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Quasi-Linear Evolution Equations in Banach Spaces.

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Quasi-linear Evolution Equations in Banach Spaces

A Dissertation

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ABSTRACT

This dissertation is concerned with studying the quasi-linear evolution equation

$$u'(t) + A(t, u(t)) u(t) = 0 \text{ in } [0, T], u(0) = x_0$$

in a Banach space setting.

The spirit of this inquiry follows that of T. Kato and his fundamental results concerning linear evolution equations. We feel that our results give a natural approach to dealing with the quasi-linear problem.

Chapter I gives the preliminaries required for our work with abstract evolution equations. This consists of calculus in Banach spaces, the analytical theory of semigroups of bounded linear operators, and background material from the theory of linear evolution equations in Banach spaces.

Chapter II gives the main result of this dissertation, an alternate proof of the main result, a related proposition, two corollaries to the main result including an application, and directions for related research.

We assume that we have a family $\{A(t, w)\}$ of operators in a Banach space X such that each $-A(t, w)$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators in X and that the family satisfies

continuity and stability conditions. We show that on a fixed subinterval of $[0, T]$ that we have a family of approximate solutions to the quasi-linear problem that converge to a candidate function. The candidate function must be the solution to the quasi-linear problem if one exists. In fact, if u is the candidate function, it is enough that the linear problem $v'(t) + A(t, u(t)) v(t) = 0$, $v(0) = x_0$, have a solution in order that u will be the unique solution to the quasi-linear problem. We also show that the candidate function depends on the initial value in a strong way. The corollaries are concerned with the existence aspect.

INTRODUCTION

This dissertation is concerned with studying the quasi-linear evolution equation

$$u'(t) + A(t, u(t)) u(t) = 0 \text{ in } [0, T], u(0) = x_0$$

in a Banach space setting.

The spirit of this inquiry follows that of T. Kato. Kato wrote a fundamental paper on linear evolution equations in 1953 [7] ; that is, investigation of

$$u'(t) + A(t) u(t) = 0 \text{ on } [0, T], u(0) = x_0.$$

He strengthened and extended his analysis of the linear problem in 1970 [9] . Kato also wrote on the quasi-linear problem in 1975 [11]. We feel that our results give a natural approach to dealing with the quasi-linear problem.

Chapter I gives the preliminaries required for our work with abstract evolution equations. This consists of calculus in Banach spaces, the analytical theory of semigroups of bounded linear operators, and background material from the theory of linear evolution equations in Banach spaces.

Chapter II gives the main result of this dissertation, an alternate proof of the main result, a related proposition,

two corollaries to the main result including an application,
and directions for related research.

CHAPTER I

PRELIMINARIES

In this chapter, the background material for the results of this dissertation is given. A basic familiarity with Banach spaces is assumed. The results given here are well known, so proofs are omitted.

From this point on, X will be a real or complex Banach space. For $x \in X$, let $\|x\|$ be the norm of x .

Let J be an interval of real numbers and f a function on J to X . We say that f is continuous at $t_0 \in J$ if $\lim_{t \rightarrow t_0} \|f(t) - f(t_0)\| = 0$. If f is continuous at each point of J , then we say that f is continuous on J . We say that f is Lipschitz continuous on J with Lipschitz constant K if $\|f(t_1) - f(t_2)\| \leq K |t_1 - t_2|$ for any choice of t_1 and t_2 in J . The function f is absolutely continuous on J if given $\epsilon > 0$ there is a $\delta > 0$ such that $\sum_{i=1}^n \|f(t_i') - f(t_i)\| < \epsilon$ for every finite collection $\{ [t_i, t_i'] : i=1, 2, \dots, n; t_i, t_i' \in J \}$ of nonoverlapping intervals with $\sum_{i=1}^n |t_i' - t_i| < \delta$. By a partition of a closed interval $J = [a, b]$, we

mean a finite sequence $t_0, t_1, \dots, t_n$ with $a = t_0 < t_1 < \dots < t_n = b$. The function f is of bounded variation on the closed interval J if $V = \sup \{ \sum_{i=1}^n \|f(t_i) - f(t_{i-1})\| \}$ is finite, where the supremum is taken over all partitions $t_0, t_1, \dots, t_n$ of J . The number V is called the total variation of f on J . We say that f is differentiable at $t_0 \in J$ with derivative $f'(t_0) \in X$ if $\lim_{t \rightarrow t_0} \left\| \frac{f(t) - f(t_0)}{t - t_0} - f'(t_0) \right\| = 0$.

The foregoing concepts of continuity and differentiability are defined in the strong sense. Let X^* denote the space of all bounded linear functionals on X . Then f is said to be weakly continuous at t_0 if the scalar function $\varphi \circ f$ is continuous at t_0 for each φ in X^* . We say that f is weakly differentiable at t_0 if there exists $f'(t_0) \in X$ such that

$$\lim_{t \rightarrow t_0} \left| \frac{\varphi(f(t)) - \varphi(f(t_0))}{t - t_0} - \varphi(f'(t_0)) \right| = 0 \text{ for each } \varphi \text{ in } X^*.$$

We say that $f'(t_0)$ is the weak derivative of f at t_0 .

We now turn to the concept of integration of vector-valued functions. Our comments are with respect to Lebesgue measure on the line, and the domain of f is assumed to be a measurable set.

The function f is said to be weakly measurable if $\varphi \circ f$ is measurable for each φ in X^* . We say that f is countably-valued if the image of f is countable and the

preimage of each point in the range of f is a measurable set. If there exists a sequence of countably-valued functions converging almost everywhere to f , then we say that f is strongly measurable. We will write a.e. for almost everywhere. If a countable set of points in the image of f is dense in the image of f , except possibly for the image of a set of measure zero, we say that f is almost separably-valued. The two notions of measurability are connected by the following theorem of B. J. Pettis.

Theorem. A vector-valued function is strongly measurable if and only if it is weakly measurable and almost separably-valued.

We now give the definition of the Bochner integral, first for countably-valued functions and then for general functions. If $f(t) = \sum_{i=1}^{\infty} a_i \chi_{E_i}(t)$, where each a_i is a scalar and the E_i are disjoint measurable sets with χ_{E_i} the characteristic function on E_i , and $\|f\|$ is integrable, then we say that f is Bochner integrable and define $\int_E f \, dm = \sum_{i=1}^{\infty} a_i m(E_i)$, where $E = \bigcup_{i=1}^{\infty} E_i$. If there is a sequence of countably-valued Bochner integrable functions $\{f_n\}$ converging a.e. to f on the measurable set E , then we say that f is Bochner integrable on E provided

$$\lim_{n \rightarrow \infty} \int_E \|f - f_n\| \, dm = 0, \text{ and we define } \int_E f \, dm = \lim_{n \rightarrow \infty} \int_E f_n \, dm. \text{ The Bochner integral is}$$

well-defined, and the following theorem characterizes

the class of Bochner integrable functions.

Theorem. The function f is Bochner integrable if and only if f is strongly measurable and $\|f\|$ is integrable.

The following seven theorems give the properties we need concerning the Bochner integral. For a more complete discussion of the Bochner integral and its properties, the reader is referred to Hille and Phillips [6] and Komura [13] .

Theorem. If f and g are Bochner integrable on E and r_1 and r_2 are scalars, then $r_1f + r_2g$ is Bochner integrable on E and

$$\int_E (r_1f + r_2g) \, dm = r_1 \int_E f \, dm + r_2 \int_E g \, dm.$$

Theorem. If f is Bochner integrable on E , then

$$\left\| \int_E f \, dm \right\| \leq \int_E \|f\| \, dm.$$

Theorem. If $\{E_n\}$ is a sequence of disjoint measurable sets and f is Bochner integrable on $\bigcup_n E_n$, then

$$\int_{\bigcup_n E_n} f \, dm = \sum_n \int_{E_n} f \, dm,$$

where the sum on the right is absolutely convergent.

Theorem. If $\{f_n\}$ is a sequence of Bochner integrable functions that converge a.e. on a measurable set E to the function f and if there exists an integrable real-valued function F such that $\|f_n(t)\| \leq F(t)$ for all $t \in E$ and all n , then f is Bochner integrable on E and

$$\int_E f \, dm = \lim_{n \rightarrow \infty} \int_E f_n \, dm.$$

Theorem. If f is Bochner integrable on $[a, b]$, then

$$\frac{d}{dt} \int_a^t f \, dm = f(t) \text{ a.e. on } [a, b].$$

Theorem. If f is absolutely continuous and weakly differentiable a.e. on $[a, b]$, then f' is Bochner integrable

and $f(b) - f(a) = \int_a^b f'$, where $\int_a^b f'$ and $\int_a^b f'(t) dt$ will be used to denote $\int_{[a,b]} f \, dm$. Thus f' is the strong

derivative of f a.e.

Theorem. If f is absolutely continuous on $[a, b]$ and X is reflexive, then f is differentiable a.e.

Now that we have a basis for calculus in Banach spaces, we turn our attention to differential equations in Banach spaces and the analytical theory of semigroups of bounded linear operators. Here we are concerned with finding an abstract exponential $\{T(t): t \geq 0\}$ that enables us to solve the initial value problem $u'(t) = A u(t)$, $t \geq 0$ and $u(0) = x_0$, where $A : D(A) \rightarrow X$ is a fixed linear operator with domain $D(A) \subset X$, $u : [0, \infty) \rightarrow X$ and $x_0 \in D(A)$. We want to find conditions on A that give us a unique continuously differentiable solution u , and such that the solution varies continuously with the initial data.

We say that a family $\{T(t): t \geq 0\}$ of bounded linear operators on X is a strongly continuous semigroup of operators if $T(t + s) = T(t) T(s)$ for $t, s \geq 0$, $T(0) = I$, and $T(t)x$ is continuous in t for each $x \in X$. Let

$\{ T(t) \}$ be a strongly continuous semigroup of operators. For $h > 0$ define A_h by $A_h x = [T(h)x - x] / h$ for $x \in X$. Let $D(A)$ be the set of all x in X for which $\lim_{h \rightarrow 0^+} A_h x$ exists, and we define the operator A on $D(A)$ by $Ax = \lim_{h \rightarrow 0^+} A_h x$ for each $x \in D(A)$. The operator A is called the infinitesimal generator of the semigroup $\{ T(t) \}$, and we say that A generates the semigroup. It is often convenient to write $\exp(tA)$ for $T(t)$. It can be shown that $D(A)$ is a dense linear subspace of X , A is a closed linear operator on $D(A)$, A commutes with each $T(t)$ on $D(A)$, $T(t)x \in D(A)$ for $x \in D(A)$, and $\frac{d}{dt} T(t)x = A T(t)x$ for $t \in [0, \infty)$ and $x \in D(A)$. To say that A is closed means that if $Ax_n \rightarrow y$ as $x_n \rightarrow x$, then $x \in D(A)$ and $Ax = y$. The following theorem brings us back to the initial value problem.

Theorem. Let A be the infinitesimal generator of a strongly continuous semigroup $\{ \exp(tA) : t \geq 0 \}$. Then the initial value problem $u'(t) = A u(t)$ for $t \geq 0$ with $u(0) = x_0 \in D(A)$ has the unique continuously differentiable solution $u(t) = \exp(tA)x_0$.

The semigroup property $\exp[(t+s)A] = \exp(tA) \cdot \exp(sA)$ tells us that the solution at time $t + s$ is determined by $\exp(tA)$ acting on the solution at time s .

The resolvent set $\rho(A)$ of an operator A is defined to be the set of all complex numbers λ for which $(\lambda I - A)^{-1}$ exists as a bounded linear operator with domain X . The

next theorem characterizes strongly continuous semigroups.

Hille-Yosida-Phillips Theorem. A necessary and sufficient

condition that a closed operator A with dense domain $D(A)$

in the Banach space X be the infinitesimal generator of

a strongly continuous semigroup $\{T(t): t \geq 0\}$ is that

there exist real numbers M and β such that for every

real number $\lambda > \beta$, $\lambda \in \rho(A)$ and

$$\| (\lambda I - A)^{-n} \| \leq M/(\lambda - \beta)^n, n = 1, 2, \dots$$

It can be shown that $\|T(t)\| \leq Me^{\beta t}$ for each

t . General references dealing with semigroups of opera-

tors are Butzer and Berens [1], Dunford and

Schwartz [4], Friedman [5], Hille and Phillips [6],

Ladas and Lakshmikantham [14], and Yosida [19].

We now consider a special topic of semigroups of

linear operators. Let $S(\theta)$ be the open sector of the

complex plane consisting of all complex numbers t with

$-\theta < \arg t < \theta$, θ fixed in $(0, \frac{\pi}{2})$. A family of

bounded linear operators $\{T(t): t \in S(\theta) \cup \{0\}\}$

is called an analytic semigroup provided $T(t)x$ is an ana-

lytic function of t in $S(\theta)$ for each x in X (i.e., $T(t)x$ is

differentiable on $S(\theta)$), $T(t+s) = T(t)T(s)$ for t and s

in $S(\theta)$, $\{\|T(t)\| : -\theta \leq \arg t \leq \theta, \theta \text{ fixed with}$

$0 < \theta < \frac{\pi}{2}\}$ is bounded in the uniform operator topology

for each θ , and $\bigcup \{T(t)x : t \in S(\theta)\}$ is dense in X .

The above definition is from Hille and Phillips [6],

where it also shows that for each $x \in X$, $T(t)x \rightarrow x$ as

$t \rightarrow 0$ with $-\theta \leq \arg t \leq \theta$ for fixed θ in $(0, \frac{\pi}{2})$, and that

$\frac{d}{dt} T(t)x = A T(t)x$ for $t \in S(\emptyset)$ and $x \in X$ with A defined exactly as it was for strongly continuous semigroups. In particular, we have a strongly continuous semigroup once we define $T(0) = I$ and restrict ourselves to the family $\{ T(t): t \in \mathbb{R}, t \geq 0 \}$. As before, we write $\{ \exp(tA): t \in S(\emptyset) \}$ for $\{ T(t) \}$. Further references for analytic semigroups are Kato [8] , Ladas and Lakshmikantham [14] , and Yosida [19] . The following theorem gives us sufficient conditions for an operator to generate an analytic semigroup.

Theorem. Assume that A is a closed operator with domain $D(A)$ dense in X , $\rho(A)$ contains $S(\emptyset)$ for some $\emptyset \in (0, \frac{\pi}{2})$, and there exists a constant M such that $\| (\lambda I - A)^{-1} \| \leq M/|\lambda|$ for $\lambda \in S(\emptyset)$. Then, there is a unique analytic semigroup $\{ \exp(tA): t \in S(\emptyset) \}$ for which A is the infinitesimal generator.

We now give background material on linear evolution equations from Kato [9] . The theorems and propositions are numbered as in Kato's paper.

If Y is a subspace of X and A is a linear operator in X , then A induces a linear operator A' in Y such that $D(A') = \{ x \in D(A) \cap Y: Ax \in Y \}$ and $A'x = Ax$. We call A' the part of A in Y . When the context is clear, we will use A for A' . If Y is a Banach space contained in X and $D(A)$ contains Y , we say (somewhat incorrectly)

that A is a bounded linear operator on Y to X if the restriction of A to Y is a bounded linear operator from Y to X , and we write $A \in B(Y, X)$. The operator norm is written $\|A\|_{Y, X}$. If A is a bounded linear operator on X , then we write $\|A\|_X$ for $\|A\|_{X, X}$. In general a norm will be subscripted for clarity.

Let Y be a Banach space densely and continuously embedded in X ; i.e., $Y \subset X$, the closure of Y in the norm of X is X , and identity map on Y is a bounded linear operator from Y to X .

Definition 1.1. If $-A$ generates a strongly continuous semigroup on X , then we say that Y is A - admissible if $\{\exp(-tA)\}$ takes Y to Y and forms a strongly continuous semigroup on Y . The following two propositions give alternate characterizations of A - admissibility.

Proposition 2.3. Y is A - admissible if and only if for sufficiently large λ ,

(1) $(A + \lambda)^{-1}Y \subset Y$ and $\|(A + \lambda)^{-n}\|_Y \leq \tilde{M}(\lambda - \tilde{\beta})^{-n}$, $n = 1, 2, \dots$ and (2) $(A + \lambda)^{-1}Y$ is dense in Y . In this case the part $-\tilde{A}$ of $-A$ in Y generates the part of $\{\exp(-tA)\}$ in Y , and $\|\exp(-tA)\|_Y \leq \tilde{M}e^{\tilde{\beta}t}$. If Y is reflexive, condition (2) is redundant.

Proposition 2.4. Let S be an isomorphism (a bicontinuous linear map) of Y onto X . In order that Y be A - admissible, it is necessary and sufficient that $A_1 = SAS^{-1}$ be the generator of a strongly continuous semigroup. In this

case we have $S \exp(-tA) S^{-1} = \exp(-tA_1)$, $t \geq 0$.

Let $G(X, M, \beta)$ be the set of all linear operators A in X such that $-A$ generates a strongly continuous semigroup $\{T(t)\}$ in X with constants M and β (i.e., such that $\|T(t)\|_X \leq M e^{\beta t}$; see the Hille-Yosida-Phillips Theorem), and let $G(X)$ be the union of the $G(X, M, \beta)$ over all possible values of M and β . We now consider families $\{A(t): t \in [0, T]\} \subset G(X)$.

Definition 1.2. The family $\{A(t): t \in [0, T]\}$ is said to be stable in X if there are constants M and β (called the constants of Stability) such that

$\|\prod_{j=1}^k (A(t_j) + \lambda)^{-1}\|_X \leq M(\lambda - \beta)^{-k}$ for $\lambda > \beta$,
for any finite family $\{t_j\}$ with $0 \leq t_1 \leq \dots \leq t_k \leq T$,
 $k = 1, 2, \dots$.

Here and in what follows the product Π is a time-ordered juxtaposition; i.e., a factor with larger t_j stands to the left of ones with smaller t_j . The following proposition gives two alternate conditions that characterize stability in X .

Proposition 3.3. Each of the following conditions is equivalent to the definition of stability above. The $\{t_j\}$ vary over all finite families as before.

- (1) $\|\prod_{j=1}^k \exp(-s_j A(t_j))\|_X \leq M e^{\beta(s_1 + \dots + s_k)}$, $s_j \geq 0$.
- (2) $\|\prod_{j=1}^k (A(t_j) + \lambda_j)^{-1}\|_X \leq M \prod_{j=1}^k (\lambda_j - \beta)^{-1}$, $\lambda_j > \beta$.

A pair of parentheses with a letter inserted following a word such as continuity will indicate the norm ; i.e., continuous (X) means continuous with respect to the norm of the space X. If an operator $U(t,s)$ is said to be strongly (weakly) continuous (X) in t,s , it means that $U(t,s)x$ is continuous (weakly continuous) in t,s for each x in X. We now give the main result of Kato's paper followed by two related propositions.

Theorem 4.1. Let X and Y be Banach spaces such that Y is densely and continuously embedded in X. Let

$A(t) \in G(X)$, $0 \leq t \leq T$, and assume that

- (i) $\{ A(t) \}$ is stable in X, say with constants M, β .
- (ii) Y is $A(t)$ - admissible for each t . If $\tilde{A}(t) \in G(Y)$ is the part of $A(t)$ in Y, $\{ \tilde{A}(t) \}$ is stable in Y, say with constants $\tilde{M}, \tilde{\beta}$.
- (iii) $Y \subseteq D(A(t))$ so that $A(t) \in B(Y,X)$ for each t , and the map $t \rightarrow A(t)$ is norm-continuous (Y,X).

Under these conditions, there exists a unique family of operators $U(t,s) \in B(X)$, defined for $0 \leq s \leq t \leq T$, with the following properties.

- (a) $U(t,s)$ is strongly continuous (X) in s,t , with $U(s,s) = I$ and $\| U(t,s) \|_X \leq M e^{\beta(t-s)}$.
- (b) $U(t,r) = U(t,s) U(s,r)$, $r \leq s \leq t$.
- (c) $D_t^+ U(t,s)y|_{t=s} = -A(s)y$, $y \in Y$, $0 \leq s \leq T$.
- (d) $\frac{d}{ds} U(t,s)y = U(t,s) A(s)y$, $y \in Y$, $0 \leq s \leq t \leq T$.

Here D^+ denotes right derivative and $\frac{d}{ds}$ means derivative, in the strong sense of X .

Definition 1.3. The family $\{ U(t,s): 0 \leq s \leq t \leq T \}$ will be referred to as the evolution operator generated by $\{ A(t) \}$.

Proposition 4.3. Let $\{ A'(t) \}$ be another family satisfying the assumptions of Theorem 4.1 with the same X and Y and with the constants of stability $M', \beta', \tilde{M}', \tilde{\beta}'$. Let $\{ U'(t,s) \}$ be constructed from $\{ A'(t) \}$ in the same way as $\{ U(t,s) \}$ was constructed from $\{ A(t) \}$. Then $\| U'(t,r)y - U(t,r)y \|_X \leq M' \tilde{M}' e^{\gamma(t-r)} \| y \|_Y \cdot \int_r^t \| A'(s) - A(s) \|_{Y,X} ds$ where $\gamma = \max(\beta', \tilde{\beta}')$.

Proposition 4.4. (ii) of Theorem 4.1 is implied by (ii') There is a family $\{ S(t) \}$ of isomorphisms of Y onto X such that $S(t)A(t)S(t)^{-1} = A_1(t) \in G(X)$, $0 \leq t \leq T$, and that $\{ A_1(t) \}$ is a stable family in X , say with constants M_1 and β_1 . Furthermore, there is a constant c such that $\| S(t) \|_{Y,X} \leq c$, $\| S(t)^{-1} \|_{X,Y} \leq c$ and the map $t \rightarrow S(t)$ is of bounded variation in $B(Y,X)$ - norm.

When (ii') is satisfied, Kato shows that the constants of stability in Y are $M_1 c^2 e^{c M_1 V}$ and β_1 , where V is the total variation of S on $[0, T]$.

The following theorem strengthens the conclusions of Theorem 4.1 when Y is reflexive.

Theorem 5.1. In Theorem 4.1 assume further that

(iv) Y is reflexive.

Then we have, in addition to (a) to (d), (e) $U(t,s)Y \subset Y$,
 $\|U(t,s)\|_Y \leq \tilde{M} e^{\tilde{\beta}(t-s)}$, and $U(t,s)$ is weakly continuous (Y) in s,t . (c') $D_t^+ U(t,s)y = -A(t)U(t,s)y$ in X for $y \in Y$ and $t \geq s$, and this derivative is weakly continuous (X) in s,t . $U(t,s)y$ is an indefinite strong integral (X) of $-A(t)U(t,s)y$ in t . In particular,
 $\frac{d}{dt} U(t,s)y$ exists for almost every t (depending on s) and equals $-A(t)U(t,s)y$.

The following is a concept that is often useful in situations where one Banach space is contained as a subspace of another Banach space. Let X and Y be as before.

Definition 1.4. The relative completion of Y in X is the set of all points in X that are the limit in X - norm of sequences from Y that are bounded in Y - norm.

We can make this new subspace a Banach space by taking the norm of a point in the relative completion to be the least bound in Y - norm of sequences in Y that converge in X - norm to the point. It can be shown that Y is continuously embedded in the relative completion which in turn is continuously embedded in X . The relative completion of the relative completion in X yields nothing new. Also, the relative completion of Y in X is just Y if Y is reflexive. A discussion of the relative completion

and further references are available in Butzer and Berens [1] .

Results have been obtained by Sobolevskii [15] , Tanabe [16] , [17] , and [18] , and Kato and Tanabe [12] concerning linear evolution equations where the operators $-A(t)$ generate analytic semigroups. Specifically, let $S=S(\emptyset) \cup \{ 0 \}$ for some fixed $\emptyset \in (0, \pi/2)$ as before and let C be a constant. If $\{ A(t): t \in [0, T] \}$ is a family of closed, densely defined linear operators in X with common domain D independent of t such that $\rho(-A(t))$ contains S and $\| (\lambda I + A(t))^{-1} \|_X \leq \frac{C}{1+|\lambda|}$ for each $\lambda \in S$ and $t \in [0, T]$, and $\| [A(t_1) - A(t_2)] A(t_3)^{-1} \|_X \leq C | t_1 - t_2 |$ for all choices of t_1, t_2 , and t_3 in $[0, T]$, then there is an evolution operator $\{ V(t,s) : 0 \leq s \leq t \leq T \}$ such that $v(t) = V(t,0)x_1$ is the unique continuously differentiable solution to $v'(t) + A(t)v(t) = 0$ on $[0, T]$ and $v(0) = x_1 \in D$. The references by Sobolevskii [15] , Ladas [14] , and Friedman [5] contain more general statements of the results of Sobolevskii as well as proofs and details about the evolution operator referred to above. A paper by Dorroh [2] utilizes the results of Tanabe to obtain results on quasi-linear evolution equations.

CHAPTER II

QUASI-LINEAR EVOLUTION EQUATIONS IN BANACH SPACES

In this chapter the major results of this dissertation are developed. After discussing the setting and method of attack, our theorem is stated and proved. We then give an alternate proof and an application of the theorem using the Sobolevskii-Tanabe theory of linear evolution equations of parabolic type. A proposition relevant to our theorem is given, and the chapter is concluded with a discussion of directions for further research in the area of quasi-linear evolution equations in Banach spaces.

Let X and Y be Banach spaces, with Y densely and continuously embedded in X . Let $x_0 \in Y$, $T > 0$, $r > r_1 > 0$, $r_2 > 0$, $W = \overline{B_X(x_0; r)}$, $Z = B_X(x_0; r_1) \cap B_Y(x_0; r_2)$, and for each $t \in [0, T]$ and $w \in W$, let $-A(t, w)$ be the infinitesimal generator of a strongly continuous semigroup of bounded linear operators in X , with $Y \subseteq D(A(t, w))$.

We consider the quasi-linear evolution equation

$$(QL) \quad v'(t) + A(t, v(t)) v(t) = 0.$$

Given a function u from $[0, T']$ into W , where $0 < T' \leq T$, we can also consider the linearized evolution equation

$$(L;u) \quad v'(t) + A(t, u(t)) v(t) = 0.$$

By a solution of (QL) or (L;u) on $[0, T']$, we mean a function v on $[0, T']$ to W which is absolutely continuous (X) and differentiable (X) a.e., such that $v(t) \in Y$ a.e., $\text{ess sup } \{ \|v(t)\|_Y \} < \infty$, and v satisfies the appropriate equation, (QL) or (L;u), a.e. on $[0, T']$.

Our method is to produce, for each $x_1 \in Z$, a candidate u with initial value x_1 on an interval $[0, T']$, where $T' \in (0, T]$ is independent of x_1 . For a partition $\Delta = \{t_0, t_1, \dots, t_N\}$ of $[0, T']$, we use an iterative procedure to produce a Lipschitz continuous (X) function u_Δ which satisfies $u'_\Delta(t) + A(t_i, u_\Delta(t_i))u_\Delta(t) = 0$ for $t \in (t_i, t_{i+1})$ and $i \in \{0, 1, \dots, N-1\}$, with $u_\Delta(0) = x_1$. This u_Δ is shown to be the time-ordered juxtaposition of the semigroups generated by the $-A(t_i, u_\Delta(t_i))$. These approximate solutions converge uniformly, as $|\Delta|$ goes to 0, to give the candidate u . We show, in particular, that if $v = w$ is a solution of (QL) or (L;u) on $[0, T']$ with initial value x_1 , then $w = u$. Thus, subject to an initial value, a solution of (QL) is unique if it exists,

and whenever the linearized equation $(L;u)$ has a solution, then so does the quasi-linear equation (QL) . There are known conditions which are sufficient in order that $(L;u)$ has a solution.

Our hypotheses form a natural extension of Kato's assumptions for linear equations [9] . When a proposition or theorem from Kato's paper is referred to below, its statement may be found in our Chapter I and numbered as in Kato's paper.

Theorem. Assume that

(i) $\{ A(t,w) \}$ is stable in X with constants of stability M, β ; i.e., $\| (A(t_k, w_k) + \lambda)^{-1} \cdot (A(t_{k-1}, w_{k-1}) + \lambda)^{-1} \dots (A(t_1, w_1) + \lambda)^{-1} \|_X \leq M(\lambda - \beta)^{-k}$, $\lambda > \beta$, for any finite family $\{ (t_j, w_j) \}$, $0 \leq t_1 \leq \dots \leq t_k \leq T, k = 1, 2, \dots$.

(ii) $Y \subset D(A(t,w))$ for each (t,w) , which implies that $A(t,w) \in B(Y,X)$, and the map $(t,w) \rightarrow A(t,w)$ is Lipschitz continuous with constant C_1 ; i.e., $\| A(t_2, w_2) - A(t_1, w_1) \|_{Y,X} \leq C_1(|t_2 - t_1| + \| w_2 - w_1 \|_X)$.

(iii) There is a family $\{ S(t,w) \}$ of isomorphisms of Y onto X such that $S(t,w) A(t,w) S(t,w)^{-1} = A_1(t,w)$ is the negative of the infinitesimal generator of a strongly continuous semigroup in X for each (t,w) , and $\{ A_1(t,w) \}$ is stable in X , with constants of stability

M_1, β_1 . Furthermore, there is a constant C_2 such that $\|S(t,w)\|_{Y,X} \leq C_2, \|S(t,w)^{-1}\|_{X,Y} \leq C_2$, and the map $(t,w) \rightarrow S(t,w)$ is Lipschitz continuous with constant C_3 (see (ii) above).

Then, there exists a T' , with $0 < T' \leq T$, such that for each $x_1 \in Z$ and partition $\Delta = \{t_0, t_1, \dots, t_n\}$ of $[0, T']$, we can find a function u_Δ which is Lipschitz continuous (X) on $[0, T']$ to W, Y -bounded, and satisfies $u'_\Delta(t) + A(t, u_\Delta(t)) u_\Delta(t) = 0$ for $t \in (t_i, t_{i+1})$ and $i \in \{0, 1, \dots, n-1\}$, with $u_\Delta(0) = x_1$. In fact, given $\epsilon > 0$, there exists $\delta > 0$ such that $|\Delta| < \delta$ implies that $\|u'_\Delta(t) + A(t, u_\Delta(t)) u_\Delta(t)\|_X < \epsilon$ except at $t_1, \dots, t_n$. Further, the u_Δ converge uniformly, as $|\Delta|$ goes to 0, to a Lipschitz continuous (X) function u on $[0, T']$ to W which has initial value x_1 and is bounded, independent of x_1 , in the relative completion of Y in X (see Definition 1.4, Chapter I).

If $x_2 \in Z$ and w is constructed as above but with initial value x_2 , then $\|u(t) - w(t)\|_X \leq C \|x_1 - x_2\|_X$ for $t \in [0, T']$, with C independent of x_1 and x_2 .

Now, if v is a solution of (QL) or $(L;u)$ on $[0, T'']$, where $0 < T'' \leq T'$, with initial value x_1 , then $v = u$ on $[0, T'']$, and thus solutions to (QL) or $(L;u)$ are unique.

Corollary 1. If Y is reflexive, then $(L;u)$ has a solution on $[0, T']$ with initial value x_1 , and thus u is a solution of (QL) on $[0, T']$ with initial value x_1 .

Remarks. 1) If $D(A(t,w)) = Y$ for each (t,w) and there is a $\lambda > \beta$ such that $\| (\lambda I + A(t,w))^{-1} \|_{X,Y} \leq C_2$ and $\| \lambda I + A(t,w) \|_{Y,X} \leq C_2$ for each (t,w) , then (iii) is satisfied with $S(t,w) = \lambda I + A(t,w)$.

2) If Y is $A(t,w)$ - admissible (see Definition 1.1, Chapter I) for each (t,w) and $\{ A(t,w) \}$ (actually the part of $A(t,w)$ in Y) is stable in Y , then (iii) is unnecessary.

Proof. Let $T^0 = \min(T, r / \|A\| M e^{\beta T} (\|x_0\|_Y + r_2))$, where $\|A\| = \sup \{ \|A(t,w)\|_{Y,X} : t \in [0, T], w \in W \}$ which is finite by (ii). Let $K = C_2 C_3 M_1 T^0$ and $T' = T^0 / (1 + \|A\| C_2^2 M_1 e^K + \beta_1 T^0 (\|x_0\|_Y + r_2))$.

The following two lemmas are essential to our proof.

Lemma A. If u is Lipschitz continuous (X) on $[0, T']$ to W with Lipschitz constant $\|A\| C_2^2 M_1 e^K + \beta_1 T'$. $\cdot (\|x_0\|_Y + r_2)$, then $\{ A(t, u(t)) : t \in [0, T'] \}$ is Y - stable with constants $C_2^2 M_1 e^K$ and β_1 .

Proof of Lemma A. We use Kato's Proposition 4.4 [9] with $S(t) = S(t, u(t))$. Then we estimate the variation of S by $V_S \leq C_3 (1 + \|A\| C_2^2 M_1 e^{K + \beta_1 T'} (\|x_0\|_Y + r_2)) T' \leq C_3 T^0$, whence $\{ A(t, u(t)) \}$ is Y - stable with constants $C_2^2 M_1 e^{C_2 M_1 C_3 T^0} = C_2^2 M_1 e^K$ and β_1 (see comment following

Proposition 4.4, Chapter I). This completes the proof of Lemma A.

By an evolution operator $\{W(t,s) : 0 \leq s \leq t \leq T'\}$ generated by $\{\mathcal{A}(t) : t \in [0, T']\} \subset \{A(t,w) : t \in [0, T'], w \in W\}$ and a partition $\Delta = \{t_0, \dots, t_N\}$ of $[0, T']$, we mean the family of operators obtained by forming a time-ordered juxtaposition of the semigroups generated at the points of the partition; e.g., for $t \in [t_i, t_{i+1}]$, $s \in [t_j, t_{j+1}]$, $s \leq t$,

$$W(t,s) = \exp(-(t-t_i) \mathcal{A}(t_i)) \exp(-(t_i-t_{i-1}) \mathcal{A}(t_{i-1})) \cdots \exp(-(t_{j+1}-s) \mathcal{A}(t_j)).$$
It follows from (i) and Kato's Proposition 3.3 [9] that $\|W(t,s)\|_X \leq M e^{\beta(t-s)}$. If $\{\mathcal{A}(t)\}$ is Y -stable with constants $\tilde{M}, \tilde{\beta}$, then $W(t,s)Y \subset Y$ and $\|W(t,s)\|_Y \leq \tilde{M} e^{\tilde{\beta}(t-s)}$ as a result of (iii) and Kato's Propositions 2.4 and 3.3 [9]. Let $\bar{t} = t_i$ if $t \in [t_i, t_{i+1})$, $i \neq N$, and $\bar{t}_N = t_N$. If $f(t) = W(t,0)x_1$ on $[0, T']$, then f satisfies $f'(t) + \mathcal{A}(\bar{t})f(t) = 0$ for $t \notin \Delta$, with $f(0) = x_1$. The construction of an evolution operator from a family of semigroup generators and a partition, the notation $\bar{t}$, and the other results above will be used from this point on without further discussion.

Lemma B. Suppose $\{\mathcal{A}(t) : t \in [0, T']\}$ is Y -stable with constants $\tilde{M}$ and $\tilde{\beta}$, and that $\{W(t,s)\}$ is generated by $\{\mathcal{A}(t)\}$ and a partition Δ of $[0, T']$. Then, $f(t) = W(t,0)x_1$ is Lipschitz continuous (X) with

Lipschitz constant $\|A\| \tilde{M} e^{\tilde{\beta} T'} (\|x_0\|_{Y+r_2})$.

The result is also true if $\{W(t,s)\}$ is the evolution operator of Kato's Theorem 4.1 [9].

Proof of Lemma B. For the partition case, since

$f'(t) = -\mathcal{Q}(\bar{t})f(t)$ except for $t \in \Delta$, we get for

$s \leq t$

$$\begin{aligned} \|f(t) - f(s)\|_X &= \left\| - \int_s^t A(\bar{\xi})f(\xi) d\xi \right\|_X \\ &\leq \|A\| \|f\|_Y |t-s| \\ &\leq \|A\| \tilde{M} e^{\tilde{\beta} T'} \|x_1\|_Y |t-s| \\ &\leq \|A\| \tilde{M} e^{\tilde{\beta} T'} (\|x_0\|_{Y+r_2}) |t-s|. \end{aligned}$$

Now, the f on $[0, T']$ obtained from Kato's evolution operator is the uniform (X) limit of the f corresponding to the partitions Δ as $|\Delta| \rightarrow 0$. This establishes the result in the second case and completes the proof of Lemma B.

Together, Lemma A and Lemma B suggest an iteration scheme. We fix x_1 and Δ , then obtain sequences $\{u_n\}$, $\{A_n(t)\}$, and $\{U_n(t,s)\}$, with $A_n(t) = A(t, u_n(t))$, $\{U_{n+1}(t,s)\}$ the evolution operator generated by $\{A_n(t)\}$ and Δ , and $u_{n+1}(t) = U_{n+1}(t, 0)x_1$. Once Lemma A is satisfied, we have $\{A_n(t) : t \in [0, T']\}$ is Y -stable with constants $C_2^2 M_1 e^K$ and β_1 ; then, Lemma B applied to $\{\mathcal{Q}(t)\} = \{A_n(t)\}$, $\tilde{M} = C_2^2 M_1 e^K$ and $\tilde{\beta} = \beta_1$, implies that u_{n+1} is Lipschitz continuous (X) on $[0, T']$ with

Lipschitz constant $\|A\| \leq C_2^2 M_1 e^{K+\beta_1 T'} (\|x_0\|_Y + r_2)$.
 Assuming $u_{n+1} \in [0, T'] \subset W$, the stage is set to apply Lemma A to $\{A_{n+1}(t): t \in [0, T']\}$ and continue the process.

We now work with a fixed partition Δ of $[0, T']$ and fixed $x_1 \in Z$.

Let $A_0(t) = A(t, x_1)$ for $t \in [0, T']$ and let $\{U_1(t, s)\}$ be the evolution operator generated by $\{A_0(t)\}$ and Δ . Define $u_1(t) = U_1(t, 0)x_1$. Then, $u_1'(t) + A_0(\bar{t})u_1(t) = 0$ except at $t_1, t_2, \dots, t_N$. Also,

$$\begin{aligned} \|u_1(t) - x_1\|_X &= \|U_1(t, t)x_1 - U_1(t, 0)x_1\|_X \\ &= \left\| \int_0^t U_1(t, s) A_0(\bar{s})x_1 ds \right\|_X \\ &\leq M e^{\beta T'} \|A\| (\|x_0\|_Y + r_2)^t \\ &\leq r \text{ by the choice} \end{aligned}$$

of T^0 and T' . So, $u_1(t) \in W$ for each $t \in [0, T']$.

This argument also works for all the following

$u_n, n = 2, 3, \dots$

To start the procedure, we apply Lemma A to $u \equiv x_1$ and then Lemma B with $\{\mathcal{A}(t)\} = \{A_0(t)\}$, $\tilde{M} = C_2^2 M_1 e^K$ and $\tilde{\beta} = \beta_1$, proving that u_1 is Lipschitz continuous (X) on $[0, T']$ with the Lipschitz constant $\|A\| \leq C_2^2 M_1 e^{K+\beta_1 T'} (\|x_0\|_Y + r_2)$.

For the next iteration, let $A_1(t) = A(t, u_1(t))$ for $t \in [0, T']$ and let $\{U_2(t, s)\}$ be the evolution operator generated by $\{A_1(t)\}$ and Δ . Define $u_2(t) = U_2(t, 0)x_1$.

Then, $u_2'(t) + A_1(\bar{t})u_2(t) = 0$ except at $t_1, t_2, \dots, t_N$. As with u_1 , $u_2(t) \in W$ for each $t \in [0, T']$.

As we commented before, we can continue in like manner. For convenience of notation, let $M_2 = C_2^2 M_1 e^K$.

Then, for $n \geq 1$, we have

$$\begin{aligned}
 \|u_{n+1}(t) - u_n(t)\|_X &= \|U_{n+1}(t, 0)x_1 - U_n(t, 0)x_1\|_X \\
 &= \left\| -\int_0^t U_{n+1}(t, s)(A_n(\bar{s}) - A_{n-1}(\bar{s})) \cdot U_n(s, 0)x_1 ds \right\|_X \\
 &\leq M e^{\beta T'} C_1 M_2 e^{\beta_1 T'} (\|x_0\|_Y + r_2) \cdot \int_0^t \|u_n(\bar{s}) - u_{n-1}(\bar{s})\|_X ds \\
 &\leq (M M_2 e^{(\beta + \beta_1)T'} C_1 (\|x_0\|_Y + r_2))^n \cdot \int_0^t \dots \int_0^t \|u_1(\bar{s}) - x_0\|_X ds \\
 &\leq \frac{(M M_2 e^{(\beta + \beta_1)T'} C_1 (\|x_0\|_Y + r_2)^{T'})^n}{n!} r
 \end{aligned}$$

It follows that there exists a continuous function u_Δ on $[0, T']$ to W such that $u_n \rightarrow u_\Delta$ uniformly on $[0, T']$ as $n \rightarrow \infty$. The rate of convergence is independent of Δ and x_1 .

Now, let $A_\Delta(t) = A(t, u_\Delta(t))$ for $t \in [0, T']$ and let $\{U_\Delta(t, s)\}$ be the evolution operator generated by $\{A_\Delta(t)\}$ and Δ . Define $\hat{u}(t) = U_\Delta(t, 0)x_1$, then $\hat{u}'(t) + A_\Delta(\bar{t})\hat{u}(t) = 0$ except at $t_1, t_2, \dots, t_N$, and

$$\begin{aligned}
 \|\hat{u}(t) - u_n(t)\|_X &= \|U_\Delta(t, 0)x_1 - U_n(t, 0)x_1\|_X \\
 &= \left\| -\int_0^t U_\Delta(t, s)(A_\Delta(\bar{s}) - A_{n-1}(\bar{s})) \cdot U_n(s, 0)x_1 ds \right\|_X \\
 &\leq M e^{\beta T'} C_1 M_2 e^{\beta_1 T'} (\|x_0\|_Y + r_2).
 \end{aligned}$$

$$\begin{aligned} & \cdot \int_0^t \|\hat{u}(\bar{s}) - u_{n-1}(\bar{s})\|_X ds \\ & \leq ((M M_2 e^{(\beta + \beta_1)T'} C_1 (\|x_0\|_Y + r_2)^{T'})^n / n!) \|\hat{u} - x_0\|_X \end{aligned}$$

which tends to 0 as $n \rightarrow \infty$. Thus $\hat{u}(t) = u_\Delta(t)$,

$u_\Delta'(t) + A_\Delta(\bar{t})u_\Delta(t) = 0$ except at $t_1, \dots, t_N$,

$u_\Delta(0) = x_1$, u_Δ is Lipschitz continuous (X) with Lipschitz constant $\|A\| C_2^2 M_1 e^{K+\beta_1 T'} (\|x_0\|_Y + r_2)$, and $\|u_\Delta(t)\|_Y = \|u_\Delta(t, 0)x_1\|_Y \leq C_2^2 M_1 e^{K+\beta_1 T'} (\|x_0\|_Y + r_2)$, independent of t and x_1 .

We now establish that $\{u_\Delta : u_\Delta(0) = x_1 \text{ and } \Delta$ is a partition of $[0, T']\}$ is a family of approximate solutions to (QL) on $[0, T']$ with initial value x_1 .

Except for $t_1, \dots, t_N$, we have

$$\begin{aligned} u_\Delta'(t) + A(t, u_\Delta(t))u_\Delta(t) &= u_\Delta'(t) + A(\bar{t}, u_\Delta(\bar{t}))u_\Delta(t) \\ &\quad + (A(t, u_\Delta(t)) - A(\bar{t}, u_\Delta(\bar{t})))u_\Delta(t) \\ &= (A(t, u_\Delta(t)) - A(\bar{t}, u_\Delta(\bar{t})))u_\Delta(t) \end{aligned}$$

$$\begin{aligned} \text{So, } \|u_\Delta'(t) + A(t, u_\Delta(t))u_\Delta(t)\|_X &\leq C_1 (|t - \bar{t}| + \|u_\Delta(t) - u_\Delta(\bar{t})\|_X) M_2 e^{\beta_1 T'} \\ &\quad \cdot (\|x_0\|_Y + r_2) \\ &\leq C_1 M_2 e^{\beta_1 T'} (\|x_0\|_Y + r_2) \cdot \\ &\quad \cdot (1 + M_2 e^{\beta_1 T'} \|A\| (\|x_0\|_Y + r_2)) \cdot \\ &\quad \cdot |t - \bar{t}| \\ &= L |t - \bar{t}|, \end{aligned}$$

where L is independent of t in $[0, T']$ and Δ . Thus

$\|u_\Delta'(t) + A(t, u_\Delta(t))u_\Delta(t)\|_X \leq L |\Delta|$ except at $t_1, t_2, \dots, t_N$. This verifies that we have a family of approximate solutions.

To show that the $\{ u_{\Delta} \}$ converge as $|\Delta| \rightarrow 0$, let Δ_1 and Δ_2 be two partitions of $[0, T']$ with $|\Delta_1|$ and $|\Delta_2|$ small enough that both $\|f'(t) + A(t, f(t))f(t)\|_X < \epsilon$ and $\|g'(t) + A(t, g(t))g(t)\|_X < \epsilon$ for $t \in [0, T'] \setminus (\Delta_1 \cup \Delta_2)$, where $f(t) = u_{\Delta_1}(t)$, $g(t) = u_{\Delta_2}(t)$, $f(0) = x_1 = g(0)$, and $\epsilon > 0$ is fixed. The preceding paragraph allows us to do this. Let $\{ V(t, s) \}$ be the evolution operator obtained from Kato's Theorem 4.1 [9] for $\{ A(t, f(t)) : t \in [0, T'] \}$. For $s, t \in [0, T'] \setminus (\Delta_1 \cup \Delta_2)$, $s \leq t$, we get

$$\begin{aligned} g'(s) - f'(s) &= (g'(s) + A(s, g(s))g(s)) - (f'(s) + A(s, f(s))f(s)) \\ &\quad - A(s, f(s))(g(s) - f(s)) \\ &\quad + (A(s, f(s)) - A(s, g(s)))g(s). \end{aligned}$$

Moving the third expression on the right to the left side of the equation and applying $V(t, s)$, we get

$$\begin{aligned} V(t, s)(g'(s) - f'(s)) + V(t, s)A(s, f(s))(g(s) - f(s)) \\ = V(t, s)(g'(s) + A(s, g(s))g(s)) - V(t, s)(f'(s) + A(s, f(s))f(s)) \\ + V(t, s)(A(s, f(s)) - A(s, g(s)))g(s). \end{aligned}$$

The left side is simply $\frac{\partial}{\partial s} V(t, s) (g(s) - f(s))$.

Integrating both sides in s from 0 to t , evaluating the left side at the endpoints, and recognizing that

$V(t, t) = I$, we get

$$\begin{aligned} g(t) - f(t) - V(t, 0)(x_1 - x_1) &= \int_0^t V(t, s)(g'(s) + A(s, g(s))g(s)) ds \\ &\quad - \int_0^t V(t, s)(f'(s) + A(s, f(s))f(s)) ds \\ &\quad + \int_0^t V(t, s)(A(s, f(s)) - A(s, g(s)))g(s) ds. \text{ So,} \end{aligned}$$

$$\|g(t)-f(t)\|_X \leq T \cdot M e^{\beta T'} \epsilon + T \cdot M e^{\beta T'} \epsilon$$

$$+ M e^{\beta T'} C_1 M_2 e^{\beta_1 T'} (\|x_0\|_{Y+r_2}) \int_0^t \|f(s)-g(s)\|_X ds$$

$$= L_1 \epsilon + L_2 \int_0^t \|g(s)-f(s)\|_X ds. \text{ This implies that}$$

$$\|u_{\Delta_1}(t) - u_{\Delta_2}(t)\|_X = \|g(t) - f(t)\|_X = O(\epsilon)$$
 independent of t in $[0, T']$. Thus, $\{u_\Delta\}$ converges uniformly to a function u on $[0, T']$ to W as $|\Delta| \rightarrow 0$. We note that u is Lipschitz continuous (X) with constant $\|A\| C_2^2 M_1 e^{K+\beta_1 T'} (\|x_0\|_{Y+r_2})$, $u(0) = x_1$, and u is bounded, independent of x_1 , by $C_2^2 M_1 e^{K+\beta_1 T'} (\|x_0\|_{Y+r_2})$ in the relative completion of Y in X (see Definition 1.4, Chapter I).

We need to know that u "corresponds" to $\{A(t, u(t)) : t \in [0, T']\}$. Let $\{U(t, s)\}$ be the evolution operator obtained from Kato's Theorem 4.1 [9] for $\{A(t, u(t))\}$, and define $\bar{u}(t) = U(t, 0)x_1$. By Lemma A, $\{A(t, u(t))\}$ is Y -stable with constants M_2 and β_1 . For any partition Δ of $[0, T']$ we have

$$\begin{aligned}
 \|\bar{u}(t) - u_\Delta(t)\|_X &= \|U(t, 0)x_1 - U_\Delta(t, 0)x_1\|_X \\
 &= \left\| - \int_0^t U(t, s) (A(s, u(s)) - A_\Delta(\bar{s})) U_\Delta(s, 0)x_1 ds \right\|_X \\
 &\leq M e^{\beta T'} C_1 (|\Delta| + \sup\{\|u(s) - u_\Delta(\bar{s})\| : s \in [0, t]\}) \cdot \\
 &\quad \cdot M_2 e^{\beta_1 T'} (\|x_0\|_{Y+r_2}) T'.
 \end{aligned}$$

Since u_Δ converges to u uniformly on $[0, T']$ as $|\Delta| \rightarrow 0$, and u_Δ is Lipschitz continuous (X) with a Lipschitz constant that is independent of Δ , we see that $\|\bar{u}(t) - u_\Delta(t)\|_X$ goes to $\|\bar{u}(t) - u(t)\|_X$ and to 0 as $|\Delta| \rightarrow 0$. Thus $u(t) = \bar{u}(t) = U(t, 0)x_1$.

Suppose $x_2 \in Z$ and that w_Δ and w are obtained in the same manner as u_Δ and u , except that the initial value for w_Δ and w is x_2 . Analogous to the technique employed to obtain u , we get

$$\frac{\partial}{\partial s} U_\Delta(t, s)(u_\Delta(s) - w_\Delta(s)) = U_\Delta(t, s)(A(\bar{s}, w_\Delta(\bar{s})) - A(\bar{s}, u_\Delta(\bar{s})))w_\Delta(s) \text{ for } s, t \in [0, T'], s \leq t, \\ s \notin \Delta.$$

Integrating both sides in s from 0 to t

$$\text{yields } u_\Delta(t) - w_\Delta(t) - U_\Delta(t, 0)(x_1 - x_2) \\ = \int_0^t U_\Delta(t, s)(A(\bar{s}, w_\Delta(\bar{s})) - A(\bar{s}, u_\Delta(\bar{s})))w_\Delta(s) ds, \text{ and so} \\ \|u_\Delta(t) - w_\Delta(t)\|_X \leq Me^{\beta T'} \|x_1 - x_2\|_X + Me^{\beta T'} \cdot C_1 M_2 e^{\beta_1 T'} (\|x_0\|_{Y+r_2}) \int_0^t \|u_\Delta(\bar{s}) - w_\Delta(\bar{s})\|_X ds.$$

Thus, $\|u_\Delta(t) - w_\Delta(t)\|_X \leq C \|x_1 - x_2\|_X$, with C independent of t in $[0, T']$, Δ , and x_1 and x_2 . It follows that $\|u(t) - w(t)\|_X \leq C \|x_1 - x_2\|_X$ for $t \in [0, T']$, with C also independent of the initial values.

We now turn to the uniqueness of solutions to (QL) or (L;u) on $[0, T'']$, where $0 < T'' \leq T'$, with initial value $x_1 \in Z$.

Suppose v is such a solution to (L;u). Then, $v'(s) + A(s, u(s))v(s) = 0$ a.e., so

$$\frac{\partial}{\partial s} U(t,s)v(s) = U(t,s)v'(s) + U(t,s)A(s,u(s))v(s) = 0 \text{ a.e.}$$

Integrating in s from 0 to t , we get $U(t,t)v(t) - U(t,0)v(0) = v(t) - U(t,0)x_1 = v(t) - u(t) = \text{constant}$. Since $v(0) - u(0) = x_1 - x_1 = 0$, we have $v(t) = u(t)$ for all t in $[0, T'']$. This makes u the unique solution of $(L;u)$ on $[0, T'']$ with initial value x_1 . In fact, this also makes u a solution of (QL) . We note that it is not necessary that $v|_{[0, T'']} \in W$.

Now suppose that v is a solution to (QL) on $[0, T'']$ with initial value x_1 . Then, $v'(s) + A(s,v(s))v(s) = 0$ a.e., and so $v'(s) + A(s,u(s))v(s) = (A(s,u(s)) - A(s,v(s)))v(s)$ a.e. Thus,
$$\frac{\partial}{\partial s} U(t,s)v(s) = U(t,s)v'(s) + U(t,s)A(s,u(s))v(s) \text{ a.e.}$$

$$= U(t,s)(A(s,u(s)) - A(s,v(s)))v(s) \text{ a.e.}$$

Integrating in s from 0 to t , we get $v(t) - u(t) = U(t,t)v(t) - U(t,0)v(0) = \int_0^t U(t,s)(A(s,u(s)) - A(s,v(s)))v(s)ds$. This implies that $\|v(t) - u(t)\|_X \leq M e^{\beta T'} \|v\|_Y \cdot C_1 \int_0^t \|u(s) - v(s)\|_X ds$, and thus $\|v(t) - u(t)\|_X = 0$ for all t in $[0, T'']$. This makes u the unique solution of (QL) on $[0, T'']$ with initial value x_1 . This completes the proof of our theorem.

If Y is reflexive, then by Kato's Theorem 5.1 [9], we have the result that $v = u$ is a solution of $(L;u)$ on $[0, T']$ with initial value x_1 , and thus u is a solution of (QL) . This gives us Corollary 1. //

The remarks following the statement of the theorem

and corollary are straightforward. We also note that Remark 1) deals with a particular case of condition (iii) of the theorem. Remark 2) contains a condition which greatly simplifies the proof of the theorem, but which would be extremely difficult to verify in the absence of conditions stronger than condition (iii).

It is interesting that there is an alternate proof of the theorem which is in some sense dual to the proof above. It is dual in the sense that the refining of the partitions to obtain a limit function is performed by Kato's Theorem 4.1 [9] at each iteration, instead of the refining process of the proof above which only arises at the last step. Of course, we can also regard the two methods of proof as giving two constructions of the function u as an iterated limit. We emphasize that the same function u is obtained by either construction. A sketch of the alternate proof is given below.

Alternate proof. Lemma A, Lemma B, and Kato's Theorem 4.1 [9] are the heart of this iteration scheme. We fix $x_1 \in Z$, let $A_0(t) = A(t, x_1)$ for $t \in [0, T']$, and apply Lemma A to $\{ A_0(t) \}$ getting that $\{ A_0(t) \}$ is Y -stable with constants $c_2^2 M_1 e^K$ and β_1 . We then apply Kato's Theorem 4.1 [9] to obtain the evolution operator $\{ U_1(t, s) : 0 \leq s \leq t \leq T' \}$ for $\{ A_0(t) \}$. Define $u_1(t) = U_1(t, 0)x_1$ for $t \in [0, T']$, and note that $u_1(0) = x_1$. Lemma B applied to $\{ \alpha(t) \} = \{ A_0(t) \}$,

$\tilde{M} = C_2^2 M_1 e^K$ and $\tilde{\beta} = \beta_1$, implies that $u_1(t)$ is Lipschitz continuous (X) on $[0, T']$ with Lipschitz constant $\|A\| \leq C_2^2 M_1 e^{K+\beta_1 T'} (\|x_0\|_{Y+r_2})$. Kato constructs $\{U_1(t, s)\}$ as the limit of the evolution operators generated by $\{A_0(t)\}$ and Δ , where Δ is a partition of $[0, T']$. Consequently, u_1 is the uniform limit of functions which, as we have seen before in the other proof, map $[0, T']$ to W and are bounded in Y on $[0, T']$ by $C_2^2 M_1 e^{K+\beta_1 T'} (\|x_0\|_{Y+r_2})$. So, u_1 maps $[0, T']$ to W and is bounded by $C_2^2 M_1 e^{K+\beta_1 T'} (\|x_0\|_{Y+r_2})$ in the relative completion of Y in X (see Definition 1.4, Chapter I). Let $A_1(t) = A(t, u_1(t))$ on $[0, T']$, then apply Lemma A to $\{A_1(t)\}$ and continue the process. In general, we obtain sequences $\{u_n\}$, $\{A_n(t)\}$, and $\{U_n(t, s)\}$, with $A_n(t) = A(t, u_n(t))$, $\{U_{n+1}(t, s)\}$ the evolution operator obtained from $\{A_n(t)\}$ by Kato's Theorem 4.1 [9], and $u_{n+1}(t) = U_{n+1}(t, 0)x_1$.

It follows from Kato's Proposition 4.3 [9] that

$$\|u_{n+1}(t) - u_n(t)\|_X \leq M e^{\beta T'} C_2^2 M_1 e^{K+\beta_1 T'} C_1 (\|x_0\|_{Y+r_2}) \cdot \int_0^t \|u_n(s) - u_{n-1}(s)\|_X ds,$$
 which implies that $\{u_n\}$ converges uniformly on $[0, T']$ to some function u . It is clear that u will have the same properties as each u_n in respect to the image of u lying in W , the Lipschitz continuity (X) of u , and the bound on u in the relative completion of Y in X . Now, let $A(t) = A(t, u(t))$

for $t \in [0, T']$, and we apply Lemma A and Kato's Theorem to obtain the evolution operator $\{U(t, s) : 0 \leq s \leq t \leq T'\}$ corresponding to $\{A(t)\}$. If $\hat{u}(t) = U(t, 0)x_1$, we apply the proposition mentioned above to get $\|\hat{u}(t) - u_n(t)\|_X \leq M e^{\beta T'} C_2^2 M_1 e^{K + \beta_1 T'} \cdot C_1(\|x_0\|_Y + r_2) \int_0^t \|u(s) - u_{n-1}(s)\|_X ds$. This implies that $\|\hat{u}(t) - u_n(t)\|_X$ goes to $\|\hat{u}(t) - u(t)\|_X$ and to 0 as $n \rightarrow \infty$, so $u(t) = \hat{u}(t) = U(t, 0)x_1$, and u corresponds to $\{A(t)\}$.

Let $\{U_\Delta(t, s) : 0 \leq s \leq t \leq T'\}$ be the evolution operator generated by $\{A(t)\}$ and Δ , where Δ is a partition of $[0, T']$, and define $u_\Delta(t) = U_\Delta(t, 0)x_1$ for $t \in [0, T']$. By Kato's construction, $u_\Delta \rightarrow u$ uniformly on $[0, T']$ as $|\Delta| \rightarrow 0$. We also have that the image of u_Δ lies in $W \cap Y$, u_Δ is Lipschitz continuous (X) on $[0, T']$ with Lipschitz constant independent of Δ , u_Δ is Y-bounded on $[0, T']$ independent of Δ , and u_Δ satisfies $u_\Delta'(t) + A(\bar{t})u_\Delta(t) = 0$ except for $t \in \Delta \setminus \{0\}$, with $u_\Delta(0) = x_1$. It is easy to show that $\{u_\Delta : \Delta \text{ is a partition of } [0, T']\}$ forms a family of approximate solutions to (L;u) and (QL).

The results concerning dependence on initial values as well as uniqueness of solutions to (L;u) or (QL) are established exactly as in the other proof. Similar techniques show that the function u obtained in each proof is the same. This completes the discussion of the alternate

proof of the theorem.

Remark. One advantage of the original construction is that it allows us to actually construct an approximation to u directly from the semigroups generated by the operators $-A(t,w)$. For $\epsilon > 0$, there exists a positive integer N and $\delta > 0$ such that $n \geq N$ and $|\Delta| < \delta$ imply that $\|u_{n,\Delta}(t) - u(t)\|_X < \epsilon$ for $t \in [0, T']$ and $\|u'_{n,\Delta}(t) + A(t, u_{n,\Delta}(t))u_{n,\Delta}(t)\|_X < \epsilon$ for $t \in [0, T'] \setminus (\Delta \setminus \{0\})$, where $u_{n,\Delta}$ is the n^{th} iterate toward finding u_Δ for the partition Δ of $[0, T']$. We note that N and δ can be determined in terms of ϵ and various known constants. A knowledge of the evolution operators corresponding to $\{A(t, u(t))\}$ is necessary in order to even carry out the iteration scheme of the alternate proof.

We now turn our attention to an application of our theorem using the Sobolevskii-Tanabe theory of linear evolution equations of parabolic type.

Corollary 2. Let S be the sector of the complex plane $\mathbb{C}$ consisting of 0 and $\{\lambda \in \mathbb{C} : -\theta \leq \arg \lambda \leq \theta\}$, where $\theta \in (\pi/2, \pi)$ is fixed. We assume that conditions (i) and (ii) of the theorem hold with $Y = D(A(t,w))$ for each t, w , and that

(iii)' The resolvent set of $-A(t,w)$ contains S and $\|(\lambda I + A(t,w))^{-1}\|_X \leq C_\mu / (1 + |\lambda|)$ for each $\lambda \in S$, $t \in [0, T]$, and $w \in W$, where C_μ is a

constant independent of λ , t , and w .

Then, the theorem holds and $(L;u)$ has a continuously differentiable (X) solution on $[0,T']$ with initial value x_1 , and thus u is a continuously differentiable (X) solution of (QL) on $[0,T']$ with initial value x_1 .

Proof. Under these conditions the theorem holds, where $S(t,w) = A(t,w)$ for each t and w . This gives us the candidate function u . The plan of attack is to produce a solution to $(L;u)$ which is continuously differentiable (X) on $[0,T']$ and has initial value x_1 . This is where the Sobolevskii-Tanabe theory enters. Let $A(t) = A(t,u(t))$ for each $t \in [0,T']$, and we see for $t_1, t_2, t_3 \in [0,T']$ that

$$\begin{aligned} & \| (A(t_1) - A(t_2))A(t_3)^{-1} \|_X \\ & \leq \| A(t_1) - A(t_2) \|_{Y,X} \| A(t_3)^{-1} \|_{X,Y} \\ & \leq C_4 \| A(t_1, u(t_1)) - A(t_2, u(t_2)) \|_{Y,X} \\ & \leq C_5 |t_2 - t_1| \quad \text{by (ii)} \end{aligned}$$

and the Lipschitz continuity of u , where C_5 is independent of the choice of t_1, t_2, t_3 . It follows from the Sobolevskii-Tanabe theory [15] that there is an evolution operator $\{ V(t,s) : 0 \leq s \leq t \leq T' \}$ such that $v(t) = V(t,0) x_1$ defines a continuously differentiable (X) function that satisfies $v'(t) + A(t)v(t) = 0$,

$v(0) = x_1$. The operator also satisfies

$$\begin{aligned} \| A(t)V(t,0)A(0)^{-1} \|_X &\leq C_6 \text{ on } [0, T'] , \text{ with } C_6 \\ \text{independent of } t \text{ [15, p. 5]} , \text{ thus } \| v(t) \|_Y &= \\ \| V(t,0)x_1 \|_Y &= \| A(t)^{-1}A(t)V(t,0)A(0)^{-1}A(0)x_1 \|_Y \\ &\leq \| A(t)^{-1} \|_{X,Y} \| A(t)V(t,0)A(0)^{-1} \|_X \cdot \\ &\quad \cdot \| A(0) \|_{Y,X} \| x_1 \|_Y \\ &\leq C_7 \| x_1 \|_Y , \end{aligned}$$

where C_7 is independent of t . So, except for the image of v lying in W , we have that v is a solution of $(L;u)$. Since the proof of the uniqueness of a solution to $(L;u)$ does not depend on $v|_{[0,T']} \subset W$, we still have that $v(t) = u(t)$ on $[0, T']$. Consequently, u is the solution of (QL) on $[0, T']$ with initial value x_1 . In fact, u is continuously differentiable (X) , without exception, on $[0, T']$. //

We note that in general an application of the theorem involves finding conditions that guarantee the existence of a solution to $(L;u)$, which then implies that u is the solution of (QL) .

It may be difficult at times to recognize that the conditions for our theorem hold. The following proposition gives criteria that obtain the Banach space Y and verify most of condition (iii) of the theorem. If, in particular, we are able to use $\lambda I + A(t,w)$, where $\lambda > \beta$ is fixed, for $S(t,w)$ in the proposition, then condition (ii) of the theorem holds as well as all of

condition (iii).

Proposition. Let Y be a dense linear subspace of X .

Suppose for each $t \in [0, T]$ and $w \in W$ that $S(t, w)$ is an isomorphism (algebraically) from Y onto X , $S(t, w)$ is a closed operator in X , $S(t, w)^{-1} \in B(X)$ with

$\| S(t, w)^{-1} \|_X \leq L_1$, and the bounded linear operator $S(t, w)S(t_0, w_0)^{-1}$ satisfies

$\| S(t_2, w_2)S(t_0, w_0)^{-1} - S(t_1, w_1)S(t_0, w_0)^{-1} \|_X$
 $\leq L_2(|t_2 - t_1| + \|w_2 - w_1\|_X)$, where L_1, L_2, t_0

from $[0, T]$, and $w_0 \in W$ are fixed. Suppose further

that Y has the graph norm induced by $S(t_0, w_0)$; i.e.,

for $y \in Y$, $\|y\|_Y = \|y\|_X + \|S(t_0, w_0)y\|_X$.

Then, (i) Y is a Banach space under this norm, and Y is continuously embedded in X .

(ii) $S(t, w)^{-1} \in B(X, Y)$ for each t and w , and

$\| S(t, w)^{-1} \|_{X, Y} \leq 1 + L_1 + L_2(T + 2r)$, where r is the radius of the ball W .

(iii) $S(t, w) \in B(Y, X)$ for each t and w , and

$\| S(t, w) \|_{Y, X} \leq 1 + L_2(T + 2r) \equiv L_3$.

(iv) $\| S(t_2, w_2) - S(t_1, w_1) \|_{Y, X} \leq L_2 L_3(|t_2 - t_1| + \|w_2 - w_1\|_X)$

Proof. Since $S(t_0, w_0)$ is a closed linear operator with domain Y , it is clear that Y is a Banach space under the indicated norm. It is also immediate that Y is continuously embedded in X .

Let $x \in X$, then

$$\begin{aligned}
\| S(t,w)^{-1}x \|_Y &= \| S(t,w)^{-1}x \|_X + \| S(t_0,w_0)S(t,w)^{-1}x \|_X \\
&\leq L_1 \| x \|_X + \| S(t_0,w_0)S(t,w)^{-1}x - x \|_X \\
&\quad + \| x \|_X \\
&\leq (L_1+1) \| x \|_X + L_2(|t-t_0| + \|w-w_0\|_X) \|x\|_X \\
&\leq (L_1+1) \|x\|_X + L_2 (T + 2r) \|x\|_X \\
&= (1 + L_1 + L_2 (T + 2r)) \| x \|_X .
\end{aligned}$$

So, $S(t,w)^{-1} \in B(X,Y)$ and $\| S(t,w)^{-1} \|_{X,Y} \leq 1 + L_1 + L_2 (T + 2r)$.

Let $y \in Y$, then

$$\begin{aligned}
\| S(t,w)y \|_X &= \| S(t,w)S(t_0,w_0)^{-1}S(t_0,w_0)y \|_X \\
&\leq (1+L_2(|t-t_0| + \|w-w_0\|_X)) \\
&\quad \cdot \| S(t_0,w_0)y \|_X \\
&\leq (1+L_2(T+2r))(\| y \|_Y - \| y \|_X) \\
&\leq (1+L_2(T+2r)) \| y \|_Y .
\end{aligned}$$

So, $S(t,w) \in B(Y,X)$ and $\| S(t,w) \|_{Y,X} \leq 1+L_2(T+2r)$.

To show (iv), let $y \in Y$, then

$$\begin{aligned}
\| S(t_2,w_2)y - S(t_1,w_1)y \|_X &= \| (S(t_2,w_2) - S(t_1,w_1)) S(t_0,w_0)^{-1}S(t_0,w_0)y \|_X \\
&\leq \| (S(t_2,w_2) - S(t_1,w_1)) S(t_0,w_0)^{-1} \|_X \cdot \| S(t_0,w_0)y \|_X \\
&\leq L_2 (|t_2 - t_1| + \|w_2 - w_1\|_X) \cdot \| S(t_0,w_0) \|_{Y,X} \| y \|_Y \\
&\leq L_2 L_3 (|t_2 - t_1| + \|w_2 - w_1\|_X) \|y\|_Y . \quad //
\end{aligned}$$

We also note that condition (i) for our theorem holds when each $A(t,w)$ is in $G(X, 1, \beta)$.

We close with a list of topics for further research in the area of quasi-linear evolution equations.

- 1) Behavior of an alternate construction of approximate solutions. For the partition $\Delta = \{ t_0, \dots, t_N \}$ of $[0, T']$, we construct u_Δ as follows: $u_\Delta(t_0) = x_1$, $u_\Delta(t) = \exp(-tA(t_0, u_\Delta(t_0)))x_1$ for $t \in [t_0, t_1]$, $u_\Delta(t) = \exp(-tA(t_1, u_\Delta(t_1)))u_\Delta(t_1)$ for $t \in [t_1, t_2]$, ..., and $u_\Delta(t) = \exp(-tA(t_{N-1}, u_\Delta(t_{N-1})))u_\Delta(t_{N-1})$ for $t \in [t_{N-1}, t_N]$. This process would generate an approximate solution in one step, which would probably be more efficient and more likely to yield effective numerical solutions.
- 2) Further continuity and differentiability results for u .
- 3) Perturbation problems; i.e., solving $u'(t) + A(t, u(t))u(t) + B(t, u(t))u(t) = 0$, where $\{ B(t, w) : t \in [0, T] \text{ and } w \in W \} \subset B(X)$.
- 4) Non-homogeneous problems; i.e., solving $u'(t) + A(t, u(t))u(t) = f(t, u(t))$, where f is a function on $[0, T] \times W$.
- 5) Dependence on initial conditions in Y -norm; i.e., conditions like $\|x_n - x_0\|_Y \rightarrow 0$ implies $\|u_n(t) - u_0(t)\|_Y \rightarrow 0$ uniformly on $[0, T']$, where u_n is a solution of (QL) with initial value x_n .
- 6) Direct applications to partial differential equations.

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