

2014

## Obstructions to Embedding Genus-1 Tangles in Links

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# OBSTRUCTIONS TO EMBEDDING GENUS-1 TANGLES IN LINKS

A Dissertation

Submitted to the Graduate Faculty of the  
Louisiana State University and  
Agricultural and Mechanical College  
in partial fulfillment of the  
requirements for the degree of  
Doctor of Philosophy

in

The Department of Mathematics

by

Susan Marie Abernathy  
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August 2014

# Acknowledgments

First, I would like to thank my advisor, Dr. Patrick Gilmer, for his guidance and encouragement throughout this process and for introducing me to some wonderful mathematics. I'm also grateful to John Harris for allowing me to use his Mathematica code which saved me a lot of time (and probably some headaches). I'd also like to thank James and Judith Oxley for their advice and encouragement throughout my time at LSU. Graduate school is tough, but the constant love and support from my family made it so much easier. Finally, I want to thank my husband, Jesse, for always believing I could do this even when I wasn't so sure myself.

# Table of Contents

Acknowledgments . . . . .	ii
Abstract . . . . .	iv
Chapter 1 Introduction . . . . .	1
Chapter 2 A homological obstruction . . . . .	4
2.1 Introduction . . . . .	4
2.2 Surgery descriptions for double-branched covers . . . . .	7
2.3 Homology of the covers . . . . .	8
Chapter 3 The Kauffman bracket ideal . . . . .	11
3.1 Introduction . . . . .	11
3.2 The Kauffman bracket and trivalent graphs . . . . .	18
3.2.1 Kauffman bracket skein modules . . . . .	18
3.2.2 Banded colored trivalent graphs . . . . .	19
3.2.3 Defining the graph basis of $K_R(S^1 \times D^2, 2)$ . . . . .	23
3.3 The almost-orthogonal basis . . . . .	24
3.4 A finite set of generators for the Kauffman bracket ideal . . . . .	28
3.5 Partial closures . . . . .	30
3.6 An example . . . . .	34
References . . . . .	38
Appendix A Mathematica Notebook for Chapter 3 . . . . .	39
Appendix B Writing $\mathcal{F}$ as a linear combinations of basis elements . . . . .	46
Appendix C Mathematica notebook used to find generators of $I_{\mathcal{F}}$ . . . . .	50
Appendix D Permission to Reprint . . . . .	57
Vita . . . . .	58

# Abstract

Given a compact, oriented 3-manifold  $M \subseteq S^3$  with boundary, an  $(M, 2n)$ -tangle  $\mathcal{T}$  is a 1-manifold with  $2n$  boundary components properly embedded in  $M$ . We say that  $\mathcal{T}$  embeds in a link  $L \subseteq S^3$  if  $\mathcal{T}$  can be completed to  $L$  by adding a 1-manifold with  $2n$  boundary components exterior to  $M$ . The link  $L$  is called a closure of  $\mathcal{T}$ . We focus on the case of  $(S^1 \times D^2, 2)$ -tangles, also called genus-1 tangles, and consider the following question: given a genus-1 tangle  $\mathcal{G}$  and a link  $L$ , how can we tell if  $L$  is a closure of  $\mathcal{G}$ ? This question is motivated by a particular example of a genus-1 tangle given by Krebs [K], which we denote by  $\mathcal{A}$ .

Krebs asks whether the unknot is a closure of  $\mathcal{A}$ . We partially answer this question in Chapter 2 using a theorem of Ruberman [Ru] and cyclic branched covers of the solid torus branched over  $\mathcal{A}$ . We prove that if Krebs' tangle  $\mathcal{A}$  embeds in the unknot, then  $\mathcal{A}$  must be completed to the unknot by an arc which passes through the hole of the solid torus containing  $\mathcal{A}$  an even number of times.

In Chapter 3, we discuss the Kauffman bracket ideal, which gives an obstruction to tangle embedding for general  $(M, 2n)$ -tangles. For each tangle  $\mathcal{T}$  in  $M$ , we define an ideal  $I_{\mathcal{T}}$  called the Kauffman bracket ideal. It is easy to see that if  $I_{\mathcal{T}}$  is non-trivial, then  $\mathcal{T}$  does not embed in the unknot. Using skein theory, we give an algorithm for computing a finite list of generators for the Kauffman bracket ideal of any genus-1 tangle, and give an example of a genus-1 tangle with non-trivial Kauffman bracket ideal. We also explore the relationship between partial closures of tangles and this ideal.

# Chapter 1

## Introduction

A  $2n$ -tangle is a 1-manifold with  $2n$  boundary components properly embedded in the 3-ball. We say that a  $2n$ -tangle  $\mathcal{T}$  embeds in a link  $L \subset S^3$  if  $\mathcal{T}$  can be completed to  $L$  via a 1-manifold with  $2n$  boundary components exterior to  $B^3$ ; that is, there exists some 1-manifold with  $2n$  boundary components in  $S^3 - \text{Int}(B^3)$  such that gluing this 1-manifold to  $\mathcal{T}$  along their boundaries results in a link isotopic to  $L$ . We refer to  $L$  as a closure of  $\mathcal{T}$ . A natural question to ask is: given a tangle  $\mathcal{T}$  and a link  $L$ , how can we tell if  $\mathcal{T}$  embeds in  $L$ ?

This question been studied in [K, PSW, Ru]. Krebs asks a more general question in [K]: does the tangle  $\mathcal{A}$  pictured in Figure 1.1 embed in the unknot? This is the question which first sparked our interest in tangle embedding and motivated this dissertation. With this question in mind, we generalize the definition of a tangle and tangle embedding.

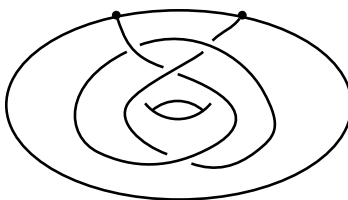


Figure 1.1: Krebs's example, which we denote by  $\mathcal{A}$ .

Rather than considering only tangles in the 3-ball, we now consider tangles inside other closed, oriented submanifolds of  $S^3$  with boundary, such as the solid torus. Given such a manifold  $M$ , we define an  $(M, 2n)$ -tangle  $\mathcal{T}$  to be a 1-manifold with  $2n$  boundary components properly embedded in  $M$ . We say that  $\mathcal{T}$  embeds in a link  $L \subset S^3$  if  $\mathcal{T}$  can be completed by a 1-manifold with  $2n$  boundary components exterior to  $M$  to form the link  $L$ ; that is, there exists some 1-manifold with  $2n$  boundary components in  $S^3 - \text{Int}(M)$  such that upon gluing this manifold to  $\mathcal{T}$  along their boundary points, we have a link in  $S^3$  which is isotopic

to  $L$ . We say that  $L$  is a closure of  $\mathcal{T}$ . According to this definition, Krebs' tangle  $\mathcal{A}$  is a  $(S^1 \times D^2, 2)$ -tangle; we refer to these as genus-1 tangles.

We focus on genus-1 tangle embedding and discuss two different obstructions to embedding. The first obstruction, which we discuss in detail in Chapter 2, relies on a theorem of Ruberman [Ru]. A genus-1 tangle  $\mathcal{G}$  is obstructed from embedding in the unknot if there is torsion in the homology of the 2-fold covers of  $S^1 \times D^2$  branched over  $\mathcal{G}$ . We outline a method for finding a surgery description of these two double-branched covers for any genus-1 tangle  $\mathcal{G}$ . Using this method, we are able to partially answer Krebs' question about the genus-1 tangle  $\mathcal{A}$ . Roughly, we show that if  $\mathcal{A}$  embeds in the unknot, then the arc which completes  $\mathcal{A}$  to the unknot must pass through the hole of the solid torus containing  $\mathcal{A}$  an even number of times. This notion of passing through the hole an even number of times is made more precise in Chapter 2. We note that Chapter 2 is substantially the same as the author's paper [A].

In Chapter 3, we take a different approach to the tangle embedding question, defining the Kauffman bracket ideal which generalizes an ideal defined by Przytycki, Silver, and Williams [PSW]. Given an  $(M, 2n)$ -tangle  $\mathcal{T}$ , we define the Kauffman bracket ideal of  $\mathcal{T}$  to be the ideal  $I_{\mathcal{T}}$  of  $\mathbb{Z}[A, A^{-1}]$  generated by the reduced Kauffman bracket polynomials of all closures of  $\mathcal{T}$ . An immediate consequence of this definition is that if  $\mathcal{T}$  embeds in a link  $L$ , then  $\langle L \rangle' \in I_{\mathcal{T}}$ . Since the reduced Kauffman bracket polynomial of the unknot is one, we have that if  $\mathcal{T}$  embeds in the unknot, then  $I_{\mathcal{T}} = \mathbb{Z}[A, A^{-1}]$ . In this case, we refer to  $I_{\mathcal{T}}$  as the trivial ideal. Thus, if  $I_{\mathcal{T}}$  is non-trivial (that is,  $I_{\mathcal{T}}$  is a proper ideal of  $\mathbb{Z}[A, A^{-1}]$ ), then  $\mathcal{T}$  does not embed in the unknot. If  $M = B^3$ , then the Kauffman bracket ideal is exactly the ideal defined in [PSW].

A brief examination shows that both the figure-eight knot and a  $-1$ -framed trefoil are closures of  $\mathcal{A}$ . It is easy to see that the Kauffman bracket polynomials of these knots generate

the trivial ideal. Thus, this ideal does not obstruct Krebs' example from embedding in the unknot.

For the Kauffman bracket ideal to be a useful obstruction, we need a way to find a finite list of generators for this ideal. Using skein theory techniques, we develop a method for finding such a list, and we use it to prove that the genus-1 tangle  $\mathcal{F}$  in Figure 1.2 does not embed in the unknot. We outline this method in Section 3.4. In particular, it utilizes two bases for the Kauffman bracket skein module of the solid torus relative to two points; we discuss these bases in detail in Sections 3.2 and 3.3. In Section 3.5, we define partial closures of  $(B^3, 2n)$ -tangles and explore how they relate to the Kauffman bracket ideal. The concept of partial closures significantly influenced our search for an example of a genus-1 tangle with non-trivial Kauffman bracket ideal, and we used Mathematica to make the search for such an example more efficient. We describe this search in more detail in Section 3.1, and the Mathematica code that we used to find the example  $\mathcal{F}$  can be found in Appendix D. Finally, in Section 3.6, we prove that the genus-1 tangle  $\mathcal{F}$  does not embed in the unknot. In Appendix B, we provide some explicit computations used to find our list of generators for  $I_{\mathcal{F}}$ . We give the Mathematica notebook with which we computed these generators, as well as an explicit list of the generators, in Appendix C.

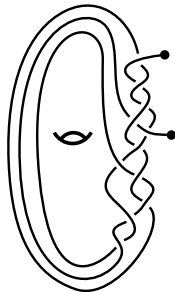


Figure 1.2: The genus-1 tangle  $\mathcal{F}$ .



# Chapter 2

## A homological obstruction<sup>1</sup>

### 2.1 Introduction

We choose a standardly embedded solid torus  $S^1 \times D^2 \subset S^3$ , denoted by  $S$ . Then a genus-1 tangle is a properly embedded arc in  $S$ . Just as we may discuss embedding ordinary tangles in  $B^3$  into knots and links (see [K], [PSW], and [Ru]), we may consider embedding genus-1 tangles in knots. We say that a genus-1 tangle  $\mathcal{G}$  embeds in a knot  $K$  if  $\mathcal{G}$  can be completed by an arc exterior to  $S$  to form the knot  $K$ ; that is, there exists some arc in  $S^3 - \text{Int}(S)$  such that upon gluing this arc to  $\mathcal{G}$  along their boundary points, we have a knot in  $S^3$  which is isotopic to  $K$ . We say that  $K$  is a closure of  $\mathcal{G}$ .

Let  $l$  denote a longitude for  $S$  which is contained in  $\partial S$  and avoids the genus-1 tangle. A closure  $K$  of  $\mathcal{G}$  is called odd (respectively, even) with respect to  $l$  if  $lk(K, l)$  is odd (respectively, even). If  $l$  is chosen to be the longitude which circles the central hole of  $S$  as in Figure 2.1, and we span the longitude  $l$  by a disk  $\Delta$  filling the hole, then  $lk(K, l)$  is the number of transverse intersections counted with sign of the arc which completes  $\mathcal{G}$  to  $K$  with  $\Delta$ . Thus, in this case we can say more colloquially that  $K$  is an odd (respectively, even) closure with respect to  $l$  if the arc which completes  $\mathcal{G}$  to  $K$  passes through the hole of  $S$  an odd (respectively, even) number of times.

In [K], Krebs asks whether the genus-1 tangle given in Figure 2.1 embeds in the unknot. We denote this tangle by  $\mathcal{A}$ , and when discussing this example, we always use the longitude  $l$  drawn in Figure 2.1. Using the following results from [Ru], we are able to partially answer the question posed by Krebs.

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<sup>1</sup>This chapter is substantially the same as the the following paper: S.M. Abernathy. On Krebs's tangle. *Topology Appl.* **160** (2013) 1379-1383. See Appendix D for the publisher's permission to reprint.

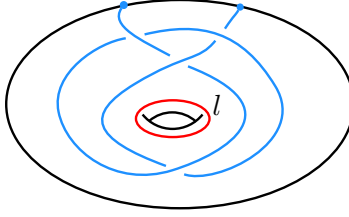


Figure 2.1: Krebs' genus-1 tangle  $\mathcal{A}$  in  $S$  together with a specified longitude  $l$

**Theorem 2.1** (Ruberman). *Suppose  $M$  is an orientable 3-manifold with connected boundary, and  $i : M \hookrightarrow N$  where  $N$  is an orientable 3-manifold with  $H_1(N)$  torsion. Then the inclusion map  $i_*$  induces an injection of the torsion subgroup  $T_1(M)$  of  $H_1(M)$  into  $H_1(N)$ .*

This theorem has a useful corollary which can easily be proved directly using a Meyer-Vietoris sequence.

**Corollary 2.2** (Ruberman). *Let  $M$  and  $N$  be as in Theorem 2.1 but suppose  $H_1(N) = 0$ . Then  $H_1(M)$  is torsion-free.*

One obtains an obstruction to embedding genus-1 tangles in knots from Theorem 2.1 by applying the result to cyclic branched covers of the solid torus  $S$  branched over genus-1 tangles.

Recall, for a given  $n$ , each  $n$ -fold cover of  $S$  branched over a genus-1 tangle  $\mathcal{G}$  is associated to a homomorphism  $\varphi : H_1(S - \mathcal{G}) \rightarrow \mathbb{Z}_n$  which maps the meridian  $m$  of  $\mathcal{G}$  to 1. The remaining generator  $l$  of  $H_1(S - \mathcal{G})$  may be sent to any element of  $\mathbb{Z}_n$ , and we use  $\varphi(l)$  to index the  $n$ -fold branched covers. So,  $Y_{\mathcal{G},i}$  denotes the  $n$ -fold cover of  $S$  branched over  $\mathcal{G}$  associated to the homomorphism  $\varphi$  which maps  $l$  to  $i$ .

If a genus-1 tangle  $\mathcal{G}$  embeds in a knot  $K$ , then the  $n$ -fold cover  $X_K$  of  $S^3$  branched over  $K$  restricts to some  $n$ -fold cover  $Y_{\mathcal{G},i}$  of  $S$  branched over  $\mathcal{G}$ . In this case, we say that the closure  $K$  induces the cover  $Y_{\mathcal{G},i}$ . Then according to Theorem 2.1, the torsion subgroup  $T_1(Y_{\mathcal{G},i})$  of  $H_1(Y_{\mathcal{G},i})$  injects into  $H_1(X_K)$ .

Note that if  $K$  is the unknot, then  $X_K$  is  $S^3$  and according to Corollary 2.2, the torsion subgroup  $T_1(Y_{\mathcal{G},i})$  is trivial. Thus, if there is any torsion in the homology of  $Y_{\mathcal{G},i}$ , then any closure of  $\mathcal{G}$  which induces the cover  $Y_{\mathcal{G},i}$  is not the unknot.

After applying this obstruction to the double-branched covers of  $S$  branched over Krebs' tangle  $\mathcal{A}$ , we prove the following results:

**Theorem 2.3.** *If a knot  $K$  in  $S^3$  is an odd closure of  $\mathcal{A}$ , then  $\det(K)$  is divisible by three.*

**Corollary 2.4.** *If  $\mathcal{A}$  embeds in the unknot, then the unknot is an even closure of  $\mathcal{A}$ .*

Before further discussion, we must make a remark about the definition of genus-1 tangles.

**Remark 2.5.** *Note that when defining genus-1 tangles, we fix a standardly embedded solid torus  $S$  in the 3-sphere. The reason that we restrict to a fixed embedding is that there are many ways to re-embed a solid torus inside  $S^3$ .*

*For instance, if we perform a meridional twist on  $S$  along the disk indicated in Figure 2.2, the image of  $\mathcal{A}$  under this twist can be easily seen to embed in an unknot via the exterior arc pictured in Figure 2.2. Thus it is necessary to specify the embedding of  $S^1 \times D^2$ , and we restrict to a fixed standardly embedded solid torus.*

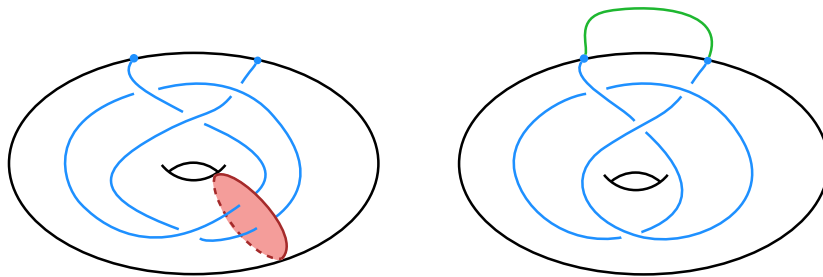


Figure 2.2: The disk in  $S$  where we perform a meridional twist, and the genus-1 tangle which results from the twist.

## 2.2 Surgery descriptions for double-branched covers

For the purposes of this paper, we restrict our attention to double-branched covers of  $S$  branched over  $\mathcal{A}$ . Since a homomorphism  $\varphi : H_1(S - \mathcal{A}) \rightarrow \mathbb{Z}_2$  must map the specified

longitude  $l$  to either 0 or 1, there are two double-branched covers,  $Y_{\mathcal{A},0}$  and  $Y_{\mathcal{A},1}$ . We call  $Y_{\mathcal{A},0}$  the even double-branched cover because it is induced by all even closures of  $\mathcal{A}$  (with respect to  $l$ ). Similarly, since  $Y_{\mathcal{A},1}$  is induced by all odd closures of  $\mathcal{A}$ , we call it the odd double-branched cover.

In this section, we adapt a technique of Rolfsen's to find surgery descriptions for these two double-branched covers.

Following [Ro], we perform surgery near a carefully selected crossing (see Figure 2.3) in such a way that after surgery we may essentially unwind  $\mathcal{A}$  (via sliding its endpoints around the boundary in the complement of  $l$ ) so that it looks trivial. This process, illustrated in Figure 2.4, results in a nice surgery description of  $\mathcal{A}$  inside  $S$ . Note that in the last drawing of Figure 2.4, we choose to draw this surgery description in a particular way because it makes constructing branched covers easier.

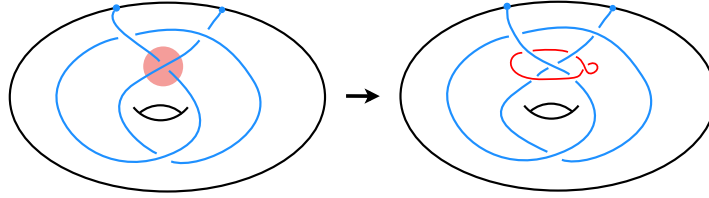


Figure 2.3: We perform surgery around a crossing, following [Ro].

Now we construct the odd cover,  $Y_{\mathcal{A},1}$ . Construction is dictated by the homomorphism  $\varphi : H_1(S - \mathcal{A}) \rightarrow \mathbb{Z}_2$  corresponding to the cover. If  $\varphi$  maps a generator of  $H_1(S - \mathcal{A})$  to a non-zero element, then we cut the solid torus along a disk transverse to that generator.

So, we have two cuts to make in the case of the odd cover. First, we cut  $S$  along a disk which is transverse to the meridian  $m$  of  $\mathcal{A}$  and whose boundary is made up of the unwound genus-1 tangle  $\mathcal{A}$  together with an arc in  $\partial S$ . Then, because  $\varphi$  sends  $l$  to 1, we cut  $S$  along a disk which is transverse to  $l$  and whose boundary is contained in  $\partial S$ . We then take two copies of the resulting manifold and glue them together carefully to obtain a surgery description for  $Y_{\mathcal{A},1}$ . This process is illustrated in Figure 2.5.

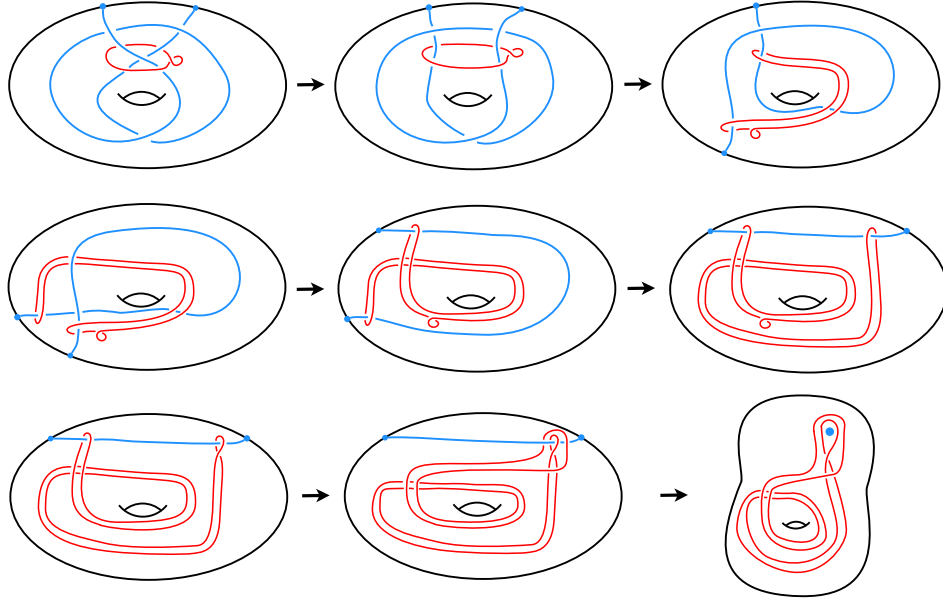


Figure 2.4: Unwinding the genus-1 tangle  $\mathcal{A}$  to make it look trivial. The surgery curve is always given the blackboard framing.

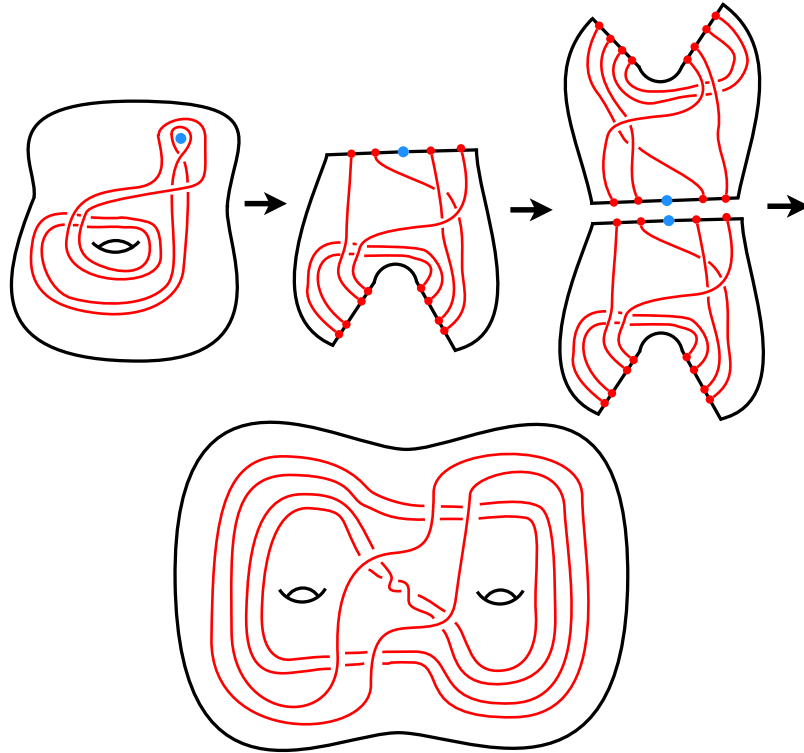


Figure 2.5: Constructing the odd double-branched cover  $Y_{\mathcal{A},1}$  of  $S$  branched over  $\mathcal{A}$ .

Although it is not needed in the proof of Theorem 2.3, we also give a surgery description of the even double-branched cover  $Y_{\mathcal{A},0}$  in Figure 2.6.

### 2.3 Homology of the covers

Now we compute the homology of the odd double-branched cover. From Figure 2.5 we see that the surgery description for  $Y_{\mathcal{A},1}$  is given by a 2-component surgery link inside a genus-2 handlebody. We denote the components of the surgery link by  $\sigma$  and  $\tau$ , and let  $H$  denote the genus-2 handlebody. The complement of  $H$  in  $S^3$  is a neighborhood of the handcuff graph  $G$ , pictured in Figure 2.7, which is composed of loops  $\alpha_1$  and  $\alpha_2$  joined together by an arc. Then the complement of  $\sigma \cup \tau$  in  $H$  can be viewed as the complement of  $\sigma \cup \tau \cup G$  in  $S^3$ . One can see that  $H_1(S^3 - (\sigma \cup \tau \cup G))$  is isomorphic to  $H_1(S^3 - (\sigma \cup \tau \cup \alpha_1 \cup \alpha_2))$  which is free on four generators: the meridians of  $\sigma$ ,  $\tau$ ,  $\alpha_1$ , and  $\alpha_2$ .

Completing the surgery by gluing in two solid tori according to  $\sigma$  and  $\tau$  introduces two relations on these four generators, which are given by the linking numbers of  $\sigma$  and  $\tau$  with each of  $\sigma$ ,  $\tau$ ,  $\alpha_1$ , and  $\alpha_2$ . So,  $H_1(Y_{\mathcal{A},1})$  is isomorphic to  $H_1(S^3 - (\sigma \cup \tau \cup \alpha_1 \cup \alpha_2))$  modulo these two relations, and we can get a presentation for  $H_1(Y_{\mathcal{A},1})$  using linking numbers. We have the following presentation matrix for  $H_1(Y_{\mathcal{A},1})$ :

$$\begin{array}{c} \sigma \quad \tau \quad \alpha_1 \quad \alpha_2 \\ \sigma \left( \begin{array}{cccc} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{array} \right) \\ \tau \end{array}.$$

Using row and columns operations we obtain the following simpler presentation matrix:

$$\begin{array}{c} \sigma \quad \tau \quad \alpha_1 \quad \alpha_2 \\ \sigma \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array} \right) \\ \tau \end{array}.$$

Therefore,  $H_1(Y_{\mathcal{A},1}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_3$  and we are now able to prove the main theorem. Corollary 2.4 follows immediately.

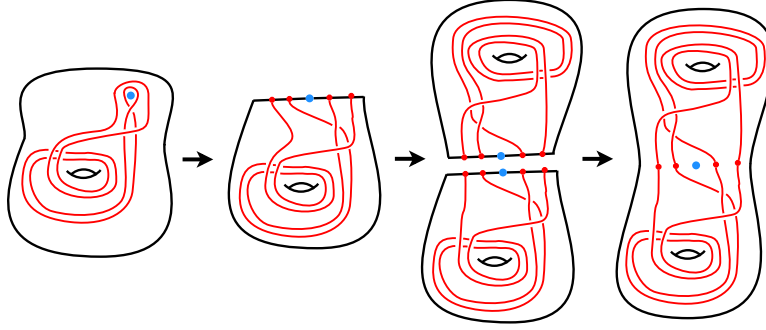


Figure 2.6: Obtaining a surgery description of  $Y_{\mathcal{A},0}$ .

*Proof of Theorem 2.3.* Let  $K$  be an odd closure of  $\mathcal{A}$ , and let  $X_K$  denote the double cover of  $S^3$  branched over  $K$ . Since  $K$  is an odd closure of  $\mathcal{A}$ , it induces a restriction from  $X_K$  to  $Y_{\mathcal{A},1}$ . Then according to Theorem 2.1, we have that  $T_1(Y_{\mathcal{A},1}) = \mathbb{Z}_3 \hookrightarrow H_1(X_K)$ . Thus  $|T_1(Y_{\mathcal{A},1})| = 3$  divides  $|H_1(X_K)| = \det(K)$ .  $\square$

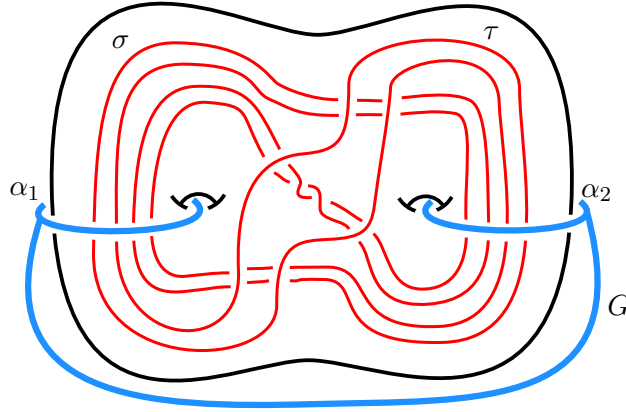


Figure 2.7: A surgery description of  $Y_{\mathcal{A},1}$

We are unable to use this method to restrict all closures of  $\mathcal{A}$  because  $Y_{\mathcal{A},0}$  has a torsion-free first homology group. Indeed, the statement in Remark 2.5 allows us to see that the even cover does embed in  $S^3$  and so must have torsion-free first homology. Of course, this can be verified by deriving a presentation for the homology of  $Y_{\mathcal{A},0}$  using the procedure above.

# Chapter 3

## The Kauffman bracket ideal

### 3.1 Introduction

Let  $M$  be any compact, oriented 3-dimensional submanifold of  $S^3$  with boundary. Then an  $(M, 2n)$ -tangle  $\mathcal{T}$  is a 1-manifold with  $2n$  boundary components properly embedded in  $M$ . We say that  $\mathcal{T}$  embeds in a link  $L \subset S^3$  if  $\mathcal{T}$  can be completed by a 1-manifold with  $2n$  boundary components exterior to  $M$  to form the link  $L$ ; that is, there exists some 1-manifold with  $2n$  boundary components in  $S^3 - \text{Int}(M)$  such that upon gluing this manifold to  $\mathcal{T}$  along their boundary points, we have a link in  $S^3$  which is isotopic to  $L$ . We say that  $L$  is a closure of  $\mathcal{T}$ .

This definition naturally gives rise to the following question: given an  $(M, 2n)$ -tangle  $\mathcal{T}$  and a link  $L \subset S^3$ , when does  $\mathcal{T}$  embed in  $L$ ?

This embedding question has been studied before in the case where  $M = B^3$  (see [K, PSW, Ru]) and discussed in the case where  $M = S^1 \times D^2$  in [K] and [Ru]. In [K], Krebs asked whether the genus-1 tangle pictured in Figure 3.1, denoted by  $\mathcal{A}$ , can be embedded into the unknot. It was this question that first motivated our interest in the topic of tangle embedding. We partially answer this question in [A] using methods different than those in this paper.

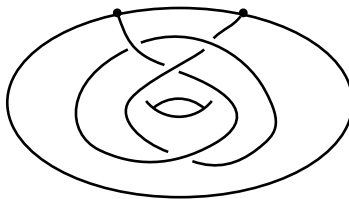


Figure 3.1: Krebs's example, which we denote by  $\mathcal{A}$ .

Though our main concern in this paper is the case where  $M$  is a solid torus, we first consider the case where  $M = B^3$ . Suppose a  $(B^3, 2n)$ -tangle  $\mathcal{T}$  embeds in a link  $L$ . Then the



complement of  $\mathcal{T}$  in  $L$  is also a  $(B^3, 2n)$ -tangle, since it is a 1-manifold with  $2n$  boundary points properly embedded in the 3-ball  $S^3 - \text{Int}(B^3)$ . Let  $\mathcal{S}$  denote this complementary tangle. We may view  $L$  as the union of  $\mathcal{S}$  and  $\mathcal{T}$  along their boundary points. In this case we refer to  $L$  as the closure of  $\mathcal{T}$  by  $\mathcal{S}$ , denoted by  $\mathcal{T}^{\mathcal{S}}$ .

In [PSW], Przytycki, Silver and Williams examine the ideal  $I_{\mathcal{T}}$  associated to a  $(B^3, 2n)$ -tangle  $\mathcal{T}$  generated by the reduced Kauffman bracket polynomials of certain closures of  $\mathcal{T}$ . The Kauffman bracket polynomial of a link (diagram)  $L$  is denoted by  $\langle L \rangle$ . From the definition given in Section 3.2.1, it is clear that the Kauffman bracket polynomial of any non-empty link  $L \subset S^3$  is a multiple of  $\delta = -A^2 - A^{-2}$ . So we define the reduced Kauffman bracket polynomial to be  $\langle L \rangle' = \langle L \rangle / \delta \in \mathbb{Z}[A, A^{-1}]$ .

The following theorem, proven in [PSW], gives an obstruction to  $(B^3, 2n)$ -tangles embedding in links. A  $2n$ -Catalan tangle  $\mathcal{C}$  is a crossingless  $(B^3, 2n)$ -tangle with no trivial components.

**Theorem 3.1** (Przytycki, Silver, and Williams). *If a  $(B^3, 2n)$ -tangle  $\mathcal{T}$  embeds in a link  $L$ , then the ideal  $I_{\mathcal{T}}$  of  $\mathbb{Z}[A, A^{-1}]$  generated by the reduced Kauffman bracket polynomials of all diagrams  $\langle \mathcal{T}^{\mathcal{C}} \rangle'$ , where  $\mathcal{C}$  is any Catalan tangle, contains the polynomial  $\langle L \rangle'$ .*

In the case where  $2n = 4$ , there are only two Catalan tangles and thus  $I_{\mathcal{T}}$  is generated by the reduced Kauffman bracket polynomials of the two tangles in Figure 3.2. These are the numerator  $n(\mathcal{T})$  and denominator  $d(\mathcal{T})$  closures of  $\mathcal{T}$ .



Figure 3.2: The numerator  $n(\mathcal{T})$  and the denominator  $d(\mathcal{T})$  of a  $(B^3, 4)$ -tangle  $\mathcal{T}$ .

In [PSW], it is noted that Theorem 3.1 may be viewed in a skein theoretic light. Recall that any closure of a  $(B^3, 2n)$ -tangle  $\mathcal{T}$  can be viewed as the union of  $\mathcal{T}$  and a complementary  $(B^3, 2n)$ -tangle  $\mathcal{S}$  along their boundary points. We may view both  $\mathcal{T}$  and  $\mathcal{S}$  as elements of the relative Kauffman bracket skein module  $K(B^3, 2n)$ . Then we can describe the closure of  $\mathcal{T}$  by  $\mathcal{S}$  in terms of a symmetric bilinear pairing  $\langle \cdot, \cdot \rangle : K(B^3, 2n) \times K(B^3, 2n) \rightarrow K(S^3) = \mathbb{Z}[A, A^{-1}]$  defined as follows:

$$\langle \text{S}, \text{R} \rangle = \langle \text{S} \cup \text{R} \rangle.$$

Any closure of  $\mathcal{T}$  may be written as  $\langle \mathcal{T}, \mathcal{S} \rangle$  for some  $(B^3, 2n)$ -tangle  $\mathcal{S}$ . Since the set of all  $2n$ -Catalan tangles forms a basis for  $K(B^3, 2n)$ , we see that any such tangle  $\mathcal{S}$  can be written as a linear combination of Catalan tangles. So the ideal  $I_{\mathcal{T}}$  is generated by pairings  $\langle \mathcal{T}, \mathcal{C} \rangle / \delta$  where  $\mathcal{C}$  is a Catalan tangle. Furthermore, this means that an equivalent way to think about  $I_{\mathcal{T}}$  is as the ideal generated by the reduced Kauffman bracket polynomials of all closures of  $\mathcal{T}$ .

We generalize this ideal to  $(M, 2n)$ -tangles. Given an  $(M, 2n)$ -tangle  $\mathcal{T}$ , let  $I_{\mathcal{T}}$  denote the ideal of  $\mathbb{Z}[A, A^{-1}]$  generated by the reduced Kauffman bracket polynomials of all closures of  $\mathcal{T}$ . We call this the Kauffman bracket ideal of  $\mathcal{T}$ . Note that if  $M = B^3$ , this is the same ideal defined in Theorem 3.1. If  $I_{\mathcal{T}} = \mathbb{Z}[A, A^{-1}]$ , we refer to  $I_{\mathcal{T}}$  as a trivial ideal. The following proposition is an immediate consequence of the definition.

**Proposition 3.2.** *If an  $(M, 2n)$ -tangle  $\mathcal{T}$  embeds in a link  $L \subset S^3$ , then  $\langle L \rangle' \in I_{\mathcal{T}}$ .*

If  $\mathcal{T}$  embeds in the unknot, then  $I_{\mathcal{T}}$  is trivial since the reduced Kauffman bracket polynomial of the unknot is one. So, Proposition 3.2 gives an obstruction tangle embedding; if  $I_{\mathcal{T}}$  is non-trivial, then  $\mathcal{T}$  does not embed in the unknot.

Our main concern in this paper is applying this obstruction to  $(S^1 \times D^2, 2)$ -tangles, which we refer to as genus-1 tangles. We apply it first to Krebs's genus-1 tangle  $\mathcal{A}$  in Figure 3.1. A

brief examination shows that both the figure-eight knot and a  $-1$ -framed trefoil are closures of  $\mathcal{A}$ , so  $f = A^{-8} - A^{-4} + 1 - A^4 + A^8$  and  $g = A^{-8} + 1 - A^4$  are two generators of  $I_{\mathcal{A}}$ . A short computation shows that  $A^{-4}f + (1 - A^{-4})g = 1$ , and thus  $I_{\mathcal{A}}$  is trivial. So Proposition 3.2 does not provide an obstruction to Krebs's example embedding in the unknot.

Obviously, we cannot always compute the Kauffman bracket ideal of a genus-1 tangle by simply examining some number of closures as we did with Krebs's tangle since the ideal has infinitely many generators by definition. One can give a finite list of generators for the ideal of a  $(B^3, 2n)$ -tangle because the Catalan tangles are a finite basis for the relative Kauffman bracket skein module  $K(B^3, 2n)$ . We generalize this method to the case of genus-1 tangles. For this we must consider the relative Kauffman bracket skein module  $K(S^1 \times D^2, 2)$  which is infinite dimensional. Nevertheless, we outline an algorithm for finding an explicit finite list of generators for the Kauffman bracket ideal  $I_{\mathcal{G}}$  of any genus-1 tangle  $\mathcal{G}$ .

We use two bases for the Kauffman bracket skein module of  $S^1 \times D^2$  relative to two points on the boundary. The first basis is for the skein module over the ring  $\mathbb{Z}[A, A^{-1}]$  localized by inverting the quantum integers, and involves banded trivalent graphs. We discuss banded trivalent graphs and define this basis in Section 3.2. Gilmer discussed this type of basis for a handlebody with colored points in a course on quantum topology in the fall of 2001. It is the generic version of the basis discussed in [BHMV2, Theorem 4.11]. The second basis is for the skein module over  $\mathbb{Z}[A, A^{-1}]$  and is related to the orthogonal basis  $\{Q_n\}$  defined in [BHMV1]. We discuss this basis in Section 3.3.

Then in Section 3.4, we outline an algorithm for finding a finite list of generators for the Kauffman bracket ideal  $I_{\mathcal{G}}$  of any genus-1 tangle  $\mathcal{G}$ .

In Section 3.6, we use this method to show that Proposition 3.2 does provide an obstruction for the genus-1 tangle  $\mathcal{F}$  pictured in Figure 3.3, and we prove the following theorem.

**Theorem 3.3.** *The Kauffman bracket ideal  $I_{\mathcal{F}}$  of  $\mathcal{F}$  is non-trivial. In fact,  $I_{\mathcal{F}} = \langle 11, 4 - A^4 \rangle$ . If a link  $L \subset S^3$  is a closure of  $\mathcal{F}$  and  $J_L(\sqrt{t})$  is the Jones polynomial of  $L$ , then  $J_L(\sqrt{t})|_{\sqrt{t}=5} = 0 \pmod{11}$ .*

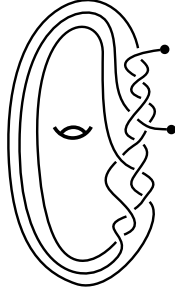


Figure 3.3: The genus-1 tangle  $\mathcal{F}$ .

Of course, one could easily give an example where the Kauffman bracket ideal is non-trivial because the genus-1 tangle contains a local knot or has a  $(B^3, 4)$ -subtangle with non-trivial ideal. The genus-1 tangle  $\mathcal{F}$  contains no local knots and does not appear to have any  $(B^3, 4)$ -subtangles with non-trivial Kauffman bracket ideals. To find this example, we used the concept of partial closures, which we discuss in Section 3.5.

The partial closure of a  $(B^3, 2n)$ -tangle  $\mathcal{T}$  is the genus-1 tangle obtained from  $\mathcal{T}$  by gluing a copy of  $D^2 \times I$  containing  $n - 1$  properly embedded arcs to  $B^3$  as indicated in Figure 3.4. We denote the partial closure by  $\hat{\mathcal{T}}$ .

If a  $(B^3, 4)$ -tangle consists of exactly two arcs embedded in  $B^3$ , then its partial closure either has a single component (if the partial closure joins boundary points from the two different arcs) or two components (if the partial closure joins boundary points of the same arc). If it has a single component, then we have the following surprising result which we prove in Section 3.5.

**Theorem 3.4.** *Let  $\mathcal{T}$  be a  $(B^3, 4)$ -tangle and let  $\hat{\mathcal{T}}$  denote the genus-1 tangle which is the partial closure of  $\mathcal{T}$ . If  $\hat{\mathcal{T}}$  has a single component, then  $I_{\hat{\mathcal{T}}} = I_{\mathcal{T}}$ .*

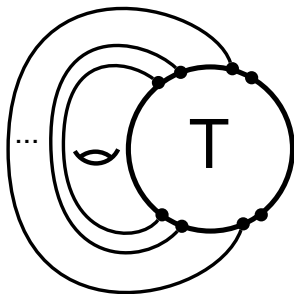


Figure 3.4: The partial closure  $\hat{\mathcal{T}}$  of a  $(B^3, 2n)$ -tangle  $\mathcal{T}$ .

This result influenced our search for an example of a genus-1 tangle with non-trivial Kauffman bracket ideal because any genus-1 tangle with one component which intersects some meridional disk of the solid torus exactly once can be viewed as the partial closure of a  $(B^3, 4)$ -tangle. Thus, its Kauffman bracket ideal can easily be computed using Theorem 3.1. So, we should consider only those genus-1 tangles which intersect every meridional disk in the solid torus at least twice. In particular, we considered partial closures of braids when looking for an example and used Mathematica to make our search more efficient.

Any braid  $B$  on  $n$  strands can be viewed as a  $(B^3, 2n)$ -tangle. So, we can obtain a genus-1 tangle from  $B$  by taking the partial closure of  $B$ . Furthermore, certain closures of any genus-1 tangle obtained from a braid are easy to describe in Mathematica.

Since  $B$  has an inverse element  $B^{-1}$  in the braid group, it is easy to see that some closure of the  $(B^3, 2n)$ -tangle consisting of  $B$  concatenated with  $B^{-1}$  is the unknot, and we have the following easy proposition.

**Proposition 3.5.** *For any  $(B^3, 2n)$ -tangle  $B$  which is a braid, we have that  $I_B = \mathbb{Z}[A, A^{-1}]$ .*

Furthermore, any subtangle of a braid  $B$  also has trivial Kauffman bracket ideal.

We do not consider 2-stranded braids, since any 2-stranded braid can be viewed as a  $(B^3, 4)$ -tangles and thus satisfies Theorem 3.4. Furthermore, it is easy to see that any 2-stranded braid embeds in either the unknot or the 2-component unlink, depending on whether

the braid has an odd or even number of twists. So we consider only partial closures of braids with at least three strands.

We wrote a Mathematica program using Bar-Natan's KnotTheory package [BN] to make detecting potential examples easier. It computes the ideal generated by certain closures of the partial closure of certain braids. This notebook is available in Appendix A. It proceeds as follows. Given the  $n$ th knot with  $m$  crossings, we obtain a braid representative  $br[m, n]$  of the knot. From  $br[m, n]$ , we obtain a genus-1 tangle by taking its partial closure  $\mathcal{G}$ . We then examine some particular closures of  $\mathcal{G}$ .

The closures of  $\mathcal{G}$  we consider are those in which the strand closing the tangle wraps around through hole  $n$  of times either front to back or back to front, for some positive integer  $n$ , as in Figure 3.5. Such a closure can be viewed as the closure of the braid  $br[m, n]$  concatenated  $n$  times with one of the following braids:

$$P = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \quad \text{or} \quad N = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} .$$

We consider eleven closures of  $\mathcal{G}$ :  $br[m, n]$  concatenated with each of  $P$  and  $N$  up to five times, along with  $br[m, n]$  itself.

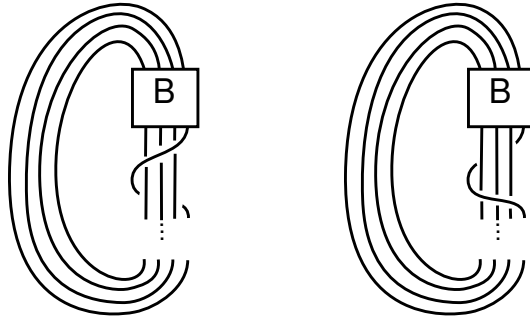


Figure 3.5: Some closures of a genus-1 tangle obtained as the partial closure of a braid  $B$  on four strands.

Our program then computes the Jones polynomials of these closures and rescales them as follows: if the smallest exponent of  $t$  appearing in the Jones polynomial is negative, then

we multiply the Jones polynomial by the power of  $t$  necessary to make that smallest degree term a constant; if all exponents of  $t$  in the Jones polynomial are positive, we do nothing. These rescaled Jones polynomials lie in  $\mathbb{Z}[t]$  and generate an ideal. Our program computes a Groebner basis for this ideal. The tangles for which this ideal was non-trivial formed our list of potential examples.

For a fixed integer  $k$ , our program does the computation described above for every knot up to 10 crossings whose braid representative has  $k$  strands. All knots whose braid representatives have three strands yielded a trivial Groebner basis. However, the ideal was non-trivial for three knots whose braid representatives have four strands:  $10_{57}$ ,  $10_{117}$ , and  $10_{162}$ . We obtained the example  $\mathcal{F}$  by taking the partial closure of the braid representative of the  $10_{57}$  knot. We chose  $10_{57}$  because its braid representative has several twist regions which make the computation in Appendix B slightly easier.

Now, a natural question is whether there exists a genus-1 tangle which is the partial closure of a braid on three strands (or more generally, a  $(B^3, 6)$ -tangle) with a trivial Kauffman bracket ideal but does not embed in the unknot. Because our search resulted in no non-trivial examples for  $k = 3$ , we must find another way to detect such an example.

## 3.2 The Kauffman bracket and trivalent graphs

### 3.2.1 Kauffman bracket skein modules

First we recall the definition of the Kauffman bracket of a link diagram  $D$ . The Kauffman bracket  $\langle D \rangle$  of a link diagram  $D$  is a polynomial in  $\mathbb{Z}[A, A^{-1}]$  given by the following relations, where  $\delta = -A^2 - A^{-2}$ :

$$(i) \quad \langle \text{crossing} \rangle = A \langle \text{positive twist} \rangle + A^{-1} \langle \text{negative twist} \rangle$$

$$(ii) \quad \langle D' \amalg \bigcirc \rangle = \delta \langle D' \rangle.$$

$$(iii) \quad \langle \text{empty diagram} \rangle = 1.$$

Furthermore, for any non-empty link  $L$  and diagram  $D$  of  $L$ , we define  $\langle D \rangle' = \langle D \rangle / \delta$  to be the reduced Kauffman bracket polynomial of  $L$ .

The Kauffman bracket skein module of a 3-manifold  $M$ , denoted by  $K(M)$ , is the  $\mathbb{Z}[A, A^{-1}]$ -module generated by isotopy classes of framed links in  $M$  modulo the Kauffman bracket relations. Note that the isotopy class of the empty link is the identity in  $K(M)$ .

Given a 3-manifold  $M$  with boundary and a set of  $m$  framed points in  $\partial M$ , the relative Kauffman bracket skein module of  $M$ , denoted  $K(M, m)$ , is the  $\mathbb{Z}[A, A^{-1}]$ -module generated by isotopy classes of framed links and arcs in  $M$  which intersect  $\partial M$  in the framed points.

Let  $R$  denote  $\mathbb{Z}[A, A^{-1}]$  localized by inverting the quantum integers,  $[k] = (A^{2k} - A^{-2k}) / (A^2 - A^{-2})$ . In addition to  $K(M, m)$ , we consider  $K_R(M, m)$  the relative Kauffman bracket skein module of  $M$  with coefficients in  $R$ . When we refer to a skein element, we mean an element of  $K_R(M, m)$ .

We must make this distinction because when we compute the Kauffman bracket ideal of an  $(M, 2n)$ -tangle, we are in fact using elements of and pairings defined on  $K_R(M, 2n)$  rather than  $K(M, 2n)$ . Since each 3-manifold  $M$  we consider in this paper has the form  $\Sigma \times I$  for some surface  $\Sigma$ , we have that  $K_R(M, 2n)$  is free on diagrams without crossings or contractable loops according to [P, Theorem 3.1]. Furthermore, according to [P, Proposition 2.2], we have that  $K_R(M, 2n) = K(M, 2n) \otimes R$ . So we may view  $K(M, 2n)$  as a subset of  $K_R(M, 2n)$ .

### 3.2.2 Banded colored trivalent graphs

Recall that for each  $n > 0$ , the  $n$ th Temperley-Lieb algebra  $TL_n = K_R(D^2 \times I, 2n)$  contains the  $n$ th Jones-Wenzl idempotent  $f_n$  defined recursively as in Figure 3.6. Here,  $\Delta_n$  denotes the  $n$ th Chebyshev polynomial. A small rectangle on an arc labelled  $n$  represents the idempotent  $f_n$ . For the rest of the paper, we drop the rectangles, and any arc labelled  $n$  represents  $n$  strands colored by  $f_n$ .



$$\begin{array}{c}
\begin{array}{c} \text{---} \\ \boxed{\phantom{x}} \\ \text{---} \end{array}^{n+l} = \begin{array}{c} \text{---} \\ \boxed{\phantom{x}} \\ \text{---} \end{array}^n - \frac{\Delta_{n-1}}{\Delta_n} \begin{array}{c} \text{---} \\ \boxed{\phantom{x}} \\ \text{---} \end{array}^{n-l} \\
\text{where } \Delta_n = \begin{array}{c} \text{---} \\ \boxed{\phantom{x}} \\ \text{---} \end{array}^n \text{ (on a circle)} .
\end{array}$$

Figure 3.6: Definition of the Jones-Wenzl idempotents.

A banded colored trivalent graph in a 3-manifold  $M$  is a framed trivalent graph equipped with a cyclic orientation of the edges incident to each vertex. The framing is given at the vertices by viewing each vertex as a disk with three bands attached (one for each edge). Away from the vertices, the framing is simply the blackboard framing.

Additionally, each edge is colored by a non-negative integer  $n$  which indicates the presence of the  $n$ th Jones-Wenzl idempotent. For the rest of this paper, any unlabelled edge is assumed to be colored one. At each vertex, the colors of the incident edges must form an admissible triple where admissibility is defined as follows.

**Definition 3.6.** *For non-negative integers  $a$ ,  $b$ , and  $c$ , if  $|a - b| \leq c \leq a + b$  and  $a + b + c \equiv 0 \pmod{2}$ , then the triple  $(a, b, c)$  is said to be admissible.*

In fact, such a vertex actually represents a linear combination of skein elements as in Figure 3.7. The inner colors,  $i$ ,  $j$ , and  $k$ , must satisfy the following conditions:  $i + j = a$ ,  $i + k = b$ ,  $j + k = c$ . For a more detailed treatment of the topic of banded colored trivalent graphs see [KL, MV].

We use the same notation as in [GH] for the evaluations of two banded colored trivalent graphs that appear frequently:

$$\begin{array}{c} \text{---} \\ \boxed{\phantom{x}} \\ \text{---} \end{array}^c = \theta(a, b, c) \text{ and } \begin{array}{c} \text{---} \\ \boxed{\phantom{x}} \\ \text{---} \end{array}^e = \text{Det} \begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix} .$$

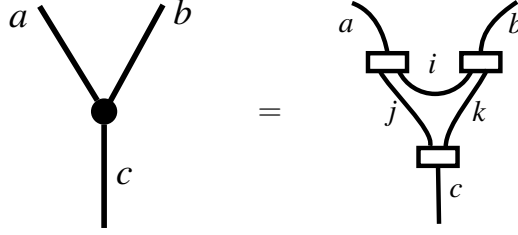


Figure 3.7: A trivalent vertex.

We use the following formulas and theorems when computing the Kauffman bracket ideal of a genus-1 tangle. For details, see [KL, MV, GH].

$$\begin{array}{c} \text{a circle with a dot at the top and bottom, labeled } a \text{ on the left, } b \text{ on the right, } c \text{ on the top, and } d \text{ on the bottom.} \end{array} = \delta_d^c \frac{\theta(a, b, c)}{\Delta_c} \left| \begin{array}{c} c \\ \hline \end{array} \right. \quad (3.2.1)$$

$$\begin{array}{c} \text{a loop with a dot at the bottom, labeled } a \text{ on the left, } b \text{ on the right, and } c \text{ on the bottom.} \end{array} = \lambda_c^{a \ b} \begin{array}{c} \text{a vertex with three incident edges labeled } a, b, \text{ and } c. \end{array} \quad \text{where } \lambda_c^{a \ b} \text{ is as given in [KL]} \quad (3.2.2)$$

$$\begin{array}{c} \text{a triangle with vertices labeled } a, b, c \text{ and edges labeled } e, d, f. \end{array} = \left\{ \begin{array}{l} \frac{Tet \begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix}}{\theta(a, d, e)} \begin{array}{c} \text{a vertex with three incident edges labeled } a, d, \text{ and } e. \\ \end{array}, \quad \text{if } (a, d, e) \text{ is admissible} \\ 0, \quad \text{otherwise} \end{array} \right. \quad (3.2.3)$$

$$\begin{array}{c} \text{a loop with a dot at the top, labeled } i \text{ on the top, and } z \text{ on the left.} \end{array} = \phi_i \left| \begin{array}{c} i \\ \hline \end{array} \right. \quad \text{where } \phi_i = -A^{2i+2} - A^{-2i-2} \quad (3.2.4)$$

$$\begin{array}{c} b \\ \diagdown \\ \bullet \\ \diagup \\ a \end{array} \begin{array}{c} j \\ \text{---} \\ \bullet \\ \text{---} \\ \end{array} \begin{array}{c} c \\ \diagup \\ \bullet \\ \diagdown \\ d \end{array} = \sum_i \left\{ \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\} \begin{array}{c} b \\ \diagdown \\ \bullet \\ \diagup \\ c \\ i \\ \text{---} \\ \bullet \\ \diagdown \\ a \\ \diagup \\ d \end{array}, \quad (3.2.5)$$

where the sum is over all admissible values  $i$  and  $\left\{ \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\} = \frac{Tet \left[ \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right] \Delta_i}{\theta(a, d, i) \theta(b, c, i)}.$

**Theorem 3.7** (Fusion Formula).

$$\left. \begin{array}{c} \\ \\ a \end{array} \right) \left( \begin{array}{c} b \\ \\ \end{array} \right) = \sum_i \frac{\Delta_i}{\theta(a, b, i)} \begin{array}{c} a \\ \diagdown \\ \bullet \\ \diagup \\ b \\ i \\ \text{---} \\ \bullet \\ \diagdown \\ a \\ \diagup \\ b \end{array}$$

where the sum is over all  $i$  such that  $(a, b, i)$  is admissible.

**Theorem 3.8.** If a sphere intersects a skein element in exactly 2 labelled arcs, then

$$\begin{array}{c} a \\ | \\ \text{---} \\ | \\ b \end{array} = \frac{\delta_b^a}{\Delta_a} \begin{array}{c} a \\ \cup \\ \text{---} \\ \cup \\ a \end{array}.$$

If a sphere intersects a skein element in exactly 3 labelled arcs, then

$$\begin{array}{c} a \\ | \\ \text{---} \\ | \\ b \\ | \\ c \end{array} = \begin{cases} \frac{1}{\theta(a, b, c)} \begin{array}{c} a \\ \cup \\ \text{---} \\ \cup \\ b \\ c \end{array}, & \text{if } (a, b, c) \text{ admissible} \\ 0, & \text{otherwise.} \end{cases}$$

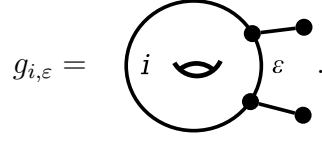
If a sphere intersects a skein element in exactly  $n > 3$  arcs, then

$$\begin{array}{c} a_1 \\ | \\ \text{---} \\ | \\ a_2 \cdots a_{n-1} \\ | \\ a_n \end{array} = \sum \frac{1}{\begin{array}{c} b_1 \\ \cup \\ \text{---} \\ \cup \\ b_{n-3} \end{array}} \begin{array}{c} a_1 \\ \cup \\ \text{---} \\ \cup \\ a_2 \cdots a_{n-1} \\ a_n \end{array},$$

where the sum is over all admissible labellings.

### 3.2.3 Defining the graph basis of $K_R(S^1 \times D^2, 2)$

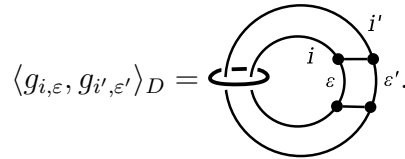
Given a pair of non-negative integers  $(i, \varepsilon)$ , let



Note that this definition implies that the triple  $(1, i, \varepsilon)$  must be admissible. Therefore, either  $\varepsilon = i + 1$  or  $\varepsilon = i - 1$ .

Since we can write any skein element as a linear combination of these  $g_{i,\varepsilon}$ 's using the fusion formula from Theorem 3.7 and Formulas 3.2.1 - 3.2.4, we have that the  $g_{i,\varepsilon}$ 's form a generating set for  $K_R(S^1 \times D^2, 2)$ .

Making use of work of Hoste-Przytycki, we see that  $K_R(S^1 \times S^2)/\text{torsion} = R$  via an isomorphism which sends the empty link to one (see [P, Theorem 2.3 (d)]). We define a pairing  $\langle \cdot, \cdot \rangle_D : K_R(S^1 \times D^2, 2) \times K_R(S^1 \times D^2, 2) \rightarrow K_R(S^1 \times S^2)/\text{torsion} = R$  as follows. First, perform a radial twist on the second solid torus (see Figure 3.8), then identify the boundaries of the two solid tori via an orientation-reversing homeomorphism to obtain  $S^1 \times S^2$ . We view the result as an element of  $K_R(S^1 \times S^2)/\text{torsion}$ . This pairing is symmetric and can be represented pictorially on the  $g_{i,\varepsilon}$ 's as follows, where the loop denotes a 0-surgery:



We call this the doubling pairing. We have the following theorem which shows that the  $g_{i,\varepsilon}$ 's are orthogonal with respect to the doubling pairing and are therefore linearly independent. So they form a basis for  $K_R(S^1 \times D^2, 2)$ .

**Theorem 3.9.**

$$\text{We have that } \langle g_{i,\varepsilon}, g_{i',\varepsilon'} \rangle_D = \begin{cases} \frac{\theta(1, i, \varepsilon)^2}{\Delta_i \Delta_\varepsilon}, & (i, \varepsilon) = (i', \varepsilon') \\ 0, & \text{otherwise.} \end{cases}$$

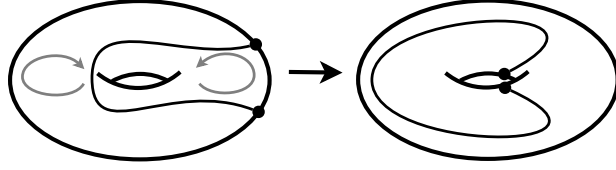


Figure 3.8: A homeomorphism given by twisting the solid torus.

*Proof.* According to Theorem 3.8 and Formula 3.2.1, we have

$$\begin{aligned}
\langle g_{i,\varepsilon}, g_{i',\varepsilon'} \rangle_D &= \frac{\delta_{i'}^i}{\Delta_i} \frac{\delta_{\varepsilon'}^\varepsilon}{\Delta_\varepsilon} \theta(1, i, \varepsilon) \\
&= \frac{\delta_{i'}^i \delta_{\varepsilon'}^\varepsilon \theta(1, i, \varepsilon)}{\Delta_i \Delta_\varepsilon} \\
&= \frac{\delta_{i'}^i \delta_{\varepsilon'}^\varepsilon \theta(1, i, \varepsilon)^2}{\Delta_i \Delta_\varepsilon}.
\end{aligned}$$

□

### 3.3 The almost-orthogonal basis

Recall, several bases for  $K(S^1 \times D^2)$  are defined in [BHMV1]. Let  $e_i$  be a non-contractable loop in  $K(S^1 \times D^2)$  colored  $i$ . The set of all such elements is a basis for  $K(S^1 \times D^2)$ . In particular,  $e_1$  is also denoted by  $z$ . The set  $\{1, z, z^2, \dots\}$  also forms a basis for  $K(S^1 \times D^2)$ , and furthermore,  $K(S^1 \times D^2) = \mathbb{Z}[A, A^{-1}][z]$  as an algebra. Finally,  $\{Q_n\}$  for  $n > 0$  is a basis where  $Q_n = (z - \phi_0)(z - \phi_1) \dots (z - \phi_{n-1})$  and  $\phi_n = -A^{2n+2} - A^{-2n-2}$ . Each of these bases is related to the others by a unimodular triangular basis change.

Recall the Hopf pairing on  $K(S^1 \times D^2)$  defined in [BHMV1]. Choose an orientation-preserving embedding of two disjoint solid tori into  $S^3$  such that each of the standard bands is sent to one component of the banded Hopf link where each component has writhe zero. Then let  $\langle \cdot, \cdot \rangle$  be given by the induced map  $K(S^1 \times D^2) \times K(S^1 \times D^2) \rightarrow K(S^3) = \mathbb{Z}[A, A^{-1}]$ . Note that  $K(S^3)$  is isomorphic to  $\mathbb{Z}[A, A^{-1}]$  via the isomorphism which sends the empty link

to one. We express this pairing pictorially as follows:

$$\langle a, b \rangle = \left\langle \begin{array}{c} \text{Diagram: A vertical link with two components. The top component is a circle with a dot labeled 'b'. The bottom component is a circle with a dot labeled 'a'. They are connected by a vertical line segment.} \end{array} \right\rangle.$$

**Lemma 3.10** (BHMV). *The set  $\{Q_n\}$  is a basis for  $K(S^1 \times D^2)$  which is orthogonal with respect to the bilinear form  $\langle \cdot, \cdot \rangle$ .*

The following formula for  $\langle Q_n, Q_n \rangle$  is stated in [GM, Section 2].

**Lemma 3.11.** *For all  $n \geq 0$ , we have  $\langle Q_n, Q_n \rangle = \Delta_n \prod_{i=0}^{n-1} (\phi_n - \phi_i)$ .*

By adapting the definition of the Hopf pairing, we define an analogous pairing  $\langle \cdot, \cdot \rangle : K_R(S^1 \times D^2, 2) \times K_R(S^1 \times D^2, 2) \rightarrow K_R(S^3) = R$  as follows:

$$\langle a, b \rangle = \left\langle \begin{array}{c} \text{Diagram: A trivalent graph with three vertices. The top vertex has a loop labeled 'b'. The bottom vertex has a loop labeled 'a'. The middle vertex has two loops. The entire graph is enclosed in a large circle.} \end{array} \right\rangle$$

where  $a$  and  $b$  are skein elements lying in a regular neighborhood of the trivalent graphs pictured. Again, we note that  $K_R(S^3)$  is isomorphic to  $R$  via the isomorphism that sends the empty link to one. We call this the relative Hopf pairing. We use the same notation for this pairing as for the Hopf pairing, but the context should make it clear which pairing is being used.

It is easy to see that this pairing is a symmetric bilinear form on  $K_R(S^1 \times D^2, 2)$ . Furthermore, this pairing restricted to  $K(S^1 \times D^2, 2) \times K(S^1 \times D^2, 2)$  is a symmetric bilinear form which takes values in  $K(S^3) = \mathbb{Z}[A, A^{-1}]$ . For simplicity, we use the same notation for the restricted pairing.

We use the basis  $\{Q_n\}$  to define a basis for  $K(S^1 \times D^2, 2)$ . For  $n \geq 0$ , let

$$x_n = \begin{array}{c} \text{Diagram: A circle containing a smaller circle with a loop labeled 'Q_n' inside it.} \end{array} \quad \text{and} \quad y_n = \begin{array}{c} \text{Diagram: A circle containing a smaller circle with a loop labeled 'Q_n' inside it, and a vertical line segment passing through the center.} \end{array}.$$

Since  $\{Q_n\}$  is a basis for  $K(S^1 \times D^2)$ , it is not hard to see that  $\{x_n, y_n\}$  is a basis for  $K(S^1 \times D^2, 2\text{pts.})$ .

Before discussing how this basis relates to the relative Hopf pairing, we must define a map on  $K(S^1 \times D^2)$ . We let  $\bar{\tau}$  denote the mirror image of the map  $\tau$  from [BHMV1]. So,  $\bar{\tau}(u)$  for  $u \in K(S^1 \times D^2)$  is given by adding a single loop as in Figure 3.9. The next two lemmas follow directly from [BHMV1, Lemmas 3.2 and 4.9] respectively.

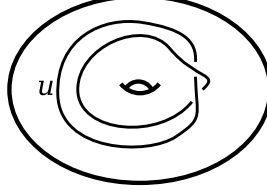


Figure 3.9:  $\bar{\tau}(u)$

**Lemma 3.12.**  $\bar{\tau}(Q_{n-1}) = A^{-2n+2}Q_n + \dots$ , where the dots indicate lower order terms; that is, terms in which the index of each  $Q_i$  appearing is at most  $n - 1$ .

**Lemma 3.13.** One has that  $\langle \bar{\tau}Q_n, Q_n \rangle = (A^{-2n}\sigma_n - A^{-2n+2}\sigma_{n-1})\langle Q_n, Q_n \rangle$  for all  $n \geq 0$  where  $\sigma_n = \sum_{i=0}^n \phi_i$ .

Using these results, we prove the following lemma which states that the basis  $\{x_n, y_n\}$  is almost orthogonal with respect to the relative Hopf pairing. In view of this lemma, we refer to  $\{x_n, y_n\}$  as the almost orthogonal basis.

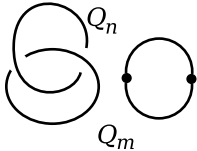
**Lemma 3.14.** We have the following formulas for pairings of elements of  $\{x_n, y_n\}$ :

$$(i) \quad \langle x_m, x_n \rangle = \begin{cases} \delta \langle Q_m, Q_m \rangle, & \text{if } m = n \\ 0, & \text{otherwise.} \end{cases}$$

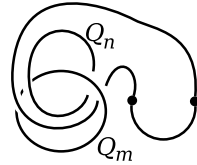
$$(ii) \quad \langle x_m, y_n \rangle = \begin{cases} \langle Q_m, Q_m \rangle, & \text{if } m = n + 1 \\ \phi_m \langle Q_m, Q_m \rangle, & \text{if } m = n \\ 0, & \text{otherwise.} \end{cases}$$

$$(iii) \quad \langle y_m, y_n \rangle = \begin{cases} A^{-2k-4} \langle Q_k, Q_k \rangle \text{ where } k = \max\{m, n\}, & \text{if } |n - m| = 1 \\ A^{-6} (A^{-2m} \sigma_m - A^{-2m+2} \sigma_{m-1}) \langle Q_m, Q_m \rangle, & \text{if } m = n \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* (i) We first consider pairing two  $x$ -elements together. Since the  $Q_i$ 's form an orthogonal basis according to Lemma 3.10, we have

$$\langle x_m, x_n \rangle = \langle \text{diagram} \rangle = \delta \langle Q_m, Q_n \rangle = \begin{cases} \delta \langle Q_m, Q_m \rangle, & \text{if } m = n \\ 0, & \text{otherwise.} \end{cases}$$


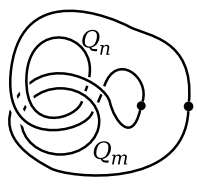

(ii) When pairing  $x_m$  with  $y_n$ , we see that

$$\langle x_m, y_n \rangle = \langle \text{diagram} \rangle = \langle Q_m, zQ_n \rangle.$$


Then, since  $Q_{n+1} = (z - \phi_n)Q_n$ , we have that

$$\langle Q_m, zQ_n \rangle = \langle Q_m, Q_{n+1} + \phi_n Q_n \rangle = \begin{cases} \langle Q_m, Q_m \rangle, & \text{if } m = n + 1 \\ \phi_m \langle Q_m, Q_m \rangle, & \text{if } m = n \\ 0, & \text{otherwise.} \end{cases}$$

(iii) Finally, we have that

$$\langle y_m, y_n \rangle = \langle \text{diagram} \rangle = A^{-6} \langle \text{diagram} \rangle = A^{-6} \langle Q_m, \bar{\tau} Q_n \rangle.$$



If  $m = n$ , then according to Lemma 3.13, we have that  $\langle y_m, y_n \rangle = A^{-6} \langle Q_m, \bar{\tau} Q_m \rangle = A^{-6} (A^{-2m} \sigma_m - A^{-2m+2} \sigma_{m-1}) \langle Q_m, Q_m \rangle$ .

Suppose  $m = n+1$ . Then Lemma 3.12 implies  $\langle y_m, y_n \rangle = A^{-6} \langle Q_m, \bar{\tau} Q_{m-1} \rangle = A^{-6} A^{-2m+2} \langle Q_m, Q_m \rangle$ .

If  $n = m + 1$ , we have  $\langle y_m, y_n \rangle = A^{-6} A^{-2n+2} \langle Q_n, Q_n \rangle$  since the relative Hopf pairing is symmetric.



Suppose  $m \geq n + 2$ . Then  $\langle y_m, y_n \rangle = A^{-6} \langle Q_m, \bar{\tau} Q_n \rangle = A^{-6} \langle Q_m, A^{-2n} Q_{n+1} + \dots \rangle = 0$  since  $m$  is greater than  $n + 1$  and the index of each lower order term. Because the relative Hopf pairing is symmetric, we also have  $\langle y_m, y_n \rangle = 0$  if  $n \geq m + 2$ . In this way, (iii) follows.  $\square$

### 3.4 A finite set of generators for the Kauffman bracket ideal

In this section, we outline an algorithm for computing a finite list of generators for the Kauffman bracket ideal  $I_{\mathcal{G}}$  of a genus-1 tangle. However, we must first discuss how the graph basis and the almost-orthogonal basis relate to one another.

**Lemma 3.15.**

$$\left. \begin{array}{c} | \\ \text{---} \circ \text{---} \\ | \end{array} \right|_i Q_j = \prod_{k=0}^{j-1} (\phi_i - \phi_k) \left| \begin{array}{c} | \\ \text{---} \circ \text{---} \\ | \end{array} \right|_i$$

where  $\phi_n = -A^{2n+2} - A^{-2n-2}$ . If  $j = 0$ , we let  $\prod_{k=0}^{j-1} (\phi_i - \phi_k) = 1$ .

*Proof.* From the definition of  $Q_j$  and Formula 3.2.4, we see that

$$\left. \begin{array}{c} | \\ \text{---} \circ \text{---} \\ | \end{array} \right|_i Q_j = \left. \begin{array}{c} | \\ \text{---} \circ \text{---} \text{---} \circ \text{---} \text{---} \circ \text{---} \\ | \end{array} \right|_i \begin{array}{c} Z-\phi_0 \\ \cdot \\ Z-\phi_1 \\ \vdots \\ Z-\phi_{j-1} \end{array} = \prod_{k=0}^{j-1} (\phi_i - \phi_k) \left| \begin{array}{c} | \\ \text{---} \circ \text{---} \\ | \end{array} \right|_i.$$

$\square$

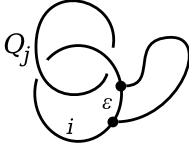
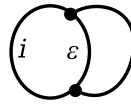
We can now compute the relative Hopf pairings of graph basis elements with almost-orthogonal basis elements, which we need to compute the generators of  $I_{\mathcal{G}}$ .

**Proposition 3.16.** We have that  $\langle g_{i,\varepsilon}, x_j \rangle = \theta(1, \varepsilon, i) \prod_{k=0}^{j-1} (\phi_i - \phi_k)$  and  $\langle g_{i,\varepsilon}, y_j \rangle =$

$\theta(1, \varepsilon, i) (\lambda_{\varepsilon}^{1 \ i})^{-1} (\lambda_{\varepsilon}^{i \ 1})^{-1} \prod_{k=0}^{j-1} (\phi_i - \phi_k)$  for all non-negative  $i, \varepsilon$ , and  $j$ . Again, if  $j = 0$ , we let  $\prod_{k=0}^{j-1} (\phi_i - \phi_k) = 1$ . Note that if  $j > i$ , then both of these pairings are zero.

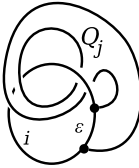
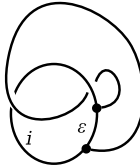
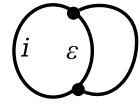
*Proof.* First, consider pairing a graph basis element with  $x_j$ . From Lemma 3.15, we have that

$$\langle g_{i,\varepsilon}, x_j \rangle = \text{Diagram 1} = \prod_{k=0}^{j-1} (\phi_i - \phi_k) \text{Diagram 2} = \theta(1, \varepsilon, i) \prod_{k=0}^{j-1} (\phi_i - \phi_k).$$

For the second case, we have from the Formula 3.2.2 that

$$\begin{aligned} \langle g_{i,\varepsilon}, y_j \rangle &= \text{Diagram 3} = \prod_{k=0}^{j-1} (\phi_i - \phi_k) \text{Diagram 4} \\ &= (\lambda_\varepsilon^{1\ i})^{-1} (\lambda_\varepsilon^{i\ 1})^{-1} \prod_{k=0}^{j-1} (\phi_i - \phi_k) \text{Diagram 5} \\ &= \theta(1, \varepsilon, i) (\lambda_\varepsilon^{1\ i})^{-1} (\lambda_\varepsilon^{i\ 1})^{-1} \prod_{k=0}^{j-1} (\phi_i - \phi_k). \end{aligned}$$

□

We are now able to outline an algorithm for explicitly computing a finite list of generators for the Kauffman bracket ideal of a genus-1 tangle  $\mathcal{G}$ . Let  $L$  be a closure of  $\mathcal{G}$ . Then  $L$  may be viewed as the relative Hopf pairing of  $\mathcal{G}$  with some complementary genus-1 tangle  $\mathcal{H}$ . Since the set  $\{x_n, y_n\}$  is a basis for  $K(S^1 \times D^2, 2)$ , we have that  $\mathcal{H}$  can be written as a linear combination of elements of  $\{x_n, y_n\}$ . Thus,  $\langle L \rangle = \langle \mathcal{G}, \mathcal{H} \rangle$  is a linear combination of pairings  $\langle \mathcal{G}, x_j \rangle$  and  $\langle \mathcal{G}, y_j \rangle$ . Since we are considering only non-empty links, we have that  $\langle L \rangle' = \langle L \rangle / \delta = \langle \mathcal{G}, \mathcal{H} \rangle / \delta \in \mathbb{Z}[A, A^{-1}]$ . So,  $\langle \mathcal{G}, x_j \rangle / \delta$  and  $\langle \mathcal{G}, y_j \rangle / \delta$  form a generating set for  $I_{\mathcal{G}}$ .

To compute these generators, we view  $\mathcal{G}$  as an element of  $K_R(S^1 \times D^2, 2)$  which allows us to write  $\mathcal{G}$  as a linear combination  $\sum c_{i,\varepsilon} g_{i,\varepsilon}$  of graph basis elements. We do this by computing the doubling pairing  $\langle \mathcal{G}, g_{i,\varepsilon} \rangle_D$  for each  $(i, \varepsilon)$  using Theorem 3.8, along with the fusion formula

given in Theorem 3.7 and Formulas 3.2.1 - 3.2.4. Since the graph basis is orthogonal with respect to the doubling pairing, we have that  $c_{i,\varepsilon} = \langle \mathcal{G}, g_{i,\varepsilon} \rangle_D / \langle g_{i,\varepsilon}, g_{i,\varepsilon} \rangle_D$  for any  $(i, \varepsilon)$ .

Then,  $\langle \mathcal{G}, x_j \rangle / \delta = \langle \sum c_{i,\varepsilon} g_{i,\varepsilon}, x_j \rangle / \delta = \sum (c_{i,\varepsilon} / \delta) \langle g_{i,\varepsilon}, x_j \rangle$  and  $\langle \mathcal{G}, y_j \rangle / \delta = \langle \sum c_{i,\varepsilon} g_{i,\varepsilon}, y_j \rangle / \delta = \sum (c_{i,\varepsilon} / \delta) \langle g_{i,\varepsilon}, y_j \rangle$ . We compute these pairings using Proposition 3.16 which states that  $\langle g_{i,\varepsilon}, x_j \rangle$  and  $\langle g_{i,\varepsilon}, y_j \rangle$  are non-zero only if  $i \geq j$ . There are only finitely many  $j$  less than or equal to a given  $i$ , and there are finitely many non-zero terms in the linear combination  $\mathcal{G} = \sum c_{i,\varepsilon} g_{i,\varepsilon}$ . Therefore, the set of all non-zero  $\langle \mathcal{G}, x_j \rangle / \delta$  and  $\langle \mathcal{G}, y_j \rangle / \delta$  is a finite generating set for the Kauffman bracket ideal  $I_{\mathcal{G}}$ .

### 3.5 Partial closures

Recall, the partial closure of a  $(B^3, 2n)$ -tangle  $\mathcal{T}$  is a genus-1 tangle obtained from  $\mathcal{T}$  by gluing a copy of  $D^2 \times I$  containing  $n - 1$  properly embedded arcs to  $B^3$  as indicated in Figure 3.4. We denote the partial closure by  $\hat{\mathcal{T}}$ . We can describe this more colloquially as partially closing off  $\mathcal{T}$  with  $n - 1$  simple arcs and placing the hole of the solid torus as indicated in Figure 3.4.

Theorem 3.4 states that in the case of a  $(B^3, 4)$ -tangle whose partial closure has a single component, the Kauffman bracket ideal of the partial closure is exactly the Kauffman bracket ideal of the original  $(B^3, 4)$ -tangle. Before proving this result, we need the following lemma.

**Lemma 3.17.** *Let  $\mathcal{T}$  be a  $(B^3, 4)$ -tangle and let  $\hat{\mathcal{T}}$  denote the genus-1 tangle which is the partial closure of  $\mathcal{T}$ . Then the Kauffman bracket ideal  $I_{\hat{\mathcal{T}}} = \langle \langle d(\mathcal{T}) \rangle', (1 - A^{-4}) \langle n(\mathcal{T}) \rangle' \rangle$ .*

*Proof.* Recall, we can think of the operation of taking closures of  $(B^3, 2n)$ -tangles as a symmetric bilinear pairing on  $K(B^3, 2n)$  as follows:

$$\left\langle \begin{array}{c} \text{S} \\ \dots \end{array} , \begin{array}{c} \text{R} \\ \dots \end{array} \right\rangle = \left\langle \begin{array}{c} \text{S} \quad \text{R} \\ \text{figure-eight} \end{array} \right\rangle.$$

For a given closure  $L$  of  $\hat{\mathcal{T}}$ ,  $L$  has  $k$  strands passing through the hole of the solid torus for some non-negative integer  $k$ . We can think of  $\langle L \rangle$  as the pairing of the  $(B^3, 2k+2)$ -tangle in Figure 3.10, denoted by  $\mathcal{T}_k$ , with some complementary  $(B^3, 2k+2)$ -tangle. Since the Catalan tangles form a basis for  $K(B^3, 2k+2)$ , we can write  $\langle L \rangle$  as a linear combination of  $\langle \mathcal{T}_k, \mathcal{C} \rangle$  where  $\mathcal{C}$  is a  $(2k+2)$ -Catalan tangle. See Figure 3.11 for an example where  $k = 2$ .

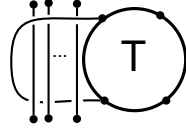


Figure 3.10: Given a  $(B^3, 4)$ -tangle  $\mathcal{T}$ , the tangle consisting of  $\hat{\mathcal{T}}$  with an additional  $k$  strands placed as above is denoted by  $\mathcal{T}_k$ .

For any non-negative integer  $k$  and Catalan tangle  $\mathcal{C}$ , we show that  $\langle \mathcal{T}_k, \mathcal{C} \rangle = f \langle d(\mathcal{T}) \rangle + g(1 - A^{-4}) \langle n(\mathcal{T}) \rangle$  for some  $f$  and  $g$  in  $\mathbb{Z}[A, A^{-1}]$ . Hence,  $\langle L \rangle$  is also a linear combination of  $\langle d(\mathcal{T}) \rangle$  and  $(1 - A^{-4}) \langle n(\mathcal{T}) \rangle$ , and we have that the reduced Kauffman bracket polynomial  $\langle L \rangle'$  is generated by  $\langle d(\mathcal{T}) \rangle'$  and  $(1 - A^{-4}) \langle n(\mathcal{T}) \rangle'$ .

$$\begin{aligned}
 & \text{Diagram of } L = \langle \text{Diagram of } \mathcal{T}_2, \text{Diagram of } \mathcal{C} \rangle \\
 &= A \langle \text{Diagram of } \mathcal{T}_2, \text{Diagram of } \mathcal{C}_1 \rangle + A^{-1} \langle \text{Diagram of } \mathcal{T}_2, \text{Diagram of } \mathcal{C}_2 \rangle
 \end{aligned}$$

Figure 3.11: The partial closure  $L$  of  $\hat{\mathcal{T}}$  represented as a linear combination of pairings of  $(B^3, 6)$ -tangles.

We proceed by induction on  $k$ . If  $k = 0$ , then  $L = \langle \mathcal{T}_0, \mathcal{U} \rangle = \langle \hat{\mathcal{T}}, \mathcal{U} \rangle$  for some  $(B^3, 2)$ -tangle  $\mathcal{U}$  and it is easy to see that  $\langle L \rangle$  is a multiple of  $\langle d(\mathcal{T}) \rangle$ .

If  $k = 1$ , then there are two possibilities for  $\langle \mathcal{T}_1, \mathcal{C} \rangle$ :

$$\langle \mathcal{T}_1, \text{diagram} \rangle = \langle \text{diagram}, \text{diagram} \rangle = \langle \text{diagram} \rangle = (-A^4 - A^{-4})\langle d(\mathcal{T}) \rangle,$$

and

$$\begin{aligned} \langle \mathcal{T}_1, \text{diagram} \rangle &= \langle \text{diagram}, \text{diagram} \rangle = \langle \text{diagram} \rangle \\ &= A \langle \text{diagram} \rangle + A^{-1} \langle \text{diagram} \rangle \\ &= A^2 \langle \text{diagram} \rangle + \langle \text{diagram} \rangle - A^{-4} \langle \text{diagram} \rangle \\ &= (1 - A^{-4})\langle n(\mathcal{T}) \rangle + A^2\langle d(\mathcal{T}) \rangle. \end{aligned} \tag{3.5.1}$$

Suppose the property holds for  $k > 1$ , and consider  $\langle \mathcal{T}_{k+1}, \mathcal{C} \rangle$ . We have two cases to consider. The first case is that the Catalan tangle  $\mathcal{C}$  connects two of the  $k + 1$  strands in  $\mathcal{T}_{k+1}$  which are adjacent. Then,

$$\langle \mathcal{T}_{k+1}, \mathcal{C} \rangle = \langle \text{diagram}, \mathcal{C} \rangle = \langle \text{diagram} \rangle$$

and we may perform a Reidemeister II move as follows:

$$\langle \text{diagram} \rangle = \langle \text{diagram} \rangle = \langle \text{diagram}, \mathcal{C}' \rangle = \langle \mathcal{T}_{k-1}, \mathcal{C}' \rangle$$


where  $\mathcal{C}'$  is some  $(2k - 2)$ -Catalan tangle. So,  $\langle \mathcal{T}_{k+1}, \mathcal{C} \rangle = \langle \mathcal{T}_{k-1}, \mathcal{C}' \rangle = f\langle d(\mathcal{T}) \rangle + g(1 - A^{-4})\langle n(\mathcal{T}) \rangle$  for some  $f$  and  $g$  in  $\mathbb{Z}[A, A^{-1}]$ .

If no adjacent strands in  $\mathcal{T}_{k+1}$  via pairing with the Catalan tangle, then

$$\begin{aligned}
\langle \mathcal{T}_{k+1}, \mathcal{C} \rangle &= \left\langle \left( \begin{array}{c} \text{Diagram: A circle labeled } \mathcal{T} \text{ with } k+1 \text{ strands entering from the left and exiting to the right. The strands are paired in a way that forms a tangle } \mathcal{C}. \end{array} \right), \mathcal{C} \right\rangle = \left\langle \left( \begin{array}{c} \text{Diagram: A circle labeled } \mathcal{T} \text{ with } k+1 \text{ strands entering from the left and exiting to the right. The strands are paired in a way that forms a tangle } \mathcal{C}. \end{array} \right), \mathcal{C} \right\rangle \\
&= (-A^4 - A^{-4}) \left\langle \left( \begin{array}{c} \text{Diagram: A circle labeled } \mathcal{T} \text{ with } k+1 \text{ strands entering from the left and exiting to the right. The strands are paired in a way that forms a tangle } \mathcal{C}'. \end{array} \right), \mathcal{C}' \right\rangle = (-A^4 - A^{-4}) \langle \mathcal{T}_k, \mathcal{C}' \rangle
\end{aligned}$$

where  $\mathcal{C}'$  is some  $(2k+2)$ -Catalan tangle.

So, given any non-negative integer  $k$  and any  $(2k+2)$ -Catalan tangle, we have that  $\langle \mathcal{T}_k, \mathcal{C} \rangle = f\langle d(\mathcal{T}) \rangle + g(1 - A^{-4})\langle n(\mathcal{T}) \rangle$  for some  $f$  and  $g$  in  $\mathbb{Z}[A, A^{-1}]$ . Thus, for any closure  $L$  of  $\hat{\mathcal{T}}$ , we see that  $\langle L \rangle$  is a linear combination of  $\langle d(\mathcal{T}) \rangle$  and  $(1 - A^{-4})\langle n(\mathcal{T}) \rangle$ . This implies that  $\langle L \rangle'$  is a linear combination of  $\langle d(\mathcal{T}) \rangle'$  and  $(1 - A^{-4})\langle n(\mathcal{T}) \rangle'$ , and so  $I_{\hat{\mathcal{T}}} \subseteq \langle \langle d(\mathcal{T}) \rangle', (1 - A^{-4})\langle n(\mathcal{T}) \rangle' \rangle$ .

Since the denominator  $d(\mathcal{T})$  is clearly a closure of  $\hat{\mathcal{T}}$ , we have that  $\langle d(\mathcal{T}) \rangle' \in I_{\hat{\mathcal{T}}}$ . Let  $\mathcal{S}$  denote the tangle . We have from Equation (3.5.1) that  $(1 - A^{-4})\langle n(\mathcal{T}) \rangle = \langle \mathcal{T}_1, \mathcal{S} \rangle - A^2\langle d(\mathcal{T}) \rangle$ . Since  $\langle \mathcal{T}_1, \mathcal{S} \rangle$  is the Kauffman bracket polynomial of a closure of  $\hat{\mathcal{T}}$ , we see that  $\langle \mathcal{T}_1, \mathcal{S} \rangle / \delta \in I_{\hat{\mathcal{T}}}$ . So,  $(1 - A^{-4})\langle n(\mathcal{T}) \rangle'$  is the difference of two elements of  $I_{\hat{\mathcal{T}}}$  and is therefore an element of  $I_{\hat{\mathcal{T}}}$  itself. Hence, the Kauffman bracket ideal  $I_{\hat{\mathcal{T}}} = \langle \langle d(\mathcal{T}) \rangle', (1 - A^{-4})\langle n(\mathcal{T}) \rangle' \rangle$ .  $\square$

We now prove Theorem 3.4.

*Proof of Theorem 3.4.* Let  $\mathcal{T}$  be a  $(B^3, 4)$ -tangle with partial closure  $\hat{\mathcal{T}}$  which has a single component. Since any closure of  $\hat{\mathcal{T}}$  is also a closure of  $\mathcal{T}$ , we have that  $I_{\hat{\mathcal{T}}} \subseteq I_{\mathcal{T}} = \langle \langle n(\mathcal{T}) \rangle', \langle d(\mathcal{T}) \rangle' \rangle$  according to Theorem 3.1. According to Lemma 3.17,  $I_{\hat{\mathcal{T}}} = \langle \langle d(\mathcal{T}) \rangle', (1 - A^{-4})\langle n(\mathcal{T}) \rangle' \rangle$ , so it remains only to show that  $\langle n(\mathcal{T}) \rangle' \in I_{\hat{\mathcal{T}}}$  to prove equality of the two ideals.

Since  $\hat{\mathcal{T}}$  has a single component, the denominator  $d(\mathcal{T})$  is a knot. Then, its Jones polynomial  $J_{d(\mathcal{T})}(t)$  evaluated at one is one by [J, Theorem 15] and so  $J_{d(\mathcal{T})}(t) = (1 - t)f(t) + 1$  for some  $f(t) \in \mathbb{Z}[t, t^{-1}]$ . Since  $J_{d(\mathcal{T})}(t) = A^{-3\omega}\langle d(\mathcal{T}) \rangle'$  where  $A^{-4} = t$  and  $\omega$  is the writhe of an

oriented diagram of the denominator, we have that

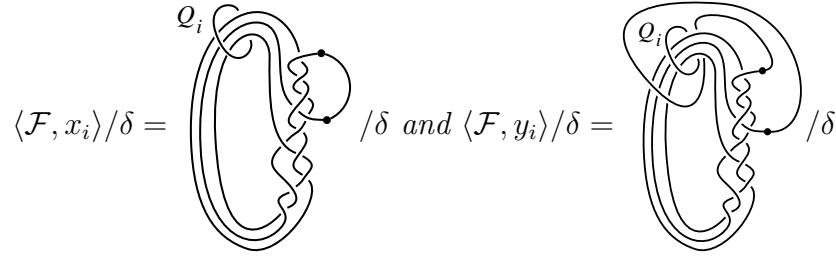
$$\langle d(\mathcal{T}) \rangle' = A^{3\omega}(1 - A^{-4})f(A^{-4}) + A^{3\omega}. \quad (3.5.2)$$

Then,  $\langle n(\mathcal{T}) \rangle' \langle d(\mathcal{T}) \rangle' \in I_{\hat{\mathcal{T}}}$  since  $\langle d(\mathcal{T}) \rangle' \in I_{\hat{\mathcal{T}}}$ . We also have from Equation 3.5.2 that  $\langle n(\mathcal{T}) \rangle' \langle d(\mathcal{T}) \rangle' = A^{3\omega}f(A^{-4})(1 - A^{-4})\langle n(\mathcal{T}) \rangle' + A^{3\omega}\langle n(\mathcal{T}) \rangle'$ . Clearly,  $A^{3\omega}f(A^{-4})(1 - A^{-4})\langle n(\mathcal{T}) \rangle' \in I_{\hat{\mathcal{T}}}$ . So,  $A^{3\omega}\langle n(\mathcal{T}) \rangle'$  and thus  $\langle n(\mathcal{T}) \rangle'$  are elements of  $I_{\hat{\mathcal{T}}}$  as well. This concludes the proof.  $\square$

### 3.6 An example

*Proof of Theorem 3.3.*

**Lemma 3.18.**  $I_{\mathcal{F}}$  is generated by the following elements:



where  $0 \leq i \leq 3$ .

*Proof.* As described in Section 3.4, we write  $\mathcal{F}$  as a linear combination  $\mathcal{F} = \sum c_{i,\varepsilon} g_{i,\varepsilon}$  of graph basis elements, and we have that

$$c_{i,\varepsilon} = \langle \mathcal{F}, g_{i,\varepsilon} \rangle_D / \langle g_{i,\varepsilon}, g_{i,\varepsilon} \rangle_D. \quad (3.6.1)$$

The formula for  $\langle g_{i,\varepsilon}, g_{i,\varepsilon} \rangle_D$  is given in Theorem 3.9. We need to compute  $\langle \mathcal{F}, g_{i,\varepsilon} \rangle_D$ .

We use Theorems 3.7 and 3.8 and Formulas 3.2.1 - 3.2.4 to find a general formula for  $\langle \mathcal{F}, g_{i,\varepsilon} \rangle_D$ . The computation of this formula is given in Appendix B. From the second line of that computation, one can see using admissibility that  $\langle \mathcal{F}, g_{i,\varepsilon} \rangle_D = 0$  unless  $i = 1$  or  $i = 3$ . So, we need only compute four coefficients:  $c_{1,0}$ ,  $c_{1,2}$ ,  $c_{3,2}$ , and  $c_{3,4}$ . Using some code from [H], we implemented the formula derived in Appendix B in Mathematica to find explicit expressions for these coefficients. As  $\langle \mathcal{F}, g_{1,0} \rangle_D$  turned out to be zero, we have that  $c_{1,0} = 0$ .

The remaining three coefficients are as follows:

$$\begin{aligned}
c_{1,2} &= \frac{\langle \mathcal{F}, g_{1,2} \rangle_D}{\langle g_{1,2}, g_{1,2} \rangle_D} = \text{Diagram} / \langle g_{1,2}, g_{1,2} \rangle_D \\
&= (1 + A^4 + A^8)^{-1} (-A^{-21} + 2A^{-17} - 4A^{-13} + 4A^{-9} - 3A^{-5} + \\
&\quad 2A^{-1} + A^3 - 4A^7 + 4A^{11} - 4A^{15} + 2A^{19} - A^{23}), \\
c_{3,2} &= \frac{\langle \mathcal{F}, g_{3,2} \rangle_D}{\langle g_{3,2}, g_{3,2} \rangle_D} = \text{Diagram} / \langle g_{3,2}, g_{3,2} \rangle_D \\
&= (A^7 + A^{11} + A^{15} + A^{19})^{-1} (1 - A^4 + A^8 - A^{16} + A^{20}), \text{ and} \\
c_{3,4} &= \frac{\langle \mathcal{F}, g_{3,4} \rangle_D}{\langle g_{3,4}, g_{3,4} \rangle_D} = \text{Diagram} / \langle g_{3,4}, g_{3,4} \rangle_D = A^3.
\end{aligned}$$

So  $\mathcal{F} = c_{1,2}g_{1,2} + c_{3,2}g_{3,2} + c_{3,4}g_{3,4}$ .

Now, we use Lemma 3.15 and Proposition 3.16 to compute  $\langle \mathcal{F}, x_i \rangle / \delta$  and  $\langle \mathcal{F}, y_i \rangle / \delta$  for any  $i$  to obtain a list of generators for  $I_{\mathcal{F}}$ . According to Proposition 3.16, since  $\mathcal{F} = c_{1,2}g_{1,2} + c_{3,2}g_{3,2} + c_{3,4}g_{3,4}$ , we have that  $\langle \mathcal{F}, x_i \rangle / \delta$  and  $\langle \mathcal{F}, y_i \rangle / \delta$  are zero if  $i > 3$ . So,  $I_{\mathcal{F}}$  is generated by  $\langle \mathcal{F}, x_i \rangle / \delta$  and  $\langle \mathcal{F}, y_i \rangle / \delta$  where  $i \leq 3$ . An explicit expression for each generator is given in Appendix C. □



Let  $g_1, \dots, g_8$  denote these eight generators rescaled by the power of  $A$  necessary to make the lowest degree term a constant.<sup>1</sup> Since  $A$  is a unit in  $\mathbb{Z}[A, A^{-1}]$ , we have that the ideal  $\langle g_1, \dots, g_8 \rangle_{\mathbb{Z}[A, A^{-1}]} = I_{\mathcal{F}}$ . Using the GroebnerBasis command in Mathematica, we see that  $\{11, 4 - A^4\}$  is a generating set for the ideal  $\langle g_1, \dots, g_8 \rangle_{\mathbb{Z}[A]}$ .

**Lemma 3.19.** *The Kauffman bracket ideal  $I_{\mathcal{F}} = \langle 11, 4 - A^4 \rangle_{\mathbb{Z}[A, A^{-1}]}$ , and  $I_{\mathcal{F}}$  is non-trivial in  $\mathbb{Z}[A, A^{-1}]$ .*

*Proof.* We first show that  $I_{\mathcal{F}} \subseteq \langle 11, 4 - A^4 \rangle_{\mathbb{Z}[A, A^{-1}]}$ . Let  $f \in I_{\mathcal{F}}$ . Then  $f = f_1 g_1 + \dots + f_8 g_8$  for some  $f_i \in \mathbb{Z}[A, A^{-1}]$ . Since  $\langle g_1, \dots, g_8 \rangle_{\mathbb{Z}[A]} = \langle 11, 4 - A^4 \rangle_{\mathbb{Z}[A]}$ , we have for each  $i$  that  $g_i = 11r_i + (4 - A^4)s_i$  for some  $r_i$  and  $s_i$  in  $\mathbb{Z}[A]$ . Then,

$$\begin{aligned} f &= f_1 g_1 + \dots + f_8 g_8 \\ &= f_1(11r_1 + (4 - A^4)s_1) + \dots + f_8(11r_8 + (4 - A^4)s_8) \\ &= 11(f_1 r_1 + \dots + f_8 r_8) + (4 - A^4)(f_1 s_1 + \dots + f_8 s_8) \end{aligned}$$

which is an element of  $\langle 11, 4 - A^4 \rangle_{\mathbb{Z}[A, A^{-1}]}$ .

It is easy to see that  $\langle 11, 4 - A^4 \rangle \subseteq I_{\mathcal{F}}$ . Since 11 and  $4 - A^4$  are elements of  $\langle g_1, \dots, g_8 \rangle_{\mathbb{Z}[A]}$ , it follows immediately they are both in  $\langle g_1, \dots, g_8 \rangle_{\mathbb{Z}[A, A^{-1}]} = I_{\mathcal{F}}$ . Therefore,  $I_{\mathcal{F}} = \langle 11, 4 - A^4 \rangle_{\mathbb{Z}[A, A^{-1}]}$ .

It remains only to show that  $I_{\mathcal{F}} = \langle 11, 4 - A^4 \rangle_{\mathbb{Z}[A, A^{-1}]}$  is non-trivial. Let  $\rho : \mathbb{Z}[A, A^{-1}] \rightarrow \mathbb{Z}_{11}$  be the map which sends  $A$  to 3. It is easy to see that  $\rho$  is a ring homomorphism. The image of  $I_{\mathcal{F}}$  under  $\rho$  is the ideal  $\langle 11, 4 - 81 \rangle = \langle 11 \rangle = \langle 0 \rangle$  in  $\mathbb{Z}_{11}$ . So,  $I_{\mathcal{F}} \subseteq \ker \rho$ . Since  $\rho$  is not the trivial homomorphism, this implies that  $I_{\mathcal{F}} \neq \mathbb{Z}[A, A^{-1}]$ .  $\square$

Recall that the Jones polynomial of an oriented link  $L$  is defined to be  $J_L(\sqrt{t}) = A^{-3\omega(D)} \langle D \rangle'$  where  $D$  is an oriented diagram of  $L$  with writhe  $\omega(D)$  and  $t = A^{-4}$ .

We show that if  $L$  is a closure of  $\mathcal{F}$ , then  $J_L(\sqrt{t})$  evaluated at  $\sqrt{t} = 5$  is 0 (mod 11). Let  $D$  be an oriented diagram for  $L$ . Then  $\langle D \rangle' \in I_{\mathcal{F}}$  and thus  $A^{-3\omega(D)} \langle D \rangle' \in I_{\mathcal{F}}$ . So,

---

<sup>1</sup>Actually, as seen in Appendix C,  $g_4$  is a multiple of  $g_3$ , and  $g_8$  is a multiple of  $g_7$ . Thus,  $g_4$  and  $g_8$  are not needed in the list of generators.

$\rho(A^{-3\omega(D)}\langle D \rangle') = 0$  in  $\mathbb{Z}_{11}$ . Note that  $\sqrt{t} = 5$  implies  $t = 25 = 3 \pmod{11}$  and  $3^{-4} = \frac{1}{81} = \frac{1}{4} = 3 \pmod{11}$ . Therefore,  $J_L(\sqrt{t})|_{\sqrt{t}=5} = \rho(A^{-3\omega(D)}\langle D \rangle') = 0 \pmod{11}$ .

□

**Proposition 3.20.** *The genus-1 tangle  $\mathcal{F}$  contains no local knots.*

*Proof.* One closure of  $\mathcal{F}$  is the knot  $10_{57}$  which is prime and therefore has no local knots. Therefore,  $\mathcal{F}$  has no local knots unless  $10_{57}$  itself is a local knot. In that case, any closure  $L$  of  $\mathcal{F}$  may be written as the connect sum of  $10_{57}$  with some knot  $K \subset S^3$ , and we have that  $\langle L \rangle' = \langle 10_{57} \rangle' \langle K \rangle'$ . Thus,  $I_{\mathcal{F}}$  is the principal ideal generated by  $\langle 10_{57} \rangle'$ . However, since  $11 \in I_{\mathcal{F}}$ , this means that 11 is a multiple of  $\langle 10_{57} \rangle'$  which is impossible. □

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# Appendix A: Mathematica Notebook for Chapter 3

In this section, we give the Mathematica notebook described in Section 3.1 which we used to identify possible examples of genus-1 tangles with non-trivial Kauffman bracket ideals. As discussed in Section 3.1, we viewed genus-1 tangles as partial closures of braids. The genus-1 tangles we consider in this notebook are partial closures of braid representatives of knots. We utilized the KnotTheory package [BN] to write this code.

`br[m, n]` gives the braid representative for `Knot[m, n]`, and `index[m, n]` gives the braid index of that braid representative; that is, the number of strands in the braid representative `br[m, n]`. So, for each knot (up to 10 crossings) we can obtain a genus-1 tangle  $\mathcal{G}$  as the partial closure of the knot's braid representative. These partial closures form our list of possible examples.

```
br[m_, n_] := BR[Knot[m, n]]  
index[m_, n_] := First@br[m, n]
```

`JonesScaled[L]` computes the rescaled Jones polynomial of a link  $L$ . If the lowest degree exponent in the Jones polynomial is negative, then the polynomial is rescaled by the minimum power of  $t$  necessary to make the lowest degree term a constant. `JonesScaled[L]` lies in  $\mathbb{Z}[t]$ .

```
JonesScaled[L_] := If[Exponent[Jones[L][t], t, Min] < 0,  
  Jones[L][t] * t^(-Exponent[Jones[L][t], t, Min]) // Expand, Jones[L][t]]
```

`FrontToBack[]` takes a braid (in our case, the braid representative of a knot) and concatenates to it the braid word which corresponds to the genus-1 tangle closure pictured on the left in Figure 3.5. `BackToFront[]` concatenates to a braid the braid word corresponding to

the genus-1 tangle closure pictured on the right in Figure 3.5. `Range[k - 1]` gives the list  $\{1, 2, \dots, k - 1\}$  while `Reverse[Range[k - 1]]` simply gives this same list in reverse order.

```
Neg[x_] := -x
FrontToBack[BR[k_, l_]] := BR[k, Join[l, Range[k - 1], Reverse[Range[k - 1]]]]
BackToFront[BR[k_, l_]] := BR[k, Join[l, Neg[Range[k - 1]], Neg[Reverse[Range[k - 1]]]]]
```

By iterating `FrontToBack[]` and `BackToFront[]`, we create a list of closures of the genus-1 tangle  $\mathcal{G}$ . `ClosureList[m, n]` gives the list (without repetition) generated by iterating `FrontToBack[]` and `BackToFront[]` five times each. So this list includes the closures formed by wrapping the closing strand around through the hole of the solid torus containing  $\mathcal{G}$  front to back and back to front  $n$  times, where  $0 \leq n \leq 5$ .

We compute `JonesScaled[]` for each of these eleven closures, forming a set of polynomials in  $\mathbb{Z}[t]$ . `Ideal[]` computes the Groebner basis of the ideal generated by these eleven polynomials.

```
Ideal[m_, n_, k_] := If[index[m, n] == k, Print[{m, n},
  GroebnerBasis[JonesScaled /@ ClosureList[m, n], t, CoefficientDomain -> Integers]]]
```

If this ideal has trivial Groebner basis, then the Kauffman bracket ideal of  $\mathcal{G}$  is trivial. Thus, any tangle which has a non-trivial Groebner basis is a potential example. For a fixed integer  $k$ , we completed this ideal computation for every genus-1 tangle obtained from a knot whose braid representative has index  $k$ . Our output lists  $\{m, n\}$  followed by the Groebner basis for the genus-1 tangle obtained from `Knot[m, n]`, for every knot with braid index  $k$ . We do not consider the case where  $k = 2$  since the partial closure of any 2-stranded braid embeds in the unknot.

```
Do[Ideal[m, n, 3], {m, 1, 10}, {n, 1, 165}]
```

$\{4, 1\}\{1\}$   
 $\{5, 2\}\{-1\}$   
 $\{6, 2\}\{1\}$   
 $\{6, 3\}\{1\}$   
 $\{7, 3\}\{-1\}$   
 $\{7, 5\}\{-1\}$   
 $\{8, 2\}\{1\}$   
 $\{8, 5\}\{1\}$   
 $\{8, 7\}\{-1\}$   
 $\{8, 9\}\{1\}$   
 $\{8, 10\}\{-1\}$   
 $\{8, 16\}\{1\}$   
 $\{8, 17\}\{-1\}$   
 $\{8, 18\}\{1\}$   
 $\{8, 19\}\{-1\}$   
 $\{8, 20\}\{1\}$   
 $\{8, 21\}\{1\}$   
 $\{9, 3\}\{-1\}$   
 $\{9, 6\}\{-1\}$   
 $\{9, 9\}\{-1\}$   
 $\{9, 16\}\{1\}$   
 $\{10, 2\}\{-1\}$   
 $\{10, 5\}\{-1\}$   
 $\{10, 9\}\{-1\}$   
 $\{10, 17\}\{-1\}$   
 $\{10, 46\}\{1\}$   
 $\{10, 47\}\{1\}$   
 $\{10, 48\}\{1\}$   
 $\{10, 62\}\{1\}$   
 $\{10, 64\}\{1\}$   
 $\{10, 79\}\{1\}$   
 $\{10, 82\}\{1\}$   
 $\{10, 85\}\{-1\}$   
 $\{10, 91\}\{1\}$   
 $\{10, 94\}\{1\}$

```

{10, 94}{1}
{10, 99}{1}
{10, 100}{1}
{10, 104}{1}
{10, 106}{1}
{10, 109}{1}
{10, 112}{1}
{10, 116}{1}
{10, 118}{1}
{10, 123}{1}
{10, 124}{1}
{10, 125}{1}
{10, 126}{1}
{10, 127}{1}
{10, 139}{1}
{10, 141}{-1}
{10, 143}{1}
{10, 148}{1}
{10, 149}{-1}
{10, 152}{1}
{10, 155}{-1}
{10, 157}{1}
{10, 159}{1}
{10, 161}{-1}

```

We see that the case where  $k = 3$  does not yield any potential examples; all tangles have ideals with trivial Groebner basis.

```

Do[Ideal[m, n, 4], {m, 1, 10}, {n, 1, 165}]
{6, 1}{1}
{7, 2}{1}
{7, 4}{1}
{7, 6}{1}
{7, 7}{-1}

```

$\{8, 4\} \{1\}$   
 $\{8, 6\} \{1\}$   
 $\{8, 8\} \{1\}$   
 $\{8, 11\} \{-1\}$   
 $\{8, 13\} \{-1\}$   
 $\{8, 14\} \{1\}$   
 $\{8, 15\} \{1\}$   
 $\{9, 4\} \{1\}$   
 $\{9, 7\} \{1\}$   
 $\{9, 10\} \{1\}$   
 $\{9, 11\} \{1\}$   
 $\{9, 13\} \{-1\}$   
 $\{9, 17\} \{1\}$   
 $\{9, 18\} \{1\}$   
 $\{9, 20\} \{-1\}$   
 $\{9, 22\} \{1\}$   
 $\{9, 23\} \{-1\}$   
 $\{9, 24\} \{-1\}$   
 $\{9, 26\} \{1\}$   
 $\{9, 27\} \{1\}$   
 $\{9, 28\} \{-1\}$   
 $\{9, 29\} \{-1\}$   
 $\{9, 30\} \{-1\}$   
 $\{9, 31\} \{1\}$   
 $\{9, 32\} \{1\}$   
 $\{9, 33\} \{1\}$   
 $\{9, 34\} \{1\}$   
 $\{9, 36\} \{-1\}$   
 $\{9, 38\} \{1\}$   
 $\{9, 40\} \{1\}$   
 $\{9, 42\} \{1\}$   
 $\{9, 43\} \{1\}$   
 $\{9, 44\} \{1\}$   
 $\{9, 45\} \{1\}$   
 $\{9, 46\} \{-1\}$   
 $\{9, 47\} \{1\}$   
 $\{9, 48\} \{1\}$   
 $\{9, 49\} \{1\}$



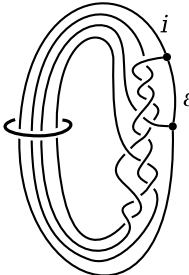
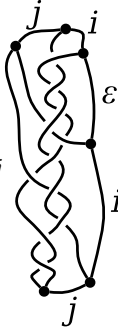
$\{10, 6\}\{1\}$   
 $\{10, 8\}\{-1\}$   
 $\{10, 12\}\{1\}$   
 $\{10, 14\}\{1\}$   
 $\{10, 15\}\{1\}$   
 $\{10, 19\}\{-1\}$   
 $\{10, 21\}\{1\}$   
 $\{10, 22\}\{1\}$   
 $\{10, 23\}\{1\}$   
 $\{10, 25\}\{-1\}$   
 $\{10, 26\}\{1\}$   
 $\{10, 39\}\{-1\}$   
 $\{10, 40\}\{1\}$   
 $\{10, 49\}\{1\}$   
 $\{10, 50\}\{1\}$   
 $\{10, 51\}\{1\}$   
 $\{10, 52\}\{-1\}$   
 $\{10, 54\}\{1\}$   
 $\{10, 56\}\{1\}$   
 $\{10, 57\}\{11, -3+t\}$   
 $\{10, 61\}\{1\}$   
 $\{10, 65\}\{1\}$   
 $\{10, 66\}\{-1\}$   
 $\{10, 72\}\{-1\}$   
 $\{10, 76\}\{-1\}$   
 $\{10, 77\}\{1\}$   
 $\{10, 80\}\{1\}$   
 $\{10, 83\}\{1\}$   
 $\{10, 84\}\{1\}$   
 $\{10, 86\}\{1\}$   
 $\{10, 87\}\{1\}$   
 $\{10, 90\}\{-1\}$   
 $\{10, 92\}\{1\}$   
 $\{10, 93\}\{1\}$   
 $\{10, 95\}\{1\}$   
 $\{10, 98\}\{-1\}$   
 $\{10, 102\}\{1\}$   
 $\{10, 103\}\{1\}$   
 $\{10, 108\}\{1\}$

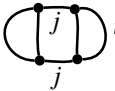
$\{10, 111\}\{1\}$   
 $\{10, 113\}\{-1\}$   
 $\{10, 114\}\{-1\}$   
 $\{10, 117\}\{11, -5+t\}$   
 $\{10, 119\}\{1\}$   
 $\{10, 121\}\{1\}$   
 $\{10, 122\}\{1\}$   
 $\{10, 128\}\{1\}$   
 $\{10, 129\}\{1\}$   
 $\{10, 130\}\{1\}$   
 $\{10, 131\}\{-1\}$   
 $\{10, 132\}\{-1\}$   
 $\{10, 133\}\{1\}$   
 $\{10, 134\}\{-1\}$   
 $\{10, 135\}\{1\}$   
 $\{10, 140\}\{1\}$   
 $\{10, 142\}\{1\}$   
 $\{10, 144\}\{1\}$   
 $\{10, 145\}\{-1\}$   
 $\{10, 146\}\{1\}$   
 $\{10, 147\}\{-1\}$   
 $\{10, 150\}\{-1\}$   
 $\{10, 151\}\{1\}$   
 $\{10, 153\}\{-1\}$   
 $\{10, 154\}\{1\}$   
 $\{10, 156\}\{1\}$   
 $\{10, 158\}\{1\}$   
 $\{10, 160\}\{1\}$   
 $\{10, 162\}\{11, 2+t\}$   
 $\{10, 163\}\{1\}$   
 $\{10, 164\}\{1\}$   
 $\{10, 165\}\{1\}$

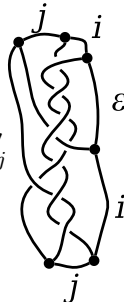
We see that partial closures of braid representatives of knots  $10_{57}$ ,  $10_{117}$ , and  $10_{162}$  have non-trivial ideals.

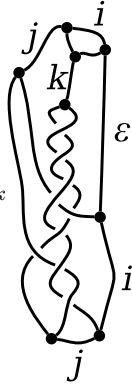
## Appendix B: Writing $\mathcal{F}$ as a linear combinations of basis elements

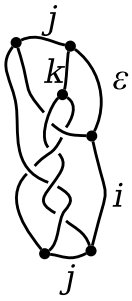
Here we give the computation illustrating how to write  $\mathcal{F}$  as a linear combination of graph basis elements  $g_{i,\varepsilon}$ . We first find a general formula for the pairing  $\langle \mathcal{F}, g_{i,\varepsilon} \rangle_D$  for any  $(i, \varepsilon)$ . Using this formula and Mathematica code from [H], we were able to compute this pairing and find the explicit formulas for non-zero  $c_{i,\varepsilon}$  given in Section 3.6. The Mathematica notebook we used to do this is available in Appendix C. Each sum in the following computation ranges over all admissible colorings of the corresponding graph. Using Theorems 3.7 and 3.8 along with Formulas 3.2.1 - 3.2.4, we have that:

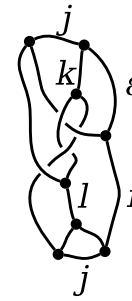
$$\langle \mathcal{F}, g_{i,\varepsilon} \rangle_D = \text{Diagram 1} = \sum_j c_j \text{Diagram 2}$$



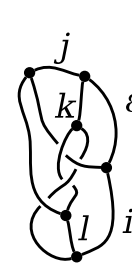
$$\text{where } c_j = \frac{1}{\text{Diagram 3}} = \frac{\Delta_j}{\theta(1, 1, j)\theta(1, i, j)}$$


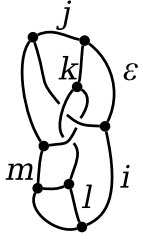
$$= \sum_j c'_j \text{Diagram 4} \quad \text{where } c'_j = c_j(\lambda_j^{11})^2$$


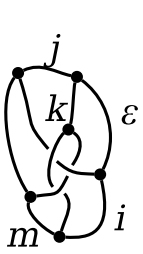
$$= \sum_{j,k} c_{j,k} \quad \text{where } c_{j,k} = c'_j \frac{\Delta_k}{\theta(1,1,k)}$$


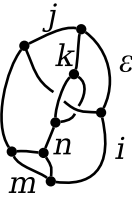
$$= \sum_{j,k} c'_{j,k} \quad \text{where } c'_{j,k} = c_{j,k} \frac{Tet \begin{bmatrix} j & i & \varepsilon \\ 1 & k & 1 \end{bmatrix} (\lambda_k^{11})^{-3}}{\theta(j,k,\varepsilon)}$$


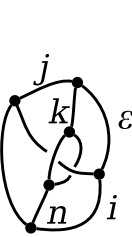
$$= \sum_{j,k,l} c_{j,k,l} \quad \text{where } c_{j,k,l} = c'_{j,k} \frac{\Delta_l (\lambda_l^{11})^{-2}}{\theta(1,1,l)}$$


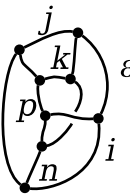
$$= \sum_{j,k,l} c'_{j,k,l} \quad \text{where } c'_{j,k,l} = c_{j,k,l} \frac{Tet \begin{bmatrix} i & j & 1 \\ 1 & l & 1 \end{bmatrix}}{\theta(i,l,1)}$$


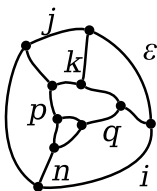
$$= \sum_{j,k,l,m} c_{j,k,l,m} \quad \text{where } c_{j,k,l,m} = c'_{j,k,l} \frac{\Delta_m \lambda_m^{11}}{\theta(1,1,m)}$$


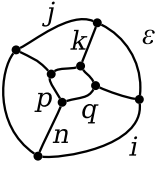
$$= \sum_{j,k,l,m} c'_{j,k,l,m} \quad \text{where } c'_{j,k,l,m} = c_{j,k,l,m} \frac{Tet \begin{bmatrix} 1 & l & i \\ 1 & m & 1 \end{bmatrix}}{\theta(1,m,i)}$$


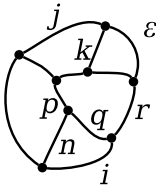
$$= \sum_{j,k,l,m,n} c_{j,k,l,m,n} \quad \text{where } c_{j,k,l,m,n} = c'_{j,k,l,m} \frac{\Delta_n (\lambda_n^{11})^{-1}}{\theta(1,1,n)}$$


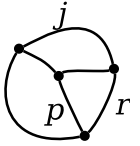
$$= \sum_{j,k,l,m,n} c'_{j,k,l,m,n} \quad \text{where } c'_{j,k,l,m,n} = c_{j,k,l,m,n} \frac{Tet \begin{bmatrix} 1 & 1 & n \\ 1 & i & m \end{bmatrix}}{\theta(1,i,n)}$$


$$= \sum_{j,k,l,m,n,p} c_{j,k,l,m,n,p} \quad \text{where } c_{j,k,l,m,n,p} = c'_{j,k,l,m,n} \frac{\Delta_p (\lambda_p^{11})^{-1}}{\theta(1,1,p)}$$


$$= \sum_{j,\dots,p,q} c_{j,\dots,p,q} \quad \text{where } c_{j,\dots,p,q} = c_{j,k,l,m,n,p} \frac{\Delta_q (\lambda_q^{11})^{-1}}{\theta(1,1,q)}$$


$$= \sum_{j, \dots, p, q} c'_{j, \dots, p, q} \quad \text{where } c'_{j, \dots, p, q} = c_{j, \dots, p, q} \frac{Tet \begin{bmatrix} n & 1 & p \\ 1 & q & 1 \end{bmatrix}}{\theta(n, q, p)}$$


$$= \sum_{j, \dots, p, q, r} c_{j, \dots, p, q, r} \quad \text{where } c_{j, \dots, p, q, r} = c'_{j, \dots, p, q} \begin{Bmatrix} q & 1 & r \\ \varepsilon & i & 1 \end{Bmatrix}$$


$$= \sum_{j, \dots, p, q, r} c'_{j, \dots, p, q, r} \quad \text{where } c'_{j, \dots, p, q, r} = c_{j, \dots, p, q, r} \frac{Tet \begin{bmatrix} 1 & n & p \\ q & r & i \end{bmatrix} Tet \begin{bmatrix} 1 & k & j \\ \varepsilon & r & 1 \end{bmatrix}}{\theta(1, r, p)\theta(1, r, j)}$$


$$= \sum_{j, \dots, p, q, r} Tet \begin{bmatrix} 1 & p & 1 \\ 1 & j & r \end{bmatrix} c'_{j, \dots, p, q, r} \cdot$$

## Appendix C: Mathematica notebook used to find generators of $I_{\mathcal{F}}$

We now give the Mathematica notebook used to compute the generators of  $I_{\mathcal{F}}$ . We also give explicit expressions for the generators themselves. The following code was written by Harris in [H], where the formulas are evaluated as in [KL].

`oddq[]` and `evenq[]` extend `OddQ[]` and `EvenQ[]` to variables.

```
oddq[a_ b_ /; oddq[a] && oddq[b]] := True;
oddq[a_ + b_ /; (oddq[a] && evenq[b]) || (evenq[a] && oddq[b])] := True;
oddq[a_] := OddQ[a];

evenq[a_ b_ /; (evenq[a] && IntegerQ[b]) || (evenq[b] && IntegerQ[a])] := True;
evenq[a_ + b_ /; (evenq[a] && evenq[b]) || (oddq[a] && oddq[b])] := True;
evenq[a_] := EvenQ[a];
```

`qi[n]` is the  $n$ th quantum integer, and `qif[n]` is the quantum integer factorial. Quantum integers and their factorials are left unevaluated. `delta[n]` is the  $n$ th Chebyshev polynomial, while `adm[a1,b1,c1]` returns “True” if  $(a1,b1,c1)$  is an admissible triple and “False” otherwise. Here,  $\text{lambda}[a,b,c] = \lambda_c^a b$  given in Formula 3.2.2, and  $\text{theta}[a,b,c] = \theta(a,b,c)$ .

```
qi[0] = 0; qi[1] = 1;

qif[0] = 1; qif[n_ /; n ≥ 1] := qif[n-1] qi[n];
qif[n_ + x_ /; n ≥ 1] := qif[n+x-1] qi[n+x];

qi[n_] := Sum[A^i, {i, 2-2n, 2n-2, 4}];

delta[n_] := (-1)^n qi[n+1];

adm[a1_, b1_, c1_] := Module[{a = Simplify[a1], b = Simplify[b1], c = Simplify[c1]},
  Simplify[a ≥ 0 && b ≥ 0 && c ≥ 0 && Abs[a-b] ≤ c && c ≤ a+b, given] && evenq[a+b+c];
```

```

lambda[a_, b_, c_] :=
  (-1) ^ ((a+b-c) / 2) A ^ ((a (a+2) + b (b+2) - c (c+2)) / 2) // Simplify // Expand;

theta[a_, b_, c_] := Module[{
  m = (a+b-c) / 2 // Simplify,
  n = (b+c-a) / 2 // Simplify,
  p = (a+c-b) / 2 // Simplify
},

  If[adm[a, b, c],
    (-1) ^ (m+n+p) qif[m+n+p+1] qif[m]
    qif[n] qif[p] / qif[m+n] / qif[n+p] / qif[m+p] // Simplify,
    0]
];

```

Here,  $\text{admtet}[a, b, c, d, e, f]$  tests whether  $(a, b, c, d, e, f)$  is an admissible coloring of the tetrahedron graph, and  $\text{tet}[a, b, c, d, e, f] = \text{Tet} \begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix}$ .

```

admtet[a_, b_, c_, d_, e_, f_] := adm[a, d, e] && adm[b, c, e] && adm[a, b, f] && adm[c, d, f];

```

```

tet[a_, b_, c_, d_, e_, f_] := Module[{
  a1 = (a+d+e) / 2 // Simplify,
  a2 = (b+c+e) / 2 // Simplify,
  a3 = (a+b+f) / 2 // Simplify,
  a4 = (c+d+f) / 2 // Simplify,
  av,

  b1 = (b+d+e+f) / 2 // Simplify,
  b2 = (a+c+e+f) / 2 // Simplify,
  b3 = (a+b+c+d) / 2 // Simplify,
  bv,

  m, M, cv, s
},

  av = {a1, a2, a3, a4}; bv = {b1, b2, b3};
  m = Max[a1, a2, a3, a4]; M = Min[b1, b2, b3];

  If[admtet[a, b, c, d, e, f],
    intfac = Product[qif[bv[[j]] - av[[i]]], {i, 1, 4}, {j, 1, 3}];
    extfac = qif[a] qif[b] qif[c] qif[d] qif[e] qif[f];
    cv = Intersection[av, bv];
    (intfac / extfac) If[Length[cv] > 0, s = cv[[1]]; (-1) ^ s
      qif[s+1] / Product[qif[s - av[[i]]], {i, 1, 4}] / Product[qif[bv[[j]] - s], {j, 1, 3}],
      Sum[(-1) ^ s qif[s+1] / Product[qif[s - av[[i]]], {i, 1, 4}] /
        Product[qif[bv[[j]] - s], {j, 1, 3}], {s, m, M}] // Simplify,
    0]
];

```



This concludes the code written by Harris.  $\text{Fusion}[c]$  is the coefficient given in Theorem 3.7 resulting from performing fusion on two strands each colored  $a = b = 1$ , where  $c$  is the color of the resulting fused strand, corresponding to  $i$  in the theorem statement.

```
Fusion[c_] := If[adm[1, 1, c], delta[c] / theta[1, 1, c], 0]
```

$\text{TetR}[a, b, c, d, e, f]$  is the coefficient resulting from reducing a tetrahedron in a graph as in Formula 3.2.3, where  $a, b, c, d, e$ , and  $f$  are as pictured in the statement of the formula.

```
TetR[a_, b_, c_, d_, e_, f_] :=  
If[admtet[a, b, c, d, e, f], tet[a, b, c, d, e, f] / theta[a, d, e], 0]
```

$\text{SixJ}[a, b, c, d, e, f]$  is the  $6j$ -symbol  $\begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix}$  given in Formula 3.2.5.

```
SixJ[a_, b_, c_, d_, e_, f_] :=  
If[admtet[a, b, c, d, e, f], tet[a, b, c, d, e, f] delta[e] / theta[a, d, e] / theta[b, c, e], 0]
```

$\text{Phi}[i]$  corresponds to  $\phi_i$  given in Formula 3.2.4.

```
Phi[i_] := -A^(2 i + 2) - A^(-2 i - 2)
```

$\text{Dpairing}[i, e] = \langle \mathcal{F}, g_{i,e} \rangle_D$ . The formula for  $\text{Dpairing}[i, e]$  is derived in Appendix B.

```
Dpairing[i_, e_] := Sum[  
  If[adm[1, i, j] && adm[1, e, r] && adm[q, r, i], (delta[j] / theta[1, 1, j] / theta[1, i, j]) *  
    lambda[1, 1, j]^2 Fusion[k] TetR[j, i, 1, k, e, 1] lambda[1, 1, k]^3 Fusion[l]  
    lambda[1, 1, l]^2 TetR[i, j, 1, l, 1, 1] Fusion[m] lambda[1, 1, m] TetR[1, l, 1, m, i, 1]  
    Fusion[n] lambda[1, 1, n]^1 TetR[1, 1, 1, i, n, m] Fusion[p] lambda[1, 1, p]^1  
    Fusion[q] lambda[1, 1, q]^1 TetR[n, 1, 1, q, p, 1] SixJ[q, 1, e, i, r, 1]  
    TetR[1, n, q, r, p, i] TetR[1, k, e, r, j, 1] tet[1, p, 1, j, 1, r], 0],  
{j, 0, 2, 2}, {k, 0, 2, 2}, {l, 0, 2, 2}, {m, 0, 2, 2}, {n, 0, 2, 2},  
{p, 0, 2, 2}, {q, 0, 2, 2}, {r, i - 2, i + 2}] // Simplify
```

Recall from Section 3.6, we have  $\mathcal{F} = c_{1,2}g_{1,2} + c_{3,2}g_{3,2} + c_{3,4}g_{3,4}$  where  $c_{i,\varepsilon} = \langle \mathcal{F}, g_{i,\varepsilon} \rangle_D / \langle g_{i,\varepsilon}, g_{i,\varepsilon} \rangle_D$ .

$\text{Coeff}[i, e]$  corresponds to  $c_{i,e}$  which is computed according to Proposition 3.16.

```

Coeff[i_, e_] := Dpairing[i, e] / (theta[i, 1, e]^2 / delta[i] / delta[e]) // Simplify

Coeff[1, 0]
0

Coeff[1, 2]

$$\left(-1 + 2 A^4 - 4 A^8 + 4 A^{12} - 3 A^{16} + 2 A^{20} + A^{24} - 4 A^{28} + 4 A^{32} - 4 A^{36} + 2 A^{40} - A^{44}\right) / \left(A^{21} \left(1 + A^4 + A^8\right)\right)$$


Coeff[3, 2]

$$\frac{1 - A^4 + A^8 - A^{16} + A^{20}}{A^7 + A^{11} + A^{15} + A^{19}}$$


Coeff[3, 4]
A^3

```

According to Lemma 3.18, generators of  $I_{\mathcal{F}}$  are:  $\langle \mathcal{F}, x_i \rangle / \delta$  and  $\langle \mathcal{F}, y_i \rangle / \delta$  for  $0 \leq i \leq 3$ . The generators are computed and labelled as follows:

$$\begin{array}{ll}
\text{g1hat} &= \langle \mathcal{F}, x_0 \rangle / \delta & \text{g5hat} &= \langle \mathcal{F}, y_0 \rangle / \delta \\
\text{g2hat} &= \langle \mathcal{F}, x_1 \rangle / \delta & \text{g6hat} &= \langle \mathcal{F}, y_1 \rangle / \delta \\
\text{g3hat} &= \langle \mathcal{F}, x_2 \rangle / \delta & \text{g7hat} &= \langle \mathcal{F}, y_2 \rangle / \delta \\
\text{g4hat} &= \langle \mathcal{F}, x_3 \rangle / \delta & \text{g8hat} &= \langle \mathcal{F}, y_3 \rangle / \delta
\end{array}$$

We rescale each gihat by  $A^k$  where  $k$  is the minimum power necessary to make the lowest degree term a constant. Upon rescaling, we relabel gihat by  $g_i = g_i$ .

```

del := -A^2 - A^-2
gihat =
(Coeff[1, 2] theta[1, 2, 1] + Coeff[3, 2] theta[1, 2, 3] + Coeff[3, 4] theta[1, 4, 3]) / del //
Simplify // Expand

```

$$\frac{1}{A^{23}} - \frac{3}{A^{19}} + \frac{7}{A^{15}} - \frac{10}{A^{11}} + \frac{12}{A^7} - \frac{14}{A^3} + 12A - 10A^5 + 6A^9 - 3A^{13} + A^{17}$$

**g1 = g1hat\*A^23 // Simplify**

$$1 - 3A^4 + 7A^8 - 10A^{12} + 12A^{16} - 14A^{20} + 12A^{24} - 10A^{28} + 6A^{32} - 3A^{36} + A^{40}$$

**g2hat =**

$$(\text{Coeff}[1, 2] \theta[1, 2, 1] (\Phi[1] - \Phi[0]) + \text{Coeff}[3, 2] \theta[1, 2, 3] (\Phi[3] - \Phi[0]) + \text{Coeff}[3, 4] \theta[1, 4, 3] (\Phi[3] - \Phi[0])) / \text{del} // \text{Simplify} // \text{Expand}$$

$$-\frac{1}{A^{27}} + \frac{1}{A^{25}} + \frac{3}{A^{23}} - \frac{2}{A^{21}} - \frac{9}{A^{19}} + \frac{4}{A^{17}} + \frac{16}{A^{15}} - \frac{3}{A^{13}} - \frac{23}{A^{11}} + \frac{2}{A^9} + \frac{30}{A^7} - \frac{2}{A^5} - \frac{31}{A^3} - \frac{2}{A} + 30A + 2A^3 - 24A^5 - 4A^7 + 17A^9 + 3A^{11} - 9A^{13} - 2A^{15} + 4A^{17} + A^{19} - A^{21}$$

**g2 = g2hat\*A^27 // Simplify // Expand**

$$-1 + A^2 + 3A^4 - 2A^6 - 9A^8 + 4A^{10} + 16A^{12} - 3A^{14} - 23A^{16} + 2A^{18} + 30A^{20} - 2A^{22} - 31A^{24} - 2A^{26} + 30A^{28} + 2A^{30} - 24A^{32} - 4A^{34} + 17A^{36} + 3A^{38} - 9A^{40} - 2A^{42} + 4A^{44} + A^{46} - A^{48}$$

**g3hat =**

$$(\text{Coeff}[3, 2] \theta[1, 2, 3] (\Phi[3] - \Phi[0]) (\Phi[3] - \Phi[1]) + \text{Coeff}[3, 4] \theta[1, 4, 3] (\Phi[3] - \Phi[0]) (\Phi[3] - \Phi[1])) / \text{del} // \text{Simplify} // \text{Expand}$$

$$\frac{1}{A^{27}} - \frac{3}{A^{23}} - \frac{1}{A^{21}} + \frac{4}{A^{19}} + \frac{2}{A^{17}} - \frac{6}{A^{15}} - \frac{1}{A^{13}} + \frac{8}{A^{11}} + \frac{2}{A^9} - \frac{9}{A^7} - \frac{1}{A^5} + \frac{10}{A^3} - \frac{1}{A} - 10A + 9A^5 - 2A^7 - 7A^9 + 2A^{11} + 6A^{13} - A^{15} - 4A^{17} + A^{19} + 2A^{21} - A^{25}$$

**g3 = g3hat\*A^27 // Simplify // Expand**

$$1 - 3A^4 - A^6 + 4A^8 + 2A^{10} - 6A^{12} - A^{14} + 8A^{16} + 2A^{18} - 9A^{20} - A^{22} + 10A^{24} - A^{26} - 10A^{28} + 9A^{32} - 2A^{34} - 7A^{36} + 2A^{38} + 6A^{40} - A^{42} - 4A^{44} + A^{46} + 2A^{48} - A^{52}$$

$$\text{g4hat} = (\text{Coeff}[3, 2] \theta[1, 2, 3] (\Phi[3] - \Phi[0]) (\Phi[3] - \Phi[1]) (\Phi[3] - \Phi[2]) + \text{Coeff}[3, 4] \theta[1, 4, 3] (\Phi[3] - \Phi[0]) (\Phi[3] - \Phi[1]) (\Phi[3] - \Phi[2])) / \text{del} // \text{Simplify} // \text{Expand}$$

$$-\frac{1}{A^{35}} + \frac{1}{A^{33}} + \frac{3}{A^{31}} - \frac{2}{A^{29}} - \frac{5}{A^{27}} + \frac{2}{A^{25}} + \frac{8}{A^{23}} - \frac{4}{A^{21}} - \frac{10}{A^{19}} + \frac{3}{A^{17}} + \frac{13}{A^{15}} - \frac{3}{A^{13}} - \frac{13}{A^{11}} + \frac{3}{A^9} + \frac{14}{A^7} - \frac{1}{A^5} - \frac{15}{A^3} + 13A + 2A^3 - 15A^5 - 2A^7 + 13A^9 + 4A^{11} - 12A^{13} - 3A^{15} + 10A^{17} + 3A^{19} - 7A^{21} - 3A^{23} + 5A^{25} + A^{27} - 2A^{29} - A^{31} + A^{33}$$

**g4 = g4hat\*A^35 // Simplify // Expand**

$$-1 + A^2 + 3A^4 - 2A^6 - 5A^8 + 2A^{10} + 8A^{12} - 4A^{14} - 10A^{16} + 3A^{18} + 13A^{20} - 3A^{22} - 13A^{24} + 3A^{26} + 14A^{28} - A^{30} - 15A^{32} + 13A^{34} + 2A^{36} - 15A^{38} - 2A^{40} + 13A^{42} + 4A^{44} - 12A^{46} - 3A^{50} + 10A^{52} + 3A^{54} - 7A^{56} - 3A^{58} + 5A^{60} + A^{62} - 2A^{64} - A^{66} + A^{68}$$

g5hat = (Coeff[1, 2] lambda[1, 1, 2] ^-2 theta[1, 2, 1] +  
 Coeff[3, 2] lambda[1, 3, 2] ^-1 lambda[3, 1, 2] ^-1 theta[1, 2, 3] + Coeff[3, 4]  
 lambda[1, 3, 4] ^-1 lambda[3, 1, 4] ^-1 theta[1, 4, 3]) / del // Simplify // Expand

$$\frac{2}{A^{21}} - \frac{5}{A^{17}} + \frac{10}{A^{13}} - \frac{14}{A^9} + \frac{16}{A^5} - \frac{17}{A} + 15A^3 - 12A^7 + 7A^{11} - 4A^{15} + A^{19}$$

g5 = g5hat \* A^21 // Simplify // Expand

$$2 - 5A^4 + 10A^8 - 14A^{12} + 16A^{16} - 17A^{20} + 15A^{24} - 12A^{28} + 7A^{32} - 4A^{36} + A^{40}$$

g6hat =  
 (Coeff[1, 2] lambda[1, 1, 2] ^-2 theta[1, 2, 1] (Phi[1] - Phi[0]) + Coeff[3, 2] lambda[1, 3, 2] ^  
 -1 lambda[3, 1, 2] ^-1 theta[1, 2, 3] (Phi[3] - Phi[0]) + Coeff[3, 4] lambda[1, 3, 4] ^  
 -1 lambda[3, 1, 4] ^-1 theta[1, 4, 3] (Phi[3] - Phi[0])) / del // Simplify // Expand

$$-\frac{1}{A^{29}} + \frac{1}{A^{25}} + \frac{2}{A^{23}} - \frac{3}{A^{19}} - \frac{5}{A^{17}} + \frac{5}{A^{15}} + \frac{11}{A^{13}} - \frac{4}{A^{11}} - \frac{18}{A^9} + \frac{2}{A^7} + \frac{24}{A^5} - \frac{1}{A^3} -$$

$$\frac{25}{A} - 2A + 25A^3 + 3A^5 - 20A^7 - 5A^9 + 14A^{11} + 3A^{13} - 7A^{15} - 3A^{17} + 3A^{19} + A^{21}$$

g6 = g6hat \* A^29 // Simplify // Expand

$$-1 + A^4 + 2A^6 - 3A^{10} - 5A^{12} + 5A^{14} + 11A^{16} - 4A^{18} - 18A^{20} + 2A^{22} + 24A^{24} - A^{26} -$$

$$25A^{28} - 2A^{30} + 25A^{32} + 3A^{34} - 20A^{36} - 5A^{38} + 14A^{40} + 3A^{42} - 7A^{44} - 3A^{46} + 3A^{48} + A^{50}$$

g7hat = (Coeff[3, 2] lambda[1, 3, 2] ^-1 lambda[3, 1, 2] ^-1 theta[1, 2, 3] (Phi[3] - Phi[0])  
 (Phi[3] - Phi[1]) + Coeff[3, 4] lambda[1, 3, 4] ^-1 lambda[3, 1, 4] ^-1  
 theta[1, 4, 3] (Phi[3] - Phi[0]) (Phi[3] - Phi[1])) / del // Simplify // Expand

$$\frac{1}{A^{37}} - \frac{3}{A^{33}} - \frac{1}{A^{31}} + \frac{5}{A^{29}} + \frac{2}{A^{27}} - \frac{7}{A^{25}} - \frac{2}{A^{23}} + \frac{9}{A^{21}} + \frac{2}{A^{19}} - \frac{11}{A^{17}} - \frac{1}{A^{15}} + \frac{12}{A^{13}} - \frac{13}{A^9} + \frac{1}{A^7} +$$

$$\frac{12}{A^5} - \frac{1}{A^3} - \frac{10}{A} + 2A + 9A^3 - 2A^5 - 7A^7 + 5A^{11} - A^{13} - 3A^{15} + 2A^{19} - A^{23} + A^{25} + A^{27} - A^{31}$$

g7 = g7hat \* A^37 // Simplify // Expand

$$1 - 3A^4 - A^6 + 5A^8 + 2A^{10} - 7A^{12} - 2A^{14} + 9A^{16} + 2A^{18} - 11A^{20} - A^{22} + 12A^{24} - 13A^{28} + A^{30} +$$

$$12A^{32} - A^{34} - 10A^{36} + 2A^{38} + 9A^{40} - 2A^{42} - 7A^{44} + 5A^{48} - A^{50} - 3A^{52} + 2A^{56} - A^{60} + A^{62} + A^{64} - A^{68}$$

g8hat = (Coeff[3, 2] lambda[1, 3, 2] ^-1 lambda[3, 1, 2] ^-1  
 theta[1, 2, 3] (Phi[3] - Phi[0]) (Phi[3] - Phi[1]) (Phi[3] - Phi[2]) +  
 Coeff[3, 4] lambda[1, 3, 4] ^-1 lambda[3, 1, 4] ^-1 theta[1, 4, 3]  
 (Phi[3] - Phi[0]) (Phi[3] - Phi[1]) (Phi[3] - Phi[2])) / del // Simplify // Expand

$$-\frac{1}{A^{45}} + \frac{1}{A^{43}} + \frac{3}{A^{41}} - \frac{2}{A^{39}} - \frac{6}{A^{37}} + \frac{3}{A^{35}} + \frac{9}{A^{33}} - \frac{4}{A^{31}} - \frac{12}{A^{29}} + \frac{4}{A^{27}} + \frac{15}{A^{25}} - \frac{4}{A^{23}} - \frac{16}{A^{21}} +$$

$$\frac{3}{A^{19}} + \frac{18}{A^{17}} - \frac{3}{A^{15}} - \frac{18}{A^{13}} + \frac{19}{A^9} + \frac{1}{A^7} - \frac{19}{A^5} - \frac{2}{A^3} + \frac{19}{A} + 4A - 18A^3 - 3A^5 + 14A^7 + 4A^9 -$$

$$13A^{11} - 3A^{13} + 8A^{15} + 3A^{17} - 6A^{19} - A^{21} + 4A^{23} + A^{25} - 2A^{27} - A^{29} + 2A^{31} - A^{35} - A^{37} + A^{39}$$

g8 = g8hat \* A^45 // Simplify // Expand

$$-1 + A^2 + 3A^4 - 2A^6 - 6A^8 + 3A^{10} + 9A^{12} - 4A^{14} - 12A^{16} + 4A^{18} + 15A^{20} - 4A^{22} - 16A^{24} + 3A^{26} +$$

$$18A^{28} - 3A^{30} - 18A^{32} + 19A^{36} + A^{38} - 19A^{40} - 2A^{42} + 19A^{44} + 4A^{46} - 18A^{48} - 3A^{50} + 14A^{52} +$$

$$4A^{54} - 13A^{56} - 3A^{58} + 8A^{60} + 3A^{62} - 6A^{64} - A^{66} + 4A^{68} + A^{70} - 2A^{72} - A^{74} + 2A^{76} - A^{80} - A^{82} + A^{84}$$

Note that  $\mathbf{g4hat} = \langle \mathcal{F}, x_3 \rangle / \delta = (\phi_3 - \phi_2) \langle \mathcal{F}, x_2 \rangle / \delta = (\phi_3 - \phi_2) \mathbf{g3hat}$  and  $\mathbf{g8hat} = \langle \mathcal{F}, y_3 \rangle / \delta = (\phi_3 - \phi_2) \langle \mathcal{F}, y_2 \rangle / \delta = (\phi_3 - \phi_2) \mathbf{g7hat}$ .

We use the `GroebnerBasis` command to find an easier to analyze generating set of the ideal of  $\mathbb{Z}[A]$  generated by  $\{\mathbf{g1}, \dots, \mathbf{g8}\}$ .

```
GroebnerBasis[{g1, g2, g3, g4, g5, g6, g7, g8}, A, CoefficientDomain  $\rightarrow$  Integers]
{11, 4 - A4}
```

We prove in Section 3.6, Lemma 3.19, that  $I_{\mathcal{F}} = \langle 11, 4 - A^4 \rangle$  as ideals in  $\mathbb{Z}[A, A^{-1}]$ .






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# Vita

Susan M. Abernathy was born in Louisiana and grew up in Texas. She completed her undergraduate studies at Trinity University in May 2007, earning a Bachelor of Arts degree in mathematics. She began her graduate studies at Louisiana State University in August 2007, and earned a Master of Science in mathematics in May 2009. She is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2014.