Exploring Rational Numbers in Middle School

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EXPLORING RATIONAL NUMBERS IN MIDDLE SCHOOL

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ABSTRACT

The move by the state of Louisiana to fully implement the Common Core State Standards (CCSS) from 2013 -2014 school year on and to align all state mandated tests to the CCSS has caused teachers to change the way they teach and how they deliver content. The overall most crucial new part of the CCSS in Mathematics is the emphasis on the "Standards for Mathematical Practice". In order to illustrate the meaning of the Mathematical Practice Standards, non routine problems must be used that allow students and teachers to "dig deeper" and practice their mathematical habits of mind. Rational numbers provide an almost ideal playground to practice the standards.

In my thesis I will define rational numbers and discuss their representation as decimals both terminating and repeating. I will look briefly at the history of rational numbers, what role the Egyptians played in the history and how they used unit fractions. I will also look at two exploratory problems that can be discussed in a middle school mathematics class in order to illustrate the "Mathematical Standard Practices" as required by the CCSS. In the first exploration students will investigate properties of Egyptian unit fractions. The second exploration will focus on investing the periods of the decimal representation of rational numbers and their connection to the distribution of the prime numbers.

INTRODUCUTION

The move by the State of Louisiana to fully implement the Common Core State Standards (CCSS) from the 2013-2014 school year on and to align all state mandated tests to the CCSS has sent many school districts and educators scrambling to find resources to implement the new standards. School districts are relying heavily on publishing companies for supposedly common core aligned materials. Due to the rush to get material into classrooms, many districts fell pressured to purchase materials which may not have been adequately researched or reviewed by school personnel. Still other districts have decided not to re-invent the wheel and are using what New York and Great Minds Inc. (in collaboration with LSU faculty and staff serving as project leaders) are providing through their open access, web-based Engage New York Math Curriculum and its commercial twin sister curriculum Eureka Math.

“Districts Using Eureka Math In Fall 2014”.

This move by the state puts significant demands and stress on teachers. Many teachers across school districts express concerns about high demands being placed on them without adequate training. Many school districts have heard their pleas and are offering more opportunities for professional development and teacher grade level collaboration. In some systems teachers are getting together to plan lessons and assessments that are being used across the districts. However, the question remains if the teachers are receiving high quality professional development that is useful in helping them to prepare to teach the new standards and create assessments that are rigorous enough for the new standards.

Although the state of Louisiana has a definite timeline on how to move forward with the full implementation of the CCSS, many people and organizations have voiced opposition to the proposed changes. Among all the disagreement, there is at least one common ground: everyone seems to be concerned about the lack of training and support provided to teachers, parents, and school administration. According to a National Education Association poll (Bidwell, 2013), the majority of its members support the standards or support them with some reservation. Their biggest concern is that teachers are not being properly trained to implement the standards. For example, the Louisiana Association of Educators believes that the CCSS can give students the opportunity to experience a challenging curriculum which would prepare them to compete globally, but they have grave concern over how the state is moving forward with its implementation of the standards and the lack of educators' input in the decision on how to implement the standards properly.

There is also growing concern that parents have not been informed as to what the new curriculum entails. The breakdown in communication between districts, schools, and parents has many parents feeling helpless in being able to assist their children in navigating the new mathematics standards and practices. The new standards and their emphasis on “conceptual understanding” in addition to the more traditional “computational fluency” requirement have caused many parents to panic and to seek the assistance of a tutor for students in first grade through high school. The uncertainty about the new curriculum has left many teachers puzzled about the changes that are taking place in education and how they can better serve their students.

With the ascent of the CCSS, significant changes are taking place in the teaching profession. Teachers for many years have been handed textbook curriculums with very little depth and simply told “teach”. Many have managed to be successful with teaching the subject matter on the surface. The changes that have come about with the new standards require teachers to teach fewer topics. However, more in-depth teaching of the content is required, with a much greater emphasis on problem solving skills and conceptual understanding. This has created a great dilemma for most middle school teachers. Many teachers lack the mathematical sophistication and experiences needed to explain the how’s and why’s of mathematics. The argument for quite some time has been that universities and educational programs do not adequately prepare teachers to teach mathematics. As a middle school teacher I can attest to the discomfort of having to explain the how’s and why’s of mathematics and having to instruct students how to solve problems that I cannot solve myself. In making a commitment to take on writing a

thesis on “Rational Numbers”, I knew that I was stepping into an area which could be uncomfortable but would provide me with great joy on completion.

As we well know, curriculum changes will come and go. However, one thing remains certain: rational numbers will always be an integral part of any mathematics curriculum and they provide an almost ideal playground to practice the type of mathematical activities, processes, and habits of mind that are emphasized in the “Standards for Mathematical Practice,” maybe the overall most crucial new part of the CCSS in Mathematics.

MATHEMATICAL PRACTICE STANDARDS	
MP 1 .	Make sense of problems and persevere in solving them
MP 2 .	Reason abstractly and quantitatively
MP 3.	Construct viable argument and critique the reasoning of others
MP 4.	Model with mathematics
MP 5 .	Use appropriate tools strategically
MP 6.	Attend to precision
MP 7.	Look for and make use of structure
MP 8.	Look for and express regularity in repeated reasoning

It is widely accepted that in order to illustrate the meaning of the Mathematical Practice Standards, non-routine problems must be used that allow students, teachers,

and parents to "dig deep" and practice their mathematical habits of mind. Unfortunately, this is easier said than done.

For the author, the two explorations presented in Chapter 2 demonstrated vividly what the "Standards for Mathematical Practice" mean in practice. Before writing this thesis on "Rational Numbers", the author had no true image what is meant by MP1: "Make sense of problems and preserve in solving them." During the work for this thesis it became abundantly clear how difficult it is to "make sense of problems," how difficult it is to "persevere", and how impossible it is to "persevere in solving them." At least when it comes to "Rational Numbers," it appears that one is never done in solving any problems. One problem seems to lead to the next, a never ending process with innocent beginnings leading to a never ending chain of new problems.

This thesis consists of two parts. In Chapter 1, I will look at how various authors define rational numbers and try to determine which definition can give a middle school student a clear understanding of what a rational number is and what it is not. The geometric construction of rational numbers will be discussed. This entails giving an explanation of how unit lengths can be divided into equal segments. I will look at how and where rational numbers are addressed in the CCSS in an attempt to show how the pieces of the puzzle fit together. Elements of the history of rational numbers will be discussed briefly, what role the Egyptians played in the history, and how they used unit fractions. Rational numbers and their representation as decimals both terminating and repeating will be addressed. Irrational numbers will be discussed and we will take a peek into the proof by contradiction that the square root of an integer that is not a

square number is irrational. Finally, the countablilty of rational numbers and the uncountablility of the irrational numbers will be discussed.

In Chapter 2 of this thesis I will look at two exploratory problems that can be discussed in a middle school mathematics class in order to illustrate the “Mathematical Standard Practices” as required by the CCSS. In Exploration 2.1, students will investigate properties of Egyptian unit fractions, including some open conjectures of Erdös and Straus and of Sierpinski. In Exploration 2.2, students will investigate the periods of the decimal representations of rational numbers and their connection to the distribution of the prime numbers. Both explorations are open-ended and in no way, shape, or form complete. As stated above, one problem seems to lead to the next, a never ending process with innocent beginning leading to a never ending chain of new problems.

CHAPTER 1. RATIONAL NUMBERS

In this chapter, I survey several definitions of rational numbers, give an example concerning their geometric construction, explain their position in the Common Core State Standards, and provide some facts about the role of unit fractions in Egyptian mathematics.

1.1 <u>The Definition of Rational Numbers</u>

In reviewing the literature, one finds several definitions for rational numbers. One of the underlying assumptions in all these definitions seems to be the implicit assumption that the student knows already what a real number is, namely, a point on the real number line.

Strangely enough, no one seems to be worried about the fact that a line is a beautiful construction of human imagination that has no equivalent in the real world. A line is an abstract construction of mind that has no width and no height – just length. Thus lines are, and always will be, invisible and therefore non-existent in a narrow sense of the word. Continuing down this path, worse than lines are points on lines. A point has mathematical dimension of zero: no width, no length, no height. Which makes one wonder: what is a point on a line, really? And, therefore, what is a real number, really?

The Australian artist and architect, Friedensreich Hundertwasser (1928 -2000) gives us his thoughts on lines in general, and the number line in particular which shows that not everyone is happy with using the number line as the guide for our definition of numbers.

“In 1953 I realized that the straight line leads to the downfall of mankind. But the straight line has become an absolute tyranny. The straight line is something cowardly drawn with a rule, without thought or feeling; it is a line which does not exist in nature. And that line is the rotten foundation of our doomed civilization. Even if there are certain places where it is recognized that this line is rapidly leading to perdition, its course continues to be plotted. The straight line is godless and immoral. The straight line is the only uncreative line, the only line which does not suit man as the image of God. The straight line is the forbidden fruit. The straight line is the curse of our civilization. Any design undertaken with the straight line will be stillborn. Today we are witnessing the triumph of rationalist know how and yet, at the same time, we find ourselves confronted with emptiness. An aesthetic void, desert of uniformity, criminal sterility, loss of creative power. Even creativity is prefabricated. We have become impotent. We are no longer able to create. That is our real illiteracy.”

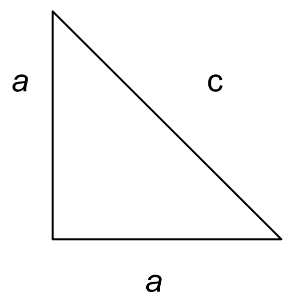
”— From “The Paradise Destroyed by the Straight Line,” Friedensreich Hundertwasser (1985)

But aside of a few skeptics like Hundertwasser (and my advisor), no one else is worried about taking the number line as the definition of real numbers. So, for us, a real number is a point on the real number line and rational numbers are real numbers with a specific property. The New South Wales Syllabus for the Australian Curriculum (“ Surds and Indices,” n.d, para.2) defines a rational number *as “*any number written as the ratio a/b of two integers a and b, where b ≠ 0*”*. In an article written by Ehrhard Behrends (2015) Freie Universitat Berlin rational numbers are defined as *“*the set of all quotients of the form m/n, where m is an integer and n is a natural number.*”* According to Paulos (1991/1992) “a rational number is one that may be expressed as a ratio of two whole numbers (as fractions are)”.

For the sixth grader, trying to understand the definition of rational numbers, this definition only gives a clear picture if the student fully understands what a ratio is and what is meant by the statement that a number P on the positive real number line (or equivalently the length of the line segment, starting at the origin and ending at P) can be better expressed as a ratio of two whole numbers.

Similar to Paulos, Professors William D. Clark and Sandra Luna McCune (2012) define a rational number as ”a number that can be expressed as a quotient of an integer divided by an integer other than 0. That is, the rational numbers are all the numbers that can be expressed as $\frac{p}{q}$, where p and q are integers, $q \neq 0$. Fractions, decimals, and percents are rational numbers”.

Unfortunately, unlike Paulos’ definition, the one of Clark and McCune is not entirely correct. That is, it is not at all clear why fractions, decimals or percents should always be rational numbers. For example, considering a right isosceles triangle



one can ask the following question. In percent, how much smaller is the segment *a* compared to the segment c? We believe the answer to be $\frac{1}{\sqrt{2}}$ * 100% or 70.71067812……….%.

Also, as is well known and as we will prove below, decimal representations of rational numbers will be terminating or repeating decimals, any other decimal representation would represent an irrational number. Therefore, despite the definition of Clark and McCune, not all decimals are rational numbers. Moreover, not all fractions are rational numbers, for example the fraction $\frac{1}{\sqrt{2}}$ is not a rational number. Despite

these short comings, Clark and McCune's definition of rational numbers improves the one by Paulos by explaining that the value of q cannot be equal to 0. This helps the sixth grader to remember that division by 0 is undefined, or at least requires some additional thought and the use of "infinity[1]."

Niven (1961) pays particular attention to the wording in his definition of rational numbers. He defines rational numbers as *"*a number which can be put in the form a/d, where a and d are integers, and d is not zero*.* He notes that he uses specifically "a number which could be put in the form …." and not "a number of the form a/d, where a and d are integers…." His reasoning is that a number can be expressed in many ways. There are many numbers (points on the real number line) that are written (represented, expressed) differently but have the same value (place) on the real number line.

However, when speaking of rational numbers (since we think of them as points on the number line) it is important that we look at what Jensen (2003) says about fractions and their relations to rational numbers.

"The fraction $\frac{\mathrm{p}}{\mathrm{q}}$ represents the point on the number line arrived at by dividing the unit interval into q equal parts and then going p of these parts to the right from 0. This point is called the value of the fraction. A rational number is the value of some fraction".

For our purposes we will define a rational number as follows.

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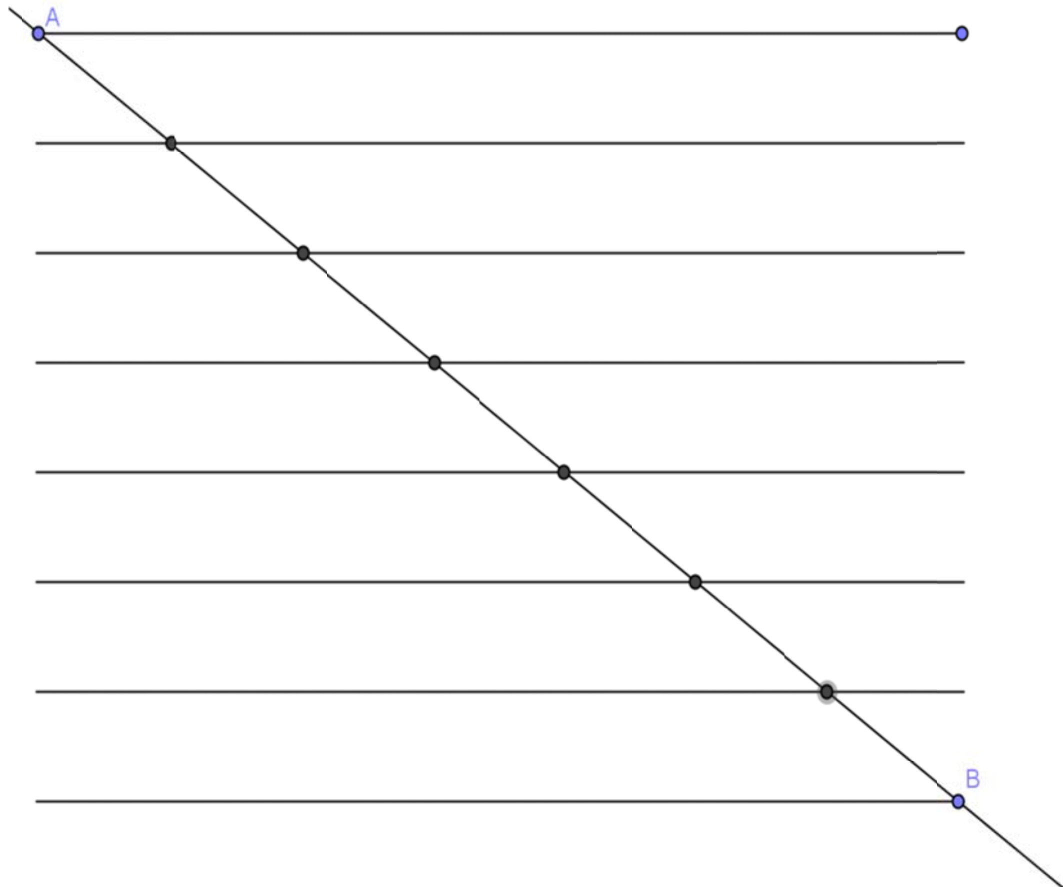
[1] When taking limits in calculus, one is confronted with problems like $\lim_{x\to 0}\frac{1}{x^2} = \frac{1}{+0} = +\infty$, where + 0 means that (in the sense of limits), the numerator is divided by smaller and smaller positive numbers resulting in larger and larger positive numbers. However, there are also situations where $\frac{1}{0}$ does not exist in any sense. For example, $\lim_{x\to 0}\frac{1}{x}$ approaches $+\infty$ or - ∞ pending if x approaches zero from the right or from the left. Therefore $\lim_{x\to 0}\frac{1}{x}$ is not a unique quantity.

Definition 1: A rational number is a point on the number line that can be written (represented, expressed) as a ratio (fraction) of a whole number p and a natural number q, where q cannot be 0. That is, a rational number is a point on the number line that can be arrived at by dividing the unit interval into q equal parts and then going p of these parts to the right from 0 if p is positive and $|p|$ of these parts to left of zero if p is negative.

1.2 The Geometric Construction of Rational Numbers

When using the definition of rational numbers given above, it is essential that students know how a unit (length) can be divided into q equal pieces. According to Jensen (2003) this basic task can be done by "laying the line segment on a grid of $n + 1$ equally spaced parallel lines". For example using a piece of string, if we wanted to divide the string into n equal parts (n not too large), we would lay the string on a lined sheet of loose leaf paper with equally spaced parallel lines. The student should place the string so that one end of the string lies on the 0^{th}-line and the other end lies on the nth- line. The student can then divide the unit (that is, the string) into n equal segments by marking the parts where the string crosses the lines. Have the student continue the process with different values of n and marking the segments with different colors. The student should be able to see that the points where the string intersects with the parallel lines divide it into equal segments. The student should also be able to identify that each segment is of equal length. A class discussion should be held to reaffirm the students understanding of how the segments can be added together.



1.3 Rational Numbers in the Common Core State Standards

In the Common Core State Standards, the term rational number is defined as "a number expressible in the form $\frac{a}{b}$ $or -\frac{a}{b}$ for some fraction $\frac{a}{b}$. The rational numbers include the integers". The term fraction appears for the first time in Grade 3. Students are expected to develop an understanding of fractions, mainly unit fractions (fractions with numerator 1). They learn parts and wholes using visual models and displays. Students should be able to determine that the denominator of a fraction tells how many parts the whole is divided into and they should learn to solve problems involving comparing fractions and equivalent fractions. They use visual models and fraction tiles to make comparisons and they understand that fractions are equivalent if they represent

the same point on a number line. In Grades 4 and 5 students build on what they have learned in 3rd grade. In Grade 4, they learn to add and subtract fractions with common denominators. They also learn how to multiply fractions by whole numbers and they build equivalent fractions by creating common denominators and comparing numerators. In Grade 5, students learn to add and subtract fractions with unlike denominator. Grade 5 is also where students extend learning multiplying and dividing fractions with whole numbers.

The term rational number appears for the first time in Grade 6 as one of the major areas on which instructional time should be focused. Although, students learn to divide fractions by fractions in Grade 6, we see a shift from the term fraction to the term rational number. The focus in Grade 6 is on how to use rational numbers in real-world problems and ordering rational numbers on the vertical and horizontal number lines. Students are introduced to integers and negative rational numbers.. By the end of Grade 6, students are able to successful apply all arithmetic operations to rational numbers.

In Grade 7, students extend their prior knowledge of rational numbers in solving more complex problems. They come to understand that fractions, decimals and some percents are different representations of rational numbers. They also continue to learn to perform arithmetic operations on rational numbers. In Grade 8, the term rational numbers is mentioned as a way to approximate irrational numbers. Therefore, students are expected to use what they learned about rational numbers in Grade 6 and Grade 7 to make approximations of irrational numbers. Students should also be able to locate irrational numbers on a number line. By Grade 8 students should be very confident in

what they know about rational numbers and are introduced to the concepts of real, imaginary, and complex numbers.

1.4 Rational Numbers in the Engage NY (Eureka) Mathematics Curriculum

The New York State Common Core mathematics curriculum defines a rational number as “a fraction or the opposite of a fraction on the number line”.

In the New York curriculum, the term fraction is introduced in Module 5 of Grade 3. They discuss fractions as numbers on the number line. The students learn what it means to take a whole and break it into equal parts. They learn what unit fractions are and their relationship to the whole. In Grade 4, they continue to build on what they learned in 3rd grade. They extend fraction comparison and fraction equivalency using multiplication and division. In Grade 5, the students start using fractions in solving word problems. They discuss line plots and interpretation of numerical expressions. Just as in the CCSS, rational numbers are mentioned for the first time in Grade 6. In Grade 6, Module 2 – Arithmetic Operations Including Division of Fractions, students complete the study of the four operations of positive rational numbers and start to learn how to locate and order negative rational number on a number line. In Grade 6, Module 3 – Rational Numbers, students use positive and negative numbers to describe quantities having opposite values, they understand absolute value of rational numbers and how to plot rational numbers on the number line. The coordinate plane is introduced in this module and students are able to graph points in all four quadrants.

The NYS curriculum also has a rational number module in Grade 7. In this module students build on their prior understanding of rational numbers to perform all

operations on signed numbers. In Grade 8, Module 7, students are introduced to irrational numbers using geometry. Students learn to further understand square roots, irrational numbers, and the Pythagorean Theorem.

1.5 Remarks on the Use of Unit Fractions in Egyptian Mathematics

When we look at rational numbers in middle school, it provides an opportunity to talk about the history of fractions (ratios of whole numbers). The Egyptians were one of the first civilizations to study fractions. They were writing fractions as early as 1800 B.C. The Egyptian number system was a base 10 system somewhat like the system we use today with the main difference being that they used pictures (called hieroglyphs) to represent their numbers. Egyptians wrote all fractions (except 2/3) as unit fractions or the sum of non-repeated unit fractions; that is, fractions with a numerator of one. Evidence of their ability to write fractions can be found in the Rhind Papyrus. It was purchased by Alexander Henry Rhind in Luxor, Egypt in 1858 and is housed in the British Museum in London. The papyrus was copied by the scribe Ahmes around 1650 B.C. and historians believe that the original papyrus on which the Rhind papyrus is based dates around 1850 B.C. The papyrus contains 87 mathematical problems of which 81 involve operations with fractions. Apparently, there was no multiplication or division in Egyptian mathematics. Egyptians used addition only. Multiplication was handled by repeated addition. Division was handled by doing the reverse of multiplication. The divisor is repeatedly doubled to give the dividend.

Example: Divide a number by 7. This is done by doubling 7 until the number is reached.

0	1
1	7
2	14
4	28
8	56
16	112
32	224

Observe that all multiples of 7 can be written as sums of the numbers 7, 14, 28, 56, 112, …

91 = 56 + 28 + 7 = 7 (8 + 4 + 1) = 7 x 13 or $\frac{91}{7}$ = 13

110 = 56 + 28 + 14 + 7 with remainder of 2, = 7 (8 + 4 +2+1) with remainder of 5. Thus $\frac{110}{7}$ = 15 with a remainder of 2 divided by 7, or $\frac{110}{7}$ = 15 $\frac{5}{7}$.

Due to the Egyptians use of only unrepeated unit fractions, they were limited in what they could do with fractions. Since they used only unit fractions (except $\frac{2}{3}$), all other fractions (except $\frac{2}{3}$) had to be written as the sum of unit fractions. Also, since they did not allow repeated use of unit fractions, when writing $\frac{6}{7}$, the Egyptians would represent $\frac{6}{7}$ as

$$\frac{6}{7}=\frac{1}{2}+\frac{1}{4}+\frac{1}{14}+\frac{1}{28} \text{ , and not } \frac{6}{7}=\frac{1}{7}+\frac{1}{7}+\frac{1}{7}+\frac{1}{7}+\frac{1}{7}+\frac{1}{7}$$

They believed that once one seventh of anything was used you could not use it again. Therefore, to write $\frac{6}{7}$ as $\frac{1}{7}+\frac{1}{7}+\frac{1}{7}+\frac{1}{7}+\frac{1}{7}+\frac{1}{7}$ would not be logical. Since the part $\frac{1}{7}$ exits only once, it cannot be used again after it is used once. To explain this in more

detail it is helpful to look at the following example of a division problem one finds in the Rhind Papyrus.

<u>Example</u>: Divide 6 loafs of bread among 7 people.

Egyptian Solution: If there were six loaves of bread that needed to be divided among seven people, each person would first receive $\frac{1}{2}$ of a loaf of bread. Once everyone received a $\frac{1}{2}$ of loaf of bread, the remaining $2\frac{1}{2}$ loaves would be divided as follows. The $\frac{1}{2}$ loaf would be divided into 7 pieces giving each person $\frac{1}{14}$ of a loaf. The remaining 2 loaves would be divided into fourths so that each person would receive $\frac{1}{4}$ of a loaf. The remaining fourth would be divided into 7 pieces and each person would receive $\frac{1}{28}$ of a loaf. Thus, $\frac{6}{7}=\frac{1}{2}+\frac{1}{14}+\frac{1}{4}+\frac{1}{28}$ ■

As it turns out Egyptian Fractions is a rich resource of interesting problems for middle school students to explore. This will be done in more detail in Section 2.1.

1.6 <u>Rational Numbers As Decimals</u>

In this section we will collect some basic facts concerning the decimal representation of rational numbers. First of all we will explain why rational numbers have a terminating or repeating decimal representation.

Observation 1.1: Let a, b be positive integers with $a < b.$ Then

$$\frac{a}{b} = 0.a_1a_2\ldots\ldots a_i\overline{a_{i+1}\ldots\ldots\ldots a_n} = 0.a_1a_2\ldots.a_ia_{i+1}\ \ldots.a_na_{i+1}\ldots..a_na_{i+1}\ldots\ldots.a_{n\ldots..},$$

where a_j are natural numbers (decimals) with $0 \le a_j \le 9$ and $0 \le i \le n < b$. In particular, the period of the repeating decimal representation of $\frac{a}{b}$ is less than $b.$

Proof: To convert a rational number $\frac{a}{b}$ to a decimal one performs long division. At each step in the division process one is left with a remainder of either 0,1,2, … , b - 1. Since we have no more than b remainders, at some point the remainder will start to repeat. Therefore, the quotient will start to repeat defining a period for the rational number. ■

Example: To find the decimal representation of $\frac{3}{4}$, we use long division and obtain

$$\begin{array}{r l}
 & \underline{0.75000\ldots} \\
4) & 3.000000\ldots \\
 & \underline{-\ 28} \\
 & \ \ 20 \\
 & \ \ \underline{-20} \\
 & \ \ \ \ 0
\end{array}.$$

Therefore, the decimal expansion of $\frac{3}{4}$ is 0.75000…. = 0.75 (terminating decimal).

Clearly, a terminating decimal can also be written as a repeating decimal by repeating zeros at the end of the number or by writing 0.75 = 0.74999… (See Observation 1.3) ■

Example: To find the decimal representation of $\frac{3}{7}$, we divide 3 by 7 using long division and obtain

$$
\begin{array}{l}
\quad\ 0\,.\,4\,2\,8\,5\,7\,1\,4\,2\ \ldots\ldots \\
7\,\overline{)3\,.\,0\,0\,0\,0\,0\ 0\,0\,0} \\
\quad\ \underline{2\,8} \\
\qquad 2\,0 \\
\quad \underline{-\,1\,4} \\
\qquad\ \ 6\,0 \\
\qquad \underline{-5\,6} \\
\qquad\quad 4\,0 \\
\qquad\ \underline{-\,3\,5} \\
\qquad\qquad 5\,0 \\
\qquad\quad \underline{-\ \ 4\,9} \\
\qquad\qquad\ \ 1\,0 \\
\qquad\qquad\ \underline{-7} \\
\qquad\qquad\quad 3\,0 \\
\qquad\qquad \underline{-\,2\,8} \\
\qquad\qquad\qquad 2\,0 \\
\qquad\qquad\quad \underline{-14} \\
\qquad\qquad\qquad\ \ 6
\end{array}
.
$$

Thus, the decimal representation of $\frac{3}{7}$ is $0.\overline{428571}$ = 0.428571428571428571…. where 428571 represent the repetend (repeating portion) of the decimal expansion. ■

The decimal representation of rational numbers will be studied further in Section 2.2. Here are a few amazing properties of the decimal representation. In the decimal representation of $\frac{3}{7} = 0.\overline{428571}$, observe that the first three numbers (428) and the last three numbers (571) add up to 999. Also observe that the first (4) and the fourth numbers (5), the second (2) and the fifth (7), and the third (8) and the sixth (1) all add

up to 9. This is not an accident as $\frac{1}{13} = 0.\overline{076923}$ shows. We return to this topic in Section 2.2.

Once students have been given to opportunity to perform long division and are familiar with what happens at each step in the process, it is counterproductive to continue to work manually on a process that can easily be handled by technology. When we speak of technology we are not talking about simply calculators. Most calculators do not have the capacity to handle more than 10-15 digits. Therefore, the decimal representation of fractions like

$$\frac{1}{23} = 0.\overline{0434782608695652173913} \quad \text{(period of 22)}$$

is beyond the scope of a simple scientific calculator. This makes it necessary to introduce students to more powerful computational tools that are readily available on the internet (e.g., Wolfram Alpha at www.wolframalpha.com/). These programs allow students to explore/compute decimal representations of rational numbers without having to do long division – an almost impossible task for a fraction like 1/113, where the period of the repeating decimal is 112 digits long – that is, the long division process repeats itself only after 112 divisions. As mentioned above, in Section 2.2, we will further discuss the periods of rational numbers and explore some of their remarkable properties.

<u>Observation 1.2:</u> A fraction $\frac{a}{b}$ (in lowest terms) has a terminating decimal representation if and only if

$$b = 2^n 5^m$$

for some integers $n, m \geq 0$.

Proof: We can assume without loss of generality that $a < b$. If $\frac{a}{b} = 0.a_1a_2 \ldots a_k$ has a terminating decimal representation, then

$$\frac{a}{b} = 0.a_1a_2 \ldots\ldots a_k = a_1 \bullet 10^{-1} + a_2 \bullet 10^{-2} + \ldots a_k \bullet 10^{-k}$$

$$= \frac{a_1 \,.\, 10^{k-1} + a_2 \,.\, 10^{k-2} \ldots + a_k}{10^k} = \frac{\tilde{a}}{10^k} = \frac{\tilde{a}}{2^k \bullet 5^k} \ ,$$

where $\tilde{\alpha} = a_1 \bullet 10^{k-1} + a_2 \bullet 10^{k-2} + \ldots + a_k$. Since $\frac{a}{b} = \frac{\tilde{a}}{2^k \bullet 5^k}$ and since $\frac{a}{b}$ is in lowest terms, it follows that there exists an integer c such that $\tilde{a} = a \bullet c$ and $2^k \bullet 5^k = b \bullet c$. This shows that $b = 2^n \bullet 5^m$ for some $0 \le n, m \le k$.

Reversely, if $\frac{a}{b} = \frac{a}{2^n \bullet 5^m}$, then

$$\frac{a}{b} = \frac{a \bullet 2^j 5^r}{2^k \bullet 5^k} = \frac{a 2^j 5^r}{10^k}$$

is a terminating decimal, where $j = \max(n,m) - n$, $r = \max(n,m) - m$, and $k = \max(n,m)$. ■

Example: To decide whether or not $\frac{21}{60}$ has a terminating decimal representation one observes first that $\frac{21}{60}$ is not yet in lowest terms. Dividing numerator and denominator by 3 yields $\frac{21}{60} = \frac{7}{20}$. Since $20 = 2^2 \bullet 5$ it follows from Observation 2 that $\frac{21}{60} = \frac{7}{20}$

has a terminating decimal representation, namely

$$\frac{21}{60}=\frac{7}{20}=\frac{7}{2\cdot 2\cdot 5}=\frac{7\cdot 5}{2\cdot 2\cdot 5\cdot 5}=\frac{35}{10^2}=0.35\,.$$ ■

<u>Example:</u> $$\frac{7}{625}=\frac{7}{5^4}=\frac{7\cdot 2^4}{2^4\cdot 5^4}=\frac{7\cdot 16}{10^4}=\frac{112}{10000}=0.0112$$ ■

<u>Example:</u> The fraction $\frac{6}{735}$ cannot have a terminating decimal representation since $735=3\cdot 5\cdot 7^2$ has prime factors other than 2's and 5's. ■

We end this section with two important remarks. The first one is about the fact that, the decimal representation of a rational number with a terminating decimal representation is not unique. This follows from the following observation.

<u>Observation 1.3:</u>	1 = 0.9999…….

<u>Proof 1:</u> Using long division one sees that $\frac{1+0.999....}{2}=\frac{1.999....}{2}$ = 0.999….. Therefore, $\frac{1+0.9999....}{2}=0.9999....$ or 1 + 0.9999… = 2 • 0.9999….. This shows that 1=0.999… ■

<u>Proof 2:</u> Let x = 0.9999…. Then 10x = 9.9999… , and therefore

10x - x = 9.9999... - 0.9999... = 9. This implies 9x = 9 or x = 1. This shows that

1 = 0.9999……. . ■

Proof 3: If 1 would be different from 0.9999….then there would be a number between

Them. However, there is no number such that 0.999…. $< a < 1.00000$ ■

Example: Since $\frac{1}{2} = 0.5 = 0.49 + 0.01$ = 0.49 + 0.00999….. = 0.499999……

it follows that the decimal representation of fractions is not unique. ■

The last observation in this section is useful when transforming repeating decimals into fractions.

> **Observation 1.4:** Let x = $0.\overline{a_1 a_2 \ldots a_n}$ be a repeating decimal. Then
>
> $$x = \frac{a}{10^n - 1},$$
>
> where $a = a_1 a_2 a_3 \ldots\ldots a_n = a_1\, 10^{n-1} + a_2\, 10^{n-2} + \cdots a_n.$

Proof: If $x = 0.a_1 a_2 a_3 \ldots a_n a_1 a_2 \ldots\ldots, a_n \ldots,$ then $10^n x = a_1 a_2 \ldots \ldots a_n.\overline{a_1 a_2 \ldots . a_n}$ and therefore, $10^n x - x = a_1 a_2 \ldots \ldots a_n.\overline{a_1 a_2 \ldots . a_n}$ - $0.\overline{a_1 a_2 \ldots . a_n}$ = $a_1 a_2 \ldots\ldots a_n$ = $a_1\, 10^{n-1} + a_2 10^{n-1} + a_2 10^{n-2} + \ldots \ldots a_n.$ Therefore, $x = \frac{a}{10^n - 1}$. ■

Example: $0.\overline{34}$ = 0.343434…………. $= \frac{34}{10^2 - 1} = \frac{34}{99}$ ■

Example: $0.12\overline{34} = \frac{12}{100} + 0.00\overline{34} = \frac{12}{100} + \frac{1}{100} \bullet 0.\overline{34} = \frac{12}{100} + \frac{1}{100} \bullet \frac{34}{99} = \frac{12 \bullet 99 + 34}{100 \bullet 99} = \frac{1222}{9900} = \frac{611}{4950}$. ■

1.7 Irrational Numbers/Countable/Uncountable

When defining an irrational number, we follow what many authors do by simply saying it is a number that is not rational, that is, a point on the number line whose distance to the origin cannot be expressed by a rational number. Havil (2012) makes it a point to say "It is a number which cannot be expressed as the ratio of two integers. It is a number the decimal expansion of which is neither finite nor recurring*"*. His statement clarifies what we already know about rational numbers. Namely, that they can be written as the ratio of two integers (a over b) where b is not 0 and that the decimal expansion of a rational number will be terminating or repeating as previously mentioned in Section 1.6. With that said, an irrational number has a decimal expansion that is infinite and non periodic (not recurring).

Observation 1.5: Let n be a positive integer. Then $\sqrt{n}$ is irrational if and only if n is not a perfect square.

Proof: We show that $\sqrt{n}$ is rational if and only if n is a perfect square of a positive integer. If $n = k^2$ for some positive integer k, *then the* $\sqrt{n} = k$ is a rational number. Conversely, if $\sqrt{n}$ is a rational number, then $\sqrt{n} = \frac{p}{q}$ (in lowest terms). But then

$$n = \sqrt{n} \bullet \sqrt{n} = \frac{p}{q} \bullet \sqrt{n} \text{ or } \sqrt{n} = \frac{n \bullet q}{p} \text{ or } \frac{p}{q} = \frac{n \bullet q}{p}.$$

Since $\frac{p}{q}$ is in lowest terms, there must be a positive integer c such that $p = c \bullet q$ and $nq = p \bullet c$ so that $\frac{p \bullet c}{q \bullet c} = \frac{nq}{p}$. But then $c = \frac{p}{q} = \sqrt{n}$ is an integer and therefore $n = c^2$ is a

perfect square. This shows that $\sqrt{n}$ is rational if and only if n is a perfect square. ■

Observation 1.6: Let n be an integer. Then $\sqrt[3]{n}$ is irrational if and only if n is not a perfect cube.

Proof: If $\sqrt[3]{n} = \frac{p}{q}$, then

$$n = \sqrt[3]{n} \bullet \sqrt[3]{n} \bullet \sqrt[3]{n} = (\sqrt[3]{n})^2 \bullet \sqrt[3]{n} = \frac{p^2}{q^2} \bullet \sqrt[3]{n}.$$

Thus, $\frac{p}{q} = \sqrt[3]{n} = \frac{n \bullet q^2}{p^2}$. Since $\frac{p}{q}$ is in lowest terms, there exists an integer c such that

$p \bullet c = nq^2$ and $q \bullet c = p^2$. Since $c = \frac{p}{q} \bullet p$ can only be an integer if $\frac{p}{q}$ is an integer, it follows that $n = (\frac{p}{q})^3$ is a perfect cube. ■

In addition to roots, numbers like

$$\Pi = 3.14159265358979323846264338327950288419716939937 3\ldots.$$

and

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = 2.71828182845904523536028747135266249775724709\ldots.$$

are known to be irrational. However, the proofs of these statements are too difficult to be included here. Although roots are often irrational, they have the nice property that they are solutions of algebraic equations. For example, $\sqrt{3}$ is the positive solution of the

equation $x^2 = 3$. This allows one to compute the irrational number $\sqrt{3}$ to any degree of precision using the following method (intermediate value theorem of calculus).

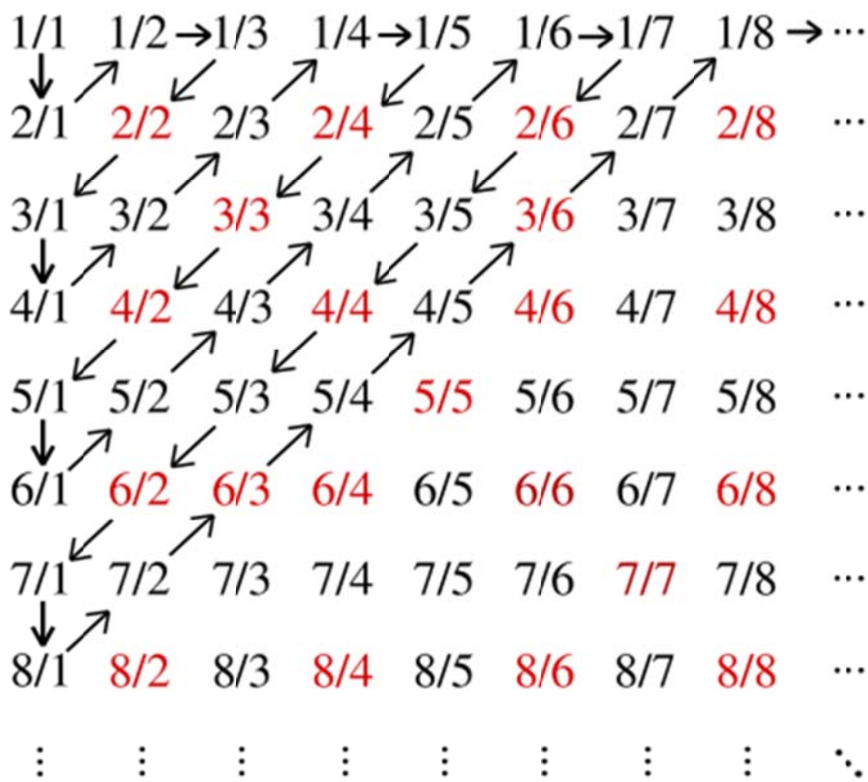
<u>Example:</u> First of all, we know that $x = \sqrt{3}$ must be between 1 and 2 since $1^2 \leq x^2 \leq 2^2$. Now $1.7 \leq x \leq 1.8$ since $(1.7)^2 = 2.89 \leq x^2 \leq (1.8)^2 = 3.24$, and then $1.73 \leq x \leq 1.74$ since $(1.73)^2 = 2.9429 \leq x^2 \leq (1.74)^2 = 3.0276$. Continuing in this fashion we find that $x = \sqrt{3}$ satisfies $1.73205 \leq x \leq 1.732051$ or $x \approx 1.73205$ since $(1.732050)^2 = 2.999997202500 \leq x^2 \leq (1.732051)^2 = 3.000000666601$. ■

One of the astonishing facts about rational numbers is the following.

<u>Observation 1.7:</u> The rational numbers are countable; that is, there are as many rational numbers as there are positive integers.

<u>Proof:</u> The rational numbers are countable since we can count them all. That is, there is a one to one function between all positive rational numbers and all positive integers.

Here is how:



Simply following the arrows we can count all of the rational numbers. All of the fractions will be listed. ■

Observation 1.8:	The real numbers are uncountable.

Proof: We will go by contradiction. Suppose that the real numbers between 0 and 1 would be countable. Then, we could make the following list of real numbers that contains all the real numbers between 0 and 1.

$$1 \longmapsto 0.a_{1,1}a_{1,2}a_{1,3}a_{1,4}a_{1,5}\ldots$$

$$2 \longmapsto 0.a_{2,1}a_{2,2}a_{2,3}a_{2,4}a_{2,5}\ldots$$

$$3 \longmapsto 0.a_{3,1}a_{3,2}a_{3,3}a_{3,4}\alpha_{3,5}\ldots$$

$4 \mapsto 0.a_{4,1}a_{4,2}a_{4,3}a_{4,4}a_{4,5}\ldots$ and so on.

Now we show that that it is impossible that this list contains all the real numbers between 0 and 1. We do this by constructing a number $b = 0.b_1b_2b_3\ldots\ldots$ that is definitely not in the list. We take b_n to be 1 if a_{nn} is not 1 and 2 otherwise. The b differs from the nth number in the list in the nthdigit. ■

Observation 1.9: The irrational numbers are uncountable.

Proof: Since the real numbers $\mathbb{R}$ are not countable and since $\mathbb{R}$ is the disjoint union of the rationals and the irrationals, the irrationals cannot be countable since the union of two countable sets is again countable. ■

CHAPTER 2. EXPLORATIONS

In this chapter, I will look at two exploratory problems that can be discussed in a middle school mathematics class. In these problems the students should be able to demonstrate the “Mathematical Standard Practices” as required by the CCSS.

2.1 Writing Fractions the Egyptian Way

As was already mentioned in the history section of rational numbers, the Egyptian used to represent all rational numbers (except 2/3) as sums of non-repeating unit fractions. This was a reasonable way to divide property (or loaves of bread) among people. For example when dividing 8 loaves of bread among 11 people, using Egyptian fractions one would start by first giving everyone $\frac{1}{2}$ loaf of bread, leaving behind 2 $\frac{1}{2}$ loaves of bread. Then the $\frac{1}{2}$ loaf of bread, would be divided among 11 people, giving each person $\frac{1}{22}$ loaf of bread. Now the two loaves would be divided in sixth, so that everyone would get $\frac{1}{6}$ of a loaf of bread, leaving $\frac{1}{6}$ of a loaf of bread. Finally, the remaining $\frac{1}{6}$ loaf would be divided among the 11 people, giving each one $\frac{1}{66}$ of a loaf of bread. Therefore,

$$\frac{8}{11} = \frac{1}{2} + \frac{1}{22} + \frac{1}{6} + \frac{1}{66},$$

or 8 loaves $= \frac{1}{2}$ loaf $\bullet$ 11 + $\frac{1}{22}$loaf $\bullet$ 11 + $\frac{1}{6}$ loaf $\bullet$ 11 + $\frac{1}{66}$ loaf $\bullet$ 11.

The teacher can then ask the students to use the Egyptian Method to write $\frac{7}{12}$ and $\frac{11}{12}$ as a sum of unit fractions. Using the same method, the students would find that

$$\frac{7}{12}=\frac{1}{2}+\frac{1}{12} \quad \text{and} \quad \frac{11}{12}=\frac{1}{2}+\frac{1}{3}+\frac{1}{12}\ .$$

The Egyptian "bread-dividing" method of representing fractions $\frac{a}{b}$ as the sum of non-repeating unit fractions is similar to the following method where they write a as the sum of factors of b. For example when writing $\frac{7}{12}$ as the sum of units factors, we know that the factors of 12 are 1, 2, 3, 4, 6, 12 and that the following is also true 7 = 1 + 6, or 7 = 3 + 4, notice that the numbers used to get a sum of 7 are indeed factors of 12.

Thus,

$$\frac{7}{12}=\frac{1+6}{12}=\frac{1}{12}+\frac{1}{2} \quad \text{and} \quad \frac{11}{12}=\frac{1+4+6}{12}=\frac{1}{12}+\frac{1}{3}+\frac{1}{2}\,.$$

However, this method does not always work. In some cases one needs a trick to make it work. For example, when finding the unit fraction decomposition of $\frac{8}{11}$, since the only factors of 11 are 1 and 11 then 8 cannot be written as the sum of the factors of 11. Similarly, the representations

$$\frac{8}{11}=\frac{16}{22}=\frac{16}{1 \cdot 2 \cdot 11} \quad \text{and} \quad \frac{8}{11}=\frac{24}{33}=\frac{24}{1 \cdot 3 \cdot 11}$$

do not work, but the representation $\frac{8}{11}=\frac{48}{66}$ works since the factors of 66 are 1, 2, 3, 11, 22, 33, 66. Therefore, 48 = 33 + 11 + 3 + 1. Thus,

$$\frac{48}{66}=\frac{33+11+3+1}{66}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{66}\ ,$$

which is the Egyptian Method of dividing 8 loafs of bread among 11 people. The Egyptian way of dividing a loaves of bread among b people works if at each step the remaining loaves of bread are of the form $k + \frac{1}{n}$. However, this is not always the case. For example, what if we have to divide 7 loaves among 131 people? Then by the above, we divide each loaf into 19 parts and divide them equally among the 131 people, leaving us with $7 - \frac{131}{19} = \frac{2}{19}$ loaf of bread. But, how can we divide $\frac{2}{19}$ loaf of bread among 131 people using only unit fractions? Similarly, if we divide 8 loaves among 131 people, we divide each loaf into 17 parts, leaving us with $8 - \frac{131}{7} = \frac{5}{17}$ loaf of bread. But how can we divide $\frac{5}{17}$ loaf of bread among 131 people? This takes us to the problem of how to write numbers of the form

$$\frac{2}{n}, \frac{3}{n}, \frac{4}{n}, \ldots\ldots\ldots\ldots \frac{n-1}{n}$$

as the sum of unit fractions. This is a good activity for students because they do not perform a standard algorithm, but they have to truly explore and learn how to dig deeper into a mathematical problem. As we will see, they tryly have to take first steps towards what is called in the Common Core State Standards the "Mathematical Practices" that should be mastered by every student to be ready for college and careers.

Step 1: Students explore how to write $\frac{2}{n}$ as the sum of unit fractions. The students should observe that if n is even ($n = 2k$), then $\frac{2}{n} = \frac{2}{2k} = \frac{1}{k}$ is already a unit fraction.

If n is odd, ($n = 2k - 1$) then $\frac{2}{n} = \frac{2}{2k-1}$ and the students should do a few examples as follows.

Table 2.1

k	$\frac{2}{2k-1}$
2	$\frac{2}{3} = \frac{1}{2} + \frac{1}{6} = \frac{1}{2} + \frac{1}{2 \bullet 3}$
3	$\frac{2}{5} = \frac{1}{3} + \frac{1}{15} = \frac{1}{3} + \frac{1}{3 \bullet 5}$
4	$\frac{2}{7} = \frac{1}{4} + \frac{1}{28} = \frac{1}{4} + \frac{1}{4 \bullet 7}$
5	$\frac{2}{9} = \frac{1}{5} + \frac{1}{45} = \frac{1}{5} + \frac{1}{5 \bullet 9}$
6	$\frac{2}{11} = \frac{1}{6} + \frac{1}{66} = \frac{1}{6} + \frac{1}{6 \bullet 11}$

Now, after some in class discussion, the students should be able to first observe that

$$\frac{2}{2k-1} = \frac{1}{k} + \frac{1}{k(2k-1)}$$

and then prove this observation using basic algebra. Namely,

$$\frac{1}{k}+\frac{1}{k(2k-1)}=\frac{2k-1+1}{k(2k-1)}=\frac{2k}{k(2k-1)}=\frac{2}{2k-1}.$$

■

This leads to the following observation.

Observation 2.1: All rational number of the form $\frac{2}{n}$ can be written as the sum of at most 2 unit fractions.

Step 2: Next have students explore fractions of the form $\frac{3}{n}$. If n is a multiple of 3 then $\frac{3}{n}$ is already a unit fraction. Now have the students observe the following.

Table 2.2

n	$\frac{3}{n}$
4	$\frac{3}{4}=\frac{1}{2}+\frac{1}{4}$
5	$\frac{3}{5}=\frac{1}{2}+\frac{1}{10}$
6	$\frac{1}{2}$
7	$\frac{3}{7}=\frac{1}{7}+\frac{2}{7}=\frac{1}{7}+\frac{1}{4}+\frac{1}{28}$ or $\frac{3}{7}=\frac{1}{3}+\frac{2}{21}=\frac{1}{3}+\frac{1}{11}+\frac{1}{11\cdot 21}$

	(by Observaton2.1)
8	$\frac{3}{8}=\frac{1}{4}+\frac{1}{8}$
9	$\frac{3}{9}=\frac{1}{3}$
10	$\frac{3}{10}=\frac{1}{5}+\frac{1}{10}$

In summary, the student should be able to formulate the following observation.

Observation 2.2: All numbers of the form $\frac{3}{n}$ can be written as the sum of at most 3 unit fractions.

Proof: This proof is divided into the three cases n = $3k$, $n = 3k - 1, n = 3k - 2$

for some integer k ≥ 1.

Case 1: If $n = 3k$, then $\frac{3}{n}=\frac{3}{3n}=\frac{1}{k}$ is a unit fraction when written in simplified form.

Case 2: If n = $3k$ – 1 then $\frac{3}{n}=\frac{3}{3k-1}=\frac{3k-1+1}{k(3k-1)}=\frac{1}{k}+\frac{1}{k(3k-1)}$ n is a sum of two distinct

unit fraction.

Case 3: If $n = 3k - 2$ then $\frac{3}{n} = \frac{3}{3k-2} = \frac{3k-2+2}{k(3k-2)} = \frac{1}{k} + \frac{2}{k(3k-2)}$. However, by Observation 2.1, we know that $\frac{2}{k(3k-2)}$, can be written as the sum of at most two unit fractions. Thus, $\frac{3}{3k-2}$ can be written as the sum of at most 3 unit fractions.

Step 3: Now let the students explore rational numbers of the form $\frac{4}{n}$.

Table 2.3

n	$\frac{4}{n}$
5	$\frac{4}{5} = \frac{1}{2} + \frac{1}{4} + \frac{1}{20}$
6	$\frac{4}{6} = \frac{2}{3} = \frac{1}{2} + \frac{1}{6}$
7	$\frac{4}{7} = \frac{1}{2} + \frac{1}{14}$
8	$\frac{4}{8} = \frac{1}{2}$
9	$\frac{4}{9} = \frac{1}{3} + \frac{1}{9}$
10	$\frac{4}{10} = \frac{1}{3} + \frac{1}{15}$
11	$\frac{4}{11} = \frac{1}{3} + \frac{1}{33}$
12	$\frac{4}{12} = \frac{1}{3}, \frac{4}{12} = \frac{1}{4} + \frac{1}{20} + \frac{1}{52}$
13	$\frac{4}{13} = \frac{1}{4} + \frac{3}{52}, \frac{3}{52} = \frac{2}{52} + \frac{1}{52} = \frac{1}{26} + \frac{1}{52}$

This leads to the following conjecture.

<u>Conjecture 1:</u> Numbers of the form $\frac{4}{n}$, can be written as the sum of at most 3 unit fractions.

This is a famous conjecture of Paul Erdös and Ernst G. Straus given around 1948 (Knott, 2015). It is believed that this statement is true for every rational number of the form $\frac{4}{n}$ where $n \geq 2$. As of today computer searches have verified the truth of the conjecture for n up to 10^{14}. However, it is still unknown if the conjecture holds for all values of n. In contrast to Conjecture 1, the following statement can be proved easily.

<u>Observation 2.3:</u> Rational numbers of the form $\frac{4}{n}$ can be written as the sum of at most 4 unit fractions.

<u>Proof:</u> We distinguish between the four cases $n = 4k$, $4k$-1, $4k$-2 and $4k$-3.

Case 1: If $n = 4k$, then $\frac{4}{n} = \frac{1}{k}$.

Case 2: If $n = 4k - 1$, then $\frac{4}{n} = \frac{4}{4k-1} = \frac{4k-1+1}{k(4k-1)} = \frac{1}{k} + \frac{1}{k(4k-1)}$.

Case 3: If n = 4k-2, then $\frac{4}{n}=\frac{4}{4k-2}=\frac{4k-2+2}{k(4k-2)}=\frac{1}{k}+\frac{2}{k(4k-2)}$ and, by Observation 2.1, $\frac{2}{k(4k-2)}$ can be written as the sum of at most 2 unit fractions.

Case 4: If $n=4k-3$, then $\frac{4}{n}=\frac{4}{4k-3}=\frac{4k-3+3}{k(4k-3)}=\frac{1}{k}+\frac{3}{k(4k-3)}$ and, by Observation 2.2, $\frac{3}{k(4k-3)}$ can be written as the sum of at most 3 unit fractions.

■

By now it should be rather obvious how this proof can be extended to $\frac{5}{n},\frac{6}{n},\frac{7}{n},\frac{8}{n}$, and so forth.

<u>Observation 2.4:</u> Numbers of the form $\frac{a}{n}$, can be written as the sum of at most a unit fractions.

Waclaw Sierpinski conjectured that numbers of the form $\frac{5}{n}$ can be written as the sum of at most 3 unit fractions (Knott, 2015). Like the conjecture of Erdös and Straus no one knows for certain if this is true for all rational numbers n.

So far, we have seen that all fractions of the form $\frac{a}{n}$ can be written in the form

$$\frac{a}{n}=\frac{1}{n_1}+\frac{1}{n_2}+\ \dots\dots+\frac{1}{n_k},$$

where $n_i \neq n_j$ for $i \neq j$ and $1 \leq k \leq a$.

However, the method used is not easily represented as an “algorithm”. In the 12th century, Leonardo of Pisa (also known as Fibonacci) popularized what is known today as the “Greedy Algorithm”.

> **<u>Greedy Algorithm</u>**: Assume that $0< \frac{a}{b} < 1$ and that $\frac{a}{b}$ is in simplest form.
> <u>Step 1:</u> Find the largest unit fraction less than $\frac{a}{b}$.
> <u>Step 2:</u> Subtract this unit fraction from $\frac{a}{b}$ and obtain $\frac{c}{d}$,
> Now repeat Step 1 and Step 2 until it results in a unit fraction.

<u>Example:</u> Let us see how the Greedy Algorithm works for $\frac{7}{17}$. The largest unit fraction less than $\frac{7}{17}$ is $\frac{1}{3}$. This yields

$$\frac{7}{17} = \frac{1}{3} + \frac{c}{d}$$

where $\frac{c}{d} = \frac{7}{17} - \frac{1}{3} = \frac{21-17}{3 \cdot 17} = \frac{4}{51}$. . Thus,

$$\frac{7}{17} = \frac{1}{3} + \frac{4}{51}.$$

The largest unit fraction less than $\frac{4}{51}$ is $\frac{1}{13}$. This yields

$$\frac{4}{51} = \frac{1}{13} + \frac{c}{d},$$

where $\frac{c}{d} = \frac{4}{51} - \frac{1}{13} = \frac{52-51}{13 \cdot 51} = \frac{1}{13 \cdot 51} = \frac{1}{663}$.

Therefore,

$$\frac{7}{17} = \frac{1}{3} + \frac{4}{51} = \frac{1}{3} + \frac{1}{13} + \frac{1}{663}. \qquad \blacksquare$$

It was shown by J.J. Sylvester (1814 – 1897) that the greedy algorithm always works in at most a steps. He argued as follows.

Proof of the Greedy Algorithm:

Assume that $0 < \frac{a}{b} < 1$ and that $\frac{a}{b}$ is in simplest form. By the division algorithm,

$$b = a \bullet q + r, or\ a \bullet q = b - r$$

where $q \geq 1$ and $1 \leq r < a$. Then

$$\frac{a}{b} = \frac{a(q+1)}{b(q+1)} = \frac{aq+a}{b(q+1)} = \frac{b-r+a}{b(q+1)} = \frac{1}{q+1} + \frac{a-r}{b(q+1)} = \frac{1}{q+1} + \frac{c}{d}$$

where $c < a$ and $\frac{1}{q+1}$ is the largest unit fraction less than $\frac{a}{b}$. Since $c < a$, this process will terminate in an Egyptian representation in at most a steps. most a steps.

$\blacksquare$

Clearly, as already explored above in many cases the Greedy Algorithm will terminate in less than a steps. A more through investigation of the actual number of steps the Greedy Algorithm takes would be well worth a follow-up project that could be done by interested students. For example, as we have seen above, the number $\frac{7}{12}$ can be written as the sum of the three unit fractions $\frac{1}{3}, \frac{1}{13},$ and $\frac{1}{663}$. It would be interesting

to see how many of the numbers $\frac{7}{n}$ $(8 \leq n \leq N)$ can be written as the sum of 1, 2, 3, 4, 5, 6, or 7 unit fractions.

Table 2.4

$\frac{7}{8} = \frac{1}{2}+\frac{1}{3}+\frac{1}{24}$	$\frac{7}{9}=\frac{1}{2}+\frac{1}{4}+\frac{1}{36}$	$\frac{7}{10}=\frac{1}{2}+\frac{1}{5}$	$\frac{7}{11}=\frac{1}{2}+\frac{1}{8}+\frac{1}{88}$
$\frac{7}{12}=\frac{1}{2}+\frac{1}{12}$	$\frac{7}{13}=\frac{1}{2}+\frac{1}{26}$	$\frac{7}{14}=\frac{1}{2}$	$\frac{7}{15}=\frac{1}{3}+\frac{1}{8}+\frac{1}{120}$
$\frac{7}{16} = \frac{1}{3}+\frac{1}{10} + \frac{1}{240}$	$\frac{7}{17}=\frac{1}{3}+\frac{1}{13}+\frac{1}{663}$	$\frac{7}{18}=\frac{1}{3}+\frac{1}{18}$	$\frac{7}{19}=\frac{1}{3}+\frac{1}{29}+\frac{1}{1653}$
$\frac{7}{20}=\frac{1}{3}+\frac{1}{60}$	$\frac{7}{21}=\frac{1}{3}$	$\frac{7}{22} = \frac{1}{4}+\frac{1}{15}+\frac{1}{660}$	$\frac{7}{23}=\frac{1}{4}+\frac{1}{19}+\frac{1}{583} + \frac{1}{1019084}$
$\frac{7}{24}=\frac{1}{4}+\frac{1}{24}$	$\frac{7}{25}=\frac{1}{4}+\frac{1}{34}+\frac{1}{1700}$	$\frac{7}{26}=\frac{1}{4}+\frac{1}{52}$	$\frac{7}{27}=\frac{1}{4}+\frac{1}{108}$
$\frac{7}{28}=\frac{1}{4}$	$\frac{7}{29}=\frac{1}{5}+\frac{1}{25}+\frac{1}{725}$	$\frac{7}{30}=\frac{1}{5}+\frac{1}{30}$	$\frac{7}{31}=\frac{1}{5}+\frac{1}{39}+\frac{1}{6045}$

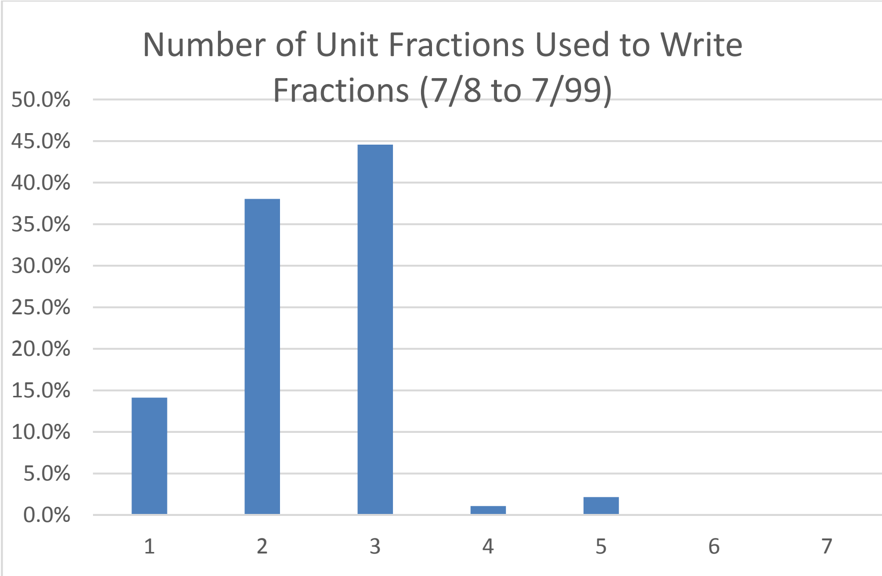
$\frac{7}{32} = \frac{1}{5} + \frac{1}{54} + \frac{1}{4320}$	$\frac{7}{33} = \frac{1}{5} + \frac{1}{83} + \frac{1}{13695}$	$\frac{7}{34} = \frac{1}{5} + \frac{1}{170}$	$\frac{7}{35} = \frac{1}{5}$
$\frac{7}{36} = \frac{1}{6} + \frac{1}{36}$	$\frac{7}{37} = \frac{1}{6} + \frac{1}{45} + \frac{1}{3330}$	$\frac{7}{38} = \frac{1}{6} + \frac{1}{57}$	$\frac{7}{39} = \frac{1}{6} + \frac{1}{78}$
$\frac{7}{40} = \frac{1}{6} + \frac{1}{120}$	$\frac{7}{41} = \frac{1}{6} + \frac{1}{246}$	$\frac{7}{42} = \frac{1}{6}$	$\frac{7}{43} = \frac{1}{7} + \frac{1}{51} + \frac{1}{3071} + \frac{1}{11785731} + \frac{1}{185204595153417}$
$\frac{7}{44} = \frac{1}{7} + \frac{1}{62} + \frac{1}{9548}$	$\frac{7}{45} = \frac{1}{7} + \frac{1}{79} + \frac{1}{24885}$	$\frac{7}{46} = \frac{1}{7} + \frac{1}{108} + \frac{1}{17388}$	$\frac{7}{47} = \frac{1}{7} + \frac{1}{165} + \frac{1}{54285}$
$\frac{7}{48} = \frac{1}{7} + \frac{1}{336}$	$\frac{7}{49} = \frac{1}{7}$	$\frac{7}{50} = \frac{1}{8} + \frac{1}{67} + \frac{1}{13400}$	$\frac{7}{51} = \frac{1}{8} + \frac{1}{82} + \frac{1}{16728}$

$\frac{7}{52} = \frac{1}{8} + \frac{1}{104}$	$\frac{7}{53} = \frac{1}{8} + \frac{1}{142} + \frac{1}{30101}$	$\frac{7}{54} = \frac{1}{8} + \frac{1}{216}$	$\frac{7}{55} = \frac{1}{8} + \frac{1}{440}$
$\frac{7}{56} = \frac{1}{8}$	$\frac{7}{57} = \frac{1}{9} + \frac{1}{86} + \frac{1}{14706}$	$\frac{7}{58} = \frac{1}{9} + \frac{1}{105} + \frac{1}{18270}$	$\frac{7}{59} = \frac{1}{9} + \frac{1}{133} + \frac{1}{70623}$
$\frac{7}{60} = \frac{1}{9} + \frac{1}{180}$	$\frac{7}{61} = \frac{1}{9} + \frac{1}{275} + \frac{1}{150975}$	$\frac{7}{62} = \frac{1}{9} + \frac{1}{558}$	$\frac{7}{63} = \frac{1}{9}$
$\frac{7}{64} = \frac{1}{10} + \frac{1}{107} + \frac{1}{34240}$	$\frac{7}{65} = \frac{1}{10} + \frac{1}{130}$	$\frac{7}{66} = \frac{1}{10} + \frac{1}{165}$	$\frac{7}{67} = \frac{1}{10} + \frac{1}{224} + \frac{1}{75040}$
$\frac{7}{68} = \frac{1}{10} + \frac{1}{340}$	$\frac{7}{69} = \frac{1}{10} + \frac{1}{690}$	$\frac{7}{70} = \frac{1}{10}$	$\frac{7}{71} = \frac{1}{11} + \frac{1}{131} + \frac{1}{20463} + \frac{1}{523397499} + \frac{1}{365259922089209169}$

$\frac{7}{72} = \frac{1}{11} + \frac{1}{159} + \frac{1}{41976}$	$\frac{7}{73} = \frac{1}{11} + \frac{1}{201} + \frac{1}{161403}$	$\frac{7}{74} = \frac{1}{11} + \frac{1}{272} + \frac{1}{110704}$	$\frac{7}{75} = \frac{1}{11} + \frac{1}{413} + \frac{1}{340725}$
$\frac{7}{76} = \frac{1}{11} + \frac{1}{836}$	$\frac{7}{77} = \frac{1}{11}$	$\frac{7}{78} = \frac{1}{12} + \frac{1}{156}$	$\frac{7}{79} = \frac{1}{12} + \frac{1}{190} + \frac{1}{90060}$
$\frac{7}{80} = \frac{1}{12} + \frac{1}{240}$	$\frac{7}{81} = \frac{1}{12} + \frac{1}{324}$	$\frac{7}{82} = \frac{1}{12} + \frac{1}{492}$	$\frac{7}{83} = \frac{1}{12} + \frac{1}{996}$
$\frac{7}{84} = \frac{1}{12}$	$\frac{7}{85} = \frac{1}{13} + \frac{1}{185} + \frac{1}{40885}$	$\frac{7}{86} = \frac{1}{13} + \frac{1}{224} + \frac{1}{125216}$	$\frac{7}{87} = \frac{1}{13} + \frac{1}{283} + \frac{1}{320073}$
$\frac{7}{88} = \frac{1}{13} + \frac{1}{382} + \frac{1}{218504}$	$\frac{7}{89} = \frac{1}{13} + \frac{1}{579} + \frac{1}{669903}$	$\frac{7}{90} = \frac{1}{13} + \frac{1}{1170}$	$\frac{7}{91} = \frac{1}{13}$

$\frac{7}{92} = \frac{1}{14} + \frac{1}{215} + \frac{1}{138460}$	$\frac{7}{93} = \frac{1}{14} + \frac{1}{261} + \frac{1}{113274}$	$\frac{7}{94} = \frac{1}{14} + \frac{1}{329}$	$\frac{7}{95} = \frac{1}{14} + \frac{1}{444} + \frac{1}{295260}$
$\frac{7}{96} = \frac{1}{14} + \frac{1}{672}$	$\frac{7}{97} = \frac{1}{14} + \frac{1}{1358}$	$\frac{7}{98} = \frac{1}{14}$	$\frac{7}{99} = \frac{1}{15} + \frac{1}{248} + \frac{1}{122760}$

The following graph shows the number of unit fractions used to write the fractions $\frac{7}{8}$ through $\frac{7}{99}$ using only non repeating unit fractions. In reviewing the data it is clear that most of the fractions could be written using 3 unit fractions. A small percentage of the numbers were written using 5 unit fractions. This leads an open ended discussion for students as to what they think would happen if we extended the fractions to $\frac{7}{200}$, $\frac{7}{2000}$, etc.



Open Problem: Let $R_7(n, j)$ be the percentage of fractions $\frac{7}{k}$ $(8 \le k \le n)$ that can be written as the sum of j unit fractions, where $1 \le j \le 7$. Find a formula for $R_7(n, j)$, or more general for $R_p(n, j)$.

At this point the questions remains what to do with a fraction $\frac{a}{b}$ where $\frac{a}{b} > 1$.

As a first step let students investigate whether or not every positive integer can be written as the sum of unit fractions. Since every integer is the sum of 1's, students should see that one way to approach this problem is to show how the number 1 can be written in different ways as the sum of non-repeating unit fractions. A key to understanding how this can be done is the formula

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)} \quad . \qquad \text{(F)}$$

According to Hoffman (1998) “Fibonacci knew, a sum of unit fractions could be continuously expanded” .This expansion would be accomplished by using the identity

$$\frac{1}{a} = \frac{1}{(a+1)} + \frac{1}{a(a+1)}.$$

Starting with the fact that

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$$

and using formula (F) for each of the terms one obtains

$$1 = \frac{1}{3} + \frac{1}{6} + \frac{1}{4} + \frac{1}{12} + \frac{1}{7} + \frac{1}{42}$$

Repeating this process once more gives us

$$1 = \frac{1}{4} + \frac{1}{12} + \frac{1}{7} + \frac{1}{42} + \frac{1}{5} + \frac{1}{20} + \frac{1}{13} + \frac{1}{156} + \frac{1}{8} + \frac{1}{56} + \frac{1}{43} + \frac{1}{1806}$$

.

Therefore,

$$2 = \left[\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right] + \left[\frac{1}{4} + \frac{1}{12} + \frac{1}{7} + \frac{1}{42} + \frac{1}{5} + \frac{1}{20} + \frac{1}{13} + \frac{1}{156} + \frac{1}{8} + \frac{1}{56} + \frac{1}{43} + \frac{1}{1806}\right].$$

If one uses the above formula a sufficient number of times, every fraction $\frac{1}{n}$, can be replaced by a sum of unit fractions of sufficiently large denominators. Therefore, the number 1 can be written in infinitely many different ways with non-repeating unit

fractions. Therefore, since every integer can be written as a sum of non-repeating unit fractions it is clear that

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}.............=\infty$$

a well known fact that can be shown in many other ways. This leads to the following observation.

Observation 2.5: Every integer *a* can be written in infinite ways in the form $a = \frac{1}{n_1} + \frac{1}{n_2} + \ldots\ldots\ldots\ldots \frac{1}{n_k}, n_i \neq n_k \ for\ i \neq k.$

Observation 2.6: Every fraction $\frac{a}{b}$ >1, can be written as a sum of non-repeating unit fractions.

Proof: $\frac{a}{b} = p + \frac{\tilde{a}}{b}$, where p is an integer and $0 \leq \frac{\tilde{a}}{b} < 1$. By the Greedy Algorithm, $\frac{\tilde{a}}{b} = \frac{1}{n_1} + \frac{1}{n_2} + \ldots\ldots\ldots\ldots \frac{1}{n_k}$ where $n_i < n_{i+1}$ and $k \leq \tilde{a}$. Obviously, by Observation 1.7, $p = \frac{1}{m_1} + \ldots\ldots \frac{1}{m_j}$ where $m_1 < m_2 < \ldots\ldots\ldots m_j$ and m_j and $m_1 > n_k$. ■

There are many follow up questions students can investigate. For example the integer 1 can be written as the sum of these unit fractions,

$$1 = \frac{1}{2}+\frac{1}{3}+\frac{1}{6} = \frac{1}{2}+\frac{1}{4}+\frac{1}{5}+\frac{1}{20} = \frac{1}{4}+\frac{1}{5}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{15}+\frac{1}{230}+\frac{1}{57960}.$$

Whereas 2 can also be written as,

$$2 = \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{15}+\frac{1}{230}+\frac{1}{57960}.$$

Since the sum $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\ldots\ldots=\infty$, one way the write every integer N as the sum of unit fractions is to find a number n such that

$$\frac{1}{2}+\ldots\frac{1}{n}<N<\frac{1}{2}+\ldots\frac{1}{n}+\frac{1}{n+1}$$

.

Then N will be a positive fraction less than $\frac{1}{n+1}$ and we can use the Greedy Algorithm to write the difference between N and $\frac{1}{n+1}$ as the sum of unit fractions $\frac{1}{n_1}+\ldots\ldots\frac{1}{n_k}$,

where $n+1\le n_1<\cdots n_{k.}$ Here is an example.

$$3=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}+\frac{1}{17}+\frac{1}{18}+\frac{1}{19}+\frac{1}{20}+\frac{1}{21}+\frac{1}{22}+\frac{1}{23}+\frac{1}{24}+\frac{1}{25}+\frac{1}{26}+\frac{1}{27}+\frac{1}{28}+\frac{1}{29}+\frac{1}{30}+\frac{1}{90}+\frac{1}{200}+\frac{1}{1000}.$$

This proposes a question. What is the best way to write integers as the sum of non-repeating unit fractions?

2.2 Decimal Periods of Rational Numbers and the Distribution of Unit Fractions with a Finite Decimal Representation

In this section we will explore the periods of unit fractions, their properties and how the ones with finite decimal reprsentation are distributed. Due to the varying lengths of the decimal expansion of some rational numbers, it would be very time consuming and tedious to perform the calculations by hand. Therefore, we can calculate the decimal expansions by using programs like Mathematica (for teachers), and Wolfram Alpha (for students). The following table of unit fractions 1/q was created using the Wolfram Alpha website at www.wolframalpha.com.

PERIODS OF DECIMALS

p/q	DECIMALS	PERIODS		p/q	DECIMALS	PERIODS
1/2	0.**5**	T		1/3	0.**3.........**	1
1/4	0.25	T		1/5	0.2	T
1/6	0.1**6.........**	1		1/7	0.**142857.........**	6
1/8	0.125	T		1/9	0.**1.........**	1
1/10	0.1	T		1/11	0.**09.........**	2
1/12	0.08**3........**	1		1/13	0.**076923.........**	6
1/14	0.0**714285.........**	6		1/15	0.0**6.........**	1
1/16	0.0625	T		1/17	0.**0588235294117647.........**	16
1/18	0.0**5..........**	1		1/19	0.**052631578947368421......**	18
1/20	0.05	T		1/21	0.**047619.........**	6
1/22	0.0**45.........**	2		1/23	0.**0434782608695652173913........**	22
1/24	0.041**6.........**	1		1/25	0.04	T
1/26	0.0**384615.........**	6		1/27	0.**037.........**	3
1/28	0.03**571428**	6		1/29	0.**0344827586206896551724137931........**	28
1/30	0.0**3..........**	1		1/31	0.**032258064516129.........**	15
1/32	0.03125	T		1/33	0.**03.........**	2
1/34	0.0**2941176470588235.........**	16		1/35	0.0**285714.......**	6
1/36	0.02**7.........**	1		1/37	0.**027.........**	3
1/38	0.0**263157894736842105... ...**	18		1/39	0.**025641.........**	6

1/40	0.025	T		1/41	0.**02439……**	5
1/42	0.0**238095……**	22		1/43	0.**023255813953488372093……….**	46
1/44	0.02**27…….**	21		1/45	0.0**2………….**	1
1/46	0.**02173913043478260869565……..**	22		1/47	0.0**2127659574468085106382978723404255319148936 17………..**	46
1/48	0.0208**3………**	1		1/49	0.0**20408163265306122448979591836734469387 7551………**	42
1/50	0.02	T		1/51	0.**0196078431372549**………	16
1/52	0.01**923076………**	6		1/53	0.**0188679245283………**	13
1/54	0.0**185………**	3		1/55	0.0**18…………**	3
1/56	0.017**857142……….**	6		1/57	0.**017543859649122807………**	18
1/58	0.0**17241379310344827586 20689655……….**	28		1/59	0.**01694915254237288135593220338983050847576271186440677966 1………**	58
1/60	0.01**6………**	1		1/61	0.0**163934426229508196721311475409836065573770491803227868852459 0……**	60
1/62	0.0**161290322580645………**	15		1/63	0.0**158730………**	6
1/64	0.0**15625**	T		1/65	0.0**153846………**	6
1/66	0.0**15………**	2		1/67	0.0**14925373134328358208955223880597 0…………**	33
1/68	0.01**4705882352941176…**	16		1/69	0.0**1449275362318840579710….**	22
1/70	0.0**142857………**	6		1/71	0.0**14084507042253521126760563380281690………**	35
1/72	0.01**38……..**	1		1/73	0.0**13698630………**	8
1/74	0.0**135………**	3		1/75	0.01**3……….**	1
1/76	0.01**315789473684210526…**	18		1/77	0.0**129870………**	6
1/78	0.0**128205………**	6		1/79	0.0**1265822784810………**	13
1/80	0.0125	T		1/81	0.0**1234567890………**	9
1/82	0.0**12195………**	5		1/83	0.0**120481927710843373493975903614457831325 30…**	41
1/84	0.01**190476………**	6		1/85	0.0**1176470588235294……**	16
1/86	0.0**116279069767441860465…**	21		1/87	0.0**1149425287356321839080459770………**	28
1/88	0.11**36………**	2		1/89	0.0**11235955056179775280898 87640449438202247191 0…..**	44
1/90	0.0**1………**	1		1/91	0.0**109890……..**	6
1/92	0.01**0869565217391304347826…**	22		1/93	0.0**107526881720430……**	15

1/94	0.0**10638297872340425531 91489 361702127659574468085…**	46		1/95	0.0**105263157894736842…… …**	18
1/96	0.01041**6………**	1		1/97	0.0**103092783505154639175 2577 3195876288659793814432 9 8969 0721649484536082474226 8 0412 371134002061855670………**	96
1/98	0.0**10204081632653061224 48979 59183673469387755………**	42		1/99	0.0**10………**	2
1/100	0.0**1**	T		1/101	0.00**9900………**	4
1/102	0.00**9803921568627450…… …**	16		1/103	0.0097087378640776699029 12621359223300………	34
1/104	0.009**615384………**	6		1/105	00.**952380……..**	6
1/106	00.**9433962264150………**	13		1/107	0.00**934579439252336448 59 8130841121495327102803 7 383 1775700………**	53
1/108	0.00**925………**	3		1/109	0.00**917431192660550458 71 5596330275229357798165 1 3761467889908256880733 9 4495412844036697247706 4 2201834862385321100………**	108
1/110	0.00**90………**	2		1/111	0.00**900………**	3
1/112	0.0089**285714………**	6		1/113	0.00**884955752212389380 53 0973451327433628318584 0 7079646017699115044247 7 8761061946902654867256 6 3716814159292035398230 0 ……..**	112
1/114	0.00**877192982456140350… ……..**	18		1/115	0.00**869565217391304347 82 60……….**	22
1/116	0.00**862068965517241379 3 103448275………**	28		1/117	0.00**854700………..**	6
1/118	0.00**847457627118644067 7 9661016949152542372881 3 5593220338983050………..**	58		1/119	0.00**840336134453781512 60 5042016806722689075630 2 521 00………**	48
1/120	0.008**3………..**	1		1/121	0.00**826446280991735537 19 00……….**	22
1/122	0.00**819672131147540983 6 0655737704918032786885 2 459016393442622950……….**	60		1/123	0.00**81300……….**	5

1/124	0.00**806451612903225……**	15		1/125	0.008	T
1/126	0.00**793650………..**	6		1/127	0.00.**78740157480314960629921259842519685039370 0…**	42
1/128	0.00**78125**	T		1/129	0.00**775193798449612403100…….**	21
1/130	0.00**769230………**	6		1/131	0.00**763358778625954198473282442748091603053435114503816793893129770992366412213740458015267175572519083969465648854961832061068702290 0……….**	130
1/132	0.00**75……….**	2		1/133	0.00**751879699248120300… ……..**	18
1/134	0.00**746268656716417910447761194029850………..**	33		1/135	0.00**740…..**	3
1/136	0.007**3529411764705882… …….**	16		1/137	0.00**72992700…………**	8
1/138	0.00**7246376811594202898550……**	22		1/139	0.00**71942446043165467625899280575539568345323741 00……**	46
1/140	0.00**714285………..**	6		1/141	0.00**7092198581560283687943262411347517730496453900…….**	46
1/142	0.00**704225352112676056338028169014084 50……….**	35		1/143	0.00**699300……………**	6
1/144	0.0069**4………..**	1		1/145	0.006896551724137931034482758620…………	28
1/146	0.00**68493150………..**	8		1/147	0.00**6802721088435374149659863945578231292517 00 ….**	42
1/148	0.00**675……….**	3		1/149	0.00**67114093959731543624161073825503355704697986577181208053691275167785234899328859060402… ……**	148
1/150	0.00**6………..**	1		1/151	0.00**66225165562913907284768211920529801324503311258278145695364238410596026490 0………**	75
1/152	0.006**578947368421052631 ……….**	18		1/153	0.00**6535947712418300…… …….**	16
1/154	0.00**649350……..**	6		1/155	0.00**645161290322580……… ..**	15

1/156	0.00641025…….	6		1/157	0.00**636942675159235668789808917197452229299363057324840764331210191082802547770700…………**	78
1/158	0.00**6329113924050…………**	13		1/159	0.00**6289308176100………..**	13
1/160	0.00**625**	T		1/161	0.00**62111801242236024844720496894409937888198757763975155279503105590 0………..**	66
1/162	0.00**617283950…………**	9		1/163	0.00**6134969325153374233128834355828220858895705521472392638036809815950920245398773 00……….**	81
1/164	0.00**60975………..**	5		1/165	0.00**60……….**	2
1/166	0.00**6024096385542168674698779518072289156626 50…..**	41		1/167	0.00**5988023952095808383233532934131736526946107784431137724550898203592814371257485029940119760479041916167664670658682634730538992215568862275449101796407185628 7425149700……….**	166
1/168	0.005**952380……….**	6		1/169	0.00**59171597633136094674556213017751479289940828402366863905325443786 9822485207100……….**	78
1/170	0.00**5882352941176470………**	16		1/171	0.00**584795321637426900………..**	18
1/172	0.00**581395348837209302325………………..**	21		1/173	0.00**5780346820809248554913294797687861271676300…**	43
1/174	0.00**5747126436781609195402298850……….**	28		1/175	0.00**571428……………**	6
1/176	0.0056**81**	2		1/177	0.00**56497175141242938531073446327683615819209039548022598870 0………**	58
1/178	0.00**5617977528089887640449438202247191011235955 0..**	44		1/179	0.00**55865921787709497206703910614525139664804469273743016759776536312849162011173184357541899441340782122905027932960893854748603351955307262569832402234636871508379888268156424581 00……….**	178
1/180	0.00**5**……….	1		1/181	0.00**552486187845303867 40**	180

					331491712707182320441988950276243093922651933701657458563535911602209944751381215469613259668508287292817679558011049723756906077348066298342541436464088397790 0………	
1/182	0.00**549450………**	6		1/183	0.00**546448087431693989071038251366120218579234972677595628415300…**	60
1/184	0.005**4347826086956521739130………**	22		1/185	0.00**540……….**	3
1/186	0.00**537634408602150……….**	15		1/187	0.00**5347593582887700………**	16
1/188	0.00**53191489361702127659574468085106382978723404 25**	46		1/189	0.00**529100……….**	6
1/190	0.00**526315789473684210……..**	18		1/191	0.00**523560209424083769633507853403141361256544502617801047120418848167539267015706806282722513 08900**	95
1/192	0.005208**3……….**	1		1/193	0.00**518134715025906735751295336787564766839378238341968911917098445595854922279792746113989637305699481865284974093264248704663212435233160621761658031088082901554404145077720207253886010362694300.**	192
1/194	0.00**51546391752577319587628865979381443298969072164948453608247422680412371134020618556701030927835 0…………**	96		1/195	0.00**512820……….**	6
1/196	0.00**5102040816326530612244897959183673469387 75….**	42		1/197	0.00**50761421319796954314720812182741116751269035532994923857868020304568527918781725888324873096446700…….**	98

When we look at the first 32 unit fractions, with the exception of a few ($\frac{1}{17}, \frac{1}{19}, \frac{1}{23}, \frac{1}{29}, \frac{1}{31}$) a middle school student would be able to calculate the decimal expansions by hand or with a basic calculator. It is important to have the student

explore the decimal expansions before giving them a printed table of the decimal expansions. The table can lead to a good discussion about periods of rational numbers.

We can start a discussion by asking the students, what is special about the fractions with terminating decimal expansions? The student should then look at the table and notice that the numbers $\frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{8}, \frac{1}{10}, \frac{1}{16}, \frac{1}{20}, \frac{1}{25}, \frac{1}{32}$ ... . are all terminating decimals, thereby finding what we summarized above in Observation 1.2. That is

> **Observation 1.2:** A fraction $\frac{a}{b}$ (in lowest terms) has a terminating decimal representation if and only if $b = 2^n 5^m$ for some integers $n, m \geq 0$.

At this point the teacher with little prompting should be able to get the students to see that the terminating decimal expansion of 2's and 5's raised to the nth power, $\frac{1}{2 \bullet 5^n}, \frac{1}{5 \bullet 2^n}$ all have length n, meaning the decimal expansion is n digits long. Whereas the numbers $\frac{1}{2^2 \bullet 5^n}$ and $\frac{1}{5^2 \bullet 2^n}$, have length n if $n \geq 2$.

> **Observaton 3.1:** The terminating decimal expansion of $\frac{1}{2^k 5^n}$ and $\frac{1}{5^k 2^n}$ seem to have length n if $n \geq k$.

This is should be clear since

$$\frac{1}{2^k * 5^n} = \frac{2^{n-k}}{2^n * 5^n} = \frac{2^{n-k}}{10^n} = 0.a_1 a_2 a_3 \ldots\ldots a_n.$$

The student should also notice that for rational numbers $\frac{1}{n}$, as n increases the number of terminating decimals seem to decrease. Therefore, we will address the following problem.

Problem 1. Let R(n) be the number of terminating unit fractions $\frac{1}{q}$, for $1 \leq q \leq n$. Find an estimate for R(n).

The following is what the student can gather by investigating Table 3.1.

Table 3.2: NUMBER R(n)OF TERMINATING DECIMALS $\frac{1}{q}$ for $1 \leq q \leq n$

n	R(n)		n	R(n)
1	1		2	2
4	3		5	4
8	5		10	6
16	7		20	8
25	9		32	10
40	11		50	12
64	13		80	14
100	15		125	16
128	17		160	18
200	19			

The information from the table can then be turned into a graph that shows the distributions of terminating decimals. This is why it is important for teachers to be knowledgeable of programs like Mathematica so that they are able to produce visuals to help students better understand the information in tables.

In Mathematica, the command

$$\mathbf{Table}[\mathbf{2}^{\boldsymbol{n}} * \mathbf{5}^{\boldsymbol{m}}, \{\boldsymbol{n}, \mathbf{0}, \mathbf{2}\}, \{\boldsymbol{m}, \mathbf{0}, \mathbf{2}\}]$$

produces the list

$$\{\{\mathbf{1}, \mathbf{5}, \mathbf{25}\}, \{\mathbf{2}, \mathbf{10}, \mathbf{50}\}, \{\mathbf{4}, \mathbf{20}, \mathbf{100}\}\}.$$

Now the command

$$[\mathbf{Flatten}[\mathbf{Table}[\mathbf{2}\hat{}\boldsymbol{n} * \mathbf{5}\hat{}\boldsymbol{m}, \{\boldsymbol{n}, \mathbf{0}, \mathbf{2}\}, \{\boldsymbol{m}, \mathbf{0}, \mathbf{2}\}]]]$$

removes the outer brackets and produces the list,

{1,5, 25, 2, 10, 50, 4, 20, 100}.

Finally the command

$$\mathbf{Sort}\left[\mathbf{Flatten}\left[\mathbf{Table}[\mathbf{2}^{\boldsymbol{n}} * \mathbf{5}^{\boldsymbol{m}}, \{\boldsymbol{n}, \mathbf{0}, \mathbf{2}\}, \{\boldsymbol{m}, \mathbf{0}, \mathbf{2}\}]\right]\right],$$

produces the list,

$$\{\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{5}, \mathbf{10}, \mathbf{20}, \mathbf{25}, \mathbf{50}, \mathbf{100}\}.$$

Since we chose to look at values of n from 0 to 2 and m from 0 to 2, we notice that some values 2^n are missing from the list (namely 8, 16, 32, and 64).

To fix this define,

$$\mathbf{DD} = \mathbf{Sort}[\mathbf{Flatten}[\mathbf{Table}[\mathbf{2}\hat{}\boldsymbol{n} * \mathbf{5}\hat{}\boldsymbol{m}, \{\boldsymbol{n}, \mathbf{0}, \mathbf{20}\}, \{\boldsymbol{m}, \mathbf{0}, \mathbf{20}\}]]];$$

$$\mathbf{RR}[\mathbf{n\_}] := \{\mathbf{Part}[\mathbf{DD}, \boldsymbol{n}], \boldsymbol{n}\};.$$

Then the command

$$\mathbf{Table}[\mathbf{RR}[\boldsymbol{n}], \{\boldsymbol{n}, \mathbf{1}, \mathbf{32}\}],$$

Produces the list

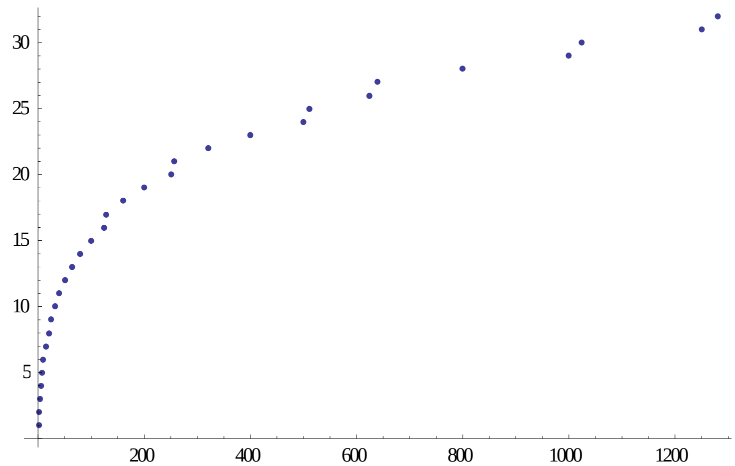
$\{\{\mathbf{1}, \mathbf{1}\}, \{\mathbf{2}, \mathbf{2}\}, \{\mathbf{4}, \mathbf{3}\}, \{\mathbf{5}, \mathbf{4}\}, \{\mathbf{8}, \mathbf{5}\}, \{\mathbf{10}, \mathbf{6}\}, \{\mathbf{16}, \mathbf{7}\}, \{\mathbf{20}, \mathbf{8}\}, \{\mathbf{25}, \mathbf{9}\}, \{\mathbf{32}, \mathbf{10}\}, \{\mathbf{40}, \mathbf{11}\}, \{\mathbf{50}, \mathbf{12}\}, \{\mathbf{64}, \mathbf{13}\},$
$\{\mathbf{80}, \mathbf{14}\}, \{\mathbf{100}, \mathbf{15}\}, \{\mathbf{125}, \mathbf{16}\}, \{\mathbf{128}, \mathbf{17}\}, \{\mathbf{160}, \mathbf{18}\}, \{\mathbf{200}, \mathbf{19}\}, \{\mathbf{250}, \mathbf{20}\}, \{\mathbf{256}, \mathbf{21}\}, \{\mathbf{320}, \mathbf{22}\},$
$\{\mathbf{400}, \mathbf{23}\}, \{\mathbf{500}, \mathbf{24}\}, \{\mathbf{512}, \mathbf{25}\}, \{\mathbf{625}, \mathbf{26}\}, \{\mathbf{640}, \mathbf{27}\}, \{\mathbf{800}, \mathbf{28}\}, \{\mathbf{1000}, \mathbf{29}\}, \{\mathbf{1024}, \mathbf{30}\}, \{\mathbf{1250}, \mathbf{31}\},$
$\{\mathbf{1280}, \mathbf{32}\}\}$

That is, 200 is the 19th number in the list and therefore there are 19 terminating fractions $\frac{1}{q}$ for $1 \le q \le 200$ (see also Table 3.1).

Finally the command

$$\mathbf{ListPlot}[\mathbf{Table}[\mathbf{RR}[\boldsymbol{n}], \{\boldsymbol{n}, \mathbf{1}, \mathbf{100}\}]]$$

produces the graph,



The graph shows that the farther we extend the n values the terminating decimals become less dense. The graph also shows that R(u) grows “logarithmically”. To test the hypothesis that the values become less dense as the n values increase, let’s look at what happens.

The command,

$$\mathbf{ListPlot}[\{\mathbf{Table}[\mathbf{RR}[\boldsymbol{n}], \{\boldsymbol{n}, \mathbf{1}, \mathbf{200}\}], \mathbf{Table}[\mathbf{Robyn}[\boldsymbol{n}], \{\boldsymbol{n}, \mathbf{1}, \mathbf{4300}\}]\}]$$

produces the list,

$\{1,1\}, \{2,2\}, \{4,3\}, \{5,4\}, \{8,5\}, \{10,6\}, \{16,7\}, \{20,8\}, \{25,9\}, \{32,10\}, \{40,11\}, \{50,12\}, \{64,13\}, \{80,14\}, \{100,15\}, \{125,16\}, \{128,17\}, \{160,18\}, \{200,19\}, \{250,20\}, \{256,21\}, \{320,22\}, \{400,23\}, \{500,24\}, \{512,25\}, \{625,26\}, \{640,27\}, \{800,28\}, \{1000,29\}, \{1024,30\}, \{1250,31\}, \{1280,32\}, \{1600,33\}, \{2000,34\}, \{2048,35\}, \{2500,36\}, \{2560,37\}, \{3125,38\}, \{3200,39\}, \{4000,40\}, \{4096,41\}, \{5000,42\}, \{5120,43\}, \{6250,44\}, \{6400,45\}, \{8000,46\}, \{8192,47\}, \{10000,48\}, \{10240,49\}, \{12500,50\}, \{12800,51\}, \{15625,52\}, \{16000,53\}, \{16384,54\}, \{20000,55\}, \{20480,56\}, \{25000,57\}, \{25600,58\}, \{31250,59\}, \{32000,60\},$

$\{32768{,}61\}, \{40000{,}62\}, \{40960{,}63\}, \{50000{,}64\}, \{51200{,}65\}, \{62500{,}66\}, \{64000{,}67\}, \{65536{,}68\},$

$\{78125{,}69\}, \{80000{,}70\}, \{81920{,}71\}, \{100000{,}72\}, \{102400{,}73\}, \{125000{,}74\}, \{128000{,}75\},$

$\{131072{,}76\}, \{156250{,}77\}, \{160000{,}78\}, \{163840{,}79\}, \{200000{,}80\}, \{204800{,}81\}, \{250000{,}82\},$

$\{256000{,}83\}, \{262144{,}84\}, \{312500{,}85\}, \{320000{,}86\}, \{327680{,}87\}, \{390625{,}88\}, \{400000{,}89\},$

$\{409600{,}90\}, \{500000{,}91\}, \{512000{,}92\}, \{524288{,}93\}, \{625000{,}94\}, \{640000{,}95\}, \{655360{,}96\},$

$\{781250{,}97\}, \{800000{,}98\}, \{819200{,}99\}, \{1{,}000{,}000{,}100\},$

The results we obtain are surprising at the least. One would think that if we looked at all of the terminating decimals $\frac{1}{q}$ up to $\frac{1}{1{,}000{,}000}$, we certainly would expect to see a large number of terminating decimals. Our result shows that there are only 100 terminating decimals $\frac{1}{q}$ for $1 \le q \le 1{,}000{,}000$.

The command

$$\mathbf{FindFit}[\mathbf{DDD}, \boldsymbol{a} + \boldsymbol{b}\mathbf{Log}[\boldsymbol{x}], \{\boldsymbol{a}, \boldsymbol{b}\}, \boldsymbol{x}]$$

produces

$$\{\boldsymbol{a} \to -\mathbf{20.66470229422352}, \boldsymbol{b} \to \mathbf{8.065188443189992}\}$$

The command

$$\mathbf{Robyn}[\mathbf{x\_}] := -\mathbf{20.7} + \mathbf{8.06} * \mathbf{Log}[\boldsymbol{x}];$$

defines the Robyn function estimating the number of terminating decimals up to 1/x.

If we substitute x with 1,000,000 for we get the following,

$$\mathbf{Robyn}[\mathbf{1,000,000}] = \mathbf{90.65}, \qquad \mathbf{Robyn}[\mathbf{100,000}] = \mathbf{72.09}$$

$$\mathbf{Robyn}[\mathbf{10,000}] = \mathbf{53.54}, \ \mathbf{Robyn}[\mathbf{1,000}] = \mathbf{35.}$$

These Robyn estimates are not great, but they are in the ball park. Thus, since

$$\mathbf{Robyn}[\mathbf{10\hat{}100}] = \mathbf{1835.18}$$

we think that it is true that the percent of unit fractions with terminating decimals among the fractions up to 1/googol is very close to zero.

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