2014

# Combinatorial minimal free resolutions of ideals with monomial and binomial generators

Trevor McGuire
*Louisiana State University and Agricultural and Mechanical College*

# COMBINATORIAL MINIMAL FREE RESOLUTIONS OF IDEALS WITH MONOMIAL AND BINOMIAL GENERATORS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
Trevor E. McGuire
B.A., New College of Florida, 2009
M.S., Louisiana State University, 2010
May 2014

This dissertation is dedicated to my late grandfather, Eugene Henry Snow Sr. I will never forget ice fishing on Parkers Pond, raking blueberries, climbing your trees and your innumerable funny sayings for any situation.

# Acknowledgments

This dissertation happened very organically, and would not have been possible without several people. I would like to thank my advisor, James Madden, for his keen insight into the interstitial points in textbooks, one of which grew into this dissertation. Additionally, I would like to thank my committe members, Charles Delzell, Milen Yakimov, Bogdan Oporowski, Frank Neubrander, and Muhammad Wahab. In particular, I would like to thank Frank Neubrander for his assistance and motivation during the time of my qualifying exams.

On the personal side of my life, I can't thank my father, Michael McGuire, enough. His assistance during my time as an undergraduate student at New College of Florida paved the way for my success in graduate school. My beautiful wife Angela has endured these years alongside me, and I can never begin to understand what it must have looked like from the other side; she supported me in ways I may never even realize. My friends at the weekly vegan potluck have saved my sanity more times than I can remember; without the joy they brought me, my studies would have suffered. Finally, I would like to thank my friend, Mika Torkkola, who introduced me to the sport of long distance cycling; I have done some of my best mathematics while pedaling down the open road, free from nearly all distractions.

Academically, I cannot thank everyone at LSU personally, but overall, I always had the freedom to explore what I wanted, teach when and what I wanted, and still obtain a thorough mathematics education.

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# Abstract

In recent years, the combinatorial properties of monomials ideals [7, 10, 14] and binomial ideals [9, 10] have been widely studied. In particular, combinatorial interpretations of minimal free resolutions have been given in both cases. In this present work, we will generalize existing techniques to obtain two new results. The first is $S[\Lambda]$-resolutions of $\Lambda$-invariant submodules of $k[\mathbb{Z}^n]$ where $\Lambda$ is a lattice in $\mathbb{Z}^n$ satisfying some mild conditions. A consequence will be the ability to resolve submodules of $k[\mathbb{Z}^n/\Lambda]$, and in particular ideals $J$ of $S/I_\Lambda$, where $I_\Lambda$ is the lattice ideal of $\Lambda$.

Second, we will provide a detailed account in three dimensions on how to lift the aforementioned resolutions to resolutions of ideals in $k[x, y, z]$ with monomial and binomial generators.

# Chapter 1: Introduction

Solving a system of equations entails finding relations among the variables. If the system is sufficiently complicated, then it might be possible to have relations among the relations. There is no good reason to stop there, though; we could have relations among relations ad infinitum. In polynomial rings with $n$ variables, though, the Hilbert Syzygy Theorem tells us that this process does terminate. Until we have all the relations among the relations, one could argue that we have not yet fully solved the initial problem. These collections of nested relations are called resolutions, and they have formed an area of strong research interest for over 100 years.

In recent decades, various groups of mathematicians have independently studied resolutions of binomial ideals, and resolutions of monomial ideals. Many beautiful results have been obtained, but resolutions of sums of such ideals remain elusive. It is exactly these types of ideals that will be studied in this present work.

In the first chapter, we will discuss the combinatorial setup we will be using for the rest of work. The objects of interest are subsets of $\mathbb{Z}^n$ that are typically infinite. (In the existing theory, researchers utilized finite subsets of $\mathbb{N}^n$.) We will draw on the language of [3] to generalize the tools from [6] and [10].

The next chapter examines subsets of $\mathbb{Z}^n$ that are groups as well as antichains. We will call them antichain lattices, and we will work intimately with them throughout the remainder of the work. Our antichain condition parallels other work where the subgroups are not allowed to intersect the positive orthant anywhere but 0; requiring that the lattice is an antichain is a more concise way to state this condition.

We will give a brief review of resolutions in the following chapters, specifically focusing on resolutions of certain types of binomial ideals that have been studied in [6] and [10].

The penultimate chapter will take us on our final step before we begin resolving our desired ideals. We will need to enter the world of Laurent monomial modules, which is the analogue of monomial ideals, but in a larger ambient space. Notice that the Laurent polynomial ring $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is actually a field, and hence there are no ideals. We will look at $k[x_1, \ldots, x_n]$-modules contained in this ring.

The final chapter, where the bulk of the work lies, will tie everything together in the full generality of $\mathbb{Z}^n$, but our final computation will actually be in $\mathbb{Z}^3$ because the proper notation for the increasingly complex computations in $\mathbb{Z}^n$ have hitherto proven to be elusive. That is, we will give the general combinatorial algorithm for the resolution of certain ideals with binomial and monomial generators in $k[x_1, x_2, x_3]$ as the main result. We will end with a detailed example outlining the full algorithm.

# Chapter 2: Subsets of $\mathbb{Z}^n$

In this chapter, we will consider certain subsets of $\mathbb{Z}^n$. In particular, we will develop the same structure of $\mathbb{Z}^n$ utilized in [10] in terms of the weak order on $\mathbb{Z}^n$, and then we will add structure from there. The ultimate object defined in this chapter is the generic antichain lattice. Preceeding that is the vital definition of neighbors in $\mathbb{Z}^n$.

The general setup we will be working with is one of $M$-sets, where $M$ is a monoid.

**Definition 2.1.** *Let $M =< M, *, 0 >$ be a monoid. Then an $M$-set is a set $S$ together with a map*

$$\begin{array}{rcl} M \times S & \to & S \\ (m, s) & \mapsto & ms \end{array}$$

*such that $(m * m')s = m(m's)$ and $0s = s$.*

## 2.1 Subsets of $\mathbb{Z}^n$ as a Poset

In this section, we will generalize some of the notions from [10] to a new notation and dictionary that will help us reform previous statements in the language of posets. In particular, we will formalize the notation associated to the weak order on $\mathbb{Z}^n$, and we will use $\mathbb{N}^n$-sets. We have the following definitions and notations for elements $\alpha, \beta$ and subsets $A$ of $\mathbb{Z}^n$:

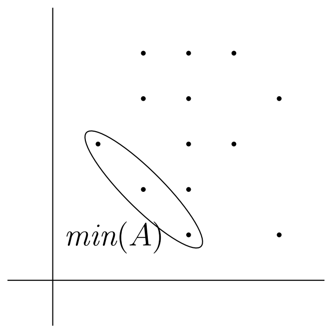
1. If $\alpha \in \mathbb{R}^n$, then $\pi_j(\alpha)$ denotes the $j^{th}$ component of $\alpha$.
2. $\alpha \leq \beta$ if $\pi_i(\alpha) \leq \pi_i(\beta)$, $i = 1, \ldots, n$
3. $\alpha < \beta$ if $\alpha \leq \beta$, and $\alpha \neq \beta$
4. $\alpha << \beta$ if $\pi_i(\alpha) < \pi_i(\beta)$, $i = 1, \ldots, n$[1]

[1] At times, we use the notation $a << b$ for $a, b \in \mathbb{R}$ to mean that $b$ is much greater than $a$, but context will prevent any notational confusion.

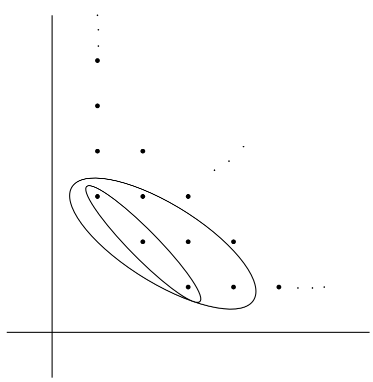
5. $\min(A) := \{\alpha \in A | \zeta < \alpha \Rightarrow \zeta \notin A\}$
6. If $A = A + \mathbb{N}^n$, then $A$ is an $\mathbb{N}^n$-set with the map being defined by $(\eta, \alpha) \mapsto \eta\alpha = \alpha + \eta$.
7. The $\mathbb{N}^n$ set generated by $A$ is $A + \mathbb{N}^n = \{\zeta \in \mathbb{Z}^n | \exists \alpha \in A \text{ with } \alpha \leq \zeta\}$
8. If $\alpha, \beta \in \mathbb{Z}^n$, then $\alpha \vee \beta = (\sup\{\alpha_1, \beta_1\}, \ldots, \sup\{\alpha_n, \beta_n\})$, and $\alpha \wedge \beta = -(-\alpha \vee -\beta)$.

**Remark 2.2.** *Notice that* $\min(\mathbb{Z}^n) = \emptyset$.

**Example 2.3.** *If $A$ is the finite set in $\mathbb{N}^2$ indicated below, then* $\min(A)$ *is circled.*




**Example 2.4.** *Suppose $A$ and $A'$ are as illustrated, where $A \subset A'$. Then* $\min(A + \mathbb{N}^n) = \min(A' + \mathbb{N}^n)$. *Notice also that because* $\min(A) = \min(A')$, *we have that* $A + \mathbb{N}^n = A' + \mathbb{N}^n = \min(A) + \mathbb{N}^n$.



**Definition 2.5.** *A descending chain in a poset $X$ is a function $f : I \to X$ where $I \subseteq \mathbb{N}$ is an interval and $f(i) > f(j)$ if $i < j$. If $A \subseteq \mathbb{Z}^n$ does not have any infinite*

*descending chains, we will say it satisfies the decending chain condition., and we call it a DCC set.*

If $A \subseteq \mathbb{Z}^n$ is a DCC $\mathbb{N}^n$-set, then $\min(A) + \mathbb{N}^n = A$. The definition of $\min(A)$ implies that it is an antichain with respect to the weak order on $\mathbb{Z}^n$.

There is a bijection between monomials in $k[x_1, \ldots, x_n]$ and vectors in $\mathbb{N}^n$. If $I =< m_1, \ldots, m_s >$, where $m_i = X^{a_i}$, then the monomials in $I$ are exactly the vectors in $\hat{A}$ where $A = \{a_1, \ldots, a_s\}$.

**Definition 2.6.** *For $\alpha \in \mathbb{Z}^n$, the support of $\alpha$ is supp$(\alpha) = \{i \mid \pi_i(\alpha) \neq 0\}$.*

**Definition 2.7.** *Let $\eta \in \mathbb{Z}^n$, and let $[n] = \{1, \ldots, n\}$. Let $T_\eta = \eta - \mathbb{N}^n = \{\eta - \alpha \mid \alpha \in \mathbb{N}^n\}$, and say that for nonempty $X \subseteq [n]$, an $X$-face of $T_\eta$ is $\{\alpha \in \mathbb{Z}^n | \pi_i(\alpha) = \pi_i(\eta)$ for all $i \in X\}$. Let $T^o_\eta = \eta - \mathbb{N}^n_{>0}$.*

**Definition 2.8.** *Let $A \subseteq \mathbb{Z}^n$. We say $A$ is generic if for all $\eta \in \mathbb{Z}^n$, such that $T^o_\eta \cap A = \emptyset$, $T_\eta$ contains at most one element of $A$ on each face.*

If $A$ is an $\mathbb{N}^n$-set that has a minimal element, it is never generic. This is because if $\alpha \in \min(A)$, then $T^o_{\alpha+(1,0,\ldots,0)} \cap A = \emptyset$, but $T^o_{\alpha+(1,0,\ldots,0)}$ contains two points on one face. Because of this, we will adopt the convention of calling a DCC $\mathbb{N}^n$-set generic if its generating antichain is generic.

### 2.2 Neighborly Sets

If $A \subset \mathbb{Z}^n$, we wish to have a way of distinguishing certain subsets of $A$ that have desirable properties. This distinction will be in the form of neighborly sets.

**Definition 2.9.** *Let $A \subset \mathbb{Z}^n$, and let $B \subset A$. We say that $B$ is neighborly in $A$ if $T^o_{\vee B} \cap A = \emptyset$. We say $B$ is maximally neighborly if $B$ is neighborly and $B' \supset B$ implies $B'$ is not neighborly.*

**Remark 2.10.** *The empty set is considered neighborly.*

**Lemma 2.11.** *If $A \subseteq \mathbb{Z}^n$, and $B \subseteq A$ is neighborly, then every subset of $B$ is neighborly.*

*Proof.*

Since $B$ is neighborly, we have that $T^o_{\vee B} \cap A = \emptyset$. Additionally, since $B' \subseteq B$, we have that $T^o_{\vee B'} \cap A \subseteq T^o_{\vee B} \cap A = \emptyset$, and hence $T^o_{\vee B'} \cap A = \emptyset$. Therefore, $B'$ is neighborly. □

**Definition 2.12.** *Let $A \subset \mathbb{Z}^n$ and let $B \subset A$. If $B' \subseteq A$ and $\vee B' = \vee B$ implies that $B' = B$ for all such $B' \subseteq A$, then $B$ is called strongly neighborly.*

**Proposition 2.13.** *Let $A \subset \mathbb{Z}^n$. Then $B \subset A$ strongly neighborly implies that $B$ is neighborly, and the converse holds if $A$ is generic.*

*Proof.*

Let $B$ be strongly neighborly. Then for any $B' \subset A$ such that $\bigvee B = \bigvee B'$, we have that $B = B'$. If $T^o_{\vee B} \cap A \neq \emptyset$, then there exists $\alpha \in A$ such that $\alpha << \bigvee B$, and hence $\bigvee B = \bigvee (B \cup \alpha)$. Then $B = B \cup \alpha$, which is a contradiction, and hence $T^o_{\vee B} \cap A = \emptyset$, so $B$ is neighborly.

Now suppose that $B$ is neighborly and $A$ is generic, then at most one element of $A$ lies on each face of $T_{\vee B}$ by definition. Now consider $B'$ such that $\bigvee B = \bigvee B'$. Each $\beta \in B$ contributes to $\bigvee B$ in some component because of genericity. If $\beta' \in B'$ contributes to $\bigvee B$ what $\beta$ did, then they lie in the same face of $T_{\vee B}$ and hence must be the same. In this manner, we conclude that each element of $B$ matches up with an element of $B'$, and vice versa, and hence $B = B'$, so $B$ is strongly neighborly.

□

**Definition 2.14.** *If $A \subseteq \mathbb{Z}^n$, let $N(A) := \{$strongly neighborly sets of $A\}$, and let $N_i(A) := \{\sigma \in N(A) | |\sigma| = i + 1\}$. We call $N(A)$ the Scarf complex of $A$.*

**Proposition 2.15.** *If $A \subseteq \mathbb{Z}^n$, then $N(A)$ is a simplicial complex.*

*Proof.*

By Lemma 2.11, neighborliness is closed under taking subsets. Hence, $\sigma \in N_i(A)$ is an $i$-face of $N(A)$, and $N_{i-1}(A) \ni \tau \subseteq \sigma$ is a face of $\sigma$. $\square$

### 2.3 Antichain Lattices

Although we will be considering a very specific type of lattice in this section and throughout, we will begin with a definition for a general lattice, and then specialize immediately.

**Definition 2.16.** *A lattice $\Lambda$ is an additive subgroup of $\mathbb{Z}^n$.*

**Remark 2.17.** *Any lattice in $\mathbb{Z}^n$ inherits the weak partial order of $\mathbb{Z}^n$ outlined in the previous section.*

**Lemma 2.18.** *If $\Lambda$ is a lattice that is an antichain, then it intersects $\mathbb{N}^n$ only at the origin.*

*Proof.*

If $\eta \in \mathbb{N}^n_{>0}$, and $\eta \in \Lambda$, then $\eta + \eta \in \Lambda$. Since $\eta << \eta + \eta$, $\Lambda$ would have two comparable elements and hence not be an antichain. $\square$

If $\Lambda \subseteq \mathbb{Z}^n$ is an antichain lattice, then we define $I_\Lambda \subset k[x_1, \ldots, x_n]$ to be the ideal generated by

$$\{X^{\lambda^+} - X^{\lambda^-} \mid \lambda \in \Lambda\}$$

Notice that any monoid morphism $\phi : \mathbb{N}^n \to \mathbb{N}^m$ extends to a group homomorphism $\overline{\phi} : \mathbb{Z}^n \to \mathbb{Z}^m$, and that $\ker(\overline{\phi})$ is an antichain lattice. Also, $\phi$ induces $\hat{\phi} : k[x_1, \ldots, x_n] \to k[y_1, \ldots, y_m]$, and $\ker(\hat{\phi}) = I_{\ker(\overline{\phi})}$.

**Example 2.19.** *Let $\phi$ be the monoid morphism*

$$\begin{array}{rccc} \phi : & \mathbb{N}^3 & \to & \mathbb{N} \\ & (\alpha,\beta,\gamma) & \mapsto & 3\alpha+4\beta+5\gamma \end{array}$$

*Then* $\phi$ *extends to*

$$\begin{array}{rccc} \overline{\phi} : & \mathbb{Z}^3 & \to & \mathbb{Z} \\ & (\alpha,\beta,\gamma) & \mapsto & 3\alpha+4\beta+5\gamma \end{array}$$

*and* $\ker(\overline{\phi}) = \{(\alpha,\beta,\gamma) \in \mathbb{Z}^3 \mid 3\alpha+4\beta+5\gamma = 0\} =< (3,-1,-1), (-1,2,-1), (-2,-1,2) >$. *Also,* $\phi$ *induces*

$$\begin{array}{rccc} \hat{\phi} : & k[x,y,z] & \to & k[t] \\ & x & \mapsto & t^3 \\ & y & \mapsto & t^4 \\ & z & \mapsto & t^5 \end{array}$$

*Then* $\ker(\hat{\phi}) = I_{\ker(\overline{\phi})} < x^3 - yz, y^2 - xz, z^2 - x^2y >$.

## 2.4 Markov Bases

Consider a lattice $\Lambda \subseteq \mathbb{Z}^n$ intersecting $\mathbb{N}^n$ at only the origin. Define the *fiber over* $u$ for $u \in \mathbb{N}^n$ to be $\mathcal{F}(u) := (u+\Lambda) \cap \mathbb{N}^n = \{v \in \mathbb{N}^n | u - v \in \Lambda\}$. Now consider an arbitary finite subset $\mathcal{B} \subseteq \Lambda$. For an arbitrary element $u \in \mathbb{N}^n$, we can define a graph denoted $\mathcal{F}(u)_{\mathcal{B}}$ where the vertices are the elements of $\mathcal{F}(u)$ and the edges are between vertices $v, w$ if $v - w$ or $w - v$ are in $\mathcal{B}$.

**Definition 2.20.** *A Markov basis of a lattice* $\Lambda \in \mathbb{Z}^n$ *is a finite set* $\mathcal{B} \subseteq \Lambda$ *such that* $\mathcal{F}_{\mathcal{B}}(u)$ *is connected for all* $u \in \mathbb{N}^n$*. We call a Markov basis* minimal *if it is such with respect to inclusion.*

**Theorem 2.21.** [Theorem 1.3.2, [4]]*If $\mathcal{B}$ and $\mathcal{B}'$ are minimal Markov bases for a lattice, then $|\mathcal{B}| = |\mathcal{B}'|$.*

**Theorem 2.22.** [Theorem 1.3.6, [4]]*A subset $\mathcal{B}$ of a lattice $\Lambda$ is a (minimal) Markov basis if and only if the set $\{X^{b^+} - X^{b^-} | b \in \mathcal{B}\} \subset k[x_1, \ldots, x_n]$ forms a (minimal) generating set of the lattice ideal $I_\Lambda =< X^{b^+} - X^{b^-} | b \in \Lambda >$.*

In the future, we will be referring to Theorem 2.22 more often than to Definition 2.20

**Definition 2.23.** *Let $\Lambda \subset \mathbb{Z}^n$ be a lattice. For any $\beta \in \mathbb{Z}^n$, the fiber over $\beta$ is $\beta + \mathbb{N}^n \cap \Lambda$.*

**Proposition 2.24.** *Let $\Lambda \subseteq \mathbb{Z}^n$ be a lattice that is an antichain. If $B$ is a Markov basis of $\Lambda$ and $\mathcal{N}$ is the set of neighbors of the origin, then $\mathcal{N} = B \cup -B$.*

*Proof.*

First, notice that $\mathcal{N} \subseteq B \cup -B$ because if $\lambda_1$ and $\lambda_2$ are neighborly, then there is a fiber of $\Lambda$ that contains only $\lambda_1$ and $\lambda_2$.

For the opposite inclusion, it suffices to show that $\mathcal{N}$ is a Markov basis. As a Markov basis, it will contain a minimal Markov basis, and because neighborliness is closed under taking negatives, it will also contain the negative of that minimal Markov basis. For any two minimal Markov bases, $B$ and $B'$, it is the case that $B \cup -B = B' \cup -B'$, so we will be finished. We proceed by proving that $\mathcal{N}$ is a Markov basis by showing that for any fiber, any two points in the fiber are connected by a path of neighborly pairs of elements.

Suppose that $F$ is a fiber of $\Lambda$ that contains only two elements. Then those two elements are neighborly, and hence there is a neighborly path between them. Now suppose that the result holds for all fibers $F$ such that $|F| < m$. Suppose $F$ is a fiber such that $|F| = m$, and suppose $\lambda_1, \lambda_2 \in F$ where $\lambda_1$ and $\lambda_2$ are not neighborly. Without loss of generality, let $F$ be the fiber over $\lambda_1 \wedge \lambda_2$. Since $\lambda_1$ and $\lambda_2$ are not neighborly, there exists $\alpha \in F$ such that $\alpha << \lambda_1 \vee \lambda_2$.

Let

$$\Delta_1 = \{i \in [1, \ldots, n] \mid \pi_i(\lambda_1) > \pi_i(\lambda_2)\}$$

Then $\pi_i(\lambda_1) > \pi_i(\alpha)$ for all $i \in \Delta_1$ and $\pi_j(\lambda_1) < \pi_j(\alpha)$ for all $j \in \Delta_1^c$. By construction, $\pi_i(\alpha) > \pi_i(\lambda_2)$ for all $i \in \Delta_1$, and $\pi_j(\alpha) < \pi_j(\lambda_2)$ for all $j \in \Delta_1^c$. Therefore, $(\alpha - \lambda_1) \wedge 0 > (\lambda_2 - \lambda_1) \wedge 0$ and hence $(\alpha \wedge \lambda_1) > (\lambda_1 \wedge \lambda_2)$.

We can draw two conclusions from this final inequality. The first is that $(\alpha \wedge \lambda_1 + \mathbb{N}^n) \cap \Lambda \subset (\lambda_1 \wedge \lambda_2 + \mathbb{N}^n) \cap \Lambda$, and the second is that $\lambda_2 \notin (\alpha \wedge \lambda_1 + \mathbb{N}^n) \cap \Lambda$. The final conclusion to draw is that the minimal fiber containing $\alpha$ and $\lambda_1$ has size less than $n$, and likewise for $\lambda_2$. Thus, by the inductive hypothesis, there is a neighborly path from $\lambda_1$ to $\alpha$ and another from $\alpha$ to $\lambda_2$, creating the desired neighborly path from $\lambda_1$ to $\lambda_2$.

$\square$

**Corollary 2.25.** *Let $\Lambda \subseteq \mathbb{Z}^n$ be a lattice that is an antichain. If $\mathcal{B}$ and $\mathcal{B}'$ are minimal Markov bases of $\Lambda$, then $\mathcal{B}$ and $\mathcal{B}'$ differ by a sign vector.*

*Proof.*

Minimal Markov bases are drawn from the set of neighbors of the origin, which occur in pairs with opposite signs. A Markov basis must have exactly one vector from each pair, and hence they differ from one another only by what sign is assigned to each vector. $\square$

### 2.5 Generic Lattices

In our quest to unite the various definitions of genericity, we will now consider generic lattices. For the sake of formalizing the reference, we have the definition given in Theorem 2.22, and a second definition, both of which come from [12].

**Definition 2.26.** *If $\Lambda \subset \mathbb{Z}^n$ is an antichain lattice, then the associated lattice ideal is $I_\Lambda =< X^\alpha - X^\beta | \alpha, \beta \in \mathbb{N}^n$ and $\alpha - \beta \in \Lambda >$.*

**Definition 2.27.** *If $\Lambda \subset \mathbb{Z}^n$ is an antichain lattice, we say $\Lambda$ is generic if there is a minimal Markov basis $L$ of $\Lambda$ such that each $\lambda \in L$ is fully supported.*

**Lemma 2.28.** *If $\Lambda \subset \mathbb{Z}^n$ is an antichain lattice, then $\Lambda$ is generic as in Definition 2.27 if and only if $\Lambda$ is generic in $\mathbb{Z}^n$ as in Definition 2.8.*

*Proof.*

By Proposition 2.24, we can first consider an identical statement: the neighbors of the origin with respect to $\Lambda$ are fully supported if and only if there are no neighborly pairs that share a component.

Let $\Lambda$ be generic by Definition 2.27. Under lattice translations, if

$$L = \{\text{neighbors of the origin with respect to } \Lambda\},$$

then

$$\alpha + L = \{\text{neighbors of } \alpha \text{ with respect to } \Lambda\}$$

If $\beta \in \alpha + L$, then $pi_1(\alpha) \neq \pi_i(\beta)$ for $i = 1, \ldots, n$ because the elements of $L$ are fully supported, and $beta = \alpha + \ell$ for some $\ell \in L$. Because of this, if there exists a $\beta$ such that $\pi_i(\beta) = \pi_i(\alpha)$, then $\alpha$ and $\beta$ are not neighborly. Therefore, there exists $\gamma \in T^o_{\alpha \vee \beta} \cap \Lambda$ by definition. That is, there exists $\gamma << \alpha \vee \beta$ and hence $\Lambda$ is generic by Definition 2.8.

Let $\Lambda$ be generic by Definition 2.8. Then for all $\alpha, \beta \in \Lambda$ such that $pi_i(\alpha) = \pi_i(\beta)$, there exists $\gamma \in \Lambda$ such that $\gamma << \alpha \vee \beta$. That is, $\gamma \in T^o_{\alpha \vee \beta} \cap \Lambda$. Therefore, if $\pi_i(\alpha) = \pi_i(\beta)$ for some $i = 1, \ldots, n$, then they are not neighborly. Hence, if $\alpha, \beta$ are to be neighborly, $\alpha - \beta$ must be fully supported. Thus, if $A$ is the set of neighbors of $\alpha$, then the vectors $\{\alpha - \beta \mid \beta \in A\}$ are fully supported, and hence $\Lambda$ is generic by Definition 2.27. $\square$

Lemma 2.28 shows us that the notion of a generic lattice from [12] matches the definition for generic we have already seen for $\mathbb{N}^n$-sets.

# Chapter 3: $\Lambda$-Sets

In this Chapter, we will generalize the lattices from the previous chapter into $\Lambda$-sets, and then reform some of the notions and definitions we had for lattices. We will close with a brief discussion of resolutions of lattice ideals, with more details to come in Chapter 2.5. If not explicitly mentioned, our lattices will continue to be subsets of $\mathbb{Z}^n$, antichains and generic.

The primary object of study in this chapter is a $\Lambda$-set, which is a specific case of an $M$-set, where $M$ is a monoid. If $A \subseteq \mathbb{Z}^n$, and $A = A + \Lambda$, then $A$ is a $\Lambda$-set under the map $A \times \Lambda \to A$ defined by $(\alpha, \lambda) \mapsto \alpha + \lambda$.

### 3.1 Structure of $\Lambda$-sets

**Definition 3.1.** *Suppose $A = A+\Lambda$. If $A_0 \subseteq A$, we call $A_0$ a set of $\Lambda$-representatives for $A$ if*

1. $A = A_0 + \Lambda$
2. $a, b \in A_0$ *implies* $a - b \notin \Lambda$

*Call $A$ $\Lambda$-finite if $A$ has a finite set of representatives.*

**Example 3.2.**

1. $\Lambda$ *is* $\Lambda$*-finite with* $A_0 = \{0\}$.
2. $\mathbb{Z}^n$ *is not* $\Lambda$*-finite unless* $\Lambda = \mathbb{Z}^n$.
3. $((1, \ldots, 1) + \Lambda) \cup \Lambda$ *is* $\Lambda$*-finite with* $A_0 = \{(1, \ldots, 1), 0\}$.
4. *If* $\Lambda$ *is generated by* $(1, -2, 1)$ *and* $(2, 3, -4)$*, and* $A_0 = \{(1, 3, 3), (2, 1, 4)\}$*, then* $A_0 + \Lambda$ *is* $\Lambda$*-finite, but* $A_0$ *is not a set of* $\Lambda$*-representatives.*

Unless $\Lambda = \{0\}$, infinitely many options for $A_0$ exist. When thinking of $\Lambda$ as a $\Lambda$-set, for example, we could choose any $\lambda \in \Lambda$ to be our $A_0$. In the second $\Lambda$-finite example, we could have chosen any $\lambda_1 \in \Lambda$ together with any $\lambda_2 \in (1, \ldots, 1) + \Lambda$. In the second case, we make no reference to the Euclidean distance between $\lambda_1$ and $\lambda_2$. It will be important later to be able to address this distance, so we will develop a method for choosing an $A_0$ that has an additional desirable property: closeness.

**Lemma 3.3.** *Let $\Lambda \subset \mathbb{R}^n$ and let $A$ be $\Lambda$-finite. Let $V$ be the subspace of $\mathbb{R}^n$ spanned by $\Lambda$, and let $\mathcal{C}$ be a fundamental region ($k$-parallelapiped, where $\Lambda$ has codimension $n-k$) of $\Lambda$ in $V$. If $\pi : \mathbb{R}^n \to V$ is the orthogonal projection map, then there is a set of $\Lambda$-representatives for $A$ contained in $\pi^{-1}(\mathcal{C})$.*

*Proof.* We have that $A$ is $\Lambda$-finite, so choose $A_0$ as a finite set of representatives. For ease, order $A_0$ as $\alpha_1 < \alpha_2 < \cdots < \alpha_s$, and consider $\pi(\alpha_1) + \mathcal{C}$. Since $\pi(\alpha_1) \neq \pi(\alpha_i)$ for all $i > 1$, and $\pi(\alpha_i) + \mathcal{C} + \Lambda$ is a division of $V$ into $k$-parallelapipeds, there exists a $\lambda_i \in \Lambda$ such that $(\pi(\alpha_i) + \mathcal{C} + \lambda_i) \cap (\pi(\alpha_i) + \mathcal{C}) \neq \emptyset$. To complete the proof, let the representative set be $\pi^{-1}(\alpha_1) \cup \{\pi^{-1}(\alpha_i + \lambda_i) | i > 1\}$. □

Although there will be many situations where this property is not needed, we will henceforth only consider sets of $\Lambda$-representatives of $\Lambda$-finite sets $A$ of the form of the conclusion of Lemma 3.3.

**Example 3.4.**

1. *If $A = A + \Lambda$, then each $\alpha \in A$ is a neighborly set.*

2. *If $\Lambda \in \mathbb{Z}^2$ is generated by $(1, -1)$, and $A$ is a $\Lambda$ set with $A_0 = \{(1, 1)\}$ a set of $\Lambda$-representatives for $A$, then $\{(i+1, i-1), (i+2, i-2)\}$ is a neighborly set of $A$.*

**Proposition 3.5.** *Let $A$ be a generic $\Lambda$-finite set, then $N_i(A)$ is $\Lambda$-finite set under the map $N_i(A) \times \Lambda \to N_i(A)$ where $(\sigma, \lambda) \mapsto \sigma + \lambda$*

*Proof.* If $\sigma \in N_i(A)$, then $\sigma + \lambda \in N_i(A)$ for all $\lambda \in \Lambda$, so $N_i(A) = N_i(A) + \Lambda$, and hence it is a $\Lambda$-set. The $\Lambda$-finiteness property will come as a corollary to Lemma 5.4. □

### 3.2 Resolutions of Lattice Ideals

Later, we will cover resolutions of lattice ideals in more generality, but for this section, we will give the basic results concerning lattice ideals.

**Definition 3.6.** [Definition 9.11, [10]] *Given a lattice $\Lambda$ whose intersection with $\mathbb{N}^n$ is 0, the lattice module, $M_\Lambda$, is the $S$-submodule of the Laurent polynomial ring $S^{\pm} = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ generated by $\{X^\lambda | \lambda \in \Lambda\}$.*

In [10], one will find that the Scarf complex of $A \in \mathbb{Z}^n$ is defined as the set of strongly neighborly sets, where we have defined it to be the set of neighborly sets. We saw in Lemma 2.13 that when $A$ is generic, strongly neighborly and neighborly are identical, and as such, the reader does not need to make any distinction going foward.

We will finish this section with a prelude to what we intend to do with the machinery we have hitherto developed. In Chapter 3.2.1, we will construct a collection of maps that we will associate to simplicial complexes. When we apply this construction to the Scarf complex of a generic $\Lambda$-set, $A$, we will obtain free a free resolution of $M_A$ as an $S$-module. Additionally, we will be able to resolve lattice ideals by considering the construction modulo the lattice. The machinery behind these ideas will be developed in later sections in more general situations. The machinery will primarily exploit the structure of the lattice, and in fact, we will use a more general version of the Scarf complex.

### 3.2.1 Lattice Ideal Resolutions in $\mathbb{Z}^3$

In $\mathbb{Z}^3$, we have a remarkable amount of control over Markov bases of lattices. In particular, the Markov bases will have three elements, $\lambda_1, \lambda_2$, and $\lambda_3$, and they can be chosen such that $\lambda_1 = -(\lambda_2 + \lambda_3)$.

**Lemma 3.7.** *If $\Lambda \subset \mathbb{Z}^3$ is a generic antichain lattice with codimension 1 and Markov basis $\lambda_1 = \{(\alpha_1, -\beta_1, -\gamma_1), \lambda_2 = (\alpha_2, -\beta_2, -\gamma_2), \lambda_3 = (\alpha_3, -\beta_3, -\gamma_3)\}$, then the minimal free resolution of $S/I_\Lambda$ is*

$$\begin{array}{ccccccc}
S/I_\Lambda & \leftarrow & S & \leftarrow & Se_{\lambda_1} \oplus Se_{\lambda_2} \oplus Se_{\lambda_3} & \leftarrow & Se_{p_1} \oplus Se_{p_2} \\
 & & b_1 & \leftarrow\!\shortmid & e_{\lambda_1} & x_3^{\gamma_2} e_{\lambda_1} + x_1^{\alpha_3} e_{\lambda_2} + x_2^{\beta_1} e_{\lambda_3} & \leftarrow\!\shortmid e_{p_1} \\
 & & b_2 & \leftarrow\!\shortmid & e_{\lambda_2} & x_2^{\beta_3} e_{\lambda_1} + x_3^{\gamma_1} e_{\lambda_2} + x_1^{\alpha_2} e_{\lambda_3} & \leftarrow\!\shortmid e_{p_2} \\
 & & b_3 & \leftarrow\!\shortmid & e_{\lambda_3} & &
\end{array}$$

*Proof.* Apply the tools from Chapter 5.2 that we will cover latter. Alternatively, [8]. □

# Chapter 4: Resolutions

This chapter will cover our primary object of study: resolutions. We will mostly address the general definitions via our specific uses, and in particular, via a constructive algorithm. We will cover the definitions associated to cellular resolutions, which encompasses the algorithm that we will apply to the scarf complex in later chapters.

**Definition 4.1.** *Let $M$ be an $S$-module, then a* resolution *of $M$ is a complex $F_\bullet$ with maps $\delta_i$ such that*

$$0 \longleftarrow M \xleftarrow{\delta_0} F_0 \xleftarrow{\delta_1} F_1 \xleftarrow{\delta_2} \cdots \longleftarrow: F_\bullet$$

*is exact. I.e., if $\ker(\delta_i) = Im(\delta_{i+1})$. The resolution is free if $F_i$ is free for all $i$. If the resolution is free, then $F_i = S^{\beta_i} := \underbrace{S \oplus \cdots \oplus S}_{\beta_i \text{ times}}$, and if it is minimal, the $\beta_i$s are collectively called the Betti numbers of the resolution.*

### 4.1 Cellular Resolutions

Consider $A \subset \mathbb{Z}^n$ and $\Lambda \subset \mathbb{Z}^n$ such that $\Lambda$ is a lattice interescting $\mathbb{N}^n$ only at 0, and that $A$ is a generic $\Lambda$-finite set. We already have that $N(A)$ is a simplicial complex; to the simplicial structure, we can add more information in the form of face labels.

**Definition 4.2.** *Let $A \subseteq \mathbb{Z}^n$, and let $B$ be a collection of finite subsets of $A$. We say that $B$ is labeled by $\mathbb{Z}^n$ if we have a map $B \hookrightarrow \mathbb{Z}^n$.*

We will be using a specific label when we work with $N(A)$ for some $A \subset \mathbb{Z}^n$. Each element $\sigma$ of $N(A)$ is a finite subset of $A$. The label of $\sigma$ is $\vee\sigma$. Formally, if $A \subset \mathbb{Z}^n$ and $B = \{B_1, B_2, \dots\}$ is a collection of finite subsets of $A$, and

$B_i = \{\alpha_{i,1}, \ldots, \alpha_{i,m}\}$, our label is a mapping

$$\begin{array}{ccc} B_i & \to & \mathbb{Z}^n \\ \{\alpha_{i,1}, \ldots, \alpha_{i,m}\} & \mapsto & \bigvee_{j=0}^{m} \alpha_{i,j} \end{array} \tag{4.1}$$

**Definition 4.3.** *Let $S = k[x_1, \ldots, x_n]$, and let $F_i(N(A)) := \bigoplus_{\sigma \in \mathbb{N}_i(A)} Se_\sigma$ be the free $S$-module with generators $\{e_\sigma \mid \sigma \in N_i(A)\}$.*

If $\sigma = \{\sigma_0, \ldots, \sigma_i\} \in N_i(A)$, then $\partial_j \sigma = \{\sigma_0, \ldots, \sigma_{j-1}, \sigma_{j+1}, \ldots, \sigma_i\}$. Let $\phi_i : F_i(N(A)) \to F_{i-1}(N(A))$ be defined as follows:

$$\begin{array}{rccc} \phi_i : & F_i(N(A)) & \to & F_{i-1}(N(A)) \\ & e_\sigma & \mapsto & \sum_{j=0}^{i} (-1)^j X^{\vee\sigma - \vee\partial_j\sigma} e_{\partial_j \sigma} \end{array} \tag{4.2}$$

**Proposition 4.4.** *With $\phi_i$ defined above, $\phi_i \phi_{i-1} = 0$.*

*Proof.* We proceed by direct computation, using the notation that $(\partial_j(\partial_i \sigma) = \partial_{ij}\sigma$ for simplicity.

$$\phi_i(\phi_{i-1}(e_\sigma)) = \phi_i(\sum_{j=0}^{i} (-1)^j X^{\vee\sigma - \vee\partial_j\sigma} e_{\partial_j\sigma})$$

$$= \sum_{j=0}^{i} (-1)^j X^{\vee\sigma - \vee\partial_j\sigma} \phi_i(e_{\partial_j\sigma})$$

$$= \sum_{j=0}^{i} (-1)^j X^{\vee\sigma - \vee\partial_j\sigma} \sum_{k=0}^{i-1} (-1)^k X^{\vee\partial_j\sigma - \vee\partial_{jk}\sigma} e_{\partial jk\sigma}$$

$$= \sum_{j=0}^{i} \sum_{k=0}^{i-1} (-1)^{j+k} X^{\vee\sigma - \vee\partial_{jk}\sigma} e_{\partial_{jk}\sigma}$$

To finish the proof, we need to show that the coefficients for $e_{\partial_{jk}\sigma}$ appear in pairs with opposite parities, so that everything cancels out. To do this, notice that

$$\partial_{jk}\sigma = \begin{cases} \partial_{k(j-1)}\sigma & k < j \\ \partial_{(k+1)j}\sigma & k \geq j \end{cases}$$

That is, we can create a function $f : \mathbb{Z}^2 \to \mathbb{Z}^2$ defined by

$$f(j,k) = \begin{cases} (k, j-1) & k < j \\ (k+1, j) & k \geq j \end{cases}$$

that is a bijection from $[0,n] \times [0,n-1]$ to itself. This gives us the desired pairing of coefficients. To show the parity argument, notice that in this matching, the image and preimage pairs under $f$ have parities $j+k$ and $j+k-1$ or $j+k$ and $j+k+1$. In either case, the parity is different, and hence the coefficient pairs annihilate each other, giving us that $\phi_i \phi_{i-1} = 0$. $\square$

The construction we did here was specifically applied to $N(A)$, but it could have been applied to any simplicial complex $\mathcal{X}$ labeled by elements of $\mathbb{Z}^n$ as in the mapping (4.1).

**Definition 4.5.** *Let $\mathcal{X}$ be a simplicial complex labeled with elements of $\mathbb{Z}^n$ as in (4.1), and let $\mathcal{X}_i$ be the set of i-faces of $\mathcal{X}$. The cellular free complex supported on $\mathcal{X}$, denoted $\mathcal{F}_{\mathcal{X}}$ is the complex of free $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$-modules generated by $e_\sigma$ for $\sigma \in \mathcal{X}_i$. If $\mathcal{X}$ is acyclic, we pair $\mathcal{X}$ together with the maps $\phi$ from (4.2) the cellular free resolution supported on $\mathcal{X}$. We denote it $\mathcal{F}_{\mathcal{X}}$.*

**Definition 4.6.** *If $\mathcal{X}$ is a simplicial complex labeled with elements of $\mathbb{Z}^n$, then for all $b \in \mathbb{Z}^n$, $\mathcal{X}_{\preceq b}$ is the subcomplex supported on all faces with labels coordinatewise at most b.*

**Proposition 4.7.** *The cellular free complex $\mathcal{F}_{\mathcal{X}}$ supported on $\mathcal{X}$ is is a cellular resolution if and only if $\mathcal{X}_{\preceq b}$ is acyclic over $k$ for all $b \in \mathbb{Z}^n$. When $\mathcal{F}_{\mathcal{X}}$ is acyclic (homology only in dimension 0), then it is a free resolution of $M_{\mathcal{X}} = \{X^{\zeta} \mid \zeta$ the label of some vertex of $\mathcal{X}\}$, the $k[x_1, \ldots, x_n]$-submodule of $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.*

*Proof.*

This is an extension of the finite case given in Proposition 4.5 in [10], but the proof runs identically. □

In Definition 3.6, we took a lattice $\Lambda$, and defined $M_\Lambda$ to be the $k[x_1, \ldots, x_n]$-submodule of $S^{\pm} = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ generated by $\{X^{\lambda} \mid \lambda \in \Lambda\}$. We used $\Lambda$ because we were working with lattices, but we can define the same object for any $\Lambda$-set.

**Definition 4.8.** *If $A \subseteq \mathbb{Z}^n$ is a $\Lambda$-set for some $\Lambda \subset \mathbb{Z}^n$, then $M_A$ is the $S$-submodule of $S^{\pm}$ generated by $\{X^{\alpha} \mid \alpha \in A\}$.*

As a special case we would like to consider, we have the following proposition.

**Proposition 4.9.** *Let $\Lambda \subset \mathbb{Z}^n$ be an antichain lattice, and let $A$ be a generic $\Lambda$-finite set. Then*

$$F_{\bullet} : \cdots \to F_i(N(A)) \xrightarrow{\phi_i} F_{i-1}(N(A)) \xrightarrow{\phi_{i-1}} \cdots F_0(N(A)) \xrightarrow{\phi_1} M_A$$

*is a resolution of $M_A$ as an $S$-module.*

# Chapter 5: Laurent Monomial Modules

## 5.1 Hull Complex

We begin with some notation. We will always assume that $t \in \mathbb{R}$ with $t > 1$ and that $A \subset \mathbb{Z}^n$. Let

$$E_t(\alpha) = (t^{\pi_1(\alpha)}, \ldots, t^{\pi_n(\alpha)})$$

for $\alpha \in \mathbb{Z}^n$ and

$$E_t(A) = \{E_t(\alpha) \mid \alpha \in A\}$$

Additionally, we will let

$$\mathcal{P}_t(A) = \mathrm{conv}(E_t(A) + \mathbb{N}^n) = \mathbb{R}^n_{\geq 0} + \mathrm{conv}(E_t(A))$$

**Lemma 5.1.** *If $A \subseteq \mathbb{Z}^n$ is a generic $\Lambda$-finite set for some antichain lattice $\Lambda \subseteq \mathbb{Z}^n$, then for $t > 1$, the vertices of $\mathcal{P}_t(A)$ are $E_t(A)$.*

*Proof.*

Clearly, $E_t(A) \subset \mathcal{P}_t(A)$. We must show that for large enough $t$, $E_t(A) \subset \partial\mathcal{P}_t(A)$ and that there are no collections of $n+1$ points of $E_t(A)$ in the same supporting hyperplane of $\mathcal{P}_t(A)$. First note that from [13], we have the following condition for convexity: a set $C \in \mathbb{R}^n$ is convex if and only if for all $x, y \in \partial C$, $< N_C(x) - N_C(y), x - y >\geq 0$, where $N_C(x)$ is the normal vector to $C$ at $x$.[1]

Let $a \in \mathbb{R}^n_{>0}$ and let $t \in \mathbb{R}_{>1}$. Let

$$H_a = \{x \in \mathbb{R}^n | a \cdot x \geq 0\}$$

[1]In [13], as here, we will consider a normal vector at a point to be any vector inside the normal cone at that point. That is, we can choose a normal vector to any plane that is tangent at the point, and the result still hols.

and

$$\partial H_a = \{x \in \mathbb{R}^n | a \cdot x = 0\}.$$

Then

$$E_t(H_a) = \{t(x) | a \cdot x \geq 0\}$$

$$= \{(t^{x1}, \ldots, t^{x_n}) | a_1 x_1 + \cdots + a_n x_n \geq 0\}$$

$$= \{(\xi_1, \ldots, \xi_n) | \xi_1^{a_1} \ldots \xi_n^{a_n} \geq 1, \xi_i = t^{x_i}\}$$

and

$$E_t(\partial H_a) = \{(\xi_1, \ldots, \xi_n) | \prod \xi_i^{a_i} = 1\}$$

To simplify notation, let $f_a(\xi) = \xi_1^{a_1} \ldots \xi_n^{a_n}$. Then we have that $t(\partial H_a)$ is the level set defined by $f_a(\xi) = 1$ and $t(H_a) = \{\xi | f_a(\xi) \geq 1\}$. Note that since $t$ is a homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$, we have that $t(\partial H_a) = \partial t(H_a)$.

We wish to show that $C = E_t(H_a)$ is convex. By the aforementioned convexity condition, we can show $< N_C(x) - N_C(y), x - y >\geq 0$ for all $x, y \in \partial C$. The (outward facing) normal vector to $\partial C$ at $\xi$ is $-\nabla f_a(\xi)$. Now $\frac{\partial f_a}{\partial \xi_i} = \frac{a_i}{\xi_i} f_a(\xi)$ and if $\xi \in \partial C$, then $f_a(\xi) = 1$. Thus $\nabla f(\xi) = (\frac{a_1}{\xi_1}, \cdots, \frac{a_n}{\xi_n})$ for all $\xi \in \partial C$.

To finish the computation, choose $\xi, \eta \in \partial C$. Then

$$N_C(\xi) = -(\frac{a_1}{\xi_1}, \cdots, \frac{a_n}{\xi_n})$$

and

$$N_C(\eta) = (\frac{a_1}{\eta_1}, \cdots, \frac{a_n}{\eta_n})$$

Now $< N_C(\xi) - N_C(\eta), \xi - \eta >=< (\dots, \frac{a_i(\xi_i - \eta_i)}{\xi_i \eta_i}, \dots), (\dots, \xi_i - \eta_i, \dots) > = a_1 \frac{(\xi_1-\eta_1)^2}{\xi_1\eta_1} + \cdots + a_n \frac{(\xi_n-\eta_n)^2}{\xi_n\eta_n} \geq 0$. So we have that $C$ is convex.

If we fail to satisfy the condition that no collection of $n+1$ points of $E_t(A)$ lie in a supporting plane of $\mathcal{P}_t(A)$, then increasing $t$ will suffice. This is because for all $\alpha, \beta \in \min(A)$, and for all $i = 1, \dots, n$, $\pi_i(\alpha) \neq \pi_i(\beta)$. Therefore, if $\alpha$ was in the supporting hyperplane of $\alpha_1, \dots, \alpha_n$ for $t$, it would not be in the supporting hyperplane of $\alpha_1, \dots, \alpha_n$ for $t + \epsilon$ for some $\epsilon > 0$. □

**Corollary 5.2.** *Let $A \subseteq \mathbb{Z}^n$ be a generic $\Lambda$-finite set for some antichain lattice $\Lambda \subseteq \mathbb{Z}^n$, and $t > 1$. If $F$ is a face of conv$(E_t(A))$, then $F \cap E_t(A) = E_t(\sigma)$ where $\sigma \in A$.*

*Proof.* We already have that $E_t(A)$ is the vertex set of $\mathcal{P}_t(A)$. Suppose that $F$ is a maximal face of $E_t(A)$ and let $F \cap E_t(A) = \{t^{\alpha_1}, \dots, t^{\alpha_r}\}$. Suppose for a contradiction that $\{\alpha_1, \dots, \alpha_r\} \notin N(A)$. Then there exists $b \in A$ such that $b << \vee\alpha_i$. Therefore, $t^b << t^{\vee\alpha_i} = \vee t^{\alpha_i}$. We have three cases to consider.

1. $t^b \in \text{conv}(t^{\alpha_1}, \dots, t^{\alpha_r}, \vee t^{\alpha_i})$.

2. $t^b \notin \text{conv}(t^{\alpha_1}, \dots, t^{\alpha_r}, \vee t^{\alpha_i})$.

3. $t^b \in F$.

Examining each case:

1. We would have that $t^b$ lies in the interior of $\mathcal{P}_t(A)$, contradicting Lemma 5.1.

2. This would imply that the hyperplane containing $F$ separates $E_t(A)$, contradicting the convexity of $\mathcal{P}_t(A)$.

3. If $t^b \in F$, increase $t$ by $\epsilon > 0$ to be back in case 2.

□

Before we cover the main concepts in this chapter, we first need a structural lemma that underlies many statements that will be made later.

**Lemma 5.3.** *Let $A \subset \mathbb{Z}^n$, and suppose that $A$ is a generic $\Lambda$-finite set for some antichain lattice $\Lambda$. Then for $t >> 0$, $\partial(conv(E_t(A) + \mathbb{N}^n)) \cong \mathbb{R}^{n-1}$.*

*Proof.*

Let $B = \{\beta \in \mathbb{R}^n \mid \pi_1(\beta) + \cdots + \pi_n(\beta) = 0\}$, and for each $\beta \in B$, let $\ell_\beta = \{\beta + s(1, \ldots, 1) \mid s \in \mathbb{R}\}$. Since $\partial(\text{conv}(E_t(A) + \mathbb{N}^n))$ is convex, and $B \cap \mathbb{N}^n = 0$, we have that each $\ell_\beta$ intersects $\partial(\text{conv}(E_t(A) + \mathbb{N}^n))$ in at most one point. To see that $\ell_\beta$ intersects $\partial(\text{conv}(E_t(A) + \mathbb{N}^n))$ at all, notice that the point of $\partial(\text{conv}(E_t(A) + \mathbb{N}^n))$ that is closest to the origin is the point of intersection with $\ell_0$. Call this point $\gamma$. Then $\mathbb{N}^n \subset \partial(\text{conv}(E_t(A) + \mathbb{N}^n)) - \gamma$. The line connecting any point $\eta$ on any coordinate face of $\mathbb{N}^n$ to the closest point on $B$ passes through $\partial(\text{conv}(E_t(A) + \mathbb{N}^n)) - \gamma$, showing that each $\ell_\beta$ intersects $\partial(\text{conv}(E_t(A) + \mathbb{N}^n))$ in exactly one point.

Therefore, we have a bijection between $B$ and $\partial(\text{conv}(E_t(A) + \mathbb{N}^n))$. For each $\beta \in B$, call this point of intersection $\beta'$. Consider the map

$$\begin{array}{rccc} f: & \partial(\text{conv}(E_t(A) + \mathbb{N}^n)) & \to & B \\ & \beta' & \mapsto & \beta \end{array}$$

Since $f$ maps different elements along lines parallel to $t(1, \ldots, 1)$, then two points that are close in $\partial(\text{conv}(E_t(A) + \mathbb{N}^n))$ remain close under $f$. This also holds mutatis mutandis under $f^{-1}$, which maps $\beta'$ to $\beta$. Therefore, we have a continuous bijection with a continuous inverse, and hence $\partial(\text{conv}(E_t(A) + \mathbb{N}^n))$ and $B$ are homeomorphic. Since $B$ is a hyperplane in $\mathbb{R}^n$, it is homeomorphic to $\mathbb{R}^{n-1}$, and hence, so is $\partial(\text{conv}(E_t(A) + \mathbb{N}^n))$. $\square$

Continuing, we need to show an important property of $A$.

**Lemma 5.4.** *Let $\Lambda$ be an antichain lattice, and let $A \in \mathbb{Z}^n$ be a generic $\Lambda$-finite set. Then for each $\alpha \in A$, the set of neighbors of $\alpha$ is finite.*

*Proof.*

Without loss of generality, assume that $\alpha = 0$. Let $R_{A,k} \in \mathbb{R}$ be a number such that for any $\alpha \in A$, the ball $B(\alpha, R_{A,k})$ contains at least $k$ points from each $\Lambda$-coset of $A$. If $\Lambda$ is codimension $d$, then $A$ has infinitely many points in $2^n - 2^d$ orthants of $\mathbb{Z}^n$, and only finitely many in the remaining orthants.

Construct $2^n - 2^d$ $n$-cubes in $\mathbb{R}^n$ of side length $2R_{A,k}$ and centers $\alpha_i$, called $C_{\alpha_i}$, satisfying the following conditions:

1. $\alpha_i \in A$
2. $\operatorname{sgn}(\alpha_i) \neq \operatorname{sgn}(\alpha_j)$ for all $i \neq j$. I.e., each $\alpha_i$ lies in a different orthant.
3. No supporting hyperplane of $C_{\alpha_i}$ intersects $C_{\alpha_j}$ for $i \neq j$.
4. $C_{\alpha_i}$ exists wholly in an orthant containing infinitely many points of $A$.

Continuing, let $\alpha_i'$ be the corner of $C_{\alpha_i}$ that is furthest from the origin, and let $Q_{\alpha_i} = \{z \in \mathbb{Z}^n \mid \operatorname{sgn}(z) = \operatorname{sgn}(\alpha_i)\}$ be the orthant containing $\alpha_i$.

Notice now, that by construction, if $\gamma \in \alpha_i' + Q_{\alpha_i}$, then $B(\alpha_i, R_{A,k}) \subseteq T_{0\vee\gamma}$. Since $A$ is generic, if we let $k = n+1$, we have that $T_{0\vee\gamma} \cap A \neq \emptyset$ because each face contains at most 1 element of $A$, and we have at least 1 extra element. Hence $\gamma$ is not a neighbor of 0.

It remains to show that

$$|(\bigcup_i (\alpha_i' + Q_{\alpha_i}))^c \bigcap A| < \infty$$

By construction, each $\alpha_i' + Q_{\alpha_i}$ contains part of a hyperplane that is parallel to $\Lambda$. This holds even if $\Lambda$ is not codimension 1; there are simply more choices for hyperplanes parallel to $\Lambda$ if $\Lambda$ has high codimension. Thus, since $Q_{\alpha_i}$ has dimension $n$, it contains part of every hyperplane parallel to $\Lambda$. In particular, each face of $Q_{\alpha_i}$ intersects the hyperplane. Since $A$ lies entirely in between two hyperplanes parallel to $\Lambda$, in each orthant, we can separate the points of $A$ as those that are in $\alpha_i' + Q_{\alpha_i}$, and those that are not. By bounding $A$ by hyperplanes parallel to $\Lambda$, we have shown that only finitely points of $A$ exist in the orthant outside of $\alpha_i' + Q_{\alpha_i}$.

Thus, we have only finitely many choices of neighbors of 0 in the orthants that have infinitely many elements of $A$, and only finitely many choices in all other orthants, so we have at most finitely many neighbors of 0.

$\square$

For $A \subseteq \mathbb{Z}^n$, let $\mathrm{hull}_t(A) = \{E_t(F) \subseteq E_t(A) \mid \mathrm{conv}(E_t(F)) \text{ is a face of } \mathcal{P}_t(A)\}$.

**Proposition 5.5.** *If $A \in \mathbb{Z}^n$ is generic, then there exists $T \in \mathbb{R}$ such that for $t, t' \geq T$, $hull_t(A) = hull_{t'}(A)$.*

*Proof.*

Let $B_i = B(0, i)$ be the ball of radius $i$ about the origin in $\mathbb{R}^n$. If $\mathcal{V}_{i,t} = B_i \cap \mathrm{hull}_t(A)$, then $\mathrm{hull}_t(A) = \varinjlim \mathcal{V}_{i,t}$. By Proposition 4.14 of [10], there exists a $T \in \mathbb{R}$ such that for $t, t' \geq T$, $\mathrm{hull}_t(\mathcal{V}_{i,t}) = \mathrm{hull}_{t'}(\mathcal{V}_{i,t})$. Specifically, the Proposition tells us that $T = (n+1)!$. Since this holds for all $\mathcal{V}_{i,t}$, it holds under the direct limit, and hence when $T > (n+1)!$, $\mathrm{hull}_t(A) = \mathrm{hull}_{t'}(A)$. $\square$

**Remark 5.6.** *Although not mentioned explicitly, if $A$ is not generic, Proposition 5.5 fails. This is because there will exist two elements that share a component without a third element dividing the supremum of the first two. Under the exponentiation, these two elements would continue to share a component for all $t$, which would imply the existence of a supporting hyperplane of $\mathcal{P}_t(A)$ that was parallel to a coordinate plane, violating Lemma 5.1.*

When $t$ is large enough, $\text{hull}_t(A)$ is independent of $t$, so we will drop the subscript and use $\text{hull}(A)$ when it is understood that $t \geq T$.

**Proposition 5.7.** *Let $A \subset \mathbb{Z}^n$ be a $\Lambda$-finite set for some antichain lattice $\Lambda \subset \mathbb{Z}^n$. For all $\alpha \in A$,*

$$|\{\sigma \in \mathit{hull}(A) \mid \alpha \in \sigma\}| < \infty$$

*Proof.*

If a face of $\text{hull}(A)$ were incident with infinitely many other faces, that would imply the existence of an edge that was incident with infinitely many other edges; up to a suitable translation, we could consider the point of incidence to be 0, contradicting Lemma 5.4. □

**Remark 5.8.** *In Lemma 5.4, we worked strictly in $A$ and $N(A)$, but Proposition 5.7 made a claim about hull($A$). However, we have a structure-preserving bijection between the two objects outlined in Corollary 5.2, so, up to notation, the claim in the lemma could have been made as a claim about hull($A$).*

**Proposition 5.9.** *If $A \subseteq \mathbb{Z}^n$ is a $\Lambda$-finite set for some antichain lattice $\Lambda \subseteq \mathbb{Z}^n$, then every face of conv($E_t(A)$) is a polyhedron.*

*Proof.*

It is clear that $\text{conv}(E_t(A))$ is the intersection of half-spaces by the definition of convexity, so it remains to show that each face is the convex hull of finitely many points. If $\text{conv}(E_t(A))$ had a supporting hyperplane that contained infinitely many points, that would imply the existence of a hyperplane containing infinitely many points of $A$. The only such hyperplanes are those that are parallel to $\Lambda$ and that contain $\alpha_0 + \Lambda$ for some $\alpha_0 \in A$. But by Theorem 9.14 of [10], these collections of points are mapped to locally finite sets under the exponentiation map, and hence no supporting hyperplane of $\mathrm{conv}(E_t(A))$ containing infinitely many points exists. $\square$

### 5.2 Cellular Resolutions Continued

Beginning this section, recall the mapping 4.1 that we applied to $N(A)$ to label the vertices with elements of $\mathbb{Z}^n$. We return to this labeling now.

**Definition 5.10.** *Let $\Delta$ be a simplicial complex with labels from $\mathbb{Z}^n$, and let $\Delta_i = \{i\text{-faces of } \Delta\}$. Let $S = k[x_1, \dots, x_n]$ and $S(e_\sigma)$ be the free $S$-module generated by $e_\sigma$. The Taylor Complex supported on $\Delta$ is*

$$\mathcal{F}_\Delta : \cdots \xrightarrow{d_{i+1}} \mathcal{F}_i \xrightarrow{d_i} \cdots \xrightarrow{d_1} \mathcal{F}_0 \xrightarrow{d_0} 0$$

*where*

$$\mathcal{F}_i = \bigoplus_{\sigma \in \Delta_i} S(e_\sigma)$$

*and if $\sigma = \{\alpha_0, \dots, \alpha_i\}$, and $\partial_j \sigma = \{\alpha_0, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_i\}$,*

$$d(e_\sigma) = \sum_{\alpha_j \in \sigma} (-1)^{j-1} (X^{\vee\sigma - \vee\partial_j\sigma}) e_{\partial_j \sigma}.$$

**Remark 5.11.** *In [10], the Taylor complex is defined on a simplicial complex $\Delta$ with $\Delta_0$ finite, but this restriction is not needed.*

**Definition 5.12.** *The Taylor resolution of $A \subseteq \mathbb{Z}^n$ is the Taylor complex supported on the simplicial complex that is full over $A$. I.e., the faces of the simplicial complex are in bijection with the finite subsets of $2^A$.*

**Definition 5.13.** *Suppose $N(A)$ is the Scarf complex of $A$ (Definition 2.14). The algebraic Scarf complex, denoted $\mathcal{F}_{N(A)}$, is the Taylor complex supported on $N(A)$.*

**Definition 5.14.** *The labeled algebraic Scarf complex is the algebraic Scarf complex labeled by mapping 4.1. We will denote the labeled algebraic Scarf complex $N(A)$, identically to the Scarf complex.*

**Remark 5.15.** *The Scarf complex is a simplicial complex, and the algebraic Scarf complex is that complex coupled with a collection of maps.*

**Proposition 5.16.** *If $A \subseteq \mathbb{Z}^n$, then every free $S$-resolution of $M_A$ contains the algebraic Scarf complex $\mathcal{F}_{N(A)}$ as a subcomplex.*

*Proof.*

The Taylor resolution is an $S$-resolution of $M_A$. By [11], it must contain a minimal resolution. Call that minimal resolution $\mathcal{F}_\bullet$. By definition, $\mathcal{F}_\bullet$ must contain all relations of $M_A$ in all dimensions. Additionally, the Taylor resolution contains the Scarf complex by construction, which in turn contains relations of $M_A$ without repitition. Since the Scarf complex does not necessarily contain all relations, it is a subcomplex of $\mathcal{F}_\bullet$. □

**Theorem 5.17.** *If $A \subset \mathbb{Z}^n$, then $\mathcal{F}_{N(A)}$ is a subcomplex of hull$(A)$.*

*Proof.*

Let $\sigma \subset A$ be a face of the Scarf complex. Then $\sigma$ is strongly neighborly. We wish to relabel the elements of $\sigma$ in a meaningful way. To do this, consider $i \in [p]$ and let

$$J(i) = \{j \in [n] \mid \pi_j(\vee\sigma \setminus i) < \pi_j(\vee\sigma)\}$$

Notice that $J(i)$ is nonempty, because if it was empty, then $\alpha_i$ would not contribute to $\bigvee \sigma$, and hence $\sigma$ could not be neighborly because $\bigvee \sigma = \bigvee(\sigma \setminus \alpha_i)$. Additionally, for similar reasons, $J(i) \nsubseteq \bigcup_{k \neq i} J(k)$. Therefore, for each $i \in [p]$, there is a $j = j(i) \in [n]$ such that $\alpha_i$ contributes to $\bigvee \sigma$ in component $j(i)$ and no other element of $\sigma$ does. Now for each $\alpha_i \in \sigma$, choose such a $j(i)$, and relabel $\alpha_i$ as $\alpha_{j(i)}$. Then $\pi_i(\alpha_i) > \pi_i(\alpha_k)$ for all $k \neq i$.

The second step of the proof is that $\{t^{\pi_i(\alpha_k)}\}$ is a nonsingular matrix for large enough $t$. It suffices to show this by showing that for large enough $t$,

$$\prod_{i=1}^{p} t^{\pi_i(\alpha_i)} > p! \prod_{i=1}^{p} t^{\pi_i(\alpha_{\rho(i)})} \tag{$*$}$$

for any non-identity permutation $\rho$ of $[p]$. If $(*)$ is satisfied, then the term $\prod_{i=1}^{p} t^{\pi_i(\alpha_i)}$ will dominate all other terms $\det(\{t^{\pi_i(\alpha_k)}\})$, and hence the matrix will be nonsingular. Assume $t > p$, then

$$\frac{\prod_{i=1}^{p} t^{\pi_i(\alpha_i)}}{\prod_{i=1}^{p} t^{\pi_i(\alpha_{\rho(i)})}} = \prod_{i=1}^{p} t^{\pi_i(\alpha_i)-\pi_i(\alpha_{\rho(i)})} \geq \prod_{i=1}^{p} t \geq \prod_{i=1}^{p} p = p^p > p!$$

Therefore, $(*)$ is satisfied for all non-identity permutations $\rho$.

This says that the points $\{t^{\alpha_1}, \ldots, t^{\alpha_p}\}$ are affinely independent. Because they are affinely independent, the convex hull of the points forms a simplex in which every point is a vertex.

By definition, $\mathrm{hull}(A)_{\preceq \vee\sigma}$ is exactly the convex hull of $\{t^{\alpha_1}, \ldots, t^{\alpha_p}\}$. Because $\sigma$ is (strongly) neighborly, there is no other subset of $A$ that has the same supremum as $\sigma$. As such, if a face of $\mathrm{hull}(A)$ is labeled with $x^{\vee\sigma}$, it necessarily came from the image of $\sigma$, and since the exponential map is injective, there can be only one such face. Proposition 5.16 says that every free resolution contains the algebraic Scarf complex as a subcomplex. This tells us that in addition to there being at most one face with label $x^{\vee\sigma}$, there also must be at least one. Therefore, every strongly neighborly set of $A$ is present as a face in $\mathrm{hull}(A)$. $\square$

**Theorem 5.18.** *If $A \subset \mathbb{Z}^n$ is a generic $\Lambda$-finite set for some lattice $\Lambda \subset \mathbb{Z}^n$ that intersects $\mathbb{N}^n$ only at 0, then $\mathcal{F}_{N(A)} = hull(A)$.*

We need a lemma to prove the theorem.

**Lemma 5.19.** *If $A \subset \mathbb{Z}^n$ is a generic $\Lambda$-finite set for some lattice $\Lambda \subset \mathbb{Z}^n$ that intersects $\mathbb{N}^n$ only at 0, and $F$ is a face of hull($A$), then for every $\alpha \in A$, there is a component $\pi_j(\alpha)$ such that $\pi_j(\alpha) \geq \pi_j(\vee F)$.*

*Proof.*

The analogous statement in [10], Lemma 6.14 has a finite $A \subset \mathbb{N}^n$, but the hypothesis is never used, and the proof runs identically for infinite $A \subset \mathbb{Z}^n$. □

*Proof.* [Theorem 5.18]

Let $F$ be a face of hull($A$) and let $\{\alpha_1, \ldots, \alpha_p\} \subset A$ be the points that correspond to the vertices of $F$. Without loss of generality, we can assume that $\pi_i(\bigvee_j \alpha_j) \neq 0$. For a contradiction, assume that $\{\alpha_1, \ldots, \alpha_p\}$ is not a face of $N(A)$. This could occur in two cases:

1. There exists $k \in \{1, \ldots, p\}$ such that $\bigvee_{j \neq k} \alpha_j = \bigvee_j \alpha_j$.

2. There exists $\beta \in A$ such that $t^\beta \notin F$ and $\beta < \bigvee_j \alpha_j$. (I.e., $\bigvee_j \alpha_j = \beta \vee \bigvee_j \alpha_j$.)

For the first case, if we apply Lemma 5.19 to $\alpha_k$, then there exists a $j$ such that $\pi_j(\alpha_k) = \pi_j(\bigvee_j \alpha_j)$, and hence there is an element $\alpha_\ell$ such that $\pi_j(\alpha_k) = \pi_j(\alpha_\ell)$. Since $A$ is generic, there exists $\gamma \in A$ such that $\gamma << \alpha_k \vee \alpha_\ell$, and hence $\gamma \leq \bigvee_j \alpha_j$, contradicting Lemma 5.19.

In the second case, if we assume we are distinct from the first case, then for any $\alpha_k \in \{\alpha_1, \ldots, \alpha_p\}$, there exists $j$ such that $\pi_j(\alpha_k) = \pi_j(\bigvee_i \alpha_i) \geq \pi_j(\beta)$. If the inequality is equality, then by genericity, there exists $\beta' << \bigvee_i \alpha_i$, which is a contradiction to Lemma 5.19 again, so we have a strict inequality. Having a strict inequality means that $\beta << \bigvee_i \alpha_i$, again contradicting Lemma 5.19.

In both cases, we reached contradictions, and hence every face of hull($A$) is a face of the Scarf complex. Coupled with Theorem 5.17, we have that hull($A$) $= N(A)$.

□

**Corollary 5.20.** *If $A \subset \mathbb{Z}^n$ is a generic $\Lambda$-finite set for some antichain lattice $\Lambda \subset \mathbb{Z}^n$, then $\mathcal{F}_{(N(A))}$ minimally resolves $M_A$ as an $S$-module.*

*Proof.*

We already have that hull$(A)$ resolves $M_A$, and Theorems 5.17 and 5.18 together give us that $\mathcal{F}_{(N(A))}$ also resolves it. The resolution is minimal because no two faces of $N(A)$ have the same degree. □

# Chapter 6: Different Module Structures

Currently, we are operating under the condition that $A \subset \mathbb{Z}^n$ is a generic $\Lambda$-finite set such that $\Lambda \subset \mathbb{Z}^n$ and $\Lambda \cap \mathbb{N}^n = \{0\}$. With these assumptions, we have constructed a minimal free resolution of the $S$-submodule $M_A = \{\sum_\alpha c_\alpha X^\alpha\}$ of the Laurent polynomial ring $S^{\pm} = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. The minimal free resolution we constructed, namely the algebraic Scarf complex of $A$ may only have finitely many nonzero dimensions, but in most dimensions, the module is infinitely generated. That is,

$$(\mathcal{F}_{N(A)})_i = \bigoplus_{\sigma \in N_i(A)} S e_\sigma$$

is nonzero for only finitely many $i$, but for the $i$'s for which it is nonzero, there are typically infinitely many $\sigma \in N_i(A)$.

An underlying structure that we haven't utilized much hitherto is the grading on $S$, and hence on the $S$-modules.

### 6.1 Gradings on S

**Definition 6.1.** *Let $M$ be a monoid with operation $+$. A ring $R$ is $M$-graded if $R = \bigoplus_{i \in M} R_i$, and $R_i R_j \subseteq R_{i+j}$. Similarly, an $R$-module $N$ is $M$-graded if $N = \bigoplus_{i \in M} N_i$, and $N_i R_j \subseteq N_{i+j}$.*

The polynomial ring $S = k[x_1, \ldots, x_n]$ is graded by $\mathbb{N}^n$, and hence all the $S$-modules we have seen have also been graded by $\mathbb{N}^n$. Because of this grading, and our ability to associate any monomial in $S$ to a vector in $\mathbb{N}^n$, it will be helpful at times to consider $S$ as the monoid algebra $k[\mathbb{N}^n]$. This notation will be used when considering gradings other than the $\mathbb{N}^n$-grading. There has been a second grading present for many examples that we have yet to consider: the $\Lambda$-grading.

Consider the rings

$$S[\Lambda] = k[\mathbb{N}^n][\Lambda] = \{\sum_{\alpha,\lambda} c_{\alpha\lambda} X^\alpha z^\lambda \mid c_{\alpha\lambda} \in k \text{ finitely non-zero }, \alpha \in \mathbb{N}^n, \lambda \in \Lambda\}$$

and

$$k[\mathbb{N}^n + \Lambda] = \{\sum_{\beta} c_\beta X^\beta \mid c_\beta \in k \text{ finitely non-zero }, \beta \in \mathbb{N}^n + \Lambda\}$$

.

**Lemma 6.2.** *Let $A \subset \mathbb{Z}^n$ be a $\Lambda$-finite set such that $\Lambda \subset \mathbb{Z}^n$ and $\Lambda \cap \mathbb{N}^n = \{0\}$. With $M_A = \{\sum_\alpha c_\alpha t^\alpha \mid c_\alpha \in k$ finitely non-zero $, \alpha \in A + \mathbb{N}^n\}$,*

1. *$M_A$ is a $k[\mathbb{N}^n + \Lambda]$-module, with action defined by:*

$$(x^\beta, t^\alpha) \mapsto t^{\alpha+\beta},\ \alpha \in A + \mathbb{N}^n, \beta \in \mathbb{N}^n + \Lambda,$$

   *and linearity.*

2. *$M_A$ is a $S[\Lambda]$-module, with action defined by:*

$$(x^\beta z^\lambda, t^\alpha) \mapsto t^{\alpha+\beta+\lambda},\ \alpha \in A, \beta \in \mathbb{N}^n, \lambda \in \Lambda,$$

   *and linearity.*

3. *The set $\{ t^\alpha \mid \alpha \in A \}$ is a minimal set of generators for $M_A$ as an $S$-module.*

4. *The set $\{ t^\alpha \mid \alpha \in A_0 \}$ is a minimal set of generators for $M_A$ as a $k[\mathbb{N}^n + \Lambda]$-module.*

5. *If $A \subseteq \Lambda + \mathbb{N}^n$ and $A = A + \Lambda + \mathbb{N}^n$, then $M_A$ is an ideal in $k[\mathbb{N}^n + \Lambda]$.*

*Proof.*

1. We have the following equalities that show the result:

(a) $(x^\beta, t^{\alpha_1} + t^{\alpha_2}) \mapsto t^{\alpha_1+\beta} + t^{\alpha_2+\beta} = x^\beta t^{\alpha_1} + x^\beta t^{\alpha_2}$.

(b) $(x^{\beta_1} + x^{\beta_2}, t^\alpha) \mapsto t^{\alpha+\beta_1} + t^{\alpha+\beta_2} = x^{\beta_1} t^\alpha + x^{\beta_2} t^\alpha$.

(c) $(x^{\beta_1} x^{\beta_2}, t^\alpha) \mapsto t^{\alpha+\beta_1+\beta_2} = x^{\beta_1} t^{\alpha+\beta_2} = x^{\beta_1}(x^{\beta_2} t^\alpha)$.

(d) $(1, t^\alpha) \mapsto t^{\alpha+0} = t^\alpha$.

2. Identical to part 1 with the realization that $\beta + \lambda \in A$, and $\alpha + A \in A$.

3. Let $M_A \ni m = \sum c_\alpha t^\alpha$ for finitely many $\alpha \in A + \mathbb{N}^n$. If some $\alpha$ is not in $A$, then there exists an $\eta \in \mathbb{N}^n$ and $\alpha_0 \in A$ such that $\alpha = \alpha_0 + \eta$. Then we have that $c_\alpha t^\alpha = c_\alpha t^{\alpha_0+\eta} = c_\alpha t^\eta t^{\alpha_0}$. But $t^\eta \in S$, so $A$ generates $M_A$ as an $S$-module.

4. Mutatis mutandis with part two, except that now every $\alpha \in A+\mathbb{N}^n$ is written as $\alpha_0 + \lambda + \eta$.

5. It suffices to show that $\alpha + \beta \in A + \mathbb{N}^n$ when $\alpha \in A + \mathbb{N}^n$ and $\beta \in \mathbb{N}^n + \Lambda$. If $\alpha = \lambda_1 + \eta_1$, and $\beta = \lambda_2 + \eta_2$, then $\alpha + \beta = \lambda_1 + \lambda_2 + \eta_1 + \eta_2$, and since $A = A + \lambda + \mathbb{N}^n$, we have that $\alpha + \beta \in A + \mathbb{N}^n$.

$\square$

We have already defined the algebraic Scarf complex to be the Taylor complex supported on $N(A)$. Implicit in this definition was the consideration of the algebraic Scarf complex as a complex of $S$-modules. We have now seen that these modules can be considered as $S[\Lambda]$-modules.

**Definition 6.3.** *If $A \subset \mathbb{Z}^n$ is a $\Lambda$-finite set for $\Lambda \subset \mathbb{Z}^n$ such that $\Lambda \cap \mathbb{N}^n = \{0\}$, then the algebraic Scarf complex supported on $N(A)/\Lambda$ considered as a complex of $S[\Lambda]$-modules is $\mathcal{F}^\Lambda_{N(A)}$.*

A typical free module in $\mathcal{F}^{\Lambda}_{N(A)}$ would be of the form

$$\bigoplus_{\sigma+\Lambda\in N_i(A)/\Lambda} Se_{\sigma+\Lambda}$$

Due to the onerous nature of this notation, we often will refrain from writing out the modules in detail.

## 6.2 The Functor $\_\_ \otimes_{S[\Lambda]} S$

Let $J =< 1 - z^{\lambda} \mid \lambda \in \Lambda >\subseteq S[\Lambda]$ be an ideal, and let $\overline{J}$ be the image of $J$ in $k[\mathbb{N}^n + \Lambda]$.

**Lemma 6.4.** *Let $M$ be an $S[\Lambda]$-module. Then $S \otimes_{S[\Lambda]} M \cong M/JM$.*

*Proof.*

Define $b : S \times M \to M/JM$ by $b(s,m) = sm + JM$. Then $b$ is surjective and $S$-bilinear. Furthermore, $b$ is $S[\Lambda]$-bilinear because $b(x^{\lambda}s,m) = \overline{sm} = b(s,x^{\lambda}m)$. Therefore, $b$ induces an $S$-algebra morphism from $S\otimes_{S[\Lambda]} M$ to $M/JM$, and we can exhibit an inverse. The kernel of the map $m \mapsto 1 \otimes m : M \to S \otimes_{S[\Lambda]} M$ contains $JM$, hence this map induces a morphism $M/JM$ to $S \otimes_{S[\Lambda]} S$. $\square$

Let $A = \Lambda$, under the usual conditions, and consider $M_A \otimes_{S[\Lambda]} S = M_{\Lambda} \otimes_{S[\Lambda]} S$. If $I_A = I_{\Lambda} =< X^{\lambda^+} - X^{\lambda^-} \mid \lambda \in \Lambda >$ as usual, then $I_{\Lambda} = \overline{J} \cap S$, and

$$M_{\Lambda} \otimes_{S[\Lambda]} S \cong k[\mathbb{N}^n + \Lambda]/\overline{J} \cong (S + \overline{J})/\overline{J} \cong S/(\overline{J} \cap S) \cong S/I_{\Lambda}$$

More generally, we can let $M_0$ be the $S$-submodule of $k[\mathbb{Z}^n]$ generated by $\{x^{\alpha} \mid \alpha \in A_0\}$, where $A = A_0 + \Lambda$, as usual. Then notice that if $\alpha \in A$, we can write $\alpha = \alpha_0 + \lambda$ for some $\alpha_0 \in A_0$ and $\lambda \in \Lambda$, and as such, we have that $x^{\alpha} = x^{\alpha_0} - (1-z^{\lambda})x^{\alpha_0}$.

With this representation of $x^\alpha$, we see that $M_A = M_0 + JM_A$. Therefore, we have

$$M_A \otimes_{S[\Lambda]} S \cong M_A/JM_A \cong (M_0 + JM_A)/JM_A \cong M_0(JM_A \cap M_0)$$

This is too general to say much about, so we will make the assumption that $A \subset \mathbb{N}^n + \Lambda$. With this assumption, we have the following useful lemma.

**Lemma 6.5.** *If $A \subset \mathbb{N}^n + \Lambda$, where $\Lambda \subset \mathbb{Z}^n$ and $\Lambda \cap \mathbb{N}^n = 0$, then for any $\alpha \in A$, there are $\alpha_0 \in \mathbb{N}^n$ and $\lambda \in \Lambda$ such that $\alpha = \alpha_0 + \lambda$.*

*Proof.*

Let $\alpha \in A$. Then there exists $\lambda \in \Lambda$ such that $\alpha \in -T_\lambda$. Let $\alpha_0 = \alpha - \lambda$. Then $\alpha_0 \in -T_{\alpha-\lambda} \subseteq \mathbb{N}^n$, completing the proof. □

If we choose a generating set for $A$ that is distinguished by being contained in $\mathbb{N}^n$, then using Lemma 6.2, we have that

$$M_A \otimes_{S[\Lambda]} S \cong M_0(JM_A \cap M_0) = M_0(I_\Lambda \cap M_0) \cong (M_0 + I_\Lambda)/I_\Lambda$$

Therefore, in this case, we can identify $M_A \otimes_{S[\Lambda]} S$ with the monomial ideal of $S/I_\Lambda$ that is generated by

$$\{x^\alpha + I_\Lambda \mid \alpha \in A_0\}$$

Additionally, we have

$$k[\mathbb{Z}^n] \otimes_{S[\Lambda]} S \cong k[\mathbb{Z}^n]/Jk[\mathbb{Z}^n] \cong k[\mathbb{Z}^n/\Lambda]$$

With this last computation, since $M_A$ is an $S$-submodule of $k[\mathbb{Z}^n]$, we make the claim that $M_A \otimes_{S[\Lambda]} S$ is the $S$-submodule of $k[\mathbb{Z}^n/\Lambda]$ generated by the image of $M_0$. The proof of this claim will come as corollary to Theorem 6.6

### 6.3 Categorical Equivalence

Let $\mathcal{A}$ be the category of $S[\Lambda]$-modules with the usual $\mathbb{Z}^n$-grading. Under the functor $\_\_ \otimes_{S[\Lambda]} S$ that we just worked with, the images are $\mathbb{Z}^n/\Lambda$-graded. With this setup, let $\mathcal{B}$ be the category of $\mathbb{Z}^n/\Lambda$-graded $S$-modules.

**Theorem 6.6.** [Theorem 9.17, [10]] *The functor* $\pi(\_\_) = \_\_ \otimes_{S[\Lambda]} S : \mathcal{A} \to \mathcal{B}$ *is an equivalence of categories.*

**Corollary 6.7.** *If* $\mathcal{F}_\bullet$ *is any* $\mathbb{Z}^n$*-graded free resolution of* $M_A$ *over* $S[\Lambda]$*, then* $\pi(\mathcal{F}_\bullet$ *is a* $\mathbb{Z}^n/\Lambda$*-graded free resolution of* $S/I_\Lambda$ *over* $S$*. Moreover,* $\mathcal{F}_\bullet$ *is minimal if and only if* $\pi(\mathcal{F}_\bullet)$ *is minimal.*

**Theorem 6.8.** *For an antichain lattice* $\Lambda \subset \mathbb{Z}^n$*, and a generic* $\Lambda$*-finite set* $A \subset \mathbb{Z}^n$*, the following complexes of* $S$*-modules are isomorphic:*

1. *The algebraic Scarf complex of* $A$*.*
2. *The hull resolution of* $A$*.*

*Additionally, they are minimal free* $S$*-resolutions of* $M_A$*.*

*Proof.*

This theorem is a generalization of Theorem 9.24 from [10]. The machinery is unchanged, but the setting is broader with the same conclusion and identical proof. □

**Corollary 6.9.** *The isomosphism in Theorem 6.8 can be chosen to commute with the* $\Lambda$*-actions and therefore the isomorphism holds for* $S[\Lambda]$*-modules and we have a minimal free* $S[\Lambda]$*-resolution of* $M_A$*.*

*Proof.* This is an identical statement to Corollary 5.2, but with a different application. □

**Corollary 6.10.** *The minimal free resolution of a generic lattice ideal* $I_\Lambda$ *is* $\pi(N(\Lambda))$*.*

## 6.4 Application of the Horseshoe Lemma

To bring everything we have worked on together, we will need the first part of the Horseshoe Lemma.

**Lemma 6.11.** *Suppose given a commutative diagram*

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
\cdots & P_2' \arrow[r, "d_2'"] & P_1' \arrow[r, "d_1'"] & P_0' \arrow[r, "d_0'"] & A' \arrow[r] \arrow[d, "i_A"] & 0 \\
 & & & & A \arrow[d, "\pi_A"] & \\
\cdots & P_2'' \arrow[r, "d_2''"] & P_1'' \arrow[r, "d_1''"] & P_0'' \arrow[r, "d_0''"] & A'' \arrow[r] \arrow[d] & 0 \\
 & & & & 0 &
\end{tikzcd}$$

*where the column is exact and the rows are projective resolutions. Set $P_n = P_n' \oplus P_n''$. Then the $P_n$ assemble to form a projective resolution $P_\bullet$ of $A$.*

In our particular case of using cyclic $S$-modules, all of our modules are free and hence projective. Likewise, our resolutions are free and hence projective. Before we arrive at a situation where we can use the Horseshoe Lemma, we need to verify a few conditions first.

**Lemma 6.12.** *Let $A \subset \mathbb{Z}^n$ be a generic $\Lambda$-finite set for some antichain lattice $\Lambda \subset \mathbb{Z}^n$ such that $A = A_0 + \Lambda$ with $A_0 \subset \mathbb{N}^n$ and $A_0 \neq \{0\}$. If for some minimal Markov basis $B$ of $\Lambda$, $\alpha \in A_0$, $\alpha \not\leq \lambda^+$ and $\alpha \not\leq \lambda^-$ for all $\lambda \in B$, then every minimal generating set of $I_\Lambda + I_{A_0} \subseteq S$ contains*

$$\{X^{\lambda^+} - X^{\lambda^-} \mid \lambda \in L\}$$

*for some minimal Markov basis $L$ of $\Lambda$.*

*Proof.*

Because of Proposition 2.24, we have that a minimal Markov bases is a subset of a finite set of positive and negative pairs of vectors. A minimal Markov basis is any subset of this set that chooses one vector from each pair. As such, the minimal bases only differ by sign patterns, and hence the property $\alpha \in A_0$, $\alpha \nleq \lambda^+$ and $\alpha \nleq \lambda^-$ for all $\lambda \in B$ holds for all Markov bases. The result is that the set of polynomials generated by $\{X^{\lambda^+} - X^{\lambda^-} \mid \lambda \in B\}$ is unaided by any of the monomials independent of the Markov bases. The fundamental theorem of Markov bases (Theorem 2.22) gives us that $\{X^{\lambda^+} - X^{\lambda^-} \mid \lambda \in \Lambda\}$ is minimally generated by a minimal Markov basis, and hence, any minimal generating set of $I_A$ must contain a Markov basis. □

So we have shown that for our generic $\Lambda$-finite sets $A \subseteq \mathbb{Z}^n$, the associated ideals in $S$ can be written as $I_\Lambda + J$.

**Proposition 6.13.** *Let $A \subset \mathbb{Z}^n$ be a generic $\Lambda$-finite set for some antichain lattice $\Lambda \subset \mathbb{Z}^n$ such that $A = A_0 + \Lambda$ with $A_0 \subset \mathbb{N}^n$ and $A_0 \neq \{0\}$. If $I_{A_0} =< X^\alpha \mid \alpha \in A_0 >$ for some choice of $A_0 \subset \mathbb{N}^n$, then the syzygy modules of the minimal free resolution of $I_\Lambda + I_{A_0}$ are submodules of the syzygy modules of $\mathcal{F}_{N(\Lambda)/\Lambda} \oplus \mathcal{F}_{N(A)/\Lambda}$.*

*Proof.*

Consider the exact sequence

$$0 \to I_\Lambda \hookrightarrow I_\Lambda + I_{A_0} \twoheadrightarrow I_{A_0} \to 0$$

By previous arguments, $\mathcal{F}_{N(\Lambda)/\Lambda}$ and $\mathcal{F}_{N(A)/\Lambda}$ are free resolutions of $I_\Lambda$ and $I_{A_0}$, respectively. By the Horseshoe Lemma, there exists maps that can be paired with the syzygy modules of $\mathcal{F}_{N(\Lambda)/\Lambda} \oplus \mathcal{F}_{N(A)/\Lambda}$ that form a resolution of $I_\Lambda + I_{A_0}$. By [11], all graded free resolutions contain a minimal graded free resolution, completing the proof. □

Unfortunately, even though $\mathcal{F}_{N(\Lambda)/\Lambda}$ and $\mathcal{F}_{N(A)/\Lambda}$ minimally resolve the binomial ideal $I_\Lambda \subset S$, and the monomial ideal $I_{A_0} \subseteq S/I_\Lambda$ respectively, the Horseshoe Lemma makes no claim as to the minimality of $\mathcal{F}_{N(\Lambda)/\Lambda} \oplus \mathcal{F}_{N(A)/\Lambda}$ as a resolution. The key to utilizing the Horseshoe Lemma is to understand the maps that are created from the separate resolutions.

## 6.5 Lifting Terms

The proof of the Horseshoe lemma provides a method for defining the new maps of the constructed resolution. In the diagram in Lemma 6.11, the horizontal maps terminating in $A$ are defined first by lifting the map $\epsilon''$ to a map $\overline{\epsilon}'' : P_0'' \to A$, and then defining $i_A \circ \epsilon' \oplus \overline{\epsilon}'' : P_0' \oplus P_0'' \to A$. Once this map is constructed, then the process is iterated. A lifting is defined when we choose a representative of $N_i(A)$ for each $i$.

### 6.5.1 Lifting Terms in $\mathbb{Z}^3$

When working with the syzygy modules of the ideal $I_A = I_\Lambda + I_{A_0}$, we have several symbols that must be handled very carefully. In particular, if we are given a face $F$ of $N(A)$ with dimension $k$, then we have a representative face $F'$ such that $F = F' + \lambda$ for some $\lambda \in \Lambda$. Additionally, each face of $F$ has its own representative that may or may not be a face of $F$. These considerations lead us to the following potential problem. In $N(A)/\Lambda$, we have generators of our modules of the from $e_{\sigma+\Lambda} = e_{\overline{\sigma}}$; in $N(A)$, it would appear that we have generators of the form $e_\sigma$, but that is only true of the representative we chose for the lifting. As such, we need a definition for $e_\sigma$ if $\sigma$ is not a representative.

**Lemma 6.14.** *Let $\Lambda \subset \mathbb{Z}^3$ be an antichain lattice with minimal Markov basis $\{\lambda_i\}$, and let $g \in \Lambda$. Then there exists $\{c_i\} \subset S$ such that*

$$X^{g^+} - X^{g^-} = \sum_i c_i(X^{\lambda_i^+} - X^{\lambda_i^-})$$

*Proof.*

By definition of $I_\Lambda$, if $g \in \Lambda$, then $X^{g^+} - X^{g^-} \in I_\Lambda$, and by the fundamental theorem of Markov bases, $\{X^{\lambda_i^+} - X^{\lambda_i^-}\}$ generates $I_\Lambda$. $\square$

**Definition 6.15.** *Let $\Lambda$ be an antichain lattice in $\mathbb{Z}^3$ with minimal Markov basis $\{\lambda_i\}$, and let $P_0 = (\bigoplus_{\sigma \in A_0} Se_\sigma) \oplus Se_{\lambda_1} \oplus Se_{\lambda_2} \oplus Se_{\lambda_3}$. Let $g \in \Lambda$ such that $X^{g^+} - X^{g^-} = \sum_i c_i(X^{\lambda_i^+} - X^{\lambda_i^-})$ and let $C = \{c_i\}$. If $e_g$ is not a module generator in $P_1$, then we define $e_g(C) = \sum c_i e_{\lambda_i}$.*

**Remark 6.16.** *For all $C \subset S$ satisfying Definition 6.15, if $d_0 : P_0 \to I_A$, then $d_0(e_g(C)) = X^{g^+} - X^{g^-}$. Because of this, we will immediately relax the notational dependence of $e_g$ on $C$.*

**Lemma 6.17.** *Let $A \subset \mathbb{Z}^3$ be a generic $\Lambda$-finite set for a codimension 1 lattice $\Lambda \subset \mathbb{Z}^3$, and $\sigma = \{B + f, C + g\} \subset N_1(A)$ oriented from $B + f$ to $C + g$. If $P_0 = (\bigoplus_{\sigma \in A_0} Se_\sigma) \oplus Se_{\lambda_1} \oplus Se_{\lambda_2} \oplus Se_{\lambda_3}$, $I_A = I_\Lambda + I_{A_0}$, and $d_0 : P_0 \to I_A$, then*

$$d_0(e_\sigma) = X^{S-(C+g)}e_C - X^{S-(B+f)}e_B + X^{S-g^+}e_g - X^{S-f^+}e_f$$

*where $S = (B + f) \vee (C + g)$.*

*Proof.*

The first two terms of the expression are $d'(e_\sigma)$ when we consider $\sigma$ as an element of $N_1(A)/\Lambda$. So we need to show that if we attempted to use this same map for $d(e_\sigma)$, then we would have the second pair of terms of the expression left over.

Computing,

$$dd(e_\sigma) = d(X^{S-(C+g)}e_C - X^{S-(B+f)}e_B) = X^{S-(C+g)}d(e_C) - X^{S-(B+f)}d(e_B)$$

$$= X^{S-g} - X^{S-f} \neq 0$$

Therefore, we need to add an expression to $X^{S-(C+g)}e_C - X^{S-(B+f)}e_B$ such that applying $d$ to that expression will give us $X^{S-g} - X^{S-f}$. That expression is exactly $X^{S-g^+}e_g - X^{S-f^+}e_f$. Applying $d$, we get

$$X^{S-g^+}(X^{g^+} - X^{g^-}) - X^{S-f^+}(X^{f^+} - X^{f^-})$$

$$= X^S - X^{S-g^++g^-} - X^S + X^{S-f^++f^-} = X^{S-g} - X^{S-f}$$

as required. $\square$

**Remark 6.18.** *In Lemma 6.17, even though we were equipped with Definition 6.15, it appears as though we did not use it. This is because if we had replaced $e_g$ with $\sum c_i e_{\lambda_i}$, all the terms would have canceled just as if we had left $e_g$ in the computation. This situation repeats itself often in similar computations, and when unnecessary, we will use the analogues of $e_g$ directly in future computations with the understanding that they are only symbolic.*

Since we are in $\mathbb{Z}^3$, need only have Definition 6.15 and a similar definition for faces to handle all possible cases we might run into.

**Lemma 6.19.** *Let $A \subset \mathbb{Z}^3$ be a generic $\Lambda$-finite set for some codimension 1 antichain lattice $\Lambda \subset \mathbb{Z}^3$ with minimal Markov basis $\{\lambda_i\}$. Let $A_1$ be a set of $\Lambda$-representatives of $N_1(A)$. Suppose $t \in N_1(A)$ with endpoints $B + f$ and $C + g$. Let $t^r \in N_1(A)$ be the representative of $t$ we have chosen such that $t = t^r + h$ for some $h \in \Lambda$ and let $c_i, c_i', d_i, d_i'$ be the coefficients described in Lemma 6.14 for*

*$g-h, g, f-h,$ and $f$, respectively. If $P_1 = (\bigoplus_{\sigma \in A_1} Se_\sigma) \oplus Se_{p_1} \oplus Se_{p_2}$ where $p_1, p_2$ are as in Lemma 3.7, and $P_0 = (\bigoplus_{\sigma \in A_0} Se_\sigma) \oplus Se_{\lambda_1} \oplus Se_{\lambda_2} \oplus Se_{\lambda_3}$ and $d_1 : P_1 \to P_0$, then, symbolically,*

$$d_1(e_t) - d_1(e_{t^r}) = \sum (c_i X^{\vee t^r - (g-h)^+} - c_i' X^{\vee t - g^+} - d_i X^{\vee t^r - (f-h)^+} + d_i' X^{\vee t - f^+}) e_{\lambda_i}$$

*Proof.*

We compute with the understanding that $d(e_t)$ is a symbolic computation.

$$d(e_{t^r}) = X^{\vee t^r - (C+g-h)} e_C - X^{\vee t^r - (B+f-h)} e_B + X^{\vee t^r - (g-h)^+} \sum c_i e_{\lambda_i} - X^{\vee t^r - (f-h)^+} \sum d_i e_{\lambda_i}$$

$$d(e_t) = X^{\vee t - (C+g)} e_C - X^{\vee t - (B+f)} e_B + X^{\vee t - g^+} \sum c_i' e_{\lambda_i} - X^{\vee t - f^+} \sum d_i' e_{\lambda_i}$$

Taking the difference and rearranging, we get

$$(X^{\vee t^r - (C+g-h)} - X^{\vee t - (C+g)}) e_C - (X^{\vee t^r - (B+f-h)} - X^{\vee t - (B+f)}) e_B$$

$$+ \sum (c_i X^{\vee t^r - (g-h)^+} - c_i' X^{\vee t - g^+}) e_{\lambda_i} - \sum (d_i X^{\vee t^r - (f-h)^+} - d_i' X^{\vee t - f^+}) e_{\lambda_i}$$

Notice now that

$$\vee t^r - (C+g-h) = (B+f-h) \vee (C+g-h) - (C+g-h) = (B+f) \vee (C+g) - h - (C+g-h)$$

$$= (B+f) \vee (C+g) - (C+g) = \vee t - (C+g),$$

so the first parenthetical expression is 0, and by an identical computation, the second parenthetical expression is also 0. This leaves us with the desired result $\square$

**Definition 6.20.** *Under the conditions of Lemma 6.19, we define* $d(e_t(\mathcal{B})) = d(e_{t^r}) + \sum b_i d(e_{p_i})$*, where the* $\mathcal{B} = \{b_i\}$*, and the* $b_i$ *satisfy*

$$\sum (c_i X^{\vee t^r - (g-h)^+} - c_i' X^{\vee t - g^+} - d_i X^{\vee t^r - (f-h)^+} + d_i' X^{\vee t - f^+}) e_{\lambda_i} = \sum b_i d(e_{p_i})$$

**Remark 6.21.** *As in Definition 6.17, our qualifying sets* $\mathcal{B}$ *have no bearing on the outcome, and hence we will not refer to the specific coefficient sets.*

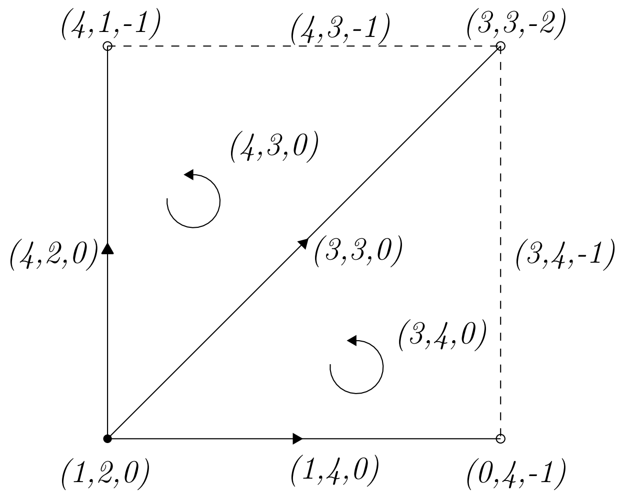
We will call the expressions computed for Definitions 6.15 and 6.20 lifting terms in their respective dimensions.

### 6.5.2 Example

To conclude, we will compute a example using the tools developed here.

**Example 6.22.** *Let* $\Lambda$ *be the lattice generated by* $\{(-1,2,-1),(3,-1,-1)\}$*in* $\mathbb{Z}^3$*, and let* $A_0 = \{\alpha\} = \{(1,2,0)\}$*. A minimal Markov basis of* $\Lambda$ *is* $\{\lambda_1, \lambda_2, \lambda_3\} = \{(-1,2,-1),(3,-1,-1),(-2,-1,2)\}$[1]*. For representatives, we will choose* $A_1 = \{r,s,t\} = \{\{(1,2,0),(4,1,-1)\},\{(1,2,0),(3,3,-2)\},\{(1,2,0),(0,4,-1)\}\}$*, and* $A_2 = \{u,v\} = \{\{(1,2,0),(0,4,-1),(3,3,-2)\},\{(1,2,0),(3,3,-2),(4,1,-1)\}\}$ *with the orientations as listed, and we obtain the following diagram for* $N(A)/\Lambda$ *where the representatives are indicated by solid lines or filled in circles, and the suprema labeled in the appropriate places.*

[1] Markov basis computations can be performed in 4ti2, [1]




*We must first compute the resolution of* $(I_\Lambda + I_{A_0})/I_\Lambda$ *using the coefficients computed from Definition 5.10. For example, the relation associated to the edge* $t$ *is* $xze_{\overline{\alpha}} - y^2 e_{\overline{\alpha}} = (xz - y^2)e_{\overline{\alpha}}$.[2] *The relation associated to the face* $u$ *is* $x^2 e_{\overline{t}} + ze_{\overline{r}} - ye_{\overline{s}}$. *Omitting the details of the remaining computations, we have that the resolution is*

$$\begin{array}{ccccccc}
Se_{\overline{u}} \oplus Se_{\overline{v}} & \to & Se_{\overline{r}} \oplus Se_{\overline{s}} \oplus Se_{\overline{t}} & \to & Se_{\overline{\alpha}} & \to & (I_\Lambda + I_{A_0})/I_\Lambda \\
e_{\overline{u}} & \mapsto & x^2 e_{\overline{t}} + ze_{\overline{r}} - ye_{\overline{s}} & & & & \\
e_{\overline{v}} & \mapsto & ze_{\overline{t}} + ye_{\overline{r}} - xe_{\overline{s}} & & & & \\
& & e_{\overline{t}} & \mapsto & (xz - y^2)e_{\overline{\alpha}} & & \\
& & e_{\overline{r}} & \mapsto & (y - x^3 z)e_{\overline{\alpha}} & & \\
& & e_{\overline{s}} & \mapsto & (z^2 - x^2 y)e_{\overline{\alpha}} & & \\
& & & & e_{\overline{\alpha}} & \mapsto & xy^2 + I_\Lambda
\end{array}$$

*Using the same diagram for the lifting computations, we will again show one example from each dimension. The edge* $t$ *is of the form* $\{\alpha, \alpha + \lambda_1\}$ *oriented from* $\alpha$ *to* $\alpha + \lambda_1$. *Making the substitutions into Lemma 6.17, we have that* $B = \alpha$, $f = 0$

[2] We are making a slight abuse of the diagram here: the diagram should only explicity be used for the resolution of $I_A$, but if we ignore the repeated edges, we can make use of it as a guide for the resolution of $(I_A + I_\Lambda)/I_\Lambda$.

*(consequently, $e_f = 0$), $C = \alpha$, and $g = \lambda_1$. Therefore, our lifted map will be*

$$d_1(e_t) = X^{S-(\alpha+\lambda_1)}e_\alpha - X^{S-\alpha}e_\alpha + X^{S-\lambda_1^+}e_{\lambda_1}$$

$$= xze_\alpha - y^2e_\alpha + xy^2e_\lambda$$

$$= (xz - y^2)e_\alpha + xy^2e_{\lambda_1}$$

*We will show the use of Lemma 6.19 for the face $u$. Notice that the edges $t$ and $s$ are already representatives, so we will only need a lifting term for our tranlsation of the edge $r$. From Lemma 6.19, we have that $f = \lambda_1$, $g = -\lambda_3$, and $h = \lambda_1$. Additionally, we already have computed that $\vee r = (3, 4, -1)$ and $\vee r^r = (4, 2, 0)$. What is left to compute are the $c_is, c_i's, d_is$, and $d_i's$. The three easy cases are $c_i', d_i$, and $d_i'$: $f = \lambda_1$ implies $d_1' = 1$ and $d_2' = d_3' = 0$; $g = -\lambda_3$ implies $c_3' = -1$ and $c_1' = c_2' = 0$; and $f = h$ implies $d_i = 0$ for all $i$. For $g - h$, we need to write $X^{(g-h)^+} - X^{(g-h)^-} = \sum c_i(X^{\lambda_i^+} - X^{\lambda_i^-})$. Since $g - h = \lambda_2$, we have that $c_2 = 1$ and $c_1 = c_3 = 0$.*

*Continuing, we have*

$$d_1(e_r^r) - d_1(e_r) = X^{(3,4,-1)-(0,2,0)}e_{\lambda_1} + X^{(4,2,0)-(3,0,0)}e_{\lambda_2} - X^{(3,4,-1)-(2,1,0)}e_{\lambda_3}$$

$$= x^3y^2z^{-1}e_{\lambda_1} + xy^2e_{\lambda_2} + xy^3z^{-1}e_{\lambda_3}$$

*To use this,*

$$d_2(e_u) = x^2e_t + ze_r - ye_s$$

$$= x^2e_{t^r} + z(e_{r^r} - xy^2(x^2z^{-1}d_1^{-1}(e_{\lambda_1}) + d_1^{-1}(e_{\lambda_2}) - yz^{-1}d_1^{-1}(e_{\lambda_3}))) - ye_{s^r}$$

$$= x^2e_{t^r} + ze_{r^r} - ye_{s^r} - xy^2e_{p_1}$$

*Omitting the remaining similar computations, we have*

$$
\begin{array}{ccccccc}
Se_u \oplus Se_v & \stackrel{d_2}{\rightarrow} & Se_{p_1} \oplus Se_{p_2} \oplus Se_r \oplus Se_s \oplus Se_t & \stackrel{d_1}{\rightarrow} & Se_{\lambda_1} \oplus Se_{\lambda_2} \oplus Se_{\lambda_3} \oplus Se_{\alpha} & \stackrel{d_0}{\rightarrow} & I_A \\
e_u & \mapsto & x^2 e_{t^r} + z e_{r^r} - y e_{s^r} - xy^2 e_{p_1} & & & & \\
e_v & \mapsto & z e_{t^r} + y e_{r^r} - x e_{s^r} - xy^2 e_{p_2} & & & & \\
 & & e_{p_1} & \mapsto & x^2 e_{\lambda_1} + z e_{\lambda_2} - y e_{\lambda_3} & & \\
 & & e_{p_2} & \mapsto & z e_{\lambda_1} + y e_{\lambda_2} - x e_{\lambda_3} & & \\
 & & e_r & \mapsto & xy^2 e_{\lambda_2} - (x^3 - yz) e_{\alpha} & & \\
 & & e_s & \mapsto & xy^2 e_{\lambda_3} - (x^2 y - z^2) e_{\alpha} & & \\
 & & e_t & \mapsto & xy^2 e_{\lambda_1} - (y^2 - xz) e_{\alpha} & & \\
 & & & & e_{\lambda_1} & & \mapsto y^2 - xz \\
 & & & & e_{\lambda_2} & & \mapsto x^3 - yz \\
 & & & & e_{\lambda_3} & & \mapsto x^2 y - z^2 \\
 & & & & e_{\alpha} & & \mapsto xy^2
\end{array}
$$

**Remark 6.23.** *During long computations, such as we have just completed, many small perturbations occur without mention, such as rearranging terms, or moving negative signs around. One notable point from the previous computation was the occurence of $z^{-1}$ during an intermediate step. Although $z^{-1} \notin S$, the end result justified the means, so we choose to ignore the phenomenon.*

# Chapter 7: Conclusion and Future Work

It is clear that this work has laid the foundation for solving the fully general problem: what is the minimal free resolution of an ideal of the form $I_\Lambda + I_{A_0} \subset k[x_1, \ldots, x_n]$. The author believes that this problem will relent sooner rather than later. The computations in the final section were very involved, and once we get a handle on what is happening combinatorially in these translations and lifting factors, we will be able to generate the $d_i$ maps for all $i$.

Furthermore, in the early part of this research, much work was done in three dimensions that lead to very surprising and exciting conjectures about higher dimensions. Because $N(A)$ can be realized as a graph when $A \subset \mathbb{Z}^3$, there are enumerable avenues that can be taken for classification theorems, structure theorems, etc, and there are now at least half a dozen projects that are perfect for undergraduate or begining graduate research that can be made available.

Early computational studies were made of two more general cases: one in which the set of binomial generators of an ideal corresponded to only a partial Markov basis, and one in which the set of binomial generators could be partitioned into partial (or full) Markov bases for different lattices. In the former case in three dimensions, a promising algorithm was developed that lead to a resolution of those ideals. In the latter case, interesting thought exercises lead to generalized combinatorial objects of which the Scarf complex is a specific case; much more work needs to be done with these complicated objects.

Additional future work that has arisen during this research is the possibility of studying Markov bases from a combinatorial standpoint. A search of the literature for Markov bases will yield a plethora of material from algebraic statistics, a large portion of which deals with studying sample sets with incomplete Markov bases.

This is because Markov bases are generally very difficult to compute. Examining them from a combinatorial standpoint may lead to new insights. Personal communication with members of the field have shown that this is an area that is ripe for research.

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# Vita

Trevor McGuire was born in 1984 in Norfolk, VA. He served in the United States Navy after high school, and finished his undergraduate studies at New College of Florida in May, 2009. He earned a master of science degree in mathematics from Louisiana State University in December, 2010. He is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in May, 2014.