Computational model for elasto-plastic and damage analysis of plates and shells

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COMPUTATIONAL MODEL FOR ELASTO-PLASTIC AND DAMAGE ANALYSIS OF PLATES AND SHELLS

A Dissertation
Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
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ABSTRACT

Shells and plates are very important for various engineering applications. Analysis and design of these structures is therefore continuously of interest to the scientific and engineering community. Accurate and conservative assessments of the maximum load carried by the structure, as well as the equilibrium path in both elastic and inelastic range are of paramount importance.

Elastic behaviour of shells has been very closely investigated, mostly by means of the finite element method. Inelastic analysis on the other hand, especially accounting for damage effects, has received much less attention from the researchers.

A computational model for finite element, elasto-plastic and damage analysis of homogenous and isotropic shells is presented here. The formulation of the model proceeds in several stages, described in the following chapters. First, a theory for thick spherical shells is developed, providing a set of shell constitutive equations. These equations incorporate the effects of transverse shear deformation, initial curvature and radial stresses.

The proposed shell equations are conveniently used in finite element analysis. A simple $C^0$ quadrilateral, doubly curved shell element is developed. By means of a quasi-conforming technique shear and membrane locking are prevented. The element stiffness matrix is given explicitly which makes this formulation computationally very efficient.

The elasto-plastic behavior of thick shells and plates is represented by means of the non-layered model, with an Updated Lagrangian method used to describe a small strain geometric non-linearity. In the treatment of material non-linearities an Iliushin’s yield function expressed in terms of stress resultants is adopted, with isotropic and kinematic hardening rules.

Finally, the damage effects modeled through the evolution of porosity are incorporated into the yield function, giving a generalized and convenient yield surface expressed in terms of the stress resultants. Since the elastic stiffness matrix is derived explicitly, and a non-layered model is employed in which integration through the thickness is not necessary, the current stiffness matrix is also given explicitly and numerical integration is not performed at any stage during the analysis. This makes this model consistent mathematically, accurate for a variety of applications and very inexpensive from the point of view of computer power.
CHAPTER 1
INTRODUCTION

1.1 Shell Structures

The current dissertation is devoted to a comprehensive analysis of homogenous and isotropic shells and also plates and beams. A beam is a structural element in which the length is substantially larger than the width and thickness. A plate is a flat surface, in which the thickness is small comparing to the other two dimensions. A shell is a curved surface, in which the thickness is much smaller than the remaining dimensions. The geometrical properties of shells, i.e. single or double curvature give rise to a tremendous advantage of these light weight structures when compared to plates. Plates and beams are mostly loaded in the direction normal to the plane, or longitudinal axis in the case of the latter, and carry the loads primarily through bending. The efficiency of shells in the load-carrying mechanism is based on their curvature, which allows multiple stress paths and an optimum form of transmission of different load types.

There are two main ways in which shells support the loads. If subjected to uniform pressures, shells can usually resist the loads by membrane (in plane) action. The most desirable situation is when a shell is subjected to a uniform load causing tensile stresses, because the material can be used to its full strength (Wadee 2001). Concentrated loads on the other hand, introduce local bending stresses, which are much more likely to cause yielding or ultimately failure of the shell. The eggshell can provide a very good illustration of these actions. If we try to squash an egg, using a uniform or approximately uniform external pressure, we will notice that it can resist a very high pressure considering its very small thickness. If we press the finger against the surface of the egg, applying a ‘point load’, the shell is going to fracture under a much smaller force. Like in the case of the eggshell, structural shells are best utilized if subjected to uniform loading.

Local reinforcement of certain critical regions of shell structures is often necessary. A possible location of these areas is at the transition from one basic surface to another. The connection between the main cylindrical vessel and its spherical ends is a very good example of the critical region, where stiffening may be required (Wadee 2001). Stiffened shells are beyond the scope of this work, and they will not be considered any further.

The analysis of shells often involves two distinct theories. A membrane theory is only capable of describing the membrane behaviour, i.e. is performed under the assumption that a curved surface is incapable of conveying the shear forces or bending moments (Ugural 1999). A bending theory includes the effects of bending in the analysis. Although for practical purposes, the membrane stresses are of far greater importance than the bending stresses, one needs a general or bending theory to account for the discontinuity effects in geometry e.g. changes in thickness, or boundary conditions, e.g. concentrated loadings. These effects cannot be approximated be means of the membrane theory only.

Most of the investigations of beams, plates and shells are performed under the assumption that the thickness is small comparing to the other two dimensions. The shell
or plate is considered thin if the effects of transverse shear deformations on the behaviour of the structure are negligible. Normal strains and stresses in the out-of plane (radial) direction are also considered negligible for thin shells. This is mostly the case for ratios of thickness $h$ to radius of curvature $R$ equal to, or less than 1/50. This limit is however not definite and some of the results in the literature show that under certain boundary conditions, the shear deformation can be significant even for very thin shells or plates. With the increase of the utilization of thick shells to various engineering applications such as cooling towers, dams, pressure vessels, etc. it is imperative to develop a simple and accurate theory for thick shells, accounting for not only transverse shears, but also radial effects and initial curvature.

Shells are often considered to act globally as a member, e.g. lighting column. In that case, a global behaviour of the component can be accurately approximated using a simplistic model. Local behaviour of shell is however often critical. Dimpling in domes, or the development of the Yoshimura pattern (Fig 1.1) due to buckling in compressed cylinders, are complex phenomena, which require an in-depth analysis with non-linear behaviour taken into account. Although buckling, as an eigenvalue problem is not considered, the local behaviour of shells is very closely approximated in the current dissertation.

![Yoshimura pattern in a compressed cylinder](image)

**Figure 1.1** Yoshimura pattern in a compressed cylinder (Wadee, K., 2005)

All the aforementioned structural elements are extensively used in various applications in many fields of engineering. Different types of shells have often been used for industrial purposes. Examples of single curvature shells are storage tanks and silos, pressure vessels, submarines, airplanes, chimneys, oilrigs or even lighting columns. A double curvature form of shell can be used to construct spherical tanks and reservoirs, roofs, stadiums, vehicles and water towers. Examples of the shell structures are shown in Figure 1.2.

### 1.2 Motivation and Scope

Shells are very important for various engineering applications. Analysis and design of these structures is therefore continuously of interest to the scientific and engineering community. Accurate and conservative assessments of the maximum load carried by the structure, as well as the equilibrium path in both elastic and inelastic range are therefore of paramount importance.
Figure 1.2 Examples of shell structures: a) Submarine Seawolf, US Navy (1996); b) Headquarters of a radio station, (RFM-FM) in Poland, Monolithic Dome Institute (2005)

Determination of the equilibrium path in the elastic and inelastic range usually involves a complex analysis. Manual calculations provide valuable information about the behaviour of shells. They are however mostly performed under simplifying assumptions and for a specific problem. Universal algorithms based on manual calculations and accurately approximating the load-displacement response for a variety of shell problems
are practically unobtainable. At the same time, recent developments in computer
technology allow us to formulate computational models capable of delivering accurate
results, while being relatively simple. By means of the Finite Element Method, we can
carry out elasto-plastic and damage analyses of both thick and thin shells of general
shape. Finite elements offer tremendous flexibility and possibility to account for nearly
every effect observed in the experimental or ‘real life’ tests of material or structural
behaviour. Attempting to investigate every experimentally observable phenomenon is
however neither necessary, nor feasible. Constitutive modeling is understood as a
reasonable choice of effects, which are the most important for explanation of the
phenomenon described (Perzyna 2005). The model formulated and presented in the
current dissertation is addressed to the engineering environment. Thus, it considers the
most important issues from the structural analysis point of view. The objective of this
work is to develop a computational model for non-linear elasto-plastic large
displacements damage analysis of isotropic shells.

One of the difficulties of non-linear calculations is the fact that they are based on
incremental and/or iterative algorithms, which may require prohibitively large storage of
the computer. Computational efficiency needs special attention in non-linear modeling of
shells. In order to formulate an algorithm delivering close approximations of the
equilibrium path in both the elastic and inelastic range, while being at the same time
simple and efficient, we need to proceed in several stages.

A refined shell theory, providing a set of shell constitutive equations, is proposed
first. The theory is universal and general, i.e. it accounts for both membrane and bending
behaviour, and is formulated for thick shells, accounting for the effects of the transverse
shear deformations, radial stresses and initial curvature. The assumptions used to derive
the shell equations are described in the following section, and detailed derivations are
given in Chapter 2.

The constitutive equations generated by the theory are conveniently used in finite
element analysis. A simple $C^0$ quadrilateral, doubly curved shell element is developed. In
order to overcome membrane and shear locking as well as other numerical deficiencies, a
quasi-conforming technique is adopted, featuring an explicit form of the stiffness matrix.
Chapter 3 is devoted to the formulation of this finite element.

Shelled structures are very often subjected to loading conditions causing very
large displacements. Geometrical non-linearities are crucial in the elasto-plastic and
damage modeling of shells. Thus, to achieve a desired accuracy, geometric non-linearities
must be accounted for. We consider here small strain problems, studied by means of the
Updated Lagrangian method. The details of the geometrically non-linear calculations are
given in Chapter 4.

A ‘non-layered’ plastic model is adopted in the treatment of material non-
linearities due to its efficiency and convenient applicability to engineering problems. The
yield function is defined in the stress resultant space and integration of the stresses over
the thickness of the shell is not necessary. Isotropic and kinematic hardening rules are
developed with the latter aimed at representing the Bauschinger effect. The definitions of
the yield surface, flow and hardening rules, with the derivation of the stiffness matrix, are
given in Chapter 5.
Description of the influence of damage on the behaviour of shells is the final stage of the formulation. The experimental results show that the degradation of material properties of ductile metals in the elastic range due to the damage effects is negligible. Hence, the damage is considered here as a phenomenon induced by the plastic strain and is represented by the scalar porosity parameter introduced into the yield function. Static loading conditions are considered here, with both plasticity and damage treated as the rate independent processes. The description of the effects of damage is introduced into the model in Chapter 6.

Chapter 7 gives a description of the computational issues. The numerical algorithms used here are outlined, along with the developed software and hardware information. In Chapter 8, we summarize the results and draw the conclusions.

Each of the components of the formulation, namely the theory, the finite element analysis, the non-linear analysis and damage description, are integral parts of the model. Nevertheless, they are universal and introduce original ideas on every level of the algorithm. This leads to the capability of the elements of this framework to be used also separately, as the ‘stand-alone’ concepts.

It is very important to note that all the concepts developed and adopted in the current dissertation, are postulated and they are formulated into a unified algorithm and verified through a series of discriminating numerical examples, given in the consecutive Chapters. In Chapter 8, we summarize all the results of the tests and draw the conclusions.

1.3 Basic Assumptions and Literature Review

The main assumptions of the computational model developed in the current dissertation are:
- Material is homogenous and isotropic;
- We consider shells of general shape, both thick and thin, with both membrane and bending actions;
- Buckling as an eigenvalue problem is not considered;
- Loading conditions are static;
- We adopt a non-layered approach in plastic analysis;
- Plasticity and damage are treated as rate-independent processes;
- Damage variable is isotropic and induced by the plastic strain;

Apart from these major assumptions, there are others, pertinent to particular components comprising the model. These will be explained in detail in the following sections.

1.3.1 Refined Theory of Thick Spherical Shells

The complete two-dimensional theory of thin shells was developed by Love over 100 years ago. Numerous contributions to this subject have been made since then. Any two-dimensional theory of shells approximates the real three dimensional problem. Researchers have been seeking better approximations for the exact three-dimensional
elasticity solutions for shells. In the last three decades, the developed refined two-
dimensional linear theories of thin shells include important contributions of Sanders
(1959), Flugge (1960), and Niordson (1978). In these refined shell theories, the initial
curvature effect is taken into consideration. Nevertheless, the deformation is based on the
Love-Kirchhoff assumption, and the radial stress effect is neglected. In the current work,
we refer to all the theories built on the Kirchhoff-Love assumption, as “the classical
theory”. The refined theories by Sanders (1959), Flugge (1960) and Niordson (1978)
provide very good results for the analysis of thin shells. The theory of Sanders-Koiter has
been widely used in the finite element analysis of shells (Ashwell and Gallagher, 1976).
However it was shown (Niordson, 1971) that Love’s strain energy expression has
inherent errors of relative order \[\frac{h}{R} + \left(\frac{h}{L}\right)^2\], where \(h\) is the thickness of the shell, \(R\)
is the magnitude of the smallest principal radius of curvature, and \(L\) is a characteristic
wave length of the deformation pattern of the middle surface. Consequently, when the
refined theories of thin shells are applied to thick shells, with \(h/R\) not small compared to
unity, the error could be quite large. Unlike the theory of thin shells, the comprehensive
theory of thick shells, with not only transverse shear strains considered, but also initial
curvature and radial stresses, has received limited attention from researchers up to now.
Voyiadjis and Shi (1991) developed a refined shell theory for thick cylindrical shells
which is very accurate and convenient for finite element analysis. The current work
presents a refined shell theory for thick spherical shells, with the shell equations based on
similar assumptions as those of Voyiadjis and Shi (1991). The proposed work can be
considered a more general formulation of the Voyiadjis and Shi theory (1991).

Thick shells have a number of distinctly different features from thin shells. One of
these features is that in the former case, the Kirchhoff-Love assumption is no longer
valid. According to this assumption, plane sections remain plane after the deformation,
and perpendicular to the middle surface. The angle of rotation of the cross-section \(\phi\) is
therefore equal to the first derivative of the vertical displacement \(\frac{\partial w}{\partial x}\), and transverse
shear deformation \(\gamma_{xz}\) can be neglected (Figure 1.3).

\[
\mathbf{u} = -z \left( \frac{\partial w}{\partial x} - \gamma_{xz} \right)
\]

**Figure 1.3** Transverse shear deformations (Voyiadjis and Woelke, 2005)
If the thickness of the shell becomes significant, the transverse shear strains $\gamma_{xz}$ are not negligible and the angle of rotation of the cross-section is altered, as shown in Figure 1.3. The current formulation is unified for both thick and thin shells and thus, the influence of the transverse shear strains is considered in the analysis.

Another important distinction between thick and thin shell analyses is that in thick shells the initial curvatures not only contribute to the stress resultants and stress couples, but also result in a nonlinear distribution of the in-plane stresses across the thickness of the shell. This is because the length of the surface away from the middle surface is different from that of the middle surface (Figure 1.4). We also account for this effect in the present formulation of the shell equations.

**Figure 1.4 Initial curvature effect**

In a number of particular cases of loadings, the radial stress distribution of thick shells is very important and needs to be incorporated in the analysis. The theory presented here is based on the analytical closed form solution of the thick spherical container subjected to external and internal pressures. These investigations were conducted by Lame (1852), who obtained the expression for the radial stresses, which serves as a base for the derivation of the shell theory.

It is not difficult to incorporate transverse shear deformations in shells. This may be accomplished following the work of Reissner (1945) for the plate theory. Many other authors directed their attention to the transverse shear strains, due to their importance in analysis of bending of thick isotropic structures (Basar et al. 1992, 1993; Bathe 1982; Bathe & Brezzi 1985; Bathe & Dvorkin 1984; Dennis & Palazotto 1989; Palazotto et al. 1991; Niordson 1978, 1985; Noor & Burton 1989; Reissner 1945, 1975; Mindlin 1951; Reddy 1984, 1989; Kratzig 1992; Kratzig and Jun 2003 and many others).

The attention in the previously developed shell theories is focused on the two-dimensional shell equations together with maintaining a linear stress distribution through the shell thickness (Flugge, 1960; Niordson, 1985). It appears that refinement of the stress distribution in thick shells has not been extensively studied with respect to the inclusion of radial stresses. The theory of thin shells may provide a good estimate of the strain energy for some problems in thick shells. However, it cannot provide an accurate distribution of the stresses through the thickness (Gupta and Khatua, 1978). This accuracy is imperative from an engineering point of view. In the current dissertation, we
incorporate radial stresses in the shell theory and obtain nonlinear stress distributions through the shell thickness.

The formulation procedure for the proposed shell theory is based on the following:

- Assumed out of plane stress components that satisfy given traction boundary conditions;
- Three-dimensional elasticity equations with an integral form of the equilibrium equations;
- Stress resultants and stress couples acting on the middle surface of the shell together with average displacements along a normal of the middle surface of the shell and the average rotations of the normal (Voyiadjis and Baluch, 1981).

The resulting constitutive equations of shells reduce to those given by Flugge (1960) when the shear deformation and radial effects are neglected. In this case, the average displacement is replaced by the middle surface displacements.

1.3.2 Finite Element Implementation of the Theory of Shells

The finite element method is arguably the most convenient way of analyzing shells. Shell finite elements are most often based on the shell constitutive equations relating stress resultants to strains. In these cases, we operate in the stress resultant space. By means of the stress-based three dimensional solid brick elements, we can also successfully analyze shells and avoid a problem of the derivation of the shell equations, which is often tedious. A term ‘shell element’ is most commonly used in description of the finite element formulation based on the shell constitutive equations. From the point of view of the computational expense, shell elements are much more attractive than solid brick elements (Figure 1.5).

![Figure 1.5 Meshes of: a) Shell finite elements and; b) 3D solid, brick finite elements](image)

Figure 1.5 shows the mesh of 400 shell finite elements. In order to solve the same problem using brick elements, we need 4000 elements (Figure 1.5b), for a similar level of accuracy. Thus, for complex geometries, the use of three dimensional stress based elements might require prohibitively large storage of the computer. A reliable shell element founded on the accurate shell theory is therefore much more convenient.
Needless to mention, this becomes even more significant for the case of the non-linear analysis of plates and shells, where the stiffness matrix has to be evaluated many times. In addition the solid element accounts for the transverse normal strain $\varepsilon_z$, which may give rise to numerical difficulties. This difficulty can be overcome in shell elements by simply neglecting the transverse normal strains, or by the appropriate choice of the strain fields in the quasi-conforming technique, adopted in the present dissertation. In summary, considerations of economy and robustness indicate that solid elements should not be used to model plates and shells (Cook et al. 1989).

Developing a shell element capable of delivering close approximations of the behaviour of both thick and thin shells poses some numerical problems. Thick shells require accounting for the transverse shear strains, which gives rise to additional modes in the strain energy expression. These modes should be negligible when the thickness decreases. Instead, they become very large and suppress the bending effects in the case of thin shells. The finite element becomes then too stiff and dominated by the shear part of the stiffness matrix, which should be negligible. We call this phenomenon ‘shear-locking’ of the mesh. Similarly, we experience ‘membrane locking’ when during the analysis of the bending dominant problem, the membrane part of the stiffness matrix suppresses the remaining modes. Locking is explained in greater detail in Chapter 2. Despite the existence of the above-mentioned numerical deficiencies, there are many ways to overcome locking and spurious energy modes, which make finite elements very suitable for the analysis of shells.

Different levels of continuity of fields can be used in the formulation of a shell finite element. A $C^0$ element is considered to be a simple and efficient way of analyzing shells. It provides continuity of the displacements across interelement boundaries, but not continuity of the first derivatives of the displacements (strains). Thus, in the mesh of $C^0$ elements the strains exhibit a jump or discontinuity on the interelement boundaries (Cook et al. 1989). Refining of the mesh usually suppresses the discontinuity and leads to convergence of the results. Many $C^0$ elements exhibit some deficiencies in the analysis of thin plates and shells, such as shear and membrane locking and spurious kinematic modes. Widely used 9-node fully integrated Lagrange $C^0$ element performs poorly in the analysis of shells, even for a relatively fine mesh. In the past years, these disadvantages of the $C^0$ elements have received increasing attention from researchers (Hughes and Hinton, 1986; Babu and Prathap, 1986; Atluri and Yagava, 1988, Stolarski and Belytschko, 1981; Belytschko et al. 1985), and many approaches were proposed to construct reliable and accurate $C^0$ elements for the analysis of both thick and thin plates and shells. Most of the $C^0$ elements are based on the degenerated shell formulation given by Ahmad et al. (1970). It follows the concept of degeneration of 3D continuum elements, to 2D shell-like kinematics, (over the thickness) by linear displacement interpolations (Kratzig & Jun, 2003). The degenerated elements have twice as many nodal points as the classical shell elements and are therefore computationally expensive. This imposes limits on their application to non-linear problems, (Yang et al. 2000).

The transverse shear-locking problem in the degenerated elements can be solved by the use of the reduced or selective integration technique (Zienkiewicz et al., 1971; Stolarski and Belytschko, 1983-84; Hughes, 1987; Yang et al. 2000). This may however result in development of spurious zero strain energy modes, decreasing the reliability of
the elements based on this approach. Most of the degenerated elements deal with the locking phenomena, being only the result of inadequate representation of the rigid body modes. Those elements that correctly incorporate rigid body motion can still encounter severe membrane locking (Belytschko 1985).

The employment of discrete Kirchhoff constraints (Wempner et al. 1968, Li et al. 1985) is another approach to avoid shear locking for C° elements. Unfortunately, the discrete Kirchhoff constraints lead to complex inversion and calculations in the formulation of the element stiffness matrix (Li et al., 1985). Another method to construct C° elements is to employ the so-called enhanced interpolations of the transverse shear strain and membrane strain (Huang and Hinton, 1986) or assumed natural-coordinate strains (Park and Stanley, 1986). The C° elements based on the enhanced strains can overcome the shear locking and spurious kinematic mode problems and give good results. However, like in all other degenerated elements, numerical integration is employed in the formulation of these elements, even in the case of flat plate elements. This process is very time consuming especially in non-linear problems where the stiffness matrix has to be evaluated numerous times during the analysis.

Locking can also be eliminated by means of the mixed formulation (Pian 1964; Lee and Pian 1978). Separate interpolations for displacements and strains are used here, (Pian 1964) or node decompositions where nodal displacements are projected to minimize parasitic stresses (Hughes and Tezduyar, 1981; Belytschko et al. 1984).

The term strain element was first introduced by Ashwell et al. (1971, 1972, 1976). The strain functions in this element were only used for curved finite element shape functions in order to satisfy the rigid body motion of a curved element. Therefore, this type of element still belongs to the conventional assumed displacement elements. In Huang and Hinton’s element (1986), only the transverse shear strains and membrane strains are interpolated. In Park and Stanley’s element (1986), the strains are interpolated along the so-called reference lines. Consequently, elements given by Huang and Hinton (1986), and Park and Stanley (1986) are not based on the general strain fields.

Tang et al. (1980, 1983) and Chen, and Liu (1980), presented the ’quasi-conforming technique’ and the use of the string net functions. This technique was named ‘quasi-conforming’ because the inter-element boundary compatibility requirements are satisfied under the integral sign (Tang et al 1983). Here, the strain fields are interpolated directly rather than obtained from the assumed displacement field. The element strains may be expressed in terms of the element nodal displacement vector by integration along the element boundaries together with the string net functions, which are similar to the edge displacement interpolations in Pian’s (1964) hybrid stress element. Based on the element strain field, the stiffness matrix may be evaluated in the usual way. This method is related to the so-called ‘generalized hybrid model’, which may also be derived using Hu-Washizu principle (Tang et al. 1983). The quasi-conforming element technique gives the explicit form of the stiffness matrix, as integrations can be done directly, without performing the numerical integration. It is a general method, inspired by Pian’s pioneering work (1964), which treats the conforming, non-conforming, and hybrid elements in a simple unified way. Many excellent quasi-conforming elements were obtained for plane stress/strain, plate bending and shell problems (Tang et al. 1980; Shi, 1980; Lu and Liu, 1981). Shi and Voyiadjis (1991) developed a very efficient and
accurate $C^0$ thick/thin shell element based on the quasi-conforming element technique. This element is unified for both curved and flat configurations, exhibits neither shear nor membrane locking and is free from spurious zero energy modes. It was successfully applied to circular arches and straight beams. However, analyzing shells by means of the Shi and Voyiadjis element is mostly suited for cylindrical configurations (Shi & Voyiadjis 1990, Voyiadjis & Shi 1991).

The current work presents a new $C^0$ finite element capable of simulating the behaviour of thin and thick shells of arbitrary shape, as well as thin and thick plates, circular arches and straight beams. Studies performed by many researchers (Pandya & Kant - 1988, Reddy et al. – 1989, Basar et al. 1992) show that the classical shell models including the Mindlin-Reissner type, do not predict the deformation with sufficient accuracy if the side-thickness ratio exceeds certain limits, (Basar et al. 1993). To achieve the desired accuracy, one needs to build a finite element based on the efficient and accurate theory of thick shells. Previously derived shell constitutive equations are adopted in the formulation of the computational model. Although spherical strains are used, i.e. the radius of curvature is the same in both directions, it will be shown that by means of finite elements the theory is applicable not only to spherical shells, but also to shells of general shape, as well as plates and beams. A strain energy density is proposed which provides the foundation for the $C^0$ assumed strain element.

The quasi-conforming technique is used to overcome locking which results in the explicit form of the element stiffness matrix. The element strain fields are interpolated directly rather than obtained from the assumed displacement field providing adequate representation of the rigid body modes. The spurious energy mode, which may be a problem in finite elements with reduced integration, is avoided by the appropriate choice of the strain fields. The compatibility equations of the displacements may also be satisfied in the assumed strain fields. This results in a more complicated formulation of the element stiffness matrix and therefore, in the current work, the compatibility is not enforced in the assumed strain fields.

The resulting $C^0$ finite element satisfies the Kirchhoff-Love hypothesis in the case of thin plates and shells, by the simple dependent displacement and rotation interpolations of a straight beam, which should successfully prevent shear locking.

1.3.3 Geometrically Non-Linear Analysis of Shells

In the elasto-plastic, finite element analysis of shells geometrical non-linearities play a very important role. Displacements at the regions of the structure, which undergo inelastic deformations, can be very large. Moreover, large deformations are crucial in modeling of damage in shells (Kleiber and Kollmann 1993). Most of the early works on geometric non-linearity was undertaken by Turner et al. (1960), Holand and Moan (1969), Gallagher et al. (1967) and many others. These were most of the time related to instability problems. The incremental procedures for modeling geometric non-linearity were originally adopted by Turner et al., (1960) and Argyris (1964, 1965) who used the geometric stiffness matrix in conjunction with the updating of coordinates and an initial displacement matrix (Mallet and Marcal, 1968; Marcal, 1969; Dupius et al., 1971).
Similar approaches were adopted for analysis of the material non-linearity (Zienkiewicz, 1971).

The Updated Lagrangian description, which has proven to be a very effective method in the treatment of geometric non-linearities (Bathe 1982; Flores et al. 2001; Horrimgoe et al. 1978; Kebahi et al. 1992) is adopted here. The element local coordinates and local reference frame are continuously updated during the deformation. We consider large rotations and rigid translations here, but small strains with the total rotations decomposed into large rigid rotations and moderate relative rotations. The relative rotations and the derivatives of the in-plane displacements from two consecutive configurations may be considered small (Shi and Atluri 1988; Shi and Voyiadjis 1991). Consequently, the quadratic terms of the derivatives of the in-plane displacement are negligible. We therefore have a non-linear analysis with large displacements and rotations, but small strains. The transformation matrix given by Argyris (1982) is employed to handle large rigid rotations. The assumed strain finite element with an explicit form of the stiffness matrix, as described above, provides the linear part of the element tangent stiffness matrix. The assumptions made here are verified through the numerical tests in Chapters 4 and 5.

1.3.4 Elasto-Plastic Analysis of Shells

Many approaches have been used in the elasto-plastic analysis of plates and shells. The finite element method has been a successful way of modeling the linear behaviour of shells and it is therefore natural to apply the same method in non-linear computations. Having a finite element procedure for linear elastic analysis of shells, we further develop the model to investigate the elasto-plastic behaviour of the structures under consideration.

In the case of non-linear modeling, the advantage of the explicit form of the stiffness matrix, obtained through the use of the quasi-conforming technique, becomes even more apparent. This is due to the fact that the element matrices are calculated many times during the analysis. Moreover, selective integration, which is the most often used remedy to overcome locking in shells, requires an explicit segregation of transverse shear terms from bending and membrane terms. This is not possible when coupling between these exists, as is mostly the case for non-linear analysis. Although, this problem was solved by a generalization of the selective integration procedure (Hughes 1980) the algorithm given in this dissertation offers far greater simplicity and lower computational cost.

Many investigators avoided the problem of shell constitutive equations by following a layered approach, also referred to as ‘through-the-thickness-integration’ (Dvorkin and Bathe 1984; Flores and Onate 2001; Kebahi and Cassell 1992; Kollmann and Sansour 1997; Onate 1999; Parish 1981). In this method, a plate or a shell is divided into layers where stresses are calculated and the yield condition is checked for each layer separately. The forces and moments are then calculated by integration through the thickness. Although this method can give very accurate results, it can also be very demanding in terms of computational power. If on the other hand a non-layered approach is adopted, the yield function is integrated through the thickness of the plate or shell and
therefore expressed in terms of stress resultants and couples. Numerical integration of the stresses is not necessary in this case, which makes the non-layered formulation much cheaper. The approximation of the yield criterion expressed in terms of forces and moments is expected to result in a loss of accuracy. It is however not the case as was shown by many authors studying the two methods (Bieniek and Funaro 1976; Owen and Hinton 1980; Shi and Voyiadjis 1992). Both models compare very well with the analytical solutions of pertinent problems available in the literature (Hodge 1959; Olszak and Sawczuk 1977; Sawczuk 1989; Sawczuk and Sokol-Supel 1993).

The non-layered model is employed in the current work with the yield function in terms of stress resultants and couples. In this case, the accuracy of the yield criterion is very important. A comparison of different yield surfaces is given by Robinson (1971). A modified Iliushin’s yield function is adopted in this work (Iliushin 1956). The first modification allows for capturing progressive development of the plastic curvatures across the thickness of the shell, as introduced by Crisfield (1981) and later applied by Shi and Voyiadjis (1992). This approach allows us to model accurately the first yield point and track the growth of plastic curvatures until a plastic hinge is developed.

The transverse shear forces may significantly affect the plastic behaviour of both thick and, for certain loading conditions, thin shells. Shear becomes even more important in the case of laminate composites. Yet the influence of transverse shear forces on the plastic behaviour of plates and shells has been covered in the literature to much less extent than in the case of elastic analysis. This effect is investigated here.

Isotropic hardening as given by Shi and Atluri (1988) is also incorporated into the yield function in the present formulation. More importantly however, a kinematic hardening rule aimed at capturing the Bauschinger effect is defined. It is well known that when a material or a structure is loaded in tension into a plastic zone, and subsequently the load is reversed, then yielding in compression will occur at a reduced value of stress. This anisotropy in the material is induced by plastic deformations. Relatively few hardening rules for non-layered plates and shells have been published, capable of correctly representing this phenomenon. As first recognized by Wempner (1973), the stress resultants and couples of the classical theory are not sufficient to describe accurately the state of stress in plastic shells. Bieniek and Funaro (1976) introduced ‘hardening parameters’, in the form of residual bending moments, allowing for description of a Bauschinger effect. However Bieniek and Funaro (1976) recognized that the ‘hardening parameters’ defined by them, do not give a full representation of the kinematic hardening phenomenon. For the appropriate representation of the rigid translation of the yield surface during non-elastic deformation in the stress resultant space, one needs not only the residual bending moments, but also residual shear and normal forces. These are equivalent parameters to the backstress in the stress space. We therefore present a new kinematic hardening rule for non-layered plates and shells here, explicitly derived from the evolution of backstress given by Armstrong and Frederick (1966).

Modeling of elasto-plastic behaviour of structural elements based on the mathematical theory of plasticity involves analysis of spread of plastic deformations in the regions where the yield condition is satisfied. Alternatively, the inelastic deformations may be considered concentrated in the plastic hinges. The latter method originates from
the analytical limit analysis of structures performed under the assumption of elastic-perfectly plastic behaviour of the material (Hodge 1959-63; Olszak and Sawczuk 1977; Sawczuk and Sokol-Supel 1993). Using the finite element method and the concept of the plastic hinges, Ueda and Yao (1982) developed a ‘Plastic Node Method’ for the plastic analysis of structures. In their formulation, the yield function is expressed in terms of stresses, as in the layered model. Shi and Voyiadjis (1992) presented a non-layered plate element with the yield function in terms of forces and moments, adopting the concept of concentration of plastic deformations in the plastic hinges. A plastic node method allows for further enhancements of the model to perform viscoplastic and damage analysis, without much complication and effort. This approach is adopted in this work, owing to its efficiency and versatility.

1.3.5 Damage in Plates and Shells

A ductile metal or structure is capable of undergoing large inelastic deformations. The plastic strains can induce the changes of the microstructure of the material, leading to its softening. These changes in the microstructure of the material are irreversible thermodynamic processes and result in the progressive degradation of the material properties (Shi and Voyiadjis 1993). The experimental investigations (Barbee et al. 1972; Seaman et al. 1971; Seaman et al. 1976) show that the softening of the material triggered by inelastic strains is mainly due to the nucleation, growth and coalescence of microvoids (sometimes thermal effects are also pronounced), (Perzyna 2005; Wray 1969). This process is called ductile plastic damage. Modeling of damage is aimed at the assessments of the influence of microvoids, microcracks and other microdamages on the degradation of material properties.

The investigations of the damage accumulation and evolution can be carried out following a micromechanical approach (micromechanical damage models) or a continuum damage theory (phenomenological damage model). The latter approach is based on the pioneering work of Kachanov (1958), who introduced the effective stress concept, as well as a scalar damage variable representing the effective surface density of microdamages per unit volume (Abu Al-Rub and Voyiadjis 2003; Venson and Voyiadjis 2001; Voyiadjis and Venson 1995). The effective stress concept involves a comparison of the actual damaged configuration with the fictitious undamaged configuration (Kachanov 1958; Voyiadjis and Kattan 1999).

Many authors used the phenomenological approach as a basis for modeling of damage (Abu Al-Rub and Voyiadjis 2003; Chaboche 1988; Krajcinovic 1979, 1984, 1989; Krajcinovic and Foneska 1981; Lemaitre 1985; Murakami 1988; Voyiadjis and Deliktas 2000a, 2000b; Voyiadjis and Kattan 1992a, 1992b, 1999; Voyiadjis and Park 1997, 1999). An isotropic scalar damage parameter, based on the concept of Kachanov is also frequently employed (Doghri 2000; Krajcinovic 1984; Krajcinovic and Foneska 1981; Lemaitre and Chaboche 1990). In this method, the stiffness of the material is reduced according to the same relation in all the directions. For a better description of the anisotropic effects, a second order damage tensor, capable of representing different levels of material degradation in different directions is often adopted (Abu Al-Rub and Voyiadjis 2003; Doghri 2000; Lubarda and Krajcinovic 1993; Murakami 1988; Seweryn
and Mroz 1998; Voyiadjis and Abu-Lebdeh 1993; Voyiadjis and Deliktas 2000a, 2000b; Voyiadjis and Kattan 1991, 1992a, 1992b, 1999; Voyiadjis and Park 1997, 1999; Voyiadjis and Venson 1995). The anisotropic damage variable poses however a problem which is not often addressed. For the appropriate depiction of directional dependency of the evolution of damage, it is necessary to determine the material constants, which define the evolution laws in different directions. Extensive experimental data is needed to calibrate these constants with sufficient accuracy and consistency. The isotropic damage formulation requires determination of fewer constants (two in the case of the current analysis), while in the same time it delivers accurate results for a variety of applications. Moreover, it would be unrealistic to include in the description all effects observed experimentally. For the current work, concerning the investigation of behaviour of isotropic plates and shells, the isotropic scalar parameter in representation of damage is deemed satisfactory. The validity of this assumption will be later tested by numerical examples.

Micromechanical damage models are based on the observations of the material at the microscale. The observations of ductile fracture in metals (Beachem 1963; Gurland and Plateau 1963) led to a conclusion that this process may involve generation of considerable porosity through nucleation and growth of voids (Gurson 1977). Gurson developed a mathematical model (1975, 1977) describing the damage effects through the evolution of porosity, which was incorporated into the yield function. He investigated a yield criterion and flow rule for porous ductile materials. Various modifications of Gurson’s formulation appeared later in the literature (Tvergaard and Needleman 1984), as well as the articles discussing the model (Li 2000; Mahnken 2002). Further investigations of ductile fracture aimed at explanation of the formation of white-etching bands, commonly referred to as shear bands. The general conclusion from the experimental results by Giovanola (1988) was that the thermomechanical strain localization and micro-damage mechanisms become the main cooperative phenomena responsible for adiabatic shear band formation and localized fracture (Perzyna 2005). Based on the microscopic observations of the shear bands (Cho et al. 1989) it was found that fracture preceded by the shear band formation, occurred through nucleation, growth and coalescence of voids. The extensive study of the shear bands and fracture phenomena, followed by the development of micro-damage model by means of the porosity function, was performed by Duszek-Perzyna (Duszek-Perzyna and Perzyna 1988a, 1988b, 1991, 1993, 1994, 1998; Duszek-Perzyna et al. 1997) and Perzyna (Perzyna 1982, 1984a, 1984b, 1985, 1986, 1990, 1994, 1998, 2001, 2004; Perzyna and Drabik 1989; Perzyna and Korbel 1998).

Duszek-Perzyna and Perzyna presented a theoretical formulation for the description of the intrinsic micro-damage process through evolution of the isotropic scalar damage variable, i.e. the porosity parameter (Duszek-Perzyna and Perzyna 1994). Similarly to the Gurson’s model (1975, 1977), the porosity variable is incorporated into the yield function, obtaining a consistent and convenient procedure for elastic-viscoplastic, damage analysis of ductile solids, with a coupling between plasticity and damage. The evolution of porosity reduced to a rate independent case, consists of three terms responsible for the cracking of the second phase particles, debonding of the second phase particles from the matrix material, and the void growth assumed to be controlled
only by plastic flow phenomena. The first term (cracking of the second-phase particles) is only dependent on the stress, which allows for variation of damage, even without the occurrence of the plastic flow. This makes the formulation universal and capable of describing correctly the material behaviour under all loading conditions, including the hydrostatic stress.

In the current dissertation damage is formulated within the framework of the micromechanical damage model. The scalar porosity parameter defined by Duszek-Perzyna and Perzyna (1994) is used to describe damage effects in shells. We only consider a rate independent case here. The evolution of porosity, which as previously mentioned, accounts for the cracking of the second phase particles, debonding of the second phase particles from the matrix material, and the void growth is reduced to represent the void growth only, as the most important phenomenon for isotropic plates and shells. Consequently, any damage occurring in the elastic region is neglected. The yield function given by Duszek-Perzyna and Perzyna (1994), which could be directly related to Gurson’s model is expressed in terms of the stress resultants and stress couples, similarly to Iliushin’s yield function (Iliushin 1956), following the procedure outlined by Bieniek and Funaro (1976). The yield surface derived here is very similar to the one presented by Voyiadjis and Woelke (2005), with kinematic hardening parameters in the form of residual normal and shear forces, and residual bending moments. It is however enhanced to account also for damage effects, leading to a reduction of stiffness, by means of the porosity parameter.

A plastic node method (Ueda and Yao 1982) is employed once again to derive the large rotation, elasto-plastic, damage tangent stiffness matrix. Since the elastic stiffness matrix is derived explicitly, with all the integrals calculated analytically, and a non-layered model is employed in which integration through the thickness is not necessary, the current stiffness matrix is also given explicitly and numerical integration is not performed. Such approach is consistent mathematically and very inexpensive from the point of view of computer time and power. Its accuracy for variety of applications is tested in the following chapters.
2.1 Introduction

A refined theory of thick spherical shells is derived in this chapter. The basic assumptions and the review of published work pertaining to the subject of thick plates and shells were given in Section 1.3.1. The most important features of the present theory are discussed here in further detail.

In the present theory, we employ the following hypothesis: plane sections originally perpendicular to the middle surface remain plane after the deformation but not perpendicular to the middle surface (Figure 1.3). From this hypothesis, we can deduce that the displacements $u$ and $v$ along $x$ and $y$ directions are:

\[
\begin{align*}
    u &= -z\phi_x \quad \text{and} \quad v = -z\phi_y
\end{align*}
\]

where $\phi_x$ and $\phi_y$ are the angles of rotation of the sections originally perpendicular to the middle section, in the $xz$ and $yz$ planes respectively, given by:

\[
\begin{align*}
    \phi_x &= \frac{\partial w}{\partial x} - \gamma_{xz} \quad \text{and} \quad \phi_y &= \frac{\partial w}{\partial y} - \gamma_{yz}
\end{align*}
\]

where $w$ is the vertical displacement in the $z$ direction and $\gamma_{xz}, \gamma_{yz}$ are the transverse shear strains in the $xz$ and $yz$ planes respectively. Shells in which the ratio of the thickness to the radius of curvature is equal or less than $1/50$ are most often considered thin. In the case of thin shells, transverse shear strains are negligible. This is true for most of the boundary conditions. There are some types of loading conditions however, which cause significant shear forces, regardless of the thickness of the structure. An example of such a loading condition may be a concentrated bending moment applied in the midspan of the beam, plate or shell (Figure 2.1).

![Figure 2.1 Concentrated couples formed by: a) The vertical forces causing significant shear forces, and; b) The horizontal forces - no significant shear forces; (Hu 1984)](image_url)
We recognize that there are two different ways of applying the concentrated bending moment. It can be formed by the vertical force couple (Figure 2.1.a), or by the horizontal force couple (Figure 2.1.b) In the former case, a very large shear force is generated at the midspan of the beam. This force increases as the distance of the force couple decreases. For the correct representation of the deformation of the beam, transverse shear strains need to be considered in this case, regardless of the thickness of the beam. We encounter the same situation if such loading conditions are applied to plates and shells.

We also include the radial stresses (the transverse normal stresses) in the formulation. They may become very important for certain engineering applications, for instance pressure vessels. The derivation given here is based on the assumed out-of-plane stress components that satisfy given traction boundary conditions. These stress components are established by means of the analytical investigation of the distribution of the radial stresses in thick spherical containers subjected to internal and external pressure, performed by Lame (1852). Thus, the distribution of the radial stresses is the starting point for the derivation.

In the proposed shell theory, all the in-plane stresses exhibit a non-linear distribution through the thickness. This is primarily due to the incorporation of the initial curvature effect in the theoretical formulation of the proposed shell theory (Figure 1.4). The nonlinear stress expressions given here are compared for specific examples to those obtained through the three-dimensional theory of elasticity.

The shell equations derived in this chapter are presented in several stages. After the introduction, the solution of the thick spherical container subjected to uniform pressure is given, which along with the assumed out-of-plane stress components provide a base for the formulation. Using the stress-strain and strain-displacement relationships, the displacement field is determined. Having the displacements, the remaining components of the strain and stress tensors are obtained. From the stresses, we calculate the stress resultant and stress couples, accounting for the initial curvature effect. In Section 2.2.5, the average displacements of the shell are defined, for identifying the proper boundary conditions. The stress resultants and stress couples are then expressed in terms of the average displacements of the shell. The current theory can be easily reduced to the equivalent plate theory, which is given in Section 2.3. Finally, the numerical examples verifying the reliability of the proposed shell constitutive equations are presented.

2.2 Theoretical Formulation of Shell Equations

2.2.1 Assumed Out-of-Plane Stress Components

We consider a thick spherical container subjected to uniform external and internal pressures, as shown in Figure 2.2. This problem was analyzed by Lame (1852), who through the analytical calculations obtained an expression for the radial stresses in the spherical vessel shown in Figure 2.2. The radial stress distribution for thick spheres subjected to constant radial loads at both surfaces $z = -h/2$ and $z = -h/2$, (Figure 2.3) is given by equation (2.3).
Figure 2.2 A spherical container under uniform pressures
(Voyiadjis and Woelke, 2004)

\[ \sigma_z = \left( \frac{r_2}{r} \right)^3 - 1 \frac{c_1}{p_i} + \left( \frac{r_1}{r} \right)^3 - 1 \frac{c_2}{p_o} \]  \hspace{1cm} (2.3)

where:

\[ c_1 = 1 - \left( \frac{r_2}{r_1} \right)^3 \quad \text{and} \quad c_2 = \left( \frac{r_1}{r_2} \right)^3 - 1 \]  \hspace{1cm} (2.4)

and:

\[ r = R + z \]  \hspace{1cm} (2.5)

Based on the solution given by the equations (2.3) and (2.4), the following out-of-plane stress components are assumed:

\[ \tau_{\theta z} = \left( 1 + \frac{z}{R} \right) \frac{3Q_\theta}{2h} \left[ 1 - \left( \frac{2z}{h} \right)^2 \right] + \frac{(r_2/r)^3 - 1}{c_1} p_{\theta i} + \frac{(r_1/r)^3 - 1}{c_2} p_{\theta o} \]  \hspace{1cm} (2.6)

\[ \tau_{\phi z} = \left( 1 + \frac{z}{R} \right) \frac{3Q_\phi}{2h} \left[ 1 - \left( \frac{2z}{h} \right)^2 \right] + \frac{(r_2/r)^3 - 1}{c_1} p_{\phi i} + \frac{(r_1/r)^3 - 1}{c_2} p_{\phi o} \]  \hspace{1cm} (2.7)

\[ \sigma_z \] - radial stresses;

\[ p_i, p_o \] - distributed radial loads on the inner and outer surfaces respectively \((z = -h/2 \text{ and } z = h/2)\);

\[ r_1, r_2 \] - radius of curvature of the inner and outer surface respectively (Figure 2.3);

\( r \) - radius of curvature of the plane away from the middle plane

\( R \) - radius of curvature of the mid-plane (Figure 2.3);
where:
$\tau_{\theta z}, \tau_{\phi z}$ - transverse shear stresses (first subscripts - $\theta$ and $\phi$ denote the direction of the normal to the plane on which stresses are acting; second subscripts - $z$ denote the direction of the stresses);
$p_{\theta i}, p_{\theta o}$ - distributed loads along the $\theta$ direction, on the inner and outer surfaces respectively;
$p_{\phi i}, p_{\phi o}$ - distributed loads along the $\phi$ direction;
$Q_{\theta}, Q_{\phi}$ - transverse shear forces;
$h$ - thickness of the shell;

Equations (2.6) and (2.7) express the assumed transverse shear stresses. The first term in both of the equations depicts the transverse shear stresses calculated for the plate cross section, modified by the term $(1 + z/R)$, which accounts for the fact that the cross section is not rectangular, but exhibits curvature. We notice that the modification applied here to account for the initial curvature is different than the one most commonly used, i.e. $(1 - z/R)$, (see Ugural 1981). This is due to a different orientation of the $z$ axis, which points outwards here. The last two terms from the equations (2.6) and (2.7) are assumed such that the stresses satisfy the boundary conditions. This is achieved through employing similar functions to those representing the distribution of the radial stresses in equation (2.3). The assumed stress field (equations (2.3)-(2.7)) satisfies the following boundary conditions:
\[
\begin{align*}
\sigma_z &= p_o \quad \text{at} \quad z = h/2 \\
\tau_{z\phi} &= p_{\phi o} \quad \text{at} \quad z = h/2 \\
\tau_{z\theta} &= p_{\theta o} \quad \text{at} \quad z = h/2 \\
\sigma_z &= -p_i \quad \text{at} \quad z = -h/2 \\
\tau_{z\phi} &= -p_{\phi i} \quad \text{at} \quad z = -h/2 \\
\tau_{z\theta} &= -p_{\theta i} \quad \text{at} \quad z = -h/2
\end{align*}
\] (2.8)

### 2.2.2 Displacement Field

We use the three dimensional elasticity constitutive equations to derive the displacement field. Using Hooke’s law for a linear elastic material, we obtain the transverse normal strain \( \varepsilon_z \) in terms of the stresses as follows:

\[
\varepsilon_z = \frac{1}{E} [\sigma_z - \nu (\sigma_\theta + \sigma_\phi)]
\] (2.9)

where \( E \) is the elastic modulus, \( \nu \) is the Poisson’s ratio and \( \sigma_\theta \) and \( \sigma_\phi \) are normal stresses in the \( \theta \) and \( \phi \) directions, respectively.

The sum of \( (\sigma_\theta + \sigma_\phi) \) can be written as indicated below:

\[
\sigma_\theta + \sigma_\phi = \frac{12(M_\theta + M_\phi)z}{h^3}
\] (2.10)

Equation (2.10) was first used by Reissner (1975) to modify the expression for the transverse displacement \( w \). It can be easily proved using the definition of the stresses expressed in terms of the transverse displacement \( w \). Substituting expressions (2.3) and (2.10) into equation (2.9) we obtain:

\[
\varepsilon_z = \frac{\partial w}{\partial z} = \frac{1}{E} \left[ \frac{(r_z r / r)^3 - 1}{c_1} p_i + \frac{(r_z r / r)^3 - 1}{c_2} p_o - \frac{12 \nu}{h^3} (M_\theta + M_\phi)z \right]
\] (2.11)

Integrating equation (2.11) with respect to \( z \) yields the following expression for the displacement \( w \):

\[
w(\theta, \phi, z) = w_0(\theta, \phi) + \frac{1}{E} \int \left[ \frac{(r_z r / r)^3 - 1}{c_1} p_i + \frac{(r_z r / r)^3 - 1}{c_2} p_o - \frac{12 \nu}{h^3} (M_\theta + M_\phi)z \right] dz
\] (2.12)

Denoting:

\[
M = (M_\theta + M_\phi)
\] (2.13)

and representing \( 1/(R + z) \) as a power series:

\[
\frac{1}{R + z} = \frac{1}{R} - \frac{z}{R^2} + \frac{z^2}{R^3} - \ldots
\] (2.14)
we have:

\[
\begin{align*}
  w(\theta, \phi, z) &= w_0(\theta, \phi) + \frac{1}{E} \left\{ \frac{p_o}{c_1} \left[-z + \frac{r_1^3}{R^3} \left(z - \frac{3z^2}{2R}\right)\right] + \\
  &+ \frac{p_o}{c_2} \left[-z + \frac{r_1^3}{R^3} \left(z - \frac{3z^2}{2R}\right)\right] v \frac{6z^2}{h^3} M \right\} \\
\end{align*}
\]

In the classical theory of bending of thin shells, the term \( z/R \) and its higher order terms are neglected. In the present formulation, the term \( z/R \) is retained, but all higher order terms are neglected. Equation (2.15) is the resulting expression for \( w(\theta, \phi, z) \).

In order to obtain consistent assumptions for the displacements \( u(\theta, \phi, z) \) and \( v(\theta, \phi, z) \), the following strain-displacement relations are used:

\[
\begin{align*}
  \frac{\partial \nu}{\partial z} + \frac{1}{R + z} \frac{\partial w}{\partial \phi} - \nu = \gamma_{\phi z} &= \frac{\tau_{\phi z}}{G} \\
  \frac{1}{(R + z)\sin \phi} \frac{\partial u}{\partial \theta} + \frac{u}{(R + z)} = \gamma_{\theta z} &= \frac{\tau_{\theta z}}{G}
\end{align*}
\]

where \( u, v, w \) are the displacements along \( \theta, \phi, z \) axes respectively. Multiplying both sides of the equations (2.16) and (2.17) by \( 1/(R + z) \) we obtain:

\[
\begin{align*}
  \frac{1}{(R + z)} \frac{\partial v}{\partial z} - \frac{\nu}{(R + z)} = \frac{1}{(R + z)} \left( \frac{\tau_{\phi z}}{G} - \frac{1}{R + z} \frac{\partial w}{\partial \phi} \right) \\
  \frac{1}{(R + z)} \frac{\partial u}{\partial z} - \frac{u}{(R + z)} = \frac{1}{(R + z)} \left( \frac{\tau_{\theta z}}{G} - \frac{1}{(R + z)\sin \phi} \frac{\partial w}{\partial \theta} \right)
\end{align*}
\]

The left hand side of both of the above equations may be rewritten:

\[
\begin{align*}
  \frac{\partial}{\partial z} \left( \frac{\nu}{R + z} \right) &= \frac{1}{(R + z)} \left( \frac{\tau_{\phi z}}{G} - \frac{1}{R + z} \frac{\partial w}{\partial \phi} \right) \\
  \frac{\partial u}{\partial z} \left( \frac{u}{R + z} \right) &= \frac{1}{(R + z)} \left( \frac{\tau_{\theta z}}{G} - \frac{1}{(R + z)\sin \phi} \frac{\partial w}{\partial \theta} \right)
\end{align*}
\]

or:

\[
\begin{align*}
  v = (R + z)^{h/2} \int_{-h/2}^{h/2} \frac{1}{(R + z)} \left( \frac{\tau_{\phi z}}{G} - \frac{1}{R + z} \frac{\partial w}{\partial \phi} \right) dz \\
  u = (R + z)^{h/2} \int_{-h/2}^{h/2} \frac{1}{(R + z)} \left( \frac{\tau_{\theta z}}{G} - \frac{1}{(R + z)\sin \phi} \frac{\partial w}{\partial \theta} \right) dz
\end{align*}
\]

Solution of the above equation in the current form will produce the logarithmic terms, which are cumbersome. In order to avoid these terms, we replace the term \( 1/(R + z) \) in the equations (2.22) and (2.23) with the power series, given by equation (2.14). Substituting for the appropriate shearing stress from expressions (2.6) and (2.7) into
equations (2.22) and (2.23), and integrating both expressions with respect to \( z \), we obtain the remaining components of the displacement field:

\[
\begin{align*}
    u(\theta, \phi, z) &= \left(1 + \frac{z}{R}\right) \left\{u_0(\theta, \phi) + \frac{Q_\phi}{2Gh} z \left[3 - \frac{4z^2}{h^2}\right] - \frac{1}{R \sin \phi} \frac{\partial w_0}{\partial \theta} \left(z - \frac{z^2}{R}\right) + \frac{2\nu}{Eh^3} \frac{1}{R \sin \phi} \frac{\partial M}{\partial \theta} z^3 \left(1 - \frac{3z}{2R}\right) - \right.
    \\
    &\left. - \frac{1}{Ec_1 R \sin \phi} \frac{\partial p_i}{\partial \theta} \left[\frac{z^2}{2} + \frac{2z^3}{3R} + \frac{r_i^3}{R^3} \left(\frac{z^2}{2} - \frac{7z^3}{6R}\right)\right] \right\} \\
    v(\theta, \phi, z) &= \left(1 + \frac{z}{R}\right) \left\{v_0(\theta, \phi) + \frac{Q_\phi}{2Gh} z \left[3 - \frac{4z^2}{h^2}\right] - \frac{1}{R} \frac{\partial w_0}{\partial \phi} \left(z - \frac{z^2}{R}\right) + \frac{2\nu}{Eh^3} \frac{1}{R} \frac{\partial M}{\partial \phi} z^3 \left(1 - \frac{3z}{2R}\right) - \right.
    \\
    &\left. - \frac{1}{Ec_1 R} \frac{\partial p_i}{\partial \phi} \left[\frac{z^2}{2} + \frac{2z^3}{3R} + \frac{r_i^3}{R^3} \left(\frac{z^2}{2} - \frac{7z^3}{6R}\right)\right] \right\}
\end{align*}
\]

(2.24)

In the shell theory that follows the variations of the distributed loads \( p_{\phi i}, p_{\phi o}, p_{bi}, p_{bo} \) are omitted for simplicity and conciseness. The reader may choose to include them by following the procedure outlined below.

### 2.2.3 Stress Components

In order to obtain the remaining stress components, the following three-dimensional stress-strain relationships are used:

\[
\begin{align*}
    \sigma_\theta &= \frac{E}{1 - \nu^2} \left[\varepsilon_\theta + \nu \varepsilon_\phi\right] + \frac{\nu}{1 - \nu} \sigma_z \\
    \sigma_\phi &= \frac{E}{1 - \nu^2} \left[\varepsilon_\phi + \nu \varepsilon_\theta\right] + \frac{\nu}{1 - \nu} \sigma_z
\end{align*}
\]

(2.26)
together with the following strain-displacement relations:

$$
\varepsilon_\theta = \frac{1}{(R+z) \sin \phi \partial \theta} \frac{\partial u}{\partial \theta} + \frac{v}{(R+z)} \operatorname{ctg} \phi + \frac{w}{R+z} \tag{2.29}
$$

$$
\varepsilon_\phi = \frac{1}{(R+z) \partial \phi} \frac{\partial v}{\partial \phi} + \frac{w}{R+z} \tag{2.30}
$$

$$
\gamma_{\theta \phi} = \frac{1}{(R+z) \sin \phi \partial \theta} \frac{\partial u}{\partial \phi} + \frac{1}{(R+z) \partial \phi} \frac{\partial u}{\partial \theta} - \frac{u}{R} \operatorname{ctg} \phi \tag{2.31}
$$

Substituting for the displacements $u, v$ and $w$ from equations (2.15), (2.24) and (2.25) respectively, into expressions (2.29), (2.30), (2.31) and substituting the resulting strains into equations (2.26), (2.27), (2.28), we obtain the following expression for the normal stresses in the $\theta$ and $\phi$ directions respectively:

$$
\sigma_\theta = \frac{E}{1-\nu^2} + \frac{1}{R \sin \phi} \frac{\partial u_0}{\partial \theta} + \frac{\cos \phi}{R} \frac{\partial v_0}{\partial \phi} \tag{2.32}
$$

where:

$$
\Delta_i^2 = \frac{1}{R \sin^2 \phi} \frac{\partial^2}{\partial \theta^2} + \frac{\cos \phi}{R \sin \phi} \frac{\partial}{\partial \phi} + \frac{v}{R} \frac{\partial^2}{\partial \phi^2} \tag{2.33}
$$
\[ \sigma_\phi = \frac{E}{1-\nu^2} \left\{ \frac{1}{R \sin \phi} \frac{\partial u_0}{\partial \theta} + \nu \cos \phi \frac{v_0}{R \sin \phi} + \frac{1}{R} \frac{\partial v_0}{\partial \phi} + \right. \\
+ \frac{z}{2Gh} \left[ 3 - \frac{4z^2}{h^2} \right] \left[ \frac{\nu}{R \sin \phi} \frac{\partial Q_\theta}{\partial \theta} + \nu \cos \phi \frac{Q_\phi}{R \sin \phi} + \frac{1}{R} \frac{\partial Q_\phi}{\partial \phi} \right] \\
+ \Delta_2^2 \left[ -\frac{1}{R} \left( z - \frac{z^2}{R} \right) w_0 + \frac{2\nu}{Eh^3} \frac{1}{R} z^3 \left( 1 - \frac{3z}{2R} \right) \right] M - \\
- \frac{1}{E_1} \left[ -\frac{z^2}{2} + \frac{2z^3}{3R} + \frac{r_3^2}{R^3} \left( z - \frac{7z^3}{6R} \right) \right] p_i + \frac{1}{E_2} \left[ -\frac{z^2}{2} + \frac{2z^3}{3R} + \frac{r_3^2}{R^3} \left( z - \frac{7z^3}{6R} \right) \right] p_o + \\
+ \nu p_i \cos \phi \frac{Gc_i \sin \phi}{R} \left[ -z + \frac{z^2}{2R} + \frac{r_3^2}{2R} \left( z - \frac{2z^2}{R} \right) \right] + \\
+ \frac{1+\nu}{R} \left[ w_0 + \frac{p_i}{E_1} \left[ -z + \frac{r_3^2}{R^3} \left( z - \frac{3z^2}{2R} \right) \right] + \\
+ \frac{p_o}{E_2} \left[ -z + \frac{r_3^2}{R^3} \left( z - \frac{3z^2}{2R} \right) \right] - \nu \frac{6z^2}{Eh^3} M \right]\right\} \\
+ \nu \left[ \frac{p_i}{c_1} \left( \frac{r_3^2}{(R+z)^3} - 1 \right) + \frac{p_o}{c_2} \left( \frac{r_3^2}{(R+z)^3} - 1 \right) \right]
\]

(2.34)

where:

\[ \Delta_2^2 = \frac{\nu}{R \sin^2 \phi} \frac{\partial^2}{\partial \theta^2} + \nu \cos \phi \frac{\partial}{R \sin \phi} \frac{\partial}{\partial \phi} + \frac{1}{R} \frac{\partial^2}{\partial \phi^2} \] 

(2.35)

The shear stresses in the \( \theta \phi \) -plane are as follows:

\[ \tau_{\theta \phi} = G \left\{ \frac{1}{R \sin \phi} \frac{\partial v_0}{\partial \theta} + \frac{1}{R} \frac{\partial u_0}{\partial \phi} - u_0 \cos \phi \frac{v_0}{R \sin \phi} + \right. \\
+ \frac{z}{2Gh} \left[ 3 - \frac{4z^2}{h^2} \right] \left[ \frac{1}{R \sin \phi} \frac{\partial Q_\theta}{\partial \theta} + \frac{1}{R} \frac{\partial Q_\phi}{\partial \phi} - \cos \phi \frac{Q_\theta}{R \sin \phi} \right] \right\} + \\
+ \Delta_3^2 \left[ -\frac{1}{R} \left( z - \frac{z^2}{R} \right) w_0 + \frac{2\nu}{Eh^3} \frac{1}{R} z^3 \left( 1 - \frac{3z}{2R} \right) \right] M - \\
- \frac{1}{E_1} \left[ -\frac{z^2}{2} + \frac{2z^3}{3R} + \frac{r_3^2}{R^3} \left( z - \frac{7z^3}{6R} \right) \right] p_i + \frac{1}{E_2} \left[ -\frac{z^2}{2} + \frac{2z^3}{3R} + \frac{r_3^2}{R^3} \left( z - \frac{7z^3}{6R} \right) \right] p_o + \\
+ \frac{p_i}{Gc_i \sin \phi} \cos \phi \left[ -z + \frac{z^2}{2R} + \frac{r_3^2}{2R} \left( z - \frac{2z^2}{R} \right) \right] + \frac{p_o}{Gc_i \sin \phi} \cos \phi \left[ -z + \frac{z^2}{2R} + \frac{r_3^2}{R^3} \left( z - \frac{2z^2}{R} \right) \right] \right\}

(2.36)
where:

\[
\Delta_z^2 = \frac{2}{R \sin \phi} \frac{\partial^2}{\partial \theta \partial \phi} + \frac{2 \cos \phi}{R \sin^2 \phi} \frac{\partial}{\partial \theta}
\]  

(2.37)

### 2.2.4 Stress Couples and Stress Resultants on the Middle Surface

The definitions of the stress couples with the initial curvature effect taken into account are:

\[
M_\phi = -\int_{-h/2}^{h/2} \sigma_\phi z \left(1 + \frac{z}{R}\right) dz
\]  

(2.38)

\[
M_\theta = -\int_{-h/2}^{h/2} \sigma_\theta z \left(1 + \frac{z}{R}\right) dz
\]  

(2.39)

\[
M_{\theta\phi} = -\int_{-h/2}^{h/2} \tau_{\theta\phi} z \left(1 + \frac{z}{R}\right) dz
\]  

(2.40)

The positive bending moment is the one that results in positive stresses in the bottom part of the shell. We now substitute the expressions for stresses form equations (2.32), (2.34) and (2.36) into the respective relations for the stress couples to obtain:

\[
M_\phi = D \left\{ -\frac{1}{R^2 \sin \phi} \frac{\partial u_0}{\partial \theta} - \frac{\cos \phi}{R^2 \sin \phi} v_0 - \frac{\nu}{R^2} \frac{\partial v_0}{\partial \phi} \right.
\]

\[
- \frac{6}{5Gh} \left[ \frac{1}{R \sin \phi} \frac{\partial Q_\phi}{\partial \theta} + \frac{\cos \phi}{R \sin \phi} Q_\phi + \frac{\nu}{R} \frac{\partial Q_\phi}{\partial \phi} \right] +
\]

\[
+ \frac{1}{R} \Delta_i^2 w_0 + \left( \frac{9\nu h}{112ER^3} - \frac{3\nu}{10ERh} \right) \Delta_i^2 M +
\]

\[
- \frac{1}{ER^2 c_1} \left[ \frac{h^2}{24} \left( 1 + \frac{12r_3^3}{5R^2} - \frac{r_2^3}{R^3} \right) \right] \Delta_i^2 p_i -
\]

\[
- \frac{1}{ER^2 c_2} \left[ \frac{h^2}{24} \left( 1 + \frac{12r_3^3}{5R^2} - \frac{r_1^3}{R^3} \right) \right] \Delta_i^2 p_o +
\]

\[
+ \frac{p_\phi \cos \phi}{R \sin \phi} Gc_1 \left[ 1 - \frac{r_3^3}{R^3} - \frac{3h^2}{40R^2} \right] + \frac{p_\phi \cos \phi}{R \sin \phi} Gc_2 \left[ 1 - \frac{r_1^3}{R^3} \right] + \frac{3h^2}{40R^2} +
\]

\[
+ \frac{1 + \nu}{ER} \left[ \frac{p_i}{c_1} \left[ 1 + \nu - \frac{r_3^2}{R^3}(1-2\nu) \right] + \frac{p_o}{c_2} \left[ 1 + \nu - \frac{r_3^2}{R^3}(1-2\nu) \right] \right].
\]  

(2.41)
\[ M_\phi = D \left\{ - \frac{v}{R^2 \sin \phi} \frac{\partial u_0}{\partial \theta} - \frac{v \cos \phi}{R^2 \sin \phi} \frac{v_0}{\partial \phi} - \frac{1}{R^2} \frac{\partial v_0}{\partial \phi} - \right. \]
\[ - \frac{6}{5Gh} \left[ \frac{v}{R \sin \phi} \frac{\partial Q_\theta}{\partial \theta} + \frac{v \cos \phi}{R \sin \phi} Q_\theta + \frac{1}{R} \frac{\partial Q_\phi}{\partial \phi} \right] + \]
\[ + \frac{1}{R} \Delta_2^2 w_0 + \left( \frac{9v_0}{112ER^3} - \frac{3v}{10ERh} \right) \Delta_2^2 M + \]
\[ - \frac{1}{ER^2 c_1} \left[ \frac{h^2}{24} \left( 1 + \frac{12 r_2^3}{5 R^2} - \frac{r_2^3}{R^3} \right) \right] \Delta_2^2 p_i - \]
\[ - \frac{1}{ER^2 c_2} \left[ \frac{h^2}{24} \left( 1 + \frac{12 r_3^3}{5 R^2} - \frac{r_3^3}{R^3} \right) \right] \Delta_2^2 p_o + \]
\[ + \frac{v p_\phi \cos \phi}{R \sin \phi} \frac{1}{Gc_1} \left[ 1 - \frac{r_2^3}{R^3} - \frac{3h^2}{40R^2} \right] + \frac{v p_\phi \cos \phi}{R \sin \phi} \frac{1}{Gc_2} \left[ 1 - \frac{r_3^3}{R^3} - \frac{3h^2}{40R^2} \right] + \]
\[ + \frac{1 + \nu}{ER} \left[ \frac{p_i}{c_1} \left[ 1 + \nu - \frac{r_2^3}{R^3}(1 - 2\nu) \right] + \frac{p_o}{c_2} \left[ 1 + \nu - \frac{r_3^3}{R^3}(1 - 2\nu) \right] \right] \right\} \] (2.42)

\[ M_{\phi \theta} = D \frac{1 - \nu}{2} \left\{ - \frac{1}{R^2 \sin \phi} \frac{\partial v_0}{\partial \theta} - \frac{1}{R^2} \frac{\partial u_0}{\partial \phi} - \frac{\cos \phi}{R^2 \sin \phi} u_0 - \right. \]
\[ - \frac{6}{5Gh} \left[ \frac{1}{R \sin \phi} \frac{\partial Q_\phi}{\partial \theta} + \frac{1}{R} \frac{\partial Q_\phi}{\partial \phi} - \right. \]
\[ \left. \cos \phi \frac{Q_\phi}{R \sin \phi} \right] + \]
\[ + \frac{1}{R} \Delta_3^2 w_0 + \left( \frac{9v_0}{112ER^3} - \frac{3v}{10ERh} \right) \Delta_3^2 M + \]
\[ - \frac{1}{ER^2 c_1} \left[ \frac{h^2}{24} \left( 1 + \frac{12 r_2^3}{5 R^2} - \frac{r_2^3}{R^3} \right) \right] \Delta_3^2 p_i - \]
\[ - \frac{1}{ER^2 c_2} \left[ \frac{h^2}{24} \left( 1 + \frac{12 r_3^3}{5 R^2} - \frac{r_3^3}{R^3} \right) \right] \Delta_3^2 p_o + \]
\[ + \frac{p_\theta \cos \phi}{R \sin \phi} \frac{1}{Gc_1} \left[ 1 - \frac{r_2^3}{R^3} - \frac{3h^2}{40R^2} \right] + \frac{p_\theta \cos \phi}{R \sin \phi} \frac{1}{Gc_2} \left[ 1 - \frac{r_3^3}{R^3} - \frac{3h^2}{40R^2} \right] \right\} \] (2.43)

Substituting for the stresses \( \sigma_{\phi}, \tau_{\phi \theta}, \tau_{\phi \phi} \) from the equations (2.32), (2.34) and (2.36) into the definitions of the stress resultants:

\[ N_\phi = \int_{-h/2}^{h/2} \sigma_{\phi} \left( 1 + \frac{z}{R} \right) dz \] (2.44)

\[ N_{\phi \theta} = \int_{-h/2}^{h/2} \sigma_{\phi \theta} \left( 1 + \frac{z}{R} \right) dz \] (2.45)
we obtain the following expression for the normal force in the $\theta$ direction:

$$N_\theta = \frac{Eh}{1 - \nu^2} \left\{ \frac{1}{R \sin \phi} \frac{\partial u_0}{\partial \theta} + \frac{\cos \phi}{R \sin \phi} v_0 + \frac{\nu}{R} \frac{\partial v_0}{\partial \phi} + \frac{h}{10GR} \left[ \frac{1}{R \sin \phi} \frac{\partial Q_\phi}{\partial \theta} + \frac{\cos \phi}{R \sin \phi} Q_\phi + \frac{\nu}{R} \frac{\partial Q_\phi}{\partial \phi} \right] \right\}$$

$$+ \frac{1}{Ec_1} \left[ \frac{h^2}{24R} \left( 1 - \frac{r_3^2}{R^3} \right) \right] \Delta_1^2 p_i + \frac{1}{Ec_2} \left[ \frac{h^2}{24R} \left( 1 - \frac{r_3^2}{R^3} \right) \right] \Delta_1^2 p_o - \frac{p_\phi \cos \phi}{\sin \phi} \frac{1}{Gc_1} \left[ \frac{h^2}{24R} \left( 1 + 2 \frac{r_3^3}{R^3} \right) \right] - \frac{p_\phi \cos \phi}{\sin \phi} \frac{1}{Gc_2} \left[ \frac{h^2}{24R} \left( 1 + 2 \frac{r_1^3}{R^3} \right) \right] +$$

$$+ \frac{1 + \nu}{R} w_0 + \frac{9v h}{112ER^3} - \frac{3v}{10ERh} \left( 1 + \nu \right) M + \frac{(1 + \nu) h^2}{ER} \left[ \frac{p_i}{c_1} \left( 1 - \frac{r_3^3}{10R^3} \right) + \frac{p_o}{c_2} \left( 1 - \frac{r_1^3}{10R^3} \right) \right] \right\}$$

The normal force in the $\phi$ direction is:

$$N_\phi = \frac{Eh}{1 - \nu^2} \left\{ \frac{\nu}{R \sin \phi} \frac{\partial u_0}{\partial \phi} + \frac{\nu \cos \phi}{R \sin \phi} v_0 + \frac{1}{R} \frac{\partial v_0}{\partial \phi} + \frac{h}{10GR} \left[ \frac{\nu}{R \sin \phi} \frac{\partial Q_\phi}{\partial \phi} + \frac{\nu \cos \phi}{R \sin \phi} Q_\phi + \frac{1}{R} \frac{\partial Q_\phi}{\partial \phi} \right] \right\}$$

$$+ \frac{1}{Ec_1} \left[ \frac{h^2}{24R} \left( 1 - \frac{r_3^2}{R^3} \right) \right] \Delta_2^2 p_i + \frac{1}{Ec_2} \left[ \frac{h^2}{24R} \left( 1 - \frac{r_3^2}{R^3} \right) \right] \Delta_2^2 p_o - \frac{p_\phi \nu \cos \phi}{\sin \phi} \frac{1}{Gc_1} \left[ \frac{h^2}{24R} \left( 1 + 2 \frac{r_3^3}{R^3} \right) \right] - \frac{p_\phi \nu \cos \phi}{\sin \phi} \frac{1}{Gc_2} \left[ \frac{h^2}{24R} \left( 1 + 2 \frac{r_1^3}{R^3} \right) \right] +$$

$$+ \frac{1 + \nu}{R} w_0 + \frac{9v h}{112ER^3} - \frac{3v}{10ERh} \left( 1 + \nu \right) M + \frac{(1 + \nu) h^2}{ER} \left[ \frac{p_i}{c_1} \left( 1 - \frac{r_3^3}{10R^3} \right) + \frac{p_o}{c_2} \left( 1 - \frac{r_1^3}{10R^3} \right) \right] \right\}$$
The normal force in the $\theta\phi$-plane is:

\[
N_{\theta\phi} = \frac{Eh}{1-\nu^2} \left(1-\frac{1}{2}\right) \left\{ \frac{1}{R \sin \phi} \frac{\partial v_0}{\partial \theta} + \frac{1}{R \sin \phi} \frac{\partial u_0}{\partial \phi} - \cos \phi \right\} u_0 +
\]

\[
+ h \left[ \frac{1}{10GR} \frac{\partial Q_\phi}{\partial \theta} + \frac{1}{R \sin \phi} \frac{\partial Q_\theta}{\partial \phi} - \cos \phi \frac{Q_\phi}{R \sin \phi} \right] +
\]

\[
+ \frac{1}{E_c_1} \left[ \frac{h^2}{24R} \left(1 - \frac{r_3}{R^3}\right) \right] \Delta_2 p_l + \frac{1}{E_c_2} \left[ \frac{h^2}{24R} \left(1 - \frac{r_3}{R^3}\right) \right] \Delta_3^2 p_o -
\]

\[
- \frac{p_{\phi}}{\sin \phi} \frac{1}{G_c_1} \left[ \frac{h^2}{24R^2} \left(1 + 2 \frac{r_3}{R^3}\right) \right] - \frac{p_{\phi}}{\sin \phi} \frac{1}{G_c_2} \left[ \frac{h^2}{24R^2} \left(1 + 2 \frac{r_3}{R^3}\right) \right].
\]

(2.49)

### 2.2.5 Average Displacements $\bar{u}, \bar{v}, \bar{w}$ and Rotations $\phi_0, \phi_\phi$

For identifying the appropriate boundary conditions for the derived shell theory, average displacements $\bar{u}, \bar{v}, \bar{w}$, and rotations $\phi_0, \phi_\phi$ are introduced. The rotations are for sections $\theta = \text{const}$ and $\phi = \text{const}$, respectively. We first define the transverse shear resultants as:

\[
Q_0 = T \gamma_{\theta z}
\]

\[
Q_\phi = T \gamma_{\phi z}
\]

(2.50)

(2.51)

where $T$ is given by:

\[
T = \frac{5}{6} Gh
\]

(2.52)

and $\gamma_{\theta z}, \gamma_{\phi z}$ are expressed similarly to equations (2.16) and (2.17):

\[
\gamma_{\theta z} = \frac{1}{(R + z) \sin \phi} \frac{\partial \bar{w}}{\partial \theta} + \frac{\partial \bar{u}}{\partial \phi} - \frac{\bar{u}}{(R + z)}
\]

(2.53)

\[
\gamma_{\phi z} = \frac{\partial \bar{v}}{\partial \phi} + \frac{1}{(R + z)} \frac{\partial \bar{w}}{\partial \phi} - \frac{\bar{v}}{(R + z)}
\]

(2.54)

The average transverse displacement $\bar{w}$ is obtained by equating the work of the transverse shear stress $\tau_{\phi z}$ due to the displacement $w$ to the work of the transverse shear resultant $Q_\phi$ due to the average displacement $\bar{w}$ (Voyiadjis and Baluch, 1981, Hu 1984).

\[
\int_{-h/2}^{h/2} \tau_{\phi z} w \left(1 + \frac{z}{R}\right) dz = Q_\phi \bar{w}
\]

(2.55)

One could choose to equate the work of the transverse shear stress $\tau_{\theta z}$ due to the displacement $w$ to the work of the transverse shear resultant $Q_\theta$ due to the average displacement $\bar{w}$ instead, which yields the same resulting expression for $\bar{w}$, given by:
\[ \bar{w} = w_0 - M \left( \frac{3v}{10Eh} - \frac{9vfh}{112ER^2} \right) - \frac{1}{10} \frac{h^2}{R} \frac{r_2}{R^2} P_i - \frac{1}{10} \frac{h^2}{R} \frac{r_1}{R^3} P_o \]  

(2.56)

Similarly, to obtain \( \bar{u}, \bar{v}, \phi_\theta, \phi_\phi \) we use the following equations:

\[
\int_{-h/2}^{h/2} \sigma_\theta u \left( 1 + \frac{z}{R} \right) dz = N_\theta \bar{u} + M_\phi \phi_\phi
\]  

(2.57)

\[
\int_{-h/2}^{h/2} \sigma_\phi v \left( 1 + \frac{z}{R} \right) dz = N_\phi \bar{v} + M_\phi \phi_\phi
\]  

(2.58)

The resulting expressions for \( \bar{u}, \bar{v}, \phi_\theta, \phi_\phi \) are given by:

\[
\bar{u} = u_0 + \frac{1}{ER \sin \phi} \frac{h^2}{24} \left[ \frac{1}{c_1} \frac{\partial p_i}{\partial \theta} \left( 1 - \frac{r_2^3}{R^3} \right) + \frac{1}{c_2} \frac{\partial p_o}{\partial \theta} \left( 1 - \frac{r_1^3}{R^3} \right) \right]
\]  

(2.59)

\[
\bar{v} = v_0 + \frac{1}{ER \sin \phi} \frac{h^2}{24} \left[ \frac{1}{c_1} \frac{\partial p_i}{\partial \phi} \left( 1 - \frac{r_2^3}{R^3} \right) + \frac{1}{c_2} \frac{\partial p_o}{\partial \phi} \left( 1 - \frac{r_1^3}{R^3} \right) \right]
\]  

(2.60)

\[
\phi_\theta = \frac{1}{R \sin \phi} \frac{\partial \bar{w}}{\partial \theta} - \frac{6}{5Gh} \frac{Q_\phi}{R} - \frac{\bar{u}}{R}
\]  

(2.61)

\[
\phi_\phi = \frac{1}{R \cos \phi} \frac{\partial \bar{w}}{\partial \phi} - \frac{6}{5Gh} \frac{Q_\phi}{R} - \frac{\bar{v}}{R}
\]  

(2.62)

Making use of the equations (2.50) and (2.51) we can rewrite equations (2.61) and (2.62):

\[
\phi_\theta = \frac{1}{R \sin \phi} \frac{\partial \bar{w}}{\partial \theta} - \gamma_{\theta z} - \frac{\bar{u}}{R}
\]  

(2.63)

\[
\phi_\phi = \frac{1}{R \cos \phi} \frac{\partial \bar{w}}{\partial \phi} - \gamma_{\phi z} - \frac{\bar{v}}{R}
\]  

(2.64)

The remaining stress resultants and stress couples may be expressed in a more concise manner in terms of \( \bar{u}, \bar{v}, \bar{w}, \gamma_{\theta z}, \gamma_{\phi z} \) as follows:

\[
M_\theta = D \left[ \frac{1}{R \sin \phi} \frac{\partial}{\partial \theta} \left( \frac{1}{R \sin \phi} \frac{\partial \bar{w}}{\partial \theta} - \gamma_{\theta z} - \frac{\bar{u}}{R} \right) + \frac{c \tan \phi}{R} \left( \frac{1}{R \sin \phi} \frac{\partial \bar{w}}{\partial \phi} - \gamma_{\phi z} - \frac{\bar{v}}{R} \right) + k_1 p_i + k_2 p_o + k_3 p_{w_1} + k_4 p_{w_0} \right] + \frac{v}{R} \frac{\partial}{\partial \phi} \left( \frac{1}{R \sin \phi} \frac{\partial \bar{w}}{\partial \phi} - \gamma_{\phi z} - \frac{\bar{v}}{R} \right)
\]  

(2.65)

\[
M_\phi = D \left[ \frac{v}{R \sin \phi} \frac{\partial}{\partial \theta} \left( \frac{1}{R \sin \phi} \frac{\partial \bar{w}}{\partial \theta} - \gamma_{\theta z} - \frac{\bar{u}}{R} \right) + \frac{v \cot \phi}{R} \left( \frac{1}{R \sin \phi} \frac{\partial \bar{w}}{\partial \phi} - \gamma_{\phi z} - \frac{\bar{v}}{R} \right) + k_1 p_i + k_2 p_o + v k_3 p_{w_1} + v k_4 p_{w_0} \right] + \frac{1}{R} \frac{\partial}{\partial \phi} \left( \frac{1}{R \sin \phi} \frac{\partial \bar{w}}{\partial \phi} - \gamma_{\phi z} - \frac{\bar{v}}{R} \right)
\]  

(2.66)
\[M_{\phi\theta} = D \frac{1 - \nu}{2} \left[ \frac{1}{R} \frac{\partial}{\partial \phi} \left( \frac{1}{R \sin \phi} \frac{\partial \bar{w}}{\partial \phi} - \gamma_{\theta z} - \frac{\bar{u}}{R} \right) \right] - \frac{ctg \phi}{R} \left( \frac{2}{R \sin \phi} \frac{\partial \bar{w}}{\partial \phi} - \gamma_{\theta z} + \frac{\bar{u}}{R} \right) + \]

\[+ \frac{1}{R \sin \phi} \frac{\partial}{\partial \theta} \left( \frac{1}{R} \frac{\partial \bar{w}}{\partial \phi} - \gamma_{\theta z} - \frac{\bar{v}}{R} \right) \right] + k_3 \frac{1 - \nu}{2} p_{\phi \theta} + k_4 \frac{1 - \nu}{2} p_{\phi \phi} \quad (2.67)\]

\[N_\theta = \frac{Eh}{1 - \nu^2} \left[ \frac{1}{R \sin \phi} \frac{\partial \bar{u}}{\partial \theta} + \frac{ctg \phi}{R} \frac{\bar{v}}{R} + \frac{v}{R \sin \phi} + \frac{1 + \nu}{R} \frac{\bar{w}}{R} \right] + \]

\[+ \frac{D}{R} \left[ \frac{1}{R \sin \phi} \frac{\partial \gamma_{\theta z}}{\partial \theta} + \frac{ctg \phi}{R} \gamma_{\theta z} + \frac{v}{R \sin \phi} \frac{\partial \gamma_{\theta z}}{\partial \phi} \right] + k_5 p_i + k_6 p_o + k_7 p_{\phi \theta} + k_8 p_{\phi \phi} \quad (2.68)\]

\[N_\theta = \frac{Eh}{1 - \nu^2} \left[ \frac{v}{R \sin \phi} \frac{\partial \bar{u}}{\partial \theta} + \frac{v c t g \phi}{R} \frac{\bar{v}}{R} + \frac{1 + \nu}{R} \frac{\bar{w}}{R} \right] + \]

\[+ \frac{D}{R} \left[ \frac{v}{R \sin \phi} \frac{\partial \gamma_{\theta z}}{\partial \theta} + \frac{v c t g \phi}{R} \gamma_{\theta z} + \frac{1 + \nu}{R} \frac{\partial \gamma_{\theta z}}{\partial \phi} \right] + k_5 p_i + k_6 p_o + v k_7 p_{\phi \theta} + v k_8 p_{\phi \phi} \quad (2.69)\]

\[N_{\phi\theta} = \frac{Eh}{1 - \nu^2} \left( \frac{1 - \nu}{2} \right) \left[ \frac{1}{R \sin \phi} \frac{\partial \bar{v}}{\partial \theta} + \frac{1}{R} \frac{\partial \bar{u}}{\partial \phi} - \frac{ctg \phi}{R} \frac{\bar{u}}{R} \right] + \]

\[+ \frac{D}{R} \left( \frac{1 - \nu}{2} \right) \left[ \frac{1}{R \sin \phi} \frac{\partial \gamma_{\theta z}}{\partial \theta} - \frac{ctg \phi}{R} \gamma_{\theta z} + \frac{1}{R \sin \phi} \frac{\partial \gamma_{\theta z}}{\partial \phi} \right] + \]

\[+ \left( \frac{1 - \nu}{2} \right) k_5 p_{\phi \theta} + \left( \frac{1 - \nu}{2} \right) k_6 p_{\phi \phi} \quad (2.70)\]

where:

\[k_1 = D \frac{1 + \nu}{ERc_1} \left[ 1 + \nu - \frac{r_2^3}{R^3} (1 - 2\nu) \right] \quad (2.71)\]

\[k_2 = D \frac{1 + \nu}{ERc_2} \left[ 1 + \nu - \frac{r_1^3}{R^3} (1 - 2\nu) \right] \quad (2.72)\]

\[k_3 = D \frac{ctg \phi}{RGc_1} \left( 1 - \frac{r_2^3}{R^3} \right) \quad (2.73)\]

\[k_4 = D \frac{ctg \phi}{RGc_2} \left( 1 - \frac{r_1^3}{R^3} \right) \quad (2.74)\]

\[k_5 = \frac{h^3}{Rc_1 (1 - \nu)} \quad (2.75)\]

\[k_6 = \frac{h^3}{Rc_2 (1 - \nu)} \quad (2.76)\]

\[k_7 = \frac{Eh}{1 - \nu^2} \left[ \frac{ctg \phi}{Gc_1} \frac{h^2}{24R^2} \left( 1 + 2 \frac{r_2^3}{R^3} \right) \right] \quad (2.77)\]
These constitutive equations reduce to those given by Flugge (1960) when the shear deformation and radial effects are neglected. In this case, the average displacements are replaced by the middle surface displacements. The transverse shear forces $Q_{\theta}, Q_{\phi}$ are obtained in this case from the equilibrium equations in terms of the stress couples.

An alternative set of expressions for the normal forces and bending moments may be obtained in terms of the strains $\varepsilon_{\theta}, \varepsilon_{\phi}, \gamma_{\theta z}, \gamma_{\phi z}$ and corresponding rotations $\phi_{\theta}, \phi_{\phi}$. These relations are given below (Voyiadjis and Woelke 2004):

\[
M_{\theta} = D \left[ \frac{1}{R \sin \phi} \frac{\partial \phi_{\theta}}{\partial \theta} + \frac{\nu}{R \sin \phi} \frac{\partial \phi_{\phi}}{\partial \phi} + \frac{\nu}{R} \frac{\partial \phi_{\phi}}{\partial \phi} \right] + k_{1} p_{i} + k_{2} p_{o} + k_{3} p_{\phi i} + k_{4} p_{\phi o}
\]

(2.79)

\[
M_{\phi} = D \left[ \frac{\nu}{R \sin \phi} \frac{\partial \phi_{\theta}}{\partial \theta} + \frac{1}{R} \frac{\partial \phi_{\phi}}{\partial \phi} + \frac{\nu}{R} \frac{\partial \phi_{\phi}}{\partial \phi} \right] + k_{1} p_{i} + k_{2} p_{o} + k_{3} p_{\phi i} + k_{4} p_{\phi o}
\]

(2.80)

\[
M_{\phi\phi} = D \left[ \frac{1}{2} \frac{\partial \phi_{\theta}}{\partial \phi} + \frac{1}{R} \frac{\partial \phi_{\phi}}{\partial \phi} \right] \left( \frac{2}{R} \frac{\partial \ddot{w}}{\partial \theta} - \gamma_{\theta z} + \frac{\ddot{u}}{R} \right) +
\]

\[
+ k_{3} \left( \frac{1}{R} \right) p_{\phi i} + k_{4} \left( \frac{1}{2} \right) p_{\phi o}
\]

(2.81)

\[
N_{\theta} = \frac{Eh}{1-v} \left[ \varepsilon_{\theta} + \nu \varepsilon_{\phi} \right] + \frac{D}{R} \left[ \frac{1}{R \sin \phi} \frac{\partial \gamma_{\theta z}}{\partial \theta} + \frac{\nu}{R} \frac{\partial \gamma_{\phi z}}{\partial \phi} + \frac{\nu}{R} \frac{\partial \gamma_{\phi z}}{\partial \phi} \right] +
\]

\[
+ k_{5} p_{i} + k_{6} p_{o} + k_{7} p_{\phi i} + k_{8} p_{\phi o}
\]

(2.82)

\[
N_{\phi} = \frac{Eh}{1-v} \left[ \varepsilon_{\phi} + \nu \varepsilon_{\theta} \right] + \frac{D}{R} \left[ \frac{\nu}{R \sin \phi} \frac{\partial \gamma_{\theta z}}{\partial \theta} + \frac{\nu}{R} \frac{\partial \gamma_{\phi z}}{\partial \phi} + \frac{1}{R} \frac{\partial \gamma_{\phi z}}{\partial \phi} \right] +
\]

\[
+ k_{5} p_{i} + k_{6} p_{o} + k_{7} p_{\phi i} + k_{8} p_{\phi o}
\]

(2.83)

\[
N_{\phi\phi} = \frac{Eh}{1-v} \left( 1-v \right) \varepsilon_{\phi\phi} + \frac{D}{R} \left( \frac{1-v}{2} \right) \left[ \frac{1}{R \sin \phi} \frac{\partial \gamma_{\phi z}}{\partial \theta} - \frac{\nu}{R} \frac{\partial \gamma_{\phi z}}{\partial \phi} + \frac{1}{R} \frac{\partial \gamma_{\phi z}}{\partial \phi} \right] +
\]

\[
+ \left( \frac{1-v}{2} \right) k_{7} p_{\phi i} + \left( \frac{1-v}{2} \right) k_{8} p_{\phi o}
\]

(2.84)

### 2.2.6 Equilibrium Equations and Boundary Conditions

For the case of small deformation analysis, the shell equilibrium equations are given by Flugge, 1960:

\[
\frac{\partial}{\partial \phi} \left( N_{\phi} \sin \phi \right) + \frac{\partial N_{\phi\phi}}{\partial \theta} - N_{\theta} \cos \phi - Q_{\phi} \sin \phi + R \sin \phi p_{\phi} = 0
\]

(2.85)

\[
\frac{\partial}{\partial \phi} \left( N_{\phi\phi} \sin \phi \right) + \frac{\partial N_{\theta}}{\partial \theta} + N_{\phi\phi} \cos \phi - Q_{\theta} \sin \phi + R \sin \phi p_{\phi} = 0
\]

(2.86)
\begin{align}
N_\phi \sin \phi + N_\theta \sin \phi + \frac{\partial Q_\theta}{\partial \theta} + \frac{\partial}{\partial \phi} \left( Q_\phi \sin \phi \right) - R \sin \phi p_z &= 0 \quad (2.87) \\
\frac{\partial}{\partial \phi} \left( M_\phi \sin \phi \right) + \frac{\partial}{\partial \theta} \left( M_{\theta \phi} \sin \phi \right) - M_\theta \cos \phi - R Q_\phi \sin \phi &= 0 \quad (2.88) \\
\frac{\partial}{\partial \phi} \left( M_{\theta \phi} \sin \phi \right) + \frac{\partial}{\partial \theta} \left( M_{\theta \theta} \sin \phi \right) + M_\theta \cos \phi - R Q_\theta \sin \phi &= 0 \quad (2.89) \\
\frac{M_{\theta \theta}}{R} - \frac{M_{\theta \phi}}{R} &= N_{\phi \theta} - N_{\theta \phi} \quad (2.90)
\end{align}

In the above equilibrium expressions \( p_\phi, p_\theta, p_z \) are the equivalent distributed loads acting on the middle surface of the shell. Equation (2.90) is identically satisfied consequently reducing the number of equilibrium equations to five. The stress resultants and couples may be expressed in terms of either \( \bar{u}, \bar{v}, \bar{w}, \gamma_\theta, \gamma_\phi \) or \( \bar{u}, \bar{v}, \bar{w}, \phi_\theta, \phi_\phi \). We therefore have five unknowns to solve from the conditions (2.85) - (2.89).

The static and kinematic boundary conditions may be expressed in terms of either \( \bar{u}, \bar{v}, \bar{w}, \gamma_\theta, \gamma_\phi \) or \( \bar{u}, \bar{v}, \bar{w}, \phi_\theta, \phi_\phi \), together with the use of the constitutive equations (2.65) - (2.78). The boundary conditions (BC’s) are given as follows:

- if the edge \( (0, \phi) \) is simply supported the BC’s may be written as:
  \( \bar{w}(0, \phi) = 0; \ \phi_\phi(0, \phi) = 0; \ M_\theta(0, \phi) = 0 \)
- if the edge \( (0, \phi) \) is clamped the BC’s may be written as:
  \( \bar{w}(0, \phi) = 0; \ \phi_\phi(0, \phi) = 0; \ \phi_\theta = (0, \phi); \ \bar{u}(0, \phi) = 0 \)
- if on the edge \( (0, \phi) \) stretching of the mid-plane is prevented, BC’s may be written as: \( u_\theta(0, \phi) = 0; \ v_\phi(0, \phi) = 0 \) and if additionally the pressures \( p_z \) are uniformly distributed, i.e. \( \frac{\partial p_z}{\partial \theta} = \frac{\partial p_z}{\partial \phi} = 0 \) then \( \bar{u}(0, \phi) = 0; \ \bar{v}(0, \phi) = 0 \)
- if the edge \( (0, \phi) \) is free to stretch in \( \theta \) direction, then:
  \( v_\phi(0, \phi) = 0; \ N_\theta(0, \phi) = 0 \)
- if the edge \( (0, \phi) \) is free the BC’s may be written as:
  \( M_\phi(0, \phi) = 0; \ Q_\phi(0, \phi) = 0; \ M_{\theta \phi}(0, \phi) = 0; \ N_\phi(0, \phi) = 0; \ N_{\phi \phi}(0, \phi) = 0 \)

### 2.2.7 The Non-Linear Nature of the Stress Distribution

The nonlinear distribution of the in-plane stresses through the thickness in the proposed thick shell theory is due to the incorporation of the initial curvature of the shell, and the three-dimensional constitutive equations obtained from relations (2.26) - (2.28). This effect becomes highly pronounced in thick shells by changing the magnitude of the maximum stress significantly when compared to the linear stress variation theory.
In the expressions for in-plane stress components $\sigma_\phi, \tau_{\phi\theta}$ given by equations (2.32) - (2.36), nonlinear terms such as $1/(R + z)$ and $z^2 / R$ are incorporated. Consequently, the stresses given by the present theory have a nonlinear distribution along the thickness of the shell. Let us consider the simple case of a constant normal pressure and investigate the corresponding stress distribution of $\sigma_\phi$ through the thickness. In this case we have:

$$\sigma_\phi = \frac{E}{1-\nu^2} \left[ \frac{1}{R \sin \phi} \frac{\partial u_0}{\partial \theta} + \frac{\nu \cos \phi}{R \sin \phi} v_0 + \frac{1}{R} \frac{\partial v_0}{\partial \phi} + \frac{z}{2Gh} \left[ 3 \frac{4z}{h^2} - \frac{3z}{R} \left( 1 - \frac{2z}{3R} - \frac{2z^3}{6R^3} \right) \right] \right] + \frac{v \nu R}{Gc_1 R \sin \phi} \left[ -z + \frac{2z^3}{2R} + \frac{r_3}{R^3} \left( z - \frac{2z^2}{R} \right) \right] + \frac{v p_o \cos \phi}{Gc_2 R \sin \phi} \left[ -z + \frac{z^2}{2R} + \frac{r_3}{R^3} \left( z - \frac{2z^2}{R} \right) \right] + \frac{1 + \nu}{R \left( 1 + \frac{z}{R} \right)} \left[ w_0 + \frac{p_i}{Ec_1} \left[ -z + \frac{r_3}{R^3} \left( z - \frac{3z^2}{2R} \right) \right] \right] + \frac{p_o}{Ec_2} \left[ -z + \frac{r_3}{R^3} \left( z - \frac{3z^2}{2R} \right) \right] - \nu \frac{6z^2}{Eh^3} M \right] + \frac{v}{1 + \nu} \left[ \frac{p_o}{c_1} \left( \frac{r_3}{(R+z)^3} - 1 \right) + \frac{p_o}{c_2} \left( \frac{r_3}{(R+z)^3} - 1 \right) \right]$$

(2.91)

In equation (2.91) all the terms are nonlinear in $z$ except for the terms associated with:

$$\frac{1}{R \sin \phi} \frac{\partial u_0}{\partial \theta}, \frac{\partial v_0}{\partial \phi}, \frac{\partial^2 w_0}{\partial \phi^2}.$$

The stress distribution obtained using the presented theory will be compared with the elasticity theory.

### 2.3 The Equivalent Formulation for Thick Plates

It is relatively simple to reduce the proposed shell theory to a thick plate theory. As $R$ approaches infinity the stress resultants and stress couples reduce to:

$$N_x = \frac{Eh}{1-\nu^2} \left( \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} \right) + k_i \left( p_i + p_o \right)$$

(2.92)
\[ N_y = \frac{Eh}{1-\nu^2} \left( \frac{\partial^2 \nu}{\partial y^2} + \nu \frac{\partial^2 \nu}{\partial x^2} \right) + k_1 (p_i + p_o) \]  

(2.93)

\[ N_x = N_y = Gh \left( \frac{\partial^2 \nu}{\partial y^2} + \frac{\partial^2 \nu}{\partial x^2} \right) \]  

(2.94)

\[ Q_x = T \left( \frac{\partial^2 \nu}{\partial x^2} - \phi_x \right) \]  

(2.95)

\[ Q_y = T \left( \frac{\partial^2 \nu}{\partial y^2} - \phi_y \right) \]  

(2.96)

\[ M_x = D \left( \frac{\partial \phi_x}{\partial x} + \nu \frac{\partial \phi_x}{\partial y} \right) + k_2 (p_i + p_o) \]  

(2.97)

\[ M_y = D \left( \frac{\partial \phi_y}{\partial y} + \nu \frac{\partial \phi_y}{\partial x} \right) + k_2 (p_i + p_o) \]  

(2.98)

\[ M_{xy} = M_{yx} = D \frac{1-\nu}{2} \left( \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right) \]  

(2.99)

where:

\[ k_1 = \frac{\nu h}{2(1-\nu)} \]  

(2.100)

\[ k_2 = -D \frac{\nu(1+\nu)}{5Eh} \]  

(2.101)

We note that the present shell theory reduces to exactly the same equivalent thick plate theory as the one given by Voyiadjis and Shi (1991). The Voyiadjis and Shi theory (1991) can therefore be regarded as the special case of the theory developed here, as was already pointed out in Chapter 1.

### 2.4 Examples

The validity of the concepts employed here is tested through the examples. The shell equations derived in this chapter are complicated and obtaining analytical closed form solutions can be very tedious. This is a consequence of considering in the formulation many important effects affecting the behaviour of shells that are often neglected. The use of the finite element method based on the present shell equations would alleviate this difficulty. At this stage, however, the main objective is to verify the formulated refined theory of shells, and not its finite element implementation. We therefore consider two problems: cylindrical and spherical tanks subjected to uniform pressures (Sections 2.4.1 and 2.4.3), for which the analytical solutions can be obtained by means of rational simplifying assumptions and manual calculations. The problems of a hemispherical dome and an arch (Sections 2.4.2 and 2.4.4), were solved with the aid of the numerical procedure, based on the constitutive equations formulated here. We compare the results of the analysis provided by the current theory with the ones obtained
from the theory of elasticity, as well as the classical theory of thin shells (Niordson 1985).

### 2.4.1 Thick Sphere Subjected to Uniform Pressures

We investigate the stress distribution of $\sigma_\phi$ for a thick spherical container subjected to uniform pressures $p_i, p_o$ as shown in Figure 2.2 ($p_i = 5 \text{ kPa}, \ p_o = 4 \text{ kPa}$).

In this case, we have:

\[ v = Q_\phi = \frac{\partial M_\phi}{\partial \phi} = 0 \quad (2.102) \]

and

\[ w = w(z) \quad (2.103) \]

The stress $\sigma_\phi$ using the proposed theory is expressed in this case as follows:

\[ \sigma_\phi = \frac{E}{R + z} \left\{ w_0 + \frac{p_i}{Ec_1} \left[ -z + \frac{r_o^3}{R} \left( z - \frac{3 z^2}{2 R} \right) \right] + \frac{p_o}{Ec_2} \left[ -z + \frac{r_o^3}{R} \left( z - \frac{3 z^2}{2 R} \right) \right] \right\} \quad (2.104) \]

The corresponding exact elasticity solution for this problem is given by Lame (1852):

\[ \sigma_\phi = -\frac{p_o}{2c_2} \left( 2 + \frac{r_i^3}{(R + z)^3} \right) - \frac{p_i}{2c_1} \left( 2 + \frac{r_i^3}{(R + z)^3} \right) \quad (2.105) \]

The distribution of $\sigma_\phi$ given by the present theory is compared to the elasticity solution by Lame (1852) in Table 2.1.

**Table 2.1** $\sigma_\phi$ distribution for spherical shell

<table>
<thead>
<tr>
<th>$r_1$</th>
<th>$r_2$</th>
<th>$R_2/R_1$</th>
<th>$h=r_2-r_1$</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>Elasticity $\sigma_\phi$ [kPa]</th>
<th>Present theory $\sigma_\phi$ [kPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$r \ = r_1$</td>
<td>$r \ = r_2$</td>
</tr>
<tr>
<td>3</td>
<td>3.9</td>
<td>1.3</td>
<td>0.9</td>
<td>-1.2</td>
<td>-0.545</td>
<td>19.7782</td>
<td>15.2782</td>
</tr>
<tr>
<td>3</td>
<td>4.5</td>
<td>1.5</td>
<td>1.5</td>
<td>-2.4</td>
<td>-0.704</td>
<td>14.1842</td>
<td>9.6842</td>
</tr>
<tr>
<td>3</td>
<td>5.1</td>
<td>1.7</td>
<td>2.1</td>
<td>-3.9</td>
<td>-0.796</td>
<td>11.95004</td>
<td>7.45004</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>2</td>
<td>3</td>
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<td>5.92857</td>
</tr>
<tr>
<td>3</td>
<td>6.6</td>
<td>2.2</td>
<td>3.6</td>
<td>-9.6</td>
<td>-0.906</td>
<td>9.899254</td>
<td>5.39925</td>
</tr>
</tbody>
</table>

Using the theory of elasticity, we have:

\[ w_0 = \frac{R}{E} \sigma_\phi \big|_{z=0} \quad (2.106) \]

\[ \sigma_\phi \big|_{z=0} = -\frac{p_o}{2c_2} \left( 2 + \frac{r_i^3}{R^3} \right) - \frac{p_i}{2c_1} \left( 2 + \frac{r_i^3}{R^3} \right) \quad (2.107) \]

Substituting for $w_0$ from equations (2.106) and (2.107) into (2.104), we obtain the following expression for $\sigma_\phi$:
\[
\sigma_\phi = \frac{R}{R+z} \left[ -\frac{p_o}{2c_2} \left( 2 + \frac{r_1^2}{R^3} \right) + \frac{p_l}{2c_1} \left( 2 + \frac{r_2^3}{R^3} \right) \right] + \frac{1}{R+z} \left[ \frac{p_i}{Ec_1} \left( -z + \frac{r_2^3}{R^3} \left( z - \frac{3z^2}{2R} \right) \right) + \frac{p_o}{Ec_2} \left( -z + \frac{r_1^3}{R^3} \left( z - \frac{3z^2}{2R} \right) \right) \right]
\]  
(2.108)

It may be easily shown that \( \sigma_\phi \) obtained from equation (2.108) for the case of \( z = 0 \) is identical to \( \sigma_\phi \) obtained from the elasticity solution expressed by equation (2.105), for the same case, i.e. \( z = 0 \). It is also worthy to mention that, as expected in the case of a sphere, \( \sigma_\phi = \sigma_o \).

Gupta and Khatua (1978) in their derivation of a thick shell superparametric finite element proposed a modification in the expression for the circumferential stress \( \sigma_\phi \). Their modified expression for the circumferential stress is given by:

\[
\sigma_\phi = \frac{R}{R+z} \sigma_o
\]  
(2.109)

where \( \sigma_o \) is the average hoop stress. We note that Gupta and Khatua’s scheme cannot distinguish the difference between the internal and external pressures.

As shown in Table 2.1, the present theory is very close to the exact elasticity solution. In order to show the improvement in the present theory versus the classical shell theory, the problem of a spherical container subject to a uniform internal pressure \( p_i = 5 \text{ kPa} \) is analyzed. Figure 2.4 shows a comparison of the solution obtained with classical theory by Niordson (1985), and the present theory.

![Figure 2.4](image-url)

**Figure 2.4** Normalized \( \sigma_\phi \) for spherical container subjected to internal pressure
As expected the results deviate from the exact solution, as the thickness of the shell increases (Figure 2.4). However, there is a significant improvement in the results obtained using the present theory when compared with the classical theory, which yields large errors for thick shells.

A close investigation of the relative errors shows that the error in the present work is much smaller than in the case of the classical thin shell theory (Figure 2.5). The latter is built on Kirchhoff-Love assumption, which as shown by Niordson (1971) has a relative error of the order of \([\frac{h}{R} + (\frac{h}{L})^2]\). We therefore expect the error of the classical theory to be very close to the expression given by Niordson: \([\frac{h}{R} + (\frac{h}{L})^2]\). As shown in Figure 2.5, the classical theory has an error that is approximately equal to the Niordson error. The present theory also shows some loss of accuracy as the thickness of the shell increases. This loss is however significantly smaller than the Niordson error.

2.4.2 Hemispherical Dome under a Uniform Gravitational Pressure

We consider a simply supported hemispherical dome of radius \(R = 10\ m\) and thickness \(t\), subject to a gravitational pressure \(p = 0.5\ kPa\) (Figure 2.6). The bending stresses reach a maximum at the top of the dome, i.e. \(\phi = 0^\circ\). If the thickness of the shell is small, than the bending stresses are considered negligible and the loading is entirely resisted by the membrane action of the shell. As the thickness increases, bending stresses with non-linear terms start to play an important role. We will investigate the \(\sigma_\theta\) stresses at \(\phi = 0^\circ\) i.e. at the top of the dome, as they vary with the thickness. The results of the
analysis given by the classical theory and the present are shown in Figure 2.7 and Table 2.2.

**Figure 2.6** Hemispherical dome

![Hemispherical dome diagram](image)

**Figure 2.7** Hemispherical dome - comparison of results

Analysis of the results of the hemispherical shell problem leads to the same conclusions as in the previous example. The present theory shows a good agreement with
the classical one for the case of thin shells, while there is an improvement in the treatment of thick shells.

Table 2.2 $\sigma_\phi$ distribution for the spherical dome

<table>
<thead>
<tr>
<th>Thickness $t$ [m]</th>
<th>Classical–Niordson $\sigma_\phi$ [kPa]</th>
<th>Present theory $\sigma_\phi$ [kPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.06</td>
<td>-25.8065</td>
<td>-25.13</td>
</tr>
<tr>
<td>0.1</td>
<td>-15.7143</td>
<td>-13.93</td>
</tr>
<tr>
<td>0.14</td>
<td>-11.3553</td>
<td>-9.409</td>
</tr>
<tr>
<td>0.18</td>
<td>-8.91473</td>
<td>-6.965</td>
</tr>
<tr>
<td>0.22</td>
<td>-7.3501</td>
<td>-5.661</td>
</tr>
<tr>
<td>0.26</td>
<td>-6.25943</td>
<td>-4.906</td>
</tr>
<tr>
<td>0.3</td>
<td>-5.45455</td>
<td>-4.375</td>
</tr>
<tr>
<td>0.4</td>
<td>-4.13462</td>
<td>-3.226</td>
</tr>
<tr>
<td>0.6</td>
<td>-2.79412</td>
<td>-1.925</td>
</tr>
<tr>
<td>1.0</td>
<td>-1.7</td>
<td>-0.85</td>
</tr>
</tbody>
</table>

2.4.3 Thick Cylinder Subjected to Uniform Pressures

The current theory may be reduced to the case of cylindrical shells, as given by Voyiadjis and Shi (1991). Therefore, the Voyiadjis and Shi (1991) formulation may be regarded as a special case of the present theory. To show this we now investigate the stress distribution of $\sigma_\phi$ for a thick cylinder subjected to uniform pressures $p_i$ and $p_o$.

Similarly to the previous example, we have:

$$v = Q_\phi = \frac{\partial M_\phi}{\partial \phi} = 0 \quad (2.110)$$

and

$$w = w(z) \quad (2.111)$$

To reduce the current theory to the case of cylindrical shells we need to adopt two distinct radii of curvature $R_\theta = R_\phi$ in the $\theta$ and $\phi$ directions respectively. In cylindrical shells we have $R_\theta = R_\phi = \infty$, and therefore we may write:

$$R_\phi \sin \phi \partial \theta = \partial x \quad \text{and} \quad \frac{u}{R_\phi} = \frac{v}{R_\phi} = \frac{w}{R_\phi} = 0 \quad (2.112)$$

Considering also the solutions due to Lame for thick cylinders, we may obtain the stress distribution for $\sigma_\phi$ as given by Voyiadjis and Shi for the case of cylindrical shells:

$$\sigma_\phi = \frac{E}{R + z} \left\{ w_0 + p_i \frac{1}{Ec_1} \left[ -z + r^2 \left( \frac{z}{R} \right) \right] + p_o \frac{1}{Ec_2} \left[ -z + \frac{r^2}{R^2} \left( \frac{z - z^2}{R} \right) \right] \right\} \quad (2.113)$$
The corresponding exact elasticity solution for this problem is given by:

$$\sigma_\phi = -\frac{P_0}{c_2} \left( 1 + \frac{r_i^2}{(R+z)^2} \right) - \frac{P_0}{c_1} \left( 1 + \frac{r_i^2}{(R+z)^2} \right)$$  \hspace{1cm} (2.114)

Table 2.3 $\sigma_\phi$ distribution for cylindrical shell

<table>
<thead>
<tr>
<th>$r^2/r_1$</th>
<th>Winkler’s theory</th>
<th>Elasticity-exact</th>
<th>Present theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r=r_1$</td>
<td>-26.971</td>
<td>20.607</td>
<td>-27.858</td>
</tr>
<tr>
<td>$r=r_2$</td>
<td>4.863</td>
<td>-7.755</td>
<td>-7.642</td>
</tr>
<tr>
<td>$r=2r_1$</td>
<td>-2.285</td>
<td>1.095</td>
<td>-2.292</td>
</tr>
<tr>
<td>$r=2r_2$</td>
<td>1.095</td>
<td>2.130</td>
<td>0.925</td>
</tr>
</tbody>
</table>

Table 2.3 shows the comparison of the results of the given problem obtained by various theories with the results obtained here. The present theory shows very good agreement with the closed-form solution of the cylindrical shell problem provided by Lame (1852). This proves the applicability of the present work to not only spherical shells but also shells with different radii of curvature in two directions. The present theory may therefore be applied to shells of general shapes.

2.4.4 Circular Arch

Another benchmark problem testing the accuracy of the shell theories is a cantilevered circular arch subject to in-plane shear (MacNeal & Harder 1985). One end of the arch is fixed against displacements and rotations, and the other end is free. The radii of the arch are: the inner radius is $r_i = 4.12 \text{ in}$, outer radius is $r_i = 4.32 \text{ in}$, thickness is $t = 0.1 \text{ in}$, Young’s modulus is $E = 10 \times 10^6 \text{ psi}$, Poisson’s ratio is $\nu = 0.25$. Two unit forces are applied at the free end of the arch (Figure 2.8). The vertical deflection of the free end is investigated here. The analytical solution of this problem stated by MacNeal and Harder (1985) is $w = 0.08734 \text{ in}$. The deflection resulting from the present theory yields the value of $w_{\text{current}} = 0.08074 \text{ in}$, which is a very good approximation of the exact solution.

The circular arch example as well as all the other examples given here show that the proposed theory is accurate and in good agreement with the exact solutions and the other existing theories. The classical theory of shells yields errors that could become large in the case of moderate to thick shells. In the present theory there is a significant reduction in error. This is clearly shown in the first example.

The current work is applicable to plates (setting the radius of curvature infinite), beams as special cases of plates, and through the use of the finite element method to shells of arbitrary shape, with the radius of curvature being different in two directions e.g. cylindrical shells as well as arches. The proposed theory is therefore general and universal and gives very good results for all of the above-discussed cases. The finite element implementation of the derived shell constitutive equations is given in Chapter 3.
Figure 2.8 Circular Arch
CHAPTER 3

SHELL ELEMENT BASED ON THE REFINED THEORY OF THICK SPHERICAL SHELLS

3.1 Introduction

There are many problems frequently encountered when formulating a computational model for thick shells, as briefly described in Section 1.3.2. The most important of these problems, namely shear and membrane locking, and mesh instabilities are discussed in the following sections along with the remedies adopted here to overcome these difficulties.

After the introduction the details of the finite element procedure are presented, leading to formulation of the stiffness matrix of the element. The reliability of the numerical algorithm is verified through a series of discriminating examples.

3.1.1 Shear Locking

Shear locking is a numerical deficiency, experienced in thick shell finite elements accounting for the transverse shear deformation. For the purpose of conciseness, we will illustrate the problem using an example of a thick beam, since the nature of the phenomenon is the same. We consider a typical thick beam element with four degrees of freedom, as shown in Figure 3.1.

![Figure 3.1 Thick beam element](image)

The finite element algorithm presented here is founded on the refined shell theory outlined in Chapter 2. The main assumption of the theory was that the plane sections remain plane after the deformation, but not perpendicular to the middle surface. The transverse shear strains are therefore not negligible, as is in the case in thin plates or shells. For a thick beam the shear strain $\gamma_{xz}$ is given by the equation (3.1):

$$\gamma_{xz} = \frac{\partial W}{\partial x} - \phi_x$$ \hspace{1cm} (3.1)

where $\phi_x$ is the angle of rotation of the section originally perpendicular to the middle surface and $W$ is a vertical displacement, (Figure 1.3). The strain energy density of the thick beam element is:
\[ U = U_b + U_s \]  
where \( U_b \) and \( U_s \) are bending and shear strain energy densities respectively, given by:

\[
U_b = \frac{1}{2} \int \frac{\sigma_x^2}{E} \, dV = \frac{1}{2} \int E \varepsilon_x^2 \, dV \quad \text{and} \quad U_s = \frac{1}{2} \int \frac{\tau_{zx}^2}{G} \, dV
\]

(3.3)

\[
E \quad \text{and} \quad G \quad \text{are the elastic and shear modules respectively;} \quad b \quad \text{is the width of the beam and} \quad h \quad \text{is the thickness of the beam. The term 5/6 in the shear strain energy expression is the "form factor", which accounts for the parabolic distribution the shear stress } \tau_{zx} \text{ over a rectangular cross section (Cook et al. 1989). From the strain energy given by the equations (3.2)-(3.4) we may obtain the stiffness matrix:}
\[
[k] = [k_b] + [k_s]
\]

(3.5)

where \([k_b]\) resists bending strain \(\varepsilon_x\) and \([k_s]\) resists shear strain \(\gamma_{zx}\). For a considered beam we can write:

\[
([k_b] + [k_s]) \{D\} = \{R\}
\]

(3.6)

where \(\{D\}\) and \(\{R\}\) are the vectors of the nodal displacements and nodal external forces respectively. The displacements of the thin beam \(\{D\}\) should be governed by \([k_b]\) only, since the transverse shear deformation \(\gamma_{zx}\) is negligible. In other words, if the shear rigidity \(5Gbt/12\) becomes much larger than \(Ebt^3/12\) in equations (3.3) and (3.4), then \([k_s]\) should enforce the constraint \(\gamma_{zx} = 0\). Instead, when the thickness of the beam decreases, \([k_s]\) grows in relation to \([k_b]\). Thus, \([k_s]\) acts as a penalty matrix causing equation (3.6) to yield \(\{D\} = \{0\}\), unless \([k_s]\) is singular, (Cook et al. 1989). A singular \([k_s]\) may be achieved through different methods described in Section 1.3.2. The most popular method of overcoming locking is selective reduced integration. This approach was extensively examined by Zienkiewicz (1971), Stolarski and Belytschko (1983-84), Hughes (1987), Yang et al. (2000) and many others. The determination of the components of the stiffness matrix, given in equation (3.6) involves most of the time numerical integration of the strain energy density expression given by equation (3.2). Numerical integration requires a certain number of sampling or Gauss integration points, at which the values of the integrated functions are determined. If the number of the sampling points is the same for all the components of the strain energy, than the integration is uniform, otherwise it is selective. The integration is also called full if there are enough sampling points to ensure the exact integration of all stiffness coefficients. If the number of Gauss points is smaller than necessary for exact integration, than such a scheme is called reduced. When selective reduced integration is applied, one may integrate a bending mode of the strain energy, which should be dominant for thin shells, with a sufficient number of sampling points to ensure the exactness of solution, and
underintegrate the shear mode of the strain energy causing $[k_s]$ to become singular, which prevents shear locking. This method, although very effective, as reported by many authors, is not consistent from the mathematical point of view.

The approach followed in the current dissertation to overcome shear locking is the previously discussed quasi-conforming technique, along with the appropriate choice of the interpolation formulas of the displacement fields. Here, we directly interpolate the strain fields rather than obtain these from the assumed displacements. The element strains may be expressed in terms of the element nodal displacement vector by explicit integration along the element boundaries. The strain energy terms are integrated analytically, and the stiffness matrix is computed by the multiplication of the already integrated matrices. The numerical integration is not performed, which effectively prevents the shear stiffness form ‘suppressing’ the bending modes.

The mesh of the finite elements may also lock due to inadequate representation of the displacement field on the boundaries of the elements. Widely used bilinear elements are characterized by the linear displacement functions. These elements are attractive because of their simplicity, but they are too stiff in bending. We illustrate the point with reference to the rectangular, bilinear element subjected to bending moments as shown in Figure 3.2.

![Figure 3.2](image)

**Figure 3.2** a) Rectangular bilinear element. b) Deformed element-edges straight. c) Correctly deformed element for pure bending

The correct deformation, shown in Figure 3.2c gives rise to storage of the energy caused by the normal strains only, while a bilinear element shown in Figure 3.2b stores the energy caused by the normal strain $\varepsilon_{xx}$ and a spurious shear strain $\gamma_{xz}$. Thus, for the same deformation we have: $M_1 > M_2$. The unwanted shear strain that produces $M_1 > M_2$ is called parasitic shear. This effect becomes decisive especially when the ratio of $a/b$ is large. In this work, we use Hu’s (1984) cubic approximation formulas for the displacements and the problem of parasitic shear is not experienced. Moreover, the compatibility equations are only enforced in the weak sense, i.e. under the integral sign, which causes a desirable effect of softening the structure of elements.

### 3.1.2 Membrane Locking

The term ‘membrane locking’ refers to an excessive stiffness in bending of the curved elements. The nature of this phenomenon is similar to shear locking. In the curved element in pure bending, the nodal displacements should be resisted by the bending action only. If this element suffers from membrane locking, the deformation will be also
resisted by the membrane action. Because the membrane stiffness is much greater than
the bending stiffness in thin shells and thin arches, the desired bending mode is
suppressed from the element response to load (Cook et al. 2002). For conciseness we use
a simple problem of a circular arch as an example of membrane locking that may arise in
the curved element with a low-order displacement field (Figure 3.3). This problem was
presented by Cook et al. (2002).

![Figure 3.3 Arch element](image)

The membrane strain \( \varepsilon_m \) and the curvature \( \kappa \) are given by:

\[
\varepsilon_m = \frac{du}{ds} + \frac{w}{R} \quad \text{and} \quad \kappa = \frac{1}{R} \frac{du}{ds} - \frac{d^2w}{ds^2}
\]  

(3.7)

where \( s \) is a tangential direction, as shown in Figure 3.3. The membrane strain \( \varepsilon_m \) is
associated with the membrane force and in the \( s \) direction. The curvature \( \kappa \) is associated
with the bending moment. By analogy with the displacement fields used for a straight
beam, we employ the radial and axial displacement functions:

\[
u = a_i + a_s s
\]

\[
w = a_1 + a_2 s + a_3 s^2 + a_4 s^3
\]

(3.8)

where \( a_i \) are the generalized degrees of freedom. Substituting the above equations into
equations (3.7) we have:

\[
\varepsilon_m = \left( a_2 + \frac{a_5}{R} \right) + \frac{a_4}{R} s + \frac{a_3}{R} s^2 + \frac{a_6}{R} s^3
\]

(3.9)

\[
\kappa = \frac{a_2}{R} - 2a_5 - 6a_6 s
\]

Under most loading conditions, a slender arch has very little membrane strain, which
implies the inextensibility condition:

\[
\varepsilon_m = \frac{du}{ds} + \frac{w}{R} = 0
\]  

(3.10)

For the infinitely slender arch, the above condition requires that:

\[
\left( a_2 + \frac{a_1}{R} \right) = 0, \quad a_4 = 0, \quad a_5 = 0, \quad a_6 = 0
\]  

(3.11)

If the equations (3.11) are enforced, the only contribution to \( \kappa \) comes from the
membrane term \( a_2 \). This is of course insufficient for the appropriate representation of the
bending mode. If on the other hand equations (3.11) are not satisfied, the membrane strain \( \varepsilon_m \) is not zero. The nonzero membrane strain produces very large energy, leading to a very large membrane stiffness. The bending modes are then suppressed and the bending deformation is ‘locked out’, where in fact the deformation should be governed by bending. The problem disappears when \( R = \infty \).

Similarly to shear locking, reduced selective integration is the most commonly used approach to overcome membrane locking. The principle is the same as discussed previously. The membrane modes must be underintegrated here, in order to enforce the inextensibility condition and free the formulation from membrane locking.

In the current dissertation, the linear interpolation functions are used for the membrane displacements and the third order polynomials given by Hu (1984), for the vertical displacement and angles of rotation. In the quasi-conforming technique, independently from the displacement approximations, the strains are assumed, such that the compatibility equations are satisfied in a weak sense, i.e. under the integral sign. The appropriate choice of the strain fields leads to the satisfaction of the inextensibility condition. This method was reported to be a very effective way of overcoming the deficiency of membrane locking (Tang et al. 1980; Shi, 1980; Shi and Voyiadjis, 1991; Lu and Liu, 1981). It also more consistent mathematically than reduced selective integration, since the integrations are performed analytically and exactly here. The stiffness matrix in the quasi-conforming technique is given explicitly, which makes the algorithm very efficient computationally. The quasi-conforming method is chosen in the current work to alleviate the numerical deficiencies discussed above and deliver an explicit stiffness matrix.

### 3.1.3 Mesh Instabilities

Mesh instabilities in the shell finite elements arise due to shortcomings in the element formulation process, such as inadequate integration scheme or inadequate approximation of strains or displacements. In the present context, an instability is not in any way related to buckling problems of structures (Cook et al. 2002). Different types of instabilities are often referred to as kinematic modes, hourglass modes or zero energy modes. Appearance of the zero energy mode is a situation in which a nodal displacement vector \( \{D\} \) that is associated with straining of the element produces zero strain energy. A spurious energy mode is a reverse situation, i.e. when a non-zero energy mode is present, despite the element moving as a rigid body, without any strains. The aforementioned, as well as the other types of mesh instabilities were discussed in detail by Zienkiewicz (1978), Cook et al. (2002) and many other authors.

A computational procedure presented in this work, relies once again on the quasi-conforming technique and the direct interpolation of the strain fields to prevent any spurious energy modes. The appropriate approximation of the strains results in a reliable algorithm free from common mesh instabilities.
3.2 Finite Element Formulation

3.2.1 Shell Constitutive Equations

In the previous chapter, the refined shell theory of thick shells was derived. The final set of constitutive equations defined in the spherical coordinates, is given by the equations (2.79)-(2.84). These expressions were given in terms of the average displacements: $\bar{u}, \bar{v}, \bar{w}, \phi_0, \phi_\rho$ given by the equations (2.56), (2.59)-(2.64). In order to formulate a spherical shell finite element with the rectangular local coordinate system, we first need to define the strains of the shell in the Cartesian coordinate system. We may express the strains and curvatures in terms of the average displacements $\bar{u}, \bar{v}, \bar{w}, \phi_x, \phi_y$ as follows:

\[
\varepsilon_x = \frac{\partial \bar{u}}{R \partial x} + \frac{\bar{w}}{R}, \quad (3.12)
\]
\[
\varepsilon_y = \frac{\partial \bar{v}}{R \partial x} + \frac{\bar{w}}{R}, \quad (3.13)
\]
\[
\varepsilon_{xy} = \frac{1}{2} \left( \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right), \quad (3.14)
\]
\[
\kappa_x = \frac{\partial \phi_x}{\partial x} = \frac{\partial}{\partial x} \left( \frac{\partial \bar{w}}{\partial x} \gamma_{xz} - \frac{\bar{u}}{R} \right), \quad (3.15)
\]
\[
\kappa_y = \frac{\partial \phi_y}{\partial y} = \frac{\partial}{\partial y} \left( \frac{\partial \bar{w}}{\partial y} \gamma_{yz} - \frac{\bar{v}}{R} \right), \quad (3.16)
\]
\[
\kappa_{xy} = \frac{1}{2} \left( \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right), \quad (3.17)
\]

The stress resultants and couples may be now expressed in terms of the strains given above:

\[
M_x = D \left[ \kappa_x + \nu \kappa_y \right], \quad (3.18)
\]
\[
M_y = D \left[ \kappa_y + \nu \kappa_x \right], \quad (3.19)
\]
\[
M_{xy} = D (1 - \nu) \kappa_{xy}, \quad (3.20)
\]
\[
N_x = S \left[ \varepsilon_x + \nu \varepsilon_y \right], \quad (3.21)
\]
\[
N_y = S \left[ \varepsilon_y + \nu \varepsilon_x \right], \quad (3.22)
\]
\[
N_{xy} = S (1 - \nu) \varepsilon_{xy}, \quad (3.23)
\]
\[
Q_x = T \gamma_{xz}, \quad (3.24)
\]
\[
Q_y = T \gamma_{yz}, \quad (3.25)
\]

where:

\[
D = \frac{E h^3}{12 (1 - \nu^2)}, \quad S = \frac{E h}{(1 - \nu^2)}, \quad T = \frac{5}{12} \frac{E h}{(1 + \nu)}, \quad (3.26)
\]
We note that in the expressions for the membrane forces, given by the equations (3.21)-(3.23), the variations of the transverse shear strains were neglected. The influence of the shear strains on the bending moments is significant in thick shells and plates, and therefore accounted for through the definition of the angles of rotations given by equations (3.15) and (3.16). The membrane forces derived in the previous chapter and given by equations (2.82)-(2.84) are functions of the membrane strains, and the variations of the shear strains. The effect of the latter is considered small and therefore disregarded (equations (3.21)-(3.23)). It is later confirmed that this approximation does not lead to substantial deterioration of the accuracy of the current model. We use the above shell constitutive equations to formulate the coupled strain energy density and derive the stiffness matrix of the element.

3.2.2 Displacements and Boundary Conditions

A simple $C^0$ thick/thin shell element based on the refined spherical shell theory and quasi-conforming technique is formulated. It satisfies the Kirchhoff-Love hypothesis for the case of thin plates or shells.

The average displacements along a point on the middle surface and the average rotations of the normal given in Chapter 2, by the equations (2.56), (2.59)-(2.64), are employed here instead of the usual middle surface displacement of the shell. (In the case of thin shells, the average displacements are replaced by the middle surface displacement). In rectangular coordinates, we have:

$$
\begin{align*}
\bar{u} &= u_0 + \frac{1}{E} \frac{h^2}{24} \left[ \frac{1}{c_1} \frac{\partial p_t}{\partial x} \left( 1 - \frac{r_2^3}{R^3} \right) + \frac{1}{c_2} \frac{\partial p_a}{\partial x} \left( 1 - \frac{r_3^3}{R^3} \right) \right] \\
\bar{v} &= v_0 + \frac{1}{E} \frac{h^2}{24} \left[ \frac{1}{c_1} \frac{\partial p_t}{\partial y} \left( 1 - \frac{r_2^3}{R^3} \right) + \frac{1}{c_2} \frac{\partial p_a}{\partial y} \left( 1 - \frac{r_3^3}{R^3} \right) \right] \\
\bar{w} &= w_0 - M \left( \frac{3v}{10Eh} - \frac{9yv}{112ER^2} \right) - \frac{1}{10} \frac{h^2}{REc_1} \frac{r_2^3}{R^3} P_t - \frac{1}{10} \frac{h^2}{REc_2} \frac{r_3^3}{R^3} P_o
\end{align*}
$$

$$
\begin{align*}
\phi_x &= \frac{\partial \bar{w}}{\partial x} - \frac{\gamma_x}{R} - \frac{\bar{u}}{R} \\
\phi_y &= \frac{\partial \bar{w}}{\partial y} - \frac{\gamma_y}{R} \frac{\bar{v}}{R}
\end{align*}
$$

The static and kinematic boundary conditions, expressed in terms of $\bar{u}, \bar{v}, \bar{w}, \phi_x, \phi_y$, together with the use of the constitutive equations (3.18) - (3.25) are similar to those given in Section 2.2.6. Boundary conditions (BC) are as follows:

- if the edge $(0, y)$ is simply supported the BC’s are given as:
  $$\bar{w}(0, y) = 0; \quad \phi_x(0, y) = 0; \quad M_x(0, y) = 0$$

- if the edge $(0, y)$ is clamped the BC’s are given as:
  $$\bar{w}(0, y) = 0; \quad \phi_x(0, y) = 0; \quad \phi_y(0, y) = 0; \quad \bar{u}(0, y) = 0$$

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- if on the edge \((0, y)\) stretching of the mid-plane is prevented, BC’s are given as:
  \[ u_0(0, y) = 0; \quad v_0(0, y) = 0 \] and if additionally the pressures \(p_z\) are uniformly distributed, i.e. \(\frac{\partial p_z}{\partial x} = \frac{\partial p_z}{\partial y} = 0\) then \(\hat{u}(0, y) = 0; \quad \hat{v}(0, y) = 0\)
- if the edge \((0, y)\) is free to stretch in the \(x\) direction, then: \(v_0(0, y) = 0; \quad N_x(0, y) = 0\)
- if the edge \((0, y)\) is free the BC’s are given as:
  \[ M_x(0, y) = 0; \quad Q_x(0, y) = 0; \quad M_{xy}(0, y) = 0; \quad N_x(0, y) = 0; \quad N_{xy}(0, y) = 0 \]

For simplicity and conciseness we now denote the average displacements \(\bar{u}, \bar{v}, \bar{w}, \phi_x, \phi_y\) as \(u, v, w, \phi_x, \phi_y\). We therefore have the quadrilateral, doubly curved finite element with five degrees of freedom per node: \(u, v, w, \phi_x, \phi_y\), and twenty degrees of freedom per element. The vector of the nodal displacements is given as follows:

\[
\mathbf{q} = \begin{bmatrix} u_1, v_1, w_1, \phi_{x1}, \phi_{y1}, u_2, v_2, w_2, \phi_{x2}, \phi_{y2}, u_3, v_3, w_3, \phi_{x3}, \phi_{y3}, u_4, v_4, w_4, \phi_{x4}, \phi_{y4} \end{bmatrix}^T
\] (3.32)

The geometry of the element with the local rectangular coordinate system and the nodal degrees of freedom is given in Figure 3.4.

**Figure 3.4 Spherical shell element**
3.2.3 Element Displacement and Strain Fields – Quasi-Conforming Method

The finite element presented here is a four-node quadrilateral, doubly curved element, as shown in Figure 3.4. The radius of curvature $R$ is constant in both directions. We assume continuity of the displacement fields, but not their derivatives. We therefore have a $C^0$ continuity problem with twenty degrees of freedom in each element. The quasi-conforming technique proposed by Tang et al. (1980, 1983) is used here to compute the element stiffness matrix. In this case, the strain field is interpolated directly, rather than evaluated from the displacement field through differentiation. The functions describing the surface are only defined on the interelement boundaries, leaving the functions inside the elements undefined explicitly. This method is related to the so-called ‘generalized hybrid model’, which may also be derived using Hu-Washizu principle, (Tang et al. 1983, Hu 1984). The quasi-conforming element technique gives an explicit form of the stiffness matrix, as integrations are carried out directly.

As previously stated the objective of the present work is to develop an element commonly applicable to thick and thin shells. The difficulty of such a model lies partly in how to assume the displacement distribution on the element boundary. Many authors adopt simple and convenient linear interpolation formulas, on the boundary of the element. The results of such approximations may be good when the shear rigidity is not very large. When the shear rigidity approaches infinity, the linear displacement interpolation leads to a contradiction with the Kirchhoff-Love assumption, which states that the shear deformation must vanish when the shell is thin. Thus, the linear displacement interpolations are not suitable for the case when the shear rigidity is very large (Hu, 1984). This shortcoming is discussed in section 3.1.1. In order to formulate a reliable and universal model for both thick and thin shells, we need the three-generalized-displacement theory, on which the element is based, to degenerate to the classical theory satisfying the Kirchhoff-Love assumption for the case of thin shells. Since the shear forces $Q_x, Q_y$ are generally finite, the shear deformations $\gamma_{xz}$ and $\gamma_{yz}$ must vanish when the shear rigidity $T$ approaches infinity, (see equations (3.24)-(3.25)). This may be achieved through the interpolation functions within the element ensuring that $w, \phi_x, \phi_y$ are in general three independent functions, but also $\phi_x, \phi_y$ depend on $w$ according to the classical theory (Hu 1984). Hu points out that in order to construct such interpolation formulas, they must contain the ratio of the flexural and shear rigidities. We use here the approximation of the displacement $w$ and tangent rotation $\phi_x$ for the straight beam of length $l$ given by Hu:

\[
w = \frac{1}{2} \left[ 1 - \xi + \frac{\lambda}{2} \left( \xi^3 - \xi \right) \right] w_i + \frac{1}{4} \left[ 1 - \xi^2 + \lambda \left( \xi^3 - \xi \right) \right] \frac{l}{2} \phi_i + \\
+ \frac{1}{2} \left[ 1 + \xi - \frac{\lambda}{2} \left( \xi^3 - \xi \right) \right] w_j + \frac{1}{4} \left[ -1 + \xi^2 + \lambda \left( \xi^3 - \xi \right) \right] \frac{l}{2} \phi_j \] (3.33)
\[
\phi_i = -\frac{3}{2l} \lambda \left[1 - \xi^2\right] w_i + \frac{1}{4l} \left[2 - 2\xi - 3\lambda \left(1 - \xi^2\right)\right] \phi_i + \frac{3}{2l} \left[1 - \xi^2\right] w_j + \frac{1}{4l} \left[2 + 2\xi - 3\lambda \left(1 - \xi^2\right)\right] \phi_j
\] (3.34)

A subscript \( s \) in equation (3.34) refers to the tangent direction to the edge of the element, as shown in Figure 3.5.

\[\phi_n = \frac{1}{2} \left[1 - \xi\right] \phi_{ni} + \frac{1}{2} \left[1 + \xi\right] \phi_{nj} \] (3.35)

where:
\[
\xi = \frac{2x}{l} 
\] (3.36)

The parameter \( \lambda \) in equations (3.33)-(3.34) is given by:
\[
\lambda = \frac{1}{1 + 12 \frac{D}{T l^2}} 
\] (3.37)

where \( D \) and \( T \) denote the flexural and shear rigidity of the shell respectively. In equation, (3.37) the parameter \( D / T l^2 \) accounts for the shear deformation effect. We notice that when shear rigidity is very large and \((h/\ell)^2 \to 0, \lambda \to 1\), \( w \) in equation (3.33) reduces to a Hermite function. When the shear rigidity is very small on the other hand \( \lambda = 0 \), and (3.33) reduces to Cook’s (1972) interpolation formula. The interpolation formulas given by equations (3.33)-(3.34) are therefore suitable for both the classical theory of shells, as well as the thick shell theory based on which the present element is formulated. For a two-dimensional problem we let \( L_x \) be the effective length in the \( x \) direction, and \( L_y \) the corresponding effective length in the \( y \) direction. The two-dimensional expressions equivalent to equation (3.37) are as follows (Woelke and Voyiadjis, 2004):

\[\phi = \frac{1}{1 + 12 \frac{D}{T l^2}} \]

Figure 3.5 Shell element on a plane, \( n \)-normal and \( s \)-tangential direction
\[
\lambda_x = \frac{1}{\left(1 + \frac{12D}{TL_x^2}\right)} \quad \text{and} \quad \lambda_y = \frac{1}{\left(1 + \frac{12D}{TL_y^2}\right)}
\]  

(3.38)

Following the argument given in the Section 3.1.2, the linear approximations for the membrane displacements \( u, v \) are used:

\[
u = \frac{1}{4}[1 - \xi]v_j + \frac{1}{4}[1 + \xi]v_j
\]  

(3.39)

where \( \xi \) is given by the equation (3.36).

The method used in the current dissertation in the linear elastic analysis of shells, was named the quasi-conforming technique (Tang et al. 1983), because the compatibility equations are satisfied in a weak sense, i.e. under the integral sign and the displacement field and the strain field are approximated independently. In order to determine the element strain fields using the quasi-conforming technique, we discretize the strains in the element:

\[
\varepsilon_b^T = \begin{bmatrix}
\frac{\partial \phi_x}{\partial x} & \frac{\partial \phi_y}{\partial y} & \frac{\partial \phi_x}{\partial y} & \frac{\partial \phi_y}{\partial x}
\end{bmatrix}
\]  

(3.41)

\[
\varepsilon_m^T = \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial u}{\partial y} & \frac{\partial v}{\partial x}
\end{bmatrix}
\]  

(3.42)

\[
\varepsilon_s^T = \begin{bmatrix}
\frac{\partial w}{\partial x} - \phi_x & \frac{\partial w}{\partial y} - \phi_y & \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y}
\end{bmatrix}
\]  

(3.43)

The derivatives may also be expressed by Taylor’s expansion or approximately by a polynomial with \( n \) truncated terms:

\[
\frac{\partial \phi_x}{\partial x} = \alpha_0 + \alpha_1 x + \alpha_2 y + \ldots = \sum_{i=1}^{n} P_i \alpha_i
\]  

(3.44)

We follow the same procedure for the remaining element strain fields. According to the given nodal variables, the compatibility equations, and the requirement for the proper rank of the element stiffness matrix (Liu et al. 1984), the strain fields are interpolated as follows:

- Linear bending strain field:

\[
\varepsilon_b = \begin{bmatrix}
\frac{\partial \phi_x}{\partial x} \\
\frac{\partial \phi_y}{\partial y} \\
\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}
\end{bmatrix} = \begin{bmatrix}
1 & x & xy & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & x & xy & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y
\end{bmatrix} \begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5 \\
\alpha_6 \\
\alpha_7 \\
\alpha_8 \\
\alpha_9 \\
\alpha_{10} \\
\alpha_{11}
\end{bmatrix} = \mathbf{P}_b \mathbf{a}_b
\]  

(3.45)
• Stretch strain field:

\[
\varepsilon_m = \begin{bmatrix}
\frac{\partial u}{\partial x} + \frac{w}{R} \\
\frac{\partial v}{\partial y} + \frac{w}{R} \\
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\end{bmatrix} = \begin{bmatrix}
1 & y & 0 & 0 \\
0 & 0 & 1 & x \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\alpha_{12} \\
\alpha_{13} \\
\alpha_{14} \\
\alpha_{15} \\
\alpha_{16}
\end{bmatrix} = P_m \alpha_m
\]  

(3.46)

• Constant transverse shear strain:

\[
\varepsilon_s = \begin{bmatrix}
\frac{\partial w}{\partial x} - \phi_x - \frac{u}{R} \\
\frac{\partial w}{\partial y} - \phi_y - \frac{v}{R}
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
\alpha_{17} \\
\alpha_{18}
\end{bmatrix} = P_s \alpha_s
\]  

(3.47)

where \(\alpha_1, \alpha_2, ..., \alpha_{18}\) are the undetermined strain parameters.

The compatibility equations of the displacement field are not enforced \textit{a priori} in the above strain interpolations.

Let \(P\) be the trial function for the assumed strain field (Equations (3.45)-(3.47)):

\[\varepsilon = P \alpha\]  

(3.48)

and \(N\) - the corresponding test function. We multiply both sides by the test function and integrate over the element domain:

\[\int \int_{\Omega} N^T \varepsilon d\Omega = \alpha \int \int_{\Omega} N^T P d\Omega\]  

(3.49)

The strain parameter \(\alpha\) is determined from the quasi-conforming technique as follows:

\[\alpha = A^{-1} C q\]  

(3.50)

where \(q\) is the element nodal displacement vector given by equation (3.32), and:

\[A = \int \int_{\Omega} N^T P d\Omega\]  

(3.51)

\[C q = \int \int_{\Omega} N^T \varepsilon d\Omega\]  

(3.52)

We may now express the strain field in terms of the nodal displacements as follows:

\[\varepsilon = P \alpha = P A^{-1} C q = B q\]  

(3.53)

In most cases, it is convenient to take \(P = N\) in order to obtain a symmetric stiffness matrix. This is the case in this work. Both matrices \(A\) and \(C\) may be easily evaluated explicitly. In order to briefly illustrate this process, we consider the transverse shear strain \(\varepsilon_s\) as an example. Substituting \(P_s\) from equation (3.47) into equations (3.51) and (3.52), setting \(N = P\) and using Green’s theorem, we obtain the following:

\[A_s = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \int \int_{\Omega} d\Omega = \Omega \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}\]  

(3.54)
and:

\[
C \Phi = \int_{\Omega} \left[ \frac{\partial w}{\partial x} \phi_x - \frac{u}{R} \frac{\partial w}{\partial y} \phi_y - \frac{v}{R} \phi_y \right] d\Omega
\]

\[
= \oint w_n ds \left\{ \frac{\phi_x}{R} \right\} - \oint \frac{\phi_y}{R} d\Omega
\]

where \( \Omega \) is the element area, \( n_x, n_y \) are the direction cosines along the element boundaries, and \( ds \) is the differential arc-length along the element boundaries. In order to evaluate \( C_\Phi, A_\Phi \), we make use of the displacement interpolations given by (3.33)-(3.34) and (3.39)-(3.40). Solving equation (3.55) as well as the determination of \( C_\Phi, C_m \) require however, the interpolation functions for rotations \( \phi_x, \phi_y \) and membrane displacements \( u, v \) to be two-dimensional. We may use the approximate methods to evaluate these integrals, as given by Hu (1984) and also Shi and Voyiadjis (1991a), which allows the direct use of the displacement approximations given by equations (3.33)-(3.34), and (3.39)-(3.40). Alternatively, the two-dimensional interpolation formulas for all the necessary displacements may be constructed, such that they reduce to the one-dimensional cases of the string net functions. The two-dimensional rotation function that reduces to the string functions given by the equations (3.34)-(3.35) is given by:

\[
\phi_x(\xi, \eta) = \frac{3}{4a} \lambda_x \left[ (1 - \xi^2)(1 - \eta) \right] w_i + \frac{3}{4a} \lambda_x \left[ (1 - \xi^2)(1 + \eta) \right] w_2 -
\]

\[
- \frac{3}{4a} \lambda_x \left[ (1 - \xi^2)(1 + \eta) \right] w_3 - \frac{3}{4a} \lambda_x \left[ (1 - \xi^2)(1 - \eta) \right] w_4 +
\]

\[
+ \frac{1}{8} \left[ 2 + 2\xi - 3\lambda_x \left( 1 - \xi^2 \right) \right] (1 - \eta) \phi_{x1} + \frac{1}{8} \left[ 2 + 2\xi - 3\lambda_x \left( 1 - \xi^2 \right) \right] (1 + \eta) \phi_{x2} +
\]

\[
+ \frac{1}{8} \left[ 2 - 2\xi - 3\lambda_x \left( 1 - \xi^2 \right) \right] (1 + \eta) \phi_{x3} + \frac{1}{8} \left[ 2 - 2\xi - 3\lambda_x \left( 1 - \xi^2 \right) \right] (1 - \eta) \phi_{x4}
\]

where:

\[
\xi = \frac{2x}{a} \quad \eta = \frac{2y}{b} \quad -1 \leq \xi, \eta \leq 1
\]

Similarly we may construct the rotation interpolation \( \phi_y(\xi, \eta) \), as well as all the other two-dimensional displacement functions, namely \( w(\xi, \eta), u(\xi, \eta) \) and \( v(\xi, \eta) \). A complete set of the expressions, by means of which the matrices \( A \) and \( C \) may be evaluated, is given in the Appendix.

Having determined the components of the strain displacement matrices \( B_m, B_b, B_s \), the strain fields may now be written in a form similar to (3.53):

\[
\varepsilon_\Phi = P_\Phi A_\Phi^{-1} C_\Phi \Phi = B_\Phi \Phi
\]

\[
\varepsilon_m = P_m A_m^{-1} C_m \Phi = B_m \Phi
\]

\[
\varepsilon_s = \frac{1}{\Omega} C_s \Phi = B_s \Phi
\]
3.2.4 Strain Energy and Stiffness Matrix

In order to determine the stiffness matrix of the element we make use of the strain energy density, expressed as follows:

\[ U = \frac{1}{2} \left( M_x^e \kappa_x + M_y^e \kappa_y + 2M_{xy}^e \kappa_{xy} + N_x^e \varepsilon_x + N_y^e \varepsilon_y + 2N_{xy}^e \varepsilon_{xy} + Q_x^e \gamma_{xz} + Q_y^e \gamma_{yz} \right) \]  

(3.61)

Substituting equations (3.12) - (3.25) into the above expression we obtain the following:

\[ U = U_b + U_m + U_s \]  

(3.62)

where \( U_b, U_m, U_s \) are respectively: the bending component of the strain energy density function (quadratic function of curvatures), the membrane component (quadratic function of membrane strains) and the transverse shear component of the strain energy. We now define the following groups of strains, namely the bending, the membrane and the shear strains separately in the form of vectors:

\[ \varepsilon_b = \{ \kappa_x, \kappa_y, 2 \kappa_{xy} \}^T \]  

(3.63)

\[ \varepsilon_m = \{ \varepsilon_x, \varepsilon_y, 2 \varepsilon_{xy} \}^T \]  

(3.64)

and

\[ \varepsilon_s = \{ \gamma_{xz}, \gamma_{yz} \}^T \]  

(3.65)

We then may write the strain energy densities \( U_b, U_m, U_s, U_{bm} \) in the matrix form as follows:

\[ U_b = \frac{1}{2} \varepsilon_b^T D \varepsilon_b \]  

(3.66)

\[ U_m = \frac{1}{2} \varepsilon_m^T S \varepsilon_m \]  

(3.67)

\[ U_s = \frac{1}{2} \varepsilon_s^T T \varepsilon_s \]  

(3.68)

The total strain energy in the element domain \( \Omega \) may be written as:

\[ \Pi_e = \iint_\Omega U \ d\Omega \]  

(3.69)

or using equations (3.66) - (3.68) as follows:

\[ \Pi_e = \frac{1}{2} \iint_\Omega \left( \varepsilon_b^T D \varepsilon_b + \varepsilon_m^T S \varepsilon_m + \varepsilon_s^T T \varepsilon_s \right) d\Omega \]  

(3.70)

We may express the strains \( \varepsilon_b, \varepsilon_m, \varepsilon_s \) in terms of the nodal displacement vector \( q \), as follows:

\[ \varepsilon_b = B_q \]  

(3.71)

\[ \varepsilon_m = B_m q \]  

(3.72)

\[ \varepsilon_s = B_s q \]  

(3.73)
where \( q \) is given by equation (3.32). Substituting expressions (3.71) - (3.73) into equation (3.70) we obtain the following:

\[
\Pi_e = \frac{1}{2} \mathbf{q}^T \mathbf{B}_b^T \mathbf{D} \mathbf{B}_b + \mathbf{B}_m^T \mathbf{S} \mathbf{B}_m + \mathbf{B}_s^T \mathbf{T} \mathbf{B}_s \ d\Omega q
\]  

(3.74)

or:

\[
\Pi_e = \frac{1}{2} \mathbf{q}^T [\mathbf{K}_b + \mathbf{K}_m + \mathbf{K}_s] \mathbf{q}
\]  

(3.75)

where \( \mathbf{K}_b, \mathbf{K}_m, \mathbf{K}_s \) are the element stiffness matrices related to bending, membrane and transverse shear deformation, given by:

\[
\mathbf{K}_b = \int_{\Omega} \mathbf{B}_b^T \mathbf{D} \mathbf{B}_b \ d\Omega
\]  

(3.76)

\[
\mathbf{K}_m = \int_{\Omega} \mathbf{B}_m^T \mathbf{S} \mathbf{B}_m \ d\Omega
\]  

(3.77)

\[
\mathbf{K}_s = \int_{\Omega} \mathbf{B}_s^T \mathbf{T} \mathbf{B}_s \ d\Omega
\]  

(3.78)

Substituting equations (3.58)-(3.60) into (3.76)-(3.78) we obtain:

\[
\mathbf{K}_b = \mathbf{C}_b^T \mathbf{A}_b^{-T} \int_{\Omega} \mathbf{P}_b^T \mathbf{D} \mathbf{P}_b \ d\Omega \mathbf{A}_b^{-T} \mathbf{C}_b
\]  

(3.79)

\[
\mathbf{K}_m = \mathbf{C}_m^T \mathbf{A}_m^{-T} \int_{\Omega} \mathbf{P}_m^T \mathbf{S} \mathbf{P}_m \ d\Omega \mathbf{A}_m^{-T} \mathbf{C}_m
\]  

(3.80)

\[
\mathbf{K}_s = \mathbf{C}_s^T \left( \frac{\mathbf{C}_s}{\Omega} \right) \mathbf{C}_s
\]  

(3.81)

The analysis of a problem by means of the finite elements formulated here involves solution of the system of linear algebraic equations:

\[
\mathbf{K} \mathbf{q} = \mathbf{R}
\]  

(3.82)

where \( \mathbf{K} \) is the element stiffness matrix given by:

\[
\mathbf{K} = \mathbf{K}_b + \mathbf{K}_m + \mathbf{K}_s
\]  

(3.83)

and \( \mathbf{R} \) is the external load vector; \( \mathbf{q} \) is the vector of nodal displacements given by equation (3.32).

3.3 Numerical Examples

Several benchmark problems selected from the literature (MacNeal & Harder 1984; Belytschko et al. 1985; Simo et al. 1989) are used here to evaluate the performance of the proposed generalized shell element. Any set of problems for shell elements should be discriminating. Inextensional bending modes of deformation must be tested, as well as rigid body modes, complex membrane state of stress and shear deformation modes (Belytschko et al. 1985). The problems were selected to challenge the aforementioned capabilities of the current formulation, as well as examine its functioning for thick and very thick shells. The convergence of the results is compared to other formulations available in the literature. Table 3.1 lists the shell elements used here as reference, and their corresponding abbreviations used later in the text.
Table 3.1 Listing of the standard shell elements used here as reference

<table>
<thead>
<tr>
<th>Name</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-ABQ</td>
<td>Doubly curved shell element used by ABAQUS, built on Mindlin/Sanders Koiter theory with reduced integration and hourglass control ('59, ‘60), (Hibbit et al. ’01)</td>
</tr>
<tr>
<td>4-DKQ</td>
<td>Discrete Kirchhoff quadrilateral element</td>
</tr>
<tr>
<td>9-GAMMA</td>
<td>Belytschko ‘85, biquadratic degenerated shell element with uniform reduced integration</td>
</tr>
<tr>
<td>4-SRI</td>
<td>Bilinear degenerated shell element with selective, reduced integration, Hughes ’86, ‘87</td>
</tr>
<tr>
<td>9-SRI</td>
<td>Biquadratic degenerated shell element with selective, reduced integration</td>
</tr>
<tr>
<td>MIXED</td>
<td>Simo ‘89, bilinear shell element with mixed formulations and full 2x2 quadrature</td>
</tr>
</tbody>
</table>

3.3.1 The Patch Test

A square plate problem is considered here, that is modeled by a single element, subjected to constant tension and bending, as indicated in Figure 3.6.

![Figure 3.6 Constant stress patch test: tension and bending (E-Young’s modulus; v-Poisson’s ratio)](image)

The displacements obtained using the present element are exact; they match the analytical solution.

3.3.2 Cantilevered Beam

Another problem frequently used as a benchmark test is the evaluation of the performance of the proposed element in straight cantilever beams (Figure 3.7). A point load applied to the free end of the beam evokes all the principal deformation modes.

![Figure 3.7. Cantilevered beam problem (E-Young’s modulus; v-Poisson’s ratio)](image)
The height and thickness of the beam are respectively, \( h = 0.4 \text{ m} \), and \( b = 0.2 \text{ m} \). We investigate the maximum displacement of the beam, which is modeled with four, eight and ten elements. The results are compared with those obtained using the engineering beam theory. Table 3.2 compares the displacement results.

Table 3.2 Vertical displacement at the free end [m] (EBT*-Engineering Beam Theory)

<table>
<thead>
<tr>
<th>Number of Elements</th>
<th>Analytical EBT* x10(^{-3})</th>
<th>Present FE x10(^{-3})</th>
<th>Normalized Present FE</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>9.52</td>
<td>9.369</td>
<td>0.98</td>
</tr>
<tr>
<td>8</td>
<td>9.52</td>
<td>9.493</td>
<td>0.997</td>
</tr>
<tr>
<td>10</td>
<td>9.52</td>
<td>9.517</td>
<td>0.999</td>
</tr>
</tbody>
</table>

Figure 3.8 Convergence of the present element for the cantilevered beam problem, displacement under the load

The results shown in Figure 3.8 indicate that the current element is in very good agreement with the analytical solution. It shows a robust performance.

3.3.3 Morley’s Hemispherical Shell (Morley & Moris 1978)

The following example is used as a standard test to evaluate the performance of the finite elements (MacNeal & Harder 1985, Simo et al. 1989). The problem represents a hemisphere with four point loads alternating in sign at 90° intervals on the equator, which is a free edge (Figure 3.9).

In the hemispherical shell problem, the membrane strains are very small, which makes this problem a discriminating test of the element’s ability to represent inextensional modes. Moreover, under these loading conditions, large sections of the
model rotate as almost rigid bodies, which allows us to check the ability of the element to model rigid body motion (Belytschko et al. 1985).

Bending strains contribute significantly to the radial displacement at the point of the application of the load $F$. The value of the displacement under the concentrated load is 0.094 in, which was obtained by MacNeal & Harder (1985). Steele (1987) and Simo et al. (1989) stated however that the analytical solution of this problem yields an answer of 0.093 in, which is used as a reference solution. The results are listed in Table 3.3. Figure 3.10 compares the proposed element’s performance to 4-DKQ, 9-SRI, 9-GAMMA, and MIXED elements.

![Figure 3.9 Morley’s Sphere (t-thickness, R-radius, E-Young’s modulus, F-load)](image)

As shown in Figure 3.10, the proposed doubly curved finite element performs very well in this test. It converges very fast, producing accurate results, even for a very coarse mesh.

We also investigate the transverse shear stresses for the above problem with different shell thicknesses. We compare the values obtained here with those of the 4-ABQ element in Table 3.4. Normal stresses $\sigma_x$ are used in Table 3.4 in order to compare:

![Table 3.3 Normalized displacement under the load for the hemispherical shell.](table)

<table>
<thead>
<tr>
<th>Nodes per side</th>
<th>Mixed</th>
<th>4-DKQ</th>
<th>9-GAMMA</th>
<th>9-SRI</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.919</td>
<td>0.663</td>
<td>1.37</td>
<td>0.002</td>
<td>1.17</td>
</tr>
<tr>
<td>5</td>
<td>1.004</td>
<td>0.928</td>
<td>1.09</td>
<td>0.01</td>
<td>1.08</td>
</tr>
<tr>
<td>9</td>
<td>0.998</td>
<td>1.001</td>
<td>1.02</td>
<td>0.05</td>
<td>1.002</td>
</tr>
<tr>
<td>17</td>
<td>0.999</td>
<td>1.003</td>
<td>1.00</td>
<td>0.3</td>
<td>1.002</td>
</tr>
</tbody>
</table>

Reference solution

Deflection under load: 0.093 in
the relative magnitudes of the normal and transverse shear stresses. The last column of the table gives the ratio of \( \frac{\tau_{xz}}{\sigma_x} \). It shows the increasing importance of the transverse shear stresses with the increase of the thickness of the shell. For the first shell analyzed, with thickness 0.04 in, \( \tau_{xz} \) is only 0.0068 of the normal stresses \( \sigma_x \), whereas for the thickness of 0.9 in the ratio increases to 0.12. It demonstrates the expected pattern of the transverse shear stresses becoming much more significant for thick shells.

![Figure 3.10 Comparison of results by different shell formulations - Morley’s Sphere, deflection under the load](image)

**Table 3.4** Transverse shear and normal stresses for the hemispherical shell

<table>
<thead>
<tr>
<th>Thickness t[in]</th>
<th>( \tau_{xz} ) [psi]</th>
<th>( \tau_{yz} ) [psi]</th>
<th>( \sigma_x ) [psi]</th>
<th>Ratio ( \tau_{xz}/\sigma_x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
<td>-38.71</td>
<td>-38.5</td>
<td>-22.38</td>
<td>-5691</td>
</tr>
<tr>
<td>0.1</td>
<td>-15.11</td>
<td>-14.98</td>
<td>-6.7</td>
<td>-965.7</td>
</tr>
<tr>
<td>0.18</td>
<td>-8.131</td>
<td>-7.96</td>
<td>-3.596</td>
<td>-305.4</td>
</tr>
<tr>
<td>0.28</td>
<td>-5.047</td>
<td>-4.97</td>
<td>-2.417</td>
<td>-127.3</td>
</tr>
<tr>
<td>0.4</td>
<td>-3.41</td>
<td>-3.19</td>
<td>-1.804</td>
<td>-62.34</td>
</tr>
<tr>
<td>0.54</td>
<td>-2.441</td>
<td>-2.3</td>
<td>-1.42</td>
<td>-33.93</td>
</tr>
<tr>
<td>0.7</td>
<td>-1.824</td>
<td>-1.642</td>
<td>-1.152</td>
<td>-19.92</td>
</tr>
<tr>
<td>0.9</td>
<td>-1.376</td>
<td>-1.27</td>
<td>-0.9376</td>
<td>-11.82</td>
</tr>
</tbody>
</table>
The present formulation provides sound and reasonable predictions of the transverse stresses, which is of a particularly great importance for the case of thick shells. It emphasizes one of the main thrusts of the proposed theory on which the current element is built. The approximations given here are better than most of the reference models, except for the MIXED shell element by Simo et al. (1989), which proves the validity of the theoretical concepts as well as the numerical procedure.

### 3.3.4 Pinched Cylinder with Diaphragms

The pinched cylinder with a diaphragm is one of the most severe tests for both inextensional bending modes and complex membrane states. An element that passes this test will also perform well if the boundary conditions are simplified to free ends. It is therefore sufficient to present only the cylinder with diaphragms (Belytschko et al. 1985).

A short cylinder, with two pinching vertical forces at the middle section and two rigid diaphragms at the ends is modeled here. We only consider one octant of the cylinder due to the symmetry and apply the appropriate boundary conditions. We investigate the radial displacement under the load, and normalize the results against the analytical solution of this problem: $1.82488 \times 10^{-5}$ in (Lindberg et al. 1969; Flugge, 1960). The geometry of the problem is shown in Figure 3.11.

**Figure 3.11** Pinched cylinder with diaphragms – geometry and material properties (t-thickness, E-Young’s modulus, ν-Poisson’s ratio)

The normalized numerical results are shown in Table 3.5 and Figure 3.12.

<table>
<thead>
<tr>
<th>Nodes per side</th>
<th>Mixed</th>
<th>4-DKQ</th>
<th>4-SRI</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.399</td>
<td>0.626</td>
<td>0.373</td>
<td>0.562</td>
</tr>
<tr>
<td>9</td>
<td>0.763</td>
<td>0.951</td>
<td>0.747</td>
<td>0.909</td>
</tr>
<tr>
<td>17</td>
<td>0.935</td>
<td>1.016</td>
<td>0.935</td>
<td>1.003</td>
</tr>
</tbody>
</table>
Figure 3.12 Comparison of results by different shell formulations – pinched cylinder with diaphragms, displacement under the load

The above problem proves to be one of the most demanding tests for shell elements. Most of the Mindlin elements, accounting for shear deformation do not converge very efficiently in this problem, except for the discrete Kirchhoff formulations. The present element once again offers a very good approximation, and fast convergence. It is known that for the pinched cylinder with diaphragms, the elements employing the discrete Kirchhoff constraints are among the best performers. The new element is however not any lesser in this case, providing results closer to the exact solution than all of the other elements considered, except for the mentioned 4-DKQ, (Discrete Kirchhoff constraints).

In spite of showing a robust performance in this test, the finite element developed here experiences mild membrane locking. When the series of cylinders with diaphragms, with a reduced radius of curvatures (R=200 in, R=100 in, R=50 in) were examined, the mesh of finite elements showed a tendency to be slightly too stiff, predicting about 80% of the reference solution, for the case of R=50 in. The problem gradually disappears when the radius of curvature of the cylinder is increased.

3.3.5 Scordellis-Lo Roof (1969)

The Scordellis-Lo Roof (1969) problem is one of the best tools in testing the accuracy of the elements in solving complex states of membrane strains. A representation of the inextensional modes is not crucial in this problem, as the membrane strain energy makes a large portion of the strain energy. Therefore, even the elements that experience
severe membrane locking may converge in this test. However, inaccuracies in the membrane stress representation in the finite element formulation will hinder the convergence process.

The Scordellis-Lo Roof is a short cylindrical section, loaded by gravity forces (Scordelis & Lo 1969). The geometry and material properties of the problem are shown in Figure 3.13.

Similarly to the case of the pinched cylinder we only consider an octant due to the symmetry of this problem. The vertical displacement at midpoint of the free edge was reported by Scordellis and Lo as 0.3024 in, which will serve here as a reference solution. Table 3.6 lists the results of the problem. Convergence of the element is shown in Figure 3.14.

**Figure 3.13** Scordellis-Lo Roof (t-thickness, E-Young’s modulus, v-Poisson’s ratio)

![Diagram of Scordellis-Lo Roof](image)

**Table 3.6** Normalized Displacement – Scordellis-Lo Roof

<table>
<thead>
<tr>
<th>Nodes per side</th>
<th>Mixed</th>
<th>4-DKQ</th>
<th>4-SRI</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1.45</td>
<td>1.391</td>
<td>1.263</td>
<td>1.58</td>
</tr>
<tr>
<td>5</td>
<td>1.083</td>
<td>1.048</td>
<td>0.964</td>
<td>1.14</td>
</tr>
<tr>
<td>9</td>
<td>1.015</td>
<td>1.005</td>
<td>0.984</td>
<td>1.022</td>
</tr>
<tr>
<td>17</td>
<td>1.00</td>
<td>0.996</td>
<td>0.999</td>
<td>1.002</td>
</tr>
</tbody>
</table>

All the elements considered here converge reasonably well. The present formulation is in good agreement with the analytical result, although, it does not show a better performance than the elements compared with. It converges completely with a reasonable mesh of 16x16 elements (Figure 3.14).

Although the vertical displacement at midpoint of the shell is very closely approximated, as seen in Figure 3.14, the deflection pattern is less accurate. The value of the vertical displacement at a distance \( L/4 \) from the midpoint is about 10% larger than displacement at midpoint. This error is, as expected, also observed in the pattern of the internal forces, which are calculated from the displacement. Since Scordelis-Lo Roof is a very demanding test of the ability of the element to model complex states of membrane strains, the deficient interpolation of these strains is most likely the reason for the loss of accuracy of the displacement patterns in this problem.
3.3.6 Pinched Cylinder

A very similar test to pinched cylinder with diaphragms (Section 3.3.4) is presented here, serving however a different purpose. The ends of the cylinder are free (no rigid diaphragms). As pointed out before, an element that gives accurate results for the cylinder with diaphragms will also perform very well if the boundary conditions are simplified to free ends. We therefore use the present example to investigate the performance of the present formulation for the case of very thick shells. The characteristics of the problem are shown in Figure 3.15.
We vary the thickness of the cylinder and investigate the displacement under point loads. The results obtained with the present finite element are compared with those obtained using 4-ABQ-doubly curved shell element used by ABAQUS, built on Mindlin/Sanders Koiter (Sanders 1959, Koiter 1960) theory with reduced integration and hourglass control, as well as an analytical solution of the problem by Timoshenko and Woinowsky-Krieger (1959). In the latter case the problem is treated as an inextensional deformation of a circular cylinder i.e. the membrane strains are zero, and the deformation is governed by the bending modes only. The values of the displacements provided by the inextensional solution are therefore slightly too low. Nevertheless, they may still be regarded as accurate, especially for the case of thin shells. The correction factor increasing the displacement under the load by 0.4%, due to extensional bending was estimated by Ashwel and Sabir (1971) for a thin shell (t=R/320). Although for very thick shells (t=R/2) this correction factor reaches 1.3%, it is still considered small. Thus, the results of the inextensional deformation of cylindrical shell by Timoshenko and Woinowsky-Krieger (1959), may still serve as a reference solution. The results of the pinched cylinder problem are shown in Table 3.7 and Figure 3.16.

<table>
<thead>
<tr>
<th>t [in]</th>
<th>Analytical</th>
<th>4-ABQ</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9.02E-05</td>
<td>1.08E-04</td>
<td>9.82E-05</td>
</tr>
<tr>
<td>1.2</td>
<td>5.22E-05</td>
<td>8.63E-05</td>
<td>6.08E-05</td>
</tr>
<tr>
<td>1.4</td>
<td>3.29E-05</td>
<td>5.38E-05</td>
<td>3.92E-05</td>
</tr>
<tr>
<td>1.6</td>
<td>2.2E-05</td>
<td>4.14E-05</td>
<td>2.84E-05</td>
</tr>
<tr>
<td>2</td>
<td>1.13E-05</td>
<td>2.39E-05</td>
<td>1.63E-05</td>
</tr>
<tr>
<td>2.5</td>
<td>5.77E-06</td>
<td>1.52E-05</td>
<td>1.13E-05</td>
</tr>
</tbody>
</table>

Although the Mindlin element with reduced integration provides a good approximation of the analytical solution, the present formulation offers superior performance. The pattern of decreasing value of the displacement with increasing thickness of the shell is in very good agreement with the analytical, inextensional bending solution. It is worthwhile to mention that the analytical solution used here as a reference is not exact, as previously discussed. The present formulation proves to be more accurate than the Mindlin type of element in the analysis of thick shells. Moreover, the 4-ABQ element, as well as all the other elements using numerical integration are dependent on the number of integration points. While it is known that with appropriate choice of the number of integration points one may obtain an accurate result, this process is cumbersome when compared with the present algorithm with an explicit stiffness matrix and numerical integration not employed.
Figure 3.16  Pinched Cylinder - deflection under the load [in]
CHAPTER 4

GEOMETRICALLY NON-LINEAR FINITE ELEMENT ANALYSIS
OF THICK PLATES AND SHELLS

4.1 Introduction

A problem of geometric non-linearity may be explained through the example of the simple beam. We consider a plane cantilever beam, subjected to an end load, as shown in Figure 4.1 (Cook 1989).

![Figure 4.1 Geometric non-linearity; cantilever beam under the end load](image)

We assume that the beam is slender and the material is linearly elastic and we seek the quasistatic deflections produced by the end load \( P \). If the deflections are small than the linear theory is adequate to simulate the behaviour of the beam. The reactant bending moment at the fixed end will be \( M = PL \), as the shortening of the moment arm \( e \) is negligible. For large displacements, \( e \) becomes significant and the reactant bending moment is \( M = P(L - e) \). If the effect of large displacement is to be taken into account, then the equilibrium equations must be written in the deformed configuration. This is because the deformation of the beam substantially alters the location of the external load \( P \). The equations describing such an effect are non-linear and the nature of non-linearity is geometric.

In the investigation of the behaviour of plates and shells large displacements play a very important role. Certain parts of the structure under given loading conditions may undergo large rigid rotations and translations. Considering these effects becomes even more important in the elasto-plastic and damage analysis of shells. The regions of the structure deforming inelastically will most likely undergo large displacements. These can only be approximated through the geometrically non-linear analysis. The objective of this dissertation is to develop a reliable computational model for the elasto-plastic and damage analysis of shells, and therefore to achieve the desired accuracy, geometrical non-linearities must be considered.

We can usually distinguish two separate types of geometrical non-linearity when analyzing shells. These are large deformations and large rotations. Large deformations
are attributed to the stretch of the middle surface of the shell undergoing big
displacements. Large rotations on the other hand, are attributed to the significant changes
of the slope during the analysis. These changes cause the transformation matrix of the
coordinates to also change during the analysis. We may also have rigid body motion
without any strains. In this case, large rigid rotations and translations are considered, but
the strains remain small.

For the purpose of simplicity and conciseness, we only regard the most significant
effects from the viewpoint of shell behaviour. This leads to including large rigid rotations
and translations but not large strains. We use the Updated Lagrangian description, with
the total rotations decomposed into large rigid and moderate relative rotations. The
relative rotations and the derivatives of the in-plane displacements from two consecutive
configurations can be considered small, (Shi and Atluri, 1988; Shi and Voyiadjis, 1991).
Consequently, the quadratic terms of the derivatives of the in-plane displacement are
negligible. We therefore have a non-linear analysis with large displacements and
rotations, but small strains. As shown later (Section 4.4, Chapter 5 & 6), such a treatment
of the geometrical non-linearities is capable of simulating the shell behaviour with
sufficient accuracy. This is also very convenient for the elasto-plastic modeling of shells
in the stress resultant space. If the strains are small, then the assumption of additive
decomposition of strains into elastic and plastic part, commonly used in modeling of
plasticity, may be extended to displacements. This allows using the plastic node method
in the elasto-plastic considerations.

Although the main motivation for including large displacements in the analysis is
validating the plastic and damage investigations of shells, the procedure outlined in this
chapter is universal and may be used as a stand-alone algorithm. The previously
developed elastic shell model with constitutive equations derived in Chapter 2, and
elastic stiffness matrix in Chapter 3, is extended here to account for the geometrical non-
linearities.

In the following sections, we briefly discuss the nature of the Updated Lagrangian
formulation, used here in the treatment of geometric non-linearities. In Section 4.3, the
kinematics of the shell is presented, followed by the derivation of the explicit tangent
stiffness matrix in Section 4.4. Finally, a numerical example is given, challenging the
adopted concepts.

4.2 Updated Lagrangian Formulation

The Updated Lagrangian description that has proven to be a very effective method
(Bathe, 1982; Flores and Onate, 2001; Horrignoe and Bergan, 1978; Kebari and Cassell,
1992) is adopted here. In the Lagrangian formulation of a mechanical problem, we study
a coordinate frame in which the body under investigation is rigidly moving and rotating,
and may also be deformed. This method is based on the calculation of the increments of
the displacements. In the Updated Lagrangian formulation the reference configuration is
in the state after the deformation, at time $t + \Delta t$, as opposed to the Total Lagrangian
formulation, in which the reference configuration is at time $t$ (Figure 4.2). In the Updated
Lagrangian approach, the element local coordinates and local reference frame are
continuously updated during the deformation. The transformation matrix given by Argyris (1982) is employed to handle large rigid rotations.

**Figure 4.2** Updated Lagrangian method

### 4.3 Shell Kinematics

As described in the previous section, the Updated Lagrangian method is employed in the present study of large displacements and rotations of the shell element. The coordinates of the nodal points are continuously updated during the deformation. The rotations are additively decomposed into large rigid rotations and moderate relative rotations (Shi and Voyiadjis 1991).

The structure under consideration is defined in the global, fixed coordinate system $\mathbf{X}$. We also have the local coordinate system $\mathbf{x}$, surface coordinates at any nodal point $\mathbf{x}$, and base coordinates, which serve as a reference frame for the global degrees of freedom (Figure 4.3).

**Figure 4.3** Local coordinate system and normal vector $\mathbf{e}_{s3}$
4.3.1 Local Coordinates

In order to obtain the unit vector in the direction normal to the plane of the element, we first define two vectors, $\overrightarrow{41}$ and $\overrightarrow{42}$ connecting the origin of the coordinate system (point 4) to points 1 and 2 respectively. The cross product of these two vectors, divided by its length, gives $e_1$, as shown in Figure 4.3 and given by equation (4.1):

$$e_1 = \frac{\overrightarrow{41} \times \overrightarrow{42}}{\overrightarrow{41} \times \overrightarrow{42}}$$  \hfill (4.1)

The unit vector $e_2$ may be similarly obtained as a cross product of $e_3$ and $e_1$.

We may now determine the relation between the global coordinates $X$ and element local coordinates in configuration $k$:

$$^k e = ^k R E$$  \hfill (4.2)

where $^k e$ is the unit base vector of the local coordinates in configuration $k$, and $E$ is the unit base vector of the global coordinates; and $R$ is a transformation matrix from local to global coordinates.

4.3.2 Surface Coordinates

The surface coordinate system $x_s$ originates at each node of the element. As defined by Shi and Voyiadjis (1991), the position and direction of this system are functions of rotations. Surface coordinates translate and rigidly rotate with the element. Consequently, $x_{s3}$ is always normal to the surface of the element.

The finite rigid body rotation vector $\mathbf{V}$ is given by:

$$\mathbf{V} = \begin{bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{bmatrix}$$  \hfill (4.3)

where $\Theta_1, \Theta_2, \Theta_3$ are rigid body rotations around $x, y, z$ axes respectively.

The transformation matrix of large rotations $T_\theta$, given by Argyris (1982) is used here:

$$T_\theta = \exp(\tilde{\theta})$$  \hfill (4.4)

with:

$$\tilde{\theta} = \tilde{\theta}_j = e_{jk}\Theta_k, \quad k = 1, 2, 3$$  \hfill (4.5)

where $\tilde{\theta}$ is a skew symmetric matrix and $e_{jk}$ is the permutation tensor. In the above equation the indicial notation is used, with Einstein’s summation convention. The transformation of the surface coordinates is therefore:

$$\mathbf{V}' = T_\theta \mathbf{V}$$  \hfill (4.6)

where $\mathbf{V}'$ is a rigid body rotation vector transformed into a new position. Similarly, we may write a transformation of the surface coordinates for a given rotation vector $\Theta_j$ resulting from configuration $k−1$ to $k$ at node $j$:
\[ k \mathbf{e}_s = \mathbf{T}_{k}^{-1} \mathbf{e}_s \]  

(4.7)

where \( k \mathbf{e}_s \) are the unit base vectors of the surface coordinates at configuration \( k \).

Defining the transformation between \( \mathbf{E} \) and \( k \mathbf{e}_s \) as:

\[ k \mathbf{e}_s = k \mathbf{R}_s \mathbf{E} \]  

(4.8)

we may rewrite equation (4.7) as:

\[ k \mathbf{e}_s = \mathbf{T}_{k}^{-1} \mathbf{R}_s \mathbf{E} = k \mathbf{R}_s k \mathbf{R}_s^T k \mathbf{e} = k \mathbf{S}_j k \mathbf{e} \]  

(4.9)

where \( k \mathbf{R}_s^T \) is the transpose of \( k \mathbf{R} \) defined in equation (4.2), and \( k \mathbf{S}_j \) is a transformation matrix from local to the surface coordinate system. It is worthy to note that \( 0 \mathbf{R}_s \) is a 3x3 identity matrix for a flat plate.

### 4.3.3 Base Coordinates

The base coordinates defined by Horrigmoe and Bergan (1978) are adopted here as a common reference frame to which all element properties are transformed, prior to the assembly of the stiffness matrices. The base coordinates are defined by the combination of the fixed global and base coordinates.

The global degrees of freedom at node \( j \) are the incremental translations: \( \Delta U_j, \Delta V_j, \Delta W_j \) in the directions of global coordinates \( X, Y, Z \) and rotations \( \Theta_{sj}, \Theta_{sj} \) around \( x_s, y_s \). The local degrees of freedom at node \( j \) are the incremental translations \( \Delta u_j, \Delta v_j, \Delta w_j \) in the directions of local coordinates \( x, y, z \) and rotations \( \phi_{sj}, \phi_{sj} \) around \( x, y \). The transformation of the increments of the displacements at node \( j \) from the local coordinate system \( \Delta \mathbf{q}_j \), to the corresponding base coordinates, \( \Delta \mathbf{q}_bj \) may be written as:

\[
\Delta \mathbf{q}_bj = \begin{bmatrix} \Delta U_j \\ \Delta V_j \\ \Delta W_j \\ \Theta_{sj} \\ \Theta_{sj} \end{bmatrix} = \begin{bmatrix} k \mathbf{R}_s^T & 0 \\ 0 & k \mathbf{S}_j \end{bmatrix} \begin{bmatrix} \Delta u_j \\ \Delta v_j \\ \Delta w_j \\ \phi_{sj} \\ \phi_{sj} \end{bmatrix} = k \mathbf{T}_{sj} \Delta \mathbf{q}_oj
\]  

(4.10)

in which \( k \mathbf{S}_j \) is the upper left 2x2 submatrix of \( k \mathbf{S}_j \) defined in equation (4.9). The transformation matrix for the nodal displacement vector can be written as:

\[
\Delta \mathbf{q}_b = k \mathbf{T}_b \Delta \mathbf{q}_e
\]  

(4.11)

where \( k \mathbf{T}_b \) is composed of \( k \mathbf{T}_{bj} \) with \( j = 1, 2, 3, 4 \).

The vector of the local increments of displacements nodal displacements is shown in Figure 4.4 and given by equation (4.12):

\[
\Delta \mathbf{q}_{oj} = \{ \Delta u_j, \Delta v_j, \Delta w_j, \Delta \phi_{sj}, \Delta \phi_{sj} \}^T \quad j = 1, 2, 3, 4
\]  

(4.12)
4.4 Explicit Tangent Stiffness Matrix

In the geometrically non-linear analysis of plates and shells presented here, we use the shell constitutive equations derived in Chapter 2. These equations, transformed into a rectangular coordinate system are given by the equations (3.12)-(3.26). In order to determine the tangent stiffness matrix of the element we define $\delta\varepsilon_b, \delta\varepsilon_m, \delta\varepsilon_s$ as virtual elastic bending, membrane and transverse shear strains respectively ($\delta$-virtual) and $\mathbf{M}, \mathbf{N}, \mathbf{Q}$ as stress couples and stress resultants of the element.

Figure 4.4 Incremental degrees of freedom of shell element

Rewriting the shell constitutive equations in the matrix form yields:

$$\mathbf{M} = \mathbf{D}\varepsilon_b$$  \hspace{1cm} (4.13)  
$$\mathbf{N} = \mathbf{S}\varepsilon_m$$  \hspace{1cm} (4.14)  
$$\mathbf{Q} = \mathbf{T}\varepsilon_s$$  \hspace{1cm} (4.15)  

where $\varepsilon_b, \varepsilon_m, \varepsilon_s$ are bending, membrane and shear strains, defined by the equations (3.45)-(3.47), rewritten here in the incremental form:

$$\varepsilon_b = \begin{pmatrix} \frac{\partial \Delta \phi_x}{\partial x} \\ \frac{\partial \Delta \phi_y}{\partial y} \\ \frac{\partial \Delta \phi_z}{\partial y} + \frac{\partial \Delta \phi_y}{\partial x} \end{pmatrix}$$  \hspace{1cm} $\varepsilon_m = \begin{pmatrix} \frac{\partial \Delta u}{\partial x} + \frac{\Delta w}{R} \\ \frac{\partial \Delta v}{\partial y} + \frac{\Delta w}{R} \\ \frac{\partial \Delta w}{\partial y} - \frac{\phi_y}{R} \end{pmatrix}$  \hspace{1cm} $\varepsilon_s = \begin{pmatrix} \frac{\partial \Delta w}{\partial x} - \phi_x - \frac{\Delta u}{R} \\ \frac{\partial \Delta w}{\partial y} - \phi_y - \frac{\Delta v}{R} \end{pmatrix}$  \hspace{1cm} (4.16)
and $D, S, T$ are bending, membrane and shear rigidities matrices respectively, given by:

$$
D = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu/2 \end{bmatrix}, \quad
S = S \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu/2 \end{bmatrix}, \quad
T = T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

(4.17)

$D, T, S$ are defined by the equation (3.26).

We also make use of the linearized equilibrium equations of the system at configuration $k+1$ in the Updated Lagrangian formulation, expressed by the principle of virtual work, which in finite element modeling takes the form:

$$
\int_{\Omega} \left( \delta \varepsilon_b^T D \varepsilon_b + \delta \varepsilon_m^T S \varepsilon_m + \delta \varepsilon_s^T T \varepsilon_s \right) dxdy + \int_{\Omega} \delta \theta^T k F \theta dxdy = k+1 R - k+1 R
$$

(4.18)

where $k+1 R$ is the total external virtual work at step $k+1$.

As previously, the quasi-conforming technique proposed by Tang et al. (1980, 1983) is used to compute the element stiffness matrix. We therefore directly interpolate the strains. The compatibility equations are only satisfied in the weak sense, i.e. under the integral sign. Once again, all the integrations are calculated analytically and the explicit form of the stiffness matrix is preserved.

The strain interpolation formulas are identical to those used in the linear elastic analysis in Chapter 3 (equations (3.45)-(3.47)). Following the procedure outlined in Chapter 3, we obtain:

$$
\varepsilon_b = P_b A_b^{-1} C_b \Delta q^e = B_b \Delta q^e
$$

(4.19)

$$
\varepsilon_m = P_m A_m^{-1} C_m \Delta q^e = B_m \Delta q^e
$$

(4.20)

$$
\varepsilon_s = \frac{1}{\Omega} C_h \Delta q^e = B_s \Delta q^e
$$

(4.21)

$\theta$ in the equation (4.18) is the slope vector and $k F$ is a membrane stress resultant matrix at step $k$ given by:

$$
\theta = \begin{bmatrix} \partial \Delta w \\ \partial \Delta W \\ \partial \Delta w \end{bmatrix}, \quad
k F = \begin{bmatrix} k N_x \\ k N_y \\ k N_y \end{bmatrix}
$$

(4.22)

The slope field $\theta$ is evaluated in a similar way to the strain fields, using the quasi-conforming technique (Tang et al., 1980, 1983). A bilinear interpolation is used as given by Shi and Voyiadjis (1991) to approximate the slope field:
The slope field $\theta$ is therefore expressed in terms of the slope-displacement matrix $G$:

$$\theta = PA^{-1}C\Delta q^e = G\Delta q^e$$  \hspace{1cm} (4.25)

The matrix $A$ appearing in the equations (4.24) may be evaluated quite easily, as was shown in Chapter 3. The matrix $C$ in equation (4.24) can be evaluated through the quasi-conforming technique as follows (Shi and Voyiadjis 1990):

$$C_i\Delta q_{ae} = \int_{\Omega} \frac{\partial \Delta w}{\partial x} dx dy = \oint \Delta w_n ds$$, \hspace{1cm} (i = 1, 2, 3, ..., $N_{ae}$)

$$C_2\Delta q_{ae} = \int_{\Omega} \frac{\partial \Delta w}{\partial x} x dx dy = \oint \Delta w x_n ds - \int_{\Omega} \Delta w dx dy =$$

$$= \oint \Delta w x_n ds - \int_{-1}^{-1} \int_{-1}^{1} \Delta w(\xi, \eta) J|d\xi d\eta$$

$$C_3\Delta q_{ae} = \int_{\Omega} \frac{\partial \Delta w}{\partial y} y dx dy = \oint \Delta w y_n ds$$

$$C_4\Delta q_{ae} = \int_{\Omega} \frac{\partial \Delta w}{\partial y} y dx dy = \oint \Delta w x_n ds - \int_{\Omega} \Delta w dx dy =$$

$$= \oint \Delta w x_n ds - \int_{-1}^{-1} \int_{-1}^{1} \Delta w(\xi, \eta) J|d\xi d\eta$$  \hspace{1cm} (4.26)

where $N_{ae}$ denotes the number of the nodal displacement variables in an element; $n_x = \cos (n, x)$ is the cosine of the angle between the normal vector to the boundary and a direction of the $x$ axis; $s$ is the tangential coordinate along an element boundary; $|J|$ is the Jacobian; $\Delta w(\xi, \eta)$ is an interpolation of the transverse displacement of the element in the isoparametric coordinates. In this dissertation the cubic interpolation of $\Delta w$ along the boundary of the elements, given by Hu (1984) will be used to evaluate the $C$ matrix:
\[
\Delta w(s) = \left[1 - \xi + \lambda \left(\xi - 3\xi^2 + 2\xi^3\right)\right] \Delta w_i \\
+ \left[\xi - \xi^2 + \lambda \left(\xi - 3\xi^2 + 2\xi^3\right)\right] \frac{L}{2} \Delta \phi_i + \\
+ \left[\xi - \lambda \left(\xi - 3\xi^2 + 2\xi^3\right)\right] \Delta w_j + \\
+ \left[-\xi + \xi^2 + \lambda \left(\xi - 3\xi^2 + 2\xi^3\right)\right] \frac{L}{2} \Delta \phi_j
\]
(4.27)

where \( l \) is the length of the side of the element; \( \Delta \phi_i, \Delta \phi_j \) are tangential rotations at nodes \( i \) and \( j \) respectively, and \( D, T \) are flexural and transverse shear rigidities respectively. The influence of parameter \( \lambda \) is explained in the previous chapter.

Using equation (4.25), the virtual work principle given by (4.18) may now be rewritten:
\[
\int \int_{\Omega} \left( \delta \varepsilon_{b}^T D \varepsilon_{b} + \delta \varepsilon_{m}^T S \varepsilon_{m} + \delta \varepsilon_{s}^T T \varepsilon_{s} \right) dx dy + \delta \Delta q^T K_e \Delta q^e = k^+ R - \\
\int \int_{\Omega} \left( \delta \varepsilon_{b}^T T \kappa M + \delta \varepsilon_{m}^T T \kappa N + \delta \varepsilon_{s}^T T \kappa Q \right) dx dy
\]
(4.28)

where \( K_e \) is the initial stress matrix defined as:
\[
K_e = \int \int_{\Omega} G^{T \kappa} F G dx dy
\]
(4.29)

Substituting equations (4.19)-(4.21) into the right hand side of the equation (4.28), we may write:
\[
\int \int_{\Omega} \left( \delta \varepsilon_{b}^T T \kappa M + \delta \varepsilon_{m}^T T \kappa N + \delta \varepsilon_{s}^T T \kappa Q \right) dx dy = \delta \Delta q^T \Delta f
\]
(4.30)

where \( f \) is the internal force vector resulting from the unbalanced forces in configuration \( k \) and is expressed as follows:
\[
f = \int \int_{\Omega} \left( B_b^{T \kappa} M + B_m^{T \kappa} N + B_s^{T \kappa} Q \right) dx dy
\]
(4.31)

Similarly, substitution of the equations (4.19)-(4.21) and (4.30) into the equation (4.28) yields:
\[
\sum_{elem} \delta \Delta q^e \left( K_e + K_g \right) \Delta q^e = k^+ R - \sum_{elem} \delta \Delta q^e \Delta f
\]
(4.32)

where \( K_e \) is a linear elastic stiffness matrix of the element given by:
\[
K_e = \int \int_{\Omega} \left( B_b^{T \kappa} D B_b + B_m^{T \kappa} S B_m + B_s^{T \kappa} T B_s \right) dx dy
\]
(4.33)

Redefining the total external virtual work as:
\[
k^+ R = k^+ R^* \delta \Delta q
\]
(4.34)
we finally obtain:

\[
\left( K_e + K_g \right) \Delta q^{e} - \Delta^\ast + \Delta f = 0
\]  (4.35)

or:

\[
K_{eg} \Delta q^{e} = \Delta^\ast - \Delta f
\]  (4.36)

where:

\[
K_{eg} = K_e + K_g
\]  (4.37)

The tangent stiffness matrix given by equation (4.37) is similar to the one presented by Shi & Voyiadjis (1990). The current formulation is however much more general as it is universal and suitable for analysis of plates, shells and beams.

One of the most important features of the derived tangent stiffness is its explicit form. This is due to the application of the quasi-conforming technique in the formulation of both the linear elastic stiffness matrix, and the initial stress stiffness matrix, which allows all the integrations to be performed analytically. This makes the present model extremely efficient from the point of view of computer time and power and in the same time mathematically consistent. All the matrices comprising the stiffness matrix of the element may be integrated analytically and exactly, without employing the numerical selective reduced integration.

### 4.5 Numerical Example

For the purpose of the analysis, a finite element code was developed in programming language Fortran 95. A modified Newton-Raphson technique was employed to solve a system of non-linear, incremental equations. The external forces in the system of the equations (4.36) are balanced by the iteration scheme. The local increments at the iterations are calculated using the arc-length method (Crisfield, 1991). The results delivered by the current model were computed using a personal computer. Further explanation of the numerical techniques employed here is provided in Chapter 7.

The example used to verify the validity and accuracy of the present model is a pinched hemispherical shell. This is a benchmark problem, commonly used to test the performance of shell elements with the influence of the large displacement taken into account. The results obtained by means of the procedure proposed here will be compared with other formulations available in the literature. Table 4.1 lists the references used here, and their corresponding abbreviations used later in the text.

<table>
<thead>
<tr>
<th>Name</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>F&amp;O</td>
<td>Numerical solution of pinched hemispherical shell by Flores and Onate (2001)</td>
</tr>
<tr>
<td>SIMO</td>
<td>Numerical solution of pinched hemispherical shell by Simo et al. (1990)</td>
</tr>
<tr>
<td>W&amp;V</td>
<td>The present formulation without shear forces included in the yield function</td>
</tr>
</tbody>
</table>
We consider a pinched hemispherical shell (Morley Sphere), with an 18° hole at the top, subjected to four point loads alternating in sign at 90° intervals on the equator. Due to the symmetry, we only model a quadrant of the shell. The geometry, deformed shape and material properties are shown in Figure 4.5. The linear elastic solution of the problem serves very often as a benchmark problem for linear analysis of shells, (Morley and Morris, 1978; Belytschko et al., 1985; MacNeal and Harder, 1985; Simo et al., 1989; Woelke and Voyiadjis, 2004). Ample sections of the shell rigidly rotate under these loading conditions, hence precise modeling of the rigid body motion is essential for good performance in this test, (Belytschko et al., 1985). Simo et al., (1990) and also Parish (1995), Hauptmann et al. (1998) and Flores et al. (2001) used the same problem with an increased load factor to examine the capabilities of their models in the description of large deformation. We only compare the results provided by the current formulation to those by Simo et al. and Flores et al. for conciseness. It is noteworthy, that in the case of geometrically non-linear analysis, the deflections under alternating forces are not equal. We therefore plot the equilibrium path for both points of application of the load - A, and B. The load displacement path is plotted in Figure 4.6.

![Figure 4.5 Pinched hemispherical shell (Morley Sphere): geometry, deformed shape and material properties](image)

The displacements calculated with W&V model (current model) are very accurate and compare very well with the reference solutions, proving that the present work gives adequate representation of large displacements.

The reliability of the present procedure will be further verified through a series of discriminating examples in Chapter 5, where the elasto-plastic behaviour of shells will be investigated. The large displacement representation is crucial for accurate predictions of the load-displacement curve for both plates and shells undergoing inelastic deformations.
Hence, close approximations of the elasto-plastic equilibrium path and collapse load, prove the robustness of the large displacement formulation.

**Figure 4.6** Pinched hemispherical shell (Morley Sphere) – Equilibrium paths (A-point under an inward load, B-point under an outward load)
CHAPTER 5

ELASTO-PLASTIC, GEOMETRICALLY NON-LINEAR FINITE ELEMENT ANALYSIS OF THICK PLATES AND SHELLS

5.1 Introduction

In this chapter, the computational model for the analysis of shells presented in the previous chapters will be further developed to simulate the elasto-plastic behaviour of plates and shells with the effect of large rotations considered. The shell constitutive equations given in Chapter 2 are again adopted here as a base for the formulation. A simple C⁰ quadrilateral, geometrically non-linear shell element presented in Chapter 4 is extended to account for material non-linearities.

Although the basic assumptions made, and the published literature pertaining to the subject of elasto-plastic analysis of shells were reviewed in Section 1.3.4, we will also briefly discuss these here, for self-completeness. In the treatment of material non-linearities the non-layered approach and the plastic node method (Ueda and Yao, 1982) are adopted. We use the Iliushin’s yield function expressed in terms of stress resultants and stress couples (Iliushin, 1956), modified to investigate the development of plastic deformations across the thickness (Crisfield, 1981), as well as the influence of the transverse shear forces on the plastic behaviour of plates and shells (Shi and Voyiadjis, 1992). Both isotropic and kinematic hardening rules are included in the yield function, with the latter derived on the basis of the Armstrong and Frederick evolution equation of backstress (Armstrong and Frederick, 1966) thus, reproducing the Bauschinger effect.

The biggest motivation for the advances in shell elements is not only accuracy but also computational efficiency. Following the argument given in Section 1.3.2, shells may be analyzed by means of solid, three-dimensional elements, defined in the stress space. This however requires sometimes prohibitively large storage of the computer. The shell elements based on the shell constitutive equations, relating stress resultants and stress couples to strains have proven to be just as accurate as the stress based elements, while being much cheaper, and therefore capable of approximating the exact solutions of problems with very complicated geometry and boundary conditions. Moreover, most of the analysis and design procedures used by structural engineers are aimed at the determination of the internal forces and bending moments in the structure and designing it to resist these forces. Thus, ideally, the analysis directed to the engineering community is performed in the stress resultant space. Nevertheless, using the stress resultant models in the elasto-plastic investigations has been cumbersome, mainly due to lack of suitable yield functions expressed in terms of the forces and moments, capable of representing the progressive plastification of the cross section, influence of all the components of the stress tensor on plastic behaviour, as well as isotropic and kinematic hardening effects. The most important feature of the framework given here is contriving an accurate, stress resultant based yield surface accounting for the gradual growth of the plastic curvatures, influence of the shear forces on yielding and isotropic and kinematic hardening rules, with the latter representing the Bauschinger effect. Application of such a yield surface is convenient from an engineering point of view, as it allows taking the full advantage of
the shell elements not only in the elastic, but also in the plastic zone, making the algorithm highly efficient. The effectiveness of this approach arises from the fact that here, unlike in the layered approach where the yield function is expressed in terms of the stresses, discretization through the thickness is not necessary. Furthermore, it leaves room for further enhancements aimed at, for instance, approximation of the effects of damage and/or rate dependence. The subject of damage of shells is discussed in Chapter 6. These advances lead to the objective of performing a full and comprehensive analysis of shells through the use of internal forces and moments.

This chapter is a continuation of the previous ones; hence, all the assumptions made before are still valid. We once again use the shell equations presented in Chapter 2. The linear elastic stiffness matrix, devised in Chapter 3 is adopted and the large displacements examined in Chapter 4 are also considered. The advantage of the explicit form of the elastic stiffness matrix, obtained by means of the quasi-conforming technique (Tang et al., 1980, 1983) is even more visible in non-linear computations where the stiffness matrix has to be evaluated many times during the analysis. Since we follow the non-layered approach, numerical integration is not performed at any stage of the analysis. All the integrals are calculated analytically, with the results later introduced into a computer code.

The order of this chapter is as follows: in Section 5.2 the yield surface with the flow and hardening rules will be derived. The elasto-plastic, large displacement stiffness matrix is later formulated in Section 5.3. The numerical examples verifying the performance of the constitutive equation, as well as the numerical procedure are given in Section 5.4.

5.2 Yield Criterion and Hardening Rule

5.2.1 Iliushin Yield Function (Iliushin, 1956)

As discussed in the Introduction, a yield criterion expressed in terms of stress resultants and couples is used here, similar to Iliushin yield function modified to account for the progressive development of the plastic curvatures and shear forces, as given by Shi and Voyiadjis (1992). The Iliushin’s yield function \( F \) can be written as:

\[
F = \frac{M^2}{M_0^2} + \frac{N^2}{N_0^2} + \frac{1}{\sqrt{3} M_0 N_0} \frac{|MN|}{\sigma_0^2} - Y(k) = 0
\]

or:

\[
F = \frac{|M|}{M_0} + \frac{N^2}{N_0^2} - \frac{Y(k)}{\sigma_0^2} = 0
\]

where \( N \) and \( M \) are the stress intensities given by:

\[
N^2 = N_x^2 + N_y^2 - N_x N_y + 3N_{xy}^2
\]

\[
M^2 = M_x^2 + M_y^2 - M_x M_y + 3M_{xy}^2
\]

\[
MN = M_x N_x + M_y N_y - \frac{1}{2} M_x N_y - \frac{1}{2} M_y N_x + 3M_{xy}^2
\]
and $M_0$ and $N_0$ are respectively the moment capacity of the cross section when the plastic hinge has formed, i.e. the cross section is fully plastic, and the normal force capacity of the cross section is given by:

$$M_0 = \frac{\sigma_0 h^2}{4}, \quad N_0 = \sigma_0 h$$  \hspace{1cm} (5.6)

The symbol $\sigma_0$ is the uniaxial yield stress, $Y(k)$ is a material parameter, which depends on the isotropic hardening parameter $k$; $h$ is the thickness of the shell, and $|.|$ denotes the absolute value; $N_x, N_y, N_{xy}$ and $M_x, M_y, M_{xy}$ are the stress resultants and stress couples defined in terms of the strains of the shell by the equations (3.18)-(3.25) and shown in Figure 5.1.

![Figure 5.1 Stress resultants on a shell element](image)

**5.2.2 The Influence of the Shear Forces**

The form of the yield condition given by the equation (5.1) may be easily derived from the von Mises function and the definition of normal stresses at top and bottom surfaces of the shell, as shown by Bieniek and Funaro (1976). In order to examine the influence of the transverse shear forces on the plastic behaviour of shells, the yield surface given by the equation (5.1) must be modified. We may include the transverse shear forces $Q_x, Q_y$ (Figure 5.1) by altering the stress intensity given by the equation (5.3) as follows (Shi and Voyiadjis, 1992):

$$N^2 = N_x^2 + N_y^2 - N_x N_y + 3\left(N_{xy}^2 + Q_x^2 + Q_y^2\right)$$  \hspace{1cm} (5.7)

It is shown later, (examples 5.4.1 and 5.4.2) that the representation of the shear forces in thick, plastic plates and shells may be very important.

**5.2.3 Development of the Plastic Hinge**

For a bending dominant situation, according to equation (5.1) or (5.2), the structure will behave linearly until the whole cross section is plastic, i.e. the plastic hinge has formed. In reality however, plastic curvature develops progressively from the outer
fibers of the shell or plate and the material behaves non-linearly as soon as the outer fibers start to yield. To account for the development of plastic curvature across the thickness, Crisfield (1981) introduced a plastic curvature parameter $\alpha\left(\bar{\kappa}^p\right)$, into the equations (5.1)-(5.2):

$$F = \frac{M^2}{\alpha^2 M_0^2} + \frac{N^2}{N_0^2} + \frac{1}{\sqrt{3}\alpha M_0 N_0} \frac{Y(k)}{\sigma_0^2} = 0$$

or:

$$F = \frac{|M|}{\alpha M_0} + \frac{N^2}{N_0^2} - \frac{Y(k)}{\sigma_0^2} = 0$$

(5.9)

where $\alpha$ was chosen such that $\alpha M_0$ follows the uniaxial moment-plastic curvature relation:

$$\alpha = 1 - \frac{1}{3} \exp\left(-\frac{8}{3} \bar{\kappa}^p\right)$$

(5.10)

and:

$$\bar{\kappa}^p = \sum \Delta \bar{\kappa}^p = \frac{Eh}{\sqrt{3}\sigma_0} \left[\left(\Delta \kappa_x^p\right)^2 + \left(\Delta \kappa_y^p\right)^2 + \Delta \kappa_x^p \Delta \kappa_y^p + \left(\Delta \kappa_{xy}^p\right)^2 / 4\right]^{1/2}$$

(5.11)

$\bar{\kappa}^p$ is the equivalent plastic curvature, $\Delta \kappa_x^p$, $\Delta \kappa_y^p$ and $\Delta \kappa_{xy}^p$ are the increments of the plastic curvatures. We note that for $\bar{\kappa}^p = 0$, $\alpha = 2/3$ and we obtain $\alpha M_0 = \frac{\sigma_y I^2}{6}$ which represents the moment capacity at first yield. If on the other hand, $\bar{\kappa}^p = \infty \rightarrow \alpha = 1$ we obtain the moment capacity of the fully plastic cross section. Therefore, through the introduction of the plastic curvature parameter $\alpha$ we account for the progressive development of the plastic curvatures and predict the first yield.

5.2.4 Bauschinger Effect and Kinematic Hardening Rule

The Bauschinger effect is a phenomenon observed in the experimental tests of metals. If a simple compression test were performed on a metal, the stress-strain curve would be almost identical as in a simple tension test. If however, the specimen is plastically prestrained in tension and then the load is reversed, the stress-strain curve in compression differs considerably from the curve, which would be obtained on reloading the specimen in tension, or on loading the undisturbed specimen in compression. As illustrated in Figure 5.2 for the specimen with the preloading $\sigma_y'$ in tension, its corresponding compression yielding occurs at the stress level $\sigma_y''$, which is less than the initial yield stress $\sigma_y$ and much less than the subsequent yield point $\sigma_y'$. This phenomenon is called the Bauschinger effect and is usually present when there is a load reversal. This proves that the strain is not a function of the stress alone, but also depends on the previous loading history. Thus, the material is load path dependent (Chen and Han, 1988). In order to model the Bauschinger effect one needs a veracious kinematic
hardening rule, which represents the rigid body motion of the yield surface in the stress or stress resultant space. The shape and orientation of the initial surface is maintained. The current dissertation is devoted to developing a stress resultant based model; hence, such a kinematic hardening rule is needed.

Figure 5.2 Bauschinger effect

Bieniek and Funaro (1976) introduced residual bending moments (‘hardening parameters’), allowing for the description of the Bauschinger effect. These were later successfully applied for dynamic (Bieniek et al., 1976), and viscoplastic dynamic analysis of shells (Atkatsh et al., 1982, 1983). In order to determine correctly the rigid translation of the yield surface in the stress resultant space, we need not only residual bending moments, but also residual normal and shear forces. These hardening parameters are related directly to backstress, representing the center of the yield surface in the stress space. We introduce a new kinematic hardening rule for plates and shells, with residual stress resultants, derived directly from the evolution of the backstress given by Armstrong and Frederick (1966). The yield surface is expressed as:

\[ F^* = \frac{|M^*|}{\alpha M_0} + \frac{(N^*)^2}{N_0^2} - \frac{Y(k)}{\sigma_0^2} = 0 \]  

where:

\[ (N^*)^2 = (N_x - N_x^*)^2 + (N_y - N_y^*)^2 - (N_x - N_x^*)(N_y - N_y^*) + \]
\[ + 3 \left[ (N_{xy} - N_{xy}^*)^2 + (Q_x - Q_x^*)^2 + (Q_y - Q_y^*)^2 \right] \]  

\[ (M^*)^2 = (M_x - M_x^*)^2 + (M_y - M_y^*)^2 - (M_x - M_x^*)(M_y - M_y^*) + \]
\[ + 3 (M_{xy} - M_{xy}^*)^2 \]  

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and \(M_x^*, M_y^*, M_{xy}^*, N_x^*, N_y^*, N_{xy}^*, Q_x^*, Q_y^*\) are previously described residual bending moments, normal and shear forces respectively. We now proceed to the definition of kinematic hardening parameters. For the purpose of conciseness, we use the indicial notation in the derivation, and only the final result is given using engineering notation. The Armstrong and Frederick’s evolution of backstress \(\rho_{ij}\) is given by:

\[
\Delta \rho_{ij} = c \Delta \varepsilon_{ij}^p - a \rho_{ij} \Delta \varepsilon_{eq}^p \tag{5.15}
\]

where \(a\) and \(c\) are material constants and the equivalent plastic strain increment is:

\[
\Delta \varepsilon_{eq}^p = \sqrt{\frac{2}{3} \Delta \varepsilon_{ij}^p \Delta \varepsilon_{ij}^p} \tag{5.16}
\]

The backstress represents the center of the translated yield surface in the stress space. It has the same dimension as the stress tensor. To compute the stress resultants we need to integrate the stresses over the thickness of the shell. We use the same definition here to derive the hardening parameters, which represent the center of the yield surface in the stress resultant space. We therefore need to integrate the backstress over the thickness of the shell or plate, to obtain residual normal and shear forces and bending moments. The definitions of the increments of hardening parameters are as follows:

\[
\Delta N_{ij}^* = \int_{-h/2}^{h/2} \Delta \rho_{ij} dz \tag{5.17}
\]

\[
\Delta M_{ij}^* = \int_{-h/2}^{h/2} \Delta \rho_{ij} dz \tag{5.18}
\]

Substituting equation (5.15) into equation (5.17) we obtain:

\[
\Delta N_{ij}^* = \int_{-h/2}^{h/2} \left( c \Delta \varepsilon_{ij}^p - a \rho_{ij} \Delta \varepsilon_{eq}^p \right) dz \tag{5.19}
\]

The increments of plastic strains \(\Delta \varepsilon_{ij}^p\) in equation (5.19) are membrane strains, due to normal forces only. These are constant across the thickness of the shell, and therefore, we may write:

\[
\Delta N_{ij}^* = c h \Delta \varepsilon_{ij}^p - a h \rho_{ij} \Delta \varepsilon_{eq}^p \tag{5.20}
\]

Defining the hardening parameters similarly to stress resultants:

\[
h \rho_{ij} = N_{ij}^* \tag{5.21}
\]

we may rewrite equation (5.20):

\[
\Delta N_{ij}^* = c h \Delta \varepsilon_{ij}^p - a N_{ij}^* \Delta \varepsilon_{eq}^p \tag{5.22}
\]

Constants \(a\) and \(c\) are given similarly to Bieniek and Funaro (1976):

\[
a = c = \beta_1 \left(1 - F\right) \frac{1}{h} \frac{N_0}{\varepsilon_0} \tag{5.23}
\]

where \(N_0\) and \(\varepsilon_0\) are given by:

\[
N_0 = \sigma_0 h \ , \ \varepsilon_0 = -\sigma_0 / E \tag{5.24}
\]

where \(F\) is a yield surface given in equation (5.9), \(h\) is a thickness of a plate and \(\beta_1\) is a constant. We therefore obtain:
\[ \Delta N_{ij}^* = \beta_i (1 - F) \frac{N_0}{\varepsilon_0} \left[ \Delta \varepsilon_{ij}^p - \frac{1}{h} N_{ij}^* \Delta \varepsilon_{eq}^p \right] \]  

(5.25)

Similarly, substituting equation (5.15) into equation (5.18) we determine the increments of the residual bending moments:

\[ \Delta M_{ij}^* = \int_{-h/2}^{h/2} \left( c \Delta \varepsilon_{ij}^p - a \rho_y \Delta \varepsilon_{eq}^p \right) zdz \]  

(5.26)

where \( \Delta \varepsilon_{ij}^p \) and \( \Delta \varepsilon_{eq}^p \) are:

\[ \Delta \varepsilon_{ij}^p = z \Delta \kappa_{ij}^p \]  

(5.27)

\[ \Delta \varepsilon_{eq}^p = \sqrt{\frac{2}{3}} z^2 \Delta \kappa_{ij}^p \Delta \kappa_{ij}^p \]  

(5.28)

Substituting equations (5.27) into equation (5.26) and integrating it we have:

\[ \Delta M_{ij}^* = c \frac{h^3}{12} \Delta \kappa_{ij}^p - a \frac{h^3}{12} \rho_y \Delta \kappa_{eq}^p \]  

(5.29)

or:

\[ \Delta M_{ij}^* = c \frac{h^3}{12} \Delta \kappa_{ij}^p - a \frac{h^3}{2} \rho_y \Delta \kappa_{eq}^p \] where \( \rho_y \frac{h^2}{6} = M_{ij}^* \)  

(5.30)

and constants \( a \) and \( c \) are expressed similarly to those in equation (5.23):

\[ a = c = \beta_2 (1 - F) \frac{12 M_0}{h^3 \kappa_o} \]  

(5.31)

which leads to:

\[ \Delta M_{ij}^* = \beta_2 (1 - F) \frac{M_0}{\kappa_o} \left[ \Delta \kappa_{ij}^p - \frac{6}{h^2} M_{ij}^* \Delta \kappa_{eq}^p \right] \]  

(5.32)

The hardening parameters may now be rewritten in engineering notation:

*If \( F^* = 1 \) and \( \forall F^* > 0 \) (plastic loading)*

\[ \Delta N_x^* = \beta_1 (1 - F) \frac{N_0}{\varepsilon_0} \left[ \Delta \varepsilon_x^p - \frac{1}{h} N_x^* \Delta \varepsilon_{eq}^p \right] \]  

\[ \Delta N_y^* = \beta_1 (1 - F) \frac{N_0}{\varepsilon_0} \left[ \Delta \varepsilon_y^p - \frac{1}{h} N_y^* \Delta \varepsilon_{eq}^p \right] \]  

\[ \Delta N_{xy}^* = \beta_1 (1 - F) \frac{N_0}{\varepsilon_0} \left[ \Delta \varepsilon_{xy}^p - \frac{1}{h} N_{xy}^* \Delta \varepsilon_{eq}^p \right] \]  

\[ \Delta Q_x^* = \beta_1 (1 - F) \frac{N_0}{\varepsilon_0} \left[ \Delta \varepsilon_{xz}^p - \frac{1}{h} Q_x^* \Delta \varepsilon_{eq}^p \right] \]  

\[ \Delta Q_y^* = \beta_1 (1 - F) \frac{N_0}{\varepsilon_0} \left[ \Delta \varepsilon_{yz}^p - \frac{1}{h} Q_y^* \Delta \varepsilon_{eq}^p \right] \]  

(5.33)
\[ \Delta M_x^* = \beta_2 (1 - F) \frac{M_0}{\kappa_0} \left[ \Delta \kappa_x^p - \frac{6}{h^2} M_x^* \Delta \kappa_{eq}^p \right] \]
\[ \Delta M_y^* = \beta_2 (1 - F) \frac{M_0}{\kappa_0} \left[ \Delta \kappa_y^p - \frac{6}{h^2} M_y^* \Delta \kappa_{eq}^p \right] \]
\[ \Delta M_{xy}^* = \beta_2 (1 - F) \frac{M_0}{\kappa_0} \left[ \Delta \kappa_{xy}^p - \frac{6}{h^2} M_{xy}^* \Delta \kappa_{eq}^p \right] \]  
(5.34)

If \( F^* < 1 \) and \( \nabla F^* \leq 0 \) (unloading or neutral loading)
\[ \Delta N_x^* = \Delta N_y^* = \Delta N_{xy}^* = \Delta Q_x^* = \Delta Q_y^* = \Delta M_x^* = \Delta M_y^* = \Delta M_{xy}^* = 0 \]  
(5.35)

Parameters \( \beta_1 \) and \( \beta_2 \) in the above formulation control the membrane force-membrane strain and moment-curvature relations. A value \( \beta_1 = \beta_2 = 2.0 \) is found to be of sufficient accuracy in the representation of behaviour of shells.

We therefore arrive at a final form of the yield function expressed in terms of stress resultants and couples, with isotropic and kinematic hardening rules. A graphic representation of the yield surface given by (5.12) on the \( N_x M_x \) plane with \( c = 1 \) and \( Y = \sigma_0^2 \) is shown in Figure 5.3. Point \( O' \) denotes the transferred center of the yield surface.

**Figure 5.3** Yield surface on \( N_x M_x \) plane – interpretation of kinematic hardening parameters \( O' \) is a center of transferred yield surface.
5.3 Explicit Elasto-Plastic Tangent Stiffness Matrix with Large Displacements

The plastic node method is adopted here (Ueda and Yao, 1982), i.e. the plastic deformations are considered to be concentrated in the plastic hinges. The yield function is only checked at each node of the finite elements. If the combination of stress resultants satisfies the yield condition, that node is considered plastic. In this method the inelastic deformations are only considered at the nodes, while the interior of the element remains always elastic.

When node \( i \) of the element becomes plastic, the yield function will take the form:

\[
F^*(N_i, Q_i, M_i, N^*_i, Q^*_i, M^*_i, k) = 0
\]  

(5.36)

where:

\[
N_i = \begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix}, \quad Q_i = \begin{bmatrix} Q_x \\ Q_y \end{bmatrix}, \quad M_i = \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix}
\]  

(5.37)

\[
N^*_i = \begin{bmatrix} N^*_x \\ N^*_y \\ N^*_{xy} \end{bmatrix}, \quad Q^*_i = \begin{bmatrix} Q^*_x \\ Q^*_y \end{bmatrix}, \quad M^*_i = \begin{bmatrix} M^*_x \\ M^*_y \\ M^*_{xy} \end{bmatrix}
\]  

(5.38)

At the same time the stress resultants must remain on the yield surface, i.e. the consistency condition must be satisfied:

\[
\frac{\partial F^*_i}{\partial M_i} dM_i + \frac{\partial F^*_i}{\partial N_i} dN_i + \frac{\partial F^*_i}{\partial Q_i} dQ_i + \\
\frac{\partial F^*_i}{\partial M^*_i} dM^*_i + \frac{\partial F^*_i}{\partial N^*_i} dN^*_i + \frac{\partial F^*_i}{\partial Q^*_i} dQ^*_i + \frac{\partial F^*_i}{\partial k} dk_i = 0
\]  

(5.39)

We assume additive decomposition of strains into elastic and plastic parts:

\[
\varepsilon = \varepsilon^e + \varepsilon^p
\]  

(5.40)

The associated flow rule is used here to determine the increments of plastic strains:

\[
\Delta \kappa^p_x = \sum_{i=1}^{NPN} \Delta \lambda_i \frac{\partial F^*_i}{\partial \kappa^e_x}
\]  

(5.41)

where \( NPN \) is the number of plastic nodes in the element and \( d \lambda_i \) is a plastic multiplier. The remaining increments of the plastic strains are obtained in the same way. The plastic strain fields are interpolated as in the linear elastic analysis, (equations (3.45)-(3.47)) rewritten here in the incremental form:

\[
\Delta \varepsilon^p_b = \begin{bmatrix} \Delta \kappa^p_x \\ \Delta \kappa^p_y \\ 2\Delta \kappa^p_{xy} \end{bmatrix}, \quad \Delta \varepsilon^p_m = \begin{bmatrix} \Delta \varepsilon^p_x \\ \Delta \varepsilon^p_y \\ 2\Delta \varepsilon^p_{xy} \end{bmatrix}, \quad \Delta \varepsilon^p_s = \begin{bmatrix} \Delta \gamma^p_{xz} \\ \Delta \gamma^p_{yz} \end{bmatrix}
\]  

(5.42)
The assumption of an additive decomposition of strains may be extended to displacements, provided that the strains are small (Ueda and Yao, 1982; Shi and Voyiadjis, 1992). Although geometric non-linearities are taken into account in the current work, we only consider large rigid rotations and translations, but small strains. We may write:

\[ q = q^e + q^p \]  

Following the work of Shi and Voyiadjis (1992) we approximate the increments of plastic displacements by the increments of plastic strains. The plastic rotation \( \Delta \phi^p \) is a function of both \( \Delta \kappa^x \) and \( \Delta \kappa^y \), as may be deduced from equations (5.43). Assuming that increment of plastic nodal rotation \( \Delta \phi^p_{xi} \) is proportional to the increment of elastic nodal rotation \( \Delta \phi_{xi} \) we may express the former as follows:

\[ \Delta \phi^p_{xi} = \lim_{\delta \Omega \rightarrow 0} \iint_{\delta \Omega} \left[ \Delta \kappa^x \frac{\partial \Delta \phi_{xi}^p}{\partial \Delta \phi_{xi}^p} \frac{\partial \Delta \phi_{xi}^p + \Delta \kappa^y}{\partial \Delta \phi_{xi}^p + \Delta \kappa^y} \right] dxdy \]

\[ = \Delta \lambda_i \left[ \frac{\partial F^*_{ix}}{\partial M_{xi}} + \frac{2\Delta \phi_{xi}^2}{\partial \Delta \phi_{xi}^2} \frac{\partial F^*_{ix}}{\partial M_{xy}} \right] \]  

(5.45)

where \( \delta \Omega \) represents the infinitesimal neighborhood of node \( i \). The vector of incremental nodal plastic displacements of the element at node \( i \) may then be expressed as:

\[ \Delta \mathbf{q}^p_i = \mathbf{a}_i \Delta \lambda_i \]  

(5.46)

with \( \mathbf{a}_i \) given by:

\[ \mathbf{a}_i = \left\{ \frac{\partial F^*_{ix}}{\partial N_{xi}} + p_u \frac{\partial F^*_{iy}}{\partial N_{yi}}; \frac{\partial F^*_{ix}}{\partial N_{xi}} + p_v \frac{\partial F^*_{iy}}{\partial N_{yi}}; \frac{\partial F^*_{ix}}{\partial Q_{xi}} + \frac{\partial F^*_{iy}}{\partial Q_{yi}}; \right\} \]

\[ \frac{\partial F^*_{ix}}{\partial M_{xi}} + p_{\phi x} \frac{\partial F^*_{ix}}{\partial M_{xy}}; \frac{\partial F^*_{ix}}{\partial M_{yi}} + p_{\phi y} \frac{\partial F^*_{ix}}{\partial M_{xy}} \]  

(5.47)
Equations (5.46) and (5.47) indicate that the plastic displacements at the nodes are only functions of the stress resultants at this node (Shi and Voyiadjis, 1992). Therefore, we may write the vector of increments of nodal plastic displacements, as follows:

$$\Delta \mathbf{q}^p = \begin{bmatrix} a_i & 0 & 0 \\
0 & a_i & 0 \\
0 & 0 & a_{\text{PN}} \end{bmatrix} \begin{bmatrix} \Delta \lambda_i \\
\Delta \lambda_j \\
\Delta \lambda_{\text{PN}} \end{bmatrix} = \mathbf{a} \Delta \lambda. \quad (5.48)$$

Similarly to the geometrically non-linear analysis presented in Chapter 4, in order to determine the tangent stiffness matrix of the element we define $\delta e_b, \delta e_m, \delta e_s$ as virtual elastic bending, membrane and transverse shear strains respectively ($\delta -$virtual) and $\mathbf{M}, \mathbf{N}, \mathbf{Q}$ as stress couples and stress resultants of the element. We also use the same linearized equilibrium equations of the system at configuration $k + 1$ in the Updated Lagrangian formulation, expressed by the principle of the virtual work, which in finite element modeling takes the form:

$$\iint_{\Omega} \left[ (\delta e_b^T \mathbf{D} e_b + \delta e_m^T \mathbf{S} e_m + \delta e_s^T \mathbf{T} e_s) \right] dx dy + \iint_{\Omega} \mathbf{\theta}^T \mathbf{F} \theta dx dy = k+1 \mathbf{R} -$$

$$\iint_{\Omega} (\delta e_b^T k \mathbf{M} + \delta e_m^T k \mathbf{N} + \delta e_s^T k \mathbf{Q}) dx dy$$

where $k+1 \mathbf{R}$ is the total external virtual work at step $k + 1$; $\theta$ is the slope vector and $k \mathbf{F}$ is a membrane stress resultant matrix at step $k$ given by the equations (4.22). Following the procedure outlined in Chapter 4, we may derive the initial stress stiffness matrix. We may rewrite equation (4.18) as follows:

$$\iint_{\Omega} \left[ (\delta e_b^T \mathbf{D} e_b + \delta e_m^T \mathbf{S} e_m + \delta e_s^T \mathbf{T} e_s) \right] dx dy + \mathbf{\Delta q}^T \mathbf{K}_g \mathbf{\Delta q}^* = k+1 \mathbf{R} - \mathbf{\Delta q}^T \mathbf{f} \quad (5.50)$$

where $\mathbf{K}_g$ is the initial stress matrix defined as in Chapter 4:

$$\mathbf{K}_g = \iint_{\Omega} \mathbf{G}^T k \mathbf{F} \mathbf{G} dx dy \quad (5.51)$$

and $\mathbf{f}$ is the internal force vector resulting from the unbalanced forces in configuration $k$ expressed as follows:

$$\mathbf{f} = \iint_{\Omega} \left( \mathbf{B}_b^T k \mathbf{M} + \mathbf{B}_m^T k \mathbf{N} + \mathbf{B}_s^T k \mathbf{Q} \right) dx dy \quad (5.52)$$

We may now rewrite equation (5.50), using equation (5.40), written in a matrix form, as follows:

$$\iint_{\Omega} \left[ (\delta e_b^T + \delta e_b^T \rho^T) \mathbf{M} + (\delta e_m^T + \delta e_m^T \rho^T) \mathbf{N} + (\delta e_s^T + \delta e_s^T \rho^T) \mathbf{Q} \right] dx dy +$$

$$\mathbf{\Delta q}^T \mathbf{K}_g \mathbf{\Delta q}^* = k+1 \mathbf{R} - \mathbf{\Delta q}^T \mathbf{f} \quad (5.53)$$

Rearranging terms and writing the above equation in incremental form, we have:

$$\iint_{\Omega} \left[ (\delta e_b^T \Delta M + \delta e_m^T \Delta N + \delta e_s^T \Delta Q) \right] dx dy +$$

$$+ \iint_{\Omega} \left[ (\delta e_b^T \rho^T \Delta M + \delta e_m^T \rho^T \Delta N + \delta e_s^T \rho^T \Delta Q) \right] dx dy + \mathbf{\Delta q}^T \mathbf{K}_g \mathbf{\Delta q}^* = k+1 \mathbf{R} - \mathbf{\Delta q}^T \mathbf{f} \quad (5.54)$$

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Substituting equations (5.41) into equation (5.54) we obtain:
\[
\int \int_{\Omega} \left( \delta \Delta \varepsilon_b^T \Delta \mathbf{M} + \delta \Delta \varepsilon_m^T \Delta \mathbf{N} + \delta \Delta \varepsilon_s^T \Delta \mathbf{Q} \right) dxdy + \\
+ \sum_{i=1}^{NPN} \delta \Delta \lambda_i \left[ \frac{\partial \mathbf{F}_i^*}{\partial \mathbf{M}_i^*} d\mathbf{M}_i + \frac{\partial \mathbf{F}_i^*}{\partial \mathbf{N}_i^*} d\mathbf{N}_i + \frac{\partial \mathbf{F}_i^*}{\partial \mathbf{Q}_i^*} d\mathbf{Q}_i \right] + \delta \Delta \mathbf{q}^T \mathbf{K}_e \delta \mathbf{q}^* = -k+1 \mathbf{R} - \delta \Delta \mathbf{q}^T \Delta \mathbf{f}
\]  
(5.55)

Making use of the equations (3.68)-(3.73), as well as the consistency condition given by equation (5.39), we may write:
\[
\delta \Delta \mathbf{q}^T \left( \mathbf{K}_e + \mathbf{K}_g \right) \delta \mathbf{q}^* - \\
- \sum_{i=1}^{NPN} \delta \Delta \lambda_i \left[ \frac{\partial \mathbf{F}_i^*}{\partial \mathbf{M}_i^*} d\mathbf{M}_i + \frac{\partial \mathbf{F}_i^*}{\partial \mathbf{N}_i^*} d\mathbf{N}_i + \frac{\partial \mathbf{F}_i^*}{\partial \mathbf{Q}_i^*} d\mathbf{Q}_i + \frac{\partial \mathbf{F}_i^*}{\partial \mathbf{k}} d\mathbf{k} \right] = -k+1 \mathbf{R} - \delta \Delta \mathbf{q}^T \Delta \mathbf{f}
\]  
(5.56)

where \( \mathbf{K}_e \) is the linear elastic stiffness matrix given by the equation (3.80).

Similarly to equation (5.47) we define:
\[
\mathbf{a}_{bi}^T = \frac{\partial \mathbf{F}_i^*}{\partial \mathbf{M}_i^*} = \left\{ \frac{\partial \mathbf{F}_i^*}{\partial \mathbf{M}_i^*}, \frac{\partial \mathbf{F}_i^*}{\partial \mathbf{M}_i^*}, \frac{\partial \mathbf{F}_i^*}{\partial \mathbf{M}_i^*} \right\};
\]
\[
\mathbf{a}_{mi}^T = \frac{\partial \mathbf{F}_i^*}{\partial \mathbf{N}_i^*} = \left\{ \frac{\partial \mathbf{F}_i^*}{\partial \mathbf{N}_i^*}, \frac{\partial \mathbf{F}_i^*}{\partial \mathbf{N}_i^*}, \frac{\partial \mathbf{F}_i^*}{\partial \mathbf{N}_i^*} \right\};
\]
\[
\mathbf{a}_{si}^T = \frac{\partial \mathbf{F}_i^*}{\partial \mathbf{Q}_i^*} = \left\{ \frac{\partial \mathbf{F}_i^*}{\partial \mathbf{Q}_i^*}, \frac{\partial \mathbf{F}_i^*}{\partial \mathbf{Q}_i^*} \right\};
\]  
(5.57)

Substituting equations (5.41) into equations (5.33) and (5.34) we obtain:
\[
d\mathbf{M}_x^* = \Delta \mathbf{M}_x^* = \\
= \beta_2 (1-F) \frac{M_0}{\kappa_0} \Delta M_x^* \\
= \beta_2 (1-F) \frac{M_0}{\kappa_0} \Delta \lambda \mathbf{A}_{mi}^T \\
\mathbf{M}_x^* - \frac{6}{h^2} \mathbf{M}_x^* \sqrt{\frac{2}{3} \left( \frac{\partial \mathbf{F}_i^*}{\partial \mathbf{M}_x} \right)^2 + \left( \frac{\partial \mathbf{F}_i^*}{\partial \mathbf{M}_y} \right)^2 + \left( \frac{\partial \mathbf{F}_i^*}{\partial \mathbf{M}_xy} \right)^2}
\]  
(5.58)

and similarly for the remaining hardening parameters. Vectors of hardening parameters will therefore yield:
\[
d\mathbf{N}_i^* = \left\{ \Delta N_i^*, \Delta N_y^*, \Delta N_{xy}^* \right\} = \Delta \lambda \mathbf{A}_{mi}^T \\
d\mathbf{Q}_i^* = \left\{ \Delta Q_x^*, \Delta Q_y^*, \Delta Q_{xy}^* \right\} = \Delta \lambda \mathbf{A}_{si}^T
\]  
(5.59)

and:
\[
d\mathbf{M}_i^* = \left\{ \Delta M_x^*, \Delta M_y^*, \Delta M_{xy}^* \right\} = \Delta \lambda \mathbf{A}_{bi}^T
\]  
(5.60)
where $A_{mi}, A_{si}, A_{bi}$ are given by:

$$A_{mi} = \left\{ \begin{align*} 
\beta_i (1 - F) \frac{N_o}{\varepsilon_0} & \left[ \frac{\partial F^*}{\partial N_x} - \frac{1}{h} N_x^* \right] \\
& \left[ \frac{2}{3} \left( \frac{\partial F^*}{\partial N_x} \right)^2 + \left( \frac{\partial F^*}{\partial N_y} \right)^2 + \left( \frac{\partial F^*}{\partial N_{xy}} \right)^2 \right] \\
\end{align*} \right\}$$

$$A_{si} = \left\{ \begin{align*} 
\beta_i (1 - F) \frac{N_o}{\varepsilon_0} & \left[ \frac{\partial F^*}{\partial Q_x} - \frac{1}{h} Q_x^* \right] \\
& \left[ \frac{2}{3} \left( \frac{\partial F^*}{\partial Q_x} \right)^2 + \left( \frac{\partial F^*}{\partial Q_y} \right)^2 \right] \\
\end{align*} \right\}$$

$$A_{bi} = \left\{ \begin{align*} 
\beta_i (1 - F) \frac{M_o}{\kappa_o} & \left[ \frac{\partial F^*}{\partial M_x} - \frac{6}{h^2} M_x^* \right] \\
& \left[ \frac{2}{3} \left( \frac{\partial F^*}{\partial M_x} \right)^2 + \left( \frac{\partial F^*}{\partial M_y} \right)^2 + \left( \frac{\partial F^*}{\partial M_{xy}} \right)^2 \right] \\
\end{align*} \right\}$$

(5.61)

Following the work of Shi and Voyiadjis (1992) we also define the isotropic hardening parameter as:

$$H \Delta \lambda = \begin{bmatrix} H_i & 0 & 0 \\ 0 & H_j & 0 \\ 0 & 0 & H_{NPN} \end{bmatrix} \begin{bmatrix} \Delta \lambda_i \\ \Delta \lambda_j \\ \Delta \lambda_{NPN} \end{bmatrix} = -\begin{bmatrix} \frac{\partial F^*_i}{\partial k_i} \ \\
\frac{\partial F^*_j}{\partial k_j} \\
\frac{\partial F^*_{NPN}}{\partial k_{NPN}} \end{bmatrix} dk_i$$

(5.62)

where $k$ is represented by the amount of plastic work, i.e. $dk_i = N_i d\varepsilon_m^p + M_i d\varepsilon_b^p$. 

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We may substitute equations (5.57), (5.59) and (5.62) into (5.56) to obtain:

\[
\delta \Delta q^T \left( K + K_g \right) \Delta q^e + \delta \Delta \lambda^T \left[ H - a_b^T A_b - a_m^T A_m - a_s^T A_s \right] \Delta \lambda =
\]

\[
= k^{+1} R - \delta \Delta q^T \Delta f
\]

(5.63)

or using (5.44) and (5.46):

\[
\left( \delta \Delta q^T - \delta \Delta q^{pT} \right) \left( K + K_g \right) \Delta q^e + \delta \Delta \lambda^T \left[ H - a_b^T A_b - a_m^T A_m - a_s^T A_s \right] \Delta \lambda -
\]

\[
- k^{+1} R + \delta \Delta q^T \Delta f = \delta \Delta q^T \left[ \left( K + K_g \right) \Delta q^e - k^{+1} R^* + \Delta f \right] +
\]

\[
+ \delta \Delta \lambda^T \left[ -a^T \left( K + K_g \right) \Delta q^e + \left( H - a_b^T A_b - a_m^T A_m - a_s^T A_s \right) \Delta \lambda \right] = 0
\]

(5.64)

with:

\[
k^{+1} R = k^{+1} R^* \delta \Delta q
\]

(5.65)

By the virtue of the variational method equation (5.64) gives:

\[
\left( K + K_g \right) \Delta q^e - k^{+1} R^* + \Delta f = 0
\]

(5.66)

Substituting (5.44) and (5.46) into the above equations we get:

\[
\left( K + K_g \right) \Delta q^e - k^{+1} R^* + \Delta f = \left( K + K_g \right) \left( \Delta q - a \Delta \lambda \right) = k^{+1} R^* - \Delta f
\]

(5.67)

Equation (5.68) leads to:

\[
\Delta \lambda = \left[ a^T \left( K + K_g \right) a + \left( H - a_b^T A_b - a_m^T A_m - a_s^T A_s \right) \right]^{-1} a^T \left( K + K_g \right) \Delta q
\]

(5.69)

Equation (5.67) becomes:

\[
K_{epg} \Delta q = k^{+1} R^* - \Delta f
\]

(5.70)

where \( K_{epg} \) is the elasto-plastic, large displacement stiffness matrix of the element, given by:

\[
K_{epg} = \left( K + K_g \right)
\]

\[
\left[ I - a \left[ a^T \left( K + K_g \right) a + \left( H - a_b^T A_b - a_m^T A_m - a_s^T A_s \right) \right]^{-1} a^T \left( K + K_g \right) \right]
\]

(5.71)

The tangent stiffness matrix given by equation (5.71) is similar to the one presented by Shi & Voyiadjis (1992). The present formulation accounts for large displacements and consequently the stiffness matrix of the element contains the initial stress matrix \( K_g \). More importantly however, the above derived stiffness matrix describes not only isotropic hardening, by means of the parameter \( H \), but also kinematic hardening, through matrices \( A_b, A_m, A_s \), which are not determined by curve fitting, but derived explicitly from the evolution equation of backstress given by Armstrong and Frederick (1966). We therefore have a non-layered finite element formulation with shell constitutive equations, yield condition, flow and hardening rules expressed in terms of membrane and shear forces and bending moments. All the variables used here, namely the stress resultants and couples, as well as the residual stress resultants and couples,
representing the center of the yield surface, are derived from stresses and backstresses in a very rigorous manner.

A very important feature of the derived tangent stiffness is its explicit form. The linear elastic stiffness matrix and initial stress matrix are determined by the quasi-conforming technique. Through the thickness integration is not employed here, since the current is the non-layered model with the yield condition expressed in terms of stress couples and resultants.

5.4 Numerical Examples

A finite element code written in the programming language Fortran 95, for the purpose of the geometrically non-linear analysis is further enhanced to model the elasto-plastic behaviour. As previously, the modified Newton-Raphson technique was employed to solve a system of non-linear, incremental equations. To overcome the singularity problem appearing at the limit point, the arc-length method (Crisfield, 1991) is adopted to determine the local load increment for each iteration. A return to the yield surface algorithm was also implemented (Crisfield, 1991). The results delivered by the current model were computed using a personal computer. Some of the reference solutions obtained with the layered approach (ABAQUS) were determined using a Silicon Graphics Onyx 3200 system. The computational and programming issues is discussed in detail in Chapter 7.

The accuracy of the present formulation is verified through a series of discriminating examples. We only solve non-linear examples here in order to test the reliability of the elasto-plastic framework presented in this chapter. The problems were chosen to challenge and demonstrate the most important features of the current model:

- Representation of progressive development of plastic deformation until the plastic hinge is formed;
- The influence of the transverse shear forces on plastic behaviour of thick plates beams and shells of general shape;
- Elasto-plastic behaviour of structures of interest upon reversal of loading (representation of Bauschinger effect through kinematic hardening);
- Description of large displacements and rotations.

The performance of the proposed procedure is compared with other formulations available in the literature. Table 5.1 lists the references used here, and their corresponding abbreviations used later in the text.

5.4.1 Simply Supported Elasto-Plastic Beam

The importance of the transverse shear forces in the approximation of the collapse load of thick beams, plates and shells is known to be significant. Neglecting transverse shears in the assessments of the maximum load carrying capacity of the structures may lead to predictions that are not conservative. Accurate and safe approximations should result in a decreasing value of the maximum load factor with increasing thickness. In order to test the accuracy of the current formulation in accounting for the shear
deformation, we consider a simply supported beam of length \(2L = 20 \text{ in}\) subjected to a concentrated load \(2P = 20 \text{ lbf}\) at its mid-point. The Young’s modulus is \(E = 10.5 \times 10^6 \text{ psi}\), yield stress \(\sigma = 500 \text{ psi}\), and width of the beam is \(b = 0.15 \text{ in}\). We compute the load factor of the beam as a function of thickness. The analytical solution of this problem given by Hodge (1959) serves here as a reference solution. The geometry of the problem as well as the material and section properties are given in Figure 5.4. The results provided by the current procedure, compared with the reference solutions are given in Figure 5.5.

**Table 5.1** Listing of the references used with abbreviations

<table>
<thead>
<tr>
<th>Name</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABQ-L</td>
<td>ABAQUS layered model with von Mises type yield criterion and Ziegler kinematic hardening rule (Hibbit et al., 2001)</td>
</tr>
<tr>
<td>C&amp;H</td>
<td>Bounds for collapse load - analytical solution of cylindrical shell Chen &amp; Han (1988)</td>
</tr>
<tr>
<td>HOD</td>
<td>Analytical solution given by Hodge (1959)</td>
</tr>
<tr>
<td>V&amp;W-Q</td>
<td>The present formulation with shear forces included in the yield function (Voyiadjis and Woelke, 2005)</td>
</tr>
<tr>
<td>V&amp;W</td>
<td>The present formulation without shear forces included in the yield function (Voyiadjis and Woelke, 2005)</td>
</tr>
</tbody>
</table>

![Figure 5.4](image-url) Simply supported beam – geometry and material properties.
As seen in Figure 5.5, the current formulation agrees very well with analytical results of this problem by Hodge (1959). We observe a substantial drop in the load factor for thick beams. It is worthy to mention, that for practical purposes only a certain range of $H$ is significant. When the thickness of the beam, plate or shell reaches 50% of its total length, we clearly enter a purely academic problem, however still valuable for illustrational purposes.

The reduction of the load factor is very significant even for moderately thick beam i.e. $H = 0.5L$ (total length of the beam is $2L$), which is very closely approximated here.

5.4.2 Simply Supported Plate

The following example corroborates the accuracy of the current formulation in the prediction of the first yield in plates, as well as the description of the load-displacement response under cyclic loads. Only material non-linearities are examined to allow for comparison with the reference solution by Owen and Hinton (1980). We consider a square ($L = 1.0$ m) simply supported plate subjected to a uniformly distributed load $q = 1.0$ kPa. Young’s modulus is $E = 10.92$ kPa, Poisson’s ratio $\nu = 0.3$, yield stress $\sigma = 1600$ kPa and thickness of the plate $t = 0.01$ m. The geometry and material properties are shown in Figure 5.6.

We compare the results obtained with the present finite element model, to those published by Owen & Hinton (1980), with the use of layered and non-layered model (O&HL, O&HNL - Table 1). The load-deflection responses are shown in Figure 5.7.
One of the objectives of the current work is to account for the progressive plasticization of the cross section by means of a non-layered model. In a layered model, used here as a reference, we track the development of the plastic deformation directly, since stresses are calculated at several different levels (layers) in the model. In a layered model we operate in a stress resultant and stress couples space. The plastic bending moment is calculated under the assumption of a fully plastic cross section. Hence, unless steps are taken to alleviate this problem, the cross section may only be either fully elastic or fully plastic, without any intermediate states.

As seen in Figure 5.7, the present approach provides a very good approximation of plastic strains growing gradually from the outer fibers to the mid-plane.

One of the main thrusts of this work is developing a physically sound kinematic hardening rule for non-layered plates and shells, correctly representing not only moment-curvature relationship, but also normal forces-normal strain and shear forces-shear strains.

Figure 5.6 Simply supported plate – geometry, material properties and deformed shape

E=10.92 kPa, ν=0.3
σ=1600 kPa, t=0.01 m,
q=1.0 kPa, L=1.0 m
relationships, upon complete reversal of loading. We therefore need to show the importance of all the hardening parameters $N^*, Q^*, M^*$. The simply supported plate under a uniformly distributed load is a problem in which the normal forces are negligible. The residual forces $N^*$ are also negligible. The influence of these is investigated in the following examples.

![Simply supported plate – Load-displacement curves](image)

**Figure 5.7** Simply supported plate – Load-displacement curves

The plate example is a bending dominant problem and the moment curvature relation is of primary importance. The load-displacement curve takes the shape of a moment-curvature relation. Figure 5.8 shows the load-deflection curves for plate in Figure 5.6, under reversing loading condition. The ABAQUS layered model with kinematic hardening rule is used as a reference. The current approximation is very close to the one by a layered approach, as seen in Figure 5.8. This proves that the definition of residual bending moments $M^*$ in the hardening rule is sound and produces accurate results.

For the thickness of the plate $t = 0.01 \, m$, the influence of the transverse shear forces on the plastic behavior is very small. In this case, the residual transverse shear forces $Q^*$ do not matter either. With increasing thickness of the plate, we observe the increasing importance of the transverse shear forces, as was shown by Shi and Voyiadjis (1992). We will show here that for thick plates, both transverse shear forces and residual transverse shear forces play a very important role.
We consider the same rectangular simply supported plate as in Figure 5.6. The thickness of the plate is however increased to \( t = 0.35 \, m \), a uniform load to \( q = 850 \, kPa \) and the yield stress reduced to \( \sigma = 1200 \, kPa \). The thickness of the plate is now 35% of its length; hence, we expect a significant reduction of the load factor of the plate, due to the influence of the shear forces. Again, we compare the results with the layered model, with the influence of transverse shears taken into account. The results are presented in Figure 5.9. A diamond line denoted by ABQ-L denoted a layered approach with shear forces considered. This approach serves as a reference solution.

As expected, the influence of the shear forces on the approximations of the collapse load is significant. The analysis in which the transverse shear forces are not considered leads to a nearly 20% higher prediction of maximum load carried by the plate. Neglecting the shear forces when analyzing thick plates, shells and beams could potentially lead to overpredicting the ultimate load carried by the structure.

When loading is reversed, and applied in the opposite direction, until yielding occurs at the top surface of the plate, the residual shear forces \( Q' \) become important.

The current model reproduces very well the lowered yield point upon reversal of loading, and offers a solution very close to the one by layered approach. We therefore conclude that the representation of the residual shear forces as kinematic hardening parameters is physically sound and capable of delivering veracious results.

The effect of the shear forces on the plastic behavior and maximum load carrying capacity is correctly recognized. As expected, the results show a reduction of the limit
load for thick plates, shells and beams, owing to the increasing significance of transverse shears.

Figure 5.9 Simply supported thick plate – Load –displacement response

5.4.3 Cylindrical Shell Subjected to a Ring of Pressure

The previous example showed the validity of the definition of the residual bending moments and residual shear forces as kinematic hardening parameters. The derivation of the residual membrane forces is based on the same assumptions, thus we expect them to be as reliable as the shear forces and bending moments. Since the membrane forces in bending of plates are negligible, the results of the former example do not prove that the formulation of the residual normal forces is sound. In order to do this, we investigate a cylindrical shell under the ring of pressure. The geometry, deformed shape of an octant of a cylinder and material parameters are shown in Figure 5.10.

The membrane forces play an important role here. If the structure is loaded into a plastic zone, then unloaded and loaded in the opposite direction, the residual membrane forces also become noteworthy. The results of the analysis compared with the ‘through the thickness integration’ (layered) method are given in Figure 5.11.
We recognize again that the present non-layered model with a new kinematic hardening rule is robust and agrees very well with the layered approach. The latter requires however many more operations for non-linear calculations, as the yield function and consistency condition need to be checked at each layer separately.

The problem presented here was originally investigated by Drucker (1954) and later by Chen et al. (1988) who analytically determined the bounds for the collapse load of the cylinder. These bounds are given by:

\[ L = 600 \text{ in} \]
\[ R = 300 \text{ in} \]
\[ t = 2 \text{ in} \]
\[ E = 3 \times 10^3 \text{ psi} \]
\[ v = 0.3 \]
\[ P = 0.85 \text{ lbf/in} \]
\[ \sigma_0 = 10 \text{ psi} \]
Assessment of the collapse load of structures is of paramount importance from an engineering point of view. We therefore examine the functioning of this model in the determination of maximum load carried by the cylinder. Equation (5.72) serves as a target solution. The collapse load as a function of thickness of the shell is shown in Figure 5.12.

\[
1.5 \leq \frac{P}{\sigma_R h^{1/2}} \leq 2.0
\]  

(5.72)

Predictions of the maximum load carried by the cylinder are accurate and fall within the analytical bounds.

It is worthy to mention, that for the case of a very thick shell, the results approach the lower bound solution. This is because the shear forces become more and more important for thick shells, causing a reduction of the load carrying capacity.

**5.4.4 Spherical Dome Subjected to a Ring of Pressure**

The problem of a spherical dome with an 18° hole at the top, subjected to a ring of pressure is solved to establish wide range of applicability of the method derived here. This is an important engineering problem, as well as a discriminating test of accuracy of the finite element representation of behavior of shells. The performance of the yield
function and the kinematic hardening rule is studied here once again. Geometrical and material data are shown in Figure 5.13.

![Spherical dome with an 180° cut-out](image)

**Figure 5.13** Spherical dome with an 180° cut-out; geometry and material properties

The structure is loaded into a plastic zone, and then the pressure is reversed. The kinematic hardening rule is applied to determine the equilibrium path. The layered approach once again serves as a reference. The load-displacement curves are plotted in Figure 5.14.

The approximation of the equilibrium path delivered by the current approach agrees very well with the adopted target solution, showing once again the validity of the assumptions made here. The lowered yield point is correctly reproduced by the yield surface defined in this dissertation.

We note that although the presented framework is robust for plates and shells of general shape, it performs best in the case of spherical shells. This is expected since the shell constitutive equations used here were derived by means of spherical strains, and later generalized through the finite element method.
Figure 5.14 Spherical dome with an 18° cut-out; Load –displacement curves
CHAPTER 6

ELASTO-PLASTIC GEOMETRICALLY NON-LINEAR FINITE ELEMENT ANALYSIS OF THICK PLATES AND SHELLS WITH DAMAGE DUE TO MICROVOIDS

6.1 Introduction

This chapter is devoted to introducing the effects of damage into the computational model for the analysis of plates and shells. All previously made assumptions pertaining to the shell theory, shell element, geometrically non-linear, and elasto-plastic investigations, presented in the preceding chapters are also employed here. The characteristics of the present damage formulation, as well as published literature were already discussed in Chapter 1. Thus, we will only briefly review the features of the damage model developed.

The experimental results (Bluhm and Morrissey, 1965; Fisher, 1980; Roy et al., 1981) show that the degradation of material properties of ductile metals in the elastic range due to the damage effects is negligible. Hence, the damage is considered here as a phenomenon induced by the plastic strain and any damage occurring in the elastic zone is disregarded.

Following the discussion given in Chapter 1, an isotropic scalar damage parameter is adopted. In the isotropic representation of damage, the stiffness of the material is reduced according to the same relation in all the directions. For a better description of the anisotropic effects, the second order damage tensor, capable of representing different levels of material degradation in different directions is often employed (Abu Al-Rub and Voyiadjis 2003; Doghri 2000; Lubarda and Krajcinovic 1993; Murakami 1988; Seweryn and Mroz 1998; Voyiadjis and Abu-Lebdeh 1993; Voyiadjis and Deliktas 2000a, 2000b; Voyiadjis and Kattan 1991, 1992a, 1992b, 1999; Voyiadjis and Park 1997, 1999; Voyiadjis and Venson 1995).

One of the disadvantages of using anisotropic damage variables is a necessity of determination of numerous material parameters which describe the directional dependency of the evolution of damage. Extensive experimental data is needed to calibrate these constants with sufficient accuracy and consistency. The isotropic damage formulation requires determination of fewer constants (two in the case of the current analysis), while at the same time it is capable of delivering very accurate results. For the current work, concerning the investigation of behaviour of isotropic plates and shells, the isotropic scalar parameter in representation of damage is deemed satisfactory. The effects of anisotropy are not accounted for here.

The isotropic porosity parameter defined by Duszek-Perzyna and Perzyna (1994) is used to describe damage effects in plates and shells. The evolution of porosity given by Duszek-Perzyna and Perzyna, reduced to a rate independent case, consists of three terms responsible for the cracking of the second phase particles, debonding of the second phase particles from the matrix material, and the void growth controlled only by plastic flow phenomena. The first term (cracking of the second-phase particles) is only dependent on the stress, which allows for variation of damage, even without occurrence of the plastic
flow. This makes the formulation universal and capable of describing correctly the material behaviour under all loading conditions, including the hydrostatic stress. In this dissertation, we only consider the most important effects from the point of view of structural analysis of isotropic homogenous plates and shells. Loading conditions are assumed to be static, and the evolution of porosity is reduced to represent the void growth only. Although the effects of the microcracks may be important for investigations of the material behaviour, the observations of ductile fracture in metals (Beachem, 1963; Gurland and Plateau, 1963) led to a conclusion that this process may involve the generation of considerable porosity through nucleation and growth of voids (Gurson, 1977). The influence of growth of microvoids is considered decisive in modeling of ductile, isotropic material. Thus, only damage due to microvoids in considered in this work.

Following the work of Gurson (1975, 1977) and also Duszek-Perzyna and Perzyna (1994) we incorporate the porosity parameter into the yield function, obtaining a yield criterion and flow rule for porous ductile materials with a strong coupling between plasticity and damage. The yield function given by Duszek-Perzyna and Perzyna (1994), which could be directly related to Gurson’s model (1975, 1977), is expressed in terms of the stress resultants and stress couples, similarly to Iliushin’s yield function (1956), following the procedure outlined by Bieniek and Funaro (1976). The yield surface derived here is very similar to the one presented by Voyiadjis and Woelke (2004) with kinematic hardening parameters in the form of residual normal and shear forces, and residual bending moments. It is however enhanced to account for the reduction of stiffness caused by the damage effects, represented by the porosity parameter.

The stiffness matrix presented in Chapter 5 was derived by means of the principle of virtual work and the plastic node method (Ueda and Yao, 1982), which assumes the inelastic deformations to be concentrated in the plastic hinges. Following the work of Shi and Voyiadjis (1992, 1993) the plastic node method is also adopted here to derive the elasto-plastic, damage stiffness matrix of the element. The explicit form of the stiffness matrix is therefore preserved i.e. numerical integration is not performed, which makes the current formulation very effective and accurate, as is shown later.

The current formulation is an attempt to deliver a very simple and convenient way of the detailed analysis of shells. It is in the same time consistent mathematically and accurate.

One of the biggest advantages of this work is its simplicity and computational efficiency. This approach is very advantageous from the point of view of structural analysis. The validity of the assumptions made here, as well as the derivation will be verified through the discriminating numerical examples.

In the following sections we first formulate a loading surface with the previously employed isotropic and kinematic hardening rules and featuring a strong coupling between plasticity and damage. An associated flow rule and evolution of porosity representing damage is defined. In Section 6.3, the explicit tangent stiffness matrix is derived, followed by the numerical examples challenging the current procedure presented in Section 6.4.
6.2 Yield and Damage Criterion

As discussed in the Introduction, a yield criterion for porous metals expressed in terms of the stress resultants and couples is derived here, similar to the yield function derived in Chapter 5, modified to account for the damage effects. The Iliushin’s yield function $F$ is given by the equations (5.1)-(5.6), which are repeated here for convenience:

$$ F = \frac{M^2}{M_0^2} + \frac{N^2}{N_0^2} + \frac{1}{\sqrt{3}} \frac{|MN|}{M_0N_0} - \frac{Y(k)}{\sigma_0^2} = 0 \quad (6.1) $$

or:

$$ F = \frac{|M|}{M_0} + \frac{N^2}{N_0^2} - \frac{Y(k)}{\sigma_0^2} = 0 \quad (6.2) $$

where:

$$ N^2 = N_x^2 + N_y^2 - N_xN_y + 3N_{xy}^2 \quad (6.3) $$

$$ M^2 = M_x^2 + M_y^2 - M_xM_y + 3M_{xy}^2 \quad (6.4) $$

$$ MN = M_xN_x + M_yN_y - \frac{1}{2}M_xN_y - \frac{1}{2}M_yN_x + 3M_{xy}^2 \quad (6.5) $$

$$ M_0 = \frac{\sigma_0 h^2}{4}, \quad N_0 = \sigma_0 h \quad (6.6) $$

and $\sigma_0$ is the uniaxial yield stress, $Y(k)$ is a material parameter, which depends on isotropic hardening parameter $k$; $h$ is the thickness of the shell, and $|.|$ denotes the absolute value.

The form of the yield condition given by equation (5.1), can be easily derived from von Mises function and the definition of normal stresses at the top and the bottom surfaces of the shell, as shown by Bieniek and Funaro, (1976). Instead, we use the yield criterion for porous ductile metals as originally proposed by Gurson (1975, 1977) and later modified by Perzyna (1984b) and Dornowski and Perzyna (2000). Although it is of the form similar to the von Mises equation, it accounts for the isotropic damage effects through the dependence of the first invariant of stress and the evolution of porosity. The plastic potential function defined by Dornowski and Perzyna (2000) may be written as:

$$ f = \frac{3}{2} \sigma_{ij} \delta_{ij} + n \xi \sigma_{ii}^2, \quad i, j = 1, 2, 3 \quad (6.7) $$

where $S_{ij}$ is deviatoric stress tensor given by:

$$ S_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij} \quad (6.8) $$

$\sigma_{ij}$ is a stress tensor given by:

$$ \sigma_{ij} = \frac{N_{ij}}{h} \pm \frac{6M_{ij}}{h^2} \quad (6.9) $$

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where $N_{ij}$ are normal forces; $M_{ij}$ are bending moments, $h$ is a thickness of the shell and $\delta_{ij}$ is a Kronecker delta. The parameter $n$ in the equation (6.7) is a material constant, determined by Perzyna (1984b): $n = 1.2587$ (for ductile metal).

The parameter $\xi$ in equation (6.7) is a porosity parameter given by Gurson (1975, 1977) and modified by Duszek-Perzyna and Perzyna (1994):

$$\Delta \xi = k_1 \Delta \sigma_{ii} + k_2 \Delta \sigma_j \Delta e_{ij}^{p} + k_3 \Delta e_{ij}^{p}$$  \hspace{1cm} (6.10)

where $k_1, k_2, k_3$ denote the material constants, $\Delta \sigma$ and $\Delta e^{p}$ are the increments of stress and plastic strain respectively.

The first two terms in the above equation are responsible for nucleation due to the cracking of the second phase particles, and debonding of the second phase particles from the matrix material. The third term depicts the growth of voids, and is controlled only by the plastic flow. The main term in the current work is the growth term. We may assume that from the metallurgical investigations of the isotropic material comprising a plate or a shell, we may determine the initial porosity $\xi(t=0) = \xi_0$, and we shall consider only the growth term in the evolution of porosity, i.e.:

$$\Delta \xi = k_3 \Delta e_{ij}^{p}$$  \hspace{1cm} (6.11)

Equations (6.7)-(6.11) are written using indicial notation and a summation convention. Rewriting equation (6.11) in engineering notation yields:

$$\Delta \xi = k_3 \left( \Delta e_x^{p} + \Delta e_y^{p} + \Delta e_z^{p} \right)$$  \hspace{1cm} (6.12)

where $\Delta e_x^{p}, \Delta e_y^{p}, \Delta e_z^{p}$ are increments of normal plastic strains due to both membrane and bending actions in $x, y, z$ directions respectively. $\Delta e_x^{p}$ and $\Delta e_y^{p}$ may be written:

$$\Delta e_x^{p} = \Delta e_{mx}^{p} + \Delta e_{bx}^{p} = \Delta e_{mx}^{p} + z\Delta k_x^{p}$$

$$\Delta e_y^{p} = \Delta e_{my}^{p} + \Delta e_{by}^{p} = \Delta e_{my}^{p} + z\Delta k_y^{p}$$  

(6.13)

where $\Delta e_{mx}^{p}$ and $\Delta e_{my}^{p}$ are the increments of plastic strains due to membrane action only, in $x, y$ directions; $\Delta e_{bx}^{p}$ and $\Delta e_{by}^{p}$ are the increments of plastic strains due to bending action only, in $x, y$ directions; $z$ is the distance from the mid-plane to the plane under consideration and $\Delta k_x^{p}, \Delta k_y^{p}$ are the increment of plastic curvatures at the midsurface in planes parallel to the $xz, yz$ planes respectively. The maximum normal plastic strain caused by bending will occur at $z = h/2$ which leads to:

$$\Delta e_x^{p} = \Delta e_{mx}^{p} + \frac{h}{2}\Delta k_x^{p}$$

(6.14)

$$\Delta e_y^{p} = \Delta e_{my}^{p} + \frac{h}{2}\Delta k_y^{p}$$

Substituting equations (6.14) into equation (6.12) and neglecting $\Delta e_z^{p}$ we obtain:

$$\Delta \xi = k_3 \left[ \Delta e_{mx}^{p} + \Delta e_{my}^{p} + \frac{h}{2}(\Delta k_x^{p} + \Delta k_y^{p}) \right]$$  \hspace{1cm} (6.15)
We now proceed to the determination of the plastic potential function expressed in terms of the stress resultants and couples. For the purpose of conciseness, we neglect radial and transverse shear stresses in the current derivation. Transverse shear forces will be later introduced into the yield condition. Equation (6.7) can be written using engineering notation:

\[
\frac{1}{\sqrt{2}} \sqrt{\left( (\sigma_x - \sigma_y)^2 + \sigma_x^2 + \sigma_y^2 + 6\tau_{xy}^2 + n\xi (\sigma_x + \sigma_y)^2 \right)} = f
\]  

(6.16)

where \(\sigma_x,\sigma_y\) are the normal stresses in the \(x,y\) directions respectively, and \(\tau_{xy}\) is a shear stress on the \(xy\) plane.

We may define the yield condition as follows:

\[
\frac{1}{\sqrt{2}} \sqrt{\left( (\sigma_x - \sigma_y)^2 + \sigma_x^2 + \sigma_y^2 + 6\tau_{xy}^2 + n\xi (\sigma_x + \sigma_y)^2 \right)} = \sigma_0
\]  

(6.17)

where \(\sigma_0\) denotes the uniaxial yield stress.

Substitution of the equations (6.9) into (6.17) and some manipulations result in:

\[
\frac{N^2}{N_0^2} + \frac{M^2}{M_{0E}^2} \pm \frac{2NM}{N_0M_{0E}} = 1
\]  

(6.18)

where:

\[
N^2 = \left( 1 + \frac{1}{2} n\xi \right) \left( N_x^2 + N_y^2 \right) - (1 - n\xi) N_x N_y + 3N_{xy}^2
\]

\[
M^2 = \left( 1 + \frac{1}{2} n\xi \right) \left( M_x^2 + M_y^2 \right) - (1 - n\xi) M_x M_y + 3M_{xy}^2
\]  

(6.19)

\[
NM = \left( 1 + \frac{1}{2} n\xi \right) \left( N_x M_x + N_y M_y \right) - \frac{1}{2} \left( 1 - n\xi \right) \left( M_x N_x + M_y N_y \right) + 3N_{xy} M_{xy}
\]

and:

\[
N_0 = \sigma_0 h, \quad M_{0E} = \frac{\sigma_0 h^2}{6}
\]  

(6.20)

Both the top and the bottom surface of the shell should be considered to obtain the larger value of the term \(\pm \frac{NM}{N_0M_{0E}}\). We may ensure representation of the most negative effect by writing the equation (6.18) in the form (Bieniek and Funaro, 1976; Bieniek et al. 1976):

\[
\frac{N^2}{N_0^2} + \frac{M^2}{M_{0E}^2} + 2\frac{|NM|}{N_0M_{0E}} = 1
\]  

(6.21)

The yield surface given above is very similar to Iliushin’s yield function (1956) given by equation (6.1). In order to derive equation (6.1) we follow the procedure outlined by Bieniek and Funaro (1976), which is essentially the surface fitting approach. We write equation (6.21) as follows:

\[
a \frac{N^2}{N_0^2} + b \frac{M^2}{M_{0E}^2} + c \frac{|NM|}{N_0M_{0E}} = 1
\]  

(6.22)
We determine the parameters $a, b, c$ by considering the special loading cases separately. If we account for membrane forces only, we see that for $a = 1$ we obtain the exact limit condition. Similarly, if we take a pure bending case, equation (6.22) will produce exact results for $b = \frac{M_{0E}^2}{M_0^2}$. To find $c$ we investigate the loading case corresponding to the maximum value of the ratio $\frac{NM}{N_0M_{0E}}$ which occurs if $N_x = N_y$, $M_x = M_y$ and $N_{xy} = M_{xy} = 0$. The stress distribution in the cross section in this case is as shown in Figure 6.1:

![Figure 6.1 Stress distribution corresponding to maximum \( \frac{NM}{N_0M_{0E}} (\eta = h/2\sqrt{3}) \)](image)

Based on the stress distribution in Figure 6.1, we may calculate the normal force:

$$N_x = \int_{-h/2}^{h/2} \sigma_x dz = \int_{-h/2}^{-h/2\sqrt{3}} -\sigma_0 dz + \int_{-h/2\sqrt{3}}^{h/2} \sigma_0 dz = \frac{\sigma_0 h}{\sqrt{3}}$$

using equation (6.20) we may write:

$$\frac{N_x}{N_0} = \frac{1}{3}$$

(6.24)

Similarly we may obtain:

$$\frac{M^2}{M_{0E}^2} = \frac{4M_0^2}{9M_{0E}^2} \text{ and } \frac{NM}{N_0M_{0E}} = 2\sqrt{3} \frac{M_0}{9M_{0E}}$$

(6.25)

Substitution of the equations (6.24)-(6.25) and the previously determined parameters $a = 1$ and $b = \frac{M_{0E}^2}{M_0^2}$ into the equation (6.22), yields:

$$\frac{1}{3} + \frac{M_{0E}^2}{M_0^2} \cdot \frac{4M_0^2}{9M_{0E}^2} + 2\sqrt{3} \frac{M_0}{9M_{0E}} = 1$$

(6.26)
which leads to:

\[ c = \frac{M_{0E}}{\sqrt{3}M_o} \]  

(6.27)

Substituting the parameters \( a,b,c \) into the equation (6.22) we arrive at the limit yield surface as defined by Iliushin:

\[ F = \frac{M^2}{M_o^2} + \frac{N^2}{N_o^2} + \frac{1}{\sqrt{3}M_oN_o} |MN| = 1 \]  

(6.28)

The stress intensities, are given by the equation (6.19) and unlike in the original Iliushin yield function, they account for the damage effects.

In Chapter 5, several other modifications were introduced to the Iliushin yield surface for a better description of the plastic behavior of shells. The damage variable is a function of the plastic flow here, which makes the accuracy of the representation of plastic behaviour very important. The same modifications of the yield function are adopted in this chapter. We may include the transverse shear forces \( Q_x, Q_y \) by expanding one of the stress intensities given in equation (6.19), as in the previous chapter, cf. (Shi and Voyiadjis, 1992):

\[ N^2 = \left( 1 + \frac{1}{2}n\xi \right) \left( N_x^2 + N_y^2 \right) - \left( 1 - n\xi \right) N_x N_y + 3 \left( N_{xy}^2 + Q_x^2 + Q_y^2 \right) \]  

(6.29)

It was shown previously (Shi and Voyiadjis, 1992) that the influence of the shear forces on plastic behaviour of thick plates and shells could be very important.

In Section 5.2.3 a plastic curvature parameter \( \alpha \left( \bar{\kappa}^p \right) \) was incorporated into the equation (6.28) in order to account for the development of plastic curvature across the thickness (Crisfield, 1981):

\[ F = \frac{M^2}{\alpha^2M_o^2} + \frac{N^2}{N_o^2} + \frac{1}{\sqrt{3}M_oN_o} \frac{Y(k)}{\sigma_o^2} - \frac{Y(k)}{\sigma_o^2} = 0 \]  

(6.30)

or:

\[ F = \frac{|M|}{\alpha M_o} + \frac{N^2}{N_o^2} - \frac{Y(k)}{\sigma_o^2} = 0 \]  

(6.31)

where \( \alpha \) and \( \bar{\kappa}^p \) were given by the equations (5.10)-(5.11).

We note that a material parameter \( Y(k) \), was employed in equations (6.30)-(6.31) which depends on isotropic hardening parameter \( k \), similarly to equations (6.1)-(6.2).

In Section 5.2.4 a stress resultant based kinematic hardening rule was derived, allowing for the correct predictions of the Bauschinger effect. Adopting that same hardening rule in the current chapter, we express the yield surface as follows:

\[ F^* = \frac{|M^*|}{\alpha M_o} + \frac{(N^*)^2}{N_o^2} - \frac{Y(k)}{\sigma_o^2} = 0 \]  

(6.32)
where:

\[
(N^*)^2 = \left(1 + \frac{1}{2} n \xi \right) \left[ (N_x - N_x^*)^2 + (N_y - N_y^*)^2 \right] - \\
-(1 - n \xi) (N_x - N_x^*) (N_y - N_y^*) + \\
+3 \left[ (N_{xy} - N_{xy}^*)^2 + (Q_x - Q_x^*)^2 + (Q_y - Q_y^*)^2 \right]
\]

\[
(M^*)^2 = \left(1 + \frac{1}{2} n \xi \right) \left[ (M_x - M_x^*)^2 + (M_y - M_y^*)^2 \right] - \\
-(1 - n \xi) (M_x - M_x^*) (M_y - M_y^*) + 3 (M_{xy} - M_{xy}^*)^2
\]

and \( M_x^* , M_y^* , M_{xy}^* , N_x^* , N_y^* , N_{xy}^* , Q_x^* , Q_y^* \) the residual bending moments, normal and shear forces respectively, derived in the previous chapter and given by the equations (5.33)-(5.35), repeated here for self-completeness:

If \( F^* = 1 \) and \( \nabla F^* > 0 \) (plastic loading)

\[
\Delta N_x^* = \beta_1 (1 - F) \frac{N_0}{\varepsilon_0} \left[ \Delta \varepsilon_x^p - \frac{1}{h} N_x^* \Delta \varepsilon_{eq}^p \right]
\]

\[
\Delta N_y^* = \beta_1 (1 - F) \frac{N_0}{\varepsilon_0} \left[ \Delta \varepsilon_y^p - \frac{1}{h} N_y^* \Delta \varepsilon_{eq}^p \right]
\]

\[
\Delta N_{xy}^* = \beta_1 (1 - F) \frac{N_0}{\varepsilon_0} \left[ \Delta \varepsilon_{xy}^p - \frac{1}{h} N_{xy}^* \Delta \varepsilon_{eq}^p \right]
\]

\[
\Delta Q_x^* = \beta_1 (1 - F) \frac{N_0}{\varepsilon_0} \left[ \Delta \varepsilon_z^p - \frac{1}{h} Q_x^* \Delta \varepsilon_{eq}^p \right]
\]

\[
\Delta Q_y^* = \beta_1 (1 - F) \frac{N_0}{\varepsilon_0} \left[ \Delta \varepsilon_z^p - \frac{1}{h} Q_y^* \Delta \varepsilon_{eq}^p \right]
\]

\[
\Delta M_x^* = \beta_2 (1 - F) \frac{M_0}{\kappa_0} \left[ \Delta \kappa_x^p - \frac{6}{h^2} M_x^* \Delta \kappa_{eq}^p \right]
\]

\[
\Delta M_y^* = \beta_2 (1 - F) \frac{M_0}{\kappa_0} \left[ \Delta \kappa_y^p - \frac{6}{h^2} M_y^* \Delta \kappa_{eq}^p \right]
\]

\[
\Delta M_{xy}^* = \beta_2 (1 - F) \frac{M_0}{\kappa_0} \left[ \Delta \kappa_{xy}^p - \frac{6}{h^2} M_{xy}^* \Delta \kappa_{eq}^p \right]
\]

If \( F^* < 1 \) and \( \nabla F^* \leq 0 \) (unloading or neutral loading)

\[
\Delta N_x^* = \Delta N_y^* = \Delta N_{xy}^* = \Delta Q_x^* = \Delta Q_y^* = \Delta M_x^* = \Delta M_y^* = \Delta M_{xy}^* = 0
\]

It is worthwhile to mention that by setting the porosity parameter to zero, i.e. \( \xi = 0 \), the yield surface given by the equations (6.32)-(6.34) reduces to the one given by the equations (5.12)-(5.14), where the damage effects were not considered.
The meaning of the material parameters $\beta_1$ and $\beta_2$ in the above formulation was explained in Chapter 5.

We arrive at a final form of the yield function for ductile porous metals, given by the equations (6.32)-(6.34) and (6.35)-(6.37), expressed in terms of the stress resultants and couples, with isotropic and kinematic hardening rules. This is a very convenient form of the yield surface for the analysis of shells accounting for the damage effects through the evolution of porosity. A graphic representation of yield surface on the $N,M_x$ plane with $\alpha = 1$ and $Y = \sigma_0^2$ is shown in Figure 5.3.

6.3 Explicit Tangent Stiffness Matrix

We follow the same procedure as in the preceding chapter to derive the tangent stiffness matrix. The plastic node method is employed in the derivation, i.e. the plastic deformations and damage are considered to be concentrated in the plastic hinges. The yield function is only checked at each node of the finite elements. If the combination of stress resultants satisfies the yield condition, that node is considered to be plastic, which triggers the void growth, as the porosity is in this work considered to be a function of the plastic flow. Thus, in this method the inelastic deformations are only considered at the nodes, while the interior of the element remains always elastic.

When node $i$ of the element becomes plastic, the yield function will take the form:

$$F^*;\left(N_i,Q_i,M_i,N'_i,Q'_i,M'_i,k_i,\xi_i\right) = 0 \quad (6.38)$$

where:

$$N_i = \begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix}; \quad Q_i = \begin{bmatrix} Q_x \\ Q_y \end{bmatrix}; \quad M_i = \begin{bmatrix} M_x \\ M_y \end{bmatrix} \quad (6.39)$$

$$N'_i = \begin{bmatrix} N'_x \\ N'_y \\ N'_{xy} \end{bmatrix}; \quad Q'_i = \begin{bmatrix} Q'_x \\ Q'_y \end{bmatrix}; \quad M'_i = \begin{bmatrix} M'_x \\ M'_y \end{bmatrix} \quad (6.40)$$

At the same time the stress resultants must remain on the loading surface, i.e. the consistency condition must be satisfied:

$$\begin{align*}
\frac{\partial F^*}{\partial M_i} dM_i + \frac{\partial F^*}{\partial N_i} dN_i + \frac{\partial F^*}{\partial Q_i} dQ_i + \frac{\partial F^*}{\partial M'_i} dM'_i + \frac{\partial F^*}{\partial N'_i} dN'_i + \frac{\partial F^*}{\partial Q'_i} dQ'_i + \\
+ \frac{\partial F^*}{\partial k_i} dk_i + \frac{\partial F^*}{\partial \xi_i} d\xi_i = 0
\end{align*} \quad (6.41)$$

We assume an additive decomposition of strains into elastic and plastic parts:

$$\varepsilon = \varepsilon^e + \varepsilon^p \quad (6.42)$$
The associated flow rule is used here to determine the increments of plastic strains:

\[
\Delta \kappa^p_x = \sum_{i=1}^{NPN} \Delta \lambda_i \frac{\partial F^*_{i}}{\partial M_{xi}} \quad \text{and} \quad \Delta \varepsilon^p_x = \sum_{i=1}^{NPN} \Delta \lambda_i \frac{\partial F^*_{i}}{\partial N_{xi}}
\]  
(6.43)

where \(NPN\) is the number of plastic nodes in the element and \(\Delta \lambda_i\) is a plastic multiplier.

The remaining increments of the plastic strains are obtained in the same way. The plastic strain fields are interpolated as in linear elastic analysis, (equations (3.45)-(3.47)) given here in the incremental form:

\[
\begin{align*}
\Delta \varepsilon^p_b &= \begin{pmatrix} \Delta \kappa^p_x \\ \Delta \kappa^p_y \\ 2\Delta \kappa^p_{xy} \end{pmatrix} , \\
\Delta \varepsilon^p_m &= \begin{pmatrix} \Delta \varepsilon^p_x \\ \Delta \varepsilon^p_y \\ 2\Delta \varepsilon^p_{xy} \end{pmatrix} , \\
\Delta \varepsilon^p_s &= \begin{pmatrix} \Delta \gamma^p_{xx} \\ \Delta \gamma^p_{yy} \\ \Delta \gamma^p_{xy} \end{pmatrix}
\end{align*}
\]

(6.44)

The evolution of the porosity parameter representing damage is given by the equation (6.15) repeated here for convenience:

\[
\Delta \xi = k_3 \left[ \Delta \varepsilon^p_x + \Delta \varepsilon^p_y + \frac{h}{2} \left( \Delta \kappa^p_x + \Delta \kappa^p_y \right) \right]
\]

(6.45)

The assumption of an additive decomposition of strains may be extended to displacements, provided that the strains are small (Shi and Voyiadjis, 1992; Ueda and Yao, 1982). Although geometric non-linearities are taken into account in the current work, we only consider large rigid rotations and translations, but small strains. Thus, we may write:

\[
q = q^e + q^p
\]

(6.46)

Following the work of Shi and Voyiadjis (1992) we approximate the increments of plastic displacements by the increments of plastic strains. The plastic rotation \(\Delta \phi^p_x\) is a function of both \(\Delta \kappa^p_x\) and \(\Delta \kappa^p_y\), as may be deduced from equation (6.44). Assuming that increment of plastic nodal rotation \(\Delta \phi^p_{xi}\) is proportional to the increment of elastic nodal rotation \(\Delta \phi_{xi}\) we may express the former as:

\[
\Delta \phi^p_{xi} = \lim_{\delta x_0 \to 0} \int_{\Omega} \left[ \Delta \kappa^p_x + \frac{\Delta \phi^2_{xi}}{\Delta \phi^2_{xi} + \Delta \phi^2_{yi}} \right] dxdy = \Delta \lambda_i \left[ \frac{\partial F^*_{\delta x_i}}{\partial M_{\delta x_i}} + \frac{2\Delta \phi^2_{xi}}{\Delta \phi^2_{xi} + \Delta \phi^2_{yi}} \frac{\partial F^*_{\delta y_i}}{\partial M_{\delta y_i}} \right]
\]

(6.47)

where \(\delta \Omega_i\) represents the infinitesimal neighborhood of node \(i\). The vector of incremental nodal plastic displacements of the element at node \(i\) may then be expressed as:

\[
\Delta \mathbf{q}_i^p = \mathbf{a}_i \Delta \lambda_i
\]

(6.48)
with \( a_i \) given by:

\[
a_i^T = \begin{bmatrix}
\frac{\partial F^*_{i+} + p_x \frac{\partial F^*_{i+}}{\partial N_{xi}} + p_y \frac{\partial F^*_{i+}}{\partial N_{yi}} + \frac{\partial F^*_{i+}}{\partial Q_{xi}} + \frac{\partial F^*_{i+}}{\partial Q_{yi}}}{\partial M_{xi} + p_x \frac{\partial F^*_{i+}}{\partial M_{xi}} + p_y \frac{\partial F^*_{i+}}{\partial M_{yi}}}
\end{bmatrix}
\]

Equations (6.48) and (6.49) indicate that the plastic displacements at the nodes are only functions of stress resultants at this node. Therefore, we may write the vector of increments of nodal plastic displacements, as follows:

\[
\Delta q^p = \begin{bmatrix}
a_1 & 0 & 0 \\
0 & a_i & 0 \\
0 & 0 & a_{\text{NPN}}
\end{bmatrix} \begin{bmatrix}
\Delta \lambda_1 \\
\Delta \lambda_i \\
\Delta \lambda_{\text{NPN}}
\end{bmatrix} = a \Delta \lambda.
\]

In order to determine the tangent stiffness matrix of the element we use the definition of the virtual elastic bending, membrane and transverse shear strains: \( \delta \varepsilon_b, \delta \varepsilon_m, \delta \varepsilon_s \) (\( \delta \) - virtual) and stress couples and stress resultants of the element \( M, N, Q \) from the preceding chapters. We once again make use of the linearized equilibrium equations of the system at configuration \( k+1 \) in the updated Lagrangian formulation, expressed by the principle of the virtual work, which in finite element modeling takes the form:

\[
\int_\Omega \left( \int_\Omega \left( \delta \varepsilon_b^T D \varepsilon_b + \delta \varepsilon_m^T S \varepsilon_m + \delta \varepsilon_s^T T \varepsilon_s \right) \right) dx dy + \int_\Omega \delta \varepsilon^T k F \delta \varepsilon dy = k+1 R - \int_\Omega \left( \delta \varepsilon_b^T k M + \delta \varepsilon_m^T k N + \delta \varepsilon_s^T k Q \right) dx dy
\]

where \( k+1 R \) is the total external virtual work at step \( k+1 \) and \( \theta \) is the slope vector and \( k F \) is a membrane stress resultant matrix at step \( k \) given by:

\[
\theta = \begin{bmatrix}
\frac{\partial \Delta w}{\partial x} \\
\frac{\partial \Delta w}{\partial y}
\end{bmatrix}, \quad k F = \begin{bmatrix}
k N_x & k N_y \\
k N_y & k N_y
\end{bmatrix}
\]

The slope field \( \theta \) is evaluated in exactly the same way as in Chapter 4. Using the equations derived in Chapter 5, we may rewrite the equation (6.51) as follows:

\[
\int_\Omega \left( \int_\Omega \left( \delta \varepsilon_b^T D \varepsilon_b + \delta \varepsilon_m^T S \varepsilon_m + \delta \varepsilon_s^T T \varepsilon_s \right) \right) dx dy + \delta \Delta q^T K \Delta q = k+1 R - \delta \Delta q^T \Delta f
\]
where $K_g$ is the initial stress matrix defined as in Chapter 4:

$$K_g = \int_G^{T_k} FG dx dy$$  \hspace{1cm} (6.54)$$

and $f$ is the internal force vector resulting from the unbalanced forces in configuration $k$ and is expressed as follows:

$$f = \int_G \left( B_b^{T_k} M + B_m^{T_k} N + B_s^{T_k} Q \right) dx dy$$  \hspace{1cm} (6.55)$$

We may now rewrite equation (6.53) using equation (6.42) as follows:

$$\int_G \left[ \left( \delta \varepsilon^T_b + \delta \varepsilon^T_e + \delta \varepsilon^T_p \right) M + \left( \delta \varepsilon^T_m + \delta \varepsilon^T_e + \delta \varepsilon^T_p \right) N + \left( \delta \varepsilon^T_s + \delta \varepsilon^T_e + \delta \varepsilon^T_p \right) Q \right] dx dy + \nabla \Delta q^T K_g \Delta q^e = k^{+1} R - \Delta q^T \Delta f$$  \hspace{1cm} (6.56)$$

Rearranging terms and writing the above equation in incremental form:

$$\int_G \left( \delta \varepsilon^T_b \Delta M + \delta \varepsilon^T_m \Delta N + \delta \varepsilon^T_s \Delta Q \right) dx dy + \int_G \left( \delta \varepsilon^T_b \Delta M + \delta \varepsilon^T_m \Delta N + \delta \varepsilon^T_s \Delta Q \right) dx dy + \nabla \Delta q^T K_g \Delta q^e = k^{+1} R - \Delta q^T \Delta f$$  \hspace{1cm} (6.57)$$

Substituting equations (6.43) into equation (6.57) we obtain:

$$\int_G \left( \delta \varepsilon^T_b \Delta M + \delta \varepsilon^T_m \Delta N + \delta \varepsilon^T_s \Delta Q \right) dx dy + \sum_{i=1}^{NPN} \nabla \Delta \lambda_i \left[ \frac{\partial F^*_i}{\partial M^*_i} dM^*_i + \frac{\partial F^*_i}{\partial N^*_i} dN^*_i + \frac{\partial F^*_i}{\partial Q^*_i} dQ^*_i \right] + \nabla \Delta q^T K_g \Delta q^e = k^{+1} R - \Delta q^T \Delta f$$  \hspace{1cm} (6.58)$$

Making use of the equations (3.68)-(3.73), as well as the consistency condition given by the equation (6.41), we may write:

$$\nabla \Delta q^T \left( K_e + K_g \right) \Delta q^e = \sum_{i=1}^{NPN} \nabla \Delta \lambda_i \left[ \frac{\partial F^*_i}{\partial M^*_i} dM^*_i + \frac{\partial F^*_i}{\partial N^*_i} dN^*_i + \frac{\partial F^*_i}{\partial Q^*_i} dQ^*_i + \frac{\partial F^*_i}{\partial k^*_i} dk^*_i + \frac{\partial F^*_i}{\partial \xi^*_i} d\xi^*_i \right]$$  \hspace{1cm} (6.59)$$

where $K_e$ is the linear elastic stiffness matrix given by the equation (3.80). Similarly to the equation (5.57) we define:

$$a_b^T = \frac{\partial F^*_i}{\partial M^*_i} = \begin{bmatrix} a_{b_1}^T & 0 & 0 \\ 0 & a_{b_i}^T & 0 \\ 0 & 0 & a_{b_NPN}^T \end{bmatrix}, \quad a_m^T = \frac{\partial F^*_i}{\partial N^*_i} = \begin{bmatrix} a_{m_1}^T & 0 & 0 \\ 0 & a_{m_i}^T & 0 \\ 0 & 0 & a_{m_NPN}^T \end{bmatrix},$$

$$a_s^T = \frac{\partial F^*_i}{\partial Q^*_i} = \begin{bmatrix} a_{s_1}^T & 0 & 0 \\ 0 & a_{s_i}^T & 0 \\ 0 & 0 & a_{s_NPN}^T \end{bmatrix}, \quad a_\xi^T = \frac{\partial F^*_i}{\partial \xi^*_i} = \begin{bmatrix} a_{\xi_1} & 0 & 0 \\ 0 & a_{\xi_2} & 0 \\ 0 & 0 & a_{\xi_NPN} \end{bmatrix}$$  \hspace{1cm} (6.60)$$
and:

\[
\begin{align*}
\mathbf{a}_{bi}^T &= \left\{ \frac{\partial F_{i}^{*}}{\partial M_{xi}^{*}}, \frac{\partial F_{i}^{*}}{\partial M_{yi}^{*}}, \frac{\partial F_{i}^{*}}{\partial M_{xyi}^{*}} \right\}; \\
\mathbf{a}_{mi}^T &= \left\{ \frac{\partial F_{i}^{*}}{\partial N_{xi}^{*}}, \frac{\partial F_{i}^{*}}{\partial N_{yi}^{*}}, \frac{\partial F_{i}^{*}}{\partial N_{xyi}^{*}} \right\}; \\
\mathbf{a}_{si}^T &= \left\{ \frac{\partial F_{i}^{*}}{\partial Q_{xi}^{*}}, \frac{\partial F_{i}^{*}}{\partial Q_{yi}^{*}} \right\}; \\
a_{qi} &= \frac{\partial F_{i}^{*}}{\partial \xi_{i}};
\end{align*}
\]

Substituting the equations (6.43) into (6.35) and (6.36) we obtain:

\[
dM_{x}^{*} = \Delta M_{x}^{*} = \beta \varepsilon (1 - F) \frac{M_{b}}{K_{0}} \Delta \lambda \left[ \frac{\partial F_{i}^{*}}{\partial M_{x}^{*}} - \frac{6}{h} M_{x}^{*} \frac{2}{3} \left( \frac{\partial F_{i}^{*}}{\partial M_{x}^{*}} \right)^{2} + \left( \frac{\partial F_{i}^{*}}{\partial M_{y}^{*}} \right)^{2} + \left( \frac{\partial F_{i}^{*}}{\partial M_{xy}^{*}} \right)^{2} \right]
\]

(6.62)

and similarly for the remaining hardening parameters. Vectors of hardening parameters therefore yield:

\[
\begin{align*}
dN_{i}^{*} &= \begin{bmatrix} \Delta N_{xi}^{*} \\ \Delta N_{yi}^{*} \\ \Delta N_{xyi}^{*} \end{bmatrix} = \mathbf{A}_{m} \Delta \lambda ; \quad dQ_{i}^{*} &= \begin{bmatrix} \Delta Q_{xi}^{*} \\ \Delta Q_{yi}^{*} \end{bmatrix} = \mathbf{A}_{s} \Delta \lambda ; \quad dM_{i}^{*} &= \begin{bmatrix} \Delta M_{xi}^{*} \\ \Delta M_{yi}^{*} \\ \Delta M_{xyi}^{*} \end{bmatrix} = \mathbf{A}_{b} \Delta \lambda.
\end{align*}
\]

(6.63)

(6.64)

where \( \mathbf{A}_{m}, \mathbf{A}_{s}, \mathbf{A}_{b} \) are given by:

\[
\begin{align*}
\mathbf{A}_{m} &= \begin{bmatrix} A_{m1} & 0 & 0 \\ 0 & A_{mi} & 0 \\ 0 & 0 & A_{mNPN} \end{bmatrix} , \\
\mathbf{A}_{s} &= \begin{bmatrix} A_{s1} & 0 \\ 0 & A_{si} \\ 0 & 0 & A_{sNPN} \end{bmatrix} , \\
\mathbf{A}_{b} &= \begin{bmatrix} A_{b1} & 0 & 0 \\ 0 & A_{bi} & 0 \\ 0 & 0 & A_{bNPN} \end{bmatrix}.
\end{align*}
\]

(6.65)

\( \mathbf{A}_{mi}, \mathbf{A}_{si}, \mathbf{A}_{bi} \) are given by the equations (5.61).

The evolution equation for porosity parameter may be written by substitution of the equations (6.43) into (6.45):
\[ d \xi = \Delta \xi = k_\xi \Delta \lambda \left[ \frac{\partial F^*}{\partial N_{xi}} + \frac{\partial F^*}{\partial N_{yi}} + h \left( \frac{\partial F^*}{\partial M_{xi}} + \frac{\partial F^*}{\partial M_{yi}} \right) \right] = A_{\xi} \Delta \lambda \]  

As previously, we apply the plastic node method to derive the matrix form of the above equation:

\[ d \bar{\xi} = \Delta \bar{\xi} = A_{\bar{\xi}} \Delta \lambda \]  

where:

\[
A_{\bar{\xi}} = \begin{bmatrix}
A_{\xi_1} & 0 & 0 \\
0 & A_{\xi_i} & 0 \\
0 & 0 & A_{\xi_NPN}
\end{bmatrix}
\]

and:

\[ A_{\xi_i} = k_\xi \left[ \frac{\partial F^*}{\partial N_{xi}} + \frac{\partial F^*}{\partial N_{yi}} + h \left( \frac{\partial F^*}{\partial M_{xi}} + \frac{\partial F^*}{\partial M_{yi}} \right) \right] \]

Following the work of Shi and Voyiadjis (1992) we also define the isotropic hardening parameter as:

\[
\mathbf{H} \Delta \lambda = \begin{bmatrix}
H_i & 0 & 0 \\
0 & H_i & 0 \\
0 & 0 & H_{NPN}
\end{bmatrix} \begin{bmatrix}
\Delta \lambda_i \\
\Delta \lambda_i \\
\Delta \lambda_{NPN}
\end{bmatrix}
\]

We now substitute equations (6.60), (6.63), (6.67) and (6.70) into (6.59) to obtain:

\[
\Delta \lambda = (k^{1+1} R - \Delta \mathbf{q}^T \Delta \mathbf{f})
\]

Using the equations (6.46) and (6.48) in equation (6.71) we may write:

\[
(k^* + K_\lambda) \Delta \mathbf{q}^c + \Delta \mathbf{\lambda}^T \left[ \mathbf{H} - \mathbf{a}^T \mathbf{A}_b - \mathbf{a}^T \mathbf{A}_m - \mathbf{a}^T \mathbf{A}_s - \mathbf{a}^T \mathbf{A}_s \right] \Delta \lambda =
\]

\[
= (k^{1+1} R - \Delta \mathbf{q}^T \Delta \mathbf{f}) + \Delta \mathbf{\lambda}^T \left[ -\mathbf{a}^T (K_e + K_g) \Delta \mathbf{q}^c + \left( \mathbf{H} - \mathbf{a}^T \mathbf{A}_b - \mathbf{a}^T \mathbf{A}_m - \mathbf{a}^T \mathbf{A}_s - \mathbf{a}^T \mathbf{A}_s \right) \Delta \lambda \right] = 0
\]

with

\[
k^{1+1} R = k^{1+1} R^* \Delta \mathbf{q}
\]

By the virtue of the variational method equation (6.72) gives:

\[
(k_e + K_g) \Delta \mathbf{q}^c - k^{1+1} R^* + \Delta \mathbf{f} = 0
\]

\[
-k^{1+1} R + \Delta \mathbf{q}^T \Delta \mathbf{f} = 0
\]

Substituting (6.46) and (6.48) into the above equations we get:

\[
(k_e + K_g) \Delta \mathbf{q}^c - k^{1+1} R^* + \Delta \mathbf{f} = (k_e + K_g) (\Delta \mathbf{q} - a \Delta \mathbf{\lambda}) = k^{1+1} R^* - \Delta \mathbf{f}
\]
\[-a^T (K_e + K_g) (\Delta q - a \Delta \lambda) + \left( H - a_b^T A_b - a_m^T A_m - a_s^T A_s - a_{\xi}^T A_{\xi} \right) \Delta \lambda = 0 \quad (6.76)\]

Equation (6.76) leads to:
\[
\Delta \lambda = \left[ \begin{array}{c} \begin{array}{c} a \left[ a^T (K_e + K_g) \right] a^T \end{array} + \left( H - a_b^T A_b - a_m^T A_m - a_s^T A_s - a_{\xi}^T A_{\xi} \right) \right]^{-1} a^T (K_e + K_g) \Delta q
\]
\[
\quad (6.77)
\]

Equation (6.75) becomes:
\[
K_{epdg} \Delta q = k^+ R^* - \Delta f
\]
\[
\quad (6.78)
\]

where \( K_{epdg} \) is the elasto-plastic, damage, large displacement stiffness matrix of the element, given by:
\[
K_{epdg} = (K_e + K_g)
\]
\[
\left\{ I - a \left[ a^T (K_e + K_g) \right] a + \left( H - a_b^T A_b - a_m^T A_m - a_s^T A_s - a_{\xi}^T A_{\xi} \right) \right]^{-1} a^T (K_e + K_g) \}
\]
\[
\quad (6.79)
\]

The tangent stiffness matrix given by the equation (6.79) is similar to the one presented by Shi and Voyiadjis (1992). The present formulation accounts for large displacements and consequently the stiffness matrix of the element contains the initial stress matrix \( K_g \). The above derived stiffness matrix describes not only isotropic hardening, by means of the parameter \( H \), but also kinematic hardening, through parameters \( A_b, A_m, A_s \), which are not determined by curve fitting, but derived explicitly from the evolution equation of backstress given by Armstrong and Frederick (1966).

The most important characteristic of this chapter is consistent and convenient incorporation of the damage effects into the yield condition and stiffness matrix, by means of \( A_{\xi} \) matrix. We therefore have a non-layered finite element formulation with shell constitutive equations, yield condition for porous ductile metals, the flow and hardening rules expressed in terms of the membrane and shear forces and bending moments. All the variables used here, namely the porosity function, the stress resultants and couples, as well as the residual stress resultants and couples, representing the center of the yield surface, are derived in a very rigorous manner.

### 6.4 Numerical Examples

A finite element computer program previously developed for the elasto-plastic considerations in programming language Fortran 95 is enhanced to account for the damage effects due to microvoids. A modified Newton-Raphson technique was employed to solve a system of non-linear, incremental equations. In order to overcome the singularity problem appearing at the limit point, the arc-length method (Crisfield, 1991) was adopted to determine local load increment for each iteration. An algorithm aiming at returning to the yield surface was also implemented (Crisfield, 1991). The results delivered by the current model were computed using a personal computer. Some of the reference solutions obtained with the layered approach (ABAQUS) were determined using a Silicon Graphics Onyx 3200 system.

The accuracy of the description of the elasto plastic and damage behaviour of shells are verified through the discriminating numerical examples. This chapter is a
continuation of the preceding ones, where linear elastic and elasto-plastic formulations were given. The most important novel feature of the present algorithm is the description of isotropic damage in plates and shells. The examples presented here were selected to challenge mainly the representation of the evolution of damage in shells and the associated reduction of stiffness.

Table 6.1 lists the references used here, and their corresponding abbreviations used later in the text.

<table>
<thead>
<tr>
<th>Name</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>W&amp;V-E</td>
<td>The present formulation – elastic analysis</td>
</tr>
<tr>
<td>W&amp;V-EP</td>
<td>The present formulation – elasto-plastic analysis</td>
</tr>
<tr>
<td>W&amp;V-EPD</td>
<td>The present formulation – elasto-plastic, damage analysis</td>
</tr>
</tbody>
</table>

6.4.1 Clamped Square Plate Subjected to a Central Point Load

In this example we consider a square plate with all the edges fixed, with an aspect ratio $L/h = 20$, where $L$ is the length of the plate and $h$ is the thickness. The plate is subjected to a central point load. Only a quarter of the plate needs to be examined due to symmetry. This problem was analyzed by Shi and Voyiadjis (1993), by means of the 4x4 mesh of finite elements. The same mesh of 4x4 elements per quarter of the plate will be employed here. The geometry of the problem and the material properties are given in Figure 6.2.

The equilibrium path is a universal curve providing most of the information about functioning of the model independently of whether the deformation of the structure is governed by the bending, membrane or shear strains. We study the equilibrium path for the problem described above. The material parameters $n$ and $k_3$ appearing in equations (6.7) and (6.11) are: $n = 1.2587$, $k_3 = 0.09$, as determined by Perzyna (1984b). The central deflection of the plate as a function of the applied load is given in Figure 6.3. Shi and Voyiadjis (1993) obtained the critical load for this problem, without the influence of damage, of $P_c = 10M_0$. They also showed the substantial reduction of stiffness of the structure when damage was considered. The critical load of the damaged plate was about $P_c = 8M_0$. The result of the current analysis with the influence of damage considered, yields approximately the same critical load $P_c = 8M_0$. As expected, we only see the damage variable becoming significant when the structure deforms plastically. This is because the evolution of damage is neglected in the elastic zone. The current formulation shows a robust performance in this test.
**Figure 6.2** Clamped square plate subjected to a central point load – geometry and material properties

**Figure 6.3** Clamped square plate subjected to a central point load – load displacement curve
6.4.2 Spherical Dome Subjected to a Ring of Pressure

A problem of a spherical dome with an $18^\circ$ hole at the top, subjected to a ring of pressure was investigated in the previous chapter. It was shown by Voyiadjis and Woelke (2004) that the stress resultant based shell model with the kinematic hardening rule given by the equations (6.35)-(6.36) is capable of correctly predicting the elasto-plastic behaviour of shells, including the Bauschinger effect. In this chapter, we revisit the problem of the spherical dome subjected to a ring of pressure, to establish the performance of the current formulation to approximate damage due to microvoids. As previously, the structure is loaded into a plastic zone, and then the pressure is reversed. We examine the elasto-plastic load-displacement curve and compare the results to the curve obtained with the influence of damage taken into account, in order to test the functioning and accuracy of the presented yield surface for ductile porous metals, defined in the stress resultant space. The material parameters $n$ and $k_3$ are the same as in the example 7.1: $n = 1.2587$, $k_3 = 0.09$. Geometrical and material data are shown in Figure 6.4 and the resulting load-displacement curves are plotted in Figure 6.5.

![Figure 6.4 Spherical dome with an $18^\circ$ cut-out; geometry and material properties](image)

The material parameters are:

- $R = 10$ in
- $t = 0.04$ in
- $E = 6.82 \times 10^7$ psi
- $\nu = 0.3$
- $\sigma_0 = 125$ psi

Through the introduction of the porosity function, which characterizes damage into the yield function we obtain a strong coupling between plasticity and damage. The damage variable is dependent on the plastic deformation and therefore through the application of a robust kinematic hardening rule we may model the evolution of damage in the structure loaded into the plastic zone in tension and then after the load is reversed, in compression (Figure 6.5). The lowered yield point due to the Bauschinger effect is again correctly approximated. The reduction of stiffness caused by damage initiated by the inelastic strains is significant. It leads to a decrease of the ultimate load carried by the structure by about 10%. It is a very substantial factor from the point of view of engineering analysis of important structures.
Figure 6.5 Spherical dome with an 18° cut-out – load displacement curve

Figure 6.6 Spherical dome with an 18° cut-out – load-porosity curve
Figure 6.6 presents a plot of porosity $\xi$ as a function of load. Since the evolution of porosity representing growth of voids is a function of the plastic strains, we only see growth of voids when plasticity occurs. In reality the porosity in the material will not be zero, even with only elastic strains. The level of porosity in the elastic range is however negligible and thus, is not accounted for in the current work. At the load level of approximately $P = 63 \text{ lbf/in}$ we observe a clear plateau in Figure 6.6. This means that this load level is the ultimate load carried by the structure. The porosity will most likely grow without application of any additional loading leading to a local fracture and ultimately collapse of the structure. The collapse will therefore occur at the load level approximately 8% lower that the load predicted by the elasto-plastic analysis. Based on the level of porosity at the first sign of unloading the fracture criterion could be postulated, which would provide additional tool for modeling of shells. Such a criterion should however be verified by the experimental results for different materials and structures, along with all the material parameters necessary for the damage description.

As we see in Figures 6.5 and 6.6, the present approach provides a good approximation of the evolution of damage of modeled structure, proving the validity of the original assumptions. Once again, the computational cost of the performed calculations is much lower than in the case of for example shell elements with a layered approach. This is due to the explicit form of the stiffness matrix, and application of the single loading surface with damage variable incorporated. A three dimensional analysis with the solid elements would be even more expensive. For some problems with complicated geometry, the computational cost of the finite element procedure may be decisive.

The reliability of the presented concept was evaluated through the solution of the two benchmark problems. In both cases, the results were accurate, which demonstrates that the model is well grounded.
CHAPTER 7

NUMERICAL METHODS AND COMPUTATIONAL ALGORITHMS

7.1 Introduction

An outline of the numerical techniques and computational algorithms used is presented in this chapter. The equations derived in the current dissertation, form a procedure for a comprehensive analysis of thick plates and shells. Considering the complexity of the constitutive equations, analytical solutions would only be possible for specific problems. Recent developments of computer technology as well as numerical methods provide us with a very powerful tool, allowing approximations of sometimes very complicated systems of equations, describing an engineering problem. In order to take advantage of this tool, a set of instructions must be provided to a computer, which defines in a suitable form the sequence of consecutive operations required to solve a given problem (Ketter and Prawel, 1969). Such a set of detailed instructions is called a computer program.

A computer program WOELKE-SHELLS developed in this work under the guidance of Professor George Z. Voyiadjis is provided in electronic form, on the back cover of the dissertation. It was gradually built, based on the shell constitutive equations presented in the preceding chapters, as well as the computer program published by Voyiadjis and Shi (1990). The programming language used was Fortran 95, along with the Compaq Visual Fortran compiler, version 6.6.C. The results of the analyses performed in this work were obtained using a personal computer. Some of the reference solutions were computed using the commercial finite element program ABAQUS installed on a Silicon Graphics Onyx 3200 Workstation.

A computer finite element code WOELKE-SHELLS is an integral part of this dissertation. It is a ‘stand-alone’ type of program, capable of carrying out full analyses of the structures under consideration, without use of any additional software. For the convenience of the user, some elements of pre-processing i.e. mesh generation and input writing were for certain problems, performed by means of the commercial finite element program ABAQUS.

The author would like to emphasize that the computer algorithm devised in this work serves a mere purpose of validating the constitutive equations derived, along with the assumptions made. In its current form, it should not be regarded as a universal software applicable to any commercial applications. Very extensive testing and further study would be imperative if the program WOELKE-SHELLS was to be adopted as a part of the commercial package intended for use in the industry.

In the following sections of this chapter, we discuss the computational issues and numerical techniques employed in the computer program. First, a method for solving a system of linear algebraic equations is covered. Next, the solution scheme of the non-linear equations is discussed, followed by the overall structure of the program.
7.2 Linear Elastic Analysis – System of Linear Algebraic Equations

A shell finite element was formulated in Chapter 3, based on the shell constitutive equations derived in Chapter 2. The stiffness matrix of the element $K$ was determined using a quasi-conforming technique. This method allows for the explicit determination of $K$, without performing numerical integration. Once the stiffness matrix is calculated, the analysis involves a solution of the system of linear algebraic equations given by:

$$ K q = R $$

(7.1)

where $K$ is the stiffness matrix of the structure given by the equation (3.83); $q$ is a vector of unknown nodal displacements of the structure and $R$ is an external load vector. In a linear problem, the coefficients of the stiffness matrix do not depend on the unknowns. There are many different methods of successive elimination of the unknowns, which is a direct way of solving simultaneous linear equations. One of the most popular elimination method i.e. the Gauss method is employed here. It is illustrated by considering a system of equations of the form:

$$
\begin{align*}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= c_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= c_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n &= c_n
\end{align*}
$$

(7.2)

In order to solve the above equations we follow two different stages, i.e. forward elimination and backward substitution. The former procedure involves eliminating the unknowns from the system of equations. For $a_{11} \neq 0$, the first of the equations (7.2) can be written as:

$$
\begin{align*}
x_1 + b_{12}x_2 + b_{13}x_3 + \ldots + b_{1n}x_n &= b_{1m}
\end{align*}
$$

(7.3)

where

$$
b_{ij} = \frac{a_{ij}}{a_{11}}, \quad j > 1 \quad \text{and} \quad m = n + 1
$$

(7.4)

An $(n-1)$ set of equations in the unknowns $x_2, x_3, \ldots, x_n$ may be obtained by successive elimination between equation (7.3) and the latter $(n-1)$ equations of (7.2). Dividing the coefficients of the first equation in this new set by the leading element $a_{22-1}$ gives the following equation (Ketter and Prawel, 1969):

$$
\begin{align*}
x_2 + b_{23-1}x_3 + \ldots + b_{2n-1}x_n &= b_{2m-1}
\end{align*}
$$

(7.5)

where

$$
b_{2j-1} = \frac{a_{2j-1}}{a_{22-1}}, \quad j > 2
$$

(7.6)

By continuing the process, the leading equation of the $(n-2)$ system may be written as:

$$
\begin{align*}
x_3 + b_{34-2}x_4 + \ldots + b_{3n-2}x_n &= b_{3m-2}
\end{align*}
$$

(7.7)
and

\[ b_{3j-2} = \frac{a_{3j-2}}{a_{33-2}}, \quad j > 3 \]  

(7.8)

When \((n-1)\) eliminations have been carried out we have:

\[ x_n = b_{nm-(n-1)} \]  

(7.9)

where:

\[ b_{nm-(n-1)} = \frac{a_{nm-(n-1)}}{a_{nn-(n-1)}} \]  

(7.10)

We follow backward substitution to determine the remaining \((n-1)\) values of the unknown \(x\).

It may be shown that a total number of multiplications and divisions necessary to solve a set of \(n\) linear simultaneous algebraic equations by the single division Gauss elimination procedure described above is (Ketter and Prawel, 1969):

\[ \frac{n}{3} \left( n^2 + 3n - 1 \right) \]  

(7.11)

Following the sequence of the above-described operations, the vector of unknown nodal displacements \(q\) is determined.

7.3 Non-Linear Analysis – System of Non-Linear Algebraic Equations

Several classes of non-linear problems of interest in many branches of science and engineering may be reduced to solution of a system of simultaneous equations in which the coefficients are dependent on some functions of the prime variables (Zienkiewicz, 1978). In this dissertation, we are only concerned with the investigations of the geometrically non-linear and elasto-plastic-damage problems. The use of finite element discretization in a large class of non-linear problems results in a system of simultaneous equations of the same form as equation (7.1):

\[ Kq = R \]  

(7.12)

The coefficients of the stiffness matrix \(K\) are however dependent on the unknowns \(q\). This is the one of the main distinctions between the non-linear problem and a linear one, in which the equation coefficients are independent. The numerical solution of the system of non-linear equations is much more complicated than the system of linear equations. Direct solution of (7.12) is generally impossible and an iterative scheme must be adopted (Owen and Hinton, 1980). This leads to the computational cost of the analysis of the non-linear problems being 10 to 100 times more than the linear approximation for the same number of degrees of freedom. Nevertheless, the advances in computer technology caused the computing cost to decline and non-linear calculations are undertaken much more often than in the past. In addition, more demands are placed on redundancy of the structures, which requires more sophisticated analysis (Cook et al., 1989).
The method of solution of the system of non-linear equation adopted in this work in a modified Newton-Raphson technique discussed in the following section.

### 7.3.1 Modified Newton-Raphson Method – Combined Incremental / Iterative Solutions

Analysis of a non-linear problem requires an iterative scheme, such as a Newton Raphson method. During any step of the iterative process of solution, expression (7.12) will not be satisfied unless convergence has occurred. A system of residual forces may be assumed to exist so that (Owen and Hinton, 1980):

\[ \Psi = Kq - R \neq 0 \]  

(7.13)

The residual forces \( \Psi \) may be interpreted as a measure of the departure of (7.12) from the equilibrium. Since \( K \) is function of \( q \) and possibly its derivatives, then at any stage of the process the residual forces are functions of the displacement vector, i.e. \( \Psi = \Psi(q) \).

If the true solution to the problem exists at \( q^* \Delta q^* \) then the Newton-Raphson approximation for the general term of the residual force vector \( \Psi^* \) is:

\[ \Psi^*_i = -\sum_{j=1}^{N} \Delta q^*_j \left( \frac{\partial \Psi_i}{\partial q_j} \right)^* \]  

(7.14)

in which \( N \) is the total number of variables in the system and the superscript \( r \) denotes the \( r^{th} \) approximation of the true solution. Substituting for \( \Psi_r \) from the equation (7.13), the complete expression for all the residual components may be written in a matrix form:

\[ \Psi(q^*) = -J(q^*) \Delta q^* \]  

(7.15)

where \( J \) is a Jacobian matrix with a typical term given by:

\[ J_{ij} = \left( \frac{\partial \Psi_i}{\partial q_j} \right)^* = k_{ij}^* + \sum_{k=1}^{m} \left( \frac{\partial k_{ik}}{\partial q_j} \right)^* q_k^* \]  

(7.16)

and \( k_{ij} \) is the general term of the stiffness matrix. The last term in the equation (7.16) gives rise to non-symmetric terms in the Jacobian matrix. These terms are retained here in order to achieve a better convergence (Owen and Hinton, 1980).

The explicit form of the non-linear terms in (7.16) will depend on the way in which the stiffness coefficients \( k_{ij} \), depend on the unknowns \( q \). The terms in the Jacobian matrix may be assembled to give the general expression:

\[ J(q) = K(q) + K'(q) \]  

(7.17)

where \( K'(q) \) contains the unsymmetric terms only. The Newton-Raphson process may finally be written using equations (7.15) and (7.17) in the following form:

\[ \Delta q^* = -\left[ J(q^*) \right]^{-1} \Psi(q^*) = \left[ K(q^*) + K'(q^*) \right]^{-1} \Psi(q^*) \]  

(7.18)
The above relation allows the correction to the vector of unknowns \( \mathbf{q} \) to be obtained from the residual force vector \( \Psi \) for any iteration. The iterative approach must be followed, with the vector \( \mathbf{q} \) being corrected at each stage according to (7.18) until convergence is achieved. The technique is illustrated schematically in Figure 7.1 for a single variable situation.

**Figure 7.1** The Newton-Raphson method for a single variable problem-convex \( K - \mathbf{q} \) relation

The solution of the non-linear problem is achieved when the residual force \( \Psi \) vanishes, since this is a direct measure of the lack of equilibrium of the governing equation. First, a trial value \( \mathbf{q}^0 \) of the unknown is assumed and the material stiffness associated with this value is calculated according to the prescribed \( K - \mathbf{q} \) relationship. The residual force is then calculated from equation (7.13) and the Jacobian matrix from equation (7.16). The correction of the first assumed value \( \Delta \mathbf{q}^0 \) may be found from (7.18), giving an improved approximation to the solution \( \mathbf{q}^1 = \mathbf{q}^0 + \Delta \mathbf{q}^0 \). The process is then repeated until the residual force \( \Psi \) vanishes or is sufficiently small. The Newton-Raphson process generally gives relatively rapid and stable convergence.

The iterative technique described above can only provide a single point solution, since we only apply a single increment of load, and iterate until convergence is achieved. In practice however, we most often need a complete load-displacement response (equilibrium path). In order to determine the equilibrium path, we combine the
incremental and iterative solution procedures. We first apply an increment of load and with a tangential stiffness matrix we obtain a starting solution $q^0$, as shown in Figure 7.1. This first step solution is called a predictor. After the first predictor is computed, we apply the iterations until the solution converges. Then another increment of loading is applied and the process is repeated until the desired load level is reached. This method is commonly referred to as a modified Newton-Raphson technique or combined incremental/iterative solutions. Figure 7.2 illustrates this process.

![Figure 7.2 A combination of incremental predictors with Newton-Raphson iterations](image)

### 7.3.2 The Arc-Length Technique

A computational model presented in the current dissertation is intended for the non-linear elasto-plastic and damage analysis of shells. Comprehensive modeling of structures requires finding the entire equilibrium path, until collapse occurs. Solution of the system of non-linear equations by means of the combined incremental/iterative algorithms discussed in Section 7.3.1, may lead to problems near the limit point, where the stiffness of the structure approaches zero. This may result in a singularity problem and potentially large errors in the results. In order to overcome this shortcoming, the arc-length method is employed in the current work. This technique in relation to structural analysis was originally proposed by Riks (1972, 1979) and Wempner (1971), with later modifications by Crisfield (1983, 1991 and 1997).

As a starting point of the illustration of the arc-length method, we write the equilibrium equations in the following form (Crisfield, 1991):

$$
g(q, \lambda) = F_i(q) - \lambda F_{ef} = 0$$

(7.19)
where $\mathbf{F}_i$ is a vector of the internal forces, which are functions of the displacements $\mathbf{q}$; the vector $\mathbf{F}_{ef}$ is a fixed internal loading vector and a scalar parameter $\lambda$ is a load level parameter that multiplies $\mathbf{F}_{ef}$. The arc-length method aims at finding the intersection of the curve described by equation (7.19) with the curve $s = \text{constant}$, where $s$ is the arc length, defined by:

$$s = \int ds$$

(7.20)

and

$$ds = \sqrt{\left(d\mathbf{q}^T d\mathbf{q} + d\lambda^2 \mathbf{G}^2 \mathbf{F}_{ef}^T \mathbf{F}_{ef}\right)}$$

(7.21)

The scaling parameter $\mathbf{G}$ in equation (7.21) is accounting for the fact that the load contribution depends on the adopted scaling between the load and displacement terms. For the arc-length method, we may replace the differential form of equation (7.21) with the incremental form:

$$a = \left(\Delta \mathbf{q}^T \Delta \mathbf{q} + \Delta \lambda^2 \mathbf{G}^2 \mathbf{F}_{ef}^T \mathbf{F}_{ef}\right) - \Delta l^2 = 0$$

(7.22)

where $\Delta l$ is a fixed radius of the desired intersection (Figure 7.3). The vector $\Delta \mathbf{q}$ and a scalar parameter $\Delta \lambda$ are incremental (not iterative) and relate to the last converged equilibrium state. The essence of the arc-length method is that the load parameter $\lambda$ is a variable. An additional constraint equation that allows us to determine that variable is (7.22).

The Newton-Raphson technique with the load parameter accounted for can be introduced via a truncated Taylor series. Using equations (7.19) and (7.22) we may write:

$$\mathbf{g}_n = \mathbf{g}_o + \frac{\partial \mathbf{g}}{\partial \mathbf{q}} \Delta \mathbf{q} + \frac{\partial \mathbf{g}}{\partial \lambda} \Delta \lambda = \mathbf{g}_o + \mathbf{K}_r \Delta \mathbf{q} - \mathbf{F}_{ef} \Delta \lambda = 0$$

(7.23)

$$a_n = a_o + 2 \Delta \mathbf{q}^T \Delta \mathbf{q} + 2 \Delta \lambda \Delta \lambda \mathbf{G}^2 \mathbf{F}_{ef}^T \mathbf{F}_{ef} = 0$$

(7.24)

where the subscript $n$ means ‘new’ and the subscript $o$ means ‘old’. We then directly introduce the constraint of equation (7.24) for displacement control for a single point. To this end, an iterative displacement $\delta \mathbf{q}$ is split into two parts. The change of the displacement at the unknown load level $\lambda_o = \lambda_o + \delta \lambda$ becomes:

$$\delta \mathbf{q} = -\mathbf{K}_r^{-1} \mathbf{g}_o \lambda_o + \delta \lambda \mathbf{F}_{ef} \lambda_o = -\mathbf{K}_r^{-1} \left(\mathbf{F}_l \left(\mathbf{q}_o\right) - \lambda_o \mathbf{F}_{ef}\right) = -\mathbf{K}_r^{-1} \left(\mathbf{g}_o \lambda_o \mathbf{F}_{ef} \lambda_o - \delta \lambda \mathbf{F}_{ef}\right)$$

(7.25)

The above equation may be written as (Crisfield, 1991):

$$\delta \mathbf{q} = -\mathbf{K}_r^{-1} \mathbf{g}_o + \delta \lambda \mathbf{K}_r^{-1} \mathbf{F}_{ef} = \delta \mathbf{q}^* + \delta \lambda \delta \mathbf{q}_t$$

(7.26)

where:

$$\delta \mathbf{q}_t = \mathbf{K}_r^{-1} \mathbf{F}_{ef} \quad \text{and} \quad \delta \mathbf{q}^* = -\mathbf{K}_r^{-1} \mathbf{g}_o$$

(7.27)

The symbol $\delta \mathbf{q}^*$ is the iterative change that would arise from the standard load-controlled Newton-Raphson method (at a fixed load level $\lambda_o$), and $\delta \mathbf{q}_t$ is the displacement vector corresponding to a fixed load vector $\mathbf{F}_{ef}$. After calculating $\delta \mathbf{q}$ from equation (7.26) we update the incremental displacements as follows:
\[ \Delta \mathbf{q}_o = \Delta \mathbf{q}_o + \delta \mathbf{q} = \Delta \mathbf{q}_o + \delta \mathbf{q} + \delta \lambda \delta \mathbf{q}_i \]  
\( (7.28) \)

where \( \delta \lambda \) is the only unknown. It can be found from equation (7.22), which may be expressed as:

\[ \left( \Delta \mathbf{q}_o \, \Delta \mathbf{q}_o + \Delta \lambda_o \, \partial^2 \mathbf{F}_{ef}^T \mathbf{F}_{ef} \right) = \left( \Delta \mathbf{q}_n \, \Delta \mathbf{q}_n + \Delta \lambda_n \, \partial^2 \mathbf{F}_{ef}^T \mathbf{F}_{ef} \right) = \Delta l^2 \]  
\( (7.29) \)

Substituting equation (7.28) into the above we obtain a scalar quadratic equation:

\[ a_1 \delta \lambda^2 + a_2 \delta \lambda + a_3 = 0 \]  
\( (7.30) \)

where:

\[ a_1 = \delta \mathbf{q}_i \, \delta \mathbf{q}_i + \partial^2 \mathbf{F}_{ef}^T \mathbf{F}_{ef} \]
\[ a_2 = 2 \delta \mathbf{q}_i \left( \Delta \mathbf{q}_o + \delta \mathbf{q} \right) + 2 \Delta \lambda_o \, \partial^2 \mathbf{F}_{ef}^T \mathbf{F}_{ef} \]
\[ a_3 = \left( \Delta \mathbf{q}_o + \delta \mathbf{q} \right)^T \left( \Delta \mathbf{q}_o + \delta \mathbf{q} \right) - \Delta l^2 + \Delta \lambda_o \, \partial^2 \mathbf{F}_{ef}^T \mathbf{F}_{ef} \]  
\( (7.31) \)

Equation (7.30) may be solved for \( \delta \lambda \) so that the change of the change of the displacement given by equation (7.28) is defined. Solution of the quadratic equation (7.30) will yield two roots \( \delta \lambda_1 \) and \( \delta \lambda_2 \). We choose the appropriate root by calculating both solutions, and substituting them into equation (7.28). We therefore have:

\[ \Delta \mathbf{q}_{n1} = \Delta \mathbf{q}_o + \delta \mathbf{q} + \delta \lambda_1 \delta \mathbf{q}_i \]
\[ \Delta \mathbf{q}_{n2} = \Delta \mathbf{q}_o + \delta \mathbf{q} + \delta \lambda_2 \delta \mathbf{q}_i \]  
\( (7.32) \)

Of the two solutions given above we choose the displacement that lies the closest to the old incremental direction \( \Delta \mathbf{q}_o \). This procedure may be implemented by finding the solution with the minimum angle between \( \Delta \mathbf{q}_o \) and \( \Delta \mathbf{q}_n \), and hence the maximum cosine of the angle, expressed as:

\[ \cos \theta = \frac{\Delta \mathbf{q}_o \, \Delta \mathbf{q}_n}{\Delta l^2} = \frac{\Delta \mathbf{q}_o \, \left( \Delta \mathbf{q}_o + \delta \mathbf{q} \right)}{\Delta l^2} + \delta \lambda \frac{\Delta \mathbf{q}_o \, \delta \mathbf{q}_i}{\Delta l^2} = \frac{a_4 + a_5 \delta \lambda}{\Delta l^2} \]  
\( (7.33) \)

The process of the determination of the load increment using the arc-length method is illustrated schematically in Figure 7.3. After convergence at the equilibrium point \( \left( \mathbf{q}_0, \lambda_0 \mathbf{F}_{ef} \right) \) an incremental, tangential predictor \( \left( \Delta \mathbf{q}_1, \Delta \lambda_1 \right) \) is calculated, leading to the point \( \left( \mathbf{q}_1, \lambda_1 \mathbf{F}_{ef} \right) \). The first iteration will use equations (7.30) and (7.31) with the old value \( \Delta \mathbf{q}_o \) as \( \Delta \mathbf{q}_1 \) and the old \( \Delta \lambda_o \) as \( \Delta \lambda_1 \) to obtain \( \delta \mathbf{q}_1 \) and \( \delta \lambda_1 \), after which the updating procedure leads to:

\[ \Delta \mathbf{q}_3 = \Delta \mathbf{q}_2 + \delta \mathbf{q}_2 \]
\[ \Delta \lambda_3 = \Delta \lambda_2 + \delta \lambda_2 \]  
\( (7.34) \)

When added to the displacements, \( \mathbf{q}_o \) and load level \( \lambda_o \), at the end of the previous increment this process leads to the point \( \left( \mathbf{q}_2, \lambda_2 \mathbf{F}_{ef} \right) \) in Figure 7.3. The next iteration again applies equations (7.30) and (7.31) with the old value \( \Delta \mathbf{q}_o \) as \( \Delta \mathbf{q}_2 \) and the old \( \Delta \lambda_o \) as \( \Delta \lambda_2 \) to obtain \( \delta \mathbf{q}_2 \) and \( \delta \lambda_2 \), after which the updating procedure leads to
$\Delta q_1 = \Delta q_2 + \delta q_2$ and $\Delta \lambda_3 = \Delta \lambda_2 + \delta \lambda_2$. The iteration process stops when convergence is reached. The flowchart for this procedure is given by Crisfield (1991).

**Figure 7.3** Spherical arc-length method (Crisfield, 1991)

The scaling parameter $\vartheta$ introduced in equation (7.21) can be for most practical problems set to zero (Crisfield, 1981, 1991). This is the case in this work.

### 7.3.3 Integrating the Rate Equations – Return to the Yield Surface

The associated flow rules given by equations (6.43) are of incremental nature. The solution of the constitutive equations is based on the predictor/corrector approach. This method however leads to errors that are not related to the lack of equilibrium, but are caused by the errors in the integration of the flow rules and their relation to the incremental/iterative solution procedure. Even if equilibrium is exactly satisfied at the beginning and end of the increment, the solution will not correspond exactly with a solution in which the increment was itself cut into a number of smaller increments for
each of which equilibrium was exactly ensured (Crisfield, 1991). If the stress and strain increments were very small we could effectively proceed with the previous tangential stiffness, without a very significant loss of accuracy. However, the strains and the subsequent stress resultant increments are not infinitesimally small, and consequently the errors will accumulate leading to a drift from the yield surface. If we follow the process outlined in Chapters 5 and 6 to determine the plastic multiplier \( \Delta \lambda \), we use equation (5.69) repeated here for convenience:

\[
\Delta \lambda = \left[ a^T \left( K + K_g \right) a + \left( H - a_b^T A_b - a_m^T A_m - a_s^T A_s \right) \right]^{-1} a^T \left( K + K_g \right) \Delta q \tag{7.35}
\]

with

\[
a_i^T = \left\{ \frac{\partial F^*}{\partial N_{si}} + p_u \frac{\partial F^*}{\partial N_{yi}}; \frac{\partial F^*}{\partial N_{yi}} + p_v \frac{\partial F^*}{\partial N_{xi}}; \frac{\partial F^*}{\partial Q_{xi}} + \frac{\partial F^*}{\partial Q_{yi}}; \frac{\partial F^*}{\partial M_{si}} + p_{\phi x} \frac{\partial F^*}{\partial M_{xi}}; \frac{\partial F^*}{\partial M_{yi}} + p_{\phi y} \frac{\partial F^*}{\partial M_{yi}} \right\} \tag{7.36}
\]

\[
p_u = \frac{2 \Delta u_i^2}{\Delta u_i^2 + \Delta v_i^2}; \quad p_v = \frac{2 \Delta v_i^2}{\Delta u_i^2 + \Delta v_i^2}; \quad p_{\phi x} = \frac{2 \Delta \phi_{xi}^2}{\Delta \phi_{xi}^2 + \Delta \phi_{yi}^2}; \quad p_{\phi y} = \frac{2 \Delta \phi_{yi}^2}{\Delta \phi_{xi}^2 + \Delta \phi_{yi}^2}
\]

The meaning of all the functions and parameters is explained in Chapter 5. In this case, we compute \( a_i \) at the beginning of increment, and we are bound to obtain the stress resultants that lie outside of the yield surface at the end of an increment, as shown in Figure (7.4), in which \( \Delta \Sigma \) denotes an increment of the stress resultants.

![Figure 7.4 Drift from the yield surface](image-url)
The situation shown in Figure 7.4 requires taking steps to return the stress resultants to the yield surface, which prevents accumulation of errors leading to overprediction of the collapse load. The procedure discussed by Crisfield (1991) is adopted here to overcome this shortcoming. We first determine the point of intersection of the elastic stress vector with the yield surface. In this case, we require that the stress resultants after application of the increment of loading remain on the yield surface. This may be written as:

\[ F(\Sigma + \beta \Delta \Sigma) = 0 \]  \hspace{1cm} (7.37)

where \( \Sigma \) is a stress resultant vector (see Figure 7.4), which is a function of both the bending moment \( M \) and normal force \( N \); \( \Delta \Sigma \) is an increment of stress resultants and \( \beta \) is a scaling parameter. Considering that the yield function is expressed in terms of the stress resultants, the function \( \Sigma + \beta \Delta \Sigma \) is:

\[ \Sigma + \beta \Delta \Sigma = \Sigma + \beta \Delta \Sigma \left( M + \beta \Delta M, N + \beta \Delta N \right) \]  \hspace{1cm} (7.38)

For the purpose of the procedure of returning to the yield surface, we use the yield surface of the form given by equation (5.1) rewritten here for convenience:

\[ F(\Sigma) = \left( \frac{M^*}{M_0} \right)^2 + \left( \frac{N^*}{N_0} \right)^2 - 1 = 0 \]  \hspace{1cm} (7.39)

where \( M_0, N_0 \) are the moment and normal force capacities of the cross section of the shell, given by:

\[ M_0 = \frac{\sigma_0 h^2}{4}, \quad N_0 = \sigma_0 h \]  \hspace{1cm} (7.40)

and \( \sigma_0 \) is a yield stress; \( h \) is the thickness of the shell; \( M^*, N^* \) are the stress resultant intensities given by equations (6.33)-(6.34).

We note that the influence of the parameter \( \alpha \) responsible for the representation of the progressive development of the plastic curvatures across the thickness of the shell, on the errors related to the integration of the rate equations, was not considered (compare equations (5.8)-(5.9)).

Multiplying both sides of the equation (7.39) by \( \sigma_0^2 \) we obtain:

\[ F(\Sigma) = \left( \frac{M^*}{M_0} \right)^2 \sigma_0^2 + \left( \frac{N^*}{N_0} \right)^2 \sigma_0^2 - \sigma_0^2 = 0 \]  \hspace{1cm} (7.41)

Substituting (7.40) into the above yields:

\[ F(\Sigma) = \frac{16M^2}{h^4} + \frac{N^2}{h^2} - \sigma_0^2 = 0 \]  \hspace{1cm} (7.42)

where \( M^*, N^* \) were replaced by \( M, N \) for brevity. Using equation (7.37) we write:

\[ F(\Sigma + \beta \Delta \Sigma) = \frac{16(M + \beta \Delta M)^2}{h^4} + \frac{(N + \beta \Delta N)^2}{h^2} - \sigma_0^2 = 0 \]  \hspace{1cm} (7.43)

simplifying equation (7.43) we obtain a quadratic expression for \( \beta \):
where we require a positive root of (7.44). Once a parameter \( \beta \) is found, we scale down the stress resultants until the yield surface \( F \) becomes zero (Ortiz and Popov, 1985).

### 7.4 Overall Program Structure

The most common way of describing the overall structure of the program is a flowchart. Because of the complexity of the program WOELKE-SHELLS, developed in this dissertation, it is difficult to create the flowchart following the algorithm in detail. An attempt however is made to illustrate the main parts of the structure of the program, as well as a summary of the procedure of the solution of the system of equations. The flowchart is given in Figure 7.5.

\[
\left( \frac{16\Delta M^2}{h^4} + \frac{\Delta N^2}{h^2} \right) \beta^2 + \left( \frac{32M\Delta M}{h^4} + \frac{2N\Delta N}{h^2} \right) \beta + \left( \frac{16M^2}{h^4} + \frac{N^2}{h^2} \right) - \sigma_0^2 = 0 \quad (7.44)
\]
Figure 7.5 Flowchart—overall program structure

- Using an updated displacement vector $\mathbf{q}$, calculate the internal forces and bending moments.
- Check the yield function – equation (6.32), $F \geq 0$?
  - No: Update coordinates of the element nodes.
  - Yes: Find a scaling parameter $\beta$, eqn. (7.44), return to the yield surface.
- Check convergence $\frac{\delta \mathbf{q}}{\Delta \mathbf{q}_n} > \text{CONCRI}$?
  - No: Start iterations, find $\delta \lambda$
    - $\delta \lambda = \text{STEP}$
    - Calculate new stiffness and unbalance force vector $\Psi$
    - Calculate an increment of displacement $\delta \mathbf{q} = \mathbf{K}_r^{-1}(\mathbf{F}_n - \Psi)$
      - RETURN
  - Yes: Update coordinates of the element nodes.
- Calculate an increment of displacement $\delta \mathbf{q} = \mathbf{K}_r^{-1}(\mathbf{F}_n - \Psi)$
  - RETURN

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CHAPTER 8
SUMMARY AND CONCLUSIONS

8.1 Introduction

The components of the formulation presented in this dissertation, namely the theory, finite element, large rotation analysis, elasto-plastic investigations and damage description, comprise a unified computational model for the elasto-plastic and damage analysis of beams, plates and shells. Nevertheless, as already discussed in Chapter 1, they are universal and introduce original ideas on every level of the algorithm. Consequently, we draw the conclusions for individual chapters separately, pointing out the most valuable and novel features introduced in a given chapter, as well as their imperfections and limitations. Section 8.7 gives the overall summary of the conclusions drawn for the formulation as a whole, followed by recommendations for the future work.

8.2 A Refined Theory of Thick Spherical Shells

A theory for thick spherical shells is developed and presented in Chapter 2. The transverse shear deformations and initial curvature effect are accounted for in the resulting shell equations, with the latter leading to a nonlinear distribution of the in-plane stresses. Since the radial stresses are also included, we obtain the stress resultants and stress couples associated not only with the displacements of the middle surface of shells, but also with the radial stresses. By means of the strain-displacement relations in spherical coordinates, stress-strain constitutive equations of the three-dimensional theory of elasticity and incorporating the above-listed effects, an accurate set of shell equations is obtained.

The constitutive equations presented here reduce to those given by Flugge (1960) when the shear deformations and the radial stress effects are neglected and the average displacements are replaced by the middle surface displacements. The proposed equations are slightly different from those given by Sanders (1959), Koiter (1960) and Niordson (1978), primarily because they use the so-called effective stress resultants and stress couples. These effective stress resultants are used in the variational derivation of the constitutive equations (Niordson, 1985). Even when both the shear deformation and the radial stresses are neglected, the stress distributions over the thickness will still be nonlinear, because the stresses are derived here from the three-dimensional constitutive equations given by expressions (2.26) – (2.28). The nonlinear distribution of the in-plane stresses through the thickness is largely ignored when investigating thick shells. This is not the case in the present dissertation. The nonlinear distribution constitutes a very important ingredient for an accurate and reliable thick shell theory.

Similarly to the shell theory by Sanders-Koiter (1959, 1960), the presented equations are convenient for use in finite element analysis. The proposed theory is not only very useful in the analysis of thick shells, but also has the potential for use in the analysis of composite shells and dynamic problems where the shear deformation and stress distribution across the thickness play an important role.
The examples given in Chapter 2, show that the proposed equations are accurate and in a good agreement with the exact solutions, and other existing theories. The classical theory of shells based on the Kirchhoff-Love assumption, yields errors that could become large in the case of thick shells. In the present work, there is a significant reduction in error. This is clearly shown in the first example (Section 2.4.1). The results presented here are very good for very thick as well as very thin shells.

A limitation of the current shell equations is use of the spherical strains in the derivation. This restrains their direct use from problems in which the radius of curvature of the shell is different in two distinct directions, e.g. cylinders. It was however shown that the theory could be easily reduced to the cylindrical case, without a substantial loss of accuracy (Section 2.4.3). Furthermore, through the finite element method the constitutive equations devised here may be employed to model shells of arbitrary shape, as well as arches (Section 2.4.4), beams and plates. In this sense, the current theory is general and universal and gives very good results for all the above-discussed cases.

Although there are numerous shell theories available in the published literature it is difficult to find the one taking into account the transverse shears, initial curvature and radial stresses. The objective of this dissertation is to develop a computational model for the non-linear analysis of plates and shells. In order to achieve this objective the shell theory is necessary. Instead of using one of the existing formulations as a base for the algorithm, the author opted for developing a new, refined and original theory of thick shells, which for certain applications is more accurate than other theories.

It is noteworthy that including all the effects considered in the present shell equations is motivated by higher demands imposed nowadays on the accuracy of structural analysis. On the other hand, higher accuracy usually means increased complexity, which is unfortunately the case here. Complicated equations are then frequently solved with the aid of simplifying assumptions, which inherently lead to a loss of accuracy. Application of the average displacements (Section 2.2.5) is an example of such an assumption. Nevertheless, it is very useful to exercise the incorporation of more and more phenomena observed physically and experimentally in modeling of materials and structures, as it ultimately drives scientific developments and technology.

8.3 Shell Element Based on the Refined Theory of Thick Spherical Shells

A simple quadrilateral doubly curved shell element based on the refined theory of thick spherical shells, strain energy density, and the quasi-conforming technique is developed in Chapter 3. The element is valid for the elastic analysis of both thick and thin shells, with the transverse shear strains and initial curvature effect included in the shell constitutive equations.

The quasi-conforming technique is adopted in construction of the element (Tang et al., 1980, 1983; Shi and Voyiadjis, 1990, 1991a). It allows for explicit derivation of the stiffness matrix, without performing numerical integration, which results in a high computational efficiency of the current procedure. This fact becomes of great importance when the shell finite element formulation is used for geometrically non-linear or inelastic analysis, where the stiffness matrix has to be evaluated many times. The current work is suited for such analyses, which was shown in Chapters 4-6. Furthermore, common
deficiencies encountered in the construction of shell elements i.e. locking, spurious energy modes and numerical ill conditioning are overcome by means of the quasi-conforming approach. The interpolation of the strain fields in the present element results in a reliable representation of the rigid body motion and prevents the spurious and zero energy modes. The shear locking is avoided by the displacement and rotation interpolations containing the ratio of the flexural and shear rigidities (Hu, 1984). These interpolation functions satisfy the Kirchhoff–Love hypothesis for thin beams, plates and shells, which prevents shear locking. Following this approach to overcome locking is more consistent mathematically than using the selective reduced integration. All the components of the stiffness matrix are integrated here analytically and exactly, as opposed to selective reduced integration where the shear part of the stiffness is underintegrated to ensure its singularity.

The performance of the proposed finite element was evaluated through a set of discriminating benchmark problems, selected from the literature. The examples shown in Chapter 3 prove that the present formulation does not experience shear locking, nor spurious energy modes for thin shells. At the same time, it offers a superior performance than most of the formulations compared with in the analysis of thick shells, providing very accurate results.

Despite very careful considerations when interpolating the strains, the finite element developed suffers from mild membrane locking. An example of a pinched cylinder with a diaphragm, which is one of the most severe tests for both inextensional bending modes and complex membrane states, is given in Section 3.3.4. Although the current formulation offered very accurate results and fast convergence in the case of the cylinder with radius R=300 in, the mesh became too stiff when the radius was decreased to R=50 in. The conclusion is that the representation of the membrane strains, may be inadequate and does not always ensure satisfaction of the inextensibility condition. This was also indicated by the results of the Scordelis-Lo roof problem in Section 3.3.5. The strain fields could be further refined to cure this problem. The author found however, that the quasi-conforming technique, although very convenient, is also very sensitive to any errors in the derivation, as well as any changes in the interpolation of both strain and displacement fields. The approximation formulas that may seem reasonable form the physical point of view, may sometimes produce results as much as 20% off the target solution. Refining the strain interpolation in order to get rid of membrane locking completely is therefore a trial and error process and may be very tedious. The best solution in the author’s opinion is to construct the membrane displacement approximations dependent on the vertical displacement function $w$. The ratio of flexural and membrane rigidities should enter the membrane displacement interpolations, similarly to adopted Hu’s functions for $w$ and $\phi$ (equations (3.33)-(3.34)), where a parameter $\lambda$ (equation (3.37)) enforced a constraint of transverse shears reducing to zero in the case of thin shells. A similar parameter, dependent on the ratio of flexural and membrane rigidities should be included in the displacement field in such a way that the inextensibility condition is explicitly satisfied when the thickness of the shell decreases. It is again the author’s opinion, that such a remedy could be successful, and the quasi-conforming technique along with carefully devised inter-related displacement
approximations would be a powerful and efficient tool for constructing curved shell elements, despite its sensitivity.

In spite of the above-discussed deficiencies, the finite element model presented in Chapter 3 performs well in all the tests undertaken. The model is also superior to many other formulations in simulating the behaviour of moderately thick and thick shells. The present procedure is therefore universal, applicable to thick and thin shells, plates and beams, showing good overall performance, and being in the same time computationally efficient.

8.4 Geometrically Non-Linear Analysis

Small strain geometrical non-linearities are considered in this dissertation. Large rigid rotations and translations are included but not large strains. The Updated Lagrangian description was used, with the total rotations decomposed into large rigid and moderate relative rotations. The linear elastic stiffness matrix developed in Chapter 3 was adopted in the description of large displacements. The quasi-conforming technique was once again employed to derive the initial stress stiffness and the total tangent stiffness matrix, which is given explicitly. The details of the representation of the large displacements are given in Chapter 4.

The ability of the model to simulate large rigid rotations and translations in shells was evaluated by a numerical example in Section 4.5. The displacements calculated with the current model are very accurate and compare very well with the reference solutions.

Since certain sections of the structure deforming inelastically usually undergo large displacements, representation of geometrical non-linearities is crucial for the accuracy of elasto-plastic and damage analysis of shells. If the computational model for the elasto-plastic analysis of shells with large rotations considered performs well in the elasto-plastic analysis, it proves also, although indirectly, the adequate representation of large displacements. The results of the elasto-plastic investigations were very accurate, as was shown in Chapter 5.

In order to thoroughly examine the functioning of the current algorithm in the description of large displacements in shells further testing would be necessary. It should be explicitly verified whether the strains produced by the rigid body motion of the structure as a whole are significant. If the large displacement description is reliable, these strains should of course be zero. Such a test is however difficult to execute by means of the in-house computer code, developed in order to validate the constitutive equations derived in the present dissertation. The sole purpose of incorporating large rotations into the analysis was to provide a sufficient accuracy to the elasto-plastic and damage simulations. It is therefore concluded, that the example presented in Section 4.5, as well as robust performance of the current procedure in the elasto-plastic and damage investigations is a sufficient proof of reliability of the formulation of geometrical non-linearities. The ability of the current computational model to simulate the plasticity and damage in plates and shells is discussed in the following sections.
8.5 Elasto-Plastic Analysis

Chapter 5 presents the elasto-plastic analysis of thick plates and shells. The previously developed algorithm is enhanced here to model the elasto-plastic behaviour of structures under consideration. We therefore have a finite element formulation for the elasto-plastic analysis of thick beams plates and shells, with large rotations.

The most important feature of the procedure devised in Chapter 5, is the non-layered yield surface with a new kinematic hardening rule. Iliushin’s yield function expressed in terms of stress resultants and couples, modified to account for progressive development of the plastic deformation and transverse shear forces was used. The kinematic hardening rule representing the rigid motion of the yield surface during loading in the stress resultant space was derived. It is capable of simulating the load-displacement response including the Bauschinger effect. Residual forces and bending moments were related to backstress using similar definition to that of primary forces and moments. All the integrals were calculated analytically, which makes the current formulation effective, as numerical integration is not employed at any stage of the computations. The yield surface with a new kinematic hardening rule outlined in Chapter 5 simplifies and speeds up the analysis, without any substantial loss of accuracy.

The reliability of the presented concepts was evaluated through a series of benchmark problems, which were carefully chosen to challenge and demonstrate the most important features of the current model. In all the cases the results were very close to the reference solutions, which demonstrates that the model is well grounded.

The effect of the shear forces on the plastic behavior and maximum load carrying capacity is correctly recognized. As expected, the results show a reduction of the load factor for thick plates, shells and beams, owing to the increasing significance of transverse shears.

The progressive plastification of the cross section is also closely approximated. Typically, in the non-layered approach, the load displacement relation is linear until the plastic hinge is developed. Any yielding occurring before the section is fully plastic is neglected. Through a modification introduced by Crisfield (1981), the first yield of the outer fibers may be predicted, as was also proven here.

A spherical dome problem (Section 5.4.4.) proves that, although presented framework is robust for plates and shells of general shape, it performs best in the case of spherical shells. This is expected since the shell constitutive equations used here were derived by means of spherical strains, and later generalized through the finite element method.

The Bauschinger effect may only be numerically observed if the method employed features a veracious kinematic hardening rule. The one proposed here was defined in a stress resultant space, which is very effective from the structural analysis viewpoint. The lowered yield point upon reversal of load was correctly determined here for both plates and shells proving that the definition of the ‘hardening parameters’ is sound and capable of delivering very accurate results. It is worthwhile to mention the importance of the material constants occurring in the definition of the hardening parameters. The correct determination of these constants is critical for the dependability of the kinematic hardening rule. In the current work, the constants were calibrated based
on the reference solutions, whereas ideally, they should be determined from extensive experimental data. Furthermore, the results of the analysis provided by the current model, which are of course approximate, should be compared to experimental results, rather than other approximations, based on a different theoretical formulation (Bieniek and Funaro, 1976) It is important to note that the elasto-plastic analysis is a continuation of formulation developed in the preceding chapters. Consequently, any limitations and deficiencies experienced in the elastic and geometrically non-linear investigations will inherently be present here.

The approximation introduced on the elasto-plastic level of the model that can be subject to critique, is employing a plastic node method, which presumes the concentration of the plastic deformation in the nodes of the elements, while the interior always remains elastic. Clearly, the spread of inelastic deformations will occur in many cases. The results of the analysis show however, that any errors arising due to following a plastic node method are not significant even for coarse meshes.

The elasto-plastic formulation given in Chapter 5, offers a redundant yield surface and a new kinematic hardening rule in the stress resultant space. It delivers precise results of the non-linear analysis of shells under cyclic loading, being at the same time relatively simple and very efficient.

8.6 Damage Due to Microvoids in Plates and Shells

The most important and novel feature introduced in Chapter 6, is a simple and convenient description of damage evolution in plates and shells. Since this work concerns the study of thick, homogenous isotropic and ductile shells, damage is modeled here as an isotropic, rate independent process, caused by the growth of microvoids only. These factors were carefully chosen as the most important ones from the point of view of the structural analysis. The evolution of porosity is introduced into the yield function leading to the strong coupling between plasticity and damage. Initial porosity is evolving due to the presence of the inelastic strains, which means that elastic damage is disregarded.

The reliability of the presented concept was evaluated through some example problems. In the example presented in Section 6.4.2 the plot of porosity versus load was given based on which the fracture criterion could be formulated. The example given in Section 6.4.2 also showed that not considering damage in the analysis of plates and shells could lead to overprediction of the ultimate load carried by the structure. Unfortunately, there is very limited amount of data about evolution of damage in plates and shells which could be used as references. Moreover, it is unrealistic to verify the damage formulation based on comparisons to other results obtained by approximate methods. As was already pointed out in the previous section, the functioning of this algorithm should be tested against experimental results, particularly in the case of damage description. The references providing the information about the damage in structures based on experiments are even more difficult to find than numerical estimates. Nevertheless, based on the limited references as well and the fact that the results presented in Chapter 6 showed the expected pattern of the reduction of stiffness caused by the evolution of
damage, it may be concluded that the current formulation provides valuable information about damage in plates and shells.

Similarly to the kinematic hardening rule, presented in Chapter 5, there are material constants $n, k_3$ defining the constitutive relations. Determination of these constants is critical to the accuracy of the model. The author found that the model is highly sensitive to changes of the values of $n, k_3$ as well as the parameters defining the kinematic hardening rule. In fact, the results of the analysis obtained using damage formulation could be closely approximated by the manipulation of the material parameters in the kinematic hardening rule and damage effects not considered. Although this is an interesting observation, such an approach cannot be regarded as constitutive modeling of the damage phenomenon.

The biggest advantage of the presented method is its simplicity and efficiency. By means of the quasi-conforming technique the elastic stiffness matrix is calculated explicitly. A non-layered approach allows for the elasto-plastic calculation without the discretization of the shell through the thickness. Thus, the tangent stiffness matrix computed here is also given explicitly and numerical integration is not employed. This makes this procedure extremely efficient computationally. The isotropic damage variable inserted into the yield criterion expressed in terms of the stress resultant and stress couples provides an additional tool for simulating the behaviour of plates and shells, without complicating the analysis substantially. The porosity parameter representing damage due to void growth only is used here. This can be regarded as a limitation of the current model, since the influence of nucleation due to microcracks is very important for certain applications. However, the current model is based on the evolution of porosity defined by Duszek-Perzyna and Perzyna (1994), who reported excellent results in modeling ductile metals. According to this reference the influence of microcracks is very important when analyzing metal matrix composites because of cracking of the reinforcing fibers. In the case of homogenous and isotropic shells the void growth is decisive and thus, is the only damage-causing phenomenon described in this work.

Only two additional material parameters need to be determined to account for damage, as opposed to higher order approximations advocated by many authors, where sometimes tremendous experimental data is necessary to calibrate all the required material constants. This would be the case if for example a second or fourth order damage tensor were used. Moreover, while a more advanced procedure would be needed to model the elasto-plastic and damage behaviour of materials, the accuracy of the current analysis form the point of view of structural analysis is satisfactory. Many variables simulating the material behaviour will lose their importance when the structure made of this material is investigated. This is confirmed by the sensitivity of the model to material parameters defining the kinematic hardening rule, as discussed in the previous paragraph. In view of the above arguments, as well as the accuracy of the numerical results presented in Chapter 6, the representation of damage in plates and shell given in this work is valuable.
8.7 Summary and Conclusions

A computational, stress-resultant based model for the elasto-plastic and damage analysis of beams, plates, and shells, with the influence of large rotations is presented. The algorithm is devised in several different stages, unified into a comprehensive shell model. The following, are the overall conclusions drawn.

- The shell constitutive equations derived in Chapter 2 are reliable. This is very important for reliability of the whole model. Developing a theory of thick shells, from the elasticity closed-form solutions for thick shells is arguably the most consistent. Such a process may sometimes become complicated but is also very accurate.

- A quasi-conforming technique employed to develop a finite element based on the refined shell theory may be a powerful tool, especially if used with carefully designed interpolation formulas for strains and displacements. The method is sensitive to any errors, but once these are eliminated, it may be very precise and efficient.

- Elastic analysis of shells by means of three-dimensional ‘brick’ elements is sometimes prohibitive due to complexity of the problem. Shell elements are ‘degenerated’, hence they require a vastly reduced number of operations executed by the computer. This allows predictions of the internal forces of complicated structures, which would have been difficult or impossible otherwise. Yet in the elasto-plastic considerations, most shell elements follow the layered approach, which is conceptually very close to three dimensional ‘brick’ elements. This is because the loading surfaces featuring dependable isotropic and kinematic hardening rules and accounting for the damage effects are expressed in terms of the stresses. While in the case of composite laminates, multi-layered shells give more accurate description of interlaminar effects, they lose their advantages in the analysis of isotropic homogenous shells. The non-layered method seems to be a natural consequence of the shell elements development, as the system of non-linear equations is expressed in terms forces and bending moments, and solved without discretization of the shell through the thickness. It allows taking the full advantage of shell elements in the investigation of elasto-plastic and damage behavior of plates and shells.

- The non-layered yield surface, accounting for the progressive development of the plastic deformation and transverse shear forces, with a new kinematic hardening rule capable of simulating the load-displacement response including the Bauschinger effect, significantly simplifies and speeds up the analysis, without a substantial loss of precision.

- The description of damage through the evolution of porosity as a function of the growth of microvoids is limited, but from the point of view of structural analysis, sufficient. It delivers valuable information about damage in isotropic plates and shells, without increasing the complexity. It is conveniently incorporated into the
stress-resultant based loading surface, resulting in a strong coupling between plasticity and damage.

- Analysis of plates and shells without considering damage effects could lead to overprediction of the ultimate load carried by the structure.

- Analytical calculation of the integrals and the explicit tangent elasto-plastic damage stiffness make the current formulation very effective. Numerical integration is not employed at any stage of the computations.

- The computer program devised in this work serves a mere purpose of validating the constitutive equations derived, along with the assumptions made. In its current form, it should not be regarded as a universal software for any commercial applications. Very extensive testing and further study would be imperative if the program WOELKE-SHELLS was to be adopted as a part of a commercial package intended for use in the industry. Such testing would most likely reveal imperfections of both the theoretical formulation and numerical procedure developed here.

- There is no ‘best’ comprehensive shell model. Any given formulation will always perform better than others, only for certain applications. The current formulation is no exception. It has its limitations and is likely to yield more accurate results for certain problems, and slightly less accurate for others. However, unlike in the layered approach, the model presented here realizes a concept of shell elements built on reliable constitutive equations, in the elasto-plastic and damage modeling of plates and shells. It offers flexibility, accuracy and tremendous efficiency. The validity of all the assumptions was verified and confirmed by a series of challenging numerical examples that proved the reliability of the model. Thus, the shell formulation developed in this dissertation is a very significant advancement.

- The nature of the equations derived here allows for further future enhancement of this algorithm, leading to the capabilities of solving viscoplastic and dynamic problems.

### 8.8 Future Work

The elasto-plastic and damage constitutive modeling of shells is a vivid research area. Dynamic and rate dependent problems seem to be of high interest to the engineering and scientific community. As already pointed out, the current model is very well suited for further enhancements leading to increasing its capabilities. Future work should therefore be directed to non-linear dynamic analysis and viscoplastic investigations.

Furthermore, the phenomena of ratcheting and buckling as an eigenvalue problem should be considered. The influence of microcracks could also be included in the description of damage, with a use of a tensorial damage variable. The determination of material parameters in kinematic hardening rule as well as damage formulation should be given further consideration.
Certain deficiencies of the current formulation should be eliminated before it is further enhanced. Considerations should be given to the representation of membrane displacements and strains, as discussed in Section 8.3. Although, the errors are not very large, they are automatically transferred to the next stage of the formulation, thus the improvements on the level of the elastic analysis are very important.

The numerical procedures and the computer program WOELKE-SHELLS, require extensive testing, if it were to be regarded as a universal software applicable to a variety of problems.
BIBLIOGRAPHY PART I: BY THE AUTHOR


BIBLIOGRAPHY PART II: BY OTHER AUTHORS


APPENDIX

INTERPOLATION FORMULAS FOR DISPLACEMENT FIELD

The two-dimensional interpolation formulas for the displacement field, discussed in Section 3.2.3 are given below:

\[
\phi_x(\xi, \eta) = \frac{3}{4a} \lambda_x \left[ (1-\xi^2)(1-\eta) \right] w_1 + \frac{3}{4a} \lambda_x \left[ (1-\xi^2)(1+\eta) \right] w_2 - \frac{3}{4a} \lambda_x \left[ (1-\xi^2)(1+\eta) \right] w_3 + \frac{3}{4a} \lambda_x \left[ (1-\xi^2)(1-\eta) \right] w_4 + \frac{1}{8} \left[ 2 + 2\xi - 3\lambda_x (1-\xi^2) \right] (1-\eta) \phi_{x1} + \frac{1}{8} \left[ 2 + 2\xi - 3\lambda_x (1-\xi^2) \right] (1+\eta) \phi_{x2} + \frac{1}{8} \left[ 2 - 2\xi - 3\lambda_x (1-\xi^2) \right] (1+\eta) \phi_{x3} + \frac{1}{8} \left[ 2 - 2\xi - 3\lambda_x (1-\xi^2) \right] (1-\eta) \phi_{x4}
\]

(A.1)

\[
\phi_y(\xi, \eta) = -\frac{3}{4b} \lambda_y \left[ (1-\eta^2)(1+\xi) \right] w_1 + \frac{3}{4b} \lambda_y \left[ (1-\eta^2)(1+\eta) \right] w_2 - \frac{3}{4b} \lambda_y \left[ (1-\eta^2)(1-\eta) \right] w_3 + \frac{3}{4b} \lambda_y \left[ (1-\eta^2)(1+\xi) \right] w_4 + \frac{1}{8} \left[ 2 + 2\eta - 3\lambda_y (1-\eta^2) \right] (1-\eta) \phi_{y1} + \frac{1}{8} \left[ 2 + 2\eta - 3\lambda_y (1-\eta^2) \right] (1+\xi) \phi_{y2} + \frac{1}{8} \left[ 2 + 2\eta - 3\lambda_y (1-\eta^2) \right] (1+\eta) \phi_{y3} + \frac{1}{8} \left[ 2 - 2\eta - 3\lambda_y (1-\eta^2) \right] (1-\xi) \phi_{y4}
\]

(A.2)

\[
u(\xi, \eta) = \frac{1}{8} [1+\xi][1-\eta] v_1 + \frac{1}{8} [1+\xi][1+\eta] v_2 + \frac{1}{8} [1-\xi][1-\eta] v_3 + \frac{1}{8} [1-\xi][1+\eta] v_4
\]

(A.3)

\[
u(\xi, \eta) = \frac{1}{8} [1+\xi][1-\eta] v_1 + \frac{1}{8} [1+\xi][1+\eta] v_2 + \frac{1}{8} [1-\xi][1-\eta] v_3 + \frac{1}{8} [1-\xi][1+\eta] v_4
\]

(A.4)
\[ w(\xi, \eta) = \frac{1}{4} \left[ 1 + \xi - \frac{\lambda_2}{2} (\xi^3 - \xi) \right] \left[ 1 - \eta + \frac{\lambda_3}{2} (\eta^3 - \eta) \right] w_1 + \]
\[ + \frac{1}{4} \left[ 1 + \eta - \frac{\lambda_2}{2} (\eta^3 - \eta) \right] \left[ 1 - \xi + \frac{\lambda_3}{2} (\xi^3 - \xi) \right] w_2 + \]
\[ + \frac{1}{4} \left[ 1 - \xi + \frac{\lambda_2}{2} (\xi^3 - \xi) \right] \left[ 1 + \eta - \frac{\lambda_3}{2} (\eta^3 - \eta) \right] w_3 + \]
\[ + \frac{1}{4} \left[ 1 - \xi - \frac{\lambda_2}{2} (\xi^3 - \xi) \right] \left[ 1 + \eta + \frac{\lambda_3}{2} (\eta^3 - \eta) \right] w_4 + \]
\[ + \frac{1}{8} \left[ -1 + \xi^2 + \lambda_2 (\xi^3 - \xi) \right] (1 - \eta) \frac{a}{2} \phi_{\xi 1} + \]
\[ + \frac{1}{8} \left[ -1 + \xi^2 + \lambda_2 (\xi^3 - \xi) \right] (1 + \eta) \frac{a}{2} \phi_{\eta 1} + \]
\[ + \frac{1}{8} \left[ 1 - \xi^2 + \lambda_2 (\xi^3 - \xi) \right] (1 + \eta) \frac{a}{2} \phi_{\xi 3} + \]
\[ + \frac{1}{8} \left[ 1 - \xi^2 + \lambda_2 (\xi^3 - \xi) \right] (1 - \eta) \frac{a}{2} \phi_{\eta 3} + \]
\[ + \frac{1}{8} \left[ 1 - \eta^2 + \lambda_3 (\eta^3 - \eta) \right] (1 + \xi) \frac{b}{2} \phi_{\xi 1} + \]
\[ + \frac{1}{8} \left[ 1 + \eta^2 + \lambda_3 (\eta^3 - \eta) \right] (1 + \xi) \frac{b}{2} \phi_{\eta 1} + \]
\[ + \frac{1}{8} \left[ 1 - \eta^2 + \lambda_3 (\eta^3 - \eta) \right] (1 - \xi) \frac{b}{2} \phi_{\xi 2} + \]
\[ + \frac{1}{8} \left[ 1 + \eta^2 + \lambda_3 (\eta^3 - \eta) \right] (1 - \xi) \frac{b}{2} \phi_{\eta 2} + \]

(A.5)

where:

\[ \xi = \frac{2x}{a}, \quad \eta = \frac{2y}{b} \quad -1 \leq \xi, \eta \leq 1 \]  

(A.6)
VITA

Pawel Woelke was born on November 15, 1976, in Poznan, Poland. After completing high school in May 1995, he was admitted to Civil and Environmental Engineering Department at Poznan University of Technology. During his fourth year of study, he was awarded a Socrates-Erasmus Scholarship to study for one semester at Polytechnic of Milan in Italy, under joint supervision of Professor Andrzej Gawecki and Professor Giulio Maier. He returned to Poznan to work towards his master of science degree under the advisors: Doctor Jacek Tasarek, Doctor Adam Glema, and his mentor, Professor Tomasz Lodygowski. After graduating from Poznan University of Technology on June 21, 2000, he was awarded a British Chevening Scholarship and enrolled in a master of science course in structural steel design at Imperial College in London, England. Having completed the course in June 2001, he returned to Poland and started working for a structural engineering company Jakon, in Tarnowo Podgorne near Poznan. On January 7, 2002, he arrived in Baton Rouge, Louisiana, and began his doctoral studies in the Civil and Environmental Engineering Department at Louisiana State University under supervision of Professor George Z. Voyiadjis. In August 2005, he graduated from Louisiana State University obtaining a doctor of philosophy degree.

Pawel Woelke has been an active judo player since the age of eleven. He participated in numerous national and international competitions in Europe and in the United States. He is a former Polish and British University Judo Champion, former member of a team of Champion of Belgium. After moving to the United States of America, he won three consecutive Louisiana State Championships.