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Characterization of the dependency across foreign exchange markets using copulas

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CHARACTERIZATION OF DEPENDENCY ACROSS FOREIGN EXCHANGE MARKETS USING COPULAS

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Abstract

Though Pearson's correlation coefficient provides a convenient approach to measuring the dependency between two variables, in the last few years, there has been a significant amount of literature warning against the use of Pearson's correlation coefficient, as it does not remain invariant under transformations of the underlying distribution functions. Since we are interested in examining the dependency pattern observed by the return on the Sterling Pound with that of the Japanese Yen, we will use the notion of a copula to approximate the joint density function between the daily returns on the Sterling Pound and the Japanese Yen. In particular, we use a result that is fundamental to the development of copula theory, namely Sklar's Theorem, to examine the observed joint density function between the daily returns on the Sterling Pound and the Japanese Yen. We will attempt to capture the approximated joint density function using a theoretical Gaussian Copula Model. This comparison is performed in the case where the underlying marginal distributions are both uniform, as well as the case where the underlying marginal distributions are both gaussian.

Chapter 1
Introduction

To study dependence in financial markets, most research has focused on the measure of correlation, using Pearson's correlation coefficient. However, there are flaws with using correlation to study dependency, as correlation can only be used to study the *overall* association between two variables. For example, Baig and Goldfajn [1998], compared the correlation between two markets for a pre-crisis and a post-crisis period determined by a shock. They found that there was indeed an increase in the cross market correlation coefficient after a crisis.

We should note here that the choice of measures used in determining the dependency structure is important. For example, to hedge against investment risk, investors in financial markets traditionally used Pearson's correlation coefficient as a measure to diversify across markets where the correlation coefficient was low. Consider the example where the annual returns in a domestic market and in a foreign market have a linear correlation coefficient of 0.2, and so under assumption that the distribution of the returns are Gaussian, the probability that the returns in both markets are in their lowest 5^{th} percentiles is less than 0.005. So with this in mind, an investor should be able to significantly reduce the underlying portfolio risk by balancing the portfolio with investments in the foreign market. However, it appears that market crashes and financial crises can often happen in different countries during the same time period, even when the correlation between these markets is fairly low. The above example shows that in addition to the importance given to the degree of dependence, the structure of the dependence is also relevant.

As pointed out by Embrechts et al [2002], pairs of markets with the same correlation coefficient could have very different dependency structures and these different structures could increase or decrease the diversification benefit.

Another approach that has been used in empirical studies of dependency involves computing conditional correlations. Ang,[1] studied the correlations between a portfolio and the market conditional on downside movements, and they found that correlations between U.S. stocks and the aggregate U.S. market were much greater for downside moves, especially for extreme downside moves, than for upside moves. Furthermore, correlations conditional on large movements were higher than those conditional on small movements. Typically, correlations computed separately for ordinary and stressful market conditions differ considerably, a pattern widely termed *correlation breakdown*, which is revealed to us through Boyer[3]. Such worries may not be justified since correlation breakdowns can easily be generated by data whose distribution is stationary and, in particular, whose correlation coefficient is constant. Therefore, although conditional correlations provide more information about the dependence than unconditional correlations, the results are sometimes misleading and need to be interpreted with caution.

A more convenient approach that can help in capturing the dependency structure of the joint distribution involves the use of a copula model. We will adopt the same approach in our analysis. In essence, a copula is a function that joins the marginal distribution functions to form the multivariate distribution functions. We will first define the concept of a copula, and then through Sklar's Theorem, we will show how a copula can be used to capture the joint bivariate distribution, if the underlying marginal distributions are known.

Since we were interested in examining the comovement of returns on foreign exchange markets, we chose to employ the copula model to capture the joint distribution of the daily returns on Sterling Pound and Japanese Yen respectively. In particular, since the distribution of the daily returns in both markets was gaussian, we chose a Gaussian copula to capture the underlying joint distribution of returns. A more detailed explanation is provided in Chapter 4. Our findings corroborated the results obtained by Hu, in that the daily returns of the Sterling Pound and the Japanese Yen moved more strongly at the tails of the distribution, in comparison to any other scenario within the distribution. However, there were a significantly large number of days where the daily returns in both markets were extremely small. We found that the theoretical Gaussian copula model successfully captured the dependency structure observed at the tails of the distribution. However, it failed to capture a 'sharp' peak that was present in the center of the distribution. It also overestimated the individual joint probabilities associated with the returns on the Sterling Pound and the Japanese Yen at the tails of the distribution.

It should be noted here that it is rather difficult to choose a copula to model the underlying joint distribution function of two variables. It is true that other copulas would have more successful at capturing the dependency structure at the tails. However, as in the case of the Gaussian copula, we would also expect these same copulas to fail in capturing the 'sharp' peak that was present in the center of the distribution.

Chapter 2
Introduction to Copulas

Copulas are functions that join or couple multivariate distribution functions to their one-dimensional marginal distributions. Our objective in this chapter is to gain a precise understanding of this. Consider a pair of random variables X and Y, with distribution functions

$$F(x) = P[X \leq x] \text{ and}$$

$$G(y) = P[Y \leq y],$$

respectively, and the joint distribution of the two variables X and Y,

$$H(x, y) = P[X \leq x, Y \leq y].$$

For any $(x, y) \in [0, 1]^2$, we associate $F(x)$, $G(y)$, and $H(x, y)$. In particular, each pair (x, y) would lead to a point $(F(x), G(y))$ in the unit square $[0, 1]^2$, and in turn, this ordered pair corresponds to a number $H(x, y)$ in $[0, 1]$. This correspondence which assigns the value $H(x, y)$ of the joint distribution function to the ordered pair $(F(x), G(y))$ is in essence, called a *copula.*

For a more detailed examination of copulas, we will need to define the notions of quasi-monotone or 2-increasing functions, and grounded functions. We will let $\mathbb{R}$ denote the set of all real numbers and

$$\bar{\mathbb{R}} = [-\infty, \infty].$$

We will denote the interval $[0, 1]$ by I. Also consider a *real function* H, whose domain, denoted, $\text{Dom}(H)$, is a subset of $\bar{\mathbb{R}}^2$ and range, denoted $\text{Ran}(H)$, is a subset of $\mathbb{R}$.

Definition 2.1. For a function $H : S_1 \times S_2 \mapsto \mathbb{R}$, where $S_1 \times S_2 \subset \bar{\mathbb{R}}^2$ and if $B = [x_1, x_2] \times [y_1, y_2] \subset \bar{\mathbb{R}}^2$, is a rectangle, with vertices in $S_1 \times S_2$, then

$$V_H(B) \stackrel{def}{=} H(x_2, y_2) - H(x_2, y_1) - H(x_1, y_2) + H(x_1, y_2). \tag{2.1}$$

We will use Definition 2.1 to define a 2-increasing or quasi-monotone function.

Definition 2.2. A function $H : \mathbb{R}^2 \mapsto \bar{\mathbb{R}}$ is *2-increasing* or *quasi-monotone*, if $V_H(B) \geq 0$ for all rectangles B.

The concept of a quasi-monotone function is useful in introducing the notion of a grounded function as shown in the following lemma and the subsequent discussion.

Lemma 2.3. *Let us consider S_1 and S_2 to be two nonempty sets on $\bar{\mathbb{R}}$, and let H be a 2-increasing function, with $Dom(H) = S_1 \times S_2$. Let x_1, x_2 be in S_1 with $x_1 \leq x_2$, and let y_1, y_2 be in S_2 with $y_1 \leq y_2$. Then the function $t \mapsto H(t, y_2) - H(t, y_1)$ is nondecreasing on S_1 and the function $t \mapsto H(x_2, t) - H(x_1, t)$ is nondecreasing on S_2.*

With additional hypothesis, we can show that a 2-increasing function H is nondecreasing in each argument. Suppose we have a set S_1 that contains a least element a_1 and in a similar manner, a set S_2 that contains a least element a_2. A function H on $[S_1 \times S_2]$ is said to be *grounded*, if $H(x, a_2) = 0 = H(a_1, y)$ for all (x, y) in $S_1 \times S_2$. As a consequence we have,

Lemma 2.4. *Let us consider S_1 and S_2, which are two nonempty sets on $\bar{\mathbb{R}}$, and let H be a grounded 2-increasing function on $S_1 \times S_2$. Then H is nondecreasing in each argument. Moreover, for any x_1, $x_2 \in S_1$ and y_1, $y_2 \in S_2$,*
$t \mapsto |H(x_2, t) - H(x_1, t)|$ and $t \mapsto |H(t, y_2) - H(t, y_1)|$ are nondecreasing on S_2 and S_1 respectively.

Proof. Let a_1, a_2 denote the least elements of the sets S_1, S_2 respectively. From Lemma 2.3, recall that $t \mapsto H(t, y_2) - H(t, y_1)$ is nondecreasing on S_1 and $t \mapsto H(x_2, t) - H(x_1, t)$ is nondecreasing on S_2. If we let $x_1 = a_1$, $y_1 = a_2$, then for any y_2, we have $H(t, y_2) - H(t, a_2)$ nondecreasing on S_1 and for any x_1, we have $H(x_2, t) - H(a_1, t)$ nondecreasing on S_2, and hence H is nondecreasing in each argument. The final claim follows by observing that

$$|H(x_2, t) - H(x_1, t)| = H(x_2 \vee x_1, t) - H(x_2 \wedge x_1, t), \text{ and similarly}$$

$$|H(t, y_2) - H(t, y_1)| = H(t, y_2 \vee y_1) - H(t, y_2 \wedge y_1).$$

□

We can use the existence of a greatest element in the sets S_1 and S_2 to define the marginal distributions F and G as shown below.

Definition 2.5. Consider the nonempty sets, S_1 and S_2 which have the greatest elements b_1 and b_2 respectively. Denote the domain of F and G by $\operatorname{Dom}(F)$ and $\operatorname{Dom}(G)$. Then a function $H : S_1 \times S_2 \mapsto \mathbb{R}$ has *marginal distributions* or *marginals* , and the marginal distributions of H are the functions F and G given by,

$$\operatorname{Dom}(F) = S_1 \text{ and } F(x) = H(x, b_2) \text{ for all } x \text{ in } S_1 \text{ and}$$

$$\operatorname{Dom}(G) = S_2 \text{ and } G(y) = H(b_1, y) \text{ for all } y \text{ in } S_2.$$

The following lemma is necessary to establish uniform continuity for copulas, which we shall see shortly and it makes use of the lemmas stated above.

Lemma 2.6. *Let us consider S_1 and S_2, which are two nonempty sets on $\bar{\mathbb{R}}$, and let H be a grounded 2-increasing function, with marginals, and domain $Dom(H) = S_1 \times S_2$. Let (x_1, y_1) and (x_2, y_2) be any two points in $S_1 \times S_2$. Then we have*

$$|H(x_2, y_2) - H(x_1, y_1)| \leq |F(x_2) - F(x_1)| + |G(y_2) - G(y_1)|.$$

Proof. Let $b_1 = max(S_1)$, $b_2 = max(S_2)$. Then

$$\begin{aligned} |H(x_2,y_2) - H(x_1,y_1)| &\leq |H(x_2,y_2) - H(x_2,y_1)| + |H(x_2,y_1)) - H(x_1,y_1)| \\ &\leq |H(b_1,y_2) - H(b_1,y_1)| + |H(x_2,b_2)) - H(x_1,b_1)| \\ &= |G(y_2) - G(y_1)| + |F(x_2) - F(x_1)| . \end{aligned}$$

Note that the second inequality follows from Lemma 2.4. □

2.1 Copulas

In this section we will provide the definition of a copula, along with properties of copulas. We note here that in the context of the preceding section that copulas are a class of grounded 2-increasing functions with marginals, with domain I^2.

Definition 2.7. A *two-dimensional* copula C is a function from $I \times I$ to I with the following properties:

1. For each u, v in I,

$$C(u,0) = 0 = C(0,v) \tag{2.2}$$

$$\text{and } C(u,1) = u \text{ and } C(1,v) = v; \tag{2.3}$$

2. For each u_1, u_2, v_1, v_2 in I with $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0. \tag{2.4}$$

In simpler terms, a two-dimensional copula C is a bivariate distribution function whose marginals are the distribution functions for two uniformly distributed variables on I. Also, it is convenient to examine the two-dimensional copula, C on the unit square $I \times I$, as outside the unit square, the values of $C(u,v)$ can be easily determined.

In particular, we have $C(u,v) = 0$ if either $u < 0$ or $v < 0$.

Also, $C(u,v) = u$ if $v > 1$ and similarly, $C(u,v) = v$ if $u > 1$.

From Definition 2.7, notice that the range of the two-dimensional copula $C(u, v)$, given by $\text{Ran}(C)$ lies in I. This makes it easier to define the n-dimensional copula, as it would simply be a multivariate distribution function whose marginals are the distribution functions for n uniformly distributed variables on I. A more precise definition of an n-dimensional copula can be obtained from Nelsen.

We also provide a theorem below that establishes the continuity of copulas on $I \times I$. The proof of the following theorem follows from Lemma 2.6.

Theorem 2.8. *Let C be a copula. Then for every u_1, u_2, v_1, v_2 in $Dom(C)$,*

$$|C(u_2, v_2) - C(u_1, v_1)| \leq |u_2 - u_1| + |v_2 - v_1| . \tag{2.5}$$

Hence C is uniformly continuous on its domain.

Chapter 3
Sklar's Theorem

Sklar's Theorem is an important result that uses the concept of a copula as a "bridge" to illustrate the relationship between the joint distribution function and its univariate marginal distribution functions. This relationship between multivariate distributions and their univariate marginal distributions through a copula function is fundamental to deriving all known copulas. To establish Sklar's Theorem, we will need to first define and discuss the notion of a distribution function and a joint distribution function, both of which are shown below.

Definition 3.1. A *distribution function* is a function F with domain $\bar{\mathbb{R}}$ such that

1. F is nondecreasing.
2. $F(-\infty) = 0$ and $F(\infty) = 1$.
3. F is right continuous.

Definition 3.2. A *joint distribution function* is a function $H : \bar{\mathbb{R}}^2 \mapsto \bar{\mathbb{R}}$ such that

1. H is 2-increasing.
2. $H(x, -\infty) = H(-\infty, y) = 0$ and $H(\infty, \infty) = 1$.
3. H is right continuous in each variable.

Thus, by definition, H is grounded, and since $\text{Dom}(H) = \bar{\mathbb{R}}^2$, H has marginal distribution F and G, which are given by,

$$F(x) = H(x, \infty) \text{ and } G(y) = H(\infty, y).$$

As a consequence, both F and G are distribution functions. We now provide Sklar's Theorem, as shown below.

Theorem 3.3. ***Sklar's Theorem.*** *Let H be a joint distribution function with marginal distribution functions F and G. Then there exists a copula C such that for all x, y in $\bar{\mathbb{R}}$,*

$$H(x) = C(F(x), G(y)). \tag{3.1}$$

If F and G are continuous, then C is either unique or can be uniquely determined on Ran$(F) \times$ RanG. Conversely, if C is a copula and F and G are distribution functions, then the function H is a joint distribution function with marginal distribution functions F and G.

Proof. from Lemma 2.6, recall that for any two nonempty subsets S_1, S_2 of $\bar{\mathbb{R}}$, if H is a grounded 2-increasing function with marginals F and G, and if (x_1, x_2) are any points in $S_1 \times S_2$, then we have

$$|H(x_2, y_2) - H(x_1, y_1)| \leq |F(x_2) - F(x_1)| + |G(y_2) - G(y_1)|\,.$$

If we let $F(x_1) = F(x_2)$ and $G(y_1) = G(y_2)$ then we have,

$$|H(x_2, y_2) - H(x_1, y_1)| \leq 0$$

$$\text{and } H(x_1, y_1) = H(x_2, y_2).$$

Therefore the set of ordered pairs given by,

$$\{((F(x), G(y))\,, H(x, y))\,|x, y \in \bar{\mathbb{R}}\}$$

defines a function C, whose domain is $\operatorname{Ran}(F) \times \operatorname{Ran}(G)$. If F and G are continuous then $\operatorname{Ran}(F) = \operatorname{Ran}(G) = I$. Observe that for each u in $\operatorname{Ran}(F)$, there is an x in $\bar{\mathbb{R}}$ such that $F(x) = u$, and that for each v in $\operatorname{Ran}(G)$, there is a y in $\bar{\mathbb{R}}$ such that $G(y) = v$.

$$\text{Hence } C(u, 1) = C(F(x), G(\infty)) = H(x, \infty) = F(x) = u$$

$$\text{and } C(1, v) = C(F(\infty), G(y)) = H(\infty, y) = G(y) = v$$

In an analogous manner,

$$C(u, 0) = C(F(x), G(-\infty)) = H(x, -\infty) = 0$$

$$\text{and } C(0, v) = C(F(-\infty), G(y)) = H(-\infty, y) = 0.$$

□

In effect, the copula can be thought of as the function that "couples" the joint distribution function to its univariate marginals. It is worth noting here that the result in Sklar's Theorem given by $H(x) = C(F(x), G(y))$, can be inverted to express the copula in terms of a joint distribution function and the inverses of the two marginals, if the inverses exist. But, if at least one marginal distribution is not strictly increasing, then it does not possess the property of an inverse. For this reason, we will need to define the concept of a quasi-inverse.

Definition 3.4. Consider a distribution function F. The *quasi-inverse* of F, denoted by $F^{(-1)}$, is any function with $\text{Dom}(F) = I$ such that

1. If t is in $\text{Ran}(F)$, then $F^{(-1)}(t)$ is any number x in $\bar{\mathbb{R}}$ such that $F(x) = t$. Specifically, for each t in $\text{Ran}(F)$, $F(F^{(-1)}(t)) = t$

2. If t is not in $\text{Ran}(F)$, then

$$F^{(-1)}(t) = \inf\{\text{x}|\text{F(x)} \geq \text{t}\} = \sup\{\text{x}|\text{F(x)} \leq \text{t}\}. \tag{3.2}$$

Through the use of quasi-inverses of distribution functions as specified by (3.2), we have the following corollary to Sklar's theorem.

Corollary 3.5. *Let H be a joint distribution function with continuous marginal distribution functions, F and G, and let C be a copula such that $Dom(C) =$*

Ran$(F) \times$ *Ran*(G), *and for all* x, y *on* $\bar{\mathbb{R}}$, $H(x,y) = C(F(x), G(y))$. *Suppose we let* $F^{(-1)}$, $G^{(-1)}$ *be quasi-inverses of* F *and* G *respectively.*

Then for any (u,v) *in* *Dom*(C), *we have*

$$C(u,v) = H(F^{(-1)}(u), G^{(-1)}(v)). \tag{3.3}$$

An example that uses the notion of a quasi-inverse is illustrated below.

Example 3.6. Consider the distribution function H given by,

$$H(x,y) = \begin{cases} \frac{(x+1)(e^y-1)}{x+2e^y-1}, & (x,y) \in [-1,1] \times [0,\infty], \\ 1 - e^{-y}, & (x,y) \in (1,\infty] \times [0,\infty], \\ 0, & \text{elsewhere.} \end{cases}$$

with marginal distribution functions F and G given by

$$F(x) = \begin{cases} 0, & x < -1, \\ \frac{(x+1)}{2}, & x \in [-1,1], \\ 1, & x > 1 \end{cases}$$

$$\text{and } G(y) = \begin{cases} 0, & y < 0, \\ 1 - e^{-y}, & y \geq 0. \end{cases}$$

Then, the quasi-inverses of F and G, which are denoted by $F^{(-1)}(u)$ and $G^{(-1)}(v)$, are given by

$$F^{(-1)}(u) = 2u - 1 \text{ and } G^{(-1)}(v) = -ln(1-v) \text{ for all } u, v \in \mathrm{I}.$$

Since, $\mathrm{Ran}(F) = \mathrm{Ran}(G) = \mathrm{I}$, through (3.1), we have

$$C(u,v) = \frac{uv}{u+v-uv}.$$

3.1 Copulas and Random Variables

We will now extend the concept of a copula to random variables. We will first need to define the notion of a random variable as shown below.

Definition 3.7. A *random variable* X is a function on the probability space Ω such that

$$[X \leq x] \stackrel{def}{=} \{\omega \in \Omega : X(\omega) \leq x\}$$

is an event, i.e. it is measurable for every $x \in \mathbb{R}$.

We will use X, Y to represent random variables, and x, y to represent the values of these random variables. We will use F as a distribution function of a random variable X, when for each x on $\bar{\mathbb{R}}$,

$$F(x) = P[X \leq x].$$

For instance, $F(-\infty) = P[X \leq -\infty] = 0$, and $F(\infty) = P[X \leq \infty] = 1$. It is worth noting here that a random variable is continuous if its corresponding distribution function is continuous. The same treatment can be extended to two or more random variables which are the components of a quantity whose values are described by a joint distribution function. Thus, it is common to define a collection of random variables on a common probability space. We will now restate Sklar's Theorem in terms of random variables as follows.

Theorem 3.8. *Let X and Y be random variables with distribution functions F and G respectively, and a joint distribution function H. Then there exists a copula C such that*

$$H(x, y) = C(F(x), G(y)).$$

If F and G are continuous, then C is unique. Alternatively, C can be uniquely determined on $Ran(F) \times Ran(G)$.

We will refer to the copula C in the above Theorem, as the copula of the random variables X and Y and denote it by C_{XY}.

In nonparametric statistics, copulas are particularly useful in obtaining the joint distribution function of two or more variables. The reason for the increase in the usage of copulas is because the copulas that are used in determining strictly monotone transformations of random variables, are usually either invariant or easy to determine as they change in predictable ways.

We will see this "invariance" exhibited by copulas in the following theorem. First, we should recall that if the distribution function of a given random variable X is continuous, and if we let α be a strictly monotone function whose domain contains $\operatorname{Ran}(X)$, then the distribution function of the random variable $\alpha(X)$ is also continuous. We will first address the case of strictly increasing transformations below.

Theorem 3.9. *Let X and Y be continuous random variables with copula, $C_{X,Y}$. Suppose α,β are strictly increasing on $Ran(X)$ and $Ran(Y)$ respectively. Then we obtain,*

$$C_{\alpha(X)\beta(Y)} = C_{XY}. \tag{3.4}$$

As a consequence, C_{XY} is invariant under strictly increasing transformations of both X and Y.

Proof. Suppose F_1, G_1, F_2, and G_2 are the distribution functions of X, Y, $\alpha(X)$, and $\beta(Y)$ respectively. Both α and β are strictly increasing on R_X and R_Y. Therefore,

$$F_2(x) = P[\alpha(X) \leq x] = P[X \leq \alpha^{-1}(x)] = F_1(\alpha^{-1}(x)).$$

Similarly,

$$G_2(y) = P[\beta(Y) \leq y] = P[Y \leq \beta^{-1}(y)] = G_1(\beta^{-1}(y)).$$

It follows that for any x, y in $\bar{\mathbb{R}}$,

$$\begin{aligned} C_{\alpha(X)\beta(Y)}(F_2(x), G_2(y)) &= P[\alpha(X) \leq x, \beta(Y) \leq y] \\ &= P[X \leq \alpha^{-1}(x), Y \leq \beta^{-1}(y)] \\ &= P[X \leq \alpha^{-1}(x), Y \leq \beta^{-1}(y)] \\ &= C_{XY}(F_1(\alpha^{-1}(x)), G_1(\beta^{-1}(y))) \\ &= C_{XY}(F_2(x)), G_2(y)). \end{aligned}$$

Since both X and Y are continuous, we have

$$\text{Ran}(F_2) = \text{Ran}(G_2) = I.$$

It follows that $C_{\alpha(X),\beta(Y)} = C_{XY}$ on I^2. □

In a similar manner we can obtain the relationship between $C_{\alpha(X)\beta(Y)}$ and C_{XY} when at least one of either α and β is strictly decreasing as shown below in the following theorems.

Theorem 3.10. *Let X and Y be continuous random variables with copula C_{XY}. Let α and β be strictly monotone on Ran(X) and Ran(Y) respectively. It follows that if α is strictly increasing and β is strictly decreasing, then*

$$C_{\alpha(X)\beta(Y)}(u, v) = u - C_{XY}(u, 1 - v).$$

Proof. Suppose F_1, G_1, F_2, and G_2 denote the distribution functions for X, Y, $\alpha(X)$, and $\beta(Y)$ respectively. Suppose α is strictly increasing on Ran(X) and β is strictly decreasing on Ran(Y). Therefore,

$$F_2(x) = P[\alpha(X) \leq x] = P[X \leq \alpha^{-1}(x)] = F_1(\alpha^{-1}(x)).$$

However,

$$
\begin{aligned}
G_2(y) = P[\beta(Y) \leq y] &= P[Y \geq \beta^{-1}(y)] \\
&= 1 - P[Y \leq \beta^{-1}(y)] \\
&= 1 - G_1(\beta^{-1}(y)).
\end{aligned}
$$

It follows that for any x, y in $\bar{\mathbb{R}}$,

$$
\begin{aligned}
C_{\alpha(X)\beta(Y)}(F_2(x), G_2(y)) &= P[\alpha(X) \leq x, \beta(Y) \leq y] \\
&= P[X \leq \alpha^{-1}(x), Y \geq \beta^{-1}(y)] \\
&= P[X \leq \alpha^{-1}(x)] - P[X \leq \alpha^{-1}(x), Y < \beta^{-1}(y)] \\
&= F_1(\alpha^{-1}(x)) - C_{XY}(F_1(\alpha^{-1}(x)), G_1\beta^{-1}(y)) \\
&= F_2(x) - C_{XY}(F_2(x), 1 - G_2(y)) \\
\text{Thus } C_{\alpha(X)\beta(Y)}(u, v) &= u - C_{XY}(u, 1 - v), \text{ for all } u, v \in [0, 1].
\end{aligned}
$$

□

By interchanging the roles of X and Y, we obtain the following theorem.

Theorem 3.11. *Let X and Y be continuous random variables with copula C_{XY}. Let α and β be strictly monotone on Ran(X) and Ran(Y) respectively. It follows that if α is strictly decreasing and β is strictly increasing, then*

$$C_{\alpha(X)\beta(Y)}(u, v) = v - C_{XY}(1 - u, v).$$

Theorem 3.12. *Let X and Y be continuous random variables with copula C_{XY}. Let α and β be strictly monotone on Ran(X) and Ran(Y) respectively. It follows that if α and β are both strictly decreasing, then*

$$C_{\alpha(X)\beta(Y)}(u, v) = u + v - 1 + C_{XY}(1 - u, 1 - v).$$

Proof. Suppose F_1, G_1, F_2, and G_2 denote the distribution functions for X, Y, $\alpha(X)$, and $\beta(Y)$ respectively. Both α and β are strictly decreasing on $\text{Ran}(X)$ and $\text{Ran}(Y)$. Therefore,

$$\begin{aligned}
F_2(x) = P[\alpha(X) \leq x] &= P[X \geq \alpha^{-1}(x)] \\
&= 1 - P[X < \alpha^{-1}(x)] \\
&= 1 - F_1(\alpha^{-1}(x)). \\
\text{Similarly, } G_2(y) &= 1 - G_1(\beta^{-1}(y)).
\end{aligned}$$

We have then, for any x, y in $\bar{\mathbb{R}}$,

$$C_{\alpha(X)\beta(Y)}(F_2(x), G_2(y)) = P[\alpha(X) \leq x, \beta(Y) \leq y]$$

$$\begin{aligned}
&= P[X \geq \alpha^{-1}(x), Y \geq \beta^{-1}(y)] \\
&= 1 - P[X \leq \alpha^{-1}(x)] + 1 - P[Y \leq \beta^{-1}(y)] - (1 - P[X \leq 1 - \alpha^{-1}(x), Y \leq 1 - \beta^{-1}(y)]) \\
&= 1 - F_1(\alpha^{-1}(x)) + 1 - G_1(\beta^{-1}(y)) - (1 - C_{XY}(1 - F_1(\alpha^{-1}(x)), 1 - G_1(\beta^{-1}(y)))) \\
&= 1 - F_1(\alpha^{-1}(x)) + 1 - G_1(\beta^{-1}(y)) - 1 + C_{XY}(1 - F_1(\alpha^{-1}(x)), 1 - G_1(\beta^{-1}(y))) \\
&= F_2(x) + G_2(y) - 1 + C_{XY}(1 - F_2(x), 1 - G_2(y))
\end{aligned}$$

Note that the third equality above used the continuity of X and Y.

So, $C_{\alpha(X)\beta(Y)}(u, v) = u + v - 1 + C_{XY}(1 - u, 1 - v)$ for all $u, v \in [0, 1]$.

$\square$

For each of the above theorems, we can see that the copula remains independent of the choices of the distribution functions, α and β respectively.

We can also use a rank correlation measure known as Kendall's Tau to measure the association between two random variables. We define Kendall's Tau using the notion of concordance, which is defined below.

Definition 3.13. Let (x_i, y_i) and (x_j, y_j) denote two observations from a vector (X, Y) of continuous random variables. We say that (x_i, y_i) and (x_j, y_j) are *concordant* if

$$x_i < x_j \text{ and } y_i < y_j,$$

$$\text{or if } x_i > x_j \text{ and } y_i > y_j.$$

In a similar manner, we can define the notion of discordance as shown below.

Definition 3.14. Let (x_i, y_i) and (x_j, y_j) denote two observations from a vector (X, Y) of continuous random variables. We say that (x_i, y_i) and (x_j, y_j) are *discordant* if

$$x_i < x_j \text{ and } y_i > y_j,$$

$$\text{or if } x_i > x_j \text{ and } y_i < y_j.$$

We can extend the notions of concordance and discordance to a correlation measure, Kendall's Tau. The importance of using Kendall's Tau is central to the theory of copulas as it is invariant under monotone transformations of the underlying distributions X and Y. We provide a definition of the sample version of Kendall's Tau below. For convenience, we will define

Definition 3.15. Let $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ denote a random sample of n observations from a vector (X, Y) of continuous random variables. There are $\binom{n}{2}$ distinct pairs (x_i, y_i) and (x_j, y_j) of observations in the sample, and each pair is either concordant or discordant. If we let c denote the number of concordant pairs and d denote the number of discordant pairs, then Kendall's Tau for the sample is defined as

$$\tau_s = \frac{c-d}{c+d} = \frac{c-d}{\binom{n}{2}} \tag{3.5}$$

Recall that Pearson's correlation coefficient is unstable under monotone transformations of functions. However, there is a way of expressing Kendall's Tau as a function of Pearson's correlation coefficient, if the underlying two-dimensional distribution is normal. We have,

$$\tau = \frac{2}{\pi} \arcsin(\rho) \tag{3.6}$$

Chapter 4
Empirical Results

Before we can begin the discussion of our theoretical model, we will need to establish several concepts as shown below.

We will often replace a market-observed variable or an underlying market variable, by a *proxy*. This will be a transformation of the market variable, so as to obtain a variable with a specified distribution such as the Gaussian distribution.

Consider the return on a security represented by a proxy variable $X(t)$ given by

$$X(t) = aM(t) + bZ_1(t), \tag{4.1}$$

with a, b as coefficients of the components $M(t)$ and $Z_1(t)$ respectively. In a similar manner, consider the return on another security represented by a proxy variable $Y(t)$ given by

$$Y(t) = cM(t) + dZ_2(t), \tag{4.2}$$

with c, d as coefficients of the components $M(t)$ and $Z_2(t)$ respectively. Note here that the component $M(t)$ is common to both $X(t)$ and $Y(t)$ respectively, while Z_1 and Z_2 are random variables affecting only $X(t)$ and $Y(t)$ respectively. Additionally, $M(t)$, $Z_1(t)$ and $Z_2(t)$ are all independent and identically distributed, with mean 0 and standard deviation 1. In fact, we will assume that they are standard Gaussian variables. Note here that $M(t)$ may be viewed as a "global" market variable factor, while Z_1 and Z_2 are idiosyncratic factors.

As a consequence, we see that

$$E[X] = E[Y] = 0,$$

$$\text{and } Var(X) = Var(Y) = 1.$$

Recall that the variance of the distribution X is given by

$$Var(X) = E[X^2] - (E[X])^2 = 1$$
$$\text{Thus, } E[X^2] = 1$$

which is equivalent to,

$$a^2E[M^2] + b^2E[{Z_1}^2] + 2abE[MZ_1] = 1 \tag{4.3}$$

Observe that $E[MZ_1] = 0$, as $E[M]E[Z_1] = 0$. Moreover, $E[M^2] = E[{Z_1}^2] = 1$. So (4.1) yields,

$$a^2 + b^2 = 1.$$

Let us assume a,b,c,$d > 0$ without loss of generalization, so that a, b and c, d are positively correlated. This means however, that X and Y are positively correlated.

Thus from (4.1), we have

$$X = aM + \sqrt{1-a^2}Z_1, \tag{4.4}$$

and similarly from (4.2),

$$\text{we have } Y = cM + \sqrt{1-c^2}Z_2. \tag{4.5}$$

Observe here that the correlation between X and Y denoted ρ_{XY}, is defined by

$$\rho_{XY} = \frac{E[XY] - E[X]E[Y]}{Var(X)Var(Y)}$$

Recall that $E[X] = E[Y] = 0$ and $Var(X) = Var(Y) = 1$, yielding

$$\rho_{XY} = E[XY].$$

From (2.4) and (2.5), it follows that

$$\rho_{XY} = E[(aM + \sqrt{1-a^2}Z_1)(cM + \sqrt{1-c^2}Z_2)]$$

$$= E[(acM^2 + a\sqrt{1-c^2}MZ_2 + c\sqrt{1-a^2}MZ_1 + \sqrt{1-a^2}\sqrt{1-c^2}Z_1Z_2)]. \quad (4.6)$$

Observe that

$$E[a\sqrt{1-c^2}MZ_2] = E[c\sqrt{1-a^2}MZ_1] = E[\sqrt{1-a^2}\sqrt{1-c^2}Z_1Z_2)] = 0.$$

So from (4.6),

$$\rho_{XY} = acE[M^2].$$

But,

$$E[M^2] = 1.$$

So we obtain

$$\rho_{XY} = E[XY] = ac.$$

Additionally, $\rho_{XM} = E[XM] = E[(aM + \sqrt{1-a^2}Z_1)M]$

$$\begin{aligned} &= E[aM^2 + \sqrt{1-a^2}MZ_1] \\ &= E[aM^2] \\ \text{Thus, } E[XM] &= a. \\ \text{Similarly, } \rho_{YM} = E[YM] &= c. \end{aligned}$$

The above equations express the correlation of the underlying returns on X and Y respectively to the returns on the market, M. Furthermore, by definition the pair (X, Y) is given by

$$(aM(t) + \sqrt{1-a^2}Z_1(t), cM(t) + \sqrt{1-c^2}Z_2(t)) \text{ with both}$$

$$aM(t) + \sqrt{1-a^2}Z_1(t) \text{ and } cM(t) + \sqrt{1-c^2}Z_2(t),$$

defined as normally distributed functions with mean 0 and variance 1.

We will now justify our usage of the Gaussian copula, as our choice of copulas to capture the joint distribution function of the returns on the Sterling Pound and Japanese Yen.

Consider the distribution of the returns of a security X, with a uniform marginal distribution denoted by $F_X(x)$, where

$$F_X(x) = P[X \leq x]$$

and $x \in X$. Let $U = F_X(x)$ and let $\tilde{X} = \Phi^{-1}(U)$, where Φ is the standard Gaussian distribution function. Then we have the distribution function of $\tilde{X}$ described as follows,

$$\begin{aligned} F_{\tilde{X}}(x) = P[\tilde{X} \leq x] &= P[\Phi^{-1}(U) \leq x] \\ &= P[\Phi^{-1}((F_X(x)) \leq x] \\ &= P[F_X(x) \leq \Phi(x)] \\ &= P[x \leq F_X^{-1}(\Phi(x))] \\ &= F_X(F_X^{-1}(\Phi(x))) \\ &= \Phi(x). \end{aligned}$$

So $\tilde{X}$ is standard gaussian.

The transformations used above can be applied to obtain the result in Sklar's Theorem in terms of a Gaussian copula as shown below. Let the marginal distribution functions of the return on securities X and Y, be F_X and G_Y respectively. Let $U = F_X(x)$ and $V = G_Y(y)$, be the *uniformized* marginal distributions of X and Y and let H be the observed joint distribution between X and Y. Then, from Sklar's Theorem, we have

$$C(F_X(x), G_Y(y)) = C(u, v) = H(x, y)$$

Recall from (3.4) that copulas are invariant under strictly linear transformations of the marginal distributions. So, if we *gaussianize* the marginal distributions U and V, then we have,

$$C(\Phi^{-1}(u), \Phi^{-1}(v)) = C(u, v) = H(x, y).$$

In essence, this gives us the choice of determining the joint distribution function of X and Y through either the "uniformized" marginal distributions or the "gaussianized" marginal distributions. In the ensuing analysis of the returns on the Sterling Pound and Japanese Yen, we shall examine both cases.

Theoretically, though Sklar's Theorem provides a desirable result by linking the joint distribution of two variables with the copula of the marginal distributions of these two variables, in practice, it is more convenient to express Sklar's theorem in terms of the density functions of the copula and joint distribution respectively. This is especially useful in scenarios where the range of either the joint distribution that is to be determined or the specified copula is $\bar{\mathbb{R}}$. This notion of using the copula density function to express the joint density function of two random variables is particularly useful in our analysis, as the density function of the Gaussian copula can easily be determined.

Consider the bivariate standard Gaussian Distribution as shown below,

$$H(x,y) = \int_{-\infty}^{\Phi^{-1}(x)} \int_{-\infty}^{\Phi^{-1}(y)} \exp\left[-\left(\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right)\right] \frac{ds\, dt}{2\pi\sqrt{1-\rho^2}},$$

If we take the partial derivative of $H(x,y)$ with respect to x,y we obtain

$$\frac{\partial^2 H(x,y)}{\partial x \partial y} = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\left(\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right)\right] \sqrt{2\pi} \exp\left[\frac{x^2}{2}\right] \sqrt{2\pi} \exp\left[\frac{y^2}{2}\right]$$

yielding,

$$\begin{aligned} &= \frac{1}{\sqrt{1-\rho^2}} \exp\left[-\left(\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right) + \frac{x^2}{2} + \frac{y^2}{2}\right] \\ &= \frac{1}{\sqrt{1-\rho^2}} \exp\left[-\left(\frac{x^2 - 2\rho xy + y^2 - x^2(1-\rho^2) - y^2(1-\rho^2)}{2(1-\rho^2)}\right)\right] \end{aligned}$$

$$\text{Thus, } h(x,y) = \frac{1}{\sqrt{1-\rho^2}} \exp\left[\frac{-1}{2(1-\rho^2)} \left(\rho^2 x^2 - 2\rho xy + \rho^2 y^2\right)\right] \tag{4.7}$$

Recall that from Sklar's Theorem, we have

$$h(x,y) = c(\Phi^{-1}(u), \Phi^{-1}(v)), \text{ yielding}$$

$$c(\Phi^{-1}(u), \Phi^{-1}(v)) = \frac{1}{\sqrt{1-\rho^2}} \exp\left[\frac{-1}{2(1-\rho^2)}\left(\rho^2 x^2 - 2\rho xy + \rho^2 y^2\right)\right] \tag{4.8}$$

Here, $h(x,y)$ is the density function of the bivariate Gaussian distribution of random variables X and Y. Notice that, through Sklar's Theorem, the above expression also describes the density function for the Gaussian copula, if the marginal distribution functions are "gaussianized". In particular, we will compare the "theoretical" Gaussian copula, $c(u,v)$ with the "observed" joint density function, denoted by $\hat{h}(\Phi^{-1}(u), \Phi^{-1}(v))$.

4.1 Determination of the "Observed" Joint Density Function

Recall that we are trying to capture the joint density function of the daily returns on the Sterling Pound and the Japanese Yen. The choice of copula is then determined by examining the distribution of the returns of the two currencies. Note that the analysis used in generating the joint probability density function can be also extended to the returns of other assets such as stocks, bonds, futures, and forwards. We start by defining the notion of the return on a currency as shown below.

Definition 4.1. Let p_t and p_{t+1} denote the intra-day exchange rate or units of B generated by one unit of A, where t represents the day. Then the *return* on currency B with respect to A is given by

$$X(t) = \frac{p_{t+1} - p_t}{p_t}.$$

Note here that if $X(t) > 0$, the value of $X(t)$ is the profit amount per unit investment observed for the t^{th} day. In an analogous manner, If $X(t) < 0$, the value of $X(t)$ is the loss amount per unit investment observed for the t^{th} day. Additionally, if $X(t) = 0$ we conclude that there was no loss or profit amount per unit investment observed for the t^{th} day. Also note that we neglect the effect of interest rates in the ensuing analysis.

Recall now, that our objective was to capture the joint probability density function between the daily return per unit investment on the Japanese Yen and the daily return per unit investment on the Sterling Pound. In lieu of this objective, we extracted the daily intra-day spot price of the above currencies in terms of US Dollars for the period $1971 - 2006$.

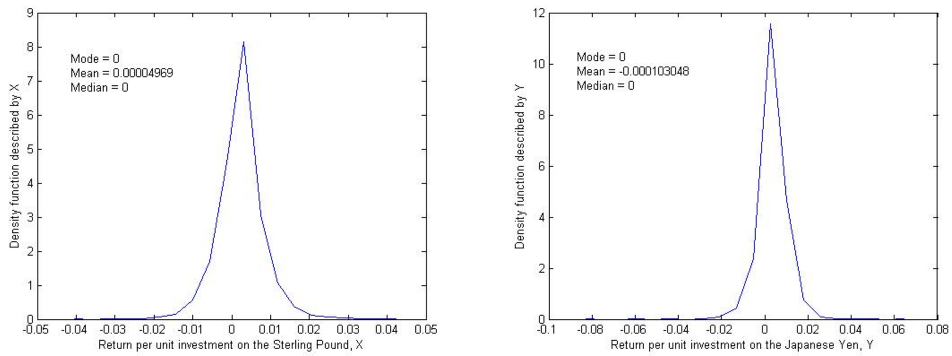
As an example, on 1/13/1971, which was the 1^{st} day in our distribution, we observed that \$1 exchanged for 0.4163 Sterling Pound and 358.44 Japanese Yen respectively.

On the following day, 1/14/1971, which was the 2^{nd} day in our distribution, we observed that \$1 exchanged for 0.4514 Sterling Pound and 358.40 Japanese Yen. If we denote the daily return on the Sterling Pound by X and the daily return on the Japanese Yen by Y, it can be seen that

$$X(1) = \frac{0.4514 - 0.4163}{0.4163} = -0.002161902 \text{ or a loss of} - 0.2161902\%$$
$$\text{and that } Y(1) = \frac{358.40 - 358.44}{358.44} = -0.000111595 \text{ or a loss of -0.0111595\%.}$$

In this manner, we calculated $X(t)$ and $Y(t)$ for 9040 days. We saw that the average return for both X and Y was approximately 0, yielding $E[X(t)] = 0$ and $E[Y(t)] = 0$. We also observed that both X and Y possessed gaussian-like distributions.

It should be noted here that this determination is consistent with the Rational Expectations Theory in Economics, where markets are deemed to be "efficient", in that the return on the market cannot be predicted in advance on a consistent basis.



(a) Distribution of X (b) Distribution of Y

FIGURE 4.1. Distribution of daily returns on the Sterling Pound and the Japanese Yen

We obtained a plot of the distributions of X and Y as shown in Figure 4.1.

To obtain the individual marginal distribution functions for X and Y, we separated and sorted the distribution of the daily returns per unit investment of the Sterling Pound and the Japanese Yen respectively, in ascending order.

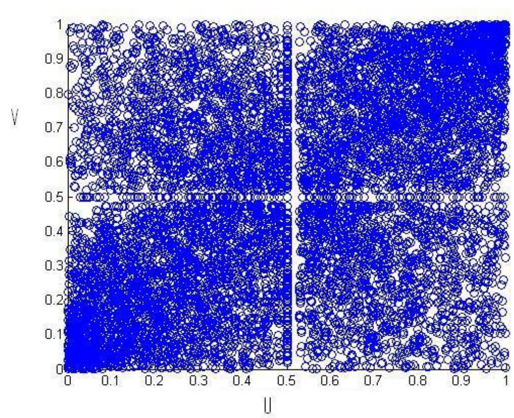


FIGURE 4.2. *Uniformized* marginal distributions of the daily returns on the Sterling Pound and Japanese Yen

We then obtained the "uniformized" individual marginal distribution functions for X and Y denoted by U and V respectively as shown in Figure 4.1. Note that the marginal distributions, U and V are both "uniformized" in that $\text{Ran}(U) = \text{Ran}(V) = I$. Observed the large concentration of pairs of returns at the lower left and upper right corners of the graph.

In determining our "observed" joint density function in terms of the "uniformized" marginals, it was necessary to approximate the joint probabilities, $P[u \leq U, v \leq V]$ on $[0,1]$. This was achieved by "sectioning" the entire distribution of U and V into intervals of equal width. We will use

$$i : i \in [1, n] \text{ and } j : j \in [1, n],$$

to denote the interval count of the distributions U and V respectively. In our example, if we let $[A, B]$ be the interval $[0, 1]$, and if we choose to have $n = 20$ intervals for both X and Y then our interval width for both U and V is given by

$$\frac{B - A}{n} = 0.05.$$

Alternatively, an interval length of 0.05 would yield $n = \frac{1}{0.05} = 20$ intervals for both U and V each.

Furthermore, if we were to observe the distribution between U and V, as shown in Figure 4.1, we can see that this would yield $20 \times 20 = 400$ squares, with each square capturing the frequency of pairs (u, v) that lie within the intervals $[u_{i-1}, u_i] \times [v_{j-1}, v_j]$ that in effect, construct that corresponding square's boundaries. For example, in examining the frequency with which $(U, V) \in [0, 0.05]^2$, we observed that there were 105 days where both U and V were less than the value 0.05. Alternatively, there were 105 days , where both U, V, the "uniformized" marginal distributions of the return on Sterling Pound and the return on the Japanese Yen were in their lowest 5 percentiles.

Moreover, in dividing the frequency of the pairs

$$(u_{i-1} \leq U \leq u_i, v_{j-1} \leq V \leq v_j),$$

that lie within a particular square's boundaries, by the total number of pairs that make up the entire distribution, we obtain the individual joint probabilities for (U, V) for each square that is formed on the joint distribution of U and V. In general, the (i^{th}, j^{th}) individual probability is represented by

$$P[u_{i-1} \leq U \leq u_i, v_{j-1} \leq V \leq v_j]$$

Reverting back to our example, we can see that

$$P[u \leq 0.05, v \leq 0.05] = \frac{105}{9040} = 0.011615044.$$

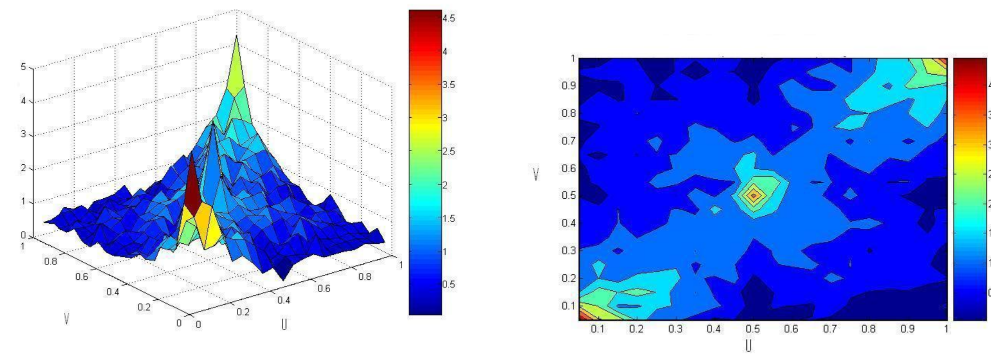
In addition, we can obtain the "observed" copula density function, $\hat{c}(u, v)$ with "uniformized" marginals, by dividing the individual joint probability associated with a particular square by the area of that square. In other words,

$$\hat{c}_{i,j}(u, v) = \frac{P[u_{i-1} < U < u_{i-1}, v_{j-1} < V < v_j]}{(u_i - u_{i-1})(v_j - v_{j-1})}.$$

In our example, we obtain

$$\hat{c}_{1,1}(u, v) = \frac{P[0 < U < 0.05, 0 < V < 0.05]}{(0.05 - 0)(0.05 - 0)} = \frac{0.011714828}{0.0025} = 4.646017699.$$

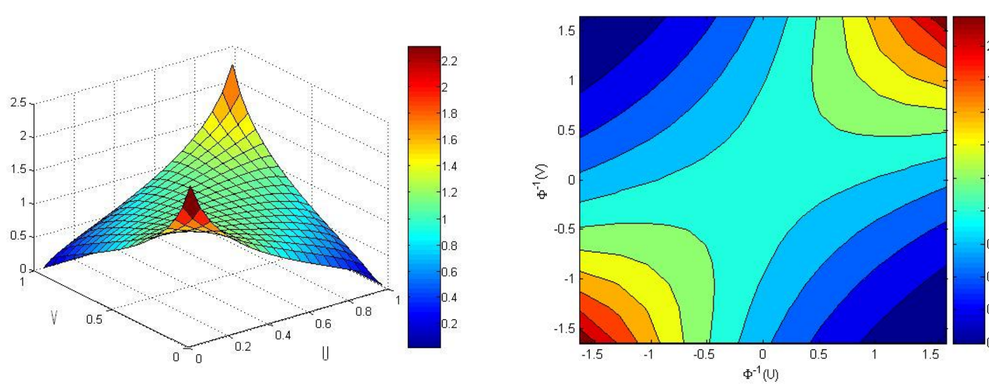
Through the above procedure, we were able to obtain the "observed" copula density function corresponding to the "uniformized" marginal distributions U and V across I^2 as well as the corresponding plot as shown in Figure 4.3. Through Sklar's Theorem, note that this "observed" copula density function is also the "observed" joint density function of the returns on the Sterling Pound and the Japanese Yen. We also provide a contour plot of the joint probability density function as depicted in Figure 4.3 as an alternative for further examination. Note here that the "theoretical" Gaussian copula density function with "uniformized" marginals was computed by replacing $\Phi^{-1}(u)$ and $\Phi^{-1}(v)$ with u and v respectively in (3.1). We can compare the plot of the "observed" Gaussian copula density function in Figure 4.3 with "uniformized marginals to that of the "theoretical" Gaussian copula with "uniformized" marginal distributions, which is depicted in Figure 4.4. The



(a) "Observed" Copula density function with uniformized marginals (b) Contour Plot of the "Observed" Copula density function with uniformized marginals

FIGURE 4.3. "Observed" Copula with *uniformized* marginal distributions

corresponding contour plot of the "theoretical" Gaussian copula density, is also provided in Figure 4.4 for further examination.



(a) "Theoretical" Copula density function with uniformized marginals (b) Contour Plot of the "Theoretical" Copula density function with uniformized marginals

FIGURE 4.4. "Theoretical" Copula with *uniformized* marginal distributions

Recall that if we were to apply a standard Gaussian transformation on U and V, we can obtain $\Phi^{-1}(U)$ and $\Phi^{-1}(V)$. Note here that in the discussion of the “theoretical” Gaussian Copula, we have

$$\begin{aligned} \tilde{X} &= \Phi^{-1}(U) \quad (4.9) \\ \text{and } \tilde{Y} &= \Phi^{-1}(V). \quad (4.10) \end{aligned}$$

From here on, for convenience purposes, we will continue to refer to the “gaussianized” marginal distributions of X and Y by $\tilde{X}$ and $\tilde{Y}$ respectively. We provide a plot of $\tilde{X}$ and $\tilde{Y}$ below in Figure 5.

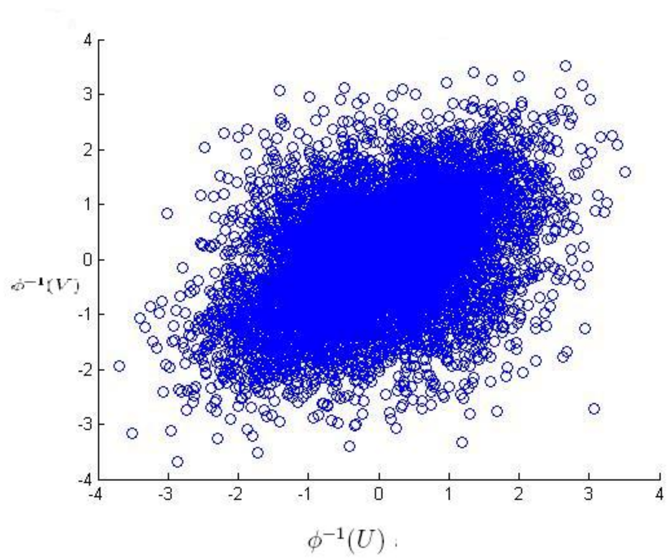


FIGURE 4.5. *Gaussianized* marginal distributions of the daily returns on the Sterling Pound and Japanese Yen

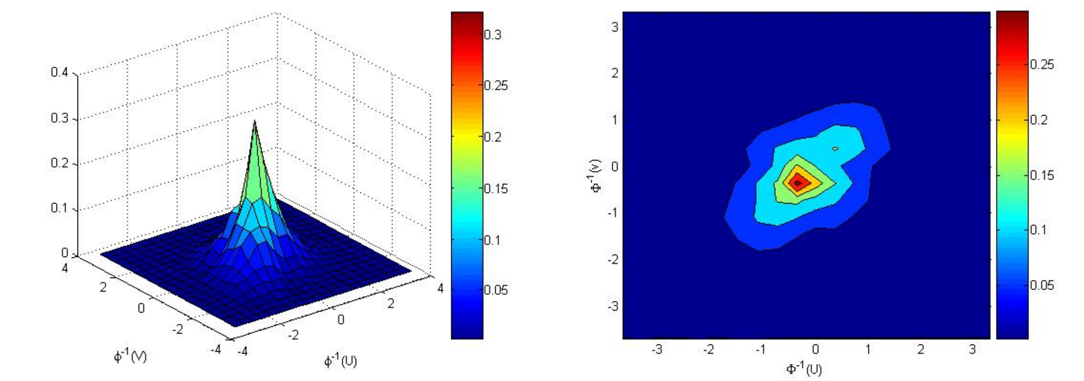
In our analysis, for both marginal distributions U and V, we adjusted the values of $\Phi^{-1}(1)$ by setting $\Phi^{-1}(1) = 3.7$, as an approximation. It follows that

$$\text{Ran}(\tilde{X}) = \text{Ran}(\tilde{Y}) = [-3.693442058, 3.7].$$

The procedure used to determine $\hat{c}(\tilde{x}, \tilde{y})$ was analogous to the one used in determining $\hat{c}(u, v)$. The only distinction made was that we restricted the domain of the interval $[A, B]$ for both $\tilde{X}$ and $\tilde{Y}$ to $[-3.693442058, 3.7]$.

For $n = 20$ intervals, it follows that the corresponding interval length was

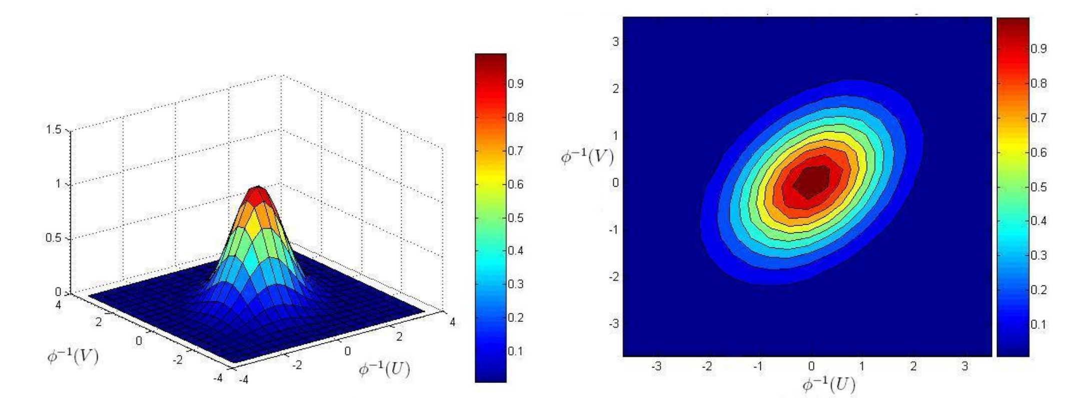
$$\frac{B - A}{n} = 0.369672103.$$



(a) "Observed" Copula density function with gaussianized marginals (b) Contour Plot of the "Observed" Copula density function with gaussianized marginals

FIGURE 4.6. "Observed" Copula with gaussianized marginal distributions

As an illustration we provide a graph and a contour plot depicting the "observed" joint density function, $h_{Gauss}(x, y)$ in Figure 4.6 respectively.



(a) "Theoretical" Copula density function with gaussianized marginals (b) Contour Plot of the "Theoretical" Copula density function with gaussianized marginals

FIGURE 4.7. "Theoretical" Copula with *gaussianized* marginals

The corresponding contour plot of the "theoretical" Gaussian copula density with "gaussianized" marginals is also provided in Figure 4.7 for further examination. Note here that the "theoretical" Gaussian copula density function with "uniformized" marginals was determined through (3.1).

4.2 Important Observations

We obtained the daily intra-day prices for the exchange rate of the US Dollar-Sterling Pound and US Dollar-Japanese Yen. For convenience, we denote the daily return on the US Dollar-Sterling Pound and the daily return on the US Dollar-Japanese Yen by X and Y respectively. We found that the range of X denoted by R_X was,

$$R_X = [-0.044799054, 0.042232558]$$

and the corresponding range of Y denoted by R_Y was,

$$R_Y = [-0.090670395, 0.064553866]$$

The Pearson's correlation coefficient between X and Y was, $\rho = 0.39452$, implying that the data was positively correlated. However, we found that the sample version of Kendall's Tau was, $\tau_s = 0.271387$, implying that the data may be less positively correlated than that suggested by Pearson's Correlation. Note here that we do not expect Pearson's correlation coefficient to change much as X and Y are both gaussian.

From Figure 6 and Figure 7, an analysis of the plot of the "observed" joint probability density function with "uniformized" marginals, $\hat{h}(x, y)$ revealed the presence of a large concentration of pairs of observations in the tails and the center of the joint density function. Notice that the "observed" joint probability density function here is also the "observed" copula density function with "uniformized" marginal distributions.

The presence of these peaks at the tails suggest that the return per unit investment on the Sterling Pound and Japanese Yen move together in extremities. Also, the presence of a peak in the center of the joint density function of $h(x, y)$ suggests that when there is little or no price movement in one exchange market, that there is little or no price movement in the other market. We should note that the period under observation is 1971-2006.

The "theoretical" Gaussian copula density function with "uniformized" marginals as illustrated in Figure 8 and Figure 9 was able to successfully capture the tails of the observed joint density function over the unit square, $[0, 1]^2$. However, it failed to capture the peak that was present in the "observed" copula density function with "uniformized" marginals.

In fact, for all $U, V \in [0.05, 0.95]$ the Least Squares residual error between the "observed" and "theoretical" Gaussian copula density functions with "uniformized" marginals was,

$$\sqrt{|(\hat{C}(U, V))^2 - (C(U, V))^2|} = 6.22$$

This high residual error was be attributed to the "theoretical" model's inability to capture the 'sharp' peak that was present in center of the "observed" density plot. Further, we observed that the reason for this 'sharp' peak in the "observed" plot, was the high frequency of days during the years 1970-1975, where the daily return per unit investment in both the Sterling Pound and Japanese Yen markets was approximately 0.

However, it is more intuitive to view the Gaussian copula density function over $\mathbb{R}$, by transforming U and V to $\Phi^{-1}(U)$ and $\Phi^{-1}(V)$. This transformation allows us to express the Gaussian copula density function in terms of the "gaussianized" marginal distributions. From Figure 7, an analysis of the plot of the "observed" joint probability density function with "gaussianized" marginals, revealed a sharp peak in the center of the "observed" density function. The center of the "observed" joint density function, which was also the mode, corresponded to approximately 0 return per unit investment in both the Sterling Pound and Japanese Yen exchange markets. This suggested that daily extreme price movements in the US Dollar-Sterling Pound and US Dollar-Japanese Yen exchange markets are rare.

The "theoretical" Gaussian copula density function with "gaussianized" marginals as illustrated in Figure 10 and Figure 11 was able to successfully capture the peak that was present in the center of the "observed" copula density function with "uniformized" marginals. However, observe that the "theoretical" Gaussian copula with "gaussianized" marginals does not possess the 'sharp' peak that is present in the "observed" Gaussian copula with "gaussianized" marginals.

In fact, for all $\tilde{X}, \tilde{Y} \in [-3.323, 3.323]$ the Least Squares residual error between the "observed" and "theoretical" Gaussian copula density functions with "gaussianized" marginals was,

$$\sqrt{|(\hat{C}(\tilde{X}, \tilde{Y}))^2 - (C(\tilde{X}, \tilde{Y}))^2|} = 6.53$$

This high residual error can be attributed to the "theoretical" model's inability to capture the 'sharp' peak that was present in the center of the "observed" density plot.

It should be noted here that the above analysis can be extended to other foreign exchange markets as well, through use of the Gaussian copula.

4.3 Results

We conclude our analysis by summarizing the above results.

1. In foreign exchange markets, significant price movement on a daily basis is rare.

2. As observed by Hu and Cholette, when price movements are extreme in one currency market, there is a similar movement observed in another currency market, if the two currencies are positively correlated. Our analysis corroborates the

results obtained by Hu and Cholette for the daily return per unit investment on the Sterling Pound and Japanese Yen respectively.

3. In general, a copula can be used as a reasonable model to construct the underlying joint probability density function between the returns of any two currencies, as it is invariant under monotone transformations, and since it captures the dependency structure of the joint probability density function better than Pearson's correlation coefficient does.

4. Though the Gaussian copula captures the dependency structure of the price movements of the Sterling Pound and the Japanese Yen at the extremities, it overestimates this dependency at the extremes.

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Vita

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