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THE INSENSITIVITY OF LEONTIEF MULTIPLIERS TO
RANDOM INPUT-OUTPUT MATRICES WITH FIXED
COLUMN SUMS.

THE LOUISIANA STATE UNIVERSITY AND
AGRICULTURAL AND MECHANICAL COL., PH.D., 1978

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THE INSENSITIVITY OF LEONTIEF MULTIPLIERS TO RANDOM
INPUT-OUTPUT MATRICES WITH FIXED COLUMN SUMS

A DISSERTATION
Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Quantitative Methods

by

Joseph Lorne Katz
B.S., Louisiana State University, 1973
M.S., Louisiana State University, 1975
August, 1978
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ABSTRACT

Input-Output analysis provides a technique to examine the interrelationships within an economic system. Although Input-Output analysis was originally developed by Wassily Leontief to measure intersectorial activity in the whole American economy, this technique has been applied to the study of economic activity within cities, states, and regions. The capability of Input-Output analysis to completely account for the complex interactions among industries makes it useful as an aid in economic planning and development for both industry and government.

One of the most useful measures that is obtained from Input-Output analysis is the Leontief multiplier. These multipliers, computed from a matrix of technical coefficients, evaluate the economic effects of each industry on the regional or national economy. While multipliers have many applications, the determination of multipliers through the use of Input-Output analysis requires a considerable amount of time and effort.

The purpose of this dissertation is to develop a mathematical formula to estimate multipliers that does not require knowledge of the complete matrix of technical coefficients. In the process of developing this formula, it is demonstrated that the Input-Output multipliers are very insensitive to the values of the individual technical coefficients, but are sensitive to the column totals of the technical coefficient matrix.
The following approach is taken in this study to prove that multipliers are sensitive to the column sums of the matrix of technical coefficients and are extremely insensitive to the individual elements. Define $R$ to be the set of $n \times n$ nonnegative matrices with column sums $w_j$, $1 \leq j \leq n$, where $0 < w_j < 1$. The assumption that each matrix $A \in R$ is equally likely makes $R$ a set of random matrices and defines a distribution on the coefficients $a_{ij}$. The distribution of the coefficients of the random matrices induces a distribution on the multipliers. This dissertation first derives the distribution of the coefficients of the matrices in $R$ and then finds the mean and variance of the distribution of multipliers. Due to the extremely small nature of the variance that results, it can be concluded that a large percentage of the matrices in $R$ will give multiplier values close to the mean. This mean, whose formula is a function of only the column sums of the matrix of technical coefficients, is proposed as an estimator of the true industry multiplier.
CHAPTER ONE
INTRODUCTION

Input-Output analysis provides a technique to examine the interrelationships within an economic system. Although Input-Output analysis was originally developed by Wassily Leontief to measure intersectorial activity in the whole American economy [13], this technique has been applied to the study of economic activity within cities, states, and regions. The capability of Input-Output analysis to completely account for the complex interactions among industries makes it useful as an aid in economic planning and development for both industry and government.

There are three tables which are the basic elements of the Input-Output model: the transactions table, the table of technical coefficients, and the direct and indirect requirements table.

To build the transactions table, the economy is divided into three sectors: the processing sector, the payments sector, and the final demand sector. The processing sector contains all the industries that produce goods and services in the region defined by the Input-Output model. Industries with similar characteristics are generally aggregated to reduce the size of the table. The payments sector contains households, government, investors, and imports, while the final demand sector contains households, government, investors, and exports. Figure One illustrates the structure of the transactions table.
**FIGURE ONE**

**TRANSACTIONS TABLE**

<table>
<thead>
<tr>
<th>Final Demand</th>
<th>H</th>
<th>G</th>
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<th>E</th>
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<tr>
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<td>Imports</td>
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<table>
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<tr>
<th>Processing Sector</th>
<th>( B_1 )</th>
<th>Interindustry Expenditures</th>
<th>( B_2 )</th>
<th>( B_4 )</th>
</tr>
</thead>
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<tr>
<td>n industries</td>
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<table>
<thead>
<tr>
<th>Payments Sector</th>
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<tr>
<td>Imports</td>
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</table>
The transactions table in Figure One is divided into blocks $B_1, B_2, B_3,$ and $B_4$. The $i,j$th element in $B_1, 1 \leq i, j \leq n,$ gives the dollar amount industry $j$ spends on goods and services from industry $i$ (that is, $B_1$ represents the interindustry expenditures). In $B_2,$ elements in the $j$th column, $1 \leq j \leq n,$ give the dollar amount industry $j$ purchases from households (wages), government (taxes), investors (interest, dividends), and foreign industries (imports). Elements in $B_3$ represent the sales of industry to households, government, investors, and foreign buyers, while elements in $B_4$ give the purchases between the payments sector and the final demand sector.

The second step in Input-Output analysis is the construction of the matrix of technical coefficients $A$. In terms of the transactions table, $a_{ij}$ is found by dividing the $i,j$th element of $B_1$ by the sum of the $j$th column for $1 \leq i, j \leq n$. A technical coefficient $a_{ij}$ indicates the amount of input industry $j$ requires from industry $i$ to produce one dollar's worth of $j$'s output. Thus, if $X$ represents the vector of outputs for each industry and $F$ represents the final demand vector, then $X = AX = F$. Solving for the column vector $X$ gives $X = (I-A)^{-1}F$.

The matrix $Q = (I-A)^{-1} = \sum_{i=0}^{\infty} A^i$ is the direct and indirect requirements table. Each element $q_{ij}$ measures the total demand after all rounds of trading for the product of industry $i$ resulting from a dollar increase in the demand for the product of industry $j$. The direct effect is the amount $A_{ij}$, while the indirect effect
is the total amount that industry i receives from all industries in the
later periods as a result of the initial dollar increase in the demand
for the product of industry j.

Let \( u_j = \sum_{i=1}^{n} q_{ij} \) be defined to be the multiplier of industry j. The multiplier \( u_j \) gives the total impact on all industries in terms of
gross output resulting from the increase of total demand for the
products of industry j by one dollar. In general, industries with
large multipliers are the ones that rely heavily upon the other
industries in the region as an outlet for their products,
while industries with small multipliers tend to have little dependence
on industries within the region. Multipliers are useful in evaluating
the effects of each industry on the regional or national economy. Such
evaluations are needed in many areas of economic policy. Two policy
areas, for example, in which they are especially essential are (1)
in the evaluation of socio-economic impacts of new plants, new mines,
and other activities as a part of environmental impact evaluations
and (2) in the evaluation of the potential impacts of alternative
prospective firms which are considering relocation into an area. All
states have active industry inducement programs designed to attract
new industry into their boundaries. The use of multipliers can assist
industrial development officials in identifying industries that would
most significantly increase economic activity in the region.

While multipliers have many applications, the determination
of multipliers through the use of Input-Output analysis requires a
considerable amount of time and effort. If, for example, there are
fifty industries in the Input-Output model, then at least twenty-five hundred pieces of data must be obtained to fill out the transactions table. Furthermore, survey based models would require considerably more information before the final transactions table were completed. This process could require millions of dollars and a few years to complete. The United States Input-Output model, compiled by the Department of Commerce, has approximately four hundred industry categories and requires five or six years to complete. Certainly, a simpler and less expensive method of estimating multipliers would be useful.

This dissertation will develop a mathematical formula to estimate multipliers without requiring knowledge of the complete matrix of technical coefficients. In the process of developing this formula, it will be demonstrated that the Input-Output multipliers are very insensitive to the values of the individual technical coefficients, but are sensitive to the column totals of the technical coefficient matrix.

There have been earlier investigations in the area of the insensitivity of multipliers. Stevens and Trainer [19] have demonstrated empirically that matrices whose coefficients are generated randomly according to a normal distribution yield multipliers with small variances. Furthermore, it was found that variation in column totals and the size of the matrix have substantially greater impacts on the value of multipliers than the specific values of the coefficients. Burford and Margrave [6] point out that, on the basis of a limited investigation, the multipliers computed from a 25 industry I-O matrix for Louisiana
appear to be relatively insensitive to possible errors in the I-O coefficients. Two papers by Drake [9, 10], which were designed to find a shortcut method of computing multipliers for states or regions not having I-O matrices, provides some theoretical as well as empirical evidence that a multiplier is a function of the average value of column totals and is affected little by the specific values in the column. Finally, in a paper by Burford (with some assistance from this author) [2], the work of Drake was extended. Burford generated random samples of fifty Input-Output matrices where the matrix dimensions and column totals were fixed. For each matrix A of the fifty matrices, (I-A)^{-1} was computed. Using this, the mean and variance of the multipliers were computed for each industry. The results showed the sample variance of the multipliers to be very small. Furthermore, Burford introduced a multiplier formula that closely approximated the sample mean of the multipliers obtained in the simulation. This formula, which is as Drake predicted, a function of the average column totals, will be placed in proper theoretical perspective by this dissertation.

The following approach will be taken in this study to prove that multipliers are sensitive to the column sums of the matrix of technical coefficients and are extremely insensitive to the individual elements. Define R to be the set of n x n nonnegative matrices with column sums w_j, 1 ≤ j ≤ n, where 0 < w_j < 1. The assumption that each matrix A ∈ R is equally likely makes R a set of random matrices and defines a distribution on the coefficients a_{ij}. The concept of random matrices, as applied to Input-Output analysis,
has been used by Quandt [16, 17] and, later, Simonovitz [18]. They allowed each coefficient of the matrix of technical coefficients to vary according to random distributions and then studied the effects of this on the inverse matrix. This general definition of random matrices, however, gave only broad conclusions about the inverse matrix and the multipliers. Given the author's definition of random matrices, the distribution of the coefficients $a_{ij}$ is found. In view of the fact that $Q = (I-A)^{-1}$, and $u_j = \sum_{i=1}^{n} q_{ij}$, the distribution of the coefficients of the matrices of $R$ induces a distribution on the multiplier $u_j$, $1 \leq j \leq n$, whose mean and variance is then derived. The resulting variance is extremely small; from this, one can conclude that a large percentage of the matrices in $R$ will give multiplier values close to the mean. This mean, whose formula is a function of only the column sums of the matrix of technical coefficients, is proposed as an estimator of the true industry multiplier.

The main body of the dissertation is developed through six chapters. Chapter One continues by presenting the fundamental well-known theorems concerning the structure of the Input-Output matrix (Theorem 1.1 through Theorem 1.8). These are included to ease referencing them at later points. After this, limits are derived for the multiplier as functions of the column sums of the Input-Output matrix. In Chapter Two, the concept of random matrices with fixed column sums is introduced, and then the distribution of their coefficients is determined. In Chapter Three, the results of Chapter Two are used to derive the mean

---

1See Dorfman, Samuelson, and Solow, Linear Programming and Economic Analysis, p. 254-257.
and variance of the distribution of multipliers. In Chapter Four, an 
appropriate distribution for the multipliers is proposed. Chapter 
Five empirically tests the results of Chapters Three and 
Four. Finally, Chapter Six presents the implications of this work 
in both a theoretical and practical framework.

This chapter will now develop the basic theorems concerning the 
Input-Output Coefficient Matrix (called the Input-Output Matrix), 
and the inverse matrix. These will be followed by the deter­
mination of bounds for multipliers derived from the column 
sums of the Input-Output matrix.

The matrix A is defined to be an Input-Output Coefficient Matrix  
if

1) A is an n x n matrix, n a positive integer.

2) a_{ij} > 0, for 1 \leq i, j \leq n.

3) \sum_{i=1}^{n} a_{ij} = w_j, \, 0 < w_j < 1, \, \text{for} \, 1 \leq j \leq n.

Thus, an Input-Output matrix is a square nonnegative matrix whose 
column sums are between zero and one.

**THEOREM 1.1:** If A is an Input-Output matrix, then (I-A) is invertible.

**PROOF:** Consider the identity

\[ I = (I-A) \left( \sum_{i=0}^{\infty} A^i \right) \]

where \( A^0 = I \).

If \( \sum_{i=0}^{\infty} A^i \) converges to the matrix Q, then I-A is invertible and
Let $Q = (I-A)^{-1}$. Let $w_j = \frac{n}{i=1} a_{ij}$ for $1 \leq j \leq n$ and let $w_{\text{max}} = \max_j \{w_j\} < 1$.

Now

$$q_{ij} = \sum_{k=0}^{\infty} a_{ik}^k$$ and

$$a_{ij} = \sum_{k=1}^{n} a_{ik} a_{kj} \leq \sum_{k=1}^{n} (w_{\text{max}})^{s-1} a_{kj} \leq (w_{\text{max}})^s$$

for $s$ a positive integer. When $s = 0$, then

$$a_{ij}^0 \leq 1 = (w_{\text{max}})^0.$$ 

Therefore,

$$q_{ij} = \sum_{k=0}^{\infty} a_{ij}^k \leq \sum_{k=0}^{\infty} (w_{\text{max}})^k = 1/(1-w_{\text{max}}).$$

Since $\sum_{k=0}^{\infty} a_{ij}^k$ is bounded above by $1/(1-w_{\text{max}})$, and $a_{ij}^s \geq 0$ for all nonnegative integers $s$, thus $q_{ij}$ exists for all $i$ and $j$. #

**THEOREM 1.2**: If $A$ is an Input-Output matrix, then each element of $Q = (I-A)^{-1}$ is nonnegative.

**PROOF**: If $A$ is an Input-Output matrix, then by Theorem 1.1, $(I-A)^{-1}$ exists and $(I-A)^{-1} = \sum_{i=0}^{\infty} A^i$. Since $a_{ij}^s$ is nonnegative for all $i, j, s, \sum_{i=0}^{\infty} a_{ij}^k$ is nonnegative. Thus $Q = (I-A)^{-1}$ is nonnegative. #

Let $A$ be an Input-Output matrix, and let $Q = (I-A)^{-1}$. Define $u_j$, the multiplier of industry $j$, to be

$$u_j = \sum_{i=1}^{n} q_{ij}.$$
The following theorem presents a system of equations that relates the coefficients of the Input-Output matrix $A$ to its set of multipliers.

**THEOREM 1.3:** Let $A$ be an Input-Output matrix, and let $\mu_j$, $1 \leq j \leq n$, represent the multipliers of $(I-A)^{-1}$. Then

$$\mu_j = 1 + \sum_{i=1}^{n} a_{ij} \mu_i \quad \text{for } 1 \leq j \leq n.$$

**PROOF:** Let $A$ be an Input-Output matrix, and let $Q = (I-A)^{-1}$. Then

$$Q(I-A) = I.$$

Let $(I-A)_j$ denote the $j^{th}$ column of $(I-A)$, and let $e_j$ denote the column unit vector. Then

$$Q(I-A)_j = e_j.$$

Multiplying and adding the rows together yields

$$\sum_{i=1}^{n} q_{ij} = \sum_{k=1}^{n} a_{ij} q_{ki} = 1 \quad \text{or}$$

$$\mu_j = 1 + \sum_{i=1}^{n} a_{ij} \mu_i.$$

**THEOREM 1.4:** Let $A$ be an Input-Output matrix, and let $Q = (I-A)^{-1}$. Define $G_j$ to be the matrix $(I-A)$ with the $i^{th}$ column replaced by $e_j$, and $H_j$ to be the matrix $(I-A)$ with the $j^{th}$ row replaced by a row of ones. Then

$$q_{ij} = \frac{\det G_j}{\det (I-A)}.$$
and

\[ \mu_j = \frac{\det H_j}{\det (I-A)} \]

**PROOF:** Since \( Q = (I-A)^{-1} \),

\[(I-A)Q = I.\]

The above yields \( n \) systems of \( n \) equations written in the notation developed in Theorem 1.3.

Therefore,

\[(I-A)Q_j = e_j \quad \text{for } 1 \leq j \leq n,\]

where the column vector \( Q_j \) represents the vector of variables. Since \( (I-A)^{-1} \) exists, \( \det(I-A) \neq 0 \). By Cramer's Rule,

\[ q_{ij} = \frac{\det G_i}{\det (I-A)} \]

for \( 1 \leq i \leq n \). It follows from \( \mu_j = \sum_{i=1}^{n} q_{ij} \) that

\[ \mu_j = \frac{\det H_j}{\det (I-A)}. \]
The following two theorems will allow a further analysis of Theorem 1.4 that will eventually yield bounds for the multipliers.

**THEOREM 1.5:** If $A$ is an Input-Output matrix, then the real eigenvalues of $A$ are strictly less than one.

**PROOF:** It is sufficient to show that the real eigenvalues of $A^T$ (A transpose) are strictly less than one.

Let $\lambda$ be a real eigenvalue of $A^T$, and let $v$ be an eigenvector for $\lambda$. Then

$$A^Tv = \lambda v.$$  

Let $v_{\max} = \max \{ v_i \} = v_k$.

**CASE I:** $v_{\max} > 0$.

Thus,

$$\lambda v_{\max} = \sum_{j=1}^{n} a_{kj} v_j = \sum_{j=1}^{n} a_{kj} v_{\max} < (v_{\max}) (1) = v_{\max}.$$  

Since $\lambda v_{\max} < v_{\max}$ and $v_{\max} > 0$, thus $\lambda < 1$.

**CASE II:** $v_{\max} < 0$.

Let $v_{\min} = \min \{ v_i \} = v_m$.

Then $v_{\min} < 0$ for if $v_{\min} = 0$, then $v = 0$ and $v$ is not an eigenvector.

Now

$$\lambda v_{\min} = \sum_{j=1}^{n} a_{mj} v_j > \sum_{j=1}^{n} a_{mj} v_{\min} > v_{\min}.$$  

Since $\lambda v_{\min} > v_{\min}$ and $v_{\min} < 0$, thus $\lambda < 1$. 

#
THEOREM 1.6: If $A$ is an Input-Output matrix, then $\det (I-A) > 0$.

PROOF: Let $\phi(X) = \det (XI-A)$ denote the characteristic polynomial of $A$. Since $\phi(X)$ is zero if $X$ is a real eigenvalue of $A$, and all real eigenvalues of $A$ are strictly less than one, then the sign of $\phi(X)$ will be unchanged for all $X > 1$.

Let $X = k+1$ where $k > n!$ and

$$\frac{\log(k+1)}{\log(k)} < \frac{n}{n-1}.$$  

Then $\phi(X) > k^n - (n! - 1)(k+1)^{n-2}$

because $k^n$ is a lower bound for the product of the main diagonal term of the matrix $(XI-A)$ and $-(k+1)^{n-2}$ represents a lower bound for the remaining $(n!-1)$ terms. Since $k > n!$,

$$k^n - (n! - 1)(k + 1)^{n-2} \geq k^n - (k + 1)^{n-1}.$$  

Furthermore, $\log k < \frac{n}{n-1}$; this gives $(k+1)^{n-1} < k^n$.

Therefore $k^n - (k+1)^{n-1} > 0$ and $\phi(X) > 0$. It follows from $\phi(X) > 0$ and $X > 1$ that

$$\phi(1) = \det(I-A) > 0.$$  

Let $F_{ij}$ be the determinant of the matrix $A$ with the $i^{th}$ row and $j^{th}$ column deleted. Define

$$D_{ij} = (-1)^{i+j} F_{ij}.$$  

THEOREM 1.7: Let $A$ be an Input-Output matrix. If $(i+j)$ is even, then $F_{ij} > 0$. If $(i+j)$ is odd, then $F_{ij} \leq 0$. 
PROOF: Recall from Theorem 1.4 that \( Q = (I-A)^{-1} \) and
\[
q_{ij} = \frac{D_{ij}}{\det(I-A)} .
\]

Since \( q_{ij} \geq 0 \) and \( \det(I-A) > 0 \), \( D_{ij} \geq 0 \). In view of the fact that
\[
D_{ij} = (-1)^{i+j} F_{ij},
\]
the theorem follows directly.

THEOREM 1.8: If \( A \) is an Input-Output matrix, and \( \mu_j \) is the multiplier of industry \( j \), then
\[
\mu_j = \frac{\sum_{i=1}^{n} D_{ji}}{\sum_{i=1}^{n} (1-w_i) D_{ji}} .
\]

PROOF: If the \( j^{th} \) row of \( (I-A) \) is replaced by the sum of all the rows, then the determinant becomes, by expanding along the \( j^{th} \) row,
\[
\det(I-A) = \sum_{i=1}^{n} (1-w_i) D_{ji} .
\]

Furthermore, the determinant of \( (I-A) \) with the \( j^{th} \) row replaced by ones is \( \sum_{i=1}^{n} D_{ji} \) by expanding along the \( j^{th} \) row. Thus, the present theorem follows immediately by applying Theorem 1.4.

The previous well-known theorems have described the multipliers of the \((I-A)^{-1}\) matrix in terms of the Input-Output matrix \( A \). Bounds on the multipliers are now considered where the column sums of the Input-Output matrix \( A \) are fixed. The following theorem is of great importance to understanding the lack of sensitivity of the multipliers to individual values of the \( a_{ij} \) coefficients.
THEOREM 1.9: Let A be an Input-Output matrix, and let the column sums of A be equal to w for all j. Then the multipliers $u_j$ of $Q = (I-A)^{-1}$ are equal and

$$u_j = 1/(1-w) \text{ for all } j.$$ 

PROOF: Consider the system of equations

$$(I-A)Q_j = e_j \quad \text{for } 1 \leq j \leq n.$$ 

Adding the rows together gives

$$\sum_{i=1}^{n} (1-w)q_{ij} = 1,$$

or

$$u_j = \frac{\sum_{i=1}^{n} q_{ij}}{n} = 1/(1-w) \quad \text{for } 1 \leq j \leq n.$$ 

The above theorem suggests the importance of the column sums to the value of the multipliers rather than the actual coefficients in the Input-Output matrix.

Let A be an Input-Output matrix with column sums $w_j$. Let

$$w_{\max} = \max \{w_j\},$$

$$w_{\min} = \min \{w_j\}.$$ 

Recall that $D_{ji} > 0$. Then for all j

$$u_j = \frac{\sum_{i=1}^{n} D_{ji}}{\sum_{i=1}^{n} (1-w_i)D_{ji}} > \frac{\sum_{i=1}^{n} D_{ji}}{\sum_{i=1}^{n} (1-w_{\min})D_{ji}} = 1/(1-w_{\min}).$$

Also,

$$u_j = \frac{\sum_{i=1}^{n} D_{ji}}{\sum_{i=1}^{n} (1-w_i)D_{ji}} < \frac{\sum_{i=1}^{n} D_{ji}}{\sum_{i=1}^{n} (1-w_{\max})D_{ji}} = \frac{1}{1-w_{\max}}.$$
Therefore, for $j = 1, 2, \ldots, n$,

$$\frac{1}{1-w_{\min}} \leq u_j \leq \frac{1}{1-w_{\max}}.$$ 

Note that if $w_i = w$ for all $i$, then

$$u_j = \frac{1}{1-w}.$$ 

Furthermore, for any individual multiplier $u_j$,

$$u_j = 1 + \sum_{i=1}^{n} a_{ij} \leq 1 + \sum_{i=1}^{n} a_{ij} \frac{1/1-w_{\max}}{1-w_{\max}} = 1 + \frac{w_j}{1-w_{\max}}.$$ 

Similarly,

$$u_j \geq 1 + \frac{w_j}{1-w_{\min}}.$$ 

Thus, for each industry $j$,

$$1 + \frac{w_j}{1-w_{\min}} \leq u_j \leq 1 + \frac{w_j}{1-w_{\max}}.$$ 

These bounds for the multipliers are important because they depend solely on the column sums of the Input-Output matrix. Although these bounds may tend to be too large to be of use, it will be shown in the succeeding chapters that a high percentage of the Input-Output matrices with the same column sums give multipliers for an industry that fall in a very small interval.
CHAPTER TWO

THE DISTRIBUTION OF COEFFICIENTS IN RANDOM I-O MATRICES

This chapter will define the concept of random matrices with fixed column sums and derive the distribution of the coefficients.

Consider the set $R$ of Input-Output Matrices of order $n$ whose column sums $w_i, 1 \leq i \leq n$, are fixed. For each matrix $A \in R$, the matrix $(I-A)^{-1}$ and the multipliers can be computed. If it is assumed that each matrix in $R$ is equally likely, then a distribution is induced for the coefficients of the Input-Output matrix and also for the multipliers. Under the above assumption, some important properties of the distribution of coefficients of the matrices in $R$ immediately follow. First, the distributions of the coefficients $a_{ij}$ and $a_{km}$ are independent, $j \neq m$, while the distribution of coefficients $a_{ij}$ and $a_{kj}$ are dependent. Secondly, coefficients in the same column must be identically distributed because each matrix in $R$ is equally likely. It is generally true that the diagonal coefficients of Input-Output models tend to be larger than the other coefficients in their column. A discussion of the effect of the assumptions of the following analysis to Input-Output analysis will be made in Chapter 6.

Define $a_{ij}, 1 \leq i, j \leq n$ to be the random variable associated with the $i,j^{th}$ element of the random matrices. The distribution of $a_{ij}$ subject to prior assumptions will be determined in Theorem 2.1.
THEOREM 2.1: Let \( a_{ij} \) be a random variable of the set of matrices of \( \mathbb{R} \). Then the density function \( f \) for the distribution of \( a_{ij} \) is given by

\[
f(x) = \frac{(n-1)(w_j-x)^{n-2}}{w_j^{n-1}} \quad x \in [0, w_j].
\]

PROOF: Due to the previous discussion, it suffices to determine the distribution of \( a_{1j} \), since all coefficients in the \( j \)th column are identically distributed. The distribution function \( F \) of \( a_{1j} \) is

\[
F(x) = P(a_{1j} \leq x \mid a_{1j} + a_{2j} + \ldots + a_{nj} = w_j) = P(a_{1j} \leq x \mid a_{1j} + \ldots + a_{n-1j} \leq w_j)
\]

\[
= \frac{\int_{0}^{x} \int_{0}^{w_j-a_{1j}} \cdots \int_{0}^{w_j-a_{1j}} \cdots \int_{0}^{w_j-a_{1j}} da_{n-1j} \cdots da_{1j}}{w_j \cdot w_j \cdot a_{1j} \cdots w_j \cdot a_{1j} \cdots a_{n-2j} \cdot \cdots \cdot a_{n-k-2j} \cdots a_{n-k-1j} \cdots da_{n-1j} \cdots da_{1j}}.
\]

The numerator represents the area of the hyperplane \( \sum_{i=1}^{n} a_{ij} = w_j \) projected on \( (n-1) \) space such that \( a_{1j} \leq x \). The denominator represents the area of the hyperplane \( \sum_{i=1}^{n} a_{ij} = w_j \) projected on \( (n-1) \) space.

The integration is the same for the numerator and denominator until the final step. To demonstrate the integration process, the multiple integral at the \( k \)th step, \( k = 0, 1, \ldots, n-2 \), for the numerator becomes

\[
\frac{\int_{0}^{x} \int_{0}^{w_j-a_{1j}} \cdots \int_{0}^{w_j-a_{1j}} \cdots \int_{0}^{w_j-a_{1j}} da_{n-k-1j} \cdots da_{1j}}{(w_j-a_{1j} \cdots a_{1j})^k} \cdot \frac{da_{n-k-1j} \cdots da_{1j}}{k!}
\]

A small example of this theorem is worked out in Appendix One.
In integrating the above directly, there is a collapsing of terms that result in

\[
F(x) = \begin{cases} 
\frac{(w_j - a_{1j})^{n-1}}{-(n-1)!} & a_{1j} = x \\
\frac{(w_j - a_{1j})^{n-1}}{-(n-1)!} & a_{1j} = 0 \\
\end{cases}
\]

or

\[
F(x) = \frac{w_j - (w_j - x)^{n-1}}{w_j^{n-1}}.
\]
Therefore, the density function for $a_{ij}$, or $a_{ij}$ for that matter, is

$$f(x) = F'(x) = \frac{(n-1)(w_j-x)^{n-2}}{w_j^{n-1}} x \in [0, w_j].$$

Note that

$$\int_0^{w_j} \frac{(n-1)(w_j-x)^{n-2}}{w_j^{n-1}} \, dx = 1.$$

The density function derived in Theorem 2.1 has the following graph.

The graph suggests that the probability of randomly obtaining a coefficient close to zero is very likely, and this probability increases as $n$ increases. It is now shown that this distribution is a scaled Beta distribution.

Recall that a Beta distribution with parameters $\alpha$ and $\beta$ has the following density function:
where $\Gamma$ is the Gamma function that is defined by
\[
\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt \quad x \in (0, \infty).
\]

If $\alpha = 1$, $\beta = n-1$, then
\[
f(x, 1, n-1) = \frac{\Gamma(n)}{\Gamma(1)\Gamma(n-1)} x^0 (1-x)^{n-2} , \quad x \in [0, 1].
\]

Since $\Gamma(n) = (n-1)!$, $\Gamma(1) = 1$, and $\Gamma(n-1) = (n-2)!$, then
\[
f(x, 1, n-1) = (n-1)(1-x)^{n-2} , \quad x \in [0, 1].
\]

Let $x^* = w_j \cdot x$ or $x = x^*/w_j$.

Then $dx^* = w_j dx$, or $dx = dx^*/w_j$.

Thus,
\[
f(x^*, 1, n-1) = \frac{(n-1)(1-x^*/w_j)^{n-2}}{w_j} ,
\]
or
\[
f(x^*, 1, n-1) = \frac{(n-1)(w_j-x^*)^{n-2}}{w_j^{n-1}} , \quad x \in [0, w_j] .
\]

It follows that the density function derived in Theorem 2.1 is a scaled Beta distribution with $\alpha = 1$ and $\beta = n-1$.

The next theorem will determine the moments of the distribution derived in Theorem 2.1.

**THEOREM 2.2:** Consider the distribution of $a_{ij}$ derived in Theorem 2.1.
1. \( E[a_{ij}] = \frac{w_i}{n} \).

2. \( E[a_{ij}^2] = \frac{2w_i^2}{n(n+1)} \).

3. \( E[a_{ij} a_{kj}] = \frac{w_j^2}{n(n+1)} \), for \( i \neq k \).

**Proof:** The proofs of 1 and 2 follow directly by integration by parts.

1. \[
E[a_{ij}] = \int_0^{w_i} \frac{x(n-1)(w_i-x)^{n-2}}{w_j^{n-1}} \, dx = \frac{w_i}{n}.
\]

2. \[
E[a_{ij}^2] = \int_0^{w_i} \frac{x^2(n-1)(w_i-x)^{n-2}}{w_j^{n-1}} \, dx = \frac{2w_i^2}{n(n+1)}.
\]

3. Since \( \sum_{i=1}^{n} a_{ij} = w_j \),

\[
\sum_{i=1}^{n} a_{ij}^2 = w_j^2 \text{ and } E[\sum_{i=1}^{n} a_{ij}^2] = E[w_j^2] = w_j^2.
\]

Since the coefficients in the same column are identically distributed,

\[
nE[a_{ij}^2] + n(n-1)E[a_{ij} a_{kj}] = w_j^2.
\]

Substituting \( E[a_{ij}^2] = \frac{2w_i^2}{n(n+1)} \) gives

\[
E[a_{ij} a_{kj}] = \frac{w_j^2}{n(n+1)}.
\]

Therefore, \( E[a_{ij} a_{kj}] = \frac{w_j^2}{n(n+1)} \), for \( i \neq k \).
CHAPTER THREE

THE MEAN AND VARIANCE OF THE DISTRIBUTION OF MULTIPLIERS

The previous chapter described the distribution of the coefficients of an Input-Output matrix with fixed column sums under the assumption that each such matrix is equally likely. This chapter will derive the mean and variance of the multipliers that are induced by the coefficients of the Input-Output matrix.

The formula that is obtained for the mean of the distribution of the multipliers is based on an additional assumption that the covariance between \( a_{ij} \) and \( \mu_i \) is zero for \( 1 \leq i, j, \leq n \). This assumes that, in general, no single coefficient of the Input-Output matrix will have a significant effect on the size of the multiplier.

**THEOREM 3.1**: Let \( A \) be an Input-Output matrix with fixed column sums \( w_k, 1 \leq k \leq n, \) \( Q = (I-A)^{-1}, \) and \( \mu_j = \frac{1}{n} \sum_{i=1}^{n} q_{ij} \) for \( 1 \leq j \leq n \). If the covariance of \( a_{ij} \) and \( \mu_i \) is zero for \( 1 \leq i, j, \leq n \), then the expected value of the distribution of multipliers is

\[
E[\mu_j] = 1 + \frac{w_j}{n} \left[ 1 - \sum_{i=1}^{n} \frac{w_i}{n} \right]
\]

\( 1 \leq j \leq n. \)

**PROOF**: The following system of equations follows from Theorem 1.3.

\[
\mu_j = 1 + \sum_{i=1}^{n} a_{ij} \mu_i \quad 1 \leq j \leq n.
\]

Since the covariance of \( a_{ij} \) and \( \mu_i \) is assumed to be zero,

\[
E[a_{ij} \mu_i] = E[a_{ij}] E[\mu_i].
\]

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Taking the expected value over the above system of equations gives

\[ E[\mu_j] = E[1 + \sum_{i=1}^{n} a_{ij} \mu_i] \]

\[ = 1 + \sum_{i=1}^{n} E[a_{ij} \mu_i] \]

\[ = 1 + \sum_{i=1}^{n} E[a_{ij}] E[\mu_i] \]

\[ = 1 + \sum_{i=1}^{n} \frac{w_i}{n} E[\mu_i]. \quad 1 \leq j \leq n. \]

Summing the above equations over \( j \) gives

\[ \sum_{j=1}^{n} E[\mu_j] = \sum_{j=1}^{n} [1 + \sum_{i=1}^{n} \frac{w_i}{n} E[\mu_i]] \]

\[ = n + \left( \sum_{j=1}^{n} \frac{w_j}{n} \right) \left( \sum_{i=1}^{n} E[\mu_i] \right). \]

Thus,

\[ \sum_{i=1}^{n} E[\mu_i] = \frac{n}{1 - \sum_{j=1}^{n} \frac{w_j}{n}}. \]

Therefore,

\[ E[\mu_j] = 1 + \frac{w_j}{n} \left[ \sum_{i=1}^{n} E[\mu_i] \right] \]

\[ = 1 + \frac{w_j}{n} \left[ \frac{n}{1 - \sum_{j=1}^{n} \frac{w_j}{n}} \right] \]

\[ = 1 + \left[ \frac{w_j}{1 - \sum_{i=1}^{n} \frac{w_i}{n}} \right] \quad 1 \leq j \leq n. \]
The covariance assumption for the previous theorem should have no effect on the final result. Essentially, the assumption
\[ \text{Cov}(a_{ij}, \mu_i) = 0 \]
means that no single coefficient in the Input-Output matrix significantly affects the final value of the multiplier.

Certainly this is true in large matrices. Although specific matrices can be constructed where there is a significant relationship between the multiplier and an individual coefficient, the probability of obtaining one of these as a random matrix is insignificant. The empirical results that are obtained in Chapter Five add further evidence to the validity of the covariance assumption.

At this point, a formula for the variance of the distribution of multipliers will be derived. This will require additional assumptions about the covariance. Again, these assumptions are believable, and the degree to which the covariance assumptions affect the variance formula will be considered in Chapter Five.

**THEOREM 3.2:** Let \( A \) be an Input-Output matrix with fixed column sums \( w_i, 1 \leq i \leq n \), \( Q = (I-A)^{-1} \) and \( \mu_j = \sum_{i=1}^{n} q_{ij} \), \( 1 \leq j \leq n \).

Suppose \( \text{Cov}(\mu_j, a_{ij}, a_{kj}) = 0 \) for \( 1 \leq i, j, k \leq n \), \( \text{Cov}(a_{ij}, \mu_i) = 0 \) for \( 1 \leq i, j \leq n \), and \( \text{Cov}(a_{ki}, a_{kj}) = 0 \) for \( 1 \leq k \leq n, 1 \leq i \neq j \leq n \).

Then,
\[
\text{Var}[\mu_j] = \frac{n^2 \left[ \sum_{k=1}^{n} w_k^2 - \left( \sum_{k=1}^{n} w_k \right)^2 \right]}{(n - \sum_{k=1}^{n} w_k)^2} \frac{n^2 - s}{n(n+1)(n^2-s) - \sum_{k=1}^{n} w_k^2 (2n^2-s)}
\]

where \( s = \left( \sum_{k=1}^{n} w_k \right)^2 - \sum_{k=1}^{n} w_k^2 \).
PROOF: Recall from Theorem 1.3 the system of equations
\[ u_j = 1 + \sum_{i=1}^{n} a_{ij} u_i, \quad 1 \leq j \leq n. \]

Let \( x = \sum_{i=1}^{n} \sum_{j=1}^{n} E[u_i u_j] \) and
\[ y = \sum_{i=1}^{n} E[u_i^2]. \]

Since
\[ u_i^2 = (1 + \sum_{j=1}^{n} a_{ji} u_j)^2, \]
then
\[ E[u_i^2] = E[(1 + \sum_{j=1}^{n} a_{ji} u_j)^2] \]
\[ = E[1 + 2 \sum_{j=1}^{n} a_{ji} u_j + \sum_{j=1}^{n} a_{ji}^2 u_j^2 + \sum_{j=1}^{n} \sum_{k=1}^{n} a_{ji} a_{kj} u_j u_k] \]
\[ = 1 + \frac{2w_i}{n} E[\sum_{j=1}^{n} u_j] + \sum_{j=1}^{n} E[a_{ji}^2] E[u_j^2] + \sum_{j=1}^{n} \sum_{k=1}^{n} E[a_{ji} a_{kj}] E[u_j u_k]. \]

Thus,
\[ E[u_i^2] = 1 + 2w_i \left[ \frac{\sum_{j=1}^{n} w_j}{n} \right] + \frac{2w_i}{n(n+1)} y + \frac{w_i^2}{n(n+1)} x. \]
Therefore,

$$E[\mu_i^2] = y = n + \frac{2n \sum_{j=1}^{n} w_j}{n(n+1)} y + \frac{\sum_{j=1}^{n} w_j^2}{n(n+1)} x.$$ 

For \( i \neq j \),

$$\mu_i \mu_j = (1 + \sum_{k=1}^{n} a_{ki} \mu_k)(1 + \sum_{k=1}^{n} a_{kj} \mu_k).$$

Therefore,

$$E[\mu_i \mu_j] = E[(1 + \sum_{k=1}^{n} a_{ki} \mu_k)(1 + \sum_{k=1}^{n} a_{kj} \mu_k)].$$

Substitution gives

$$E[\mu_i \mu_j] = 1 + (w_i + w_j) \left[ \frac{n}{n - \sum_{k=1}^{n} w_k} \right] + \frac{w_i w_j}{n(n-1)} (x + y).$$

Therefore,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} E[\mu_i \mu_j] = \sum_{i=1}^{n} \sum_{j=1, i \neq j}^{n} (w_i + w_j) \left[ \frac{n}{n - \sum_{k=1}^{n} w_k} \right] + \sum_{i=1}^{n} \sum_{j=1, i \neq j}^{n} \left[ \frac{n}{n - \sum_{k=1}^{n} w_k} \right] + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{w_i w_j}{n^2} (x+y),$$

Equations (1) and (2) represent two equations in the unknowns \( x \) and \( y \).
\[ x = \frac{n^2(n + \sum_{i=1}^{n} w_i) \left[(n-1)n^2(n+1)+(-2n^2+n-1)\left(\sum_{i=1}^{n} w_i^2\right)\right] + (n+1)\left(\sum_{i=1}^{n} w_i\right)^2}{(n - \sum_{i=1}^{n} w_i) \left[n^4 - n^2(\sum_{i=1}^{n} w_i)^2 + \left[n^2 - (\sum_{i=1}^{n} w_i)^2\right]\left(n - \sum_{i=1}^{n} w_i\right)\right]} \]

and

\[ y = \frac{n^2(n + \sum_{i=1}^{n} w_i) \left[n^2(n+1) - (\sum_{i=1}^{n} w_i)^2\right] + (n+1)\left(\sum_{i=1}^{n} w_i\right)^2}{(n - \sum_{i=1}^{n} w_i) \left[n^4 - n^2(\sum_{i=1}^{n} w_i)^2 + \left[n^2 - (\sum_{i=1}^{n} w_i)^2\right]\left(n - \sum_{i=1}^{n} w_i\right)\right]} \]

Since
\[ E[u_i^2] = 1 + 2w_i \frac{n}{n(n+1)} + \frac{w_i^2}{n(n+1)} x + \frac{2w_i^2}{n(n+1)} y \quad \text{and} \]

\[ \text{Var}[u_i] = E[u_i^2] - [E[u_i]]^2 \]

the appropriate substitutions give

\[ \text{Var}[u_i] = \frac{nw_i^2\left[n\sum_{k=1}^{n} w_k^2 - \left(\sum_{k=1}^{n} w_k\right)^2\right]n^2-s}{(n - \sum_{k=1}^{n} w_k)^2 \left[n(n+1)(n^2-s) - \sum_{k=1}^{n} w_k^2 \left(2n^2-s\right)\right]} \]

where \( s = \left(\sum_{k=1}^{n} w_k\right)^2 - \sum_{k=1}^{n} w_k^2 \). #

Let \( \overline{w} = \frac{\sum_{i=1}^{n} w_i}{n} \) and

\[ \text{Var}[w] = \frac{\sum_{i=1}^{n} (w_i - \overline{w})^2}{n} = \frac{n \sum_{i=1}^{n} w_i^2 - \left(\sum_{i=1}^{n} w_i\right)^2}{n^2} \]

Furthermore, note that
\[ (E[u_i] - 1)^2 = \left(\frac{nw_i}{n - \sum_{j=1}^{n} w_j}\right)^2 \]
Thus,

\[ \text{Var}[\mu_j] = \frac{n(E[\mu_j] - 1)^2 \text{Var}[w][n^2-s]}{[n(n+1)[n^2-s] - \sum_{k=1}^{n} w_k^2(2n^2 - s)]} \]

Clearly, \( \text{Var}[\mu_j] > 0 \) because the term \( n(n+1)[n^2-s] \) is greater than \( \sum_{k=1}^{n} w_k^2 (2n^2-s) \) and of a higher order of magnitude.

Omitting the term \( \sum_{k=1}^{n} w_k^2 (2n^2-s) \) allows the following approximate form of the variance.

\[ \text{Var}[\mu_j] = \frac{n(E[\mu_j] - 1)^2 \text{Var}[w][n^2-s]}{n(n+1)[n^2-s]} \]

\[ = \frac{(E[\mu_j] - 1)^2 \text{Var}[w]}{n+1} \]

The above approximation of \( \text{Var}[\mu_j] \) explains the major determinants of the size of the variance. The \( \text{Var}[\mu_j] \) varies directly with the square of the column sum \( w_j \) and the variance of the column sums, and inversely with the size of the matrix. It is noteworthy in both the variance formulae that if the column sums are all the same, \( \text{Var}[w] = 0 \) and \( \text{Var}[\mu_j] = 0, 1 < j < n. \)
Chapter Four presents a distribution that closely approximates the distribution of multipliers. It has been shown to this point that,

1. \[ 1 + \frac{w_j}{1-w_{\min}} \leq \mu_j \leq 1 + \frac{w_j}{1-w_{\max}} \quad 1 \leq j \leq n. \]

2. \[ E[\mu_j] = 1 + \left[ \frac{\sum_{i=1}^{n} w_i}{1 - \sum_{i=1}^{n} w_i} \right] \quad 1 \leq j \leq n. \]

3. \[ \text{Var}[\mu_j] = \frac{n w_j^2 \left[ n \sum_{i=1}^{n} w_i^2 - \left( \sum_{i=1}^{n} w_i \right)^2 \right] [n^2-s]}{(n-\sum_{i=1}^{n} w_i)^2 [n(n+1)(n^2-s) - \sum_{i=1}^{n} w_i^2 (2n^2-s)]} \quad 1 \leq j \leq n. \]

Furthermore, recall from Theorem 1.3 that

\[ \mu_j = 1 + \sum_{i=1}^{n} a_{ij} \mu_i \quad \text{or} \]

\[ \mu_{j-1} = \sum_{i=1}^{n} a_{ij} \mu_i \quad 1 \leq j \leq n. \]

Dividing both sides by \( w_j \) gives

\[ \frac{\mu_{j-1}}{w_j} = \sum_{i=1}^{n} \left( \frac{a_{ij}}{w_j} \right) \mu_i, \quad 1 \leq j \leq n. \]
Recall that $a_{ij}$, $1 \leq j \leq n$, is Beta distributed between 0 and $w_j$ with density function $f$ defined by

$$f(x) = \frac{(n-1)(w_j-x)^{n-2}}{w_j^{n-1}}.$$ 

Therefore, $\frac{a_{ij}}{w_j}$ is Beta distributed between 0 and 1 with density function $g$ given by

$$g(x) = (n-1)(1-x)^{n-2}.$$ 

Also, $\frac{a_{ij}}{w_j}$ is identically distributed for all $i$ and $j$, and the joint distribution of $(a_{ij}, \ldots, a_{nj})$ is identically distributed for $1 \leq j \leq n$. Therefore, if $\mu_k$, $1 \leq k \leq n$, is independent of each $\frac{a_{ij}}{w_j}$ for all $i$ and $j$, then the distribution $\frac{\mu_{j-1}}{w_j}$ is identically distributed for all $j$.

From the formula below, one can see that $\frac{\mu_{j-1}}{w_j}$ measures the indirect economic effect that results from spending a dollar, when $w_j$ is the proportion that industry $j$ will respond in the economy of the region.

$$\frac{\mu_{j-1}}{w_j} = \sum_{i=1}^{n} \frac{a_{ij}}{w_j} \mu_i.$$ 

These column coefficient proportions have identical Beta distributions with density function $g$. Given that a single column of Input-Output coefficients do not significantly affect the multipliers, it follows that $\frac{\mu_{j-1}}{w_j}$ is identically distributed for all $j$. Note also that for all $\frac{\mu_{j-1}}{w_j}$:
1. \[ \frac{1}{1-w_{\text{min}}} \leq \frac{\mu_j^{-1}}{w_j} \leq \frac{1}{1-w_{\text{max}}}, \quad 1 \leq j \leq n. \]

2. \[ E \left[ \frac{\mu_j^{-1}}{w_j} \right] = \frac{1}{1 - \frac{\sum_{i=1}^{n} w_i}{n}}, \quad 1 \leq j \leq n. \]

3. \[ \text{Var} \left[ \frac{\mu_j^{-1}}{w_j} \right] = \frac{n[n \sum_{i=1}^{n} w_i^2 - \left( \sum_{i=1}^{n} w_i \right)^2] [n^2-s]}{(n - \sum_{i=1}^{n} w_i)^2 [n(n+1)(n^2-s) - \sum_{i=1}^{n} w_i^2 (2n^2-s)]}, 1 \leq j \leq n. \]

Since all the parameters above are independent of the individual column sum \( w_j \), it seems reasonable to expect, in view of the previous assumptions used to derive the mean and variance formulae, that \( \frac{\mu_j^{-1}}{w_j} \) is identically distributed for \( 1 \leq j \leq n \). Hence the problem is to determine the distribution of \( \frac{\mu_j^{-1}}{w_j} \) (which is identical for all \( j \)), rather than to find the distribution of \( \mu_j \).

A derivation of the theoretical distribution of multipliers is difficult because of the inherent dependence of the random variables in each column of the Input-Output matrix (i.e., \( \sum_{i=1}^{n} a_{ij} = w_j, 1 \leq j \leq n \)). Therefore, the remainder of the chapter will consider a distribution that should closely approximate the distribution of the multiplier \( \mu_j \).

Since the variance of \( \mu_j \) tends to be very small, one could assume the distribution of \( \mu_j \), and \( \frac{\mu_j^{-1}}{w_j} \), to be unimodal. Given this assumption, along with the limits, mean, and variance of \( \frac{\mu_j^{-1}}{w_j} \) that have been derived, a unimodal Beta distribution seems appropriate.

\(^1\text{See Table II, page 39.}\)
To transform $\frac{\mu_j - 1}{w_j}$ to the interval $[0, 1]$, let

$$k = \frac{1}{1 - w_{\text{max}}} - \frac{1}{1 - w_{\text{min}}}$$

$$= \frac{w_{\text{max}} - w_{\text{min}}}{(1 - w_{\text{max}})(1 - w_{\text{min}})}.$$

Then define

$$X_j = \frac{1}{k} \left[ \frac{\mu_j - 1}{w_j} - \frac{1}{1 - w_{\text{min}}} \right]$$

$$= \frac{(1 - w_{\text{max}})(1 - w_{\text{min}})}{w_j w_{\text{max}} - w_{\text{min}}} \left[ \mu_j - \left(1 + \frac{w_j}{1 - w_{\text{min}}} \right) \right],$$

on the interval $[0, 1]$. To fit $X_j$ to a Beta distribution $f$ of the form

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}, \quad x \in [0,1],$$

one must solve for $\alpha$ and $\beta$ from the system of equations

1. \( \frac{\alpha}{\alpha+\beta} = E[X_j], \)

2. \( \frac{\alpha \beta}{(\alpha+\beta+1)(\alpha+\beta)^2} = \text{Var}[X_j], \)

where

$$E[X_j] = E \left[ \frac{(1 - w_{\text{max}})(1 - w_{\text{min}})}{w_j w_{\text{max}} - w_{\text{min}}} \left[ \mu_j - \left(1 + \frac{w_j}{1 - w_{\text{min}}} \right) \right] \right]$$

$$= \frac{(1 - w_{\text{max}})(\bar{w} - w_{\text{min}})}{(w_{\text{max}} - w_{\text{min}})(1 - \bar{w})}$$
and
\[ \text{Var}[X_j] = \frac{(1-w_{\text{max}})^2(1-w_{\text{min}})^2}{(w_{\text{max}} - w_{\text{min}})^2} \left( \frac{n\left[\sum_{i=1}^{n} w_i^2 - \left( \sum_{i=1}^{n} w_i \right)^2 \right] [n^2-s]}{(n - \sum_{i=1}^{n} w_i)^2 \left[ n(n+1)(n^2-s) - \sum_{i=1}^{n} w_i^2 \right] (2n^2-s)} \right). \]

Solving the above system gives
\[ \alpha = E[X_j]\left[ \frac{E[X_j](1-E[X_j])}{\text{Var}[X_j]} - 1 \right] \]

and
\[ \beta = (1-E[X_j])\left[ \frac{E[X_j](1-E[X_j])}{\text{Var}[X_j]} - 1 \right]. \]

The distribution of \( X_j \), which is a scaled translation of \( \frac{\mu_{ij}}{w_j} \), can be expected to be closely approximated by the Beta distribution with the parameters \( \alpha \) and \( \beta \) derived above. This would mean that the distribution of multipliers \( \mu_{ij} \) is approximately Beta distributed with the same \( \alpha \) and \( \beta \) parameters but with its appropriate translation and scale factors.
CHAPTER FIVE

SOME EMPIRICAL INVESTIGATIONS

This chapter presents empirical evidence to validate the results of the previous chapters. The analysis is conducted using the Input-Output model developed by Burford and Hargrave [5] for Louisiana. This model was chosen for the empirical work because it has only twenty-five sectors and was easily accessible to the author. Table I presents the column sums for each sector of the Louisiana model, along with the multipliers that the model obtains. In the third column, the formula mean of the distribution of multipliers is given. This is obtained solely through the use of the column sums of the technical coefficients matrix. It should be noted that only in three cases does the estimate of the sector multiplier of the Louisiana model differ with the estimate obtained by the formula mean by more than 0.1. Thus, the mean formula gives essentially the same estimate as the one obtained by using the Input-Output model.

It will next be determined how well the formula mean, formula variance, and Beta distribution fit the actual distribution of multipliers under the assumption that each Input-Output matrix with the same fixed column sums is equally likely. To run this experiment, twelve hundred matrices were randomly generated having the column sums of the Burford-Hargrave model. The coefficients in these matrices are
| Sector | Coefficient Matrix | Louisiana Model  | Expected Value
<table>
<thead>
<tr>
<th></th>
<th>Column Totals</th>
<th>Multiplier</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
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<td>2.5272</td>
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<td>1.9662</td>
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<td>1.7972</td>
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<tr>
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<td>1.5231</td>
<td>1.6134</td>
</tr>
<tr>
<td>25</td>
<td>0.333770</td>
<td>1.5392</td>
<td>1.5963</td>
</tr>
</tbody>
</table>
of the distribution derived in Theorem 2.1. The following theorem explains how these coefficients were generated.

Define $G(\lambda, \alpha)$ to be a Gamma distribution with density function

$$f(x) = \frac{1}{\Gamma(\alpha)} \lambda^\alpha x^{\alpha-1} e^{-\lambda x}, \quad x \in [0, \infty).$$

Note that $G(\lambda, 1)$ is an exponential distribution. Furthermore, define $B(\alpha, \beta)$ to be a Beta distribution with density function

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}, \quad x \in [0, 1].$$

**Theorem 5.1**: Let $x_i$ be identically distributed independent $G(1,1)$, $1 \leq i \leq n$ (i.e., exponential with $\lambda=1$). Then the density function of the distribution of $\frac{\sum_{j=1}^{n} w x_j}{\sum_{j=1}^{n} x_j}$ is

$$f(x) = \frac{(n-1)(w-x)^{n-2}}{w^{n-1}}, \quad x \in [0, w].$$

**Proof**: Fix $i$ and let $y = \sum_{j=1}^{n} x_j$. Then $y$ is $G(1, n-1)$ and $x_i$ and $y$ are independent. Therefore,

$$\frac{x_i}{x_i + y}$$

is $B(1, n-1)$.

The density function of $B(1, n-1)$ is

$$f(x) = (n-1)(1-x)^{n-2}, \quad x \in [0, 1].$$
To determine the distribution of 
\[ w \left( \frac{x_i}{x_i+y} \right), \]
let \( F(t) \) be the cumulative distribution of \( B(1,n-1) \), and let \( F^*(t) \) be the cumulative distribution function of \( w \cdot B(1,n-1) \). Then
\[ F^*(t) = P(w \cdot B(1,n-1) \leq t) = P(B(1,n-1) \leq \frac{t}{w}) = F\left(\frac{t}{w}\right). \]

Thus, \( f^*(t) = F^{'\prime}(t) = F'\left(\frac{t}{w}\right) = \frac{1}{w} f\left(\frac{t}{w}\right), \)
or \( f^*(t) = \frac{1}{w} (n-1)(1 - \frac{t}{w})^{n-2}. \)

Therefore,
\[ f^*(t) = \frac{(n-1)(w-t)^{n-2}}{w^{n-1}}, \quad t \in [0,w]. \]

For each matrix \( A \) of the twelve hundred matrices generated, \( (I-A)^{-1} \) was computed and the multipliers were then obtained (the random number generator ADRAND[7] was used to generate the unit uniform random variables that were converted to exponential and then Beta random variables). The sample mean and sample variance of the distribution of multipliers were computed for each industry and are displayed in Table II. The formula mean and formula variance are also given. Since the matrices are randomly generated, the sample mean and sample variance are obtained without the restriction of the covariance assumptions that were needed to derive the formula mean and formula variance. A comparison of the formula mean with the sample mean, and the formula
<table>
<thead>
<tr>
<th>Sector</th>
<th>Formula Mean</th>
<th>Sample Mean</th>
<th>Formula Variance</th>
<th>Sample Variance</th>
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<td>.00020</td>
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<td>.00013</td>
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<td>1.5971</td>
<td>.00027</td>
<td>.00030</td>
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</table>
variance with the sample variance demonstrates the accuracy of the formulae. The formula mean estimates the sample mean exactly to two or three decimal places, while the formula variance estimates the sample variance accurately to three or four decimal places. Further empirical testing of this kind with different Input-Output models and different matrices has been done by Burford and Katz [4]. Those results are similar to the ones obtained here.

The remainder of this chapter will be given to an empirical investigation of how well the Beta distribution fits the distribution \( X_j \). Consider the following data from the Louisiana model:

\[
\begin{align*}
\wmax &= .7539 \\
\wmin &= .2314 \\
\bar{w} &= .44027.
\end{align*}
\]

Then

\[
\begin{align*}
E[X_j] &= \frac{(1-\wmax)(\bar{w} - \wmin)}{(\wmax - \wmin)(1-\bar{w})} = .195346 \\
\text{and} \\
\Var[X_j] &= \frac{(1-\wmax)^2(1-\wmin)^2}{(\wmax - \wmin)^2} \\
&= \frac{n[n(n+1)(n^2-s)-\sum_{i=1}^{n} w_i^2(2n^2-s)]}{(\sum_{i=1}^{n} w_i)^2[n(n+1)(n^2-s)-\sum_{i=1}^{n} w_i^2(2n^2-s)]} \\
&= .000392.
\end{align*}
\]

Therefore,
\[ \alpha = \frac{E[X_j] (1-E[X_j])}{\text{Var}[X_j]} - 1 \] = 78.1

and
\[ \beta = (1-E[X_j]) \left( \frac{E[X_j] (1-E[X_j])}{\text{Var}[X_j]} - 1 \right) = 321.7. \]

To test whether the distribution \( X_j \), which is simulated from the sample of twelve hundred matrices, can be judged \( B(78.1, 321.7) \), the Chi-Square Goodness of Fit test was used. The interval \([0,1]\) was broken down into six subintervals as follows:

- 0-.16
- .16-.18
- .18-.20
- .20-.22
- .22-.24
- .24-1.00.

To determine the probabilities for the above intervals for \( B(78.1, 321.7) \), the approximation method devised by Feizer and Pratt [15] was used.

Let \( n = \alpha + \beta - 1 \) and
\[ d = -0.5 + \frac{1}{6} - (n + \frac{1}{2}) (1-x) + \frac{1}{50} \left[ \frac{x}{\beta} - \frac{(1-x)}{\alpha} + \frac{(x - .5)}{\alpha + \beta} \right]. \]

Define \( g \) to be the function
\[ g(u) = (1-u)^{-2} \left( 1 - u^2 + 2u \log_e u \right). \]

Then
\[ P(B(\alpha, \beta) \leq x) = \phi(z) \]

where
\[ z = d \left[ \frac{1 + xg\left(\frac{a-\frac{5}{n(1-x)}}{\mu}\right) + (1-x) g\left(\frac{a-\frac{5}{nX}}{\mu}\right)}{(n + \frac{1}{6}) x(1-x)} \right]^{\frac{1}{2}} \]

and \( \Phi(z) \) denotes the cumulative distribution function for the standard normal distribution.

The above technique with \( \alpha = 78.1 \) and \( \beta = 321.7 \) gives the following probabilities for the subintervals and the critical values.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Subinterval</th>
<th>Probability</th>
<th>Critical Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0-.16</td>
<td>.0323</td>
<td>Under-1.7852</td>
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<tr>
<td>2</td>
<td>.16-.18</td>
<td>.1901</td>
<td>-1.7852-.7751</td>
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<td>.18-.20</td>
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<td>-.7751+.2351</td>
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<td>.20-.22</td>
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<td>6</td>
<td>.24-1.00</td>
<td>.0153</td>
<td>Over+2.2554</td>
</tr>
</tbody>
</table>

For each sample observation \( u_j \), the transformation

\[
x_j = \frac{(1-w_{\max})(1-w_{\min})}{w_j(w_{\max}-w_{\min})} (u_j - (1 + \frac{w_j}{1-w_{\min}}))
\]

was made. Then the value

\[
C = \frac{x_j - E[x_j]}{\sqrt{\text{Var}[x_j]}}
\]

was used to determine which of the six intervals the observation fell.
The critical values were obtained by using

\[ E[X_j] = -1.95346 \quad \text{and} \quad \text{Var}[X_j] = 0.000392 \]

on the subinterval endpoints. The critical values are computed as shown below.

\[ \frac{0.16 - 1.95346}{\sqrt{0.000392}} = -1.7852 \]

\[ \frac{0.18 - 1.95346}{\sqrt{0.000392}} = -0.7751 \]

\[ \frac{0.20 - 1.95346}{\sqrt{0.000392}} = 0.2351 \]

\[ \frac{0.22 - 1.95346}{\sqrt{0.000392}} = 1.2452 \]

\[ \frac{0.24 - 1.95346}{\sqrt{0.000392}} = 2.2554. \]

The critical value approach was used because the actual calculation of the critical value may be simplified by noting that

\[ C = \frac{X_j - E[X_j]}{\sqrt{\text{Var}[X_j]}} = \frac{\mu_j - E[\mu_j]}{\sqrt{\text{Var}[\mu_j]}} \]

where \( \mu_j \) is the multiplier value corresponding to \( X_j \).

Table III gives the results of testing the sample distribution of each \( X_j \), \( 1 \leq j \leq n \), against \( B(78.1, 321.7) \). For each sample value, the formula mean and variance were used to determine the interval in which the observation occurs. That is, for each \( X_j \)
### TABLE III

<table>
<thead>
<tr>
<th>Sector</th>
<th>( x^2 ) value for ( X_j )</th>
</tr>
</thead>
<tbody>
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<td>3.82</td>
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<tr>
<td>25</td>
<td>10.79</td>
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</tbody>
</table>
\[ \frac{X_j - E[X_j]}{\sqrt{\text{Var}[X_j]}} = \frac{\mu_j - E[\mu_j]}{\sqrt{\text{Var}[\mu_j]}} \]

or

\[ \frac{X_j - .195346}{\sqrt{.000392}} \]

was computed and compared to the critical value to determine into what interval the observation should be counted. To test the hypothesis that the distribution \( X_j \) is \( B(78.1, 321.7) \) at the 95% confidence level, the Chi Square distribution with five degrees of freedom is used since the formula mean and formula variance are employed for each observation. The critical value is \( \chi^2_{.95, 5} = 11.1 \).

The results of Table III are promising, but five of the sectors not only exceed the critical value but greatly exceed it. It was noticed, however, that the sectors that exceeded the critical Chi-Square value had sample variance values that are significantly smaller than the formula variance which was used to determine the interval where each observation fell. Further investigation revealed that other sectors with high variance differences, as indicated in Table II, also gave high Chi-Square values, although they did not exceed the critical value.

The test was rerun using the sample mean and sample variance of the distribution of \( X_j \) (or \( \mu_j \)) instead of the formula mean and formula variance. The critical values for the subintervals, as derived with the theoretical mean and variance formulae, were
maintained. Each observation of $X_j$ was transformed by

$$\frac{X_j - \bar{X}_j}{s_{X_j}} = \frac{\mu_j - \bar{\mu}_j}{s_{\mu_j}}$$

where $\bar{X}_j$ is the sample mean of $X_j$, $s_{X_j}$ is the sample standard deviation of $X_j$, $\bar{\mu}_j$ is the sample mean of $\mu_j$ and $s_{\mu_j}$ is the sample standard deviation. Due to the use of the sample mean and sample variance, two degrees of freedom are lost and so the new critical value is $\chi^2_{0.95,3} = 7.8$. The results, given in Table IV, show that two sectors have Chi-Square values that exceed the critical value. This is one more than what is theoretically expected.

This chapter has demonstrated, through simulation of the Louisiana Input-Output model, that the assumptions used in deriving the mean and variance formulae are acceptable. Furthermore, the multiplier estimates that are obtained from both methods, as shown in Table I, are very close. Additional empirical work to determine how well the derived formulae hold up under different conditions, such as the size of the Input-Output matrix or the range of values of the column sums, would be desirable. To this end, results of empirical tests using another Input-Output model are shown in Appendix Two.
<table>
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<tr>
<th>Sector</th>
<th>Chi-Square $X_j$</th>
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CHAPTER SIX

CONCLUSION

It was shown in the previous chapters that, if each Input-Output matrix with a given fixed set of column sums is equally likely, then the resulting distribution of multipliers is approximately Beta distributed with

\[ E[\mu_j] = 1 + \left( \frac{w_j}{1 - \sum_{i=1}^{n} w_i} \right) \]

and

\[ \text{Var}[\mu_j] = \frac{m w_j^2 \left[ n \sum_{i=1}^{n} w_i^2 - \left( \sum_{i=1}^{n} w_i \right)^2 \right] \left[ n^2 - s \right]}{(n - \sum_{i=1}^{n} w_i)^2 \left[ n(n+1)(n^2-2) - \frac{n}{2} \sum_{i=1}^{n} w_i^2 (2n^2-s) \right]} \]

Since \( \text{Var}[\mu_j] \) tends to be very small, it can be concluded that multipliers are insensitive to individual technical coefficients, but are sensitive only to the column totals.

With the insensitivity question settled, we will now proceed to justify the use of the expected value formula as an estimator of the true multiplier on both statistical and economic grounds. Since multipliers tend to be Beta distributed with extremely small variances, one can conclude that a high percentage of Input-Output models with
fixed column totals will give multiplier values that are very close to the formula mean. Indeed, Table V displays the intervals of multiplier values that are obtained from approximately 95% of the random Input-Output matrices of the Louisiana model. The limits are computed for each sector as follows:

\[(\text{LOWER LIMIT})_j = E[u_j] + (-1.7852)(\text{Var}[u_j])\] and
\[(\text{UPPER LIMIT})_j = E[u_j] + (2.2554)(\text{Var}[u_j]).\]

The formula mean can also be interpreted on economic grounds. It can be written as

\[E[u_j] = 1 + \frac{w_j}{1 - \bar{w}} = 1 + w_j + w_j\bar{w} + w_j\bar{w}^2 + \ldots\]

where \(\bar{w} = \frac{1}{n} \sum_{i=1}^{n} \frac{w_i}{n}\). The "1" represents the dollar that is initially spent on industry \(j\), while \(w_j\) is the exact amount industry \(j\) returns to the economy of the initial dollar in the first period. The remaining terms indicate that the return in period \(k+1\) results from each industry spending its return of period \(k\) according to the average column sum. This averaging effect is reasonable because by the second time period, the money is completely dispersed. In other words, because industry \(j\) fragments its expenditures over all industries, the estimate of the total expenditures for all industries in the second time period will be approximately \(w_j\) times the average proportion of interindustry expenditures \(\bar{w}\). Because this fragmentation of expenditures

\(^1\)The numbers -1.7852 and 2.2554 are the critical values that were derived in Chapter Five. See page 42.
TABLE V

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<td>25</td>
<td>1.5670</td>
<td>1.5963</td>
<td>1.6334</td>
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</table>
remains in effect throughout the remaining time periods, \((w_j \bar{w})\bar{w}\) estimates the total expenditures of all industries in the third time period, while \((w_j \bar{w}^{-n})\bar{w}\) estimates the total expenditure of all industries in the \(n^{th}\) time period.

Not only is the expected value formula approach much simpler to use than Input-Output analysis in determining multipliers, but also it has the flexibility of finding multipliers for firms or industries not considered in the Input-Output model. Once the average proportion of interindustry to total expenditures, \(w_{\frac{i}{n}}\), has been determined, the multiplier for any firm or industry can be found using his proportion of interindustry to total expenditures. This proportion can be found simply by computing one minus the proportions spent to government (taxes), households (wages), investors (interest and dividends), and foreign industries. Furthermore, industries would be more willing to provide their proportion of interindustry to total expenditures, rather than a detailed account of their expenditures to specific industries.

As mentioned in Chapter Two, Input-Output models generally have larger coefficients on the main diagonal of the matrix of technical coefficients. The main reason for the larger coefficients on the main diagonal of the matrix of technical coefficients is the aggregation of industries. If the industries in an Input-Output model were broken down further, the assumptions of Chapter Two would become more appropriate, partly because the diagonal elements would be reduced and partly because the covariance assumptions would become more valid. Furthermore, Burford [2] showed that random matrices, generated with the main diagonal coefficients significantly larger than the other coefficients, yielded multipliers that
differed little from the results obtained from random matrices whose coefficients in a given column were identically distributed.

The results in this study can also be used to show the effect of aggregation of industries on the multipliers. In Input-Output analysis, many industries must be aggregated together to reduce the size of the Input-Output matrix and the information required for it. It is assumed that common industries, (i.e., industries producing the same general class of products or using the same resources), have similar column coefficients and that the aggregation will not affect the multipliers significantly. However, this study has demonstrated that the main determinant of an industry multiplier is the column sum of the industry. Because the aggregation of industries has been done in terms of the homogeneity of the product or resource class of the industries rather than the homogeneity of the column sums, the resulting sum of the aggregated industries is an average over all industries in that group. Thus, it is unclear whether or not the resulting multiplier will realistically represent any of the specific industries or firms in the group.

To clarify the above point, suppose that two similar industries, $y_1$ and $y_2$ have column sums of .2 and .6, respectively. Assume furthermore, that the average column sum for the economy is .5. Treating each industry separately gives multiplier values for $y_1$ of approximately $1 + \frac{.2}{1-.5} = 1.4$ and for $y_2$ of approximately $1 + \frac{.6}{1-.5} = 2.2$. If the two industries are aggregated, the column sum will be $\frac{.2 + .6}{2} = .4$. This will yield a multiplier value of approximately $1 + \frac{.4}{1-.5} = 1.8$. This multiplier does not accurately represent the multiplier effect for either industry.
Furthermore, consider the following example, taken from the National Input-Output model of 1967. In February 1974, the Department of Commerce gave a preliminary report on the Input-Output model of 1967, publishing a reduced form of the full model. This reduced model, which was created by aggregating homogeneous industries together, has only eighty industry categories, whereas the full model has three hundred and sixty-seven. Later in 1974, the Department of Commerce published the full model. To consider the aggregation effect on the multipliers, the industry "2. Other Agricultural Products" was selected from the reduced model along with two of the subcategories "2.01 Cotton" and "2.07 Forest, Greenhouse, and Nursery Products" from the full model. The results are tabulated below.

REDUCED MODEL (80)

<table>
<thead>
<tr>
<th>Industry</th>
<th>I-O Column Total</th>
<th>Multiplier</th>
</tr>
</thead>
<tbody>
<tr>
<td>2. Other Agricultural Products</td>
<td>.51111</td>
<td>2.00666</td>
</tr>
</tbody>
</table>

FULL MODEL (367)

<table>
<thead>
<tr>
<th>Industry</th>
<th>I-O Column Total</th>
<th>Multiplier</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.01 Cotton</td>
<td>.68583</td>
<td>2.37251</td>
</tr>
<tr>
<td>2.07 Forest, Greenhouse, and Nursery Products</td>
<td>.33865</td>
<td>1.65937</td>
</tr>
</tbody>
</table>

Clearly, aggregation can severely affect the multipliers.

There has been further progress in the development of simple formulae to estimate economic parameters that have previously required Input-Output analysis. Burford and Katz in [3] developed formulae that estimate direct, indirect, and induced income multipliers using only a minimal amount of information. Define $A^*$ to be the matrix of technical coefficients with the household row and column included.

Let $Q^* = (I-A^*)^{-1}$, and define $\mu_j^* = \sum_{i=1}^{n} q_{ij}^*$. The income multiplier, $I_j$, is generally computed as

$$I_j = \sum_{i=1}^{n} q_{ij}^* a_{n+1, i}^*$$

The above can be estimated by

$$I_j = a_{n+1, j}^* + \frac{a_{n+1}^*}{a_{n+1}^*} \cdot (E[u_j^*]-1)$$

where $a_{n+1, j}^*$ represents the proportion of total expenditures of industry $j$ to households, and $a_{n+1}^*$ represents the average proportion of total expenditures over all industries that is paid to households. The total income multiplier can then be broken down in terms of the direct, indirect and induced effects. The following are estimates of these effects.

(1) DIRECT EFFECTS $a_{n+1, j}^*$

(2) INDIRECT EFFECTS $(E[u_j^*]-1) \frac{a_{n+1}^*}{a_{n+1}^*}$

(3) INDUCED EFFECTS $(E[u_j^*] - E[u_j]) \frac{a_{n+1}^*}{a_{n+1}^*}$

Therefore, the total income multiplier becomes

$$I_j = \text{DIRECT EFFECTS} + \text{INDIRECT EFFECTS} + \text{INDUCED EFFECTS}$$

$$= a_{n+1, j}^* + (E[u_j^*]-1) \frac{a_{n+1}^*}{a_{n+1}^*} + (E[u_j^*] - E[u_j]) \frac{a_{n+1}^*}{a_{n+1}^*}$$

$$= a_{n+1, j}^* + \frac{a_{n+1}^*}{a_{n+1}^*} (E[u_j^*]-1).$$

The income multiplier, $I_j$, is the change in total income that results from a one dollar change in the final demand for the product of industry $j$. 

---

3The income multiplier, $I_j$, is the change in total income that results from a one dollar change in the final demand for the product of industry $j$. 

The same approach can be used for estimating employment multipliers, given information from available standard sources and the ratios of employment to output for each industry. Similarly, tax and other related multipliers can be estimated using the equations developed here, data from secondary sources on taxes paid by industry at the local, state, and federal levels, and the total output of industries in the region. Finally, this approach can be extended to the estimation of pollution and energy consumption multipliers.

The results of this dissertation can also be applied to areas other than Input-Output analysis. For example, if $A$ is an absorbing Markov chain

$$A = \begin{bmatrix} I & 0 \\ K & Q \end{bmatrix}$$

then $(I-Q)^{-1}$ gives the expected number of visits before absorption. Since $Q^T$ has the same structure as an Input-Output matrix, it can be shown that the expected number of visits to the transient set of states before absorption depends mainly on the row sums of $Q$ rather than the detailed values of the elements of $Q$. In essence, the results in this dissertation apply to any model whose structure is of the form of the Input-Output matrix defined in Chapter One.
BIBLIOGRAPHY


APPENDIX ONE

In this appendix, Theorem 2.1 will be established for the case $n = 3$. Let $a_{11}$, $a_{21}$, and $a_{31}$ be the coefficients of column one such that $a_{11} + a_{21} + a_{31} = w$.

Consider the diagram above where

- $A = (0, 0, w)$
- $B = (w, 0, 0)$
- $C = (0, w, 0)$
- $D = (0, 0, 0)$
- $E = (x, 0, 0)$
- $F = (x, w-x, 0)$
- $G = (x, 0, w-x)$. 
The plane ABC contains all possible column values for
\(a_{11}, a_{21}, \) and \(a_{31}\). Because every possible column value is weighted
the same, the distribution function \(F\) of \(a_{11}\) is

\[
F(x) = P(a_{11} < x \mid a_{11} + a_{21} + a_{31} = w)
= \frac{\text{Area AGFC}}{\text{Area ABC}}
= \frac{\text{Area ABC} - \text{Area BGF}}{\text{Area ABC}}.
\]

Since ABC and BGF are equilateral triangles with lengths \(\sqrt{2}w\) and
\(\sqrt{2}(w-x)\) respectively, it follows that

\[
\text{Area ABC} = \frac{\sqrt{3}w^2}{2}
\text{ and }
\text{Area BGF} = \frac{\sqrt{3}(w-x)^2}{2}.
\]

Therefore,

\[
F(x) = \frac{\frac{\sqrt{3}w^2}{2} - \frac{\sqrt{3}(w-x)^2}{2}}{\frac{\sqrt{3}w^2}{2}}
= \frac{\frac{w^2 - (w-x)^2}{w^2}}{w^2}.
\]

This is precisely the result that is obtained in Theorem 2.1.

Alternatively, the distribution function for \(a_{11}\) can be computed
by

\[
P(a_{11} < x \mid a_{11} + a_{21} < w) = \frac{\text{Area DEFC}}{\text{Area DBC}}
= \frac{\text{Area DBC} - \text{Area BEF}}{\text{Area DBC}}
= 1 - \frac{\text{Area BEF}}{\text{Area DBC}}.
\]

Because EF is parallel to DC, then

\[
\frac{\text{Area BEF}}{\text{Area DBC}} = \frac{BE^2}{BD^2} = \frac{(w-x)^2}{w^2}.
\]
Therefore,

\[ P(a_{11} \leq x \mid a_{11} + a_{21} \leq w) = \frac{w^2 - (w-x)^2}{w^2} \]

\[ = F(x). \]

The above may be computed directly through integration because

\[ F(x) = P(a_{11} \leq x \mid a_{11} + a_{21} \leq w) \]

\[ = \frac{\int_{0}^{x} \int_{0}^{w-a_{11}} da_{21} da_{11}}{\int_{0}^{w} \int_{0}^{w-a_{11}} da_{21} da_{11}} \]

\[ = \frac{w^2 - (w-x)^2}{w^2}. \]
Appendix Two presents additional empirical evidence to support the results that were derived in Chapters Two and Three by considering Schaffer's Input-Output model of Georgia.¹

In this experiment, random samples of 500 Input-Output matrices were generated with the matrix dimensions and column totals of the Georgia model. For the set of 500 sample matrices, multipliers were computed, along with mean and variance vectors. The purpose of these experiments has been to determine how closely the random multipliers cluster around their expected values in a sampling situation, how closely the expected value formula approximates multipliers from empirical Input-Output matrices, and how closely the variance formula approximates the sample variance. Table IB compares the multiplier estimates that are obtained from both techniques. There is little difference in the estimates. Table IIB gives a comparison of the formula mean and formula variance to the sample mean and sample variance. Again, the closeness of the values adds credence to the covariance assumptions that were used to develop the formulae.

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VITA

Joseph Lorne Katz was born in Montreal, Canada on February 13, 1952. He graduated from West Hill High School in June 1969, and entered Louisiana State University in September of 1969. In May 1972, he received his Bachelor of Science degree in mathematics, in May 1973, he received his Bachelor of Science degree in computer science, and in August 1975, he received his Master of Science degree in mathematics.

In August 1975 he began graduate study in Quantitative Methods at the Louisiana State University and Agricultural and Mechanical College where he is now a candidate for the degree of Doctor of Philosophy.
EXAMINATION AND THESIS REPORT

Candidate: Joseph Lorne Katz

Major Field: Quantitative Methods

Title of Thesis: THE INSENSITIVITY OF LEONTIEF MULTIPLIERS TO RANDOM INPUT-OUTPUT MATRICES WITH FIXED COLUMN SUMS

Approved:

[Signature]
Major Professor and Chairman

[Signature]
Dean of the Graduate School

EXAMINING COMMITTEE:

[Signature]

[Signature]

[Signature]

Date of Examination:

July 7, 1978