak{M}), (N, \mathfrak{N})$ via one enhanced stabilization. We have*

$$i^{(k+1)^2} s_{\Sigma_2 \# T^2}(\hat{\mathfrak{M}}_* \lambda_1, \hat{\mathfrak{N}}^* \lambda_3) = i^{k^2} s_{\Sigma_2}(\mathfrak{M}_* \lambda_1, \mathfrak{N}^* \lambda_3). \tag{3.59}$$

Here $k = \mathrm{genus}(\Sigma_2) - \mathrm{genus}(\Sigma_3)$ is the number of 2-handles involved in Heegaard splitting $M \cup_{\Sigma_2} N$.

Proof. As in lemma 3.11, we denote $\hat{\mathfrak{M}} = \mathfrak{M} \oplus (0, m)$ and $\hat{\mathfrak{N}} = \mathfrak{N} \oplus (\varepsilon l, 0)$, where $\varepsilon = (-1)^{\dim \hat{\mathfrak{N}}}$. From lemmas 3.2 and 3.3, we have $\hat{\mathfrak{M}}_* \lambda_1 = \mathfrak{M}_* \lambda_1 \oplus m$ and $\hat{\mathfrak{N}}^* \lambda_3 = \mathfrak{N}^* \lambda_3 \oplus \varepsilon l$. Then we have

$$i^{(k+1)^2} s_{\Sigma_2 \# T^2}(\hat{\mathfrak{M}}_* \lambda_1, \hat{\mathfrak{N}}^* \lambda_3) \quad = \quad i^{(k+1)^2} s_{\Sigma_2 \# T^2}(\mathfrak{M}_* \lambda_1 \oplus m, \mathfrak{N}^* \lambda_3 \oplus \varepsilon l)$$

Now we apply the definition of Lion and Vergne's s map. To abbreviate notations, we use $g(\Sigma)$ for genus(Σ). Then we have

$$\begin{aligned}
& i^{(k+1)^2} s_{\Sigma_2 \# T^2}(\mathfrak{M}_* \lambda_1 \oplus m, \mathfrak{N}^* \lambda_3 \oplus \varepsilon l) \\
= \; & i^{(k+1)^2} i^{g(\Sigma_2 \# T^2) - \dim((\mathfrak{M}_* \lambda_1 \oplus m) \cap (\mathfrak{N}^* \lambda_3 \oplus \varepsilon l))} \epsilon(\mathfrak{M}_* \lambda_1 \oplus m, \mathfrak{N}^* \lambda_3 \oplus \varepsilon l) \\
= \; & i^{(k+1)^2} i^{g(\Sigma_2) + 1 - \dim(\mathfrak{M}_* \lambda_1 \cap \mathfrak{N}^* \lambda_3)} \epsilon(\mathfrak{M}_* \lambda_1, \mathfrak{N}^* \lambda_3) \epsilon(m, \varepsilon l) \\
= \; & i^{(k+1)^2 + 1} s(\mathfrak{M}_* \lambda_1, \mathfrak{N}^* \lambda_3) \epsilon(m, \varepsilon l)
\end{aligned}$$

Recall $\varepsilon = (-1)^{\dim \mathfrak{N}}$,we have

$$\begin{aligned}
& i^{(k+1)^2+1} s(\mathfrak{M}_* \lambda_1, \mathfrak{N}^* \lambda_3) \epsilon(m, \varepsilon l) \\
= \; & i^{(k+1)^2+1} (-1)^{\dim \hat{\mathfrak{N}}} s(\mathfrak{M}_* \lambda_1, \mathfrak{N}^* \lambda_3) \\
= \; & i^{(k+1)^2+1} (-1)^{g(\Sigma_2 \# T^2) + g(\Sigma_3)} s(\mathfrak{M}_* \lambda_1, \mathfrak{N}^* \lambda_3) \\
= \; & i^{(k+1)^2+1} (-1)^{g(\Sigma_2) + 1 - g(\Sigma_3)} s(\mathfrak{M}_* \lambda_1, \mathfrak{N}^* \lambda_3) \\
= \; & i^{(k+1)^2+1} (-1)^{k+1} s(\mathfrak{M}_* \lambda_1, \mathfrak{N}^* \lambda_3) \\
= \; & i^{k^2} s(\mathfrak{M}_* \lambda_1, \mathfrak{N}^* \lambda_3)
\end{aligned}$$

□

A well-known result due to Reidemeister and Singer says that any two Heegaard splittings of a closed 3-manifold become the same after a finite number of stabilizations. A similar conclusion can be made to compact 3-manifold. Following theorem is cited from Scharlemann's survey paper [15].

Theorem 3.13 ([15], theorem 7.1)**.** *Any two Heegaard splittings of the same compact 3-manifold have a common stabilization.*

The following theorem will complete the verification that $j(C)$ is well-defined.

Theorem 3.14. *Let* $(C, \mathfrak{C}, w(C)) : (\Sigma_1, \lambda_1) \to (\Sigma_3, \lambda_3)$ *be an enhanced cobordism. Let* $(M_1, \mathfrak{M}_1), (N_1, \mathfrak{N}_1)$ *and* $(M_2, \mathfrak{M}_2), (N_2, \mathfrak{N}_2)$ *be two enhanced Heegaard decom-*

position of C. Then

$$i^{k_1^2} s_\Sigma(\mathfrak{M}_{1*}\lambda_1, \mathfrak{N}_1^*\lambda_3) = i^{k_2^2} s_{\Sigma'}(\mathfrak{M}_{2*}\lambda_1.\mathfrak{N}_2^*\lambda_3), \tag{3.60}$$

where k_1 is the number of 2-handles involved in Heegaard splitting $M_1 \cup_\Sigma N_1$, k_2 is the number of 2-handles involved in Heegaard splitting $M_2 \cup_{\Sigma'} N_2$. Therefore, $j(C)$ is well-defined.

Proof. Assume $M \cup N$ is a common stabilization of $M_1 \cup N_1$ and $M_2 \cup N_2$. Carefully choosing orientations on $\mathfrak{M}$ and $\mathfrak{N}$, we have $(M, \mathfrak{M}) \circ (N, \mathfrak{N}) = (C, \mathfrak{C})$. Applying theorem 3.12, we conclude

$$i^{k_1^2} s(\mathfrak{M}_{1*}\lambda_1, \mathfrak{N}_1^*\lambda_3) = i^{k^2} s(\mathfrak{M}_*\lambda_1, \mathfrak{N}^*\lambda_3) = i^{k_2^2} s(\mathfrak{M}_2*\lambda_1, \mathfrak{N}_2^*\lambda_3).$$

Then it follows immediately that $j(C)$ calculated from one enhanced Heegaard decomposition agrees with $j(C)$ calculated from any other enhanced Heegaard decomposition. ☐

3.7 Some Interesting Properties of j(C)

In this section we prove two theorems. The first theorem states that invariant j, when applied to enhanced mapping class group, agrees with Gilmer and Masbaum's $n_\lambda(f)$, see [6], modulo 4. In this sense we may say that the invariant j generalizes n_λ. The second theorem states an interesting relation between invariant j and the weight w associated with an enhanced cobordism.

In [6, section 3], Gilmer and Masbaum studied the central extension $\tilde{\Gamma}(\Sigma)$ of mapping class group $\Gamma(\Sigma)$, see also [18]. Here Σ is a closed, connected, oriented surface with genus at least one, and is equipped with a fixed unoriented lagrangian λ. Each element in $\tilde{\Gamma}(\Sigma)$ can be given by an extended cobordism

$$C(f, w(f)) : (\Sigma, \lambda) \to (\Sigma, \lambda),$$

where the underlying manifold is the mapping cylinder of $f \in \Gamma(\Sigma)$. The group multiplication is given by composition of extended cobordisms. To be specific, it is defined by following equation, see [6, eq 3.1]

$$C(g,n) \circ C(f,m) = C(g \circ f, n + m - \mu_\Sigma(f_*\lambda, \lambda, g_*^{-1}\lambda)). \tag{3.61}$$

One obtains the following short exact sequence

$$0 \to \mathbb{Z} \to \tilde{\Gamma}(\Sigma) \to \Gamma(\Sigma) \to 1. \tag{3.62}$$

Here the map $\tilde{\Gamma}(\Sigma) \to \Gamma(\Sigma)$ is given by $C(f,n) \to f$. The kernel $\mathbb{Z}$ is generated by $C(\mathrm{Id}_\Sigma, 1)$. We can see this kernel is in the center of $\tilde{\Gamma}(\Sigma)$.

Furthermore, Gilmer and Masbaum studied an index four subgroup $\tilde{\Gamma}(\Sigma)^{++}$ of $\tilde{\Gamma}(\Sigma)$. For mapping class f in $\Gamma(\Sigma)$, they considered a non-singular bilinear form $\star_f$ on $(f-1)H_1(\Sigma)$ defined as

$$a \star_f b = (f-1)^{-1}(a) \cdot b. \tag{3.63}$$

Here $(f-1)^{-1}(a) \cdot b$ means $x \cdot b$ where x is any element in $(f-1)^{-1}(a)$. For well-definedness for $\star_f$, see [6, lemma 6.1]. This form was first introduced by Turaev in [17]. We have following lemma about $\star_f$.

Lemma 3.15 ([6],Lemma 6.4)**.** *For every lagrangian $\lambda \subset H_1(\Sigma)$, the restriction of the form $\star_f$ to $\lambda \cap (f-1)H_1(\Sigma)$ is symmetric.*

Denote $\star_{f,\lambda}$ as the restriction of $\star_f$ to $\lambda \cap (f-1)H_1(\Sigma)$. Lemma 3.15 allows us to talk about the signature of $\star_{f,\lambda}$. Denote this signature as $\mathrm{Sign}(\star_{f,\lambda})$. We also let $\mathrm{sgn}[\det(\star_f)]$ denote the sign of the determinant of the matrix of $\star_f$ with respect to a basis of $(f-1)H_1(\Sigma)$. Since $\star_f$ is non-singular, $\mathrm{sgn}[\det(\star_f)]$ takes value in $\{\pm 1\}$. Let

$$n_\lambda(f) = \mathrm{Sign}(\star_{f,\lambda}) - \dim((f-1)H_1(\Sigma)) - \mathrm{sgn}[\det(\star_f)] + 1, \tag{3.64}$$

the subgroup $\tilde{\Gamma}(\Sigma)^{++}$ can be describe as following, see [6, theorem 6.6]

$$\tilde{\Gamma}(\Sigma)^{++} = \{C(f,n) | f \in \Gamma(\Sigma), n \equiv n_\lambda(f) \mod 4\}, \tag{3.65}$$

In an unpublished work, Gilmer proved the following result which relates $n_\lambda(f)$ and s map of Lion and Vergne.

Theorem 3.16 (Gilmer)**.** *Let $f \in \Gamma(\Sigma)$, λ be a lagrangian on Σ. We have*

$$i^{n_\lambda(f)} = s(f_*\lambda, \lambda). \tag{3.66}$$

Here $s(f_*\lambda, \lambda)$ is computed using an arbitrary orientation on λ.

We remark that the right-hand side of equation 3.66 is independent of the choosing orientation of λ. Had we started with $-\lambda$, we have

$$s(f_*(-\lambda), -\lambda) = s(-f_*\lambda, -\lambda) = s(f_*\lambda, \lambda). \tag{3.67}$$

The first equality follows lemma 3.10. The second equality follows from definition of s map and lemma 3.6.

Let $(C(f), \mathfrak{f}, w(f)) : (\Sigma, \lambda) \to (\Sigma, \lambda)$ be an enhanced mapping cylinder of $f \in \Gamma(\Sigma)$. Recall that enhanced structure assigns orientations on lagrangian relation $\mathfrak{f}$ and lagrangian λ. Here we take the orientation of $\mathfrak{f}$ as defined in lemma 3.4. Then from definition 3.33, we have

$$i^{j(f)} = s(f_*\lambda, \lambda); \tag{3.68}$$

from theorem 3.16, we have

$$i^{n_\lambda(f)} = s(f_*\lambda, \lambda). \tag{3.69}$$

Thus, we immediately obtain the following theorem.

Theorem 3.17. *For enhanced mapping cylinder* $(C(f), \mathfrak{f}, w(f)) : (\Sigma, \lambda) \to (\Sigma, \lambda)$ *of* $f \in \Gamma(\Sigma)$*, where* $\mathfrak{f}$ *is oriented as in lemma* 3.4*. We have*

$$j(f) \equiv n_\lambda(f) \mod 4.$$

For the second part of this section, we first cite the following theorem from Lion and Vergne.

Theorem 3.18 ([12], Theorem 1.7.6)**.** *Given* $\lambda_1, \lambda_2, \lambda_3$ *being ordered triple of oriented lagrangians in symplectic space* V*, we have*

$$i^{\tau(\lambda_1,\lambda_2,\lambda_3)} = s(\lambda_1, \lambda_2)s(\lambda_2, \lambda_3)s(\lambda_3, \lambda_1). \tag{3.70}$$

Here $\tau(\lambda_1, \lambda_2, \lambda_3)$ *is the maslov index associated with a triple lagrangians* $(\lambda_1, \lambda_2, \lambda_3)$ *with their orientation forgotten.*

Remark 3.19. *We remark that the right-hand side of equation* 3.70 *is independent of orientation of* $\lambda_i, i = 1, 2, 3$*. Actually, if one changes the orientation of one of* $\lambda_i (i = 1, 2, 3)$*, two numbers on the right change their signs.*

Remark 3.20. *The definition of maslov index used by Lion and Vergne [12, p39] is due to M. Kashiwara and differs from the one used by Turaev. We can check that the two definition agrees, using [4, Theorem 8.1].*

Now we can prove the following theorem, which roughly says that the difference between weight w and invariant j is modulo 4 additive for composition of certain types of enhanced cobordisms.

Theorem 3.21. *Let* $(M, \mathfrak{M}, w(M)) : (\Sigma_1, \lambda_1) \to (\Sigma_2, \lambda_2)$, $(N, \mathfrak{N}, w(N)) : (\Sigma_2, \lambda_2) \to (\Sigma_3, \lambda_3)$ *be enhanced increasing cobordism and decreasing cobordism respectively, then we have*

$$w(M) - j(M) + w(N) - j(N) \equiv w(N \circ M) - j(N \circ M) \mod 4 \tag{3.71}$$

Proof. From definition of $j(M), j(N)$, we have $i^{j(M)} = s(\mathfrak{M}_*\lambda_1, \lambda_2)$, and $i^{j(N)} = i^{k^2} s(\lambda_2, \mathfrak{N}^*\lambda_3)$. Apply theorem 3.18, we have

$$i^{\mu(\mathfrak{M}_*\lambda_1, \lambda_2, \mathfrak{N}^*\lambda_3)} = s(\mathfrak{M}_*\lambda_1, \lambda_2) s(\lambda_2, \mathfrak{N}^*\lambda_3) s(\mathfrak{N}^*\lambda_3, \mathfrak{M}_*\lambda_1).$$

Apply lemma 3.8, we have

$$i^{-\mu(\mathfrak{M}_*\lambda_1, \lambda_2, \mathfrak{N}^*\lambda_3)} = s(\lambda_2, \mathfrak{M}_*\lambda_1) s(\mathfrak{N}^*\lambda_3, \lambda_2) s(\mathfrak{M}_*\lambda_1, \mathfrak{N}^*\lambda_3).$$

Therefore we have

$$\begin{aligned}
& i^{j(N)+j(M)-\mu(\mathfrak{M}_*\lambda_1, \lambda_2, \mathfrak{N}^*\lambda_3)} \\
= \ & i^{k^2} s(\lambda_2, \mathfrak{N}^*\lambda_3) s(\mathfrak{M}_*\lambda_1, \lambda_2) s(\lambda_2, \mathfrak{M}_*\lambda_1) s(\mathfrak{N}^*\lambda_3, \lambda_2) s(\mathfrak{M}_*\lambda_1, \mathfrak{N}^*\lambda_3) \\
= \ & i^{k^2} s(\mathfrak{M}_*\lambda_1, \mathfrak{N}^*\lambda_3) \\
= \ & i^{j(N \circ M)} \qquad (3.72)
\end{aligned}$$

Combining equations 3.72 and the gluing formula 3.3, we obtain

$$\begin{aligned}
w(N \circ M) &= w(N) + w(M) - \mu(\mathfrak{M}_*\lambda_1, \lambda_2, \mathfrak{N}^*\lambda_3), \\
j(N \circ M) &\equiv j(N) + j(M) - \mu(\mathfrak{M}_*\lambda_1, \lambda_2, \mathfrak{N}^*\lambda_3) \mod 4.
\end{aligned}$$

Take their difference, we obtain that

$$w(M) - j(M) + w(N) - j(N) \equiv w(N \circ M) - j(N \circ M) \mod 4$$

□

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Vita

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