2015

Left-orderability, Cyclic Branched Covers and Representations of the Knot Group

Ying Hu
Louisiana State University and Agricultural and Mechanical College

LEFT-ORDERABILITY, CYCLIC BRANCHED COVERS AND REPRESENTATIONS OF THE KNOT GROUP

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
Ying Hu
B.S., Beijing Normal University, 2008
M.S., Louisiana State University, 2010
August 2015

Acknowledgments

Firstly I want to thank my thesis advisor, Oliver Dasbach, for providing constant support and encouragement throughout my graduate studies, and for introducing me to the problem that this dissertation is based upon.

I also want to acknowledge all of the great teachers, colleagues and friends whose conversations, emails, papers and expository notes have aided in the writing of this dissertation.

I would like to extend my gratitude to the Louisiana State University Department of Mathematics for their generous support from the VIGRE grant. This has enabled me to attend plenty of conferences, meet people from other institutions and discuss exciting new mathematics.

Finally and foremost, I want to thank my husband, Jacob, whose insights have influenced me so much in my life and have helped me get to where I am today. I would never have made it without his support and love. This dissertation is dedicated to him.

Table of Contents

List of Figures

Abstract

A group G is called left-orderable if one can find a total order on G, which is preserved under left multiplication. In this paper we first give a sufficient condition for the fundamental group of the n^{th} cyclic branched cover of S^3 over a prime knot K to be left-orderable, in terms of $PSL(2,\mathbb{C})$ representations of the knot group. Then we make use of this criterion to study the left-orderability of fundamental groups of cyclic branched covers over two-bridge knots and satellite knots.

Chapter 1
Introduction

In Section 1.1, we introduce the notion of left-orderable groups and list some important properties regarding orderability of groups. A brief discussion on left-orderability of these groups that arise in the study of low-dimensional topological spaces, especially 3-manifold groups, is given in Section 1.2.

1.1 Left-orderable groups

A *total order* on a set S is a binary relation, denoted by $\leq$, satisfying conditions:

1.) If $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry);

2.) If $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity);

3.) Given any two elements a, b in S, either $a \leq b$ or $b \leq a$ (totality).

A binary relation $<$ on a set S is *asymmetric* if $a < b$ then $b \not< a$ for any a, b in S. A *strict total order* $<$ on a set S is an asymmetric, transitive binary relation satisfying that given any two elements a, b in S exactly one of the three $a < b$, $a = b$ and $a > b$ is true. Based on a total order $\leq$, a strict total order $<$ can be defined by saying $a < b$ if and only if $a \leq b$ and $a \neq b$. Conversely, a total order on a set S can also be constructed from a strict total order in a similar way.

Definition 1.1.1. A group G is called *left-orderable* (LO) if there exists a strict total order $<$ on the set of group elements such that $a < b$ if and only if $c \cdot a < c \cdot b$ for any a, b, c in G.

Some authors require a group to only have a left-invariant total order to be a left-orderable group. Since any total order can be made to be strict, these two definitions are equivalent. A strict total order on a group G that is invariant under left multiplication is called a *left-order*. A *right-order* on a group G can also be defined by mimicking the definition of a left-order.

Definition 1.1.2. A group G is called *right-orderable* if there exists a total order $<$ on the set of group elements such that $a < b$ if and only if $a \cdot c < b \cdot c$ for any a, b, c in G.

As one might have guessed, left-orderability and right-orderability are equivalent group properties. Suppose that $<$ is a left-order on a group G. We define a new order $\prec$ by setting $a \prec b$ if and only if $b^{-1} < a^{-1}$. It is easy to check that the new order $\prec$ is also a strict total order and is invariant under multiplication on the right. Some groups admit a strict total order that is preserved under multiplication on both sides simultaneously; these groups are called *bi-orderable groups* or simply called *orderable groups*. We point out that in general $a < b$ does not implies $b^{-1} < a^{-1}$ and thus $<$

and $\prec$ as mentioned above are two different orders. In fact, given any two elements a, b in a group G, $a < b$ implies $b^{-1} < a^{-1}$ if and only if $<$ is a bi-order [Con59].

Example 1.1.3. The usual order used everyday to compare two numbers makes the group of all real numbers $(\mathbb{R}, +)$ a left-orderable group. Since the group $(\mathbb{R}, +)$ is abelian, it is also a bi-order on $(\mathbb{R}, +)$.

Example 1.1.4. We equip $\mathbb{R}$ with a fixed left-order. Let $\mathrm{Homeo}_+(\mathbb{R})$ be the group of order-preserving homeomorphisms from $\mathbb{R}$ to itself. We show that $\mathrm{Homeo}_+(\mathbb{R})$ is a left-orderable group.

First, one list the set of rational numbers $\mathbb{Q} = \{x_1, x_2, \cdots, x_i, \cdots\}$. Since rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$, given any two distinct functions $f, g \in \mathrm{Homeo}_+(\mathbb{R})$, there exists at least one x_i in $\mathbb{Q}$ such that $f(x_i) \neq g(x_i)$. We set $f < g$ if $f(x_k)$ is less than $g(x_k)$ as two real numbers, where x_k is the first number in the sequence $\mathbb{Q} = \{x_1, x_2, \cdots, x_n, \cdots\}$ that has distinguished images under homeomorphisms f and g. It is a routine to verify that the order we just defined is a left-order on the group $\mathrm{Homeo}_+(\mathbb{R})$.

The following theorem characterizes the left-orderable groups from the point of view of dynamics.

Theorem 1.1.5 ([GHY01, Far76])**.** *A countable group G is left-orderable if and only if it is isomorphic with a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$.*

Let G be a left-orderable group with a left-order $<$. Suppose that G has a non-trivial torsion element denoted by g, i.e. $g^n = 1$ for some positive integer n. Without lose of generality, we can assume that $g > 1$. Since the order is invariant under left multiplication, we have a chain of inequalities

$$1 < g < g^2 < \cdots < g^n = 1,$$

which leads to a contradiction.

Proposition 1.1.6. *Left-orderable groups are torsion-free.*

Hence, all finite groups are not left-orderable. In this paper, we consider the trivial group as a non-left-orderable group.

The existence of a left-order also implies a strong algebraic property on its group ring. It is known that left-orderable groups obey the zero-divisor conjecture of Kaplansky. That is, the group ring of a torsion-free group over a field does not have zero-divisor. The zero-divisor conjecture is still unsolved for torsion-free groups in general.

Orderability of groups is preserved under taking free product and group extension.

Theorem 1.1.7 ([Vin49])**.** *A free product $G = G_1 * G_2 * \cdots * G_n$ is left-orderable if and only if each G_i is a left-orderable group*

Since finitely generated free groups are free product of infinite cyclic groups, we have all finite generated free groups are left-orderable. Note that the alphabetical order on free group is not preserved under the group multiplication.

Proposition 1.1.8. *Suppose that there is a short exact sequence of groups:*

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1.$$

Then G is left-orderable if both K and H is left-orderable.

We point out that the converse of the above statement is not true. In particular, a quotient of a left-orderable group is not necessary left-orderable in general.

If the group in consideration is a fundamental group of a manifold, then its left-orderability is also encoded in its universal covering space.

Theorem 1.1.9 ([Far76])**.** *The fundamental group of a manifold M is left-orderable if and only if its universal cover $\widetilde{M}$ can be embedded into $M \times \mathbb{R}$ such that the composite map $\widetilde{M} \to M \times \mathbb{R} \to M$ is the covering map .*

1.2 Left-orderability of 3-manifold groups

It has recently been shown that many of the groups which arise in low-dimensional topology are left-orderable. Dehornoy first showed that the braid groups B_n are left-orderable [Deh94, Deh97]. Later on, this result was reinterpreted from a more geometric approach [FGR+99] and also generalized to mapping class groups of surfaces with non-empty boundary (finitely many marked points on the surface are also allowed) [SW00, RW00]. Note that the mapping class group of a closed surface has periodic (torsion) element and hence is not left-orderable by Proposition 1.1.6.

It is known that all surface groups are left-orderable except for the obvious counterexample $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$. In fact, besides the projective space $\mathbb{R}P^2$ and the Klein bottle, all surfaces have bi-orderable fundamental groups [BRW05].

The interaction between the topological properties of a manifold and the left-orderability of its fundamental group is mostly evident in the study of compact 3-manifolds and their fundamental groups.

Given two connected 3-manifolds M_1 and M_2, one can define a connect sum $M_1 \# M_2$ by removing a three ball B_i from each M_i and gluing the resulting manifolds together along two-spherical boundaries ∂B_i. This operation is independent on the choice of B_i but in general does rely on the gluing map. There are essentially two different ways of identifying ∂B_1 with ∂B_2: one is through an orientation-preserving homeomorphism of S^2 and the other is through an orientation-reversing homeomorphism of S^2. For the purpose of this paper, it is unnecessary for us to make an effort to differ these two, so we denote both of them by $M_1 \# M_2$. A 3-manifold M is *prime* if $M = M_1 \# M_2$ implies one of M_i is the 3-sphere S^3.

A compact, connected 3-manifold M can always be decomposed into *prime* pieces. That is,

$$M = M_1 \# M_2 \# \cdots \# M_n,$$

where all M_i are prime. With a minor modification for non-orientable 3-manifolds, this decomposition is unique up to a permutation of indices.

It follows from the van Kampen Theorem that

$$\pi_1(M) = \pi_1(M_1) * \pi_1(M_2) \cdots * \pi_1(M_n).$$

Hence $\pi_1(M)$ is left-orderable if and only if each nontrivial factor $\pi_1(M_i)$ is left-orderable by Theorem 1.1.7, which means that to understand the left-orderability of 3-manifold groups in general, it is sufficient to study the left-orderability of fundamental groups of prime 3-manifolds.

A 3-manifold M is *irreducible* if every 2-sphere in M bounds a three ball. An irreducible 3-manifold is apparently prime. On the other hand, if a 3-manifold M is prime but not irreducible, then there exists a non-separating 2-sphere in M, which implies M is homeomorphic to a sphere bundle over a circle. So $\pi_1(M)$ is isomorphic to $\mathbb{Z}$ and is obviously left-orderable.

The main tool we use to prove that an irreducible 3-manifold has a left-orderable fundamental group is the following theorem proved by Boyer, Rolfsen and Wiest.

Theorem 1.2.1 ([BRW05])**.** *Suppose that M is a compact, connected, P^2-irreducible 3-manifold. Then $\pi_1(M)$ is left-orderable if and only if there exists a surjective from $\pi_1(M)$ onto a left-orderable group.*

Recall that a compact 3-manifold is P^2-irreducible if it is irreducible and does not contain a 2-sided P^2, the real projective plane. When applying the above theorem to orientable 3-manifolds, P^2-irreducible can be replaced by irreducible since a non-orientable surface can not be 2-sided in an orientable 3-manifold.

Denote by $b_1(M)$ the first Betti number of a 3-manifold M, which by definition is equal to the dimension of the vector space $H_1(M, \mathbb{Q})$ over $\mathbb{Q}$. Our discussion on the left-orderability of 3-manifold groups will be divide into the following two cases: the case when $b_1(M) > 0$ and the case when $b_1(M) = 0$.

Case 1: The first Betti number is positive.

M is a compact, connected, P^2-irreducible 3-manifold. Since the first Betti number $b_1(M)$ is positive, we have surjective maps:

$$\pi_1(M) \longrightarrow H_1(M, \mathbb{Z}) \longrightarrow \mathbb{Z},$$

and by Theorem 1.2.1 the fundamental group $\pi_1(M)$ is left-orderable.

The theorem below follows from an argument by Howie and Short.

Theorem 1.2.2 ([HS85] Lemma 2)**.** *Suppose that M is a compact, P^2-irreducible and connected 3-manifold. The fundamental group $\pi_1(M)$ is locally indicable if and only if its first Betti number $b_1(M) > 0$.*

A group is *locally indicable* if every nontrivial finitely-generated subgroup has the integer group $\mathbb{Z}$ as its quotient. The condition of being locally indicable is strictly stronger than being left-orderable in general. For instance, as we will see later, many compact 3-manifolds whose first Betti numbers are zero have left-orderable fundamental groups, but none of these fundamental groups is locally indicable according to Theorem 1.2.2.

Case 2: The first Betti number is zero.

Assume that a 3-manifold M has the first Betti number $b_1(M) = 0$. The following lemma asserts that M must be orientable.

Lemma 1.2.3 ([BRW05] Lemma 3.3). *If M is a non-orientable, P^2-irreducible 3-manifold, then $b_1(M) > 0$.*

Moreover, if an orientable 3-manifold M with a nonempty boundary has the first Betti number $b_1(M) = 0$, then its boundary can only contain 2-spheres S^2. Let $\hat{M}$ be the closed 3-manifold obtained by attaching three-balls B^3 to M along its boundary spheres. Hence $\hat{M}$ is a closed 3-manifold with exactly same fundamental group with $\pi_1(M)$. On the other hand, closed 3-manifolds with first Betti number equal to zero are rational homology spheres.

Therefore, the study of the left-orderability of fundamental groups of 3-manifolds is reduced to the case when M is an irreducible rational homology sphere. Within this class of three manifolds, the left-orderability of 3-manifold groups is fully understood and characterized for non-hyperbolic and geometric rational homology 3-spheres [BRW05].

In the remainder of this section, we give a brief survey on the connections among Left-orderability, taut-foliation and L-spaces to motivate the study of the left-orderability of fundamental groups of rational homology spheres.

Left-orderability and L-spaces

Let M be a closed, connected, oriented 3-manifold and $\widehat{HF}(M)$ denote the Heegaard Floer homology of M, introduced by Ozsváth and Szabó in [OS04a]. If M is a rational homology sphere, then the dimension of $\widehat{HF}(M)$ as a vector space over $\mathbb{Z}_2$ is greater than or equals to $|H_1(M, \mathbb{Z})|$, the number of elements in $H_1(M, \mathbb{Z})$ [OS04c]. If equality is achieved, then M is called an L-space [OS05b]. Hence, L-spaces can be understood as spaces with simplest Heegaard Floer homology. Len spaces are L-spaces. More generally, 3-manifolds with finite fundamental groups are all L-spaces [OS05b].

An interesting question asked by Ozsváth and Szabó says if L-spaces can be characterized without referring Heegaard Floer homology [OS05a]. To this end, Boyer, Gordon and Watson suggested the following conjecture:

Conjecture 1.2.4 ([BGW13]). *An irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable.*

They proved the conjecture for the case when the 3-manifold M is non-hyperbolic, geometric.

Theorem 1.2.5 ([BGW13]). *Suppose that M is a closed, connected, geometric, non-hyperbolic 3-manifold. Then M is an L-space if and only if $\pi_1(M)$ is not left-orderable.*

This conjecture has also been confirmed for classes of hyperbolic rational homology spheres. These includes, among others, twofold branched covers of S^3 over non-splitting alternating links.

Left-orderability and taut foliations

A codimension-1 foliation $\mathcal{F}$ of a closed orientable 3-manifold M is a decomposition of M into connected surfaces $\{F_\alpha\}$ called *leaves* such that M is covered by a collection of charts $\varphi : U \to \mathbb{R}^2 \times \mathbb{R}$ such that for each leave F_α, the image $\varphi(F_\alpha \cap U)$ is a union of affine planes in the form of $\mathbb{R}^2 \times *$. The space of leaves of a foliated manifold is the quotient space by collapsing each leaf to a point.

Through a use of the *Reeb foliation* on a solid torus and the fact that all closed orientable 3-manifolds can be obtained by performing Dehn surgeries on a braid, one can show that a codimension-1 foliation on a closed orientable 3-manifold always exists [Lic65, Nov65]. Hence, it makes sense to restrict ourselves to a "nicer" class of codimension-1 foliations. A codimension-1 foliation on a compact 3-manifold is taut if there exists a single loop that intersects each leaf transversely at least once.

A codimension-1 foliation $\mathcal{F}$ on a closed 3-manifold M is called $\mathbb{R}$-covered if the space of leaves of the pullback foliation $\widetilde{\mathcal{F}}$ of the universal cover $\widetilde{M}$ is homeomorphic to the real line $\mathbb{R}$. In this case, the fundamental group $\pi_1(M)$ acts on the space of leaves and thus acts on the real line $\mathbb{R}$.

Proposition 1.2.6 ([CD03])**.** *If an orientable closed 3-manifold M has a co-orientable $\mathbb{R}$-covered foliation, then $\pi_1(M)$ is left-orderable.*

A $\mathbb{R}$-covered foliation is taut. On the other hand, it is known that an L-space does not admit any co-oriented taut foliation [KR14, OS04b]. As a result, Conjecture 1.2.4 would imply an affirmative answer to the following question:

Question 1 ([BGW13])**.** *Given an irreducible rational homology sphere, does the existence of a co-oriented taut foliation imply the left-orderability of its fundamental group?*

If an orientable 3-manifold M has a taut foliation with hyperbolic leaves, then its fundamental group $\pi_1(M)$ acts on a circle, Thruston's universal circle [CD03]. In addition, if we assume that the 3-manifold M is an integer homology sphere, then

$$H^2(\pi_1(M), \mathbb{Z}) \cong H^2(M, \mathbb{Z}) = 0$$

and hence an action of $\pi_1(M)$ on the circle S^1 can always be lifted to an action on the real line $\mathbb{R}$. Note that on an integer homology sphere, any foliation is automatically co-orientable.

Lemma 1.2.7 ([BB13])**.** *Suppose that M is an integer homology sphere admitting a taut foliation. Then its fundamental group $\pi_1(M)$ is left-orderable.*

1.3 Summary of the Dissertation

In Chapter 2, we give a sufficient condition for the fundamental group of the n^{th} cyclic branched cover of S^3 along a prime knot K to be left-orderable in terms of $PSL(2,\mathbb{R})$ representations of the knot group.

Some basic knot theory terminologies and notations in this paper are given in Section 2.1. We introduce cyclic branched covers in Section 2.2. And in Section 2.3 we prove Lemma 2.3.1, which is essential in our proof of Theorem 2.4.1.

Lemma (Lemma 2.3.1)**.** *Given a knot K in S^3, denote by μ a meridian curve on the boundary torus ∂X_K. Suppose that there exists a group homomorphism ρ from $\pi_1(X_K)$ to a group G and $\rho([\mu]^n)$ is in the center of G. Then ρ induces a group homomorphism from $\pi_1(\Sigma_n(K))$ to G. In particular, if ρ is non-abelian, then the induced homomorphism is nontrivial.*

Theorem (Theorem 2.4.1)**.** *Given any prime knot K in S^3, denote by $[\mu]$ a meridional element of $\pi_1(X_K)$. If there exists a non-abelian representation $\pi_1(X_K)$ to $PSL(2,\mathbb{R})$ such that $[\mu]^n$ is sent to the identity matrix I, then the fundamental group $\pi_1(\Sigma_n(K))$ is left-orderable.*

In Chapter 3, we give an application of Theorem 2.4.1 to the study of the left-orderability of the fundamental group of the n^{th} cyclic branched cover over a two-bridge knot. We first show some properties of $SL(2,\mathbb{C})$ representation space of two-bridge knot groups in Section 3.1. We use these properties to prove Theorem 3.2.1 in Section 3.2 .

Theorem (Theorem 3.2.1)**.** *Given a (p,q) two-bridge knot K, with $p \equiv 3$ mod 4, there are only finitely many cyclic branched covers, whose fundamental groups are not left-orderable.*

We also present two specific examples, showing that the n^{th} cyclic branched cover over the knot 5_2 (resp. the knot 7_4) has a left-orderable fundamental group as $n \geq 9$ (resp. $n \geq 13$). At the end, in Section 3.3, we show a stronger statement than Theorem 3.2.1.

Theorem (Theorem 3.3.1)**.** *Let K be a (p,q) two-bridge knots. Assume that the knot group $\pi_1(X_K)$ has a real parabolic representation $\rho : \pi_1(X_K) \to PSL(2,\mathbb{R})$. Then the n^{th} cyclic branched covers has left-orderable fundamental group as n sufficiently large.*

In Chapter 4, after a short introduction of satellite knots, we first prove two lemmas in Section 4.1 and then we show the following result in Section 4.2.

Theorem (Theorem 4.2.3)**.** *Let $P(K)$ be a proper satellite knot. Assume that pattern P as a knot in S^3 is a (p,q) two-bridge knot with either $p = 3$ mod 4 or $p/q = (2k+1)+\frac{1}{2n}$ and $k > 0$, $n \neq 1$. Then $\pi_1(\Sigma_n(P(K)))$ is left-orderable as n sufficiently large.*

Chapter 2

Left-Orderability and Cyclic Branched Covers

In this chapter, we introduce a class of 3-manifolds that are so-called cyclic branched covers of the 3-sphere. Roughly speaking, these spaces are almost covering spaces of S^3 except for a set of branched loci in S^3, which forms a knot or link in S^3. The goal of this chapter is to derive an connection between the left-orderability of fundamental groups of cyclic branched covers over a knot K and the representations of the knot group $\pi_1(S^3 \setminus K)$.

2.1 Knots in S^3

In this section, we briefly discuss some background material in knot theory while setting up the notations for our later discussion.

Throughout this paper, the three sphere S^3 is equipped with a fixed orientation. A link L of n components in S^3 is a smooth embedding from n copies of disjoint union of S^1 to S^3. Two links L_1 and L_2 are equivalent if there exists an orientation-preserving homeomorphism f from S^3 to itself such that $f(L_1) = L_2$. We can also assign an orientation on each component of the link, and if we do so, the link is called oriented. A (oriented) knot K is a (oriented) link with only one component. A knot K is called *unknot* if it bounds an smoothly embedded disk in S^3.

Let $S^3 = \mathbb{R}^3 \cup \infty$ be an one-point compactification of $\mathbb{R}^3$, links can be also viewed as a smooth embedding of circles into $\mathbb{R}^3$. A projection of a link on a plane $p : \sqcup S^1 \to \mathbb{R}^3 \to \mathbb{R}^2$ is called regular, if all the self-intersections are double points and the image of the link under the projection transversely intersect at these double points. To get a *link diagram* from a regular projection, one break the under-crossing arc at each double point so that the link itself can be visualized on a plane. Figure 2.1 illustrates the sum of two knots by using their diagrams. Note that the same link can have very different looking diagrams. Two link diagrams represent the same link type if and only if they are related by a finite sequence of *Reidemeister moves*.

A link diagram is called *alternating* if as one travels along each component of the link diagram the over- and under-crossing arcs alternate . A link is called *alternating* if there exists an alternating link diagram representing that link. In Figure 2.1, two diagrams on the left-hand side of the equality are alternating, while the one on the right-hand side is not.

In the following context, we will continue our discussion for knots in S^3. Although most of the concepts can be generalized to links with or without modification but

---

Some of the results in this chapter previously appeared as [Ying Hu, *Left-orderability and cyclic branched coverings.* Algebr. Geom. Topol., 15(1):399-413, 2015]. They are reprinted by permission of Mathematical Sciences Publishers.

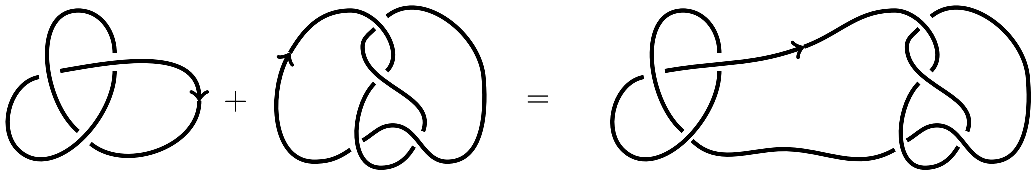

FIGURE 2.1. Sum of two knots

some of the theorems will fail when applied to links. For more details, readers are referred to [BZ03, Lic97, Rol76].

Given two oriented knots K_1 and K_2, intuitively they can be connected together to form another knot denoted by $K_1 + K_2$ as shown in Figure 2.1. More precisely, regard K_1 and K_2 as being in distinct copies of S^3, remove from each S^3 a ball that meets the given knot in an unknotted arc, and then identify together the resulting boundary spheres and their intersections with the knots so that all orientations match up. The sum of two knots is well-defined; that is, this whole procedure is independent with the location of removed unknotted arc in each knot. A knot K is a *prime knot* if it is not the unknot, and any decomposition of $K = K_1 + K_2$ implies that K_1 or K_2 is the unknot.

Let $v(K)$ be an open tubular neighborhood of a knot K in S^3 and so its closure is homeomorphic to a solid torus. Denote by X_K the complement space $S^3 \setminus v(K)$. The *knot group* is defined to be the fundamental group $\pi_1(X_K)$. It is known that two unoriented knots K_1 and K_2 are equivalent if there exists an orientation-preserving homeomorphism between their complement spaces, X_{K_1} and X_{K_2} [GL89]; For a prime knot K, its complement space X_K is determined by the knot group $\pi_1(X_K)$ up to a possible orientation-reversing homeomorphism [FW78].

Given an oriented knot K, a *meridian* μ is a simple closed curve in ∂X_K that bounds a disc in $v(K)$; a *preferred longitude* λ is a simple closed curve in ∂X_K that is homologous to the knot K in $v(K)$ and equals to zero in $H_1(X_K)$. Both μ and λ are unique up to a homotopy in ∂X_K and their orientations are inherited naturally from the orientation on the knot K and the 3-sphere S^3. Hence, each of them represents a generator in the $\pi_1(\partial X_K) \cong H_1(\partial X_K)$, which will be denoted by $[\mu]$ and $[\lambda]$ accordingly. Throughout this paper, if c is a simple closed curve in some space Y, then we use $[c]$ to denote the element in $H_1(Y)$ or the conjugacy class in $\pi_1(Y)$ represented by the curve c.

A *Seifert surface* for an oriented knot is a connected compact orientable surface in S^3 that has the knot K as its boundary. For example for the unknot, the disk that it bounds is a Seifert surface of the unknot. For any knot in S^3, Seifert surfaces always exist. In fact, there is an algorithm to construct a Seifert surface from a knot diagram, called Seifert's algorithm. Let F be a Seifert surface of a given knot K, then the intersection $F \cap \partial X_K$ gives us a curve on ∂X_K, which is the preferred longitude as defined above and it also bounds a surface $F \setminus \mathrm{int}(F \cap X_K)$ in $X_K \subset S^3$ that is homeomorphic to F. In this paper, we don't differ between F and $F \setminus \mathrm{int}(F \cap X_K)$; both of them will be referred as a Seifert surface of the knot K.

2.2 Cyclic branched covers of S^3

Let $D^2 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit disk on the complex plane $\mathbb{C}$. Consider the map f_n from D^2 to D^2 that sends z to z^n for some n in $\mathbb{Z}^+$. The map f_n fixes the origin $z = 0$ and induces an n-folded covering space from $D^2 \setminus 0$ to $D^2 \setminus 0$. This makes the map f_n an n^{th} cyclic branched cover of D^2 branched over the point $z = 0$. We call the branched cover is cyclic, since the group of deck transformations of the covering map $f_n|_{D^2\setminus 0}$ is the order n cyclic group. An example for $n = 4$ is depicted in Figure 2.2.

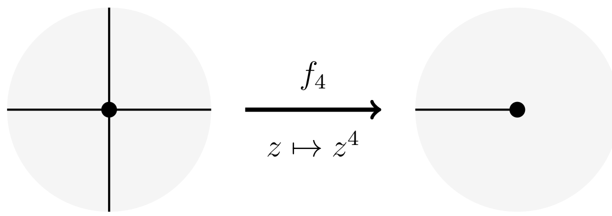

FIGURE 2.2. The 4^{th} cyclic branched cover of the unit disk branched

In general, given two compact manifolds M and N with proper codimension-2 submanifolds $A \subset M$ and $B \subset N$, a continuous map $f : M \to N$ is called a *branched cover* with branched set $B = f(A)$ if 1) components of the preimage of open sets of N are a basis for the topology of M and 2) the restriction map $f : M \setminus A \to N \setminus B$ is a covering space of $N \setminus B$. Note that since M is required to be compact, the covering map $f|_{M\setminus A}$ is finite-sheeted. The branched cover M is completely determined by specifying N, the branched set B and a finite cover of $N \setminus B$ (see [Rol76]).

By the Alexander Duality, the first homology of the knot complement $H_1(S^3 \setminus K)$ is isomorphic to $\mathbb{Z}$, generated by the meridian element $[\mu]$. Let $\widetilde{\Sigma}_n(K)$ be the n-fold cyclic covering space of $S^3\setminus K$ associated with the kernel of the following composition map:

$$\pi_1(S^3 \setminus K) \longrightarrow H_1(S^3 \setminus K) \longrightarrow \mathbb{Z}_n,$$

where the second homomorphism is the standard quotient map from $\mathbb{Z}$ to $\mathbb{Z}_n$. Denote $\Sigma_n(K)$ the corresponding n^{th} cyclic branched cover of S^3 with the branched set being the knot K. Sometimes we may abuse the terminology and call $\Sigma_n(K)$ the n^{th} cyclic branched cover of the knot K.

A construction of cyclic branched covers.

The n^{th} cyclic branched cover Σ_n can be constructed explicitly through a use of Seifert surfaces. Let K be an oriented knot in S^3. Note that the knot complement X_K has the same homotopy type as $S^3 \setminus K$ and hence $H_1(X_K) \cong H_1(S^3 \setminus K)$ is isomorphic to $\mathbb{Z}$ generated by $[\mu]$. Let's first construct the n^{th} cyclic covering space of X_K, whose fundamental group is isomorphic to $\mathrm{Ker}(\pi_1(X_K) \to \mathbb{Z}_n)$.

Let F be a Seifert surface of the knot K. A regular neighborhood of F is homeomorphic to $F \times [-1, 1]$, where the positive direction is chosen so that the induced

orientation on the boundary ∂F is the same as the orientation on the knot K and the preferred longitude λ. The complement space $X_K \setminus F \times (-1,1)$ has two copies of Seifert surface F on the boundary, $F \times -1$ and $F \times 1$. Now take n copies of $X_K \setminus F \times (-1,1)$ and glue them together by identify the surface $F \times 1$ in k^{th} copy of $X_K \setminus F \times (-1,1)$ with the surface $F \times -1$ in the $(k+1)^{th}$ copy of $X_K \setminus F \times (-1,1)$, where $k = 0, 1, 2, \cdots, n-1$ taken in $\mathbb{Z}_n$. The resulting manifold is the n^{th} cyclic cover space of X_K. We denote it by $\widetilde{\Sigma}_n(K)$.

From the above construction, it is easy to see that the covering space $\widetilde{\Sigma}_n(K)$ is also a torus. To get the n^{th} cyclic branched cover $\Sigma_n(K)$, we need to glue a solid torus $D^2 \times S^1$ back to $\widetilde{\Sigma}_n(K)$ such that the meridian curve $\partial D^2 \times *$ is identified with the preimage of the meridian μ under the covering map from $\widetilde{\Sigma}_n(K)$ to X_K.

Figure 2.3 illustrates the procedure for the 4^{th} cyclic covers $\widetilde{\Sigma}_4$ of the complement space of the unknot by using a disk as its Seifert surface. Now we glue a solid torus to $\widetilde{\Sigma}_4$ such that the meridian of the solid torus is identified with the preimage of μ under the covering map, i.e. the red curve in Figure 2.3. It is not too hard to see that the resulting manifold is S^3, which as we described above is precisely the the 4^{th} cyclic branched cover of S^3 branched over the unknot. In fact, by the Smith Conjecture, the n^{th} cyclic branched cover $\Sigma_n(K)$ is homeomorphic to S^3 only if the knot K is the unknot.

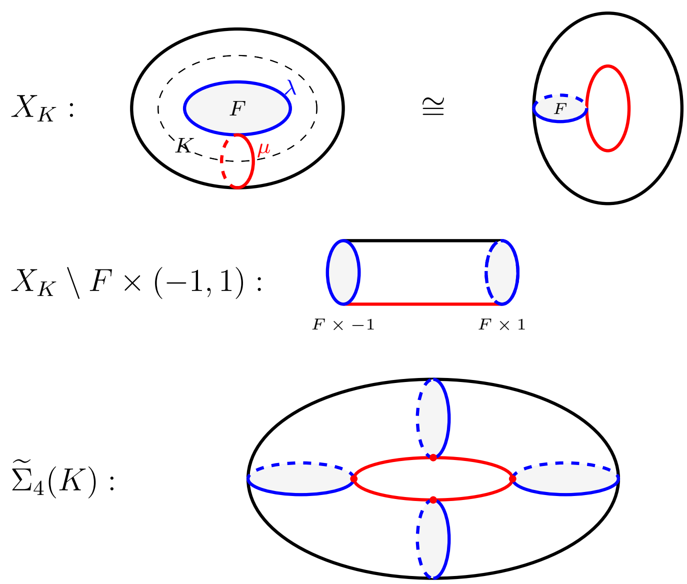

FIGURE 2.3. Construct $\widetilde{\Sigma}_4(K)$ when K is the unknot.

Properties of cyclic branched covers.

The Alexander polynomial of a knot K, usually denoted by $\Delta_K(t)$, is a polynomial invariant of a knot in $\mathbb{Z}[t^{\pm 1}]$ and is well defined up to $\pm t^k$.

Theorem 2.2.1 (Fox, see [BZ03, Gor78]). *The n^{th} cyclic branched cover $\Sigma_n(K)$ branched over a knot K is a rational homology sphere if and only if no root of the Alexander polynomial $\Delta_K(t)$ is an n^{th} root of unity.*

Since $|\Delta_K(1)| = 1$, it is not hard to show that when n is a prime power, the n^{th} cyclic branched cover $\Sigma_n(K)$ is a rational homology sphere for any knot K. On the other hand, the Alexander polynomial $\Delta_K(t) = 0$ only has finitely many roots. In this sense, given a knot K, we conclude that most of its cyclic branched covers are rational homology spheres if not all.

Theorem 2.2.2 ([Plo84]). *The n^{th} cyclic branched cover $\Sigma_n(K)$ branched over a prime knot K is irreducible.*

A knot K is called hyperbolic, if its complement space $S^3 \setminus K$ admits a complete hyperbolic structure.

Theorem 2.2.3 ([BP01]). *The n^{th} cyclic branched cover of S^3 branched over a hyperbolic knot K is hyperbolic for any $n \geq 3$, except when $n = 3$ and K is the figure-eight knot.*

As we discussed in the previous chapter, when studying the left-orderability of 3-manifold groups, the most interesting case is the fundamental groups of irreducible rational homology spheres, especially of these that are hyperbolic. The above results show that most of the cyclic branched covers belong to this class of 3-manifolds.

2.3 Proof of Lemma 2.3.1

Based on the construction of $\Sigma_n(K)$ as we described above, the fundamental group of the n^{th} cyclic branched cover $\pi_1(\Sigma_n(K))$ is isomorphic to the quotient of the fundamental group of the n^{th} cyclic covering space $\pi_1(\widetilde{\Sigma}_n(K))$ by its normal subgroup generated by the element represented by preimage of a meridian on ∂X_K. The fundamental group $\pi_1(\widetilde{\Sigma}_n(K))$ is isomorphic to the index n subgroup $\mathrm{Ker}(\pi_1(X_K) \to \mathbb{Z}_n)$; the element in $\pi_1(\widetilde{\Sigma}_n(K))$ represented by the lifting of the meridian curve μ is equal to $[\mu]^n$ up to a conjugation. Hence, we have that

$$\pi_1(\Sigma_n(K)) \cong \mathrm{Ker}(\pi_1(X_K) \to \mathbb{Z}_n)/ \ll [\mu]^n \gg . \tag{2.1}$$

The goal of this section is to derive specific group presentations of fundamental groups $\pi_1(\widetilde{\Sigma}_n(K))$ and $\pi_1(\Sigma_n(K))$, which we will use to prove Lemma 2.3.1.

An HNN decomposition of the knot group $\pi_1(X_K)$.

First let's start with a presentation of the knot group $\pi_1(X_K)$. Let F be a Seifert surface of an oriented knot K. As in Section 2.2, the Seifert surface F has a regular neighborhood that is homeomorphic to $F \times [-1, 1]$ and the positive direction is chosen so that the induced orientation on the boundary ∂F is the same as the orientation on the knot K. We denote $F \times -1$ by F_- and $F \times 1$ by F_+ as depicted in Figure 2.4. In addition, the point P_+ (resp. P_-) is the intersection point of the meridian μ and F_+ (resp. F_-).

Note that the Seifert surface F is a surface with one boundary component, so its fundamental group $\pi_1(F)$ is a free group of rank $2g$, where g is the genus of the Seifert surface F. Let $\{a_i^-\}_{i=1,\ldots,2g}$ be the free generators of $\pi_1(F_-, P_-)$ and $\{a_i^+\}_{i=1,\ldots,2g}$ be the free generators of $\pi_1(F_+, P_+)$.

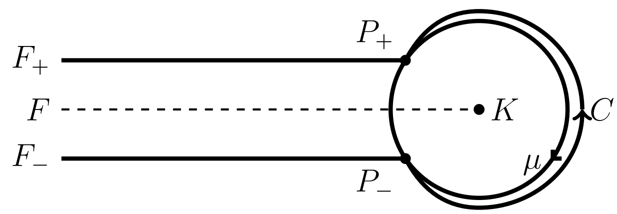

FIGURE 2.4. A cross-sectional view of a collar neighborhood of F

We denote by α_i^- the image of a_i^- under the inclusion map

$$\pi_1(F_-, P_-) \to \pi_1(S^3 \setminus F, P_-)$$

and denote by α_i^+ the image of a_i^+ in $\pi_1(S^3 \setminus F, P_-)$ under the composition map

$$\pi_1(F_+, P_+) \to \pi_1(S^3 \setminus F, P_+) \to \pi_1(S^3 \setminus F, P_-),$$

where the second map from $\pi_1(S^3 \setminus F, P_+)$ to $\pi_1(S^3 \setminus F, P_-)$ is the isomorphism induced by the arc C connecting P_- to P_+ as in Figure 2.4. By the van Kampen Theorem, after some simplification, we have

$$\pi_1(X_K, P_-) = \pi_1(S^3 \setminus F, P_-) * < [\mu] > / \ll [\mu]\alpha_i^+[\mu]^{-1} = \alpha_i^-, i = 1, \ldots, 2g \gg . \quad (2.2)$$

If the complement of the Seifert surface F in S^3 is also a handlebody, which is always the case when F is constructed through Seifert's algorithm, then the group $\pi_1(S^3 \setminus F, P_-)$ is also free and we assume that

$$\pi_1(S^3 \setminus F, P_-) =< x_1, \ldots, x_{2g} > .$$

In this case, from (2.2), we obtain Lin's presentation for the knot group $\pi_1(X_K, P_-)$ [Lin01, Lemma 2.1] as follows:

$$\pi_1(X_K, P_-) =< x_1, x_2, \ldots, x_{2g-1}, x_{2g}, [\mu] : [\mu]\alpha_i^+[\mu]^{-1} = \alpha_i^-, i = 1, \ldots, 2g >, \quad (2.3)$$

where $\alpha_i^{\pm}$ are words in x_i as described above.

Group presentations of $\pi_1(\widetilde{\Sigma}_n(K))$ and $\pi_1(\Sigma_n(K))$

We first build a 2-complex denoted by Γ, associated with presentation 2.3. Start with a single 0-cell and attach a 1-cell to the 0-cell for each generator by identifying both endpoints to it. The 1-cells are labeled by their associated generators. Then

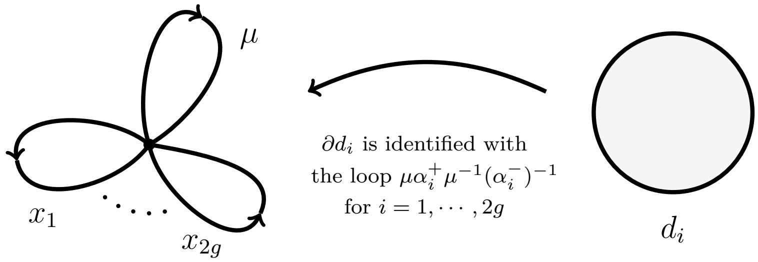

FIGURE 2.5. The 2-complex associated with presentation 2.3

we attach a 2-cell say d_i for each relator r_i such that the boundary circle of d_i is identified with $\mu\alpha_i^+\mu^{-1}(\alpha_i^-)^{-1}$, the loop determined by the i^{th} relator in presentation 2.3 (see Figure 2.5).

By construction, we have that $\pi_1(\Gamma)$ is isomorphic to the knot group $\pi_1(X_K)$. Denote $\widetilde{\Sigma}_n(\Gamma)$ the n-fold cover of the 2-complex Γ associated with the subgroup

$$\mathrm{Ker}(\pi_1(X_K) \to \mathbb{Z}_n) \cong \pi_1(\widetilde{\Sigma}_n(\Gamma)).$$

The single 0-cell in Γ gives us n vertices in $\tilde{\Sigma}_n(\Gamma)$, say $\tilde{c}_1, \cdots, \tilde{c}_n$. We pick $\tilde{c}_1$ as the basepoint.

The 1-skeleton of the complex $\widetilde{\Sigma}_n(\Gamma)$ is a covering space of the 1-skeleton of Γ, the bouquet of circles μ and $x_1, x_2, \cdots, x_{2g}$. Note that in presentation 2.3 for each i, the generator x_i is contained in the subgroup $\mathrm{Ker}(\pi_1(X_K) \to \mathbb{Z}_n)$ and its preimage under the covering map consists of n copies of its trivial lifts, denoted by $\tilde{x}_i^k$ for $k = 1, \cdots, n$ and $i = 1, \cdots, 2g$. On the other hand, the meridian element $[\mu]$ is not in the subgroup $\mathrm{Ker}(\pi_1(X_K) \to \mathbb{Z}_n)$ and thus the loop μ in Γ is lifted to an arc in $\widetilde{\Sigma}_n(\Gamma)$. Moreover, under the group homomorphism $\pi_1(X_K) \to \mathbb{Z}_n$, the meridian $[\mu]$ is mapped to the generator 1 in $\mathbb{Z}_n$. Hence, its preimage is in fact a single circle consisting of n copies of the lifting of μ denoted by μ^k for $k = 1, \cdots, n$. The 1-skeleton of the covering complex $\widetilde{\Sigma}_n(\Gamma)$ when $n = 3$ is depicted in Figure 2.6 below. The chosen generator loops of $\pi_1(\widetilde{\Sigma}_n(\Gamma))$ are: $\tilde{x_i}^1$, $\tilde{\mu}^1\tilde{x}_i^2(\tilde{\mu}^1)^{-1}$, $\tilde{\mu}^1\tilde{\mu}^3\tilde{x}_i^2(\tilde{\mu}^2)^{-1}(\tilde{\mu}^1)^{-1}$ for $i = 1, \cdots 2g$ and $\tilde{\mu}^1\tilde{\mu}^2\tilde{\mu}^3$.

Now to build the 2-complex $\tilde{\Sigma}_n(\Gamma)$, we just need to figure out how to glue the 2-cells. Each 2-cell d_i in Γ gives us n 2-cells in $\tilde{\Sigma}_n(\Gamma)$, denoted by $\tilde{d}_i^k$ for $k = 1, \cdots, n$. For each k, the boundary of the $\tilde{d}_i^k$ is identified with the k^{th} lift of the loop $\mu\alpha_i^+\mu^{-1}(\alpha_i^-)^{-1}$, which is the loop

$$\tilde{\mu}^1 \cdots \tilde{\mu}^k \cdot (\tilde{\alpha}_i^+)^k \cdot (\tilde{\mu}^k)^{-1} \cdot (\tilde{\alpha}_i^-)^{k-1} \cdot (\tilde{\mu}^{k-1})^{-1} \cdots (\tilde{\mu}^1)^{-1},$$

which is also homotopic to the loop

$$\tilde{\mu}^1 \cdots \tilde{\mu}^k \cdot (\tilde{\alpha}_i^+)^k \cdot (\tilde{\mu}^k)^{-1} \cdots (\tilde{\mu}^1)^{-1} \cdot \tilde{\mu}^1 \cdots \tilde{\mu}^{k-1} \cdot (\tilde{\alpha}_i^-)^{k-1} \cdot (\tilde{\mu}^{k-1})^{-1} \cdots (\tilde{\mu}^1)^{-1}. \quad (2.4)$$

Here $(\tilde{\alpha}_i^+)^k$ (resp. $(\tilde{\alpha}_i^-)^k$) is the k^{th} lift of α_i^+ (resp. α_i^-) based at $\tilde{c}_k$, which can be written as a product involving $\{\tilde{x_i}^k\}_{i=1,\cdots 2g}$.

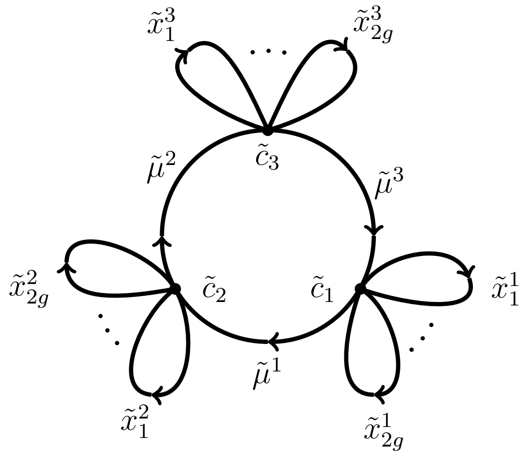

FIGURE 2.6. The 1-skeleton of $\widetilde{\Sigma}_n(\Gamma)$ when $n = 3$.

From the covering complex $\widetilde{\Sigma}_n(\Gamma)$, we can write down a presentation of its fundamental group $\pi_1(\widetilde{\Sigma}_n(\Gamma), \tilde{c}_1)$, in which generators are coming from the 1-skeleton of $\tilde{\Sigma}_n(\Gamma)$ and relations are given by the boundary of 2-cells.

Note that the fundamental group of the 1-skeleton of $\tilde{\Sigma}_n(\Gamma)$ is a free group of rank $2gn + 1$. We will pick the specific loops based at $\tilde{c}_1$ as generators of the fundamental group of the 1-skeleton. They are $\tilde{\mu}_1\tilde{\mu}_2 \cdots \tilde{\mu}_n$ and $\tilde{\mu}^1\tilde{\mu}^2 \cdots \tilde{\mu}^k \cdot \tilde{x}_i^k \cdot (\tilde{\mu}^k)^{-1} \cdots (\tilde{\mu}^2)^{-1}(\tilde{\mu}^1)^{-1}$for $i = 1, \cdots, 2g$ and $k = 0, \cdots, n-1$, where $(\tilde{\mu}^k)^{-1}$ denotes the arc $\tilde{\mu}^k$ with opposite orientation (See Figure 2.6 for $n = 3$).

Note that the group $\pi_1(\widetilde{\Sigma}_n(\Gamma))$ is isomorphic with the subgroup $\mathrm{Ker}(\pi_1(\Gamma) \to \mathbb{Z}_n)$ of $\pi_1(\Gamma)$ and the isomorphism is induced by the covering map from $\widetilde{\Sigma}_n(\Gamma)$ to Γ. Under this isomorphism, element $[\tilde{\mu}_1\tilde{\mu}_2 \cdots \tilde{\mu}_n]$ is mapped to $[\mu^n]$ and element $[\tilde{\mu}^1\tilde{\mu}^2 \cdots \tilde{\mu}^k \cdot \tilde{x}_i^k \cdot (\tilde{\mu}^k)^{-1} \cdots (\tilde{\mu}^2)^{-1}(\tilde{\mu}^1)^{-1}]$ is mapped to $[\mu^k x_i \mu^{-k}]$ for $i = 1, \cdots, 2g$ and $k = 1, \cdots, n$. Hence we obtain a presentation of the group $\pi_1(\widetilde{\Sigma}_n(K))$ with

- generators: μ^n and $\mu^k x_1 \mu^{-k}$, ... , $\mu^k x_{2g} \mu^{-k}$ for $k = 0, \cdots, n-1$;
- relators from (2.4):

$$\mu^{k+1}\alpha_i^+\mu^{-(k+1)} = \mu^k\alpha_i^-\mu^{-k}, \text{ for } k = 0, \cdots, n-2 \text{ and } i = 1, \ldots, 2g, \tag{2.5}$$

$$\mu^n \cdot \alpha_i^+ \cdot \mu^{-n} = \mu^{n-1}\alpha_i^-\mu^{-(n-1)}, \text{ for } i = 1, \ldots, 2g. \tag{2.6}$$

In the presentation above, $\mu^k x_i \mu^{-k}$ and μ^n should be viewed as abstract symbols rather than products of μ and x_i. Thus, words $\mu^k\alpha_i^+\mu^{-k}$ as in (2.5) are products of the generators $\mu^k x_i \mu^{-k}$ and the word $\mu^n \cdot \alpha_i^+ \cdot \mu^{-n}$ in (2.6) is the product of $\mu^{\pm n}$ and x_i. The notation is chosen to emphasize the fact that the isomorphism between the presented group and the subgroup $\mathrm{Ker}(\pi_1(X_K) \to \mu_n)$ is given by sending the abstract symbol $\mu^k x_i \mu^{-k}$ in the presentation to the element $[\mu^k x_i \mu^{-k}]$ of the knot group $\pi_1(X_K)$ for $k = 0, \ldots, n-1$ and $i = 1, \ldots, 2g$.

Remark: In general, a subgroup H of finite index in a finitely presented group G is also finitely presented. Given a presentation of G, one can always derive a

presentation for the subgroup H through a purely algebraic algorithm, called the Reidemeister-Schreier method [LS01]. In the case when the covering space that given by H is obtainable, we can write down a presentation of H by constructing the 2-complex as we described above, which is essentially same with the Reidemeister-Schreier method, but more intuitive.

Now let's look at the fundamental group of the n^{th} cyclic branched cover $\Sigma_n(K)$. Recall that

$$\pi_1(\Sigma_n(K)) \cong \mathrm{Ker}(\pi_1(X_K) \to \mathbb{Z}_n)/ \ll [\mu]^n \gg .$$

Therefore, the group $\pi_1(\Sigma_n(K))$ inherits the presentation with

- generators: $\mu^k x_1 \mu^{-k}$, ... , $\mu^k x_{2g} \mu^{-k}$ for $k = 0, \cdots, n-1$;
- relators:

$$\mu^{k+1} \alpha_i^+ \mu^{-(k+1)} = \mu^k \alpha_i^- \mu^{-k}, \text{ for } k = 0, \cdots, n-2 \text{ and } i = 1, \ldots, 2g, \quad (2.7)$$

$$\alpha_i^+ = \mu^{n-1} \alpha_i^- \mu^{-(n-1)}, \text{ for } i = 1, \ldots, 2g. \quad (2.8)$$

Lemma 2.3.1. *Given a knot K in S^3, denote by μ a meridian curve on the boundary torus ∂X_K. Suppose that there exists a group homomorphism ρ from $\pi_1(X_K)$ to a group G and $\rho([\mu]^n)$ is in the center of G. Then ρ induces a group homomorphism from $\pi_1(\Sigma_n(K))$ to G. In particular, if ρ is non-abelian, then the induced homomorphism is nontrivial.*

Proof. Let $\rho|_{ker}$ be the restriction of ρ to the subgroup $\mathrm{Ker}(\pi_1(X_K) \to \mathbb{Z}_n)$. We are going to show that the assignment

$$\mu^k x_i \mu^{-k} \mapsto \rho|_{ker}(\mu^k x_i \mu^{-k}) \text{ for } i = 1, \ldots, 2g \text{ and } k = 0, \ldots, n-1$$

also defines a homomorphism from $\pi_1(\Sigma_n(K))$ to G.

First of all, the relations in (2.5) which are the same as the relations in (2.7) automatically hold. It follows from (2.6) that

$$\rho|_{ker}(\mu^n) \cdot \rho|_{ker}(\alpha_i^+) \cdot \rho|_{ker}(\mu^{-n}) = \rho|_{ker}(\mu^{n-1} \alpha_i^- \mu^{-(n-1)}).$$

Since by assumption $\rho|_{ker}(\mu^n) = \rho(\mu^n)$ is in the center of G, we have

$$\rho|_{ker}(\alpha_i^+) = \rho|_{ker}(\mu^n) \cdot \rho|_{ker}(\alpha_i^+) \cdot \rho|_{ker}(\mu^{-n}) = \rho|_{ker}(\mu^{n-1} \alpha_i^- \mu^{-(n-1)}).$$

That is, the relations in (2.8) hold as well.

In addition, if ρ is a non-abelian homomorphism, since the commutator subgroup $[\pi_1(X_K), \pi_1(X_K)]$ is the normal subgroup generated by $\{x_1, \ldots, x_{2g}\}$, we have that $\rho(x_i)$ is not equal to the identity in G for some i. Therefore, the induced homomorphism from $\pi_1(\Sigma_n(K))$ to G is nontrivial. $\square$

2.4 Left-orderability and cyclic branched covers

In this section, We finish the proof of Theorem 2.4.1.

Theorem 2.4.1. *Given any prime knot K in S^3, denote by μ a meridional element of $\pi_1(X_K)$. If there exists a non-abelian representation $\pi_1(X_K)$ to $PSL(2,\mathbb{R})$ such that μ^n is sent to the identity matrix I, then the fundamental group $\pi_1(\Sigma_n(K))$ is left-orderable.*

Note that the group $PSL(2,\mathbb{R})$ itself is not left-orderable, but its universal covering group, denoted by $\widetilde{SL}(2,\mathbb{R})$, is left-orderable [Ber91].

Let E be the covering map from $\widetilde{SL}(2,\mathbb{R})$ to $PSL(2,\mathbb{R})$. Since $\widetilde{SL}(2,\mathbb{R})$ and $PSL(2,\mathbb{R})$ are both connected, we have

$$\mathcal{Z}(\widetilde{SL}(2,\mathbb{R})) = E^{-1}(\mathcal{Z}(PSL(2,\mathbb{R}))),$$

where $\mathcal{Z}(\widetilde{SL}(2,\mathbb{R}))$ and $\mathcal{Z}(PSL(2,\mathbb{R}))$ are the centers of the Lie groups $\widetilde{SL}(2,\mathbb{R})$ and $PSL(2,\mathbb{R})$ respectively [HN12, p. 336]. Therefore, $\mathcal{Z}(\widetilde{SL}(2,\mathbb{R})) = \mathrm{Ker}(E)$.

Lemma 2.4.2. *Given any knot K in S^3, let μ be a meridian curve on the boundary torus ∂X_K. Suppose that there exists a non-abelian $PSL(2,\mathbb{R})$ representation of $\pi_1(X_K)$ such that $[\mu]^n$ is sent to the identity I. Then this representation induces a nontrivial $\widetilde{SL}(2,\mathbb{R})$ representation of the fundamental group of the n^{th} cyclic branched cover $\pi_1(\Sigma_n(K))$.*

Proof. The kernel of the covering map $Ker(E)$ is isomorphic to $\pi_1(PSL(2,\mathbb{R})) \cong \mathbb{Z}$ and we have the following central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{SL}(2,\mathbb{R}) \longrightarrow PSL(2,\mathbb{R}) \longrightarrow I.$$

Suppose that ρ is a representation of $\pi_1(X_K)$ into $PSL(2,\mathbb{R})$. Then the pullback

$$\widetilde{SL}(2,\mathbb{R}) \times_{PSL(2,\mathbb{R})} \pi_1(X_K) = \{(M,x) \in \widetilde{SL}(2,\mathbb{R}) \times \pi_1(X_K) : E(M) = \rho(x)\},$$

is a central extension of $\pi_1(X)$ by $\mathbb{Z}$. On the other hand,

$$H^2(\pi_1(X_K),\mathbb{Z}) \cong H^2(X_K,\mathbb{Z}) = 0,$$

so every central extension of $\pi_1(X_k)$ by $\mathbb{Z}$ splits. Hence, ρ can be lifted to a representation into $\widetilde{SL}(2,\mathbb{R})$. That is, the composition of a splitting map with the projection from $\widetilde{SL}(2,\mathbb{R}) \times_{PSL(2,\mathbb{R})} \pi_1(X_K)$ to $\widetilde{SL}(2,\mathbb{R})$ is a lifting of ρ [Wei95] (also see [GHY01]).

Now assume that the representation ρ of the knot group $\pi_1(X_K)$ satisfies the property $\rho([\mu]^n) = I$. We denote by $\tilde{\rho}$ a lifting of ρ. Since $\rho([\mu]^n) = I$, we have $\tilde{\rho}([\mu]^n)$ is inside $E^{-1}(I)$, which is equal to $\mathcal{Z}(\widetilde{SL}(2,\mathbb{R}))$, the center of $\widetilde{SL}(2,\mathbb{R})$.

$$\begin{tikzcd}
 & & \widetilde{SL}(2,\mathbb{R}) \arrow[d, "E"] \\
\pi_1(X_K, P_-) \arrow[urr, "\tilde{\rho}"] \arrow[rr, "\rho"] & & PSL(2,\mathbb{R})
\end{tikzcd}$$

In addition, if ρ is a non-abelian representation, then $\tilde{\rho}$ is non-abelian. By Lemma 2.3.1, the representation $\tilde{\rho}$ induces a nontrivial $\widetilde{SL}(2,\mathbb{R})$ representation of $\pi_1(\Sigma_n(K))$. □

Proof of Theorem 2.4.1 . Let ρ be a non-abelian $PSL(2,\mathbb{R})$ representation of the knot group $\pi_1(X_K)$, with $\rho([\mu]^n) = I$. By Lemma 2.4.2, this representation induces a nontrivial $\widetilde{SL}(2,\mathbb{R})$ representation of the group $\pi_1(\Sigma_n(K))$.

The group $\widetilde{SL}(2,\mathbb{R})$ can be embedded inside the group of order-preserving homeomorphisms of $\mathbb{R}$, so it is left-orderable [Ber91]. Moreover, the n^{th} cyclic branched cover $\Sigma_n(K)$ is irreducible if K is a prime knot [Plo84]. Thus, Theorem 2.4.1 follows from Theorem 1.2.1. □

Here we make two remarks in comparison to Theorem 2.4.1 with the following Theorem in [BGW13].

Theorem (Theorem 6 in [BGW13])**.** *Let K be a prime knot in S^3 and suppose that the fundamental group of its twofold branched cyclic cover is not left-orderable. If $\rho : \pi_1(S^3 \setminus K) \to Homeo_+(S^1)$ is a homomorphism such that $\rho([\mu]^2) = 1$ for some meridional class μ in $\pi_1(S^3 \setminus K)$, then the image of ρ is either trivial or isomorphic to $\mathbb{Z}_2$.*

Remark 2.4.3. The proof of [BGW13, Theorem 6] naturally extends to the n^{th} cyclic branched cover for arbitrary n. Since $PSL(2,\mathbb{R})$ is a subgroup of $Homeo_+(S^1)$, the group of orientation preserving homeomorphisms of S^1, Theorem 2.4.1 is contained in [BGW13, Theorem 6] in this sense. On the other hand, if we replace the central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{SL}(2,\mathbb{R}) \longrightarrow PSL(2,\mathbb{R}) \longrightarrow 1$$

that we used in the proof of Theorem 2.4.1 by the extension below [GHY01]

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{Homeo}_+(S^1) \longrightarrow Homeo_+(S^1) \longrightarrow 1$$

the same statement with [BGW13, Theorem 6] can be achieved, where

$$\widetilde{Homeo}_+(S^1) = \{f \in Homeo_+(\mathbb{R}) : f(x+1) = f(x)+1\}.$$

Chapter 3
Applications to Two-Bridge Knots

The *bridge number* of a knot K in S^3 is the minimal number of the overpassing arcs among all knot diagrams of K. Two-bridge knots are these with bridge number equal to two. They are a well-known class of knots. They appear very frequently in studies of knot theory; often a property that is suspected to hold for all knots is first tested for this class of knots.

Every two-bridge knot can be constructed from a continued fraction expansion of a rational number p/q, where p is a positive odd number and $(p,q)=1$. We call the knot a (p,q) two-bridge knots. One can write a rational number as a continued fraction in different ways, but it turns out that the resulting two-bridge knot is independent on the continued fraction expression one decides to use. On the other hand, the correspondence between two-bridge knots and rational numbers by no means is one-to-one but well understood. In fact, two rational numbers p/q and p'/q' correspond to the same knot type if and only if $p=p'$ and $q'=q^{\pm 1} \bmod p$. Hence p is an invariant of knots which equals to the determinant $|\Delta_K(-1)|$, the absolute value of the Alexander polynomial at $t=-1$. We also point out that the (p,q) two-bridge knot is the mirror image of the $(p,-q)$ two-bridge knot, which means one can map one to the other by an orientation-reversing homeomorphism of S^3. As a results, n^{th} cyclic branched covers of these two knots are homeomorphic to each other. Hence, for the purpose of this paper, we can assume q is between 0 and p.

It is known that all two-bridge knots are prime, alternating and mostly hyperbolic. As we mentioned in Section 2.2, this means that most of the cyclic branched covers of two-bridge knots are irreducible hyperbolic rational homology spheres, which makes the left-orderability of fundamental groups of n^{th} cyclic branched covers for two-bridge knots an interesting case to investigate.

For this class of rational homology spheres, the Conjecture 1.2.4 has been verified in the following cases, where they are all L-spaces and have non-left-orderable fundamental groups:

1. The twofold branched cover of any non-split alternating link [BGW13, Gre11, Ito13, OS05c];

2. The n^{th} cyclic branched cover of a (p,q) two-bridge knot with $p/q = 2m + \frac{1}{2k}$, $mk > 0$ and n arbitrary [DPT05, Pet09];

Some of the results in this chapter previously appeared as [Ying Hu, *Left-orderability and cyclic branched coverings.* Algebr. Geom. Topol., 15(1):399-413, 2015]. They are reprinted by permission of Mathematical Sciences Publishers.

3. The 3^{rd} and 4^{th} cyclic branched cover of a (p, q) two-bridge knot with $p/q = n_1 + \frac{1}{1+\frac{1}{n_2}}$ and n_1, n_2 are positive odd integers (i.e. $p/q = 2m + \frac{1}{2k}$, $mk < 0$) [DPT05, GL14, Pet09, Ter14].

3.1 $SL(2, \mathbb{C})$-representations of two-bridge knot groups

Let K be a (p, q) two-bridge knot. From the Schubert normal form [Kaw96, p. 21], the knot group has a presentation of the following form:

$$\pi_1(X_K) =< x, y : wx = yw >,$$

where $w = (x^{\epsilon_1} y^{\epsilon_2}) \dots (x^{\epsilon_{p-2}} y^{\epsilon_{p-1}})$ and $\epsilon_i = \pm 1$.

Set $\rho : \pi_1(X_K) \to SL(2, \mathbb{C})$ be a non-abelian representation of the knot group into $SL(2, \mathbb{C})$. Up to conjugation, we can assume that

$$\rho(x) = \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} m & 0 \\ s & m^{-1} \end{pmatrix}. \tag{3.1}$$

Hence, $\rho(w) = \rho(x)^{\epsilon_1} \rho(y)^{\epsilon_2} \dots \rho(x)^{\epsilon_{p-2}} \rho(y)^{\epsilon_{p-1}}$ is a matrix with entries in $\mathbb{Z}[m^{\pm 1}, s]$. Denote $\rho(w) = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$, $w_{ij} \in \mathbb{Z}[m^{\pm 1}, s]$.

From the group relation $wx = yw$, we have

$$\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix} = \begin{pmatrix} m & 0 \\ s & m^{-1} \end{pmatrix} \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}.$$

This is equivalent to

$$\begin{pmatrix} 0 & w_{11} + (m^{-1} - m) w_{12} \\ (m - m^{-1}) w_{21} - s w_{11} & w_{21} - s w_{12} \end{pmatrix} = 0 \tag{3.2}$$

and hence s and m must satisfy the equation

$$w_{11} + (m^{-1} - m) w_{12} = 0.$$

In [Ril84], it is shown that the above equation is also a sufficient condition.

Proposition 3.1.1 (Theorem 1 of [Ril84]). *The assignment of x and y as in (3.1) defines a non-abelian $SL(2, \mathbb{C})$ representation of the knot group*

$$\pi_1(X_K) =< x, y : wx = yw >$$

if and only if

$$\varphi(m, s) \triangleq w_{11} + (m^{-1} - m) w_{12} = 0. \tag{3.3}$$

We need to make use of several properties of the polynomial $\varphi(m, s)$. All of these properties are either proven or claimed throughout Riley's paper [Ril84]. For readers' convenience, we organize them and provide a proof in the following lemma.

Lemma 3.1.2 (cf. [Ril84])**.** *The polynomial $\varphi(m,s)$ in $\mathbb{Z}[m^{\pm 1},s]$ satisfies the following:*

1. *As a polynomial in s with coefficients in $\mathbb{Z}[m^{\pm 1}]$, $\varphi(m,s)$ has s-degree equal to $\frac{p-1}{2}$, with the leading coefficient ± 1.*
2. *$\varphi(1,0) \neq 0$.*
3. *$\varphi(m,s)$ does not have repeated factors.*
4. *$\varphi(m,s) = \varphi(m^{-1},s)$ and thus $\varphi(m,s) = f(m+m^{-1},s)$ where f is a two-variable polynomial with coefficients in $\mathbb{Z}$.*

Proof. 1. Since we assign

$$\rho(x) = \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} m & 0 \\ s & m^{-1} \end{pmatrix},$$

through a direct computation we have

$$\rho(xy) = \begin{pmatrix} m^2+s & m^{-1} \\ m^{-1}s & m^{-2} \end{pmatrix}, \quad \rho(x^{-1}y) = \begin{pmatrix} 1-s & -m^{-1} \\ ms & 1 \end{pmatrix},$$

$$\rho(xy^{-1}) = \begin{pmatrix} 1-s & m \\ -m^{-1}s & 1 \end{pmatrix}, \quad \rho(x^{-1}y^{-1}) = \begin{pmatrix} m^{-2}+s & -m \\ -ms & m^2 \end{pmatrix}.$$

Say a matrix A in $M_2(\mathbb{Z}[m^{\pm 1},s])$ has s-degree equal to n if

$$A = \begin{pmatrix} \pm s^n + f_{11}(m,s) & f_{12}(m,s) \\ f_{21}(m,s) & f_{22}(m,s) \end{pmatrix}, \text{ where}$$

the s-degrees of f_{11}, f_{12} and f_{22} are strictly less than n and the s-degree of f_{21} is less than or equal to n. Hence the matrices $\rho(xy)$, $\rho(x^{-1}y)$, $\rho(xy^{-1})$ and $\rho(x^{-1}y^{-1})$ all have s-degrees equal to 1. Moreover, the product of an s-degree n matrix and an s-degree m matrix is an s-degree $m+n$ matrix. Since

$$w = (x^{\epsilon_1}y^{\epsilon_2})\dots(x^{\epsilon_{p-2}}y^{\epsilon_{p-1}}), \text{ with } \epsilon_i = \pm 1,$$

we have that the matrix

$$\rho(w) = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$$

is a product of $\frac{p-1}{2}$ s-degree 1 matrices. Therefore, the matrix $\rho(w)$ has s-degree equal to $\frac{p-1}{2}$. That is, the entry w_{11} has $\pm s^n$ as the leading term and the s-degree of w_{12} is strictly less than $\frac{p-1}{2}$. As a result, $\varphi(m,s) = w_{11} + (m^{-1}-m)w_{12}$ has leading term equal to $\pm s^n$.

2. Notice that as $m = 1$ and $s = 0$, we have

$$\rho(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

   This assignment can not define a representation of the knot group

$$\pi_1(X_K) =< x, y : wx = yw >,$$

   because these two matrices $\rho(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\rho(y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are not conjugate to each other. Therefore, $\varphi(1, 0) \neq 0$ by the Proposition 3.1.1.

3. Let $\Delta_K(t)$ be the Alexander polynomial of the knot K. It is shown in [Nag08, Proposition 1.1, Theorem 1.2] (also see [Lin01, BF08]) that any knot group has $\frac{|\Delta_K(-1)|-1}{2}$ irreducible $SL(2, \mathbb{C})$ metabelian representations up to conjugation and that these metabelian representations send meridional elements to matrices of eigenvalues $\pm i$. For a (p, q) two-bridge knot, p equals $|\Delta_K(-1)|$. This implies that the degree $\frac{p-1}{2}$ polynomial equation $\varphi(i, s) = 0$ has $\frac{p-1}{2}$ distinguished roots. Therefore $\varphi(i, s)$ does not have repeated factors and so is $\varphi(m, s)$.

   Note that we can also use the fact that $\varphi(1, s)$ does not have any repeated factors to prove that $\varphi(m, s)$ has no repeated factors [Ril72, Theorem 3].

4. Assume that the assignment

$$\rho(x) = \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} m & 0 \\ s & m^{-1} \end{pmatrix}$$

   defines a representation of the knot group

$$\pi_1(X_K) =< x, y : wx = yw > .$$

   Then

$$\rho'(x) = P \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix} P^{-1} = \begin{pmatrix} m^{-1} & 1 \\ 0 & m \end{pmatrix}$$

$$\rho'(y) = P \begin{pmatrix} m & 0 \\ s & m^{-1} \end{pmatrix} P^{-1} = \begin{pmatrix} m^{-1} & 0 \\ s & m \end{pmatrix}$$

   also defines a representation, where

$$P = \begin{pmatrix} 1 & (m^{-1} - m)/s \\ m - m^{-1} & 1 \end{pmatrix}.$$

   The matrix P is well-defined and invertible whenever (m, s) is not in the finite set

$$S \triangleq \{(m, s) : s = 0, \varphi(m, s) = 0\} \cup$$

$$\{(m,s) : s = -(m-m^{-1})^2, \varphi(m,s) = 0\}.$$

The set S is finite because neither $\varphi(m,0)$ nor $\varphi(m, -(m-m^{-1})^2)$ is a zero polynomial. Otherwise, $(1,0)$ will be a solution for $\varphi(m,s)$, which contradicts part (2).

Denote by $V(g)$ the solution set of a polynomial g. As we described above,

$$V(\varphi(m,s)) - S \subset V(\psi(m,s)),$$

where $\psi(m,s) = \varphi(m^{-1}, s)$. Points in S are not isolated, since they are embedded inside the algebraic curve $V(\varphi(m,s))$. By continuity, we have

$$V(\varphi(m,s)) \subset V(\psi(m,s)).$$

By part (3), neither of $\varphi(m,s)$ and $\psi(m,s)$ has repeated factors, so the ideal $< \psi(m,s) >$ is contained inside the ideal $< \varphi(m,s) >$ in $\mathbb{Z}[m^{\pm 1}, s]$. On the other hand, both $\varphi(m,s)$ and $\psi(m,s)$ have the same leading term, which is either $s^{(p-1)/2}$ or $-s^{(p-1)/2}$, so $\varphi(m,s) = \psi(m,s) = \varphi(m^{-1}, s)$.

$\square$

3.2 Proof of Theorem 3.2.1

In this section, we finish the proof of Theorem 3.2.1. At the end of this section, we present two specific examples.

Theorem 3.2.1. *Given a (p,q) two-bridge knot K, with $p \equiv 3$ mod 4, there are only finitely many cyclic branched covers, whose fundamental groups are not left-orderable.*

Proof. We are going to show that for sufficiently large n, the group $\pi_1(X_K)$ has a non-abelian $SL(2,\mathbb{R})$ representation with x^n sent to $-I$.

As before, we assign

$$\rho(x) = \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} m & 0 \\ s & m^{-1} \end{pmatrix}.$$

Let $m = e^{i\theta}$. Since $p = 3$ mod 4, by Lemma 4.2, we have that $\varphi(e^{i\theta}, s)$ is an odd degree real polynomial in s. So for any given θ, the equation $\varphi(e^{i\theta}, s) = 0$ has at least one real solution for s. We assume that s_0 is a real solution of the equation $\varphi(1,s) = 0$. It is known that the polynomial $\varphi(1,s)$ does not have repeated factors [Ril72, Theorem 3]. Hence, $\varphi_s(e^{i\theta}, s)|_{\theta=0, s=s_0} \neq 0$ and locally there exists a real function $s(\theta)$ such that $\varphi(e^{i\theta}, s(\theta)) = 0$ and $s(0) = s_0$.

Consider the following one-parameter family of non-abelian representations.

$$\rho\{\theta\}(x) = \begin{pmatrix} e^{i\theta} & 1 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \rho\{\theta\}(y) = \begin{pmatrix} e^{i\theta} & 0 \\ s(\theta) & e^{-i\theta} \end{pmatrix}.$$

As $\theta \neq 0$, the representations $\rho\{\theta\}$ can be diagonalized to the following forms which we still denote by $\rho\{\theta\}$,

$$\rho\{\theta\}(x) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \rho\{\theta\}(y) = \begin{pmatrix} e^{i\theta} - \frac{s(\theta)}{2\sin(\theta)}i & -1 + \frac{s(\theta)}{4\sin^2(\theta)} \\ s(\theta) & e^{-i\theta} + \frac{s(\theta)}{2\sin(\theta)}i \end{pmatrix}. \tag{3.4}$$

To get $SL(2,\mathbb{R})$ representation, we use the following claim of Khoi [Kho03, p. 786], though no details are given in the original paper. Because of the importance of this result to our argument, we provide a proof of it below.

Lemma 3.2.2 ([Kho03])**.** *The representation as in (3.4) can be conjugated to an $SL(2,\mathbb{R})$ representation if*

$$\textit{either } s(\theta) < 0 \textit{ or } s(\theta) > 4\sin^2(\theta). \tag{3.5}$$

Proof. We claim that when $s < 0$ or $s > 4\sin^2(\theta)$, the representation $\rho\{\theta\}$ is conjugate to an $SU(1,1)$ representation by the matrix

$$T \triangleq \begin{pmatrix} \sqrt{\frac{1}{\sqrt{t}} + t} & t \\ \sqrt{t} & \sqrt{\sqrt{t} + t^2} \end{pmatrix}, \text{ where } t = \frac{1}{4\sin^2(\theta)} - \frac{1}{s} \text{ is positive,}$$

and $SU(1,1)$ is conjugate to $SL(2,\mathbb{R})$ via the matrix $\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ in $GL(2,\mathbb{C})$.

It is not hard to see that easy to see $T\rho(x)T^{-1}$ is in $SU(1,1)$. Here we provide a detailed computation to verify that $T\rho(y)T^{-1}$ is also in $SU(1,1)$.

Denote $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By the assumption that $s < 0$ or $s > 4\sin^2(\theta)$, we have $t > 0$ and hence the $det(T)$ is real and never zero. Let $T\rho(y)T^{-1} \cdot det(T) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

To show that $T\rho(y)T^{-1} \in SU(1,1)$, it is sufficient to verify that $\bar{A} = D$ and $\bar{B} = C$.

$$A = ad \cdot e^{i\theta} + ac - bc \cdot e^{-i\theta} + bds - \frac{1}{2}i(bc + ad) \cdot s \cdot \csc(\theta) - \frac{1}{4}ac \cdot s \cdot \csc^2(\theta),$$

$$D = ad \cdot e^{-i\theta} - ac - bc \cdot e^{i\theta} - bds + \frac{1}{2}i(bc + ad) \cdot s \cdot \csc(\theta) + \frac{1}{4}ac \cdot s \cdot \csc^2(\theta).$$

$$A = \bar{D} \Longleftrightarrow 4ac + 4bds - ac \cdot s \cdot \csc^2(\theta) = 0 \tag{3.6}$$

Since $b = t = \frac{1}{4\sin^2(\theta)} - \frac{1}{s}$, we have $s \cdot \csc^2(\theta) = 4bs + 4$. Hence (3.6) is equivalent to

$$bds - acbs = 0. \tag{3.7}$$

Note that $ac = d$ and so (3.7) holds.

$$B = -a^2 - b^2 s + \frac{1}{4}a^2 \cdot s \cdot \csc^2(\theta) + \big(-2 + s \cdot \csc^2(\theta)\big) \cdot ab \cdot \sin(\theta) \cdot i,$$
$$C = c^2 + d^2 s - \frac{1}{4}c^2 \cdot s \cdot \csc^2(\theta) - \big(-2 + s \cdot \csc^2(\theta)\big) \cdot cd \cdot \sin(\theta) \cdot i.$$

Since $ab = cd$, we have $B = \bar{C}$ is equivalent to

$$-a^2 - b^2 s + \frac{1}{4}a^2 \cdot s \cdot \csc^2(\theta) = c^2 + d^2 s - \frac{1}{4}c^2 \cdot s \cdot \csc^2(\theta) \tag{3.8}$$

Use the fact $s \cdot \csc^2(\theta) = 4bs + 4$ again. (3.8) is equivalent to

$$-b^2 + a^2 b = d^2 - c^2 b,$$

which can be easily checked. □

Now let's continue the proof of Theorem 3.2.1. Note that

$$\lim_{\theta \to 0} s(\theta) = s_0, \text{ where } s_0 \text{ is not equal to 0 by Lemma 3.1.2 part (2).}$$

Hence, when θ is small enough, either $s(\theta) < 0$ or $s(\theta) > 4\sin^2(\theta)$. Now let $\theta = \pi/n$. For sufficiently large n, the non-abelian representation $\rho\{\theta\}$ as in (3.4) satisfies $\rho\{\theta\}(x)^n = -I$ and conjugates to an $SL(2, \mathbb{R})$ representation. Therefore, by Theorem 2.4.1, the conclusion follows. □

We end this section by computing two specific examples.

1. **Two bridge knot $(7, 4)$, the knot 5_2 in Rolfsen's table.**

   Claim: The group $\pi_1(\Sigma_n(5_2))$ is left-orderable when $n \geq 9$.

   Proof. The fundamental group $\pi_1(X_{5_2})$ has a presentation

   $$\pi_1(X_{5_2}) =< x, y : wx = yw >,$$

   where $w = xyx^{-1}y^{-1}xy$.

   From this presentation, we have

   $$\begin{aligned}\varphi(m, s) = s^3 + (2(m^2 + m^{-2}) - 3)s^2 + ((m^4 + m^{-4}) - \\ 3(m^2 + m^{-2}) + 6)s + 2(m^2 + m^{-2}) - 3.\end{aligned}$$

   as defined in (3.3). And

   $$\varphi(e^{i\theta}, s) = s^3 + (4\cos(2\theta) - 3)s^2 + (2\cos(4\theta) - 6\cos(2\theta) + 6)s + 4\cos(2\theta) - 3,$$

   which is a real polynomial in s with degree 3. Hence, we can solve a closed formula for real solutions $s(\theta)$ such that $\varphi(e^{i\theta}, s(\theta)) = 0$. Figure 3.1 is the graph of the solution $s(\theta)$ on the interval $\theta \in [0, 1]$.

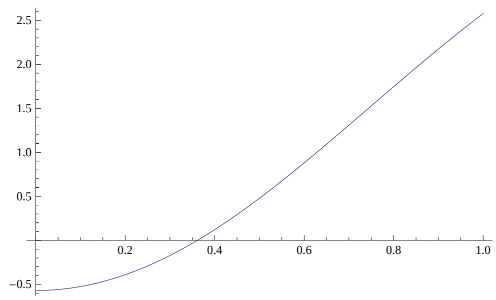

FIGURE 3.1. The graph of $s(\theta)$ for the knot 5_2

In particular, when $n = 9$, we have that $\frac{\pi}{9} \approx 0.349$ and $s(\frac{\pi}{9}) \approx -0.03667$. Hence, the group $\pi_1(\Sigma_n(5_2))$ is left-orderable when $n \geq 9$. For cyclic branched covers $\Sigma_n(5_2)$ with $n < 9$, the other known cases are $n = 2, 3$ [DPT05] and $n = 4$ [GL14], none of which has a left-orderable fundamental group. □

2. **Two bridge knot** $(15, 11)$**, the knot** 7_4 **in Rolfsen's table.**

   *Claim:*The group $\pi_1(\Sigma_n(7_4))$ is left-orderable when $n \geq 13$.

   Proof. The fundamental group $\pi_1(X_{7_4})$ has a presentation

$$\pi_1(X_{7_4}) =< x, y : wx = yw >,$$

where $w = xy^{-1}xy^{-1}x^{-1}yx^{-1}y^{-1}xy^{-1}xyx^{-1}y$.

From this presentation, we can compute the polynomial $\varphi(m, s)$ as defined in (3.3). It has two factors

$$\varphi(m, s) = \varphi_1(m, s)\varphi_2(m, s)$$

where

$$\begin{aligned}\varphi_1(m, s) =& 4(m^2 + m^{-2}) - 7 + (12 - 4(m^2 + m^{-2}))s+ \\ & (-6 + (m^2 + m^{-2}))s^2 + s^3, \\ \varphi_2(m, s) =& 1 + (-4 + 2(m^2 + m^{-2}))s + (8 - 3(m^2 + m^{-2}))s^2+ \\ & (-5 + (m^2 + m^{-2}))s^3 + s^4.\end{aligned}$$

Note that

$$\varphi_1(e^{i\theta}, s) = 8\,\cos(2\theta) - 7 + (12 - 8\,\cos(2\theta))s + (-6 + 2\,\cos(2\theta))s^2 + s^3$$

is a real polynomial in s with degree 3. Hence, we can solve a closed formula for $s(\theta)$ such that $\varphi_1(e^{i\theta}, s(\theta)) = 0$. Figure 3.2 is the graph of the function $s(\theta)$ on the interval $\theta \in [0, 1]$.

In particular, when $n = 13$, we have that $\frac{\pi}{13} \approx 0.241661$ and $s(\frac{\pi}{13}) \approx -0.0167714$. Hence, the group $\pi_1(\Sigma_n(7_4))$ is left-orderable when $n \geq 13$. For rational homology spheres $\Sigma_n(7_4)$ with $n < 13$ the only known cases are $n = 2$ and $n = 3$, neither of which has a left-orderable fundamental group [DPT05]. □

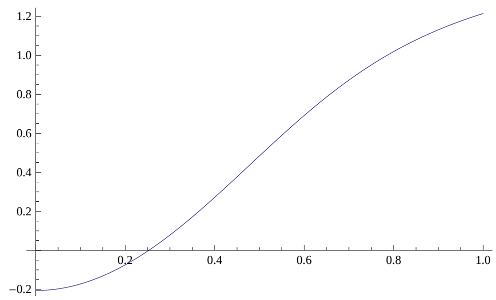

FIGURE 3.2. The graph of $s(\theta)$ for the knot 7_4

3.3 Further discussions

The left-orderability of fundamental groups of cyclic branched covers was first studied by Dabrowski, Przytycki and Togha in their paper [DPT05]. They showed that for certain families of knots and links the fundamental group of their n^{th} cyclic branched covers $\pi_1(\Sigma_n(K))$ are never left-orderable for any $n > 0$. These includes the two-bridge knots with $p/q = 2m + \frac{1}{2k}$ and $mk > 0$. Motivated by these results, they posed the following question: Given a two-bridge knot K, is $\pi_1(\Sigma_n(K))$ always non-left-orderable whenever the first Betti number $b_1(\Sigma_n(K))$ is zero? Theorem 3.2.1 certainly answers this question negatively and also shows that the situation can be quite the opposite to what one might expected.

Also by considering $SL(2, \mathbb{C})$ representations of the knot group Tran showed that, among other results, the n^{th} cyclic branched cover of a (p, q) two-bridge knots with $p/q = 2m + 1 + \frac{1}{2k}$, $m > 0$ has a left-orderable fundamental group as n sufficiently large [Tra]. Note that if k is a positive odd number, $k \neq 1$ or k is a negative even number, we have $p = 1 \bmod 4$ and is not covered in Theorem 3.2.1.

It is natural to ask the following:

Question 2. *Let K be a knot in S^3. Is the left-orderability of $\pi_1(\Sigma_n(K))$ eventually stabilized as n sufficiently large?*

Theorem 3.2.1 and results in [DPT05, Tra] already gave an affirmative answer to this question for large class of two-bridge knots. In fact, this is also the case if K is a torus knot and conjecturally should be true for all satellite knots [GL14].

The following theorem relates this question, at least for two-bridge knots, to the existence of real *parabolic representations* of the knot group, which is pointed out to the author by Ahn Tran. We call a non-abelian $PSL(2, \mathbb{C})$ representation of a knot group $\pi_1(X_K)$ is parabolic, if the meridian element $[\mu]$ is sent to a parabolic element in $PSL(2, \mathbb{C})$, i.e. the trace of the matrix is 2.

Theorem 3.3.1. *Let K be a (p, q) two-bridge knots. Assume that the knot group $\pi_1(X_K)$ has a real parabolic representation $\rho : \pi_1(X_K) \to PSL(2, \mathbb{R})$. Then the n^{th} cyclic branched covers has left-orderable fundamental group as n sufficiently large.*

Proof. We follow the notations used in the proof of Theorem 3.2.1 and assign

$$\rho(x) = \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} m & 0 \\ s & m^{-1} \end{pmatrix}.$$

As before $\varphi(m, s)$ denotes Riley's polynomial and by Lemma 4.2 the equation

$$\varphi(e^{i\theta}, s) = 0$$

is a real equation in two variables θ and s.

Note that in general a $PSL(2, \mathbb{C})$ representation of the knot group can always be lifted to an $SL(2, \mathbb{C})$ representation. There are essentially two different lifts and are parameterized by $H_1(\pi_1(X_K), \mathbb{Z}_2)$. To this end, the assumption that knot group $\pi_1(X_K)$ has a parabolic representation in $PSL(2, \mathbb{R})$ is equivalent to the existence of an $SL(2, \mathbb{R})$ presentation that maps the meridian element to a matrix in $SL(2, \mathbb{R})$ with trace equal to 2.

For two-bridge knots, from the point of view of Riley's polynomial, this is also equivalent to the existence of a real root for the polynomial equation $\varphi(1, s) = 0$. In fact, $\mathrm{tr}\rho(x) = \mathrm{tr}\rho(y) = 2$ implies $m = 1$. If s is a real number, then both generators x and y are mapped to a real matrix and hence the whole representation is real. On the other hand, if s is in $\mathbb{C} \setminus \mathbb{R}$, then

$$\mathrm{tr}(\rho(xy)) = s + 2$$

can not be real. Therefore, ρ can not be conjugate to an $SL(2, \mathbb{R})$ representation.

In summary, by assumption the knot group $\pi_1(X_K)$ has a real parabolic representation, so there is a real number s_0 such that $\varphi(1, s_0) = 0$. Also s_0 is not equal to 0 by Lemma 3.1.2 part (2). The polynomial $\varphi(1, s)$ does not have repeated factors [Ril72, Theorem 3]. Hence, $\varphi_s(e^{i\theta}, s)|_{\theta=0, s=s_0} \neq 0$ and thus locally there exists a real function $s(\theta)$ such that $\varphi(e^{i\theta}, s(\theta)) = 0$ and $s(0) = s_0$.

The rest of the argument is exactly the same with the proof of Theorem 3.2.1 and we do not repeat it here. □

Chapter 4
Application to Satellite Knots

In this chapter, we study the left-orderability of the fundamental group of the n^{th} cyclic branched cover over a satellite knot.

Satellite knots.

Let S^1 and D^2 be the unit circle and the unit disk in the complex plane $\mathbb{C}$ respectively. A knot in the solid torus $S^1 \times D^2$, i.e. a smooth embedding of S^1 into the interior of $S^1 \times D^2$, is called nontrivial if the embedded circle is not contained in a 3-ball B^3 in $S^1 \times D^2$. Here we consider the solid torus $S^1 \times D^2$ is sitting inside the three sphere S^3 by an specific unknotted embedding. Given a nontrivial knot P in $S^1 \times D^2$ and a knot K in S^3, a satellite knot $P(K)$ in S^3 is the image of P under a smooth embedding $e : S^1 \times D^2 \to S^3$ such that the core of the solid torus is mapped to the knot K and $e(S^1 \times D^2)$ is a tubular neighborhood of K, denoted by $v(K)$. The knot P in $S^1 \times D^2$ is called the pattern and the knot K in S^3 is the companion knot. Intuitively, to form the satellite knot $P(K)$, we tie up the solid torus into the companion knot K and the pattern knot P sitting inside the tied solid torus forms a satellite knot $P(K)$ (See Figure 4.1).

Note that even if both pattern P and companion knot K are given, there are still different possibilities for the satellite knot $P(K)$, because the solid torus can be twisted as it embeds around K. We call that a satellite knot $P(K)$ is *untwisted* if $e(S^1 \times 1)$ is the preferred longitude of K. The satellite knot in Figure 4.1 is an example of untwisted satellite knots.

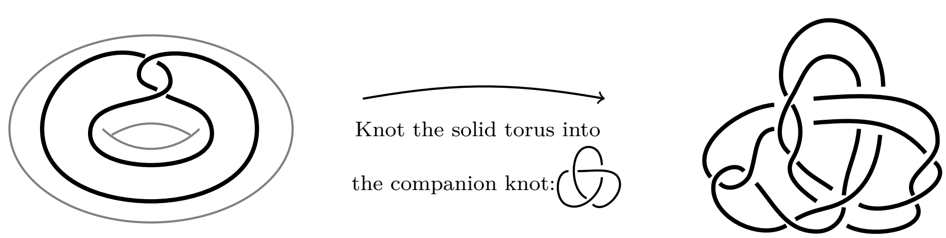

FIGURE 4.1. A (untwisted) satellite knot

There is a special class of satellite knots called cable knots, which are defined as follows. A (p, q)-torus knot is the simple closed curve

$$\{(e^{2\pi ipt}, e^{2\pi iqt}) : t \in [0, 1]\}$$

on the torus $S^1 \times S^1$, where the torus $S^1 \times S^1$ is embedded in S^3 in a unknotted fashion and both p, q are positive with $(p, q) = 1$. The (p, q)-cable of a knot K,

usually denoted by $C_{p,q}(K)$, is the untwisted satellite knot with companion K, whose pattern is the (p,q)-torus knot embedded in a solid torus in its natural way.

4.1 Lemmas.

Let $P(K)$ be the untwisted satellite knot with pattern P, satellite K and let e be the embedding of the solid tours $S^1 \times D^2$ into S^3 that takes the core of the solid torus $S^1 \times 0$ to the knot K. Denote $v(P)$ to be an open tubular neighborhood of P in S^3 that contains in $S^1 \times D^2$. Let μ be a meridian curve and λ denote a preferred longitude curve on the boundary of $S^3 \setminus v(P)$. Since $e : S^1 \times D^2 \to S^3$ is a smooth embedding, we have that $e(v(P))$ is a tubular neighborhood of $P(K)$ in S^3, call it $v(P(K))$, and both $e(\mu)$ and $e(\lambda)$ are simple closed curves on the boundary of $S^3 \setminus v(P(K))$.

Lemma 4.1.1. *$e(\mu)$ is a meridian on $\partial S^3 \setminus v(P(K))$ and $e(\lambda)$ is a preferred longitude on $\partial S^3 \setminus v(P(K))$, where $e(\mu)$ and $e(\lambda)$ are defined as above.*

Proof. Since μ is a meridian curve on the boundary of $S^3 \setminus v(P)$, it bounds a disc, say B^2, in $v(P)$. Then $e(\mu)$ bounds the disc $e(B^2)$ in $v(P(K))$, which shows that $e(\mu)$ is a meridian curve. Also the induced isomorphism $e_* : H_1(v(P)) \to H_1(v(P(K)))$ takes the generator $[\lambda]$ to a generator $e_*([\lambda]) = [e(\lambda)]$. So $e(\lambda)$ a longitude.

The rest is to show that $e(\lambda)$ is a preferred longitude, i.e. $[e(\lambda)] = 0$ in the homology group $H_1(S^3 \setminus v(P(K)))$. First of all, $H_1(S^1 \times D^2 \setminus v(P))$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, generated by $[S^1 \times 1]$ and $[\mu]$. Since λ is the preferred longitude of P, we have $[\lambda] = k[S^1 \times 1]$ in $H_1(S^1 \times D^2 \setminus v(P))$ for some $k \in \mathbb{Z}$. On the other hand, by assumption, e maps the curve $S^1 \times 1$ to a preferred longitude on the boundary of $S^3 \setminus v(K)$, so $e(S^1 \times 1)$ bounds a surface in $S^3 \setminus v(K)$, which is contained in $S^3 \setminus v(P(K))$. Hence $[e(S^1 \times 1)] = e_*[S^1 \times 1]$ is zero in $H_1(S^3 \setminus v(P(K)))$. Therefore, $e_*(\lambda) = ke_*[S^1 \times 1]$ is also trivial in $H_1(S^3 \setminus v(P(K)))$. $\square$

The knot complement of $P(K)$ can be decomposed as follows:

$$S^3 \setminus v(P(K)) = (S^1 \times D^2 \setminus v(P)) \cup_{e|_{S^1 \times \partial D^2}} S^3 \setminus v(K).$$

Hence, its fundamental group $\pi_1(S^3 \setminus v(P(K)))$ is isomorphic to the amalgamated product

$$\pi_1(S^3 \setminus v(P(K))) \cong \pi_1(S^1 \times D^2 \setminus v(P)) *_{\pi_1(S^1 \times \partial D^2)} \pi_1(S^3 \setminus v(K)),$$

together with the inclusion map $i_* : \pi_1(S^1 \times \partial D^2) \to \pi_1(S^1 \times D^2 \setminus v(P))$ and the homomorphism $e_* : \pi_1(S^1 \times \partial D^2) \to \pi_1(S^3 \setminus v(K))$. In other words,

$$\pi_1(S^3 \setminus v(P(K))) \cong \pi_1(S^1 \times D^2 \setminus v(P)) * \pi_1(S^3 \setminus v(K))/N,$$

where N is the normal subgroup of $\pi_1(S^1 \times D^2 \setminus v(P)) * \pi_1(S^3 \setminus v(K))$ generated by two elements $i_*([S^1 \times 1])e_*^{-1}([S^1 \times 1])$ and $i_*([1 \times \partial D^2])e_*^{-1}([1 \times \partial D^2])$.

To get the complement space $S^3 \setminus v(P)$ from $S^1 \times D^2 \setminus v(P)$, we glue another solid torus V to $S^1 \times D^2$ along their boundary tori such that the meridian curve of the solid torus V is identified with the longitude $S^1 \times 1$. Hence, we have

$$\pi_1(S^3 \setminus v(P)) \cong \pi_1(S^1 \times D^2 \setminus v(P))/ \ll i_*[S^1 \times 1] \gg .$$

Let ι be the quotient map from $\pi_1(S^1 \times D^2 \setminus v(P))$ to $\pi_1(S^3 \setminus v(P))$. Note that if P is a proper pattern, ι naturally induces an isomorphism when restricted to the subgroup $< [\mu], [\lambda] >$ of the fundamental group $\pi_1(S^1 \times D^2 \setminus v(P))$ generated by meridian $[\mu]$ and longitude $[\lambda]$. Hence we don't differ $\iota[\mu]$ and $\iota[\lambda]$ in $\pi_1(S^3 \setminus v(P))$ with $[\mu]$ and $[\lambda]$ in $\pi_1(S^1 \times D^2 \setminus v(P))$.

Lemma 4.1.2. *Let $P(K)$ be the untwisted satellite knot with pattern P and companion K and let G be an arbitrary group. Consider P as a knot in S^3. Then given a nontrivial group homomorphism $\rho : \pi_1(S^3 \setminus v(P)) \to G$, one can define a nontrivial group homomorphism $\xi : \pi_1(S^3 \setminus v(P(K))) \to G$. In addition, we have $\xi \circ e_*|_{<[\mu],[\lambda]>} = \rho|_{<[\mu],[\lambda]>}$ and $Im(\xi) = Im(\rho)$.*

Proof. Given a homomorphism $\rho : \pi_1(S^3 \setminus v(P)) \to G$, the quotient map ι from $\pi_1(S^1 \times D^2 \setminus v(P))$ to $\pi_1(S^3 \setminus v(P))$ induces a homomorphism from $\pi_1(S^1 \times D^2 \setminus v(P))$ to the group G, namely the composition $\rho \circ \iota$. By construction, we have $\rho \circ \iota \circ i_*[S^1 \times 1] = 1$.

Define α to be the abelian representation of $\pi_1(S^3 \setminus v(K))$ such that

$$\alpha \circ e_*[1 \times \partial D^2] = \alpha[e(1 \times \partial D^2)] = \rho \circ \iota \circ i_*[1 \times \partial D^2].$$

Note that $e(S^1 \times 1)$ is a preferred longitude on the boundary of $S^3 \setminus v(K)$, so we have

$$\alpha \circ e_*[S^1 \times 1] = \alpha[e(S^1 \times 1)] = 1.$$

Hence, $\alpha \circ e_*[S^1 \times 1] = \rho \circ \iota \circ i_*[S^1 \times 1] = 1$.

$$\begin{tikzcd}
\pi_1(S^1 \times \partial D^2) \arrow[r, "e_*"] \arrow[d, "i_*"'] & \pi_1(S^3 \setminus v(K)) \arrow[d, "i_*"] \arrow[ddr, bend left, "\alpha"] & \\
\pi_1(S^1 \times D^2 \setminus v(P)) \arrow[r, "e_*"] \arrow[dr, "\iota"'] & \pi_1(S^3 \setminus v(P(K))) \arrow[dr, dashed, "\xi"] & \\
& \pi_1(S^3 \setminus v(P)) \arrow[r, "\rho"] & G
\end{tikzcd}$$

Therefore, by the universal property of amalgamated product, there exist a unique map $\xi : \pi_1(S^3 \setminus v(P(K))) \to G$ such that the diagram above commutes. In particular, we have $\xi \circ e_* = \rho \circ \iota$ and thus

$$\xi \circ e_*|_{<[\mu],[\lambda]>} = \rho \circ \iota|_{<[\mu],[\lambda]>} = \rho|_{<[\mu],[\lambda]>}.$$

What remains is to show that ξ and ρ have same images in the group G. Note that $Im(\alpha)$ in G is the cyclic group generated by $\rho \circ \iota \circ i_*[1 \times \partial D^2]$ and hence $Im(\alpha) \subset Im(\rho)$. Therefore, $Im(\xi) = Im(\alpha) \cup Im(\rho) = Im(\rho)$. □

4.2 Left-orderability and Cyclic branched covers over satellite knots

In this section, we use lemmas that we proved in the previous section to investigate the left-orderability of fundamental groups of cyclic branched covers branched over a satellite knot.

We define the *wrapping number* of a pattern P to be the minimal geometric intersection number between P and a meridional disc $* \times D^2$ up to isotopy. It is not hard to see that if a pattern P has wrapping number equal to one, then the satellite construction becomes taking sum of two knots as defined in Section 2.1, and hence it is not a prime knot. We call a pattern P is *proper*, if it is nontrivial and its wrapping number is strictly greater than one.

Lemma 4.2.1 ([Cro04]). *A proper satellite knot is prime if its pattern, considered as a knot in S^3, is prime.*

Proof. Let $P(K)$ be a proper satellite knot with pattern P, satellite K and let e denote the associated embedding of the solid torus $S^1 \times D^2$ into S^3. Suppose that P is a prime knot in S^3. We want to prove that $P(K)$ is prime. Now assume that $P(K)$ is not prime, i.e. $P(K) = K_1 \# K_2$, where neither of K_i is the unknot. We derive a contradiction from this assumption.

By the definition of the sum of two knots, there is a 2-sphere S^2 in S^3 that intersects $P(K)$ at two points and decompose it into two parts, i.e. K_i with a trivial arc removed for each $i = 1, 2$. We simply the notation and denote the torus $e(S^1 \times \partial D^2)$ in S^3 by T. Note that the torus T is compressible in $S^3 \setminus v(P(K))$ only if the satellite K is the unknot. Up to isotopy, we assume that the 2-sphere S^2 and the torus T intersect each other transversely and hence the intersection is a collection (possibly empty) of simple closed curves.

If the intersection $S^2 \cap T = \emptyset$, then since $S^2 \cap P(K)$ is not empty, we have S^2 must be contained in the solid torus $e(S^1 \times D^2)$. Then the preimage $e^{-1}(S^2)$, also a 2-sphere, decompose P into $P = P_1 \# P_2$, where both P_i are in the solid torus $S^1 \times D^2$ in S^3. Since P is a prime knot in S^3, one of the P_i is the unknot. Without lose of generality, we assume that P_2 is the unknot in S^3. Up to isotopy, there are only two unknot types in a solid torus: meridian and longitude. If P_2 is isotopic to a meridian curve, then$P = P_1 \# P_2$ lies in a three ball in the solid torus and by definition it is a trivial pattern. If P_2 in $S^1 \times D^2$ is isotopic to the longitude, then pattern P has wrapping number equal to 1 . In either case, it leads to a contradict to our assumption that P is a proper pattern. Therefore $S^2 \cap T$ must be nonempty

Let λ be an innermost intersection curve in $S^2 \cap T$, which means that λ bounds a disk, call it D, in S^2. Suppose that $D \cap P(K) = \emptyset$. In this case, the disk D lies in the complement space $S^3 \setminus P(K)$. On the other hand, $\partial D^2 = \lambda$ is also a simple curve on the torus T. If λ is not an essential curve on T, i.e. λ bounds a disk on T,

we can isotope S^2 and T to remove the intersection. Hence we may assume that λ is an essential curve on T, so T is compressible in $S^3 \setminus P(K)$ and it follows that the satellite K must be the unknot. So the satellite knot $P(K)$ is the knot P in S^3 and $P = P(K) = K_1 \# K_2$ contradict the fact that P is prime in S^3.

Assume that satellite K is not the unknot. After removing these avoidable innermost intersection curves, we have two innermost curves λ_1 and λ_2. Each λ_i bounds a disk D_i on S^2 and D_i intersect $P(K)$ at exact one point. Note that $D_i \cap P(K) \neq \emptyset$ implies both D_i are inside the solid torus $S^1 \times D^2$ and hence λ_i are meridian curves and D_i are meridian disks. Hence we have a geometric intersection between $P(K)$ and a meridian disk D_i is one, which leads to a contradiction. □

In what follows, we extend the known results on the fundamental group of the n^{th} cyclic branched cover of a (p, q)-two bridge knot to the n^{th} cyclic branched cover of a satellites knot whose pattern is a two-bridge knot in S^3. It has been shown that for certain classes of two-bridge knots the n^{th} cyclic branched covers have left-orderable fundamental groups for sufficiently large n; these two-bridge knots are:

- (p, q) two-bridge knots with p= 3 mod 4 by Theorem 3.2.1;
- (p, q) two-bridge knots with $p/q = (2k + 1) + \frac{1}{2n}$ with $k > 0$ and $n \neq 1$ [Tra].

More precisely, let K be a (p, q) two-bridge knot satisfying one of above conditions. We showed that there exist a $PSL(2, \mathbb{R})$ representation ρ of the knot group $\pi_1(X_K)$ such that $\rho([\mu]^n) = 1$ as n sufficiently large and by Theorem 2.4.1, the fundamental group of the n^{th} cyclic branched cover $\pi_1(\Sigma_n(K))$ is left-orderable.

Remark 4.2.2. In the case of $p/q = (2k + 1) + \frac{1}{2n}$ with $k > 0$, we have $p = 1$ mod 4 only if n is a positive odd number or a negative even number. When $n = 1$, the two-bridge knot $C[2k + 1, 2]$ is isotopic to two-bridge knot $C[2k, -2]$. By the results in [DPT05], the fundamental group $\pi_1(\Sigma_n(C[2k, -2]))$ is not left-orderable.

Theorem 4.2.3. *Let $P(K)$ be a proper satellite knot. Assume that pattern P as a knot in S^3 is a (p, q) two-bridge knot with either $p = 3$ mod 4 or $p/q = (2k+1) + \frac{1}{2n}$ and $k > 0$, $n \neq 1$. Then $\pi_1(\Sigma_n(P(K)))$ is left-orderable as n sufficiently large.*

Proof. Let $P(K)$ be a satellite knot with assumed properties. In the proof of Theorem 2.4.1 and results in [Tra], we showed that there exist a nonabelian $PSL(2, \mathbb{R})$ representation

$$\rho : \pi_1(S^3 \setminus v(P)) \to PSL(2, \mathbb{R})$$

such that $\rho([\mu]^n) = 1$ for sufficiently large n, where as before μ is a meridian curve on $\partial S^3 \setminus v(P)$. By Lemma 4.1.2, for such n, there exists a nonabelian representation

$$\xi : \pi_1(S^3 \setminus v(p(K))) \to PSL(2, \mathbb{R})$$

such that $\xi(e_*([\mu]^n)) = \rho([\mu]^n) = 1$, where e is the embedding of the solid torus and $e(\mu)$ is a meridian of $P(K)$. Moreover, since all two-bridge knots are prime, by Lemma 4.2.1, the proper satellite knot $P(K)$ is also prime. Therefore, according to Theorem 2.4.1 the fundamental group $\pi_1(\Sigma_n(P(K)))$ is left-orderable for sufficiently large n. □

It is conjectured in [GL14] that the n^{th} cyclic branched cover of a proper satellite knots has a left-orderable fundamental group for sufficiently large n. This is the case for (p, q) cable knot $C_{p,q}(K)$ and K is not the unknot. Theorem 4.2.3 also provides evidence to support the conjecture.

References

[BB13] M. Boileau and S. Boyer. Graph manifolds $\mathbb{Z}$-homology 3-spheres and taut foliations. *arXiv:1303.5264*, 2013.

[Ber91] G. Bergman. Right orderable groups that are not locally indicable. *Pacific J. Math.*, 147(2):243–248, 1991.

[BF08] H. Boden and S. Friedl. Metabelian $SL(n, \mathbb{C})$ representations of knot groups. *Pacific J. Math.*, 238(1):7–25, 2008.

[BGW13] S. Boyer, C. Gordon, and L. Watson. On L-space and left-orderable fundamental groups. *Mathematische Annalen*, 356(4):1213–1245, 2013.

[BP01] M. Boileau and J. Porti. Geometrization of 3-orbifolds of cyclic type. *Astérisque No. 272*, 2001. Appendix A by Heusener and Porti.

[BRW05] S. Boyer, D. Rolfsen, and B. Wiest. Orderable 3-manifold groups. *Annales de l'institut Fourier*, 55(1):243–288, 2005.

[BZ03] G. Burde and H. Zieschang. *Knots*. Walter de Gruyter, second edition, 2003.

[CD03] D. Calegari and N. Dunfield. Laminations and groups of homeomorphisms of the circle. *Inventiones Mathematicae*, 152:149–207, 2003.

[Con59] P. Conrad. Right-ordered groups. *The Michigan Mathematical Journal*, 6(3):267 – 275, 1959.

[Cro04] P. Cromwell. *Knots and links*. Cambridge University Press, 2004.

[Deh94] P. Dehornoy. Braid groups and left distributive operations. *Trans. Amer. Math. Sco.*, 345(1):115 – 151, 1994.

[Deh97] P. Dehornoy. A fast method for comparing braids. *Adv. Math.*, 125(2):200 – 235, 1997.

[DPT05] M. Dabkowski, J. Przytycki, and A. Togha. Non-left-orderable 3-manifold groups. *Canadian Math. Bull.*, 48(1):32–40, 2005.

[Far76] F. T. Farrell. Right-orderable deck transformatoin groups. *Rocky Mountain J. Math.*, 6(3):441 – 447, 1976.

[FGR+99] R. Fenn, M. Greene, D. Rolfsen, C. Rourke, and B. Wiest. Ordering the braid groups. *Pacific J. Math.*, 191(1):49 – 74, 1999.

[FW78] C. Feustel and W. Whitten. Groups and complements of knots. *Canad. J. Math.*, 30:1284–1295, 1978.

[GHY01] É. GHYS. Groups acting on the circle. *Enseignement Mathematique*, 47(3/4):329–408, 2001.

[GL89] C. Gordon and J. Luecke. Knots are determined by their complements. *Journal of The American Mathematical Society*, 2(2):371–415, 1989.

[GL14] C. Gordon and T. Lidman. Taut foliations, left-orderability, and cyclic branched covers. *Acta Math. Vietnam.*, 39:599–635, 2014.

[Gor78] C. Gordon. *Some aspects of classical knot theory*. Knot theory. Springer Berlin Heidelberg, 1978.

[Gre11] J. Greene. Alternating links and left-orderability. *arXiv:1107.5232 [math.GT]*, 2011.

[HN12] J. Hilgert and K. Neeb. *Structure and geometry of Lie groups*. Springer, 2012.

[HS85] J. Howie and H. Short. The band-sum problem. *J. London Math. Soc.*, 31(2):571–576, 1985.

[Hu15] Y. Hu. Left-orderability and cyclic branched coverings. *Algebr. Geom. Topol.*, 15(1):399–413, 2015.

[Ito13] T. Ito. Non-left-orderable double branched coverings. *Algebraic & Geometric Topology*, 13:1937 –1965, 2013.

[Kaw96] A Kawauchi. *Survey on Knot theory*. Springer, 1996.

[Kho03] V. Khoi. A cut-and-paste method for computing the Seifert volumes. *Mathematische Annalen*, 326:759–801, 2003.

[KR14] W. Kazez and R. Roberts. Approximating c^0-foliations. *arXiv:1404.5919 [math.GT]*, 2014.

[Lic65] R. Lickorish. A foliation for 3-manifold. *Ann. of Math.*, 82:414–420, 1965.

[Lic97] WB Raymond Lickorish. *An introduction to knot theory*, volume 175. Springer, 1997.

[Lin01] X. Lin. Representations of knot groups and twisted Alexander polynomials. *Acta Mathematica Sinica, English Series*, 17(3):361–380, July 2001.

[LS01] R. Lyndon and P. Schupp. *Combinatorial Group Theory*. Springer-Verlag, New York, 2001.

[Nag08] F. Nagasato. Finiteness of a section of the $SL(2,\mathbb{C})$-character variety of the knot group. *Kobe J. Math.*, 24(2):223–240, 2008.

[Nov65] S. P. Novikov. Topology of foliations. *Trans. Moscow Math. Soc.*, pages 268–304, 1965.

[OS04a] P. Ozsváth and Z. Szabó. Holomorphic disks aand topological invariants for closed three-manifolds. *Ann. of Math.*, 159:1027 – 1158, 2004.

[OS04b] P. Ozsváth and Z. Szabó. Holomorphic disks and genus bounds. *Geom. Topol.*, 8:311 – 334, 2004.

[OS04c] P. Ozsváth and Z. Szabó. Holomorphic disks and three-manifold invariants: properties and applications. *Ann. of Math.*, 159:1159 – 1245, 2004.

[OS05a] P. Ozsváth and Z. Szabó. On Heegaard diagrams and holomorphic disks. *European Congress of Mathematics, Eur. Math. Soc., Zürich*, pages 769–781, 2005.

[OS05b] P. Ozsváth and Z. Szabó. On knot Floer homology and lens space surgeries. *Topology*, 44:1281–1300, 2005.

[OS05c] P. Ozsváth and Z. Szabó. On the Heegaard Floer homology of branched double-covers. *Adv. Math.*, 194:1–33, 2005.

[Pet09] T. Peters. On L-spaces and non left-orderable 3-manifold groups. *arXiv:0903.4495 [math.GT]*, 2009.

[Plo84] S. Plotnick. Finite group actions and nonseparating 2-spheres. *Proc. Amer. Math. Soc.*, 90(3):430–432, 1984.

[Ril72] R. Riley. Parabolic representations of knot groups. I. *Proc. London Math. Soc.*, 24(3):217– 242, 1972.

[Ril84] R. Riley. Nonabelian representations of 2-bridge knot groups. *The Quarterly Journal of Mathematics*, 35(2):191–208, 1984.

[Rol76] D. Rolfsen. *Knots and links.* Berkeley: Publish or Perish, 1976.

[RW00] C. Rourke and B. Wiest. Order automatic mapping class groups. *Pacific J. Math.*, 194:209 – 227, 2000.

[SW00] H. Short and B. Wiest. Orderings of mapping class groups after Thurston. *Enseignement Mathematique*, 46(3 - 4):279 – 312, 2000.

[Ter14] M. Teragaito. Fourfold cyclic branched covers of genus one two-bridge knots are L-spaces. *Boletín de la Sociedad Matemática Mexicana*, pages 1–13, June 2014.

[Tra] A. Tran. On left-orderablility and cyclic branched coverings. *J. Math. Soc. Japan.* to appear.

[Vin49] A. Vinogradov. On the free product of ordered groups. *Mat. Sbornik N.S.*, 25(67):163–168, 1949.

[Wei95] C. Weibel. *An introduction to homological algebra.* Cambridge university press, 1995.

Appendix: Permission for Use

PAPER ID: 140203-Hu
(e.g., 110223-Doe)

msp
mathematical sciences publishers

Consent to Publish

Title of Contribution (hereafter, the "Work"): Left-orderability and cyclic branched coverings
(please also fill out the paper ID code in the upper right corner of this form)

Author(s): Ying Hu

Name of Publication: *Algebraic & Geometric Topology.*

Publisher: Mathematical Sciences Publishers.

I PREAMBLE

1 **Mathematical Sciences Publishers** (hereafter, the "Publisher"), a California nonprofit corporation and the publishers of *Algebraic & Geometric Topology*, requires authors of articles in its publications to provide a formal written Consent to Publish. The signed Consent to Publish gives the Publisher the permission of the Author(s) to publish the Work.

2 The Publisher hereby requests that the Author(s) complete and return this form promptly so that the Work may be readied for publication.

II BASIC PROVISIONS

3 The Author(s) hereby consents that the Publisher publish the Work.

4 The Author(s) warrants that the Work has not been published before in any form except as a preprint, that the Work is not being concurrently submitted to and is not under consideration by another publication, that all Authors are properly credited, and generally that the Author(s) has the right to make the grants made to the Publisher complete and unencumbered. The Author(s) also warrants that the Work does not libel anyone, infringe anyone's copyright, or otherwise violate anyone's statutory or common law rights.

5 The Author(s) gives the Publisher unlimited rights throughout the world for all terms of copyright:

(a) to publish and distribute the Work in any form and in all media now known or hereafter discovered;
(b) to translate the Work and exercise all rights in all media in the resulting translations;
(c) to transfer or sublicense the foregoing rights in whole or in part to third parties; and
(d) to accept and retain payment for these.

III COPYRIGHT

6 The Author(s) retains copyright of the content and image of this Work, in the Author(s)' name(s) or the name of a third party (e.g., employer). If copyright is retained in the name of a third party, please indicate who the copyright owner(s) should be:

....

7 The copyright holder retains the right to duplicate the Work by any means and to permit others to do the same with the exception of reproduction by services that collect fees for delivery of documents, which may be licensed only by

1

the Publisher. In each case of authorized duplication of the Work in whole or in part, the Author(s) must still ensure that the original publication by the Publisher is properly credited, in the following manner: "First published in *Algebraic & Geometric Topology* in [volume and number, or year], published by Mathematical Sciences Publishers," and the copyright notice in proper form must be placed on all copies.

IV Signature

8 This form is to be signed by the Author(s) or, in the case of a "work made for hire," by the employer. If there is more than one Author, then either all must sign the Consent to Publish, or one Author may sign for all provided the signer appends a statement signed by all the Authors that attests that each Author has approved this agreement and has agreed to be bound by it.

9 This Agreement will be governed by the laws of California and will be binding on, and inure to the benefit of, the Author(s)' heirs and personal representatives and the Publisher's successors and assigns.

Date: 07/09/2014

Signature: Ying Hu

Author Name: Ying Hu

Address: 375 W. Roosevelt. St.
Apt 3221
Baton Rouge LA 70802

Please email a scan of the signed form, or mail a signed hardcopy to:

consent+agt@msp.org

Mathematical Sciences Publishers
798 Evans Hall #3840
c/o Department of Mathematics
University of California
Berkeley, CA 94720-3840
United States

Vita

Ying Hu was born in 1986 in Dujiangyan, Sichuan, China. She finished her undergraduate studies at Beijing Normal University in May 2008. In August 2008, Ying came to Louisiana State University to pursue graduate studies in mathematics and earned her Master of Science in Mathematics in May 2010. Currently, she is a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2015.