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Exponentially Convergent Generalized Finite Element Method for Multi-scale Problems

Xu Huang
Louisiana State University and Agricultural and Mechanical College

EXPONENTIALLY CONVERGENT GENERALIZED FINITE ELEMENT METHOD FOR MULTI-SCALE PROBLEMS

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Xu Huang
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Table of Contents

List of Figures

Abstract

The overall approach I take in the thesis falls into the category of multiscale finite element methods(MsFEM). I work to identify a new class of local approximation spaces with good approximation properties. This is carried out for the equilibrium problem of linear elasticity. The choice of local approximation spaces is motivated by the kolmogorov n-width. Part of my thesis work develops an estimate to show that it is possible to achieve a local approximation error of τ with respect to the energy norm using at most $ln^{d+1}(\frac{1}{\tau})$ local basis functions. The global approximation error τ is controlled by the local approximation error and I recover a global approximation to the actual solution with exponential accuracy.

Chapter 1
Introduction

A broad range of scientific and engineering problems involve multi-scale phenomena. Large disparities in spatial and temporal scales appear in all areas of modern science and engineering, for example, composite materials. The dispersed particles or fibers, which usually are randomly distributed in the matrix, may cause fluctuations in the electrical conductivity or elastic property; moreover, the material properties are usually discontinuous across the phase boundaries.

A complete analysis of these problems is extremely hard. Finding the direct numerical solution of multiple scale problems is computationally expensive and time consuming, even with the advent of supercomputers. A large amount of computer memory and CPU time are required, and this can easily exceed the limit of today's computing resources. In this thesis we take a domain decomposition approach. Hence we seek to carry out local computations in parallel. The results of these local computations are combined through a global computation involving significantly smaller degrees of freedom than the original discrete problem.

The overall approach we take in the thesis falls into the category of multiscale finite element methods (MsFEM). The scheme behind (MsFEM) is to use a finite dimensional space of local solutions to the problem over each element of the coarse mesh instead of using linear or polynomial FE basis functions.

This method has two components to it. The first is to create a local approximation space over each coarse element of the mesh. The second step is to "combine" these elements together. This idea is suggested by the Partition of Unity Method (PUM) for combining local approximations. This approach is analyzed in one dimension in [10] and for higher dimensional problems it is carried out using special local solutions, in [7]. It was generalized broadly in [11], [31], [47], [21] and the citations given there. To fix ideas, let ω be a triangular or quadrilateral element. We have the following three basic choices to for constructing the local function space inside ω.

1. Linear boundary data. Here we generate local shape functions as solutions of the homogeneous multiscale equation inside the element with boundary conditions on $\partial\omega$ associated with linear or bilinear FE. There is a loss of accuracy in this approach because the trace of the exact solution over the boundary of the element is not well approximated by a linear function when modeling heterogeneous media.

2. Improved boundary approximation. In order to improve the boundary approximation over $\partial\omega$ one can first solve a one dimensional problem on every edge of ω with boundary condition on the vertices taken to be the nodal values of FE functions. Typically the equation on each edge of the element is the restriction of the two dimensional problem to that edge. When modeling heterogeneous media, this approach has better accuracy than the first approach.

3. The oversampling method. Let ω^* be a triangle or quadrilateral element and take $\omega \in \omega^*$ to be a concentric element. Here the diameter of ω^* is taken to be twice that of ω. We solve the local problem over ω^* with boundary data prescribed

as traces of the linear FE shape functions on the boundary of ω^*. The local approximation is then obtained on restricting these solutions to ω. This oversampling approach of constructing local bases is presented in [21]. Several variants of these ideas have been applied and developed for multiscale problems and an overview of recent literature is given in [21].

MsFEM can also be interpreted as a domain decomposition method. Here the elements ω appearing in the MsFEM can be understood as a domain decomposition of the computational domain Ω, over which the problem is formulated. The methods 1) and 2) listed above are domain decomposition without overlap. While 3) has overlaps between neighboring domains. Further developments of this approach to multiscale problems and significant generalizations along these lines are given in [2].

We now consider the domain decomposition given by a partition of unity. Here Ω is covered with domains ω_i such that $\cup_i \omega_i = \Omega$. On each ω_i we define functions $\nabla\phi_i \geq 0$ with support inside ω_i. Here we suppose that ϕ_i and ω_i are chosen to give the partition of unity $1 = \sum \phi_i(x)$ for x in Ω. We now describe the multiscale approach based on domain decomposition using the partition of unity. The multiscale approach has two components:

1. Local Approximation. On every ω_i we introduce m dimensional spaces $V_i \subset H^1(\omega_i)$ such that the exact solution u for the multiscale problem can be well approximated by a function $v_i \in V_i$ such that $\|u - v_i\|_{\varepsilon(\omega_i)} \leq \varepsilon_i$.

2. Construction of an $H^1(\Omega)$ global approximation from local approximations. Here we paste the functions v_i together to construct a "continuous" function v belonging to $H^1(\Omega)$ given by

$$v = \sum_i \phi_i v_i \tag{1.1}$$

and

$$\sum_i \|u - v\|^2_{\varepsilon(\omega_i)} \leq C \sum \varepsilon_i^2 \tag{1.2}$$

where $\|\cdot\|_{\varepsilon(\omega_i)}$ is the energy norm over the element ω_i and C is independent of u and v_i.

As in the case of MsFEM one must also address the dual issues of finding accurate local approximations and the problem of combining these in an appropriate way to obtain a global approximation to the solution u. Here the local functions are pasted according to the partition of unitity to form a global function $v = \sum_i \phi_i v_i$ using overlapping elements.

The Generalized Finite Element Method (GFEM) described above can be thought of as an overlapping domain decomposition method. Here global approximation are obtained by pasting together local approximations through a partition of unity. Partition of Unity Methods (PUM) originated in [7] and were further extended and analyzed in [5],[31],[11],[3]. It was applied to multiscale problems in [4],[47],[44],[42],[24]. As described above the GFEM basis is constructed by partitioning the computational domain Ω into to a collection of preselected patches $\omega_i, i = 1, 2, ...m$ and constructing finite dimensional approximation spaces Ψ_i over each patch using local information. Since each space Ψ_i is computed independently, the full "global" solution is obtained by solving a global (macro) system which is an order of magnitude smaller than the system corresponding to a direct application the finite element method to the full structure. This provides an opportunity

for the significant reduction of the computational work involved in the numerical modeling of large heterogeneous problems appearing in composite materials. Several advantages for applying this strategy including structural composites are listed below:

1. Solution of a global problem with drastically reduced degrees of freedom.
2. Independent local mesh generation versus the generation of a globally defined mesh.
3. Completely independent parallel computation of local problems.
4. Ability to handle multiple right hand sides.

Crucial to the success of this method is the ability to construct a low dimensional local approximation space with good approximation properties.

In this thesis, we work to identify a new class of local approximation spaces with good approximation properties. This is carried out for the equilibrium problem of linear elasticity. The choice of local approximation spaces is motivated by the Kolmogorov n-width see, e.g., [6]. Consider any finite dimensional subspace $S(n)$ of a Banach Space B. The Kolmogorov n-width measures the relative error in approximating functions from the Banach Space by elements of $S(n)$ see, e.g., [38]. The results of this thesis provide a generalization of the results presented in [6] to linear elastic systems. The work reported here has been published in [9].

Here this concept is applied to identify and prove the existence of spectrally defined optimal local approximation spaces. In doing so, an explicitly defined optimal

local approximation space can be developed for this problem. Part of my thesis work develops an estimate to show that it is possible to achieve a local approximation error of τ with respect to the energy norm over the patch ω_i using at most $(\ln \tau^{-1})^{d+1}$ local basis functions. In fact since GFEM is a Galerkin method, the global approximation error τ is controlled by the local approximation error and we recover a global approximation to the actual solution with exponential accuracy. This type of approach is called Multiscale Spectral GFEM. See [6], [9].

Chapter 2

Mathematical Formulation for the GFEM

The over all methodology formulated here is quite general and applies to mathematical formulations of linear elasticity described by measurable tensor valued coefficients. Such generality is required for the development of mathematically rigorous solution strategies. On the other hand the machine computation of displacement fields inside engineering materials requires a precise description of the heterogeneous material properties. We begin with the general formulation of the equilibrium problem for an anisotropic heterogeneous linearly elastic medium and then specialize our treatment to two dimensional plane strain problems for unidirectional fiber reinforced composites.

Let $\Omega \in \mathbb{R}^d$, $d = 2, 3$ be a bounded domain with Lipschitz boundary $\partial\Omega$. We start by formulating the problem for the system of linear elasticity used for the determination of elastic displacement fields $\mathbf{u} : \Omega \mapsto \mathbb{R}^d$. The equilibrium equation of linear elasticity is given by

$$-div(\mathbb{A}(\mathbf{x})e(\mathbf{u}(\mathbf{x}))) = f(\mathbf{x}),\ \mathbf{x} \in \Omega \tag{2.1}$$

Where $e(\mathbf{u})$ is the elastic strain and is the symmetric part of the gradient of the displacement $\nabla\mathbf{u}$ given by $e(\mathbf{u}) = (\nabla\mathbf{u} + \nabla\mathbf{u}^T)/2$. The elasticity tensor of an anisotropic heterogeneous medium is characterized by a measurable tensor valued field $\mathbb{A}_{ijkl}(x) \in L^\infty(\Omega)$ for $i, j, k, l = 1, d$. $\mathbb{A}_{ijkl} = \mathbb{A}_{ijlk} = \mathbb{A}_{jikl} = \mathbb{A}_{klij}$. We suppose the tensor satisfies the standard coercivity and boundedness conditions for any

symmetric $d \times d$ matrix e

$$\alpha|e|^2 \leq \mathbb{A}e : e \leq \beta|e|^2, \mathbf{x} \in \Omega \tag{2.2}$$

where $\mathbb{A}e : e = \mathbb{A}_{ijkl}e_{ij}e_{kl}$, and $|e|^2 = \sum e_{ij}^2$. For this choice of elasticity coefficient the solution is sought in the Sobolev space $H^1(\Omega; \mathbb{R}^d)$ and the right-hand side (the body force) lies in the dual space $H^1(\Omega; \mathbb{R}^d)^*$.

The problem formulation is completed on prescribing traction and (or) displacement conditions on the boundary of the domain $\partial\Omega$. The traction boundary condition is given by

$$\mathbf{n} \cdot \mathbb{A}e(\mathbf{u}) = \mathbf{g}, \ \mathbf{x} \in \partial\Omega \tag{2.3}$$

where $\mathbf{n}$ is the outer unit normal vector. Here $\mathbf{g} \in H^{-1/2}(\partial\Omega; \mathbb{R}^d)$, $\int_{\partial\Omega} \mathbf{g} ds = 0$, and the weak solution $\mathbf{u} \in H^1(\Omega; \mathbb{R}^d)$ exists and is unique up to a rigid motion. Here the linear space of rigid motions is given by

$$\mathcal{R} = \{\mathbf{a} + \mathbf{b} \wedge \mathbf{x}; \ \mathbf{a} \text{ and } \mathbf{b}, \text{ in } \mathbb{R}^d\}. \tag{2.4}$$

If in addition a displacement boundary condition is prescribed on part of the boundary $\Gamma \subset \partial\Omega$, i.e., $\mathbf{u} = \mathbf{U} \in H^{1/2}(\Gamma; \mathbb{R}^d)$ then the solution exists and is uniquely specified.

The solution of a particular physical problem requires the specialization of the general formulation to the case at hand. Here we focus on the physical problem of calculating stresses and strains inside a uni-directional carbon fiber epoxy resin composite. This type of structural composite is commonly used in commercial aircraft and wind turbines primarily due to its high specific stiffness and strength. The principle objective of this paper is to describe the problem of machine computation of local fields inside engineering composite systems. The goal is to highlight

the issues and problems related to machine computation of local fields inside structural composites typically requiring 10^8 degrees of freedom per square centimeter of fibrous composite material.

To fix ideas we formulate a deterministic two dimensional elasticity problem for a uni-directional fiber reinforced composite.

In the two dimensional formulation we make the idealization and assume that the fiber cross sections are circular. See Figure 2.1. In actuality the fibers are more like deformed cylinders.

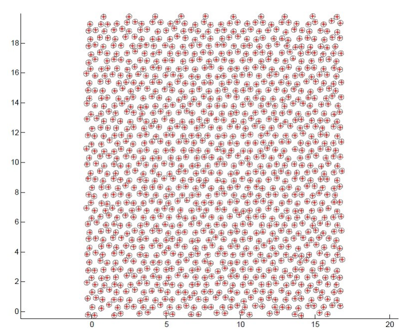

FIGURE 2.1. Composite cross section

Here we emphasize that the fiber distribution is not periodic, so that the theory and computational analysis based on the assumption of a periodicity or almost periodicity characterized by an elastic tensor field of the type $\mathbb{A}(\mathbf{x}, \mathbf{x}/\varepsilon)$ where the tensor field $\mathbb{A}$ is smooth in the first variable periodic and oscillatory in the second variable cannot be used.

We will consider the square domain $\Omega = (-\kappa, \kappa) \times (-\kappa, \kappa)$ where κ is dimension in meters denoted by m. The boundary of Ω is denoted $\partial\Omega$. We have $\mid \Omega \mid = 4\kappa^2[m]^2$. We assume that the fiber volume fraction is W$\simeq$50% . The fiber cross sections are

circular with diameters given by $d = 5\mu m$. Here $1\mu m = 10^{-6}m$. The fibers provide structural stiffness and their stiffness greatly exceeds the matrix. The domain Ω contains approximately $(2.10^5\kappa)^2$ fibers. As indicated we assume that the locations and the diameters of the fibers are known from image data. Last we assume that no fiber is touching the boundary of the sample $\partial\Omega$.

For this case the elasticity tensor field is piecewise constant and of the form

$$\mathbb{A}_{ijkl}(\mathbf{x}) = \frac{E(\mathbf{x})}{2(1-\nu(\mathbf{x}))}\delta_{ij}\delta_{kl} + \frac{E(\mathbf{x})}{2(1+\nu(\mathbf{x}))}\{\delta_{il}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl}\} \tag{2.5}$$

where

$$\begin{aligned} E(\mathbf{x}) = &\ 24Gpa \text{ for } \mathbf{x} \text{ in the fibers, } 3.6Gpa \text{ for } \mathbf{x} \text{ in the epoxy matrix} \\ \nu(\mathbf{x}) = &\ 0.24 \text{ for } \mathbf{x} \text{ in the fibers, } 0.3 \text{ for } \mathbf{x} \text{ in the epoxy matrix} \end{aligned} \tag{2.6}$$

The mathematical formulation of the two dimensional plane strain problem is given by boundary value problem for the Lame differential equation

$$L(\mathbf{u}(\mathbf{x})) = -div(\mathbb{A}(\mathbf{x})e(\mathbf{u}(\mathbf{x}))) = \mathbf{f}(\mathbf{x}), \tag{2.7}$$

with boundary conditions given by traction in units of GPa or displacement in units of meters m. Here $\mathbf{x} = (x_1, x_2)$, $\mathbf{u} = (u_1, u_2)$, where both quantities are expressed in units of meters m. The elasticity tensor field $\mathbb{A}(\mathbf{x})$ defines the isotropic elastic properties of the matrix and the fibers. It is evident that the tensor $\mathbb{A}(\mathbf{x})$ is varies rapidly across the structure and hence is very "rough". The two dimensional problem of plane strain is intended to model a section of cylindrical fiber composite of unit thickness $d = 1\ m$. The assumption that the fibers are represented by cylinders is an idealization. However it serves to illustrate the ideas and challenges behind machine computation of fields inside complex media.

Remark 2.1. *Here we will assume that the location of the fibers are known exactly. Although the position of the fibers shown in Figure 1 are determined from image data, it is not possible to map the position of all the fibers in a typical structural composite. Instead the image data may be used in a stochastic characterization of the fiber distribution within the structural composite. However in this paper we consider a deterministic description of fiber positions and view it as necessary preparation for stochastic formulations of local structure in heterogeneous media.*

The mathematical formulation of our physical problem is a boundary value problem for the strongly elliptic system given by the Lame equation 2.7. The weak solution is formulated in the standard variational way. We introduce the "energy" bilinear form

$$B(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \mathbb{A}e(\mathbf{u}) : e(\mathbf{v}) d\mathbf{x}, \mathbf{u}, \mathbf{v} \in H^1(\Omega; \mathbb{R}^2), \tag{2.8}$$

and the energy norm

$$\|\mathbf{u}\|_{\mathcal{E}(\Omega)} = (B(\mathbf{u}, \mathbf{u}))^{1/2} = \mathbf{E}(\mathbf{u})^{1/2}, \text{ in the units } [GPa]^{1/2}[m]^{3/2} \tag{2.9}$$

where $\mathbf{E}(\mathbf{u}, \mathbf{u})$ is the energy. We define the energy space given by the quotient space $H^1(\Omega; \mathbb{R}^d)/\mathcal{R}$ equipped with the energy norm $\mathcal{E}(\Omega)$. In what follows we will write $B(\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \mathbf{v})_{\mathcal{E}(\Omega)}$.

Let $F_1(\mathbf{v}) = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{g} \, d\mathbf{x}$ be the functional of the tractions and $F_2(\mathbf{v}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}$ be the load functional. We assume that the natural consistency condition between F_1 and F_2 given by

$$F_1(\mathbf{v}) + F_2(\mathbf{v}) = 0, \text{ for all } \mathbf{v} \in \mathcal{R} \tag{2.10}$$

is satisfied.

The elastic displacement field $\mathbf{u}_0$ in Ω is the solution of the problem, $\mathbf{u}_0 \in H^1(\Omega;\mathbb{R}^d)/\mathcal{R}$,

$$B(\mathbf{u}_0, \mathbf{v}) = F(\mathbf{v}) = F_1(\mathbf{v}) + F_2(\mathbf{v}), \forall \mathbf{v} \in H^1(\Omega;\mathbb{R}^d), \tag{2.11}$$

and is uniquely specified up to a rigid motion.

If additionally on $\Gamma \subset \partial\Omega$ the boundary condition is $\mathbf{u} = \mathbf{u}_\Gamma$ then the unique solution $\mathbf{u}_0 \in \mathcal{E}(\Omega), \mathbf{u}_0 = \mathbf{u}_\Gamma$ satisfies $B(\mathbf{u}_0, \mathbf{v}) = F(\mathbf{v}), \forall \mathbf{v} \in H^1(\Omega;\mathbb{R}^d)$, $\mathbf{v} = 0$ on Γ.

If $F_2 = 0$ then $\mathbf{u}_0$ is the $\mathbb{A}$-harmonic function satisfying $B(\mathbf{u}_{0,}\mathbf{v}) = 0, \forall \mathbf{v} \in C_0^\infty(\Omega)$

The basic goal of this paper is to formulate and develop a multi-scale and highly parallelisable numerical method for obtaining approximate solutions $\mathbf{u}_{0,appr}$ to elastic deformations inside composite structures that satisfy $\|\mathbf{u}_0 - \mathbf{u}_{0,appr}\|_{\mathcal{E}(\Omega)} \leq \tau \|\mathbf{u}_0\|_{\mathcal{E}(\Omega)}$ where τ is a given tolerance.

Chapter 3

An application: The mathematical formulation of the fiber composite problem

Let $V^{N,m}$ be the $N \times m$ dimensional subspace of $H_0^1(\Omega; \mathbb{R}^d)$ given by

$$V^{N,m} = span\{\sum_{i=1}^{m} \phi_i \xi_i; \sum_{i=1}^{m} \phi_i = 1, \xi_i \in V_{\omega_i}\} \tag{3.1}$$

where ϕ_i is a partition of unity subordinate to the covering $\cup_{i=1}^{m} \omega_i = \Omega$, and V_{ω_i} is the finite dimensional span of the $N-$width functions on ω_i.

Let $\omega_i^* \supset \omega_i$ be domains containing ω_i such that for interior ω_i We have $\omega_i^* \supset\supset \omega_i$ and for ω_i such that $\partial\omega_i \cap \partial\Omega_i \neq \emptyset$: $\omega_i^* \supset \omega_i, \partial\omega_i^* \cap \partial\Omega \supset\supset \partial\omega_i \cap \partial\Omega$ and $dist(\partial\omega_i \cap \Omega, \partial\omega_i^* \cap \Omega) > \alpha > 0$, see figure 4.1.

For ω_i within the interior of Ω, the $N-$ width functions are defined by Theorem 4.11.

For the case of boundary domains ω_i. i.e $\partial\omega_i \cap \partial\Omega_i \neq \emptyset$ then the boundary $N-$ width functions are zero on $\partial\omega_i \cap \partial\Omega$ and defined by Theorem 4.17.

Consider the local solutions χ_i of

$$-div(\mathbb{A}e(\chi_i)) = f, \text{ on } \omega_i \tag{3.2}$$

and $\chi_i|_{\partial\omega_i^*} = 0$.

We form $\mathbf{u}_p = \sum_{i=1}^{m} \phi_i \chi_i$. And introduce the convex set given by

$$K^{N,m} = V^{N,m} + \mathbf{u}_p \subset H_0^1(\Omega; \mathbb{R}). \tag{3.3}$$

Elements of $K^{N,m}$ are written as $\mathbf{v}_N^G$ and we seek a Galerkin solution $\mathbf{u}_N^G \in K^{N,m}$ of the variational problem

$$(\mathbf{u}_N^G, \mathbf{v}_N^G - \mathbf{u}_N^G)_{\mathcal{E}(\Omega)} \equiv \int_\Omega \mathbb{A}e(\mathbf{u}_N^G) : e(\mathbf{v}_N^G - \mathbf{u}_N^G)\, dx \tag{3.4}$$

$$(\mathbf{u}_N^G, \mathbf{v}_N^G - \mathbf{u}_N^G)_{\mathcal{E}(\Omega)} = \int_\Omega f \cdot (\mathbf{v}_N^G - \mathbf{u}_N^G),\ \forall \mathbf{v}_N^G \in K^{N,m} \tag{3.5}$$

note here that $\mathbf{v}_N^G - \mathbf{u}_N^G \in V_{N,m}$.

The theory of variational inequalities and convex analysis guarantees the existence and uniqueness of solution expressed in the following theorems.

Theorem 3.2. *There exists a unique solution $\mathbf{u}_N^G \in K^{N,m}$ of problem (3.5).*

Theorem 3.3. *There exists a unique solution $\mathbf{u} \in H_0^1(\Omega; \mathbb{R}^d)$ of the variational problem*

$$(\mathbf{u}, \mathbf{v})_{\mathcal{E}(\Omega)} = \int_\Omega f \cdot \mathbf{v},\ \forall \mathbf{v} \in H_0^1(\Omega; \mathbb{R}^d)\ . \tag{3.6}$$

Next we have a version of Cea's Lemma for Galerkin approximations.

Theorem 3.4.

$$\|\mathbf{u} - \mathbf{u}_N^G\|_{\mathcal{E}(\Omega)} \le \inf_{\mathbf{v}_N^G \in K^{N,m}} \|\mathbf{u} - \mathbf{v}_N^G\|_{\mathcal{E}(\Omega)}. \tag{3.7}$$

Proof. 1st note $\mathbf{v}_N^G - \mathbf{u}_N^G \in V^{N,m} \subset H_0^1(\Omega; \mathbb{R}^d)$.

So from Theorem 3.3 we have

$$(\mathbf{u}, \mathbf{v}_N^G - \mathbf{u}_N^G)_{\mathcal{E}(\Omega)} = \int_\Omega f \cdot (\mathbf{v}_N^G - \mathbf{u}_N^G),\ \forall \mathbf{v}_N^G \in K^{N,m}\ . \tag{3.8}$$

Now subtract 3.8 from 3.5 to get

$$(\mathbf{u}_N^G - \mathbf{u}, \mathbf{v}_N^G - \mathbf{u}_N^G)_{\mathcal{E}(\Omega)} = 0,\ \forall \mathbf{v}_N^G \in K^{N,m}\ . \tag{3.9}$$

Now we write

$$\|\mathbf{v}_N^G - \mathbf{u}\|^2_{\mathcal{E}(\Omega)} = \|\mathbf{v}_N^G - \mathbf{u}_N^G + \mathbf{u}_N^G - \mathbf{u}\|^2_{\mathcal{E}(\Omega)} \tag{3.10}$$

$$= \|\mathbf{v}_N^G - \mathbf{u}_N^G\|^2_{\mathcal{E}(\Omega)} + \|\mathbf{u}_N^G - \mathbf{u}\|^2_{\mathcal{E}(\Omega)} \tag{3.11}$$

$$+ \; 2(\mathbf{v}_N^G - \mathbf{u}_N^G, \mathbf{u}_N^G - \mathbf{u})_{\mathcal{E}(\Omega)}. \tag{3.12}$$

From 3.9 and 3.10 we conclude

$$\|\mathbf{v}_N^G - \mathbf{u}\|^2_{\mathcal{E}(\Omega)} = \|\mathbf{v}_N^G - \mathbf{u}_N^G\|^2_{\mathcal{E}(\Omega)} + \|\mathbf{u}_N^G - \mathbf{u}\|^2_{\mathcal{E}(\Omega)} \tag{3.13}$$

$$\geq \|\mathbf{u}_N^G - \mathbf{u}\|^2_{\mathcal{E}(\Omega)}. \tag{3.14}$$

And Theorem 3.4 is proved. □

Remark 3.5. *More generally, Theorems 3.2 and 3.4 apply for convex sets* $K^{N,m} \subset H_0^1(\Omega;\mathbb{R}^d)$ *of the form* $K^{N,m} = V^{N,m} + \mathbf{u}_p$ *where* $\mathbf{u}_p \in H_0^1(\Omega;\mathbb{R}^d)$

Consider now the Neumann problem and the construction of an appropriate convex set $K^{N,m} \subset H^1(\Omega;\mathbb{R}^m)$. Here we consider boundary domains ω_i of the partition of unity that satisfy $\partial\omega_i \cap \partial\Omega \neq \emptyset$. The local particular solutions χ_i^N satisfy

$$-div(\mathbb{A}e(\chi_i^N)) = 0, \text{ on } \omega_i^* \tag{3.15}$$

$$\mathbf{n} \cdot \mathbb{A}e(\chi_i^N)|_{\partial\omega_i^*} = g \tag{3.16}$$

$$\chi_i^N|_{\partial\omega_i^* \cap \Omega} = 0 \tag{3.17}$$

Here (3.15) and (3.16) are equivalent to

$$\int_{\omega_i^*} \mathbb{A}e(\chi_i^N) : e(\phi)\, d\mathbf{x} = \int_{\partial\omega_i^* \cap \partial\Omega} g \cdot \phi \, ds \tag{3.18}$$

$\forall \phi \in H^1(\omega_i^*; \mathbb{R})$ s.t $\phi|_{\partial\omega_i^* \cap \Omega} = 0$

Now consider the convex set given by $K^{N,m} = V^{N,m} + \mathbf{u}_p$, here $\mathbf{u}_p \in H^1(\Omega; \mathbb{R}^d)$

Again the theory of variational inequalities and convex analysis guarantees the existence and uniqueness of solution expressed in the following theorem.

Theorem 3.6. *There exists a unique solution $\mathbf{u}_N^G \in K^{N,m}$ of the variational problem*

$$(\mathbf{u}_N^G, \mathbf{v}_N^G - \mathbf{u}_N^G)_{\mathcal{E}(\Omega)} = \int_{\partial\Omega} g \cdot (\mathbf{v}_N^G - \mathbf{u}_N^G)\, ds,\ \ \forall \mathbf{v}_N^G \in K^{N,m} \tag{3.19}$$

Theorem 3.7. *Given $g \in H^{-1/2}(\partial\Omega; \mathbb{R}^d)$ such that $\int_{\partial\Omega} g \cdot \mathbf{v}\, ds = 0$ for $\mathbf{v}$ being a rigid translation or rotation. Then there exists $\mathbf{u} \in H^1(\Omega; \mathbb{R}^d)$ s.t.*

$$(\mathbf{u}, \mathbf{v})_{\mathcal{E}(\Omega)} = \int_{\partial\Omega} g \cdot \mathbf{v}\, ds,\ \ \forall \mathbf{v} \in H^1(\Omega; \mathbb{R}^d) \tag{3.20}$$

And as before we have the analogue of Cea's Lemma given by

Theorem 3.8.

$$\|\mathbf{u} - \mathbf{u}_N^G\|_{\mathcal{E}(\Omega)} \leq \inf_{\mathbf{v}_N^G \in K^{N,m}} \|\mathbf{u} - \mathbf{v}_N^G\|_{\mathcal{E}(\Omega)} \tag{3.21}$$

Theorem 3.9 (GFEM Approximation Theorem [2])**.** *Given $\mathbf{u}$ and $K^{N,m}$ as elements (resp. subsets) of $H_0^1(\Omega; \mathbb{R}^d)$ and $H^1(\Omega; \mathbb{R}^d)$. Then for $\mathbf{u}_N^G \in K^{N,m}$*

$$\|\mathbf{u} - \mathbf{u}_N^G\|_{\mathcal{E}(\Omega)} \leq (2k)^{1/2} (\sum_{i=1}^{m} (\frac{c_2}{diam(\omega_i)})^2 \epsilon_i^2 + c_1^2 \epsilon_i^2)^{1/2} \tag{3.22}$$

where local approximation is

$$\|\mathbf{u} - (\mathbf{u}_p|_{\omega_i} + \xi^i)\|_{L^2(\omega_i)} \leq \epsilon_i \tag{3.23}$$

$$\|\mathbf{u} - (\mathbf{u}_p|_{\omega_i} + \xi^i)\|_{\mathcal{E}(\omega_i)} \leq \epsilon_i \tag{3.24}$$

This theorem shows that control of local error can be used to the control of global error.

Chapter 4
Optimal Local Approximation Space

1 Introduction to Optimal Local Approximation Spaces

In this section we introduce optimal local approximation spaces for use in the multi-scale scheme. These spaces are distinguished by their exponential approximation properties. Given the solution $\mathbf{u}_0$ to the global problem and a prescribed tolerance τ we seek to find a local approximation on the patch ω_j. In this section we show that is is possible to find a local approximation $\mathbf{w}$ from the optimal local space Ψ_j of dimension $(\ln \tau^{-1})^{d+1}$, $d = 2, 3$ for which the error satisfies

$$\|\mathbf{u}_0 - \mathbf{w}\|_{\mathcal{E}(\omega_j)} \leq \tau. \tag{4.1}$$

In what follows we will establish the optimal local approximation properties in the general context of heterogeneous media for two and three dimensional elasticity problems characterized by measurable elasticity tensors $\mathbb{A}(\mathbf{x})$ satisfying the coercivity and boundedness conditions 2.2. We will also develop approximation properties for local domains that border the boundary of Ω. Here we will establish exponential approximation results when Ω has reentrant corners and for general Lipschitz domains.

2 Local approximation on the interior

To fix ideas we will assume that the patch ω is a cube of a given side length surrounded by a larger cube ω^*. We will distinguish two cases depending on if the set ω, lies within the interior of Ω or if it intersects the boundary, i.e., $\overline{\omega} \cap \partial\Omega \neq \emptyset$. It will be shown that the overall approach to constructing optimal local approximation spaces for these two cases is the same. We consider concentric cubes $\omega \subset \omega^*$

with side lengths given by σ and $\sigma^* = (1+\rho)\sigma$ respectively. In order to introduce the ideas I suppose first that ω lies in the interior of Ω so that $\omega \subset \omega^* \subset \Omega$.

We shall utilize ω^* to construct a finite dimensional approximation space over ω. For any open subset S of the computational domain Ω we introduce the space of functions $H_{\mathbb{A}}(S;\mathbb{R}^d)$ defined to be the functions in $H^1(S;\mathbb{R}^d)$ that are $\mathbb{A}$-harmonic on S, i.e., $\mathbf{v} \in H^1(S;\mathbb{R}^d)$ and

$$(\mathbf{v}, \varphi)_{\mathcal{E}(S)} = \int_S \mathbb{A}e(\mathbf{v}) : e(\varphi)\, dx, \quad \forall \varphi \in C_0^\infty(S;\mathbb{R}^d). \tag{4.2}$$

Here $H_{\mathbb{A}}(\omega;\mathbb{R}^d)$ and $H_{\mathbb{A}}(\omega^*;\mathbb{R}^d)$ contain local information on the heterogeneities and will be used in the construction of the optimal local basis. Let $\mathcal{R} = \{\mathbf{a} + \mathbf{b} \wedge \mathbf{x};\ \mathbf{a} \text{ and } \mathbf{b}, \text{ in } \mathbb{R}^d\}$ be the linear space of rigid motions. We introduce the quotient of $H_{\mathbb{A}}(\omega^*;\mathbb{R}^d)/\mathcal{R}$ with respect to $\mathcal{R}$ denoted by $H_{\mathbb{A}}(\omega^*;\mathbb{R}^d)/\mathcal{R}$. It is clear that the solution $\mathbf{u}$ lies in this space. We also introduce the subspace $H^0_{\mathbb{A}}(\omega^*;\mathbb{R}^d)$ given by elements of $H_{\mathbb{A}}(\omega^*;\mathbb{R}^d)$ perpendicular to $\mathcal{R}$ with respect to the $L^2(\omega^*;\mathbb{R}^d)$ inner product. Here we recall that $H_{\mathbb{A}}(\omega^*;\mathbb{R}^d)/\mathcal{R}$ equipped with the energy inner product is isometric to $H^0_{\mathbb{A}}(\omega^*;\mathbb{R}^d)$. The closure of smooth functions with compact support in the energy norm is denoted by $H^1_0(\omega^*;\mathbb{R}^d)$ and for future reference we introduce the decomposition of $H^1(\omega^*;\mathbb{R}^d)$ given by

$$H^1(\omega^*;\mathbb{R}^d) = H^0_{\mathbb{A}}(\omega^*;\mathbb{R}^d) \oplus H^1_0(\omega^*;\mathbb{R}^d) \oplus \mathcal{R}.$$

Here $H^0_{\mathbb{A}}(\omega^*;\mathbb{R}^d)$ and $H^1_0(\omega^*;\mathbb{R}^d)$ are orthogonal with respect to the energy inner product on ω^*.

In this method we choose to approximate elements in the space of functions $H_{\mathbb{A}}(\omega^*;\mathbb{R}^d)/\mathcal{R}$ restricted to ω. Let $\mathcal{T} : H_{\mathbb{A}}(\omega^*;\mathbb{R}^d)/\mathcal{R} \to H_{\mathbb{A}}(\omega;\mathbb{R}^d)/\mathcal{R}$ be the restriction operator such that $\mathcal{T}(\mathbf{u})(x) = u(x)$ for all $\mathbf{x} \in \omega$ and $\mathbf{u} \in H_{\mathbb{A}}(\omega^*;\mathbb{R}^d)/\mathcal{R}$.

We will show that this operator $\mathcal{T}$ is compact in the Appendix B. For the scalar problems, this follows immediately from an application of the the Caccioppoli inequality Lemma 4.13 together with the Rellich Kondrachov embedding theorem on ω^* see [6].

Now we approximate by "n" dimensional subspaces $S(n) \subset H_{\mathbb{A}}(\omega;\mathbb{R}^d)/\mathcal{R}$. The accuracy of a particular increasing sequence $\{S(n)\}_{n=1}^{\infty}$ of local approximation spaces is measured by

$$d(S(n),\omega) = \sup_{\mathbf{u}\in H_{\mathbb{A}}(\omega^*;\mathbb{R}^d)/\mathcal{R}} \inf_{\chi\in S(n)} \frac{\|\mathcal{T}\mathbf{u}-\chi\|_{\mathcal{E}(\omega)}}{\|\mathbf{u}\|_{\mathcal{E}(\omega^*)}}. \tag{4.3}$$

A sequence of approximation spaces $\hat{S}(n)$ is said to be optimal if it has an accuracy $d(\hat{S}(n),\omega)$ that satisfies $d(\hat{S}(n),\omega) \leq d(S(n),\omega)$, $n = 1,2,\ldots$, when compared to any other sequence of approximation spaces $S(n)$. The problem of finding the family of optimal local approximation spaces is formulated as follows. Let

$$d_n(\omega,\omega^*) = \inf_{S(n)} \sup_{\mathbf{u}\in H_{\mathbb{A}}(\omega^*;\mathbb{R}^d)/\mathcal{R}} \inf_{\chi\in S(n)} \frac{\|\mathcal{T}\mathbf{u}-\chi\|_{\mathcal{E}(\omega)}}{\|\mathbf{u}\|_{\mathcal{E}(\omega^*)}}. \tag{4.4}$$

Then the optimal family of approximation spaces $\{\Psi_n(\omega)\}_{n=1}^{\infty}$ satisfy

$$d_n(\omega,\omega^*) = \sup_{\mathbf{u}\in H_{\mathbb{A}}(\omega^*;\mathbb{R}^d)/\mathcal{R}} \inf_{\chi\in \Psi_n(\omega)} \frac{\|\mathcal{T}\mathbf{u}-\chi\|_{\mathcal{E}(\omega)}}{\|\mathbf{u}\|_{\mathcal{E}(\omega^*)}}. \tag{4.5}$$

The quantity $d_n(\omega,\omega^*)$ is known as the Kolmogorov n-width of the compact operator $\mathcal{T}$, see [38].

The optimal local approximation space $\Psi_n(\omega)$ for GFEM follows from general considerations. We introduce the adjoint operator $\mathcal{T}^* : H_{\mathbb{A}}(\omega;\mathbb{R}^d)/\mathcal{R} \to H_{\mathbb{A}}(\omega^*;\mathbb{R}^d)/\mathcal{R}$ and the operator $\mathcal{T}^*\mathcal{T}$ is a compact, self adjoint, non-negative operator mapping $H_{\mathbb{A}}(\omega^*;\mathbb{R}^d)/\mathcal{R}$ into itself. We denote the eigenfunctions and eigen-

values of the problem

$$\mathcal{T}^*\mathcal{T}\mathbf{u} = \lambda\mathbf{u} \tag{4.6}$$

by $\{\varphi_i\}$ and $\{\lambda_i\}$ and the optimal subspace Ψ_n is given by the following theorem.

Theorem 4.10. *The optimal approximation space is given by* $\Psi_n(\omega) = span\{\psi_1, \ldots, \psi_n\}$, *where* $\psi_i = \mathcal{T}\varphi_i$ *and* $d_n(\omega, \omega^*) = \sqrt{\lambda_{n+1}}$.

For the case considered here the definitions of $\mathcal{T}$ and $\mathcal{T}^*$ show that the optimal subspace and eigenvalues are given by the following explicit eigenvalue problem.

Theorem 4.11. *The optimal approximation space is given by* $V_\omega = \Psi_n(\omega) = span\{\psi_1, \ldots, \psi_n\}$ *where* $\psi_i = \mathcal{T}\varphi_i$ *and* φ_i *and* λ_i *are the first* n *eigenfunctions and eigenvalues that satisfy*

$$(\varphi_i, \delta)_{\mathcal{E}(\omega)} \quad = \quad \lambda_i(\varphi_i, \delta)_{\mathcal{E}(\omega^*)}, \; \forall \delta \in H_{\mathbb{A}}(\omega^*; \mathbb{R}^d)/\mathcal{R}. \tag{4.7}$$

The next theorem provides an upper bound on the rate of convergence for the optimal local approximation.

Theorem 4.12. Exponential convergence for interior approximations.

For $\epsilon > 0$ *there is an* $N_\epsilon > 0$ *such that for all* $n > N_\epsilon$

$$d_n(\omega, \omega^*) \leq e^{-n^{\left(\frac{1}{1+d} - \epsilon\right)}}. \tag{4.8}$$

The index N_ϵ is constructed explicitly in the proof of Theorem 4.12 given in the next section. Theorem 4.12 shows that the asymptotic convergence rate associated with the optimal approximation space is nearly exponential for the general class of $L^\infty(\Omega)$ coefficients satisfying the coercivity and boundedness conditions 2.2.

3 Proof of exponential decay for interior subdomains

In this section we give the proof of Theorem 4.12. The construction of the local approximation space is done iteratively. We start by introducing the the first n eigenfunctions $\mathbf{v}_i \in H^1(\omega^*; \mathbb{R}^d)$, that are orthogonal to $\mathcal{R}$, in the $L^2(\omega^*; \mathbb{R}^d)$ inner product, of the the eigenvalue problem

$$(\mathbf{v}_i, \mathbf{w})_{\mathcal{E}(\omega^*)} = \lambda_i \int_{\omega^*} \mathbf{v}_i \cdot \mathbf{w} \, dx, \ \forall \ \mathbf{w} \in H^1(\omega^*; \mathbb{R}^d)$$

posed over ω^*, $i = 1, \ldots, n$. The subspace spanned by these functions is denoted by $S_n(\omega^*)$. Next we introduce the span of $\mathbb{A}$ harmonic functions given by

$$W_n(\omega^*) = \text{span}\{\mathbf{w}_i \in H_{\mathbb{A}}(\omega^*; \mathbb{R}^d) : \ w_i = \mathbf{v}_i, \ \text{on } \partial\omega^*, \ i = 1, \ldots n\}. \tag{4.9}$$

The orthogonal projection from $H^1(\omega^*; \mathbb{R}^d)$ onto $H^0_A(\omega^*; \mathbb{R}^d)$ is denoted by $\mathcal{P}^{\mathbb{A}}$. And one readily checks that $W_n(\omega^*) = \mathcal{P}^{\mathbb{A}} S_n(\omega^*)$.

We define the family of approximation spaces $\mathcal{F}_n(\omega, \omega^*)$ given by the restriction of the elements of $W_n(\omega^*)$ to ω. In what follows we first show that $\mathcal{F}_n(\omega, \omega^*)$ is a family of local approximation spaces with a rate of convergence on the order of $n^{-1/d}$, $d = 2, 3$. To show this I introduce a suitable version of the Cacciappoli inequality that bounds functions in the energy norm over any measurable subset $\mathcal{O} \subset \omega^*$ for which $dist(\partial\mathcal{O}, \partial\omega^*) > \delta > 0$ in terms of the L^2 norm over ω^*.

Lemma 4.13. *Let* $\mathbf{u}$ *be* $\mathbb{A}$*-harmonic in* ω^* *and belong to* $L^2(\omega^*;\mathbb{R}^d)\cap H^1_{loc}(\omega^*;\mathbb{R}^d)$*. Then*

$$\|\mathbf{u}\|_{\mathcal{E}(\mathcal{O})} \leq (2\beta^{1/2}/\delta)\|\mathbf{u}\|_{L^2(\omega^*;\mathbb{R}^d)}. \tag{4.10}$$

where β *is defined by equation (2.2).*

The proof is given in the Appendix A.

Next we introduce the approximation theorem associated with the space $W_n(\omega^*)$ given by

Lemma 4.14. *Let* $\mathbf{u} \in H^0_A(\omega^*;\mathbb{R}^d)$ *then there exists a* $\mathbf{v}_\mathbf{u} \in W_n(\omega^*)$ *such that*

$$\begin{aligned}\|\mathbf{u}-\mathbf{v}_\mathbf{u}\|_{L^2(\omega^*;\mathbb{R}^d)} &= \inf_{\mathbf{v}\in W_n(\omega^*)} \|\mathbf{u}-\mathbf{v}\|_{L^2(\omega^*;\mathbb{R}^d)} &(4.11)\\ &\leq C_n\sigma^*\theta_d\alpha^{-1/2}\|\mathbf{u}\|_{\mathcal{E}(\omega^*)}\ d=2,3 &(4.12)\end{aligned}$$

where σ^* *is the side length of the cube* ω^**,* $C_n = n^{-1/d}(1+o(1))$*,* $d=2,3$*. For* $d=2$*,* $\theta_2 = \sqrt{2/(3\pi)}$ *and for* $d=3$*,* $\theta_3 = [(1+4\sqrt{2})/(6\pi^2)]^{1/3}$*.*

Proof. The lemma follows immediately from an upper bound on the quotient

$$T = \sup_{\mathbf{u}\in\in H^0_{\mathbb{A}}(\omega^*;\mathbb{R}^d)} \inf_{\mathbf{w}\in W_n(\omega^*)} \frac{\|\mathbf{u}-\mathbf{w}\|_{L^2(\omega^*;\mathbb{R}^d)}}{\|\mathbf{u}\|_{\mathcal{E}(\omega^*)}}. \tag{4.13}$$

Fix $\mathbf{u}\in H^0_{\mathbb{A}}(\omega^*;\mathbb{R}^d)$ and denote the projection of $\mathbf{u}$ onto $W_n(\omega^*)$ with respect to the energy norm $\|\cdot\|_{\mathcal{E}(\omega^*)}$ by $\mathcal{P}^{\mathcal{E}}\mathbf{u}$. Choosing $\mathbf{w}=\mathcal{P}^{\mathcal{E}}\mathbf{u}$ and noting that $\|(I-\mathcal{P}^{\mathcal{E}})\mathbf{u}\|_{\mathcal{E}(\omega^*)} \leq \|\mathbf{u}\|_{\mathcal{E}(\omega^*)}$ gives the upper bound

$$T \leq \sup_{\mathbf{u}\in H^0_A(\omega^*;\mathbb{R}^d)} \frac{\|(I-\mathcal{P}^{\mathcal{E}})\mathbf{u}\|_{L^2(\omega^*;\mathbb{R}^2)}}{\|(I-\mathcal{P}^{\mathcal{E}})\mathbf{u}\|_{\mathcal{E}(\omega^*)}} = \sup_{\mathbf{u}\in H^0_{\mathbb{A}}(\omega^*;\mathbb{R}^d)\perp W_n(\omega^*)} \frac{\|\mathbf{u}\|_{L^2(\omega^*;\mathbb{R}^2)}}{\|\mathbf{u}\|_{\mathcal{E}(\omega^*)}}. \tag{4.14}$$

Since $W_n(\omega^*) = \mathcal{P}^{\mathbb{A}}S_n(\omega^*)$ it follows that

$$\{\mathbf{u}\in H^0_{\mathbb{A}}(\omega^*;\mathbb{R}^d)\perp \mathcal{P}^{\mathbb{A}}S_n(\omega^*)\} = \{\mathbf{u}\in H^0_A(\omega^*;\mathbb{R}^d)\perp S_n(\omega^*)\},$$

$$\{\mathbf{u}\in H^0_{\mathbb{A}}(\omega^*;\mathbb{R}^d)\perp S_n(\omega^*)\} \subset \{u\in H^1(\omega^*;\mathbb{R}^d)^2 \perp_{L^2} (S_n(\omega^*)\oplus\mathcal{R})\}, \tag{4.15}$$

where $\perp_{L^2}$ in the second line of (4.42) denotes orthogonality with respect to the $L^2(\omega^*;\mathbb{R}^d)$ inner product.

Hence

$$T \leq \sup_{\mathbf{u}\in H^0_{\mathbb{A}}(\omega^*;\mathbb{R}^d)\perp S_n(\omega^*)} \frac{\|\mathbf{u}\|_{L^2(\omega^*;\mathbb{R}^d)}}{\|\mathbf{u}\|_{\mathcal{E}(\omega^*)}} \tag{4.16}$$

$$\leq \sup_{\mathbf{u}\in H^1(\omega^*;\mathbb{R}^d)^2\perp_{L^2}(S_n(\omega^*)\oplus R)} \frac{\|\mathbf{u}\|_{L^2(\omega^*)}}{\|\mathbf{u}\|_{\mathcal{E}(\omega^*)}} = \frac{1}{\sqrt{\mu_{n+1}}}, \tag{4.17}$$

where μ_{n+1} is the largest eigenvalue associated with $S_{n+1}(\omega^*)$. One has the elementary lower bound $\mu_{n+1} \geq \alpha\nu_{n+1}$ where ν_{n+1} is the largest corresponding eigenvalue for $\int_{\omega^*} e(\mathbf{v}_{n+1}) : e(\mathbf{w})\, dx = \nu_{n+1}\int_{\omega^*} \mathbf{v}_{n+1} w\, dx$, $\forall \mathbf{w} \in H^1(\omega^*;\mathbb{R}^d)$. Here we can appeal to the generalization of Weyl's theorem for elastic problems [14] and $\nu_{n+1} = \frac{4\pi}{8(\sigma^*)^{2/3}}(n+1)(1+o(1))$, for $d=2$ and $\nu_{n+1} = [6\pi^2/(1+4\sqrt{2})(\sigma^*)^3/]^{1/3}(n+1)^{2/3}(1+o(1))$ for $d=3$. The required upper bound on T now follows and the theorem is proved.

Now we apply Lemma 4.13 together with Lemma 4.14 to obtain the following convergence rate associated with the family of approximation spaces $\mathcal{F}_n(\omega,\omega^*)$ given by

Theorem 4.15. *Let $\mathbf{u} \in H_{\mathbb{A}}(\omega^*;\mathbb{R}^d)/\mathcal{R}$ then there exists an approximation $\mathbf{v}_u \in \mathcal{F}_n(\omega,\omega^*)$ for which*

$$\|\mathbf{u}-\mathbf{v}_u\|_{\mathcal{E}(\omega)} = \inf_{\mathbf{v}\in\mathcal{F}_n(\omega,\omega^*)} \|\mathbf{u}-\mathbf{v}\|_{\mathcal{E}(\omega)} \leq I(\omega,\omega^*)\, C_n \|\mathbf{u}\|_{\mathcal{E}(\omega^*)} \tag{4.18}$$

where

$$I(\omega,\omega^*) = 4\,\theta_d\,\frac{1+\rho}{\rho}(\beta/\alpha)^{1/2} \quad \textit{and} \quad C_n = n^{-1/d}(1+o(1)),\ d=2,\ 3. \tag{4.19}$$

Proof of Theorem 4.12.

Next we proceed iteratively to construct a family of local approximation spaces with a rate of convergence that is nearly exponential. For any pair of two concentric cubes $Q \subset Q^*$ we define $\mathcal{F}_n(Q, Q^*)$ to be the space given by the restriction of $W_n(Q^*)$ on Q. We suppose that ω^* is of side length σ^*. Let $N > 1$ be an integer and we suppose that ω is of side length σ and $\sigma^* = \sigma(1+\rho)$. Choose ω_j, $j = 1, 2, ...N$ to be the nested family of concentric cubes with side length $\sigma(1+\rho(N+1-j)/N)$ for which $\omega = \omega_{N+1} \subset \omega_N \subset \omega_{N-1} \subset \cdots \subset \omega_1 = \omega^*$. We introduce the local spaces, $\mathcal{F}_n(\omega, \omega_N)$, $\mathcal{F}_n(\omega, \omega_{N-1})$,...,$\mathcal{F}_n(\omega, \omega_1)$. Put $m = N \times n$ and we define the approximation space given by

$$\mathcal{T}(m, \omega, \omega^*) = \mathcal{F}_n(\omega, \omega_1) \oplus \cdots \oplus \mathcal{F}_n(\omega, \omega_N). \tag{4.20}$$

Theorem 4.16. *Let* $\mathbf{u} \in H_{\mathbb{A}}(\omega^*)/\mathcal{R}$ *and* N *be an integer such that* $1 \leq N \leq n^\gamma$, *with* $0 < \gamma < 1/d$. *Then there exists* $\mathbf{z}_u \in \mathcal{T}(m, \omega, \omega^*)$ *such that*

$$\|\mathbf{u} - \mathbf{z}_u\|_{\mathcal{E}(\omega)} \leq \varsigma^N \|\mathbf{u}\|_{\mathcal{E}(\omega^*)} \tag{4.21}$$

and $\varsigma = 4\,\theta_d \, \frac{1+\rho}{\rho} \, N(\beta/\alpha)^{1/2} C_n$, $\; d = 2, 3.$

Proof. In what follows we make the identification $\omega_1 = \omega^*$ and $\omega_{N+1} = \omega$. From Theorem 4.15 we have that there exists $\mathbf{v}_1 \in \mathcal{F}_n(\omega_2, \omega^*)$ such that

$$\|\mathbf{u} - \mathbf{v}_1\|_{\mathcal{E}(\omega_2)} \leq \varsigma \|\mathbf{u}\|_{\mathcal{E}(\omega^*)} \tag{4.22}$$

Suppose next that for $m = 1, \ldots, j$ there are functions $\mathbf{v}_m \in \mathcal{F}_n(\omega_{m+1}, \omega_m)$ such that

$$\|\mathbf{u} - \sum_{m=1}^{j} \mathbf{v}_m\|_{\mathcal{E}(\omega_{j+1})} \leq \varsigma^j \|\mathbf{u}\|_{\mathcal{E}(\omega^*)} \tag{4.23}$$

Applying Theorem 4.15 we see that there exists a $\mathbf{v}_{j+1} \in \mathcal{F}_n(\omega_{j+2}, \omega_{j+1})$ for which

$$\|\mathbf{u} - (\sum_{m=1}^{j} \mathbf{v}_m) - \mathbf{v}_{j+1}\|_{\mathcal{E}(\omega_{j+2})} \leq \varsigma \|\mathbf{u} - \sum_{m=1}^{j} v_m\|_{\mathcal{E}(\omega_{j+1})} \tag{4.24}$$

and the induction step goes through. Choosing $\mathbf{z}_u = \sum_{m=1}^{N} \mathbf{v}_m$ delivers

$$\|\mathbf{u} - \mathbf{z}_u\|_{\mathcal{E}(\omega)} \leq \varsigma^N \|\mathbf{u}\|_{\mathcal{E}(\omega^*)} \tag{4.25}$$

and the theorem follows noting that $\mathbf{z}_u$ belongs to $\mathcal{T}(m, \omega, \omega^*)$.

To finish the proof of Theorem 4.12 we choose N to be the largest integer less than or equal to n^γ. Thus $m \leq n^{\gamma+1}$ and $m^{\frac{1}{\gamma+1}} \leq n$ and it follows that $n^{-\frac{1}{d}} \leq m^{-\frac{1}{d(\gamma+1)}}$, and $N \leq m^{\frac{\gamma}{\gamma+1}}$. On applying these inequalities we obtain

$$\varsigma^N \leq \exp\left\{-m^{\frac{\gamma}{\gamma+1}}\left(-\ln K + \frac{1/d-\gamma}{\gamma+1}\ln m\right)\right\} \tag{4.26}$$

where $K = 4\,\theta_d(\frac{\beta}{\alpha})^{1/2}(\frac{1+\rho}{\rho})$. It is evident that decay occurs for the choice $0 \leq \gamma < \frac{1}{d}$ and

$$\varsigma^N < e^{-m^{\frac{\gamma}{\gamma+1}}} \tag{4.27}$$

for $m > N = (Ke)^{(\gamma+1)/(1/2-\gamma)}$. We set $\ell = \text{dimension}\{\mathcal{T}(m, \omega, \omega^*)\}$ and Theorem 4.16 implies

$$d_\ell(\omega, \omega^*) \leq \sup_{u \in H_{\mathbb{A}}(\omega^*)/\mathcal{R}} \inf_{\chi \in \mathcal{T}(m,\omega,\omega^*)} \frac{\|u - \chi\|_{\mathcal{E}(\omega)}}{\|u\|_{\mathcal{E}(\omega^*)}} \leq e^{-\ell^{\frac{\gamma}{\gamma+1}}}. \tag{4.28}$$

for $\ell > (Ke)^{(\gamma+1)/(1/2-\gamma)}$ and Theorem 4.12 is proved.

3.1 Optimal Local Approximation Spaces at the Boundary

We present optimal local approximation spaces for domains bordering the boundary of Ω. These results hold for domains Ω of general shape including bounded Lipschitz regions. The essential assumption is that the Rellich Kondrachov embedding theorem holds in Ω. To fix ideas we consider the L shaped domain Ω with a reentrant corner and introduce optimal local approximation spaces for domains that intersect the boundary $\partial\Omega$. To illustrate the method consider concentric L

shaped subdomains $\omega \subset \omega^*$ of Ω containing the reentrant corner. See Figure 4.1. The arguments presented here naturally apply to other choices of ω and ω^* that touch the boundary. The outer domain is denoted by ω^* and the concentric inner domain is denoted by ω. The side lengths of $\omega \subset \omega^* \subset \sigma$ are given by σ and $\sigma^* = (1+\rho)\sigma$ respectively. For future reference we set $\delta = \frac{1}{2}\rho\sigma$.

Given a function $\mathbf{u} \in H_{\mathbb{A}}(\Omega;\mathbb{R}^d)$, $d = 2,3$, the goal is to provide a approximation to $\mathbf{u}$ in ω. To this end we form a local particular solution $\mathbf{u}_p$ given by the $\mathbb{A}$-harmonic function that satisfies $\mathbb{A}e(\mathbf{u}_p)\mathbf{n} = \mathbf{g}$ on $\partial\omega^* \cap \partial\Omega$ and $\mathbf{u}_p = 0$ on $\partial\omega^* \cap \Omega$. Writing $\mathbf{u} = \mathbf{u}_p + \mathbf{u}_0$ we see that $\mathbb{A}e(u_0)\mathbf{n} = 0$ on $\partial\omega^* \cap \partial\Omega$ and $\mathbf{u}_0 = \mathbf{u}$ on $\partial\omega^* \cap \Omega$. The objective of this section is to find the optimal family of local approximation spaces that give the best approximation to $\mathbf{u}_0 = \mathbf{u} - \mathbf{u}_p$ in the energy norm over the set ω.

We introduce the space of functions $H_{\mathbb{A},0}(\omega^*;\mathbb{R}^d)$ given by all functions $\mathbf{v} \in H^1(\omega^*,\mathbb{R}^d)$ that are $\mathbb{A}$-harmonic on ω^* and for which $\partial_\nu \mathbf{v} \equiv \mathbb{A}e(\mathbf{v})\mathbf{n} = 0$ on $\partial\omega^* \cap \partial\Omega$. The analogous space of functions defined on ω is denoted $H_{\mathbb{A},0}(\omega;\mathbb{R}^d)$. Since we approximate functions with respect to the energy norm we introduce the quotient space of $H_{\mathbb{A},0}(\omega^*;\mathbb{R}^d)$ to the subspace of rigid translations $\mathcal{R}$ denoted by $H_{\mathbb{A},0}(\omega^*;\mathbb{R}^d)/\mathcal{R}$.

Now we introduce $\mathcal{T} : H_{\mathbb{A},0}(\omega^*;\mathbb{R}^d)/\mathcal{R} \to H_{\mathbb{A},0}(\omega;\mathbb{R}^d)/\mathcal{R}$ given by the restriction operator defined by $\mathcal{T}(\mathbf{u})(\mathbf{x}) = \mathbf{u}(\mathbf{x})$ for all $\mathbf{x} \in \omega$ and $\mathbf{u} \in H_{\mathbb{A},0}(\omega^*;\mathbb{R}^d)/\mathcal{R}$. The operator $\mathcal{T}$ is compact. As before this follows immediately from an application of a suitable Caccioppoli inequality (Lemma 4.20) together with the Rellich Kondrachov embedding theorem on ω^*. Let $S(n)$ be any finite dimensional subspace of

$H_{\mathbb{A},0}(\omega;\mathbb{R}^d)/\mathcal{R}$ and the problem of finding the family of optimal local approximation spaces is formulated in terms of the n-width of $\mathcal{T}$. Let

$$d_n(\omega,\omega^*) = \inf_{S(n)} \sup_{\mathbf{u}\in H_{\mathbb{A},0}(\omega^*;\mathbb{R}^d)/\mathcal{R}} \inf_{\chi\in S(n)} \frac{\|\mathcal{T}\mathbf{u}-\chi\|_{\mathcal{E}(\omega)}}{\|\mathbf{u}\|_{\mathcal{E}(\omega^*\cap\Omega)}}. \tag{4.29}$$

Then the optimal family of boundary approximation spaces $\{\Psi_n(\omega)\}_{n=1}^{\infty}$ over ω satisfy

$$d_n(\omega,\omega^*) = \sup_{\mathbf{u}\in H_{\mathbb{A},0}(\omega^*;\mathbb{R}^d)/\mathcal{R}} \inf_{\chi\in \Psi_n(\omega)} \frac{\|\mathcal{T}\mathbf{u}-\chi\|_{\mathcal{E}(\omega)}}{\|\mathbf{u}\|_{\mathcal{E}(\omega^*\cap\Omega)}}. \tag{4.30}$$

The quantity $d_n(\omega,\omega^*)$ is known as the Kolmogorov n-width of the compact operator $\mathcal{T}$, see [38].

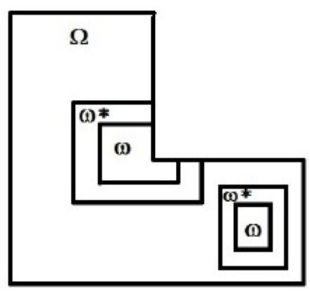

FIGURE 4.1. Local domains

Proceeding as before we introduce the adjoint operator $P^* : H_{\mathbb{A}}(\omega;\mathbb{R}^d)/\mathcal{R} \to H_{\mathbb{A}}(\omega^*)/\mathcal{R}$ and the operator $\mathcal{T}^*\mathcal{T}$ is a compact operator mapping $H_{\mathbb{A},0}(\omega^*)/\mathcal{R}$ into itself. The optimal approximating spaces are described in the following theorem.

Theorem 4.17. *The optimal approximation space is given by $V_\omega = \Psi_n(\omega) = span\{\psi_1,\ldots,\psi_n\}$ where $\psi_i = \mathcal{T}\varphi_i$ and $\varphi_i \in H_{\mathbb{A},0}(\omega^*;\mathbb{R}^d)/\mathcal{R}$ and λ_i are the first n eigenfunctions and eigenvalues that satisfy*

$$(\varphi_i,\delta)_{\mathcal{E}(\omega)} \;=\; \lambda_i(\varphi_i,\delta)_{\mathcal{E}(\omega^*\cap\Omega)},\ \forall\delta\in H_{\mathbb{A},0}(\omega^*;\mathbb{R}^d)/\mathcal{R}, \tag{4.31}$$

and $d_n(\omega,\omega^*) = \sqrt{\lambda_{n+1}}$.

The next theorem provides an upper bound on the rate of convergence for the optimal local boundary approximation.

Theorem 4.18. Exponential convergence at the boundary.

For $\epsilon > 0$ there is an $N_\epsilon > 0$ such that for all $n > N_\epsilon$

$$d_n(\omega, \omega^*) \le e^{-n^{\left(\frac{1}{d+1} - \epsilon\right)}}. \tag{4.32}$$

Theorem 4.18 shows that the asymptotic convergence rate associated with the optimal boundary approximation space is nearly exponential for the general class of $L^\infty(\omega^*)$ coefficients $\mathbb{A}(\mathbf{x})$ satisfying the coercivity and boundedness conditions (2.2).

3.2 Proof of exponential decay for boundary subdomains

The subspace of $\mathbb{A}$ harmonic functions defined on ω^* orthogonal in the $L^2(\omega^*; \mathbb{R}^d)$ inner product to rigid motions is denoted by $H^0_{\mathbb{A}}(\omega^*; \mathbb{R}^d)$ and the subspace of elements belonging to $H_{\mathbb{A},0}(\omega^*; \mathbb{R}^d)$, orthogonal in the $L^2(\omega^*; \mathbb{R}^d)$ inner product, to rigid motions is denoted by $H^0_{\mathbb{A},0}(\omega^*; \mathbb{R}^d)$. We introduce the $L^2(\omega^*; \mathbb{R}^d)$ norm closure of $H^0_{\mathbb{A},0}(\omega^*; R^d)$ denoted by $\overline{H}^0_{\mathbb{A},0}(\omega^*; \mathbb{R}^d)$. It is shown in Lemma 6.25 of the Appendix that $\mathbf{v} \in \overline{H}^0_{\mathbb{A},0}(\omega^*; \mathbb{R}^d)$ satisfies the boundary condition $\partial_\nu \mathbf{v} = 0$ on any compact subset of $\partial\omega^* \cap \partial\Omega$ and the orthogonality condition $\mathbf{v} \perp_{L^2} \mathcal{R}$.

To construct a local basis we first introduce the first n eigenfunctions $\mathbf{v}_i \in H^1(\omega^*; \mathbb{R}^d)$, that are orthogonal in $L^2(\omega^*; \mathbb{R}^d)$ to rigid motions, of

$$(\mathbf{v}_i, \mathbf{w})_{\mathcal{E}(\omega^*)} = \lambda_i \int_{\omega^*} \mathbf{v}_i \mathbf{w}\, dx, \;\; \forall\; \mathbf{w} \in H^1(\omega^*, \mathbb{R}^d)),$$

$i = 1, \dots, n$. The subspace spanned by these functions is denoted by $S_n(\omega^*)$. Next we introduce the span of $\mathbb{A}$ harmonic functions given by

$$W_n(\omega^*) = \mathrm{span}\{\mathbf{w}_i \in H^0_{\mathbb{A}}(\omega^*; R^d) : \mathbf{w}_i = \mathbf{v}_i, \text{ on } \partial\omega^* \cap \Omega \cup \partial\omega^* \cap \partial\Omega, \, i = 1, \dots n\} \tag{4.33}$$

For future reference we introduce the decomposition of $H^1(\omega^*; \mathbb{R}^d)$ given by

$$H^1(\omega^*; \mathbb{R}^d) = H^0_{\mathbb{A}}(\omega^*; \mathbb{R}^d) \oplus H^1_0(\omega^*; \mathbb{R}^d) \oplus \mathcal{R}.$$

The spaces $H^0_{\mathbb{A}}(\omega^*; \mathbb{R}^d)$ and $H^1_0(\omega^*, \mathbb{R}^d)$ are orthogonal with respect to the energy inner product $(\cdot, \cdot)_{\mathcal{E}(\omega^*)}$. We recall that the spaces $H_{\mathbb{A}}(\omega^*; \mathbb{R}^d)/\mathcal{R}$ and $H_{0,\mathbb{A}}(\omega^*; \mathbb{R}^d)/\mathcal{R}$ equipped with the energy inner product are isomorphic to $H^0_{\mathbb{A}}(\omega^*; \mathbb{R}^d)$ and $H^0_{0,\mathbb{A}}(\omega^*; \mathbb{R}^d)$ respectively.

The orthogonal projection from $H^1(\omega^*; \mathbb{R}^d)$ onto $H^0_{\mathbb{A}}(\omega^*; \mathbb{R}^d)$ is denoted by $\mathcal{P}^{\mathbb{A}}$. One readily checks that $W_n(\omega^*) = \mathcal{P}^{\mathbb{A}} S_n(\omega^*)$. Next we construct a subspace satisfying the homogeneous Neumann boundary conditions on $\partial\omega^* \cap \partial\Omega$ by L^2 projecting $W_n(\omega^*)$ onto $\overline{H}^0_{A,0}(\omega^*)$. The projection operator is denoted by $\mathcal{P}_0$ and

$$\|\mathbf{v} - \mathcal{P}_0\mathbf{v}\|_{L^2(\omega^*;\mathbb{R}^d)} = \inf_{\mathbf{w} \in \overline{H}^0_{\mathbb{A},0}(\omega^*)} \| \mathbf{v} - \mathbf{w} \|_{L^2(\omega^*;\mathbb{R}^d)}$$

and

$$\|\mathbf{v}\|^2_{L^2(\omega^*;\mathbb{R}^d)} = \|\mathcal{P}_0\mathbf{v}\|^2_{L^2(\omega^*;\mathbb{R}^d)} + \|(I - \mathcal{P}_0)\mathbf{v}\|^2_{L^2(\omega^*;\mathbb{R}^d)} \tag{4.34}$$

In what follows the local approximations will be chosen from the local function space $\mathcal{P}_0 W_n(\omega^*)$ restricted to the set ω. Next we introduce the approximation theorem associated with the space $\mathcal{P}_0 W_n(\omega)$ given by

Lemma 4.19. *Let $\mathbf{u} \in H^0_{A,0}(\omega^*;\mathbb{R}^d)$ then there exists a $\mathbf{v}_u \in \mathcal{P}_0 W_n(\omega^*)$ such that*

$$\|\mathbf{u} - \mathbf{v}_u\|_{L^2(\omega^*;\mathbb{R}^d)} = \inf_{\mathbf{v} \in \mathcal{P}_0 W_n(\omega^*)} \|\mathbf{u} - \mathbf{v}\|_{L^2(\omega^*);\mathbb{R}^d} \leq C_n \sigma^* \theta_d \alpha^{-1/2} \|\mathbf{u}\|_{\mathcal{E}(\omega^*)} \quad (4.35)$$

where σ^* is the side length of the cube ω^* and $C_n = n^{-1/d}(1 + o(1))$ for $d = 2, 3$ and θ_d is as in Lemma 4.14.

Proof. The lemma follows immediately from an upper bound on the quotient

$$T = \sup_{\mathbf{u} \in H^0_{\mathbb{A},0}(\omega^*;\mathbb{R}^d)} \inf_{\mathbf{w} \in \mathcal{P}_0 W_n(\omega^*)} \frac{\|\mathbf{u} - \mathbf{w}\|_{L^2(\omega^*;\mathbb{R}^d)}}{\|\mathbf{u}\|_{\mathcal{E}(\omega^*)}}. \quad (4.36)$$

Fix $\mathbf{u} \in H^0_{\mathbb{A},0}(\omega^*;\mathbb{R}^d)$ and for every $\mathbf{w} \in \mathcal{P}_0 W_n(\omega^*)$ one has $\mathbf{g} \in W_n(\omega^*)$ such that $\mathbf{w} = \mathcal{P}_0 \mathbf{g}$ and

$$\begin{aligned} \inf_{\mathbf{w} \in \mathcal{P}_0 W_n(\omega^*)} \|\mathbf{u} - \mathbf{w}\|_{L^2(\omega^*;\mathbb{R}^d)} &= \inf_{\mathbf{g} \in W_n(\omega^*)} \|\mathcal{P}_0(\mathbf{u} - \mathbf{g})\|_{L^2(\omega^*;\mathbb{R}^d)} && (4.37) \\ &\leq \inf_{\mathbf{g} \in W_n(\omega^*)} \|\mathbf{u} - \mathbf{g}\|_{L^2(\omega^*;\mathbb{R}^d)} && (4.38) \end{aligned}$$

Thus

$$T \leq \sup_{\mathbf{u} \in H^0_{\mathbb{A},0}(\omega^*;\mathbb{R}^d)} \inf_{\mathbf{g} \in W_n(\omega^*)} \frac{\|\mathbf{u} - \mathbf{g}\|_{L^2(\omega^*;\mathbb{R}^d)}}{\|\mathbf{u}\|_{\mathcal{E}(\omega^*)}}. \quad (4.39)$$

Denote the projection of $\mathbf{u}$ onto $W_n(\omega^*)$ with respect to the energy norm $\|\cdot\|_{\mathcal{E}(\omega^*)}$ by $\mathcal{P}^{\mathcal{E}}\mathbf{u}$. Choosing $\mathbf{g} = \mathcal{P}^{\mathcal{E}}\mathbf{u}$ and noting that $\|(I - \mathcal{P}^{\mathcal{E}})\mathbf{u}\|_{\mathcal{E}(\omega^*)} \leq \|\mathbf{u}\|_{\mathcal{E}(\omega^*)}$ gives the upper bound

$$T \leq \sup_{\mathbf{u} \in H^0_{\mathbb{A},0}(\omega^*;\mathbb{R}^d) \perp W_n(\omega^*)} \frac{\|\mathbf{u}\|_{L^2(\omega^*;\mathbb{R}^d)}}{\|\mathbf{u}\|_{\mathcal{E}(\omega^*)}} \quad (4.40)$$

Since $H^0_{\mathbb{A},0}(\omega^*;\mathbb{R}^d) \subset H^0_{\mathbb{A}}(\omega^*;\mathbb{R}^d)$ and from (4.40) we have

$$T \leq \sup_{\mathbf{u} \in H^0_{A}(\omega^*;\mathbb{R}^d) \perp W_n(\omega^*)} \frac{\|\mathbf{u}\|_{L^2(\omega^*;\mathbb{R}^d)}}{\|\mathbf{u}\|_{\mathcal{E}(\omega^*)}}. \quad (4.41)$$

Writing $W_n(\omega^*) = \mathcal{P}^A S_n(\omega^*)$ it follows that

$$\begin{aligned} \{\mathbf{u} \in H^0_{\mathbb{A}}(\omega^*;\mathbb{R}^d) \perp \mathcal{P}^A S_n(\omega^*)\} &= \{\mathbf{u} \in H^0_{\mathbb{A}}(\omega^*;\mathbb{R}^d) \perp S_n(\omega^*)\}, \\ \{\mathbf{u} \in H^0_{\mathbb{A}}(\omega^*;\mathbb{R}^d) \perp S_n(\omega^*)\} &\subset \{\mathbf{u} \in H^1(\omega^*;\mathbb{R}^d) \perp_{L^2} (S_n(\omega^*) \oplus \mathcal{R})\}, \quad (4.42) \end{aligned}$$

where the $\perp_{L^2}$ in the second line of (4.42) is orthogonality with respect to the $L^2(\omega^*; \mathbb{R}^2)$ inner product. Hence

$$T \leq \sup_{\mathbf{u} \in H^0_{\mathbb{A}}(\omega^*;\mathbb{R}^d) \perp S_n(\omega^*)} \frac{\|\mathbf{u}\|_{L^2(\omega^*;\mathbb{R}^d)}}{\|\mathbf{u}\|_{\mathcal{E}(\omega^*)}} \tag{4.43}$$

$$\leq \sup_{\mathbf{u} \in H^1(\omega^*;\mathbb{R}^d)^2 \perp_{L^2} (S_n(\omega^*) \oplus \mathcal{R})} \frac{\|\mathbf{u}\|_{L^2(\omega^*;\mathbb{R}^d)}}{\|\mathbf{u}\|_{\mathcal{E}(\omega^*)}} = \frac{1}{\sqrt{\mu_{n+1}}}, \tag{4.44}$$

where μ_{n+1} is the largest eigenvalue associated with $S_{n+1}(\omega^*)$. One has the elementary lower bound $\mu_{n+1} \geq \alpha \nu_{n+1}$ where ν_{n+1} is the largest eigenvalue for the eigenvalue problem $\int_{\omega^*} e(\mathbf{v}_{n+1}) : e(\mathbf{w})\, dx = \nu_{n+1} \int_{\omega^*} \mathbf{v}_{n+1} \mathbf{w}\, dx$, $\forall \mathbf{w} \in H^1(\omega^*; \mathbb{R}^d)$. And from the generalization of Weyl's Theorem, we have $\nu_{n+1} = \frac{4\pi}{8(\sigma^*)^2/3}(n+1)(1+o(1))$, for $d = 2$ and $\nu_{n+1} = [6\pi^2/(1+4\sqrt{2})(\sigma^*)^3/]^{1/3}(n+1)^{2/3}(1+o(1))$ for $d = 3$.

Motivated by Lemma 4.19 we define the family of approximation spaces $\mathcal{F}_n(\omega, \omega^*)$ given by the restriction of the elements of $\mathcal{P}_0 W_n(\omega^*)$ to ω. In what follows we show that $\mathcal{F}_n(\omega, \omega^*)$ is a family of local approximation spaces with a rate of convergence on the order of $n^{-1/d}$, $d = 2, 3$.

To proceed we introduce a suitable Caccioppoli inequality for boundary domains. Suppose $\omega \subset \omega^*$ satisfy $\partial\omega \cap \partial\omega^* \cap \partial\Omega \neq \emptyset$ and $\operatorname{dist}(\partial\omega^* \cap \Omega, \partial\omega \cap \Omega) = \delta > 0$, see, e.g., Figure 4.1. Then we have the following lemma.

Lemma 4.20. *Let* $\mathbf{u}$ *be in* $\overline{H}^0_{\mathbb{A},0}(\omega^*; \mathbb{R}^d)$. *Then*

$$\|\mathbf{u}\|_{\mathcal{E}(\omega)} \leq (2\beta^{1/2}/\delta)\|\mathbf{u}\|_{L^2(\omega^*;\mathbb{R}^d)}. \tag{4.45}$$

where β *is defined in equation (2.2).*

The proof is given in the Appendix A.

Application of Lemma 4.20 to $\mathbf{u} - \mathbf{v}_u$ on $\omega \subset \omega^*$ together with Lemma 4.19 delivers the following convergence rate associated with the family of approximation spaces $\mathcal{F}_n(\omega, \omega^*)$ given by

Theorem 4.21. *Let $\mathbf{u} \in H_{\mathbb{A},0}(\omega^*; \mathbb{R}^d)/\mathcal{R}$ then there exists an approximation $\mathbf{v}_u \in \mathcal{F}_n(\omega, \omega^*)$ for which*

$$\|\mathbf{u} - \mathbf{v}_u\|_{\mathcal{E}(\omega)} = \inf_{\mathbf{v} \in \mathcal{F}_n(\omega,\omega^*)} \|\mathbf{u} - \mathbf{v}\|_{\mathcal{E}(\omega)} \leq I(\omega, \omega^*)\, C_n \|\mathbf{u}\|_{\mathcal{E}(\omega^*)} \tag{4.46}$$

where

$$I(\omega, \omega^*) = 4\,\theta_d\,\frac{1+\rho}{\rho}(\beta/\alpha)^{1/2} \quad \textit{and} \quad C_n = n^{-1/d}(1 + o(1)), \tag{4.47}$$

Now we proceed as before to construct a family of local approximation spaces with a rate of convergence that is nearly exponential. For any pair of two concentric L-shaped domains $Q \subset Q^*$ we define $\partial\mathcal{F}_n(Q, Q^*)$ to be the space given by the restriction of $\mathcal{P}_0 W_n(Q^*)$ on Q. Let $N > 1$ be an integer and consider the nested family of concentric L-shaped domain Q_j, $j = 1, 2, ...N + 1$ with $Q_{j+1} \subset Q_j$ and $Q_1 = \omega^*$ and $Q_{N+1} = \omega$. The longest side lengths of Q_j are given by $\sigma(1 + \rho(N + 1 - j)/N)$. Set $\omega_j = Q_j$ to obtain $\omega = \omega_{N+1} \subset \omega_N \subset \omega_{N-1} \subset \cdots \subset \omega_1 = \omega^*$. We introduce the local spaces $\partial\mathcal{F}_n(\omega, \omega_N)$, $\partial\mathcal{F}_n(\omega, \omega_{N-1})$,...,$\partial\mathcal{F}_n(\omega, \omega_1)$. Put $m = N \times n$ and we define the approximation space given by

$$\partial\mathcal{T}(m, \omega, \omega^*) = \partial\mathcal{F}_n(\omega, \omega_1) + \cdots + \partial\mathcal{F}_n(\omega, \omega_N). \tag{4.48}$$

We now proceed as before to estimate convergence rate associated with the local approximation space $\partial\mathcal{T}(m, \omega, \omega^*)$ and Theorem 4.18 follows.

Chapter 5

Application of optimal local approximation to GFEM

Recall GFEM Approximation Theorem in Chpater2.

Theorem 5.22 (GFEM Approximation Theorem [2])**.** *Given* $\mathbf{u}_0$ *the unique global solution and* $K^{N,m}$ *as elements (resp. subsets) of* $H_0^1(\Omega;\mathbb{R}^d)$ *and* $H^1(\Omega;\mathbb{R}^d)$*. Then for* $\mathbf{u}_N^G \in K^{N,m}$

$$\|\mathbf{u}_0 - \mathbf{u}_N^G\|_{\mathcal{E}(\Omega)} \leq (2k)^{1/2}(\sum_{i=1}^{m}(\frac{c_2}{diam(\omega_i)})^2\epsilon_i^2 + c_1^2\epsilon_i^2)^{1/2} \tag{5.1}$$

where local approximation is

$$\|\mathbf{u}_0 - (\mathbf{u}_p|_{\omega_i} + \xi^i)\|_{L^2(\omega_i)} \leq \epsilon_i \tag{5.2}$$

$$\|\mathbf{u}_0 - (\mathbf{u}_p|_{\omega_i} + \xi^i)\|_{\mathcal{E}(\omega_i)} \leq \epsilon_i \tag{5.3}$$

This theorem shows that control of local error can be used to the control of global error. Here $\mathbf{u}_p|_{\omega_i}$ is a particular solution on ω_i and $\mathbf{u}_0 - \mathbf{u}_p|_{\omega_i}$ is $\mathbb{A}$-harmonic function on ω_i, $i = 1, 2, ..., m$.

In the previous chapter, we have proved the following exponential convergence theorem for both interior approximations and the boundary approximations.

For $\epsilon > 0$ there is an $N_\epsilon > 0$ such that for all $n > N_\epsilon$

$$d_n(\omega, \omega^*) \leq e^{-n^{\left(\frac{1}{d+1}-\epsilon\right)}}. \tag{5.4}$$

So if we choose ξ^i to be local optimal local approximation to $\mathbf{u}_0 - \mathbf{u}_p|_{\omega_i}$, then

$$\| \, (\mathbf{u}_0 - \mathbf{u}_p|_{\omega_i}) - \xi^i \, \|_{\mathcal{E}(\omega_i)} \leq e^{-n^{\left(\frac{1}{d+1}-\epsilon\right)}} \tag{5.5}$$

i.e

$$\| \mathbf{u}_0 - (\mathbf{u}_p|_{\omega_i} + \xi^i) \|_{\mathcal{E}(\omega_i)} \leq e^{-n^{\left(\frac{1}{d+1}-\epsilon\right)}} \tag{5.6}$$

Suppose $\zeta_i(x) = \mathbf{u}_p|_{\omega_i} + \xi^i$, then we can summarize our results in the following theorem

Theorem 5.23 (Nearly exponential approximation for GFEM)**.** *For $\varepsilon > 0$ there is an $N_\varepsilon > 0$ such that for all $N > N_\varepsilon$, there exist ζ_i and a constant $\mathbf{K}$ independent of N such that the approximation $\zeta \in H^1(\Omega)$ given by*

$$\zeta(x) = \sum_{i=1}^{m} \zeta_i(x)\phi_i(x) \tag{5.7}$$

satisfies

$$\|\mathbf{u}_0 - \zeta\|_{L^2(\Omega)} \leq \mathbf{K}e^{-N^{\left(\frac{1}{1+d}-\varepsilon\right)}} \tag{5.8}$$

and

$$\|\mathbf{u}_0 - \zeta\|_{\mathcal{E}(\Omega)} \leq \mathbf{K}e^{-N^{\left(\frac{1}{1+d}-\varepsilon\right)}} \tag{5.9}$$

Chapter 6

Homogenization of local bases for multiscale problems in random media

1 Introduction

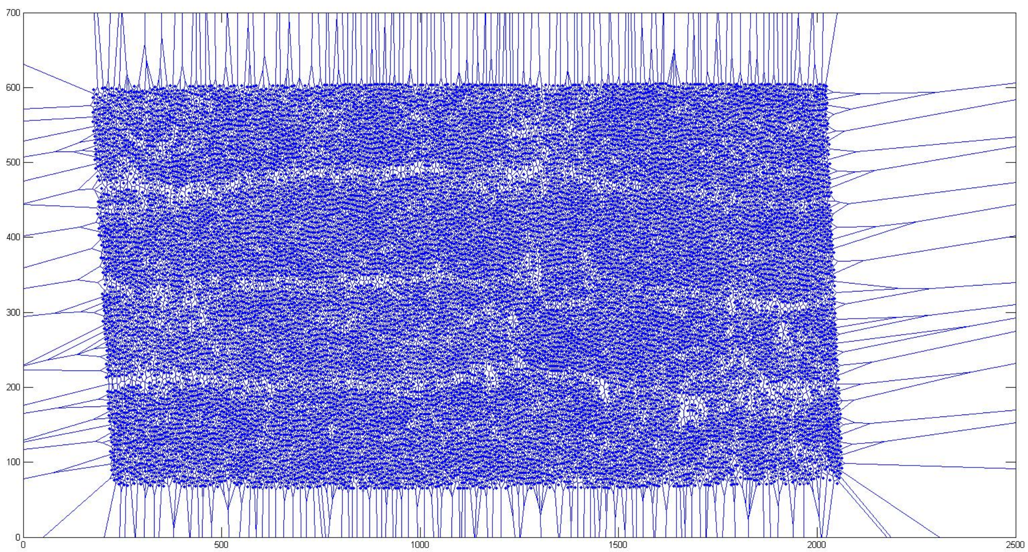

FIGURE 6.1. Voronoi diagram of a large sample of composite

In general it is not possible to completely know the coefficient of elasticity describing a composite structure. Instead we suppose we have image data for a typical subdomain of the composite structure. Here we use the image data to construct a random medium that captures the variation of the local fiber configuration across the structure. It is carried out as follows:

1. Take image data from a representative large sample of material and identify the coordinate of the centers of the cross sections of the fibers.

2. Construct a Voronoi tessellation from the centers of the fiber cross sections see, Figure 6.1.

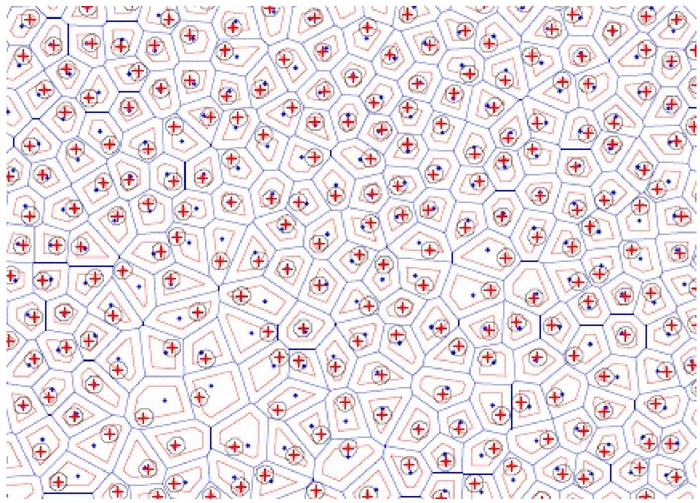

FIGURE 6.2. Voronoi subdomain

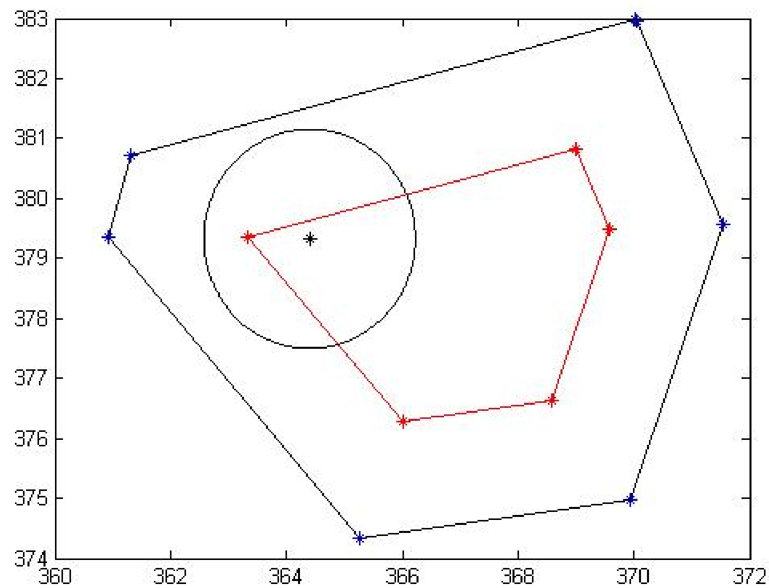

FIGURE 6.3. Voronoi cell and subdomain

3. The variation in the location of the centers of the fiber cross sections across large samples is approximated by jiggling the centers about a subdomain of each cell of the Voronoi tessellation, see Figure 6.2. A particular subdomain of each Voronoi cell is constructed such that the disk with center jiggled in the subdomain will never cross the boundary of the Voronoi cell. The centers are jiggled according to the uniform probability distribution over the subdomain see, Figure 6.3.

Remark 6.24. *Here disks are distributed inside the voroni cells so there are no overlaps between neighboring spheres. This is a method for generating high concentration random suspensions of nonoverlapping disks from image data. This algorithm is implemented by programming in Matlab. The matlab code and explanation is given in the next section.*

The Voronoi tessellation for a large sample is given in Figure 6.1. A "close up" of a typical subset of the sample shown in Figure 6.1 is presented in Figure 6.2. There each Voronoi cell is visible together with its subdomain and its disks. Every disk is located in its Voronoi cell and has no overlaps with other disks. The polygon within each Voronoi cell is the subdomain where we can jiggle the centers of disks such that the boundaries of disks would not cross the boundary of the cell. The choice of center within the subdomain is taken from a uniform distribution. A particular Voronoi cell with subdomain and realization of particle center is shown in Figure 6.3.

We replace the details of the disk configuration over the subset given by $\mathbb{A}(\mathbf{x})$ with its effective elasticity tensor $\mathbb{A}^h$ coming from the theory of homogenization. Our approach is motivated by the work provided in [13] for particles inside hexag-

onal cells.

We use the effective elasticity tensors evaluated over each typical subdomain to calculate the solution of the macroscopic or global homogenized problem. The fine scale field fluctuations are recovered using the optimal approximation space for a given realization of coefficients of $\mathbb{A}(\mathbf{x})$ over the subdomain.

For each realization of the random medium we can solve the Boundary value problem. To determine quantities of interest like local strength and strains inside the medium. The goal would be to understand the variation of the local strength and strain over the ensemble of realizations. The goal of future work is to numerically implement these ideas.

2 Construction of generating Voronoi cells and nonoverlapping disks from image data using Matlab

Firstly, the coordinates of the centers of the cross sections of the fibers are used as Voronoi seeds and we construct the Voronoi diagram.

Secondly, we construct a polygon inside each Voronoi cell such that disks with radius r and with centers on the boundary of this polygon can only touch the boundary of the Voronoi cell. This makes sure that when disks are generated with centers selected randomly in the subdomain of each voronoi cell there will be no overlaps between neignboring disks.

In the third step, for each voronoi cell, we select a center within subdomain from a uniform distribution and generate a circle.

The matlab code for generating a realization for the random medium is provided in Appendix A.

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Appendix A: Proof of Cacciappoli Inequality

We provide a proof of the Cacciappoli inequality given in Lemma (4.13). We introduce the cut off function $\eta \in (\omega^*)$ such that $0 \leq \eta \leq 1$, $\eta = 1$ for points inside $\mathcal{O}$ and $|\nabla \eta| \leq 1/\delta$ for points in ω^*. Given the function $\mathbf{u} \in H_{\mathbb{A}}(\omega^*; \mathbb{R}^d)$ We see that $\eta^2 \mathbf{u}$ belongs to $H_0^1(\omega^*; \mathbb{R}^d)$ hence

$$\int_{\omega^*} \mathbb{A} e(\mathbf{u}) : e(\eta^2 \mathbf{u}) d\mathbf{x} = 0. \tag{6.1}$$

Expanding (6.1) gives

$$\int_{\omega^*} \mathbb{A}_{ijkl} e_{kl}(\mathbf{u})(\eta^2 e_{ij}(\mathbf{u}) + \eta(\partial_i \eta u_j + \partial_j \eta u_i)) d\mathbf{x} = 0, \tag{6.2}$$

hence

$$\begin{aligned}
\int_{\omega^*} \mathbb{A}_{ijkl} e_{kl}(\mathbf{u}) e_{ij}(\mathbf{u}) \eta^2 dx &= -\int_{\omega^*} \eta \mathbb{A}_{ijkl} e_{kl}(\mathbf{u})(\partial_i \eta u_j + \partial_j \eta u_i) d\mathbf{x} && (6.3)\\
&= -2 \int_{\omega^*} \eta \sqrt{\mathbb{A}} e(\mathbf{u}) : \sqrt{\mathbb{A}} \nabla \eta \odot \mathbf{u} dx && (6.4)\\
&\leq 2 \sqrt{\int_{\omega^*} \eta^2 \mathbb{A} e(\mathbf{u}) : e(\mathbf{u}) d\mathbf{x}} \cdot && (6.5)\\
&\quad \sqrt{\int_{\omega^*} \mathbb{A} \nabla \eta \odot \mathbf{u} : \nabla \eta \odot \mathbf{u} d\mathbf{x}} && (6.6)
\end{aligned}$$

where $\nabla \eta \odot \mathbf{u} = (\frac{\partial_i \eta u_j + \partial_j \eta u_i}{2})$. Now we have

$$\begin{aligned}
\sqrt{\int_{\omega^*} \eta^2 \mathbb{A} e(\mathbf{u}) : e(\mathbf{u}) d\mathbf{x}} &\leq 2 \sqrt{\int_{\omega^*} \mathbb{A} \nabla \eta \odot \mathbf{u} : \nabla \eta \odot \mathbf{u} d\mathbf{x}} && (6.7)\\
&\leq 2\sqrt{\beta} \sqrt{\int_{\omega^*} \nabla \eta \odot \mathbf{u} : \nabla \eta \odot \mathbf{u} d\mathbf{x}} && (6.8)\\
&\leq 2\sqrt{\beta} \sqrt{\int_{\omega^*} \frac{1}{2}[|\nabla \eta|^2 |\mathbf{u}|^2 + (\nabla \eta \cdot \mathbf{u})^2]} && (6.9)\\
&\leq 2\sqrt{\beta} \sqrt{\int_{\omega^*} |\nabla \eta|^2 |\mathbf{u}|^2} && (6.10)\\
&\leq \frac{2\sqrt{\beta}}{\delta} \|\mathbf{u}\|_{L^2(\omega^*; \mathbb{R}^d)} && (6.11)
\end{aligned}$$

and Lemma 4.13 follows.

We now establish the Caccioppoli inequality, Lemma 4.20 for boundary domains. We begin by establishing the inequality for functions $\mathbf{u}$ belonging to $H^0_{\mathbb{A},0}(\omega^*;\mathbb{R}^d)$. We introduce the cut off function $\eta \in C^1(\omega^*)$ such that $0 \leq \eta \leq 1$ and $\eta = 1$ for $x \in \omega$, $\overline{\text{supp}(\eta)} \subset \omega^*$, $\text{supp}(\eta) \cap (\Omega \cap \partial\omega^*) = \emptyset$ and $\| \nabla\eta \| \leq \frac{1}{\delta}$ for points in ω^*. For functions $\mathbf{u} \in H_{\mathbb{A},0}(\omega^*;\mathbb{R}^d)$ we claim that

$$\int_{\omega^*} \mathbb{A}e(\mathbf{u}) : e(\eta^2\mathbf{u})\, d\mathbf{x} = 0.$$

To see this integrate by parts to find

$$\begin{aligned}
\int_{\omega^*} \mathbb{A}e(\mathbf{u}) : e(\eta^2\mathbf{u})\, d\mathbf{x} &= \int_{\partial\omega^*} \mathbb{A}e(\mathbf{u})\mathbf{n}\cdot\eta^2\mathbf{u}\, ds && (6.12)\\
&= \int_{\partial\omega^*\cap\Omega} \mathbb{A}e(\mathbf{u})\mathbf{n}\cdot\eta^2\mathbf{u}\, ds && (6.13)\\
&+ \int_{\partial\omega^*\cap\partial\Omega} \mathbb{A}e(\mathbf{u})\mathbf{n}\cdot\eta^2\mathbf{u}\, ds && (6.14)\\
&= 0 && (6.15)
\end{aligned}$$

Here the first term vanishes since the support of η does not intersect with $\partial\omega^* \cap \Omega$. The second term vanishes since $\mathbf{u} \in H_{A,0}(\omega^*;\mathbb{R}^d)$ implies $\mathbb{A}e(\mathbf{u})\mathbf{n} = 0$ on $\text{supp}(\eta) \cap \partial\Omega$. The proof then proceeds as for the case of interior domains and the Caccioppoli inequality (4.45) holds for $\mathbf{u}$ in $H_{\mathbb{A},0}(\omega^*;\mathbb{R}^d)$, hence it also holds for $\mathbf{u} \in H^0_{\mathbb{A},0}(\omega^*;\mathbb{R}^d)$.

The Caccioppoli inequality is also evident for $\mathbf{u} \in \overline{H}^0_{\mathbb{A},0}(\omega^*;\mathbb{R}^d)$ once we establish the following Lemma.

Lemma 6.25. *A function* $\mathbf{u}$ *belonging to the subspace* $\overline{H}^0_{\mathbb{A},0}(\omega^*;\mathbb{R}^d)$ *has the following properties.*

1. $\mathbf{u}$ *is* $\mathbb{A}$*-harmonic on* ω^**.*

2. $\mathbf{u}$ *is orthogonal to the space of rigid motions with respect to the* $L^2(\omega^*;\mathbb{R}^d)$ *inner product.*

3. *For any set* $\mathcal{O} \subset \omega^*$ *for which* $\partial\mathcal{O} \cap \partial\omega^* \cap \Omega = \emptyset$ *and* $\partial\mathcal{O} \cap \partial\omega^* \cap \partial\Omega \neq \emptyset$ *and* $\overline{\partial\mathcal{O}} \cap \partial\Omega \subset \partial\omega^* \cap \partial\Omega$*, then* $\mathbb{A}e(\mathbf{u})\mathbf{n} = 0$ *on* $\partial\mathcal{O} \cap \partial\omega^* \cap \partial\Omega$*.*

Proof. Given $\mathbf{u} \in \overline{H}^0_{A,0}(\omega^* \cap \Omega)$ then there is a sequence $\mathbf{u}_n \in H^0_{A,0}(\omega^*;\mathbb{R}^d)$ such that $\mathbf{u}_n \to \mathbf{u}$ in $L^2(\omega^*;\mathbb{R}^d)$. We show first that $\mathbf{u}$ is A-harmonic on ω^* and belongs to $H^1_{loc}(\omega^*;\mathbb{R}^d)$. To see this pick any ball $B(x_0,r) \subset\subset \omega^*$ centered at x_0 of radius r. We apply the basic Cacciappoli inequality Lemma 4.13 to deduce that $\mathbf{u}_n$ is Cauchy with respect to the energy norm in $B(x_0,r/2)$. From the completeness of $H^1(B(x_0,r/2)$ we see that there is a limit $\mathbf{u}_\infty \in H^1(B(x_0,r/2))$ and $\mathbf{u}_n \to \mathbf{u}_\infty$ in $H^1(B(x_0,r/2))$. From this we conclude that $\mathbf{u}_\infty \in H^1_{loc}(\omega^*;\mathbb{R}^d)$ and $\mathbf{u}_\infty = \mathbf{u}$.

The weak formulation of the boundary value problem together with the strong convergence of the sequence shows that $\mathbf{u}$ is $\mathbb{A}$-harmonic in ω^* and $\mathbf{u}$ is orthogonal to the space of rigid motions with respect to the $L^2(\omega^*;\mathbb{R}^d)$ inner product.

Next consider any open subset $\mathcal{O} \subset \omega^*$ as in (3). Given $\mathbf{u} \in \overline{H}^0_{A,0}(\omega^* \cap \Omega)$ then there is a sequence $\mathbf{u}_n \in H^0_{A,0}(\omega^*;\mathbb{R}^d)$ such that $\mathbf{u}_n \to \mathbf{u}$ in $L^2(\omega^*;\mathbb{R}^d)$. From the Cacciappoli inequality for $H^0_{\mathbb{A},0}(\omega^*;\mathbb{R}^d)$ functions we see that $\mathbf{u}_n$ is a Cauchy sequence in $H^1(\mathcal{O})$ with respect to the energy norm. From completeness there is a limit $\mathbf{u}_\infty \in H^1(\mathcal{O})$ and $\mathbf{u}_n$ converges strongly to $\mathbf{u}_\infty = \mathbf{u}$ in the energy norm on $\mathcal{O}$. Observe that since $\partial_\nu \mathbf{u}_n \equiv \mathbb{A}e(\mathbf{u}_n)\mathbf{n}$ vanishes on $\partial\mathcal{O} \cap \partial\omega^*$ we can write

$$\begin{aligned}&\|\partial_\nu \mathbf{u}\|_{H^{-1/2}(\partial\mathcal{O}\cap\partial\omega^*)} = \|\partial_\nu \mathbf{u}_n - \partial_\nu \mathbf{u}\|_{H^{-1/2}(\partial\mathcal{O})\cap\partial\omega^*)}\\ &\leq C\|\mathbf{u}_n - \mathbf{u}\|_{H^1(\mathcal{O})}\end{aligned} \tag{6.16}$$

where C is independent of n. Property (3) now follows on noting that

$$\lim_{n\to\infty} \|\mathbf{u}_n - \mathbf{u}\|_{H^1(\mathcal{O})} = 0. \tag{6.17}$$

Appendix B: Proof that $\mathcal{T}$ is a Compact Operator

We now show that the restriction operator $\mathcal{T}$ introduced in Chapter 4 is compact. The restriction operator $\mathcal{T} : H_{\mathbb{A}}(\omega^*; \mathbb{R}^d)/\mathcal{R} \to H_{\mathbb{A}}(\omega; \mathbb{R}^d)/\mathcal{R}$ is defined by $\mathcal{T}(\mathbf{u})(x) = \mathbf{u}(x)$ for all $\mathbf{x} \in \omega$ and $\mathbf{u} \in H_{\mathbb{A}}(\omega^*; \mathbb{R}^d)/\mathcal{R}$.

Lemma 6.26. *Given any sequence $\{\mathbf{u}_n\}_{n=1}^{\infty} \in H_{\mathbb{A}}(\omega^*; \mathbb{R}^d)/\mathcal{R}$ that is bounded in the energy norm over ω^*. One can extract a subsequence that converges in $H^1(\omega)$ to an element of $H_{\mathbb{A}}(\omega; \mathbb{R}^d)/\mathcal{R}$*

Proof. We apply the Poincare inequality first and get that $\{\mathbf{u}_n\}_{n=1}^{\infty}$ is also bounded in $H^1(\omega^*)$ norm. So there exists a subsequence $\{\mathbf{u}_{k_n}\}_{n=1}^{\infty}$ converging weakly to an element $\mathbf{u} \in H_{\mathbb{A}}(\omega^*; \mathbb{R}^d)/\mathcal{R}$ with respect to $H^1(\omega^*)$ norm. In fact this subsequence $\{\mathbf{u}_{k_n}\}_{n=1}^{\infty}$ is convergent to $\mathbf{u}$ in $L^2(\omega^*)$ with the Rellich compactness theorem. From Caccioppoli inequality Lemma, $\|\mathbf{u}_{k_n} - \mathbf{u}\|_{\mathcal{E}(\mathcal{O})} \leq (2\beta^{1/2}/\delta)\|\mathbf{u}_{k_n} - \mathbf{u}\|_{L^2(\omega^*;\mathbb{R}^d)}$.So this subsequence is Cauchy with respect to the energy norm over ω, and the convergence in $H^1(\omega)$ follows. The lemma is proved.

Appendix C: Sample Matlab Code

1. Generate Voronoi cells

```
load FIBERS16275.mat
% Load the image data coordinates file.

X_whole=A(:,1); Y_whole=A(:,2);radius=sqrt(A(:,3)/pi);
% X_whole and Y_whole are coordinates of centers of disks from the image.

voronoi(X_whole,Y_whole);
% Construct Vonoroi cells.

r=min(radius);
% Suppose radius is the same for every disks.

[Vertices,R] = voronoin([X_whole(:) Y_whole(:)]);
%[Vertices,R] returns Voronoi vertices and the Voronoi cells of the Voronoi
%diagram of image data.
%Each row of  "Vertices" corresponds to a Voronoi vertex.
%R is a vector cell array where each element contains the indices into
%Vertices of the vertices of the corresponding Voronoi cell.
```

2. Construct Subdomains of centers inside Voronoi cells

```
vx1=Vertices(R{In(i)},1); vy1=Vertices(R{In(i)},2);
% vx1 is the matrix of x-coordinates of the ith Voronoi cell.
% vy1 is the matrix of y-coordinates of the ith Voronoi ceel.

   [vx2,vy2]=poly2ccw(vx1,vy1);
% Arranges the vertices in the polygonal contour (vx1, vy1)
% in counterclockwise order, returning the result in vx2 and vy2.

n=length(vx2);
   ov=zeros(n+1,2);
   for j=1:n;
       ov(j,:)=[vx2(j)-X(i),vy2(j)-Y(i)];
   end;
% vector vertices to center
```

```
for i=1:N;
vx1=Vertices(R{In(i)},1); vy1=Vertices(R{In(i)},2);
% vx1 is the matrix of x-coordinates of the ith Voronoi cell.
% vy1 is the matrix of y-coordinates of the ith Voronoi ceel.

   [vx2,vy2]=poly2ccw(vx1,vy1);
% Arranges the vertices in the polygonal contour (vx1, vy1)
% in counterclockwise order, returning the result in vx2 and vy2.

   n=length(vx2);
   ov=zeros(n+1,2);
   for j=1:n;
       ov(j,:)=[vx2(j)-X(i),vy2(j)-Y(i)];
   end;
   % vector vertices to center
   ov(end,:)=[vx2(1)-X(i),vy2(1)-Y(i)];
   VL=zeros(n,2);
   UL=zeros(n+1,2);
   v_old=zeros(n,2); % vertices of inner polygon
   cos_2theta=zeros(n,1);
   sin_theta=zeros(n,1);
   cos_theta=zeros(n,1);
   W=zeros(n,2);
   for j=1:n;
       VL(j,:)=ov(j+1,:)-ov(j,:);
       UL(j,:)=VL(j,:)/norm(VL(j,:));
   end;
   UL(end,:)=UL(1,:);
   for j=1:n;
       cos_2theta(j)=dot(-UL(j,:),UL(j+1,:));
       sin_theta(j)=sqrt((1-cos_2theta(j))/2);
       cos_theta(j)=sqrt(1-(sin_theta(j))^2);
       W(j,:)=(-UL(j,:))*[ cos_theta(j),-sin_theta(j);sin_theta(j),cos_theta
       v_old(j,:)=W(j,:)*r/sin_theta(j)+ov(j+1,:)+[X(i),Y(i)];
   end;

 Dist_old=zeros(n,n);
 Q_old=zeros(n+1,3);
 Q_old(1:end-1,:)=[vx2 vy2 zeros(n,1)]; %vertices of voronoi cell
 Q_old(end,:)=Q_old(1,:);
 Edge_old=zeros(n,3);
 vx_double=([vx2' vx2'])';vy_double=([vy2' vy2'])';
 v_double=([v_old' v_old'])';
```

```
P_old=[v_old zeros(n,1)];


for t=1:n;
Edge_old(t,:)=Q_old(t+1,:)-Q_old(t,:);
end;

for t=1:n;
  for k=1:n;
    Dist_old(t,k)=norm(cross(Edge_old(k,:),P_old(t,:)-Q_old(k,:)))/norm(Edg
  end;
end;

h=1;
while min(Dist_old(h,:))- r < (-minEbsilon) && h <= n;
    h=h+1;
 end;

v=v_double(h:h+n,:);
vx=vx_double(h:h+n-1,:);% vertices of voronoi cell after reorder
vy=vy_double(h:h+n-1,:);% vertices of voronoi cell after reorder

Dist=zeros(n+1,n+1);
Q=zeros(n+1,3);
Q(1:end-1,:)=[vx vy zeros(n,1)];
Q(end,:)=Q(1,:);
Edge=zeros(n+1,3);
P=[v zeros(n+1,1)];

for t=1:n;
Edge(t,:)=Q(t+1,:)-Q(t,:);
end;
Edge(n+1,:)=Edge(1,:);
for t=1:n;
  for k=1:n+1;
    Dist(t,k)=norm(cross(Edge(k,:),P(t,:)-Q(k,:)))/norm(Edge(k,:));
  end;
end;
Dist(n+1,:)=Dist(1,:);

v_fixed=zeros(n,2);
v_fixed(1,:)=v(1,:);

for j=2:n;
```

```
     if min(Dist(j,:))>= r - minEbsilon
         v_fixed(j,:)=v(j,:);
     else
         a=2;
         while Dist(j,a)>min(Dist(j,:))&& a<=n+1
             a=a+1;
         end;
         % Edge(a,1:2) vertices verticej-1,j intersection point.
          l1_x1 = v(j-1,1);
          l1_y1 = v(j-1,2);
          l1_x2 = v(j,1);
          l1_y2 = v(j,2);
          l2_x1 = v(a-1,1);
          l2_y1 = v(a-1,2);
          l2_x2 = v(a,1);
          l2_y2 = v(a,2);

          dx12 = l1_x1-l1_x2;
          dx34 = l2_x1-l2_x2;
          dy12 = l1_y1-l1_y2;
          dy34 = l2_y1-l2_y2;
          dx24 = l1_x2-l2_x2;
          dy24 = l1_y2-l2_y2;

      ts = [dx12 -dx34; dy12 -dy34] \ [-dx24; -dy24];
      v(j,:)=ts(1)*[l1_x1; l1_y1] + (1-ts(1))*[l1_x2; l1_y2];
      v_fixed(j,:)=ts(1)*[l1_x1; l1_y1] + (1-ts(1))*[l1_x2; l1_y2];

     end;
 end;

v_fixed=floor(v_fixed*10^(10))/(10^10);
[Vx,iv_fixed,iVx]=unique(v_fixed(:,1));
V=v_fixed(iv_fixed,:);
x=V(:,1);
y=V(:,2);
K=convhull(x,y);
%returns the 2-D convex hull of the points (x,y), where x and y are column
%vectors. The convex hull K is expressed in terms of a vector of point
%indices arranged in a counterclockwise cycle around the hull.

m=length(V(:,1));
V_center=zeros(m+1,2);
V_center(1:end,:)=[x(K),y(K)];
```

```
%V_center is matrix of the subdomain vertices inside the ith Voronoi cell.

plot (V_center(:,1),V_center(:,2),'r-');
% Draw subdomains.
```

3. Generate circles in Voronoi cells randomly

```
xp_random=zeros(n,1);yp_random=zeros(n,1);
  for j=1:m;
  random1=sqrt(rand(1));random2=rand(1);
  xp_random(j) = (1-random1)*X(i)+random1*((1-random2)*V_center(j,1)
               +random2*V_center(j+1,1));
  yp_random(j) = (1-random1)*Y(i)+random1*((1-random2)*V_center(j,2)
               +random2*V_center(j+1,2));
  end;
%Generate point in each trangles with subdomain vertices and center.

a=randi(m);
X_p(i)=xp_random(a);
Y_p(i)=yp_random(a);
% Select one point from m points generated to be center of the circle.

ang=0:0.01:2*pi;
circlex=r*cos(ang)+X_p(i);
circley=r*sin(ang)+Y_p(i);
plot(circlex,circley,'k-');
hold on;
plot(X_p(i),Y_p(i),'r*');
% Draw circle in the ith cell.
```

Vita

Xu Huang was born in Xianyou City, Fujian, China. She finished her undergraduate studies at University of Science and Technology of China, June 2009. She came to Louisiana State University to pursue graduate studies in mathematics. She earned a master of science degree in mathematics from Louisiana State University in May 2012. She is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in May 2015.