# A characterization of near outer-planar graphs

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# A CHARACTERIZATION OF NEAR OUTER-PLANAR GRAPHS

A Thesis

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# Abstract

This thesis focuses on graphs containing an edge whose removal results in an outer-planar graph. We present partial results towards the larger goal of describing the class of all such graphs in terms of a finite list of excluded graphs. Specifically, we give a complete description of those members of this list that are not 2-connected or do not contain a subdivision of a three-spoke wheel. We also show that no members of the list contain a five-spoke wheel.

# Chapter 1
# Introduction

In short, a graph in this thesis may contain parallel edges, but not loops. More specifically, a graph $G$ is a triple $(V, E, \mathscr{I})$ where $V$ is a finite set whose elements are called *vertices*; $E$ is a finite set disjoint from $V$ whose elements are called *edges*; and $\mathscr{I}$, called the *incidence relation*, is a subset of $V \times E$ in which each edge is in relation with exactly two vertices, $u$ and $v$, called its *endpoints*. Any two vertices connected by an edge are *adjacent*. The *degree* of a vertex is the number of edges incident to the vertex. The number of vertices of a graph $G$ is the *order* of $G$ and is indicated by $|G|$. If two edges are incident to the same pair of vertices, then we call them *parallel edges*. A graph without parallel edges is called a *simple graph*. A graph $H$ is a *subgraph* of $G$ if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and $\mathscr{I}(H) \subseteq \mathscr{I}(G)$.

A *trail* is a sequence $v_0, e_0, v_1, e_1, \ldots, e_n, v_n$ where each edge, $e_i$, is incident with vertices, $v_{i-1}$ and $v_i$, and no edge is repeated. A *path* is a trail with no repeated vertices. The length of a path is the number of edges it contains. The first and the last vertices of a path are its *endpoints*. All other vertices of a path are its *internal vertices*. Two paths are *independent* if no vertex of one is an internal vertex of the other. An *isomorphism* between two graphs $G$ and $H$ is a pair of bijections, $\varphi$ and $\psi$, such that $\varphi : V(G) \to V(H)$ and $\psi : E(G) \to E(H)$, where $(u, e) \in \mathscr{I}(G)$ if and only if $(\varphi(u), \psi(e)) \in \mathscr{I}(H)$.

We call a graph *connected* if every pair of its vertices is connected by a path, and *disconnected*, otherwise. The maximal connected subgraphs of a graph are its *components*. A *cut vertex* in a graph is a vertex whose removal results in an increase in the number of components. The *connectivity* of a graph $G$ is zero if a graph is disconnected, $|G| - 1$ if $G$ is connected but has no pair of distinct non-adjacent vertices, or the size of the smallest set of vertices that disconnects $G$ if $G$ is connected and has a pair of non-adjacent vertices.

An acyclic graph is called a *forest*. A *tree* is a connected forest. A vertex of degree one or zero of a tree is called a *leaf*.

There are several classes of graphs that we will use in this thesis. A *complete graph* is a simple graph in which every pair of vertices is connected by an edge. We denote complete graphs by $K^n$, where $n$ is the number of vertices. A *bipartite graph* is composed of two disjoint sets of vertices such that each edge is incident to one vertex in each set. A *complete bipartite graph* is a simple bipartite graph in which each vertex is adjacent to every vertex in the other set. We denote a complete bipartite graph by $K_{r,s}$ where $r$ and $s$ denote the number of vertices in the disjoint sets. A *cycle* on $n$ vertices, denoted $C_n$, is a trail of $n$ vertices in which no vertices are repeated except the first equals the last. A *wheel*, $W_n$, is obtained from $C_n$ by adding a new vertex, called the *hub* and joining every vertex of $C_n$ to the new vertex. The cycle, viewed as a subgraph of $W_n$ is called the *rim*. The vertices of the rim are the *rim vertices*. The edges that connect the hub to the rim vertices are called *spokes*. Three examples of graphs listed in Figure 1.1 will play important roles in this paper.

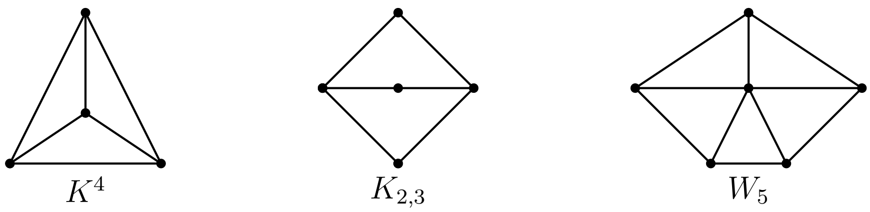



FIGURE 1.1. Examples of a complete graph, complete bipartite graph, and wheel.

A graph is *planar* if it can be drawn on the plane so that its edges only intersect at common vertices. The embedding a planar graph in the plane divides the plane into regions called *faces*. One face is unbounded; we call this the *outer face*. A graph is called *outer-planar* (OP) if it has a plane embedding in which all of the vertices lie on the boundary of the outer face. The focus of this thesis is to investigate graphs one edge away from being outer-planar graphs.

**Definition 1.1.** A graph is *near outer-planar*, or NOP, if it is edgeless or has an edge whose deletion results in an outer-planar graph.

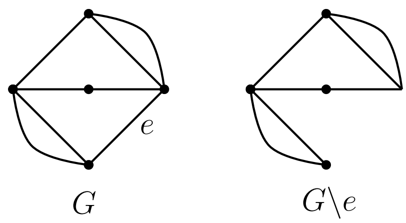


FIGURE 1.2. An example of a graph that is NOP.

The graph $G$, shown in Figure 1.2, is NOP, but $G\backslash e$ is OP. To describe the class of NOP graphs, we will provide a list of graphs which are not NOP and are minimal, in a sense that we will describe later. We will call the members of this list *excluded near outer-planar* or XNOP. First, we shall define some relations on graphs. *Edge contraction* is an operation where an edge $e$, and all edges parallel to it are removed from a graph and the two endpoints are identified to form a new vertex $v$. Any edges not parallel to $e$, but adjacent to $e$ before the contraction are incident to $v$ after the contraction. We denote an edge contraction of $G$ by $G/e$. *Edge deletion* is an operation in which an edge is removed from a graph and *vertex deletion* is an operation in which a vertex and its incident edges are removed, denoted by $G\backslash e$ and $G - v$, respectively. A graph $H$ is a *minor* of $G$ if a graph isomorphic to $H$ can be obtained from $G$ by a sequence (possibly null) of operations, each of which is one of the following three operations: contracting an edge, deleting an edge, or deleting a vertex. We denote that a graph $H$ is a minor of $G$ by $G \geqslant_{\mathrm{m}} H$ or $H \leqslant_{\mathrm{m}} G$. Similarly, a *topological minor* is obtained by a sequence (possibly null) of operations each of which is one of the following: contracting an edge incident to a vertex of degree two, deleting an edge, or deleting a vertex. An edge, $uv$, is *subdivided* if it is replaced with a path, $uwv$ of length two through a new vertex, $w$. A graph $G$ is a subdivision of another graph $H$, if a graph isomorphic to $G$ can be obtained by a sequence of subdivisions (possibly zero) of edges of $H$. An alternate way to describe $H$ as a topological minor of $G$ is to say that $G$ contains a subdivision of $H$ as a subgraph.

We are motivated by the following well-known theorem and corollary.

**Theorem 1.2.** *(Kuratowski 1930) A graph is planar if and only if it does not contain $K^5$ or $K_{3,3}$ as a topological minor.*

The following corollary can be easily derived from this theorem.

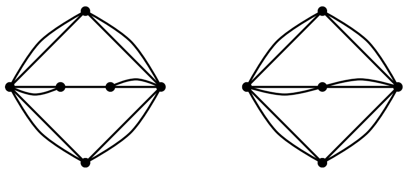
**Corollary 1.3.** *A graph is outer-planar if and only if it does not contain $K^4$ or $K_{2,3}$ as a topological minor.*

The corollary gives us a starting point for some XNOP graphs. The graphs $K^4$ and $K_{2,3}$ are NOP since each contains an edge whose removal results in a graph that is OP. (In fact, the removal of any edge from these two graphs results in a graph that is OP). We want to describe the class of NOP graphs in a finite way. Theorem 1.2 and Corollary 1.3 both describe infinite classes by excluding a finite list of graphs. We would like to give a similar description of NOP graphs. If we could use minors, we could be sure that our list is finite because of the following theorem.

**Theorem 1.4.** *(Robertson, Seymour 2004) Every infinite set of finite graphs contains two graphs, such that one is a minor of the other.*

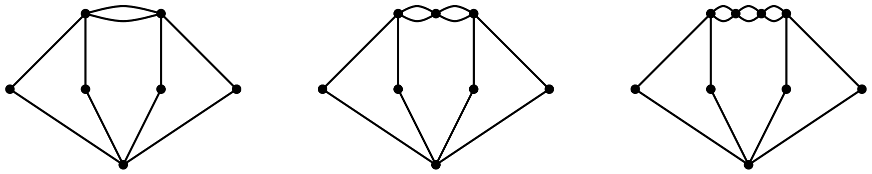
But, the excluded list of NOP graphs cannot be formed by the taking of minors, since the class of NOP graphs is not closed under the taking of minors. For example the graph $DE_1'$ shown in Figure 1.3 is NOP, but the graph $DE_1$, a minor of $DE_1'$ is not NOP.

Although the class of NOP graphs is closed by the taking of topological minors, the resulting list of XNOP graphs would not be finite. The graphs in Figure 1.4, which can be extended to form an infinite set, should be on the list of XNOP graphs since they are not NOP and not topological minors of one another. The graphs are very similar to one another



(a) $DE'_1$ (b) $DE_1$

FIGURE 1.3. $DE'_1$ is NOP, but its minor, $DE_1$ is not NOP.



$S_1$

FIGURE 1.4. Examples of graphs that have a sequence of mutiple edges.

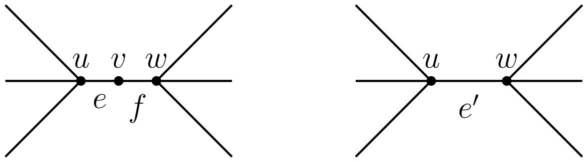
except the sequences of parallel edges have different lengths. To make our list finite, we would like to represent all graphs in this infinite set by $S_1$. We achieve this by introducing a new operation and a new relation on graphs.

**Definition 1.5.** Suppose $v$ is a vertex of $G$ with exactly two neighbors $u$ and $w$, which may or may not be adjacent to each other. Let $n$ denote the minimum of the number of $uv$ edges and the number of $vw$ edges in $G$. *Suppressing* the vertex $v$ in $G$ is the operation of replacing $v$ and all its incident edges with $n$ new $uw$ edges. An example is given in Figure 1.5.

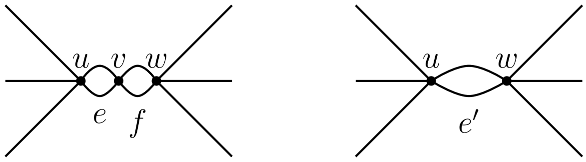
With this operation, we can define another relation.

**Definition 1.6.** A graph $H$ *dominates* a graph $G$, written $G \preceq H$, if $G$ can be obtained from $H$ by a sequence of operations each of which is one of the following:

- deleting an edge,
- deleting a vertex and all its incident edges, and
- suppressing a vertex with exactly two neighbors.




$(a)$ A suppression of $v$ incident with single edges.




$(b)$ A suppression of $v$ incident with parallel edges.

FIGURE 1.5. Suppression of $v$.

If $H$ dominates $G$ and is not isomorphic to $G$, then we say that it *properly dominates* $G$ and write $G \prec H$. Note that if $G$ is a topological minor of $H$, then $G \preceq H$.

The following proposition establishes that domination may be used in defining XNOP graphs.

**Proposition 1.7.** *The class of NOP graphs is closed under domination.*

*Proof.* Let $G$ and $H$ be graphs such that $H \preceq G$ and $G$ is NOP. We want to show that $H$ is also NOP. Since $G$ is NOP, it has an edge $e$ such that $G\backslash e$ is OP. If $H$ is isomorphic to $G$, then the conclusion follows. We assume $H \prec G$.

We examine edge- and vertex-deletions first. If a series of edge- or vertex- deletions of $H$ result in a graph isomorphic to $H$, then either $e$ is removed or not. If $e$ is removed in the series of deletions, then $H\backslash e$ is OP since $G\backslash e$ is OP. Hence, we assume that $e$ is an edge of $H$. Then $M\backslash e$ is not OP and the conclusion follows. If in the process of obtaining $H$ from $G$, the edge was deleted, then $H$ is OP and the conclusion follows.

It remains to consider the case where $e$ is an edge incident to a vertex of $G$ that is suppressed in the process of obtaining $H$ from $G$. In the suppression process, $e$ is replaced by $f$ and $H\backslash f$ is OP, hence $H$ is NOP. $\square$

We have already used the term XNOP, but in an informal way. The following defines the term XNOP precisely, and from now on, we will refer to an XNOP graph as described below.

**Definition 1.8.** A graph $H$ is *excluded near outer-planar* or *XNOP* if it is not NOP, but every graph properly dominated by $H$ is NOP.

In Appendix A, we list the 43 known graphs that are XNOP. Verifying that each of these is XNOP is tedious, so for the sake of brevity, we will only show the technique of verifying that a graph is XNOP with one example. Appendix B details the verification that $WT_4$ is XNOP. We have verified the other 42 graphs, but do not present the details here.

The set of all XNOP graphs may be divided into these sets (with the known graphs listed in parentheses):

I. Nonplanar graphs ($K_{3,3}$)

II. Disconnected graphs ($D_1$, $D_2$, $D_3$)

III. Graphs with a cut vertex ($CV_1$, $CV_2$, $CV_3$, $CV_4$, $CV_5$, $CV_6$)

IV. 2–connected graphs that do not dominate $W_3$ ($DE_1$, $K_{2,4}$, $S_1$, $S_2$, $S_3$, $S_4$, $S_5$, $S_6$)

V. 2–connected graphs that dominate $W_3$, but not $W_4$ ($WT_1$, $WT_2$, $WT_3$, $WT_4$, $WT_5$, $WT_6$, $WT_7$, $WT_8$, $WT_9$, $WT_{10}$, $WT_{11}$, $WT_{12}$, $WT_{13}$, $WT_{14}$, $WT_{15}$, $WT_{16}$, $WT_{17}$, $DE_2$, $CUBE$)

VI. 2–connected graphs that dominate $W_4$ ($WF_1$, $WF_2$, $WF_3$, $WF_4$, $K_5\backslash e$, $WF_5$).

We will devote a chapter to each of I–IV to show that each set is finite and to present the complete list of its elements. Although we strongly believe that the sets of V and VI are finite as well, we do not prove this here, although we do provide the known members of those sets, since we will use some of them in Chapters 5 and 6. The Chapters 5 and 6 present partial evidence that the sets V and VI are finite.

A common construction that we will need is that of *bridges*.

**Definition 1.9.** Let $G$ be a graph and let $J$ be a subgraph of $G$. A *bridge* of $J$ in $G$ is a subgraph $B$ of $G$ that satisfies the following:

- $B$ is not a subgraph of $J$.
- Every vertex of $B$ that is incident in $G$ with an edge not in $B$ lies in $J$.
- No proper subgraph of $B$ satisfies both (1) and (2).

Throughout this thesis, we will look at subdivisions of $K_{2,3}$ and $K^4$. Observe that $K_{2,3}$ and $K^4$ both have vertices of degree three. We refer to these vertices in $K_{2,3}$ and $K^4$ and the corresponding vertices in the subdivisions as *branch vertices*. In $K_{2,3}$, the branch vertices are the endpoints of three distinct paths, which we call the *legs* of $K_{2,3}$ or of the subdivision of $K_{2,3}$. A vertex on a leg of $H$ that is not a branch vertex is called an *internal vertex*.

The following lemma is easy to verify.

**Lemma 1.10.** *A subdivision of $K_{2,3}$ or $K^4$ has three pairwise independent paths between any two branch vertices.*

# Chapter 2
# Nonplanar Graphs

**Theorem 2.1.** *If $G$ is XNOP and nonplanar, then $G$ is $K_{3,3}$.*

*Proof.* Since $G$ is nonplanar, by Theorem 1.2, $G$ must contain $K_{3,3}$ or $K^5$ as a topological minor. But $G$ cannot contain $K^5$ since $K^5 \backslash e$ is XNOP. Then $G$ must contain $K_{3,3}$ as a topological minor and hence must dominate it. But $K_{3,3}$ is XNOP, so $G$ must be isomorphic to $K_{3,3}$. $\square$

# Chapter 3
# Disconnected Graphs

**Theorem 3.1.** *If $G$ is XNOP and disconnected, then $G$ consists of 2 components, each of which is $K_{2,3}$ or $K^4$, as in $D_1$, $D_2$, and $D_3$.*

*Proof.* If $G$ is disconnected, then it contains at least two components. We begin by proving the following.

(1) *$G$ cannot contain any components that are OP.*

Suppose $C_1$ is an OP component. Let $e$ be an edge of $C_1$. Since $G$ is XNOP, it follows that there is an edge $f$ that makes for which $G\backslash e\backslash f$ is an OP graph. Since $G$ is XNOP, it follows that $G\backslash f$ is also NOP. But, $G\backslash f$ is OP, a contradiction. This proves (1). We now focus on the number of components.

(2) *$G$ consists of exactly two components, each of which is not OP.*

Suppose $G$ has at least three components $C_1$, $C_2$, and $C_3$, each of which is not OP by (1). Clearly the removal of only two edges from $G$ always leaves a component that is not OP; a contradiction. This proves (2).

By (1) and (2), the two components of $G$ are not OP and, by Corollary 1.3 each of the two components must have $K_{2,3}$ or $K^4$ as a topological minor. Hence $G$ must dominate $D_1$, $D_2$, or $D_3$ as topological minors, and the conclusion follows. □

# Chapter 4
# Graphs with a Cut Vertex

**Theorem 4.1.** *If $G$ is XNOP and contains a cut vertex, then $G$ must be isomorphic to one of the following graphs: $CV_1$, $CV_2$, $CV_3$, $CV_4$, $CV_5$, and $CV_6$.*

*Proof.* If $G$ has a cut vertex $v$, then the removal of $v$ and its incident edges results in at least two components. Let $n$ be the number of components that results when $v$ is removed, and let $A_i$, for $i = 1, 2, \ldots, n$, be the components of $G - v$. Let $P_i$, for $i = 1, 2, \ldots, n$, be the subgraph induced by $A_i$ and $v$. We begin by observing the following fact.

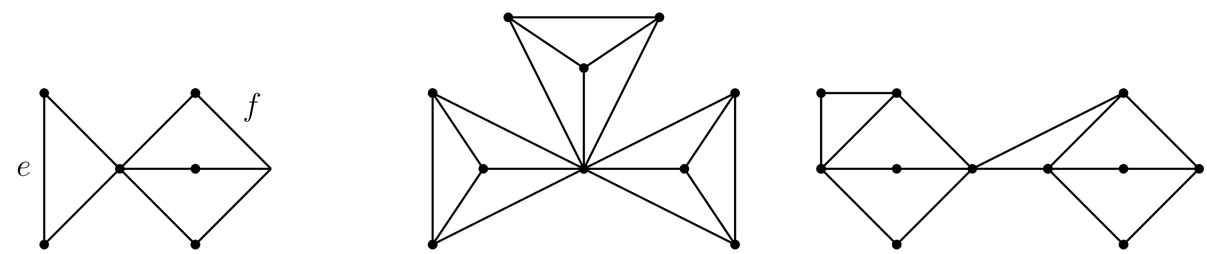
(1) *The graphs $K_{2,3}$ and $K^4$ do not have cut vertices.*

This implies that if $K_{2,3}$ or $K^4$ is a topological minor of $G$, then $K_{2,3}$ or $K^4$ is a topological minor of one of the $P_i's$. We can use this fact to prove the following.

(2) *If any $P_i$ is OP, then $G$ is not XNOP.*

Suppose $P_1$ is OP. Let $e$ be an edge of $P_1$. Since $G$ is XNOP, it follows $G \backslash e$ is NOP. Since $P_1$ is OP, then $\bigcup_{i \neq 1} P_i$ must be NOP. So there is an edge $f$ in $\bigcup_{i \neq 1} P_i$ such that $\bigcup_{i \neq 1} P_i \backslash f$ is OP. Since $G$ is XNOP, it follows that $G \backslash f$ is also NOP. But $\bigcup_{i \neq 1} P_i \backslash f$ is OP and $P_1$ is OP. So, $G \backslash f$ is OP; a contradiction. This proves (2). We now focus on the number of components.

(3) *The graph $G - v$ contains exactly two components.*




(a) OP $P_1$ and non-OP $P_2$ (b) Three $P_i$. (c) $P_i$ with bridges

FIGURE 4.1. Examples of graphs with cut vertices that are not XNOP

Suppose $G$ consists of at least three $P_i's$, say $P_1$, $P_2$, and $P_3$, each of which is not OP by (2). Clearly, the removal of only two edges from $G$ always leaves a $P_i$ that is not OP; a contradiction. This proves (3).

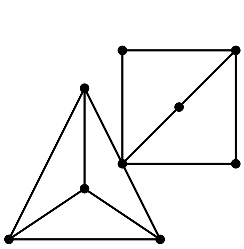
So $G$ is the union of $P_1$ and $P_2$, each of which is not OP, such that $P_1$ and $P_2$ share a single vertex $v$. By Corollary 1.3, each of $P_1$ and $P_2$ must have $K_{2,3}$ or $K^4$ as a topological minor. In fact, we can prove the following.

(4) *Each of $P_1$ and $P_2$ is a subdivision of $K_{2,3}$ or $K^4$.*

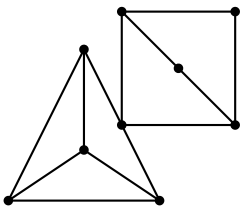
Suppose not. By (3), each of $P_1$ and $P_2$ contains a subdivision of $K_{2,3}$ or $K^4$, and so, at least one of $P_1$ or $P_2$ must properly contain a subdivision of $K_{2,3}$ or $K^4$. Let $K_1$ and $K_2$ be the subdivisions of $K_{2,3}$ or $K^4$, which are subgraphs of $P_1$ and $P_2$, respectively. If all of the edges and vertices that are not of $K_1$ or $K_2$ are removed, then the resulting graph $G'$ is either connected or disconnected. If $G'$ is disconnected, $G$ must dominate $D_1$, $D_2$, or $D_3$, a contradiction. We now assume that $G'$ is connected, and $K_1$ and $K_2$ share a common vertex $v$. But, then $G'$ dominates $CV_1$, $CV_2$, $CV_3$, $CV_4$, $CV_5$, or $CV_6$; a contradiction. See Figure 4.2. This proves (4).

So $G$ consists of two non-OP graphs, $P_1$ and $P_2$, each of which is a subdivision of $K_{2,3}$ or $K^4$, and $P_1$ and $P_2$ share a common vertex $v$. We have two types of vertices in $P_1$ and $P_2$ – vertices that are $K_{2,3}$ or $K^4$ without suppression and vertices that may be suppressed to leave a $K_{2,3}$ or $K^4$. Let the former be the K-vertices and the latter be the S-vertices. If $v$ is a K-vertex in both $P_1$ and $P_2$, then $G$ dominates $CV_1$, $CV_2$, $CV_3$, $CV_4$, $CV_5$, or $CV_6$. We assume that $v$ must be an S-vertex in at least one of $P_1$ or $P_2$, say $P_1$. If $P_1$ is a subdivision of $K^4$, then $G$ dominates $CV_3$ or $CV_4$ since every nontrivial subdivision of $K^4$ dominates a subdivision of $K_{2,3}$.

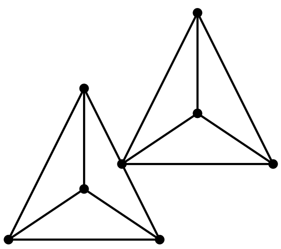
We assume that $P_1$ is a subdivision of $K_{2,3}$. Since $v$ is an S-vertex, it must be adjacent to another vertex of degree 2 in $P_1$. So $G$ dominates $CV_3$ or $CV_4$. Hence, the only XNOP graphs with a cut vertex are $CV_1$, $CV_2$, $CV_3$, $CV_4$, $CV_5$, or $CV_6$. □



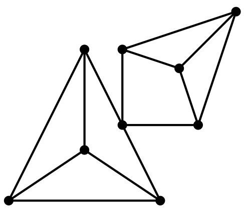
(a) This graph dominates $CV_4$.



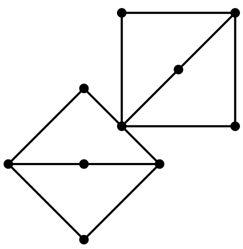
(b) This graph dominates $CV_3$.



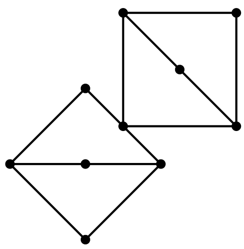
(c) This graph dominates $CV_3$.



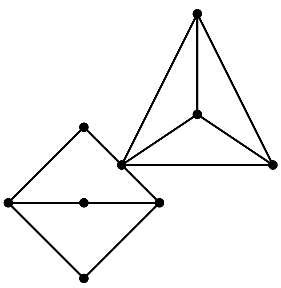
(d) This graph dominates $CV_3$.



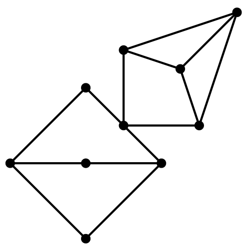
(e) This graph dominates $CV_4$.



(f) This graph dominates $CV_3$.



(g) This graph dominates $CV_3$.



(g) This graph dominates $CV_3$.

FIGURE 4.2. Examples of graphs with cut vertices with S-vertices.

# Chapter 5

## 2-Connected Graphs that Do Not Dominate $W_3$

In this chapter, we will prove a result about 2-connected graphs that do not dominate $W_3$, using the following well-known theorem.

**Corollary 5.1.** *(Menger 1927) If $G$ is a 2-connected graph and $A$, $B$ are two disjoint subsets of $V(G)$, each containing at least two elements, then $G$ contains two disjoint $A$–$B$ paths.*

In Chapters 2–4, we have exhibited complete graphs that are not 2-connected. The following theorem presents the complete list of XNOP graphs that are 2-connected, but do not dominate $W_3$.

**Theorem 5.2.** *If $G$ is XNOP, is 2-connected and does not dominate $W_3$, then $G$ is isomorphic to one of $DE_1$, $K_{2,4}$, $S_1$, $S_2$, $S_3$, $S_4$, $S_5$, and $S_6$.*

*Proof.* To make the following proof more understandable, we will break it into 7 smaller parts, numbered (1)–(7). Since $K_{3,3}$ dominates $W_3$, but $G$ does not, Theorem 2.1 implies that $G$ may be assumed to be a plane graph. It is easy to verify, following the method outlined in Appendix B, that each of $DE_1$, $K_{2,4}$, $S_1$, $S_2$, $S_3$, $S_4$, $S_5$, and $S_6$ is indeed XNOP.

Since $G$ is not OP and does not dominate $K^4$, it follows from Corollary 1.3 that $G$ must dominate $K_{2,3}$. Let $H$ be a subgraph of $G$ such that $H$ is a subdivision of $K_{2,3}$. We prove the following.

(1) *Every bridge of $H$ in $G$ has all vertices of attachment on a single leg of $H$.*

Suppose not. Let $B$ be a bridge of $H$, and let $L_1$, $L_2$, and $L_3$ be the legs of $H$. If $B$ has at least two vertices of attachment that are internal vertices of two distinct legs, say $L_1$ and $L_2$ of $H$, then the union of $B$, $L_1$, $L_2$, and $L_3$ contains a subdivision of $K^4$; a contradiction. This proves (1).

(2) *Every bridge, $B$, of $H$ in $G$ has exactly two vertices of attachment.*

Suppose not. If $B$ has fewer than two vertices of attachment, then $G$ is not 2-connected. So $B$ must have at least three vertices of attachment. We know from (1) that $B$ has all vertices of attachment on a single leg, say $L_1$, of $H$. But then the union of $B$, $L_1$, and one of the other legs of $H$ contains a subdivision of $K^4$; a contradiction. This proves (2).

Assume $n$ is an integer exceeding 2 and $K_{2,n} \leqslant_{\mathrm{m}} G$. The *skeleton* of $K_{2,n}$ in $G$ is a minimal subgraph of $G$ that contains $K_{2,n}$ as a minor. Let $F$ be a skeleton of $K_{2,n}$ in $G$, and let $D$ be a minimal set of edges of $F$ such that $F/D$ is a subdivision of $K_{2,n}$. Then the edges of $F$ that are not in $D$ form $n$ paths, each of length at least two, which are called the *legs* of the skeleton. Since $K_{2,n}$ has only two vertices of degree exceeding two, each of these vertices corresponds either to a vertex of degree $n$ in $F$, or to a connected acyclic subgraph of $F$ induced by some edges in $D$. Hence each of the two degree-$n$ vertices of $K_{2,n}$ corresponds to a subtree of $F$, which is called a *branch tree*. If a branch tree consists of a single vertex only, it is called *trivial*. Otherwise, it is called *nontrivial.* Recall that a vertex of degree one or zero of a branch tree is called a *leaf*. A vertex of a branch tree that is incident to legs of the skeleton of $F$ is called a *toe.*

(3) *Every leaf of a branch tree is adjacent to at least two legs.*

Let the branch trees of $F$ be $BT_1$ and $BT_2$. If $BT_1$ and $BT_2$ are trivial, then each leaf of $F$ is incident with at least three legs. We may assume that at least one branch tree, say $BT_1$, is nontrivial. Suppose one leaf, $w_l$, of $BT_1$ is incident with exactly one leg, $L_1$. Let $w_m$ be the vertex of $BT_1$ that is a toe and is nearest in $BT_1$ to $w_l$. Since there are no toes on the path of $BT_1$ between $w_m$ and $w_l$, it follows that $L_1$ contradicts the minimality of $D$ in the definition of a skeleton. This proves (3). See Figure 5.1.

With (3), we can also observe the following statement about leaves.

(4) *If $F$ has* 5 *or fewer legs, each of its branch trees has at most* 2 *leaves.*

Now we can prove the following statement about legs and their toes.

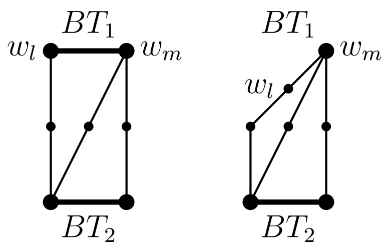



FIGURE 5.1. A contradiction of the minimality of the branch tree.

(5) *If $F$ has 5 or fewer legs and two legs share the same toe of one nontrivial branch tree, then they share a toe on the other branch tree.*

If $F$ is a skeleton of $K_{2,5}$, and $BT_1$ and $BT_2$ are the two branch trees, then by (3) each of $BT_1$ and $BT_2$ is one of the following:

($a$) a single vertex;

($b$) a nontrivial path with exactly 2 toes;

($c$) a nontrivial path with exactly 3 toes.

Now we will investigate all of the combinations of ($a$)–($c$) for each of $BT_1$ and $BT_2$. Since one branch tree is nontrivial by assumption, we need not look at the case where both branch trees are of type ($a$).

Case ($i$) Suppose one branch tree, $BT_1$, is of the type ($a$) and the other branch tree, $BT_2$, is of the type ($b$). It is obvious that if the legs of $F$ share toes at $BT_2$, then the legs share toes at $BT_1$.

Case ($ii$) Suppose both branch trees are of type ($b$). Then, all of the toes are leaves and each is incident to at least two legs. Let $w_l$ and $w_r$ be toes of $BT_1$, such that $w_l$ is incident to legs $L_1$ and $L_2$. Suppose further that $L_1$ and $L_2$ are incident to two different toes of $BT_2$, respectively, $x_l$ and $x_r$. The toe $w_r$ is incident to at least two legs, $L_3$ and $L_4$, whose other endpoints are contained in the set $\{x_l, x_r\}$ of $BT_2$. If $L_3$ and $L_4$ are not incident to the same toe of $BT_2$, then $G$ dominates $K^4$. If $L_3$ and $L_4$ are incident to the same toe of $BT_2$, then

$L_5$ must be incident to the other toe of $BT_2$ and $G$ dominates $S_4$. So, (5) holds in Case (*ii*). See Figure 5.2.

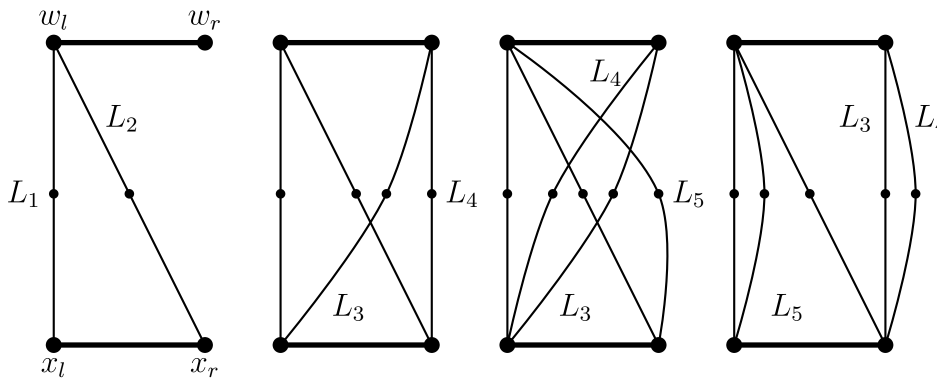



FIGURE 5.2. Subgraphs of $G$ arising in Case (*ii*) in the proof of (5).

Case (*iii*) Suppose $BT_1$ is of type (*b*) and $BT_2$ is of type (*c*). As above, suppose that $w_l$ is incident to legs $L_1$ and $L_2$ such that $L_1$ and $L_2$ are also incident to different toes of $BT_2$. Since $BT_2$ has three toes, two of which are leaves, then the legs $L_1$ and $L_2$ have toes that are both leaves or that is one leaf and one internal vertex of $BT_2$. If the incident toes of both $L_1$ and $L_2$ are the leaves, $x_l$ and $x_r$, respectively, then, by (3), the vertex $w_r$, a leaf of $BT_2$, is incident to two adjacent legs, $L_3$ and $L_4$, which also are also incident to two different toes of $BT_2$. The toes incident to $L_3$ and $L_4$ of $BT_2$ are two of $x_l$, $x_m$, or $x_l$. But, in each of the cases, the union of $L_3$, $L_4$, $BT_1$, and $BT_2$, results in a graph that dominates $K^4$. See Figure 5.3.

So, in Case (*iii*), one of $L_1$ or $L_2$, say $L_2$, is incident to $w_m$, an internal vertex of $BT_2$, and $L_1$ is incident to a leaf of $BT_2$, say $x_l$. It follows that the toe $w_r$ has two incident legs, $L_3$ and $L_4$. If both of these legs are incident to different toes of $BT_2$, then $G$ dominates $K^4$. But, if both $L_3$ and $L_4$ are incident to the same toe, $x_r$, then $G$ dominates $S_5$. So, we assume that $BT_1$ has three toes. See Figure 5.4.

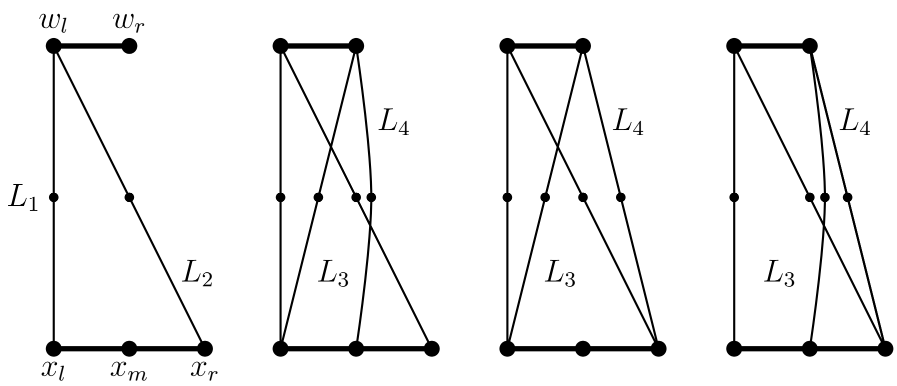



FIGURE 5.3. Graphs of Case $(iii)$ with toes of $L_1$ and $L_2$ at leaves of $BT_2$.

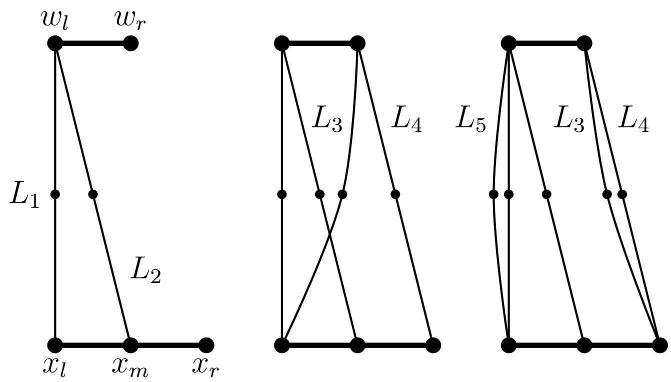



FIGURE 5.4. Graphs of Case$(iii)$ with the toe of $L_2$ at an internal vertex of $BT_2$.

Case $(iv)$ Suppose that $BT_1$ is of type $(c)$ and $BT_2$ is of type $(b)$. Then, $BT_2$ has exactly two vertices, $x_l$ and $x_r$, incident to legs. Since $w_l$ is incident to two legs which are incident to different toes of $BT_2$, then $w_m$, an internal vertex of $BT_1$, is incident to one leg, $L_3$, and $w_r$ is incident to two legs, $L_4$ and $L_5$. The leg $L_3$ is incident to either $x_l$ or $x_r$. It follows from (3), that $L_3$ and $L_4$ are not adjacent. The two possible cases that arise result in graphs, each of which dominate $K^4$. See Figure 5.5.

Case $(v)$ Suppose $BT_1$ and $BT_2$ are both of type $(c)$. So, $w_l$ is incident to legs, $L_1$ and $L_2$, which are also incident to distinct elements of $\{x_l,\ x_m,\ \text{or}\ x_r\}$. If the toes of both $L_1$ and $L_2$

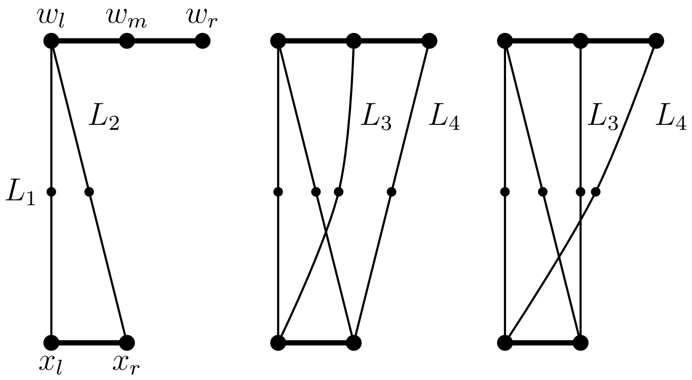



FIGURE 5.5. Graphs of Case $(iv)$.

are the leaves of $BT_2$, then the cases that arise results in graphs that dominate a $K^4$. See Figure 5.6.

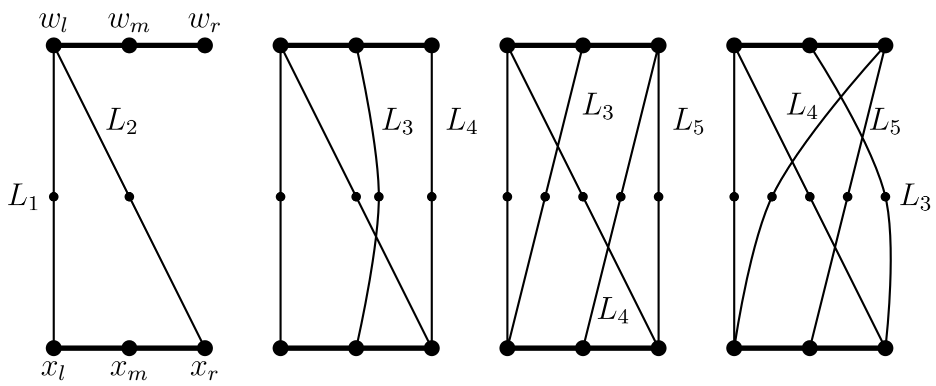



FIGURE 5.6. Graphs of Case $(v)$ with toes of $L_1$ and $L_2$ at leaves of $BT_2$.

So, we assume that one of $L_1$ and $L_2$, say $L_2$, is incident to $x_m$. It follows from (3) that $x_l$ is incident to a leg $L_3$ which is also incident to either $w_m$ or $w_r$. In either case, $w_r$ is incident to another leg, $L_4$, which is incident to $x_r$, and $G$ dominates $K^4$. See Figure 5.7 for examples of the above graphs. This proves (5).

(6) $G \not\geq_m K_{2,5}$.

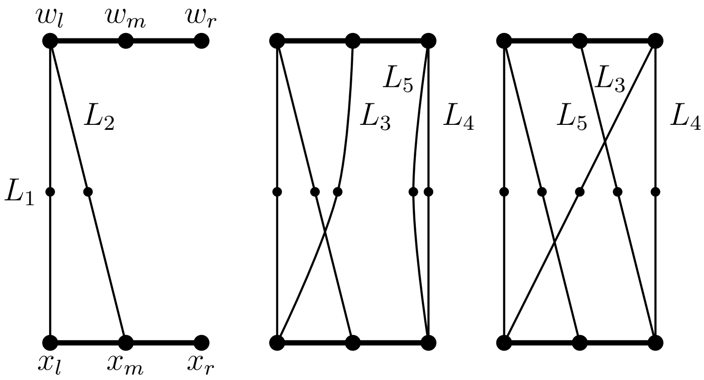



FIGURE 5.7. Graphs of Case ($v$) with the toe of $L_2$ on an internal branch vertex.

Suppose $G \geqslant_{\mathrm{m}} K_{2,5}$. Let $F$ be a skeleton of $K_{2,5}$, and let $BT_1$ and $BT_2$ be the two branch trees. It follows from (3) that $BT_1$ and $BT_2$ each can be one of ($a$)–($c$) from the proof of statement (5). We will examine each of the cases and show that each of these is a contradiction.

Case ($i$) Suppose both $BT_1$ and $BT_2$ are as described in ($a$). it follows that $G \succ K_{2,4}$; a contradiction showing that Case ($i$) cannot occur.

Case ($ii$) Suppose $BT_1$ is of the type ($a$) and $BT_2$ is of the type ($b$). It follows from (3) that $BT_2$ has two toes, $x_l$ and $x_r$, which are incident to two and three legs, respectively. Let one of the legs incident to $x_l$ be $L_2$. Since $G$ is XNOP, it follows that if we delete an internal vertex from $F$ of $L_2$, then the resulting graph $F'$ must be NOP since $G$ is XNOP. But, $G$ and $G'$ properly dominate $F'$, which dominates $K_{2,4}$. So, in fact, Case ($ii$) cannot occur. See Figure 5.8.

Case ($iii$) Suppose $BT_1$ is of the type ($a$) and $BT_2$ is of the type ($c$). It follows that $BT_2$ has three toes, two of which are leaves which are incident to two legs and one internal vertex which is incident to one leg as depicted in Figure 5.9. Let the leg incident to the internal toe of $BT_2$ be $L_3$. Since $G$ is XNOP, it follows that the graph obtained from $F$ by suppressing an internal vertex of $L_3$ is NOP. But $F' \succ S_3$ as depicted in Figure 5.9.

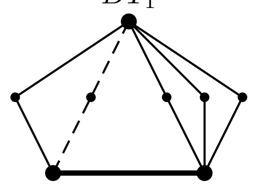



FIGURE 5.8. An example of $F$ for Case $(ii)$.

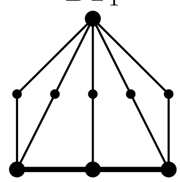



FIGURE 5.9. An example of $F$ for Case $(iii)$.

By symmetry, we have the same results if $BT_1$ and $BT_2$ were reversed in Cases $(i)$–$(iii)$. Now we may assume that neither $BT_1$ nor $BT_2$ is trivial.

Case $(iv)$ Suppose both $BT_1$ and $BT_2$ are as described in $(b)$. Let $w_l$ and $w_r$ be the toes of $BT_1$, and let $x_l$ and $x_r$ be the toes of $BT_2$. It follows from (5) that at least three legs are incident to the same toes of each branch tree, $w_l$ and $x_l$, and at least two other legs are incident to the two other toes, $w_r$ and $x_r$. Let one of the legs incident to both $w_r$ and $x_r$ be $L_2$. Since $G$ is XNOP, it follows that if we suppress an internal vertex of $L_2$ then the new graph $G'$ is NOP. But $F'$ dominates $K_{2,4}$ as depicted in Figure 5.10.

Case $(v)$ Suppose one of the branch trees is of type $(b)$ and the other is of type $(c)$. It follows from (5) that this case does not arise.

Case $(vi)$ Suppose both branch trees are as described in $(c)$. It follows from (3) that the two leaves of each branch tree are incident to two legs each and that the internal toe from each branch tree, $w_m$ and $x_m$, is incident to one leg, and from (5), the legs are incident to

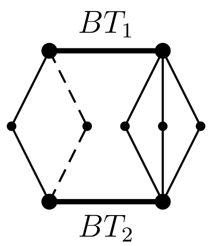



FIGURE 5.10. An example of $F$ in Case $(iv)$

the same toes on each branch tree as shown in Figure 5.11. Let the leg which is incident to toes $w_m$ and $x_m$ be $L_3$. Since $G$ is XNOP, it follows that if we suppress an internal vertex of $L_3$ in $F$, then the new graph $G'$ is NOP. But $F'$ dominates $S_6$. This proves (6). If $n$ is the largest value for which $K_{2,n}$ is a minor of $G$, then $n$ is 3 or 4.

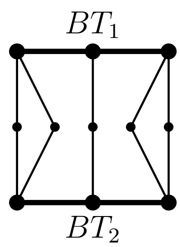



FIGURE 5.11. An example of $F$ in Case $(vi)$

(7) *If $G \geqslant_m K_{2,4}$, then $G$ is isomorphic to one of $K_{2,4}$, $S_1$, $S_2$, $S_3$, $S_4$, $S_5$, and $S_6$.*

Let $F$ be a skeleton of $K_{2,4}$ with branch trees of $BT_1$ and $BT_2$. It follows from (4) that each of $BT_1$ and $BT_2$ is described as in $(a)$ or $(b)$, as listed in the proof of (5). This gives us three cases.

Case $(i)$ Suppose both $BT_1$ and $BT_2$ are of type $(a)$. Then $G \succ K_{2,4}$.

So at least one of $BT_1$ or $BT_2$ has two vertices. It follows from (1) and (2) above, that bridges are attached in one of the following three ways: from $BT_1$ to $BT_2$, on a single leg, or only on one branch tree.

Case $(ii)$ Suppose $BT_1$ is of the type $(a)$ and $BT_2$ is of the type $(b)$. If $F$ has a bridge from $BT_1$ to an internal vertex of $BT_2$, then $G$ dominates $S_3$ as shown in Figure 5.12.

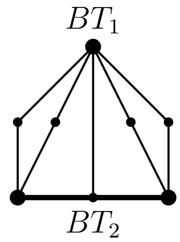



FIGURE 5.12. An example of $F$ in Case $(ii)$.

Now we consider the subgraph, $Z$, which consists of $BT_2$ and all bridges of $F$ in $G$ whose vertices of attachment lie only in $BT_2$. From (4) and (5), we know that the graph $G$ is a union of three graphs, $R_1$, $R_2$, and $Z$, such that $R_1$ and $R_2$ have two legs of $K_{2,4}$ each and share $BT_1$.

Suppose there is an edge $z$ in $Z$ whose deletion separates the endpoints of $BT_2$ in $Z$ into $Z_1$ and $Z_2$, where $Z_i$ has a vertex in common with $R_i$ for $i = 1, 2$. Consider $G\backslash z$. It is the union of two graphs $R'_1$ and $R'_2$, where $R'_i$ is the union of $R_i$ and $Z_i$. Since $G$ is XNOP, $G\backslash z$ is not OP. Since $G \not\succeq W_3$, $G\backslash z$ dominates $K_{2,3}$, and a subdivision of $K_{2,3}$ is in either $R'_1$ or $R'_2$. Let $K$ be the subdivision of $K_{2,3}$. Without loss of generality, suppose $K$ is in $R'_1$. Since $G$ is 2-connected, Theorem 5.1 implies that there are two disjoint paths in $G$ from $R_2$ to $K$. One of these paths is through $BT_1$ and the other through the endpoint of $BT_2$ shared by $R_2$. Consider the ends of these two paths in $K$. By (1), these endpoints must lie on the same leg of $K$. If the endpoints of the paths are the branch vertices of $K$, then $G \succ K_{2,4}$. So, at least one path's endpoint must be an internal vertex of a leg of $K$. Since $G$ is 2-connected, the endpoints of both paths cannot be at the same vertex, or else $G$ has a cut-vertex. If both of the endpoints of the paths are two different internal vertices of a leg of $K$, then $G \succ S_6$. We assume that one endpoint must be a branch vertex and the other an internal vertex. But $G \succ S_5$. See Figure 5.13. Hence, $G \backslash z$ does not dominate $K_{2,3}$.

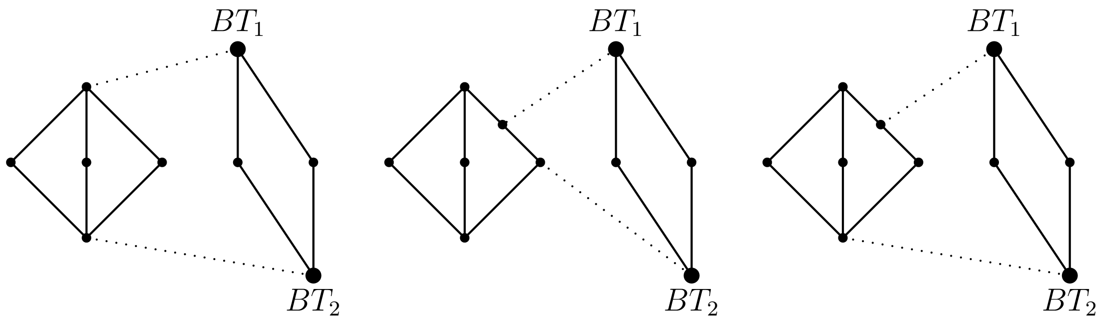



FIGURE 5.13. Examples of Case $(ii)$ with $G \backslash z$

It suffices to consider the case where no edge separates $Z$. So, there are two distinct paths, $P_1$ and $P_2$, in $Z$, which do not share edges, from one endpoint of $BT_1$, $x_l$, to the other endpoint, $x_r$. It follows that $G \succeq S_1$. Hence, neither $BT_1$ or $BT_2$ is trivial.

Case $(iii)$ Suppose both $BT_1$ and $BT_2$ are both of type $(b)$. Bridges are attached in one of the following three ways: from $BT_1$ to $BT_2$, on a single leg, or on a single branch tree. If a bridge $B$ has vertices of attachment on both $BT_1$ and $BT_2$, then the vertices of attachment can either be internal vertices or endpoints of the branch trees. We can eliminate some graphs with certain types of bridges. If both of the vertices of attachment of $B$ are each internal vertices of both branch trees, then $G$ dominates $S_6$. If one of the vertices of attachment of $B$ is an endpoint of a branch tree and the other is an internal vertex of the other branch tree, then $G$ dominates $S_5$. If both vertices of attachment are endpoints of each branch tree, but do not share any adjacent legs, then $G$ dominates $S_4$. See Figure 5.14.

So, a bridge $B$ is attached in one of three ways: to a single leg or to a single branch tree. Let $A$ be the union of $BT_1$ and all bridges of $F$ in $G$ which lie only on $BT_1$ and $Z$ be the subgraph $BT_2$ and all bridges which lie only on $BT_2$. Then $G$ is the union of $A$, $Z$, $R_1$ and $R_2$, where $R_1$ and $R_2$ have two legs of $K_{2,4}$ each. Suppose there is an edge $z$ in $Z$ whose deletion

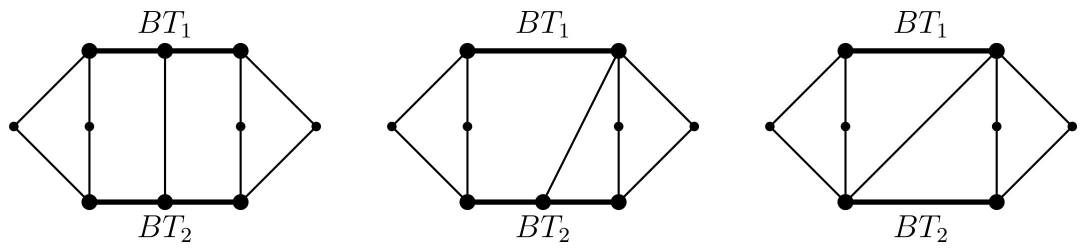



FIGURE 5.14. Case $(iii)$ graphs that dominate $S_4$, $S_5$, or $S_6$

separates the endpoints of $BT_2$ into two subgraphs $Z_1$ and $Z_2$ such that $Z_i$ has a vertex in common with $R_i$ for $i = 1, 2$. Consider $G\backslash z$. It is the union of three graphs $A$, $R_1'$ and $R_2'$, where $R_i'$ is the union of $R_i$ and $Z_i$. Since $G$ is XNOP, $G\backslash z$ is not OP and dominates $K_{2,3}$. So, a subdivision of $K_{2,3}$ is in either $R_1$ or $R_2$. Let $K$ be the subdivision of $K_{2,3}$. Without loss of generality, suppose $K$ is in $R_1$. Since $G$ is 2-connected, Theorem 5.1 implies that there are two disjoint paths from $R_2$ to $K$. One of these paths is through $BT_1$ and the other through $BT_2$. Consider the ends of these two paths on $K$. By (1), these endpoints must lie on the same leg of $K$. If the endpoints of the paths are the branch vertices of $K$, then $G \succ K_{2,4}$. So, at least one path's endpoint must be an internal vertex of a leg of $K$. If both of the endpoints of the paths are internal vertices of a leg of $K$, then $G \succ S_6$. So, one endpoint must be a branch vertex and the other an internal vertex. But $G \succ S_5$. See Figure 5.15. Hence, $G\backslash z$ does not dominate $K_{2,3}$ and because of symmetry, $G\backslash a$ does not dominate $K_{2,3}$, where $a$ is an edge whose deletion separates the endpoints of $BT_1$.

We may assume that no single edge separates $A$ and no single edge separates $Z$. So, each branch tree has two distinct paths from one of its endpoints to the other, and $G \succeq S_2$. This exhausts the cases of $G \geqslant_{\mathrm{m}} K_{2,4}$ and proves (7).

We assume that $G \not\succcurlyeq K_{2,4}$. By (3) and (4) above, both branch trees of $G$ are trivial, and there is a skeleton $F$ which is a subdivision of $K_{2,3}$. Let the legs of the subdivision of $K_{2,3}$ be $L_1$, $L_2$, and $L_3$. By (1) and (2), every bridge of $F$ has 2 vertices of attachment on a single

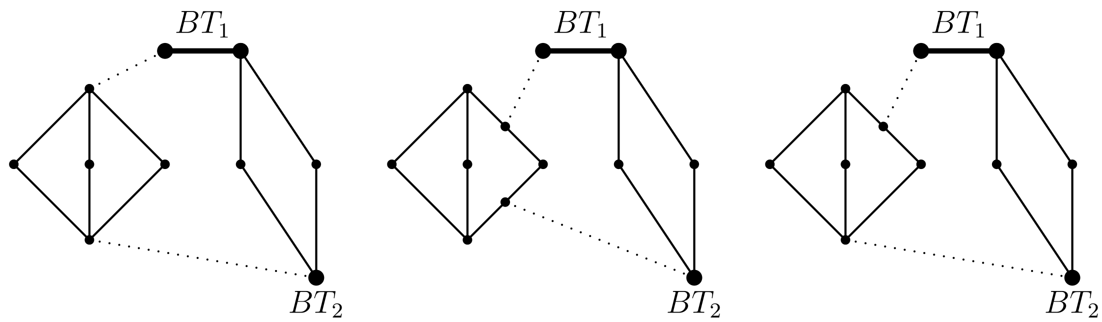



FIGURE 5.15. Graphs of Case $(iii)$ with $G\backslash j_2$.

leg of $K_{2,3}$. So, $G$ is the union of 3 graphs, $M_1$, $M_2$, and $M_3$, such that $M_i$ is $L_i$ along with any of its bridges. Any two of the subgraphs $M_1$, $M_2$, and $M_3$ meet only at $BT_1$ and $BT_2$.

Suppose there is an edge $m$ of $M_1$ such that $M_1\backslash m$ separates $BT_1$ from $BT_2$ in $M_1$. Consider $G\backslash m$. Since $G$ is XNOP, $G\backslash m$ is not OP and dominates $K_{2,3}$. So a subdivision of $K_{2,3}$ is in the union of $M_2$ and $M_3$. Let $K$ be the subdivision of $K_{2,3}$. By Menger's, Theorem 5.1, there are 2 disjoint paths to $K$ through $BT_1$ and $BT_2$. So, $K$ is either entirely in $M_2$, entirely in $M_3$, or meets edges in both. But, if $K$ is entirely in $M_2$ or $M_3$, then $G \geqslant_{\mathrm{m}} K_{2,5}$. So, $K$ has at least one leg each in $M_2$ and $M_3$. The third leg of $K$ cannot be in $M_1$, so it must be a bridge of $M_2$ or $M_3$. But, then $G \geqslant_{\mathrm{m}} K_{2,4}$.

We may assume that no single edge separates $BT_1$ from $BT_2$ in $M_1$. Similarly, no single edge separates $BT_1$ from $BT_2$ in $M_2$ or $M_3$. So, there are two paths from $BT_1$ to $BT_2$ in each of $M_1$, $M_2$, and $M_3$. Each of these paths must share a vertex in each of $M_1$, $M_2$, and $M_3$ since $G \not\geqslant K_{2,4}$. But, then $G \succ DE_1$. $\square$

# Chapter 6
# Graphs that Dominate $W_5$

**Theorem 6.1.** *No XNOP graph dominates* $W_5$.

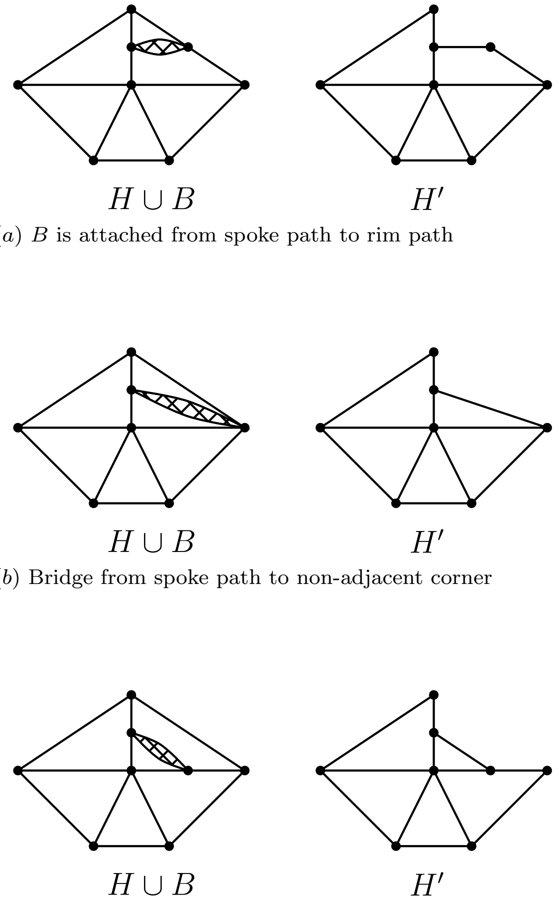
*Proof.* To make the following proof more understandable, we will break it into 7 smaller parts, numbered (1)–(7). If $G$ is XNOP and nonplanar, then Theorem 2.1 implies that $G$ is isomorphic to $K_{3,3}$, and $G \not\succ W_5$. Hence we assume $G$ is a plane graph and that is $G$ is XNOP. The graph $H$ is a subgraph of $G$ that is a subdivision of $W_n$, for $n \geq 5$, such that $G$ contains no subdivision of $W_{n+1}$. The spokes and the rim edges of $W_n$ correspond to *spoke paths* or *rim paths* in $H$. The hub of $H$ corresponds to the hub of $W_n$. A vertex of $H$ that corresponds to a vertex on $W_n$ and is incident to both the rim and a spoke is called a *corner*. The union of all rim paths of $H$ is the *rim*. It is easy to see that, without loss of generality, we may assume the hub of $H$ lies in the finite region $R$ of the plane that is homeomorphic to a disk and bounded by the rim. Moreover, we chose $H$ so that no other subdivision of $W_n$ is contained in $R$. The plane embedding induces a cyclic order on the spoke paths. Two spoke paths are *consecutive* if they are adjacent in that cyclic order.

(1) *If a bridge $B$ of $H$ in $G$ has one vertex of attachment that is an internal vertex of a spoke path of $H$, then all other vertices of attachment of $B$ are on the same spoke path, possibly including the hub and the corner.*

Suppose not. If $B$ has vertices of attachment one of which is a vertex of spoke path and the other of which is an internal vertex of a rim path, then this contradicts the choice of $H$. See Figure 6.1 ($a$).

If $B$ has vertices of attachment one of which is an internal vertex of a spoke path and the other of which is a non-adjacent corner vertex on an adjacent rim path, then $H$ is not a minimal subdivision of $W_n$. See Figure 6.1 ($b$).

Similarly, if $B$ has vertices of attachment of internal vertices on two distinct spoke paths, then $H$ is not a minimal subdivision of $W_5$. See Figure 6.1 ($c$) for an example. So, $B$ has vertices of attachment of a internal vertex on the spoke path, a corner on the spoke path, or the hub. This proves (1).



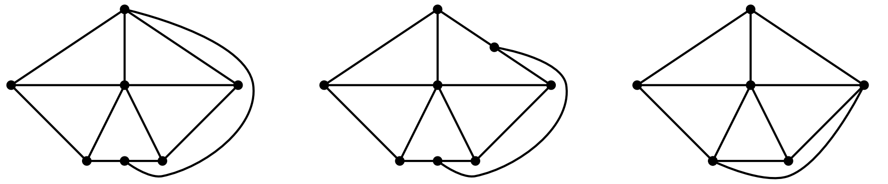

($a$) $B$ is attached from spoke path to rim path

($b$) Bridge from spoke path to non-adjacent corner

($c$) Bridge from spoke path to spoke path

FIGURE 6.1. $H \cup B$ contains $H'$, which contradicts the choice of $H$.

The *span* of a path, $P$, which is a subgraph of the rim, between two vertices, $u$ and $v$, is a pair of numbers of which the first is the number of corners other than $u$ and $v$ in $P$ and the second is the number of the vertices, $u$ and $v$, that are corners. Spans are ordered lexicographically. Since the rim of $H$ is a cycle, it contains two independent $u$–$v$ paths. The



($a$) Span of $(2,1)$ ($b$) Span of $(2,0)$ ($c$) Span of $(1,2)$

FIGURE 6.2. $H \cup B$ with spans of $B$ greater than or equal to $(1,2)$.

span of two vertices of the rim is the minimum span of the two independent $u$–$v$ paths contained in the rim. A bridge all of whose vertices of attachment are on the rim of $H$ is called an *outer bridge*, and an *inner bridge*, otherwise. The span of an outer bridge of the rim is the maximum span of all of the pairs of vertices of attachment. We prove the following.

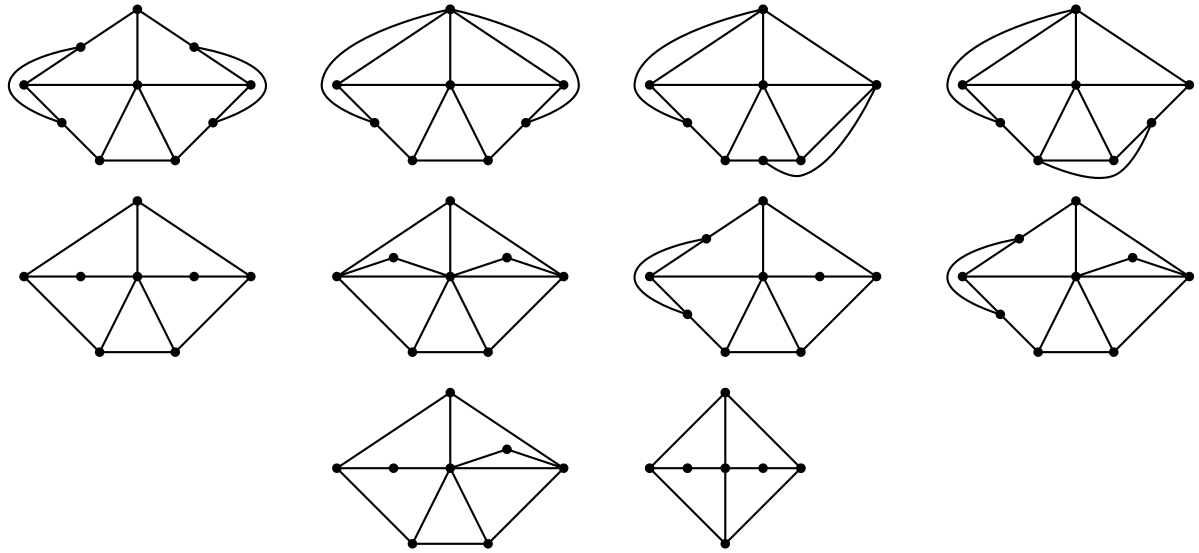
(2) *An outer bridge $B$ of $H$ in $G$ has span less than* $(1,2)$.

Let $k$ be the number of corners of $H$. The outer bridge $B$, with span $(m,n)$, where $0 \leq m \leq k$ and $0 \leq n \leq k$, has corresponding vertices of attachment $u$ and $v$. There are two disjoint paths on the rim of $H$ from $u$ to $v$. One path has $m$ corners and the other path has $m'$ corners, where $m \leq m' \leq k-m$. Clearly $n \leq 2$. If $B$ has a span of $(q,1)$ or $(q,0)$, where $q \geq 2$, then $G \succeq WF_6$. See Figure 6.2 ($a$) and ($b$). If $B$ has a bridge span of $(p,2)$, where $p \geq 1$, then $H \succeq K_5 \backslash e$. See Figure 6.2 ($c$). So, the span of $B$ is smaller than $(1,2)$. This proves (2).

It follows from (2) that all vertices of attachment of a bridge $B$ lie on a path whose span is less than $(1,2)$. The minimal such path is the *base* of the bridge. Without loss of generality, we assume that the embedding of $G$ on the plane is such a way that the vertices of the rim that do not lie on the base of $B$ are on the boundary of the outer face of $H \cup B$. A bridge *spans a corner* if the corner is a vertex of the base. A bridge *spans an edge* is it is an edge of the base. A bridge is *weak* if it consists of a single edge, and *strong* otherwise.

Let $c_i$ for $i = 1,2,\ldots,n$ be the corners of $W_n$ of $H$. We can prove the following.

(3) *There are three consecutive corners $c_1$, $c_2$, and $c_3$ of $H$ that satisfy the following conditions:*



$WF_1$

FIGURE 6.3. Graphs of Statement (3) that dominate $WF_1$.

($a$) *none of* $c_1$*,* $c_2$*, or* $c_3$ *is spanned by an outer bridge of* $H$*;*

($b$) *each of* $c_1$*,* $c_2$*, and* $c_3$ *are adjacent to the hub; and*

($c$) *none of* $c_1$*,* $c_2$*, or* $c_3$ *is a vertex of attachment of a strong inner bridge of* $H$*.*

If two or more non-adjacent corners each violate one of the conditions, then clearly $G \succ WF_1$. If exactly one corner or two adjacent corners violate any of these conditions, then there are three remaining consecutive corners that do not violate the conditions. See Figure 6.3. This proves (3).

So, $H$ has 3 consecutive corners which lie on the boundary of the outer face whose corresponding spoke paths of length 1 are $SP_1$, $SP_2$, $SP_3$. Of the other two corners, $c_4$ and $c_5$, only one can be spanned. So, one of $c_4$ or $c_5$ must be on the boundary of the outer face. The corresponding spoke paths may be subdivided or not.

Since $G$ is XNOP, every graph that $G$ properly dominates is NOP. Since $SP_2$ is a single edge, it follows that $G \backslash SP_2$ is NOP. So there is an edge $f$ such that $G \backslash SP_2 \backslash f$ is OP; a contradiction.

(4) *The edge* $f$ *is on the rim of* $H$*.*

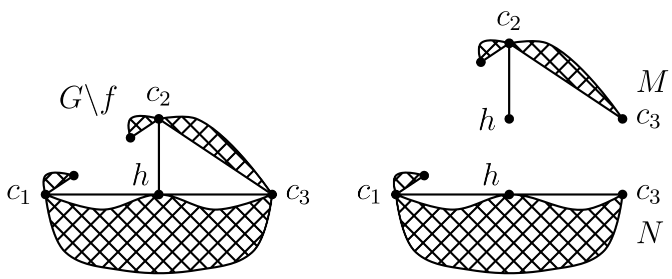



FIGURE 6.4. A representation of $G\backslash f$ and the subgraphs $M$ and $N$ of $G\backslash f$.

If $f$ is not an edge of the rim, then $G\backslash SP_2\backslash f$ has at least 3 spoke paths, which, together with the rim of $H$, form $K^4$; a contradiction. This proves (4).

(5) *No bridge spans $f$.*

Suppose a bridge spans $f$. By (2), the bridge must have vertices of attachment on the same rim path or on two adjacent rim paths. It follows that $G\backslash f \succ W_4$. But, $G\backslash f\backslash SP_2$ is not OP since $G\backslash f\backslash SP_2 \succ K^4$; a contradiction. This proves (5).

We now focus on the rim paths of $H$. Let $P_{i,i+1}$ be the rim path between $c_i$ and $c_{i+1}$ for $i = 1, 2, \ldots 4$ and let $P_{5,1}$ be the rim path between $c_5$ and $c_1$.

(6) *The edge $f$ is not an edge of $P_{1,2} \cup P_{2,3}$.*

Suppose $f$ is an edge of $P_{1,2}$. Consider $G\backslash f$. By (3) and (5), $G\backslash f$ is a graph with the hub $h$ and three corners, $c_1$, $c_2$, and $c_3$, on the boundary of the outer face. The graph $G\backslash f$ has two subgraphs, $M$ and $N$, each of which is connected, and such that $M \cap N = \{h, c_3\}$, $M \cup N = G\backslash f$, $c_2 \in V(M)$, and all edges between $h$ and $c_3$ lie in $N$. See Figure 6.4. Since $G$ is XNOP, it follows that $G\backslash f$ is not OP and so contains $K$, a subdivision of $K_{2,3}$ or $K^4$. Since $G\backslash SP_2\backslash f$ is OP, the edge $SP_2$ must be an edge of $K$.

We will now examine the location of the branch vertices of $K$ in relation to $M$ and $N$. Suppose two of the branch vertices of $K$ are $h$ and $c_3$. Observe that $K_{2,3}$ with a path between two internal vertices of two different legs of $K_{2,3}$ is a subdivision of $K^4$. By the structure of $W_n$ and Lemma 1.10, there are two independent paths in $N$ from $c_3$ to $h$. By (3), $SP_3$ cannot have a strong bridge with vertices of attachment at $h$ and $c_3$. See Figure 6.5 ($a$). So,

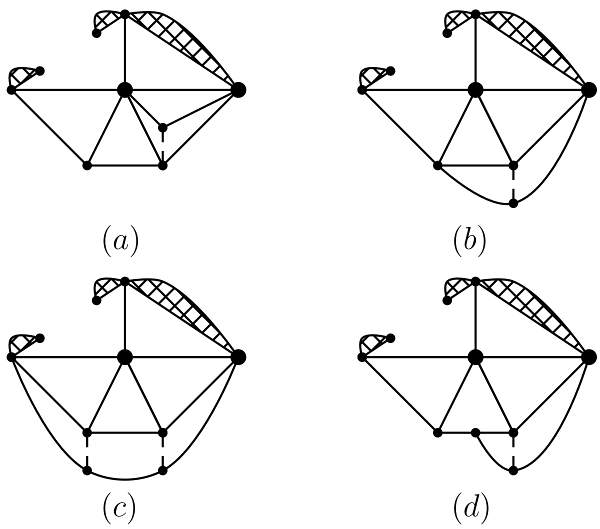



FIGURE 6.5. The subgraph $K$ of $G\backslash f$ cannot have branch vertices at $h$ and $c_3$.

at least one leg of $K$ contains an edge in an outer bridge that spans $c_4$. But, then $N \succ K^4$; a contradiction. See Figure 6.5 $(b)$–$(d)$. Thus, we assume that not both $h$ and $c_3$ are branch vertices.

Suppose at least one branch vertex of $K$ is in $N - \{c_3, h\}$. It follows from Lemma 1.10 that upon replacing the path of $K$ between $c_3$ and $h$ containing $SP_2$ with $SP_3$, the subgraph $N$ contains another subdivision of $K_{2,3}$ or $K^4$, which does not involve $SP_2$ and $G\backslash f\backslash SP_2$ is not OP; a contradiction.

Hence, $M - \{h, c_3\}$ contains at least one of the branch vertices of $K$. Let $M'$ be the subgraph of $M$ composed of $P_{2,3}$ together with the bridges of $H$ in $G$ that attach to $P_{2,3}$, and let $K \cap M' = K^-$. It follows that $M$ contains two of the legs of $K$. So, $K^-$ is a subdivision of $K_{2,3}$ minus a leg, or a subdivision of $K^4$ minus an edge. See Figure 6.6.

Suppose $M'$ contains an edge $g$ that separates $c_2$ from $c_3$ in $M'$. The graph $G\backslash g$ is not OP, so it contains $K''$, a subdivison of $K_{2,3}$ or $K^4$, such that $K''$ meets $K^-$ only possibly at $c_2$ or $c_3$. Since $G$ is 2-connected, it contains two disjoint paths, one of which may be edgeless and the other of which contains $g$, from $K^-$ to $K''$. It follows that $G$ dominates one of $K_{2,4}$, $S_3$, $S_4$, $S_5$, $S_6$, $WT_{10}$, $WT_{11}$, $WT_{12}$, and $WT_{13}$; a contradiction. See Figure 6.7.

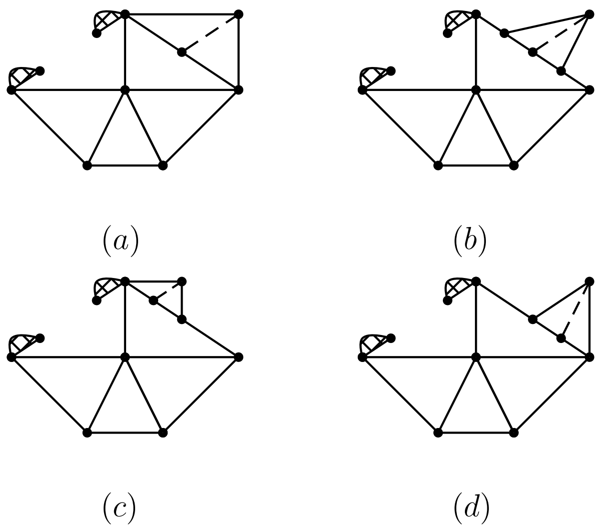


FIGURE 6.6. The subgraph $K^{-}$ of $K$ .

Hence, we now assume that no single edge of $M'$ separates $c_2$ from $c_3$ in $M'$. But, then $G$ dominates $K_{2,4}$, $S_1$, or $S_2$; a contradiction. This shows that $f$ cannot lie on $P_{1,2}$. By symmetry, $f$ cannot lie on $P_{2,3}$ either. This proves (6).

We now consider the case where $f$ is an edge of one of $P_{3,4}$, $P_{4,5}$, and $P_{5,1}$. By (1) and (6), no bridge of $H$ has vertices of attachment that are internal vertices of two spoke paths or internal vertices of rims, and no bridge spans $f$.

Since $G$ is XNOP, it follows that $G \backslash f$ is not OP and so contains $K$, a subdivision of $K_{2,3}$ or $K^4$. Since $G \backslash SP_2 \backslash f$ is OP, the edge $SP_2$ must be an edge of $K$. Let $P_{3,1}$ be the union of $P_{3,4}$, $P_{4,5}$, and $P_{5,1}$. Then, $f$ separates the endpoints of $P_{3,1}$ in $P_{3,1}$ and therefore the hub $h$ is incident to the boundary of the outer face of $G \backslash f$.

The graph $G \backslash f$ has two subgraphs, $M$ and $N$, each of which is connected, and such that $M \cap N = \{h, c_2\}$, $M \cup N = G \backslash f$, $c_1 \in V(M)$, $c_3 \in V(N)$, and all edges between $h$ and $c_2$ lie in $N$. See Figure 6.8. So, by Lemma 1.10 $K$ must lie either entirely in $N$ or entirely in $M \cup SP_2$. First we assume that $K$ is in $N$. But, then $G \backslash SP_2$ is not OP, since the union of $K \backslash SP_2$, $SP_1$, and $P_{1,2}$ is also a subdivision of $K_{2,3}$ or $K^4$; a contradiction. now, if $K$ is in $M \cup SP_2$, then $G \backslash SP_2$ is not OP since the union of $K \backslash SP_2$, $SP_3$, and $P_{2,3}$ is also a subdivision

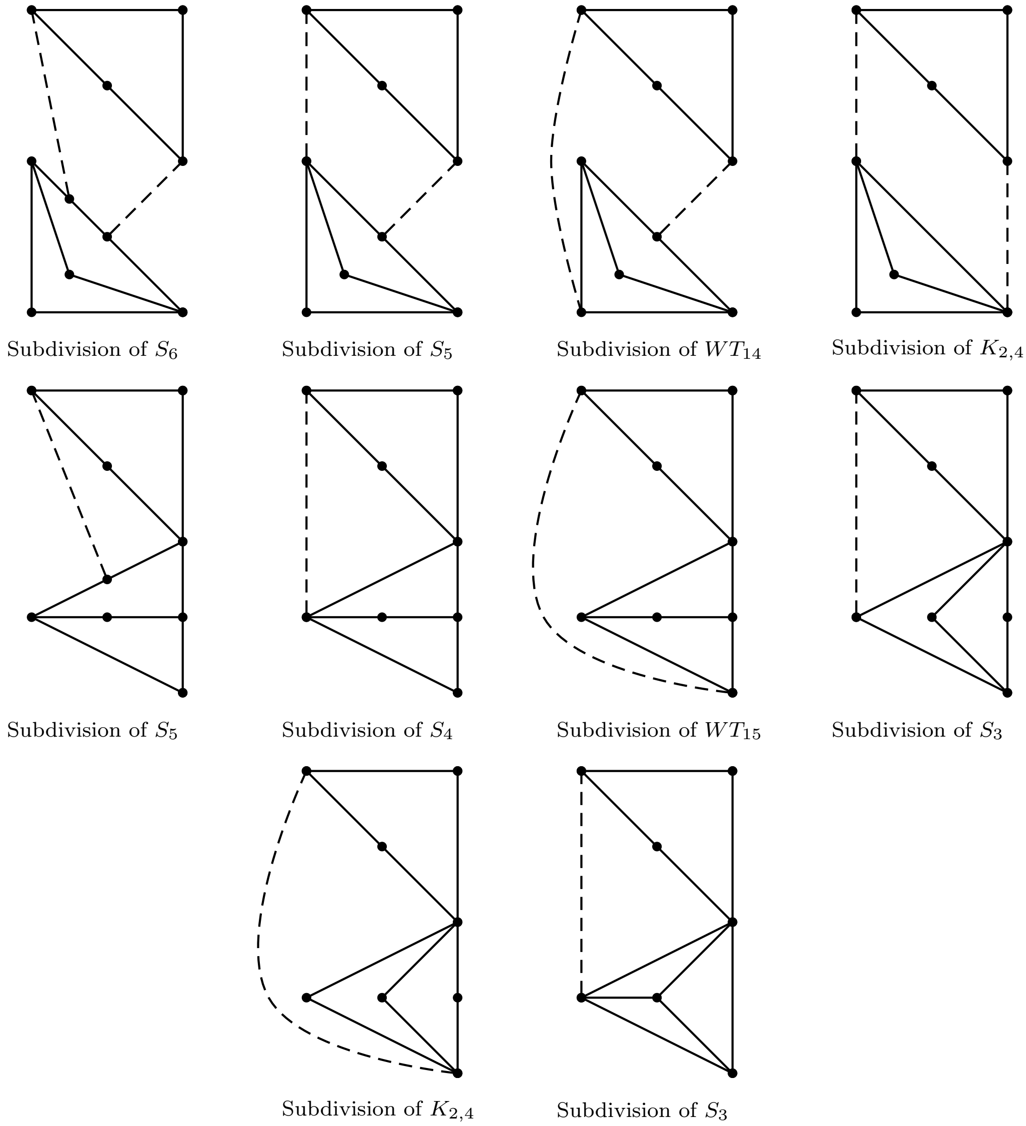



FIGURE 6.7. No single edge in $M'$ separates $c_2$ from $c_3$ in $H$.

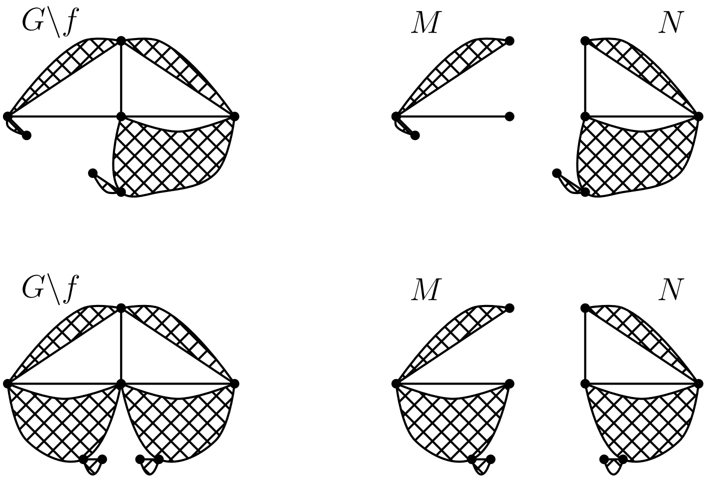



FIGURE 6.8. Graphs of $G\backslash f$, where $f$ is an edge of $P_{3,1}$, and the subgraphs $M$ and $N$.

of $K_{2,3}$ or $K^4$; a contradiction. Therefore, $f$ is not an edge of the rim contradicting (4) and so $G \not\succeq W_5$. $\square$

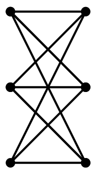
# References

[1] R. Diestel, *Graph Theory, 3rd Edition*, Graduate Texts in Mathematics, Springer-Verlag, Heidelberg 2006.

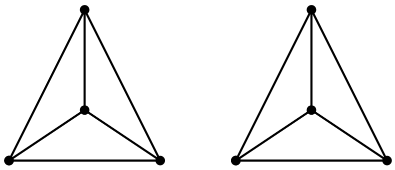
[2] D. West, *Introduction to Graph Theory, 1st Edition*, Prentice-Hall, Inc., New Jersey, 1996.

[3] N. Robertson, P. D. Seymour, *Graph Minors. XX. Wagner's Conjecture*, J. Combin. Theory, Ser. B **92** (2004): 325-357.

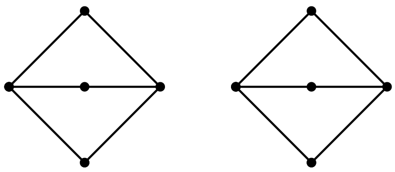
# Appendix A: List of XNOP Graphs



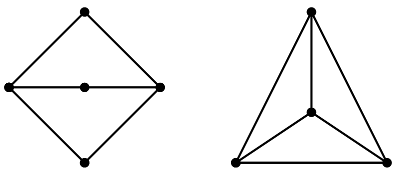
$K_{3,3}$



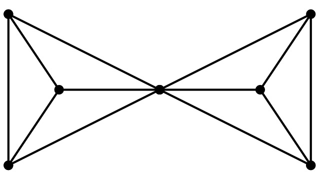
$D_1$



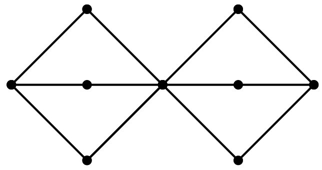
$D_2$



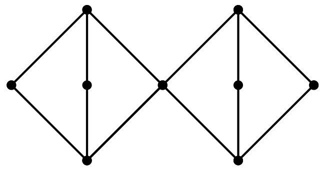
$D_3$



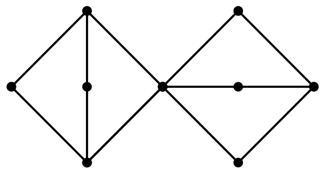
$CV_1$



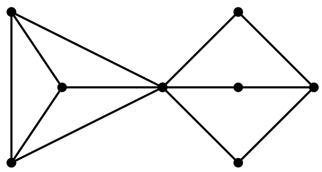
$CV_2$



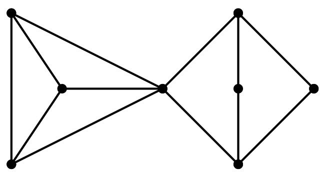
$CV_3$



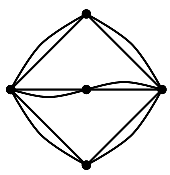
$CV_4$



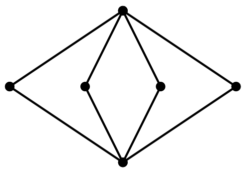
$CV_5$



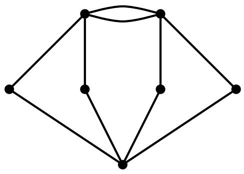
$CV_6$



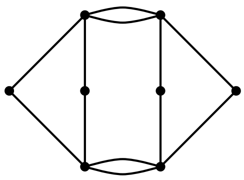
$DE_1$



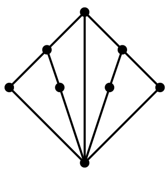
$K_{2,4}$



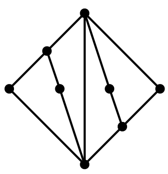
$S_1$



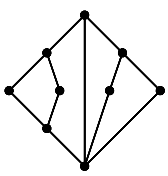
$S_2$



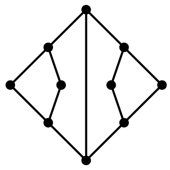
$S_3$



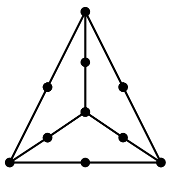
$S_4$



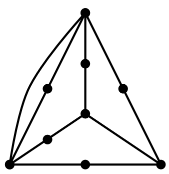
$S_5$



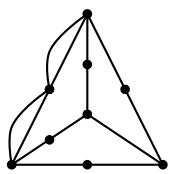
$S_6$



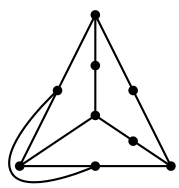
$WT_1$



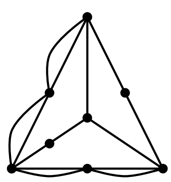
$WT_2$



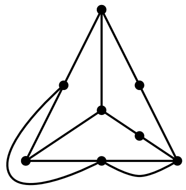
$WT_3$



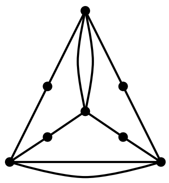
$WT_4$



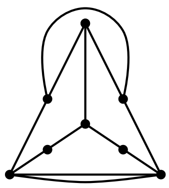
$WT_5$



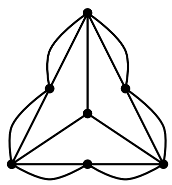
$WT_6$



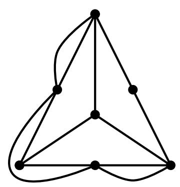
$WT_7$



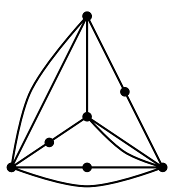
$WT_8$



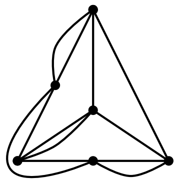
$WT_9$



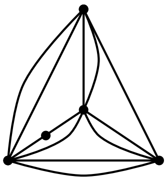
$WT_{10}$



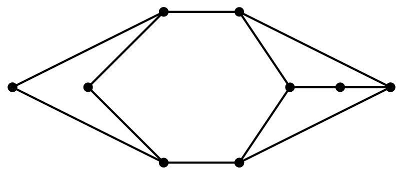
$WT_{11}$



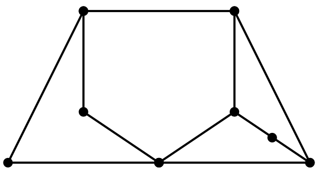
$WT_{12}$



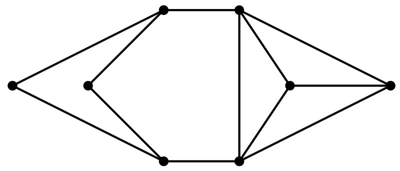
$WT_{13}$



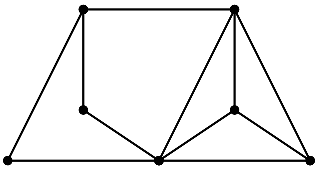
$WT_{14}$



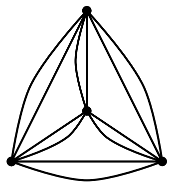
$WT_{15}$



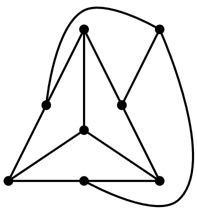
$WT_{16}$



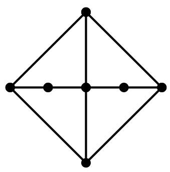
$WT_{17}$



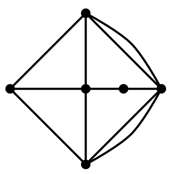
$DE_2$



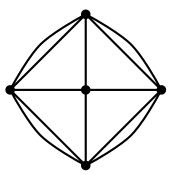
$CUBE$



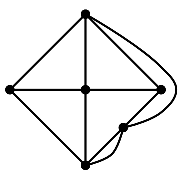
$WF_1$



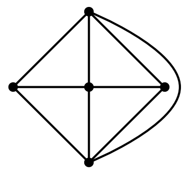
$WF_2$



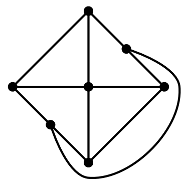
$WF_3$



$WF_4$



$K_5 \backslash e$



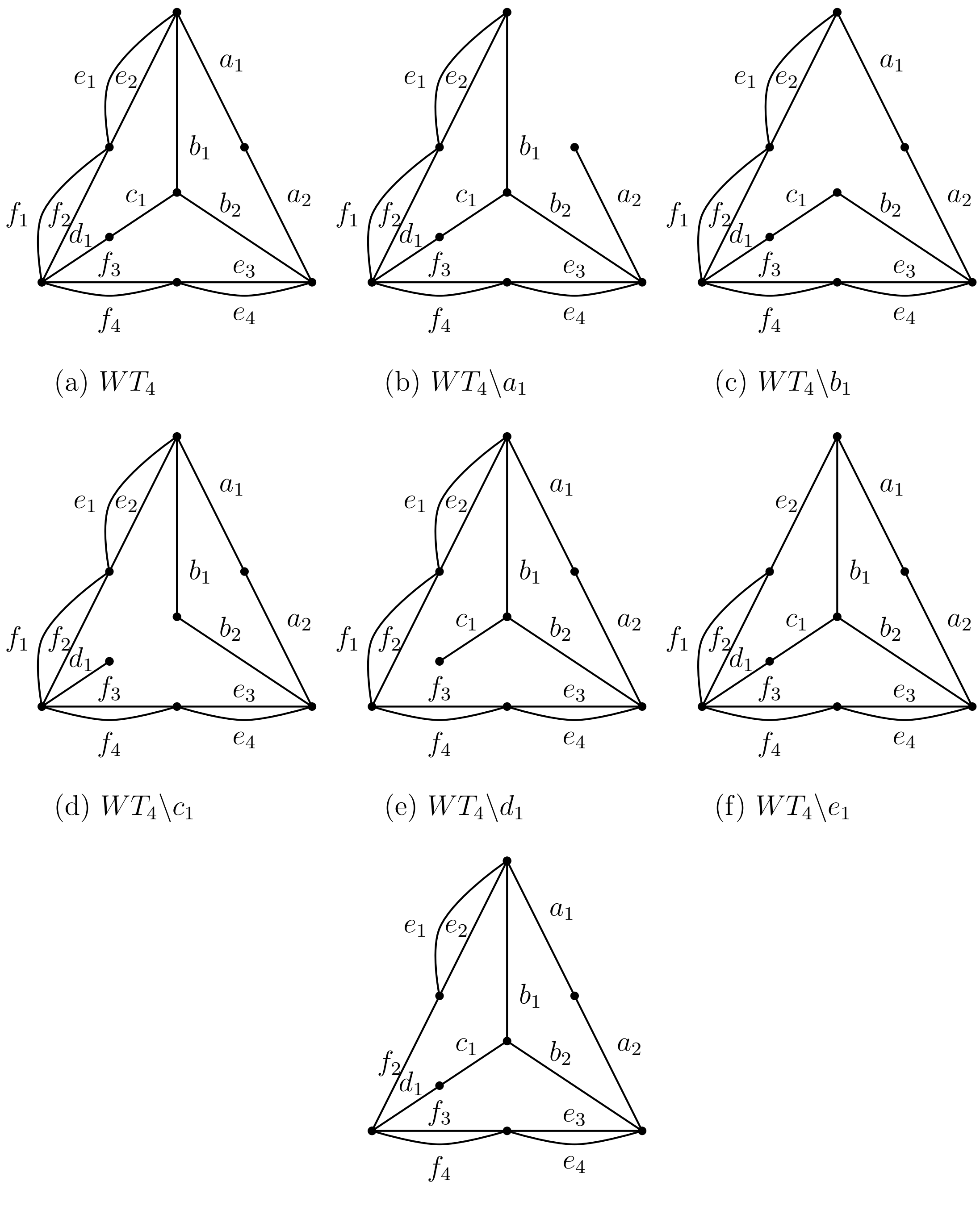
$WF_6$

# Appendix B: Verification of $WT_4$

To verify that a graph is XNOP, we must first check that the graph is not NOP and then check that every graph that $G$ properly dominates is NOP. The list of graphs is long, so for brevity, we only show one example, $WT_4$, here. The verification of the others is left to the curious reader.

We begin by looking at the graphs that result when one edge is removed and verifying that the resulting graph is not OP. Since $WT_4$ has many symmetries, we will look at the edge deletions only for edges which are in different orbits of the automorphism. The edge orbits as determined by action of the automorphism group on $WT_4$ are indicated by different letters, while indices just enumerate the element of each orbit as in Figure 6.9 ($a$). We want to look at $WT_4 \backslash k$ where $k$ is one of the edges in each of the automorphic groups. If $k$ is $a_1$ or $a_2$, then $WT_4 \backslash a_1 = A'$ has a subgraph of $K_{2,3}$ (see edges $d_2$, $e_2$, $e_3$, $d_3$, $b_2$, $b_1$, $c_1$, and $c_2$). If $b_1$ is removed, then $WT_4 \backslash b_1 = B'$ has a subgraph of $K_{2,3}$ using the edges $d_2$, $e_2$, $e_3$, $d_3$, $a_1$, $a_2$, $b_2$, $c_1$, and $c_2$. If $c_1$ is removed, then $WT_4 \backslash c_1 = C'$ has a subgraph of $K_{2,3}$ using the edges $d_2$, $e_2$, $e_3$, $d_3$, $a_1$, $a_2$, $b_1$, and $b_2$. When $d_1$ or $e_1$ is removed, then $WT_4 \backslash d_1 = D'$ and $WT_4 \backslash e_1 = E'$ each have a subgraph of $K_4$ using the edges $d_2$, $e_2$, $e_3$, $d_3$, $a_2$, $a_1$, $b_1$, $b_2$, $c_1$, and $c_2$. Hence, if any edge of $WT_4$ is deleted, $WT_4 \backslash k$ is NOP.

Now, we want to confirm that $WT_4$ is minimal XNOP by domination, that is we want to show that all graphs that are properly dominated by $WT_4$ are NOP. The graphs that $WT_4$ properly dominates are found by edge deletions, vertex deletions, and vertex suppressions. We begin with edge deletions. If $k$ is $a_1$ or $a_2$, then the subsequent removal of $b_1$, $b_2$, $c_1$, or $c_2$ gives a graph that is OP. If $b_1$ is removed, then the subsequent removal of $a_1$, $a_2$, $b_2$, $c_1$, or $c_2$ gives a graph that is OP. If $c_1$ is removed, then the subsequent removal of $a_1$, $a_2$, $b_1$, or $b_2$ gives a graph that is OP. When $d_1$ or $e_1$ is removed, the subsequent removal of $a_1$, $a_2$, $b_1$, $b_2$, $c_1$, $c_2$, $d_2$ (in $D'$), or $e_2$ (in $E'$) gives a graph that is OP. Hence, if any edge of $WT_4$ is deleted, $WT_4 \backslash k$ is NOP. All vertices in $WT_4$ are of degree 2 or greater, so any vertex deletion also means at least two edge deletions. Since one edge deletion results in graphs that are NOP, a vertex deletion with at least two edge deletions must also be NOP. So, we should look at vertex suppression. There are four vertices in $WT_4$ with exactly 2 neighbors. Label the vertex whose only incident edges are $a_1$ and $a_2$ as $v_a$. The vertex $v_{cd}$ is the one incident to only $c_1$ and $d_1$. The vertex with only $e_1$, $e_2$, $f_1$, and $f_2$ as incident edges is $v_{ef1}$. Similarly, $v_{ef2}$ has $e_3$, $e_4$, $f_3$, and $f_4$ as its only incident edges and is in the same orbit as $v_{ef1}$. We will only consider $v_{ef1}$. The graph $WT_4$ with the suppressible vertices labeled is shown in Figure 6.10 ($a$) and the results of suppressing these vertices are shown in Figure 6.10 ($b$)-($d$). The graph $WT_4 \cdot v_a$ is NOP since the deletion of $c_1$ or $d_1$ results in a graph that is OP. Similarly, the deletion of $a_1$ or $a_2$ from $WT_4 \cdot v_{cd}$ and the deletion of $b_2$ from $WT_4 \cdot v_{ef1}$ result in graphs that are OP. So all graphs that are properly dominated by $WT_4$ are NOP and $WT_4$ is minimal XNOP by domination.

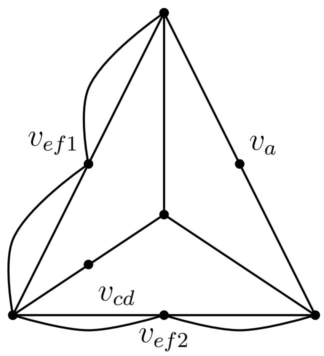



(a) $WT_4$ (b) $WT_4 \backslash a_1$ (c) $WT_4 \backslash b_1$

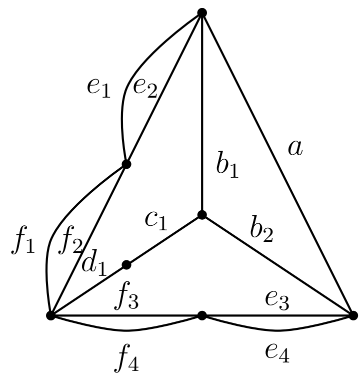
(d) $WT_4 \backslash c_1$ (e) $WT_4 \backslash d_1$ (f) $WT_4 \backslash e_1$

(g) $WT_4 \backslash f_1$

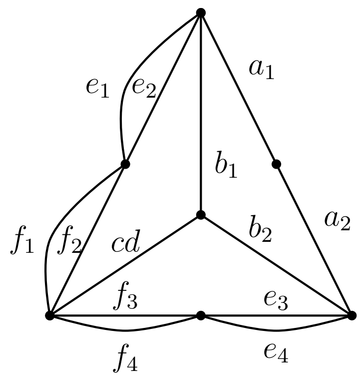
FIGURE 6.9. $WT_4$ and $WT_4$ with one edge removed.



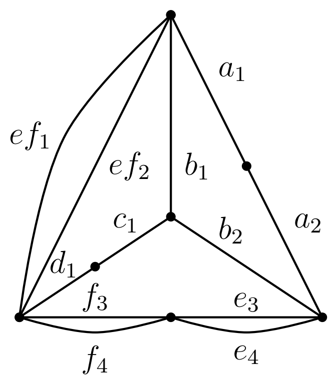

(a) Suppressible vertices of $WT_4$




(b) $WT_4 \cdot v_a$




(c) $WT_4 \cdot v_{cd}$




(d) $WT_4 \cdot v_{ef1}$

FIGURE 6.10. Suppression of vertices of $WT_4$

# Vita

Tanya Allen Lueder genannt Luehr was born on August 31, 1974, in Lafayette, Louisiana. She finished her undergraduate studies in chemical engineering at Louisiana State University in December 1996. She worked for six years as an engineer for International Paper Company. In August 2005 she began work at Louisiana State University in mathematics. She is currently a candidate for the degree of Master of Science in mathematics, which will be awarded in May 2010.