Multiscale Analysis of Contact in Smooth and Rough Surfaces: Contact Characteristics and Tribo-Damage

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MULTISCALE ANALYSIS OF CONTACT IN SMOOTH AND ROUGH SURFACES: CONTACT CHARACTERISTICS AND TRIBO-DAMAGE

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mechanical and Industrial Engineering

by

Ali Beheshti
B.S., Isfahan University of Technology, 2004
M.S., Isfahan University of Technology, 2007
December 2013
To my beloved mother and father,

    Shahla and Ahmad

and my supportive brothers

    Behrooz and Shahram.
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Abstract

This dissertation is comprised of two major interrelated foci. The first focus is to investigate the effect of surface roughness on the behavior of dry contacting bodies through both deterministic and statistical approaches. In the current research, different statistical micro-contact models are employed together with the bulk deformation of the bounding solids to predict the characteristics of the dry rough line-contact and elliptical point-contact including the apparent pressure profile, contact dimensions and real area of contact. Further, based on the results of numerical simulations, useful relationships are provided for the contact characteristics. In addition, a robust approach for the deterministic prediction of pressure and tangential traction distributions in dry rough contact configuration subjected to stick-slip condition is presented with provision for elastic-fully plastic asperity effects.

The second focus of this research involves the assessment of three of the most common types of degradation processes that are observed in contact mechanics. The first contact failure mechanism studied is the rolling/sliding contact fatigue wear. In this research, the principles of continuum damage mechanics (CDM) are applied to predict the rolling/sliding contact fatigue crack initiation, and the effect of variable loading on the fatigue behavior of rolling contact with provision for non-linear damage evolution is investigated. The estimated numbers of cycles to crack initiation are compared to the available experimental results revealing good agreement. The second contact degradation phenomenon involves the study of the adhesive wear for unlubricated and lubricated contacts. A method is presented that applies the principles of CDM to predict the Archard adhesive wear coefficient for unlubricated contacts. By carrying out pin-on-disk experiments, wear coefficients for a specific material are obtained and compared with the predicted values showing good agreement. Further, the load sharing concept is applied to develop an engineering model for lubricated wear with the consideration of the thermal effects. The third type of degradation studied is the so-called fretting fatigue which is a failure phenomenon observed in contacting bodies subjected to very small amplitude oscillatory motion. Using the deterministic model developed for stick-slip contact condition, the effect of surface roughness on the crack initiation risk in a fretting contact is investigated and compared with experimental observations. In order to investigate the last two degradation phenomena, the results obtained from the first objective are directly utilized.
Chapter 1: Overview

1.1. Introduction

Almost all mechanical and bio-mechanical systems involve contacts between their components. In general, tribology—the science of lubrication, friction and wear—encompasses the study of contact interfaces under static loading or relative motion. Tribology attempts to reduce energy consumption, provide easier motion and prolong interacting components life in order to conserve the energy and materials and reduce maintenance cost. In particular, understanding the surface contact characteristics as well as the contact failure phenomena in tribological components, ranging from bearings and gears to hard disk drives and biomedical transplants, has long been of keen research interest in engineering tribology.

Real engineering surfaces are rough at microscopic level and their interactions involve the contact of surface peaks at discrete micro protrusions called asperity tips. Surface roughness affects the contact behavior such as the pressure distribution, sub-surface stress distribution, contact width, real area of contact and contact resistance. These factors directly affect the load-carrying capacity, friction (traction) force, electro-thermo-mechanical and the contact damage (e.g., wear and fatigue) behavior of tribo-components. Broadly, the surface roughness modeling approaches fall in two categories: statistical or deterministic. Depending on the application and the desired results, each of these two approaches has its own advantages and disadvantages. The statistical approach is typically more convenient to formulate since it only requires the specification of a few surface parameters. In particular, the statistical approach lends itself to generalization of predictions for different geometries, surface properties and loading conditions. Deterministic simulation of the surfaces, on the other hand, provides a more detailed description of the pressure, deformation and the sub-surface stress distribution. Deterministic approach is required in problems involving micro-scale phenomena with surface damage—e.g., micro-pitting or surface crack nucleation in rough surface fretting or any near surface failure—in order to determine the sub-surface stress distribution for the contact region with real surface profile. However, the deterministic approach requires direct measurement of the surface profile and extensive numerical calculations. The first major task in the current research includes the assessment of contact characteristics in rough surface contact of curved bodies using both statistical and deterministic approaches. Indeed, the well-known Hertzian formulas are only valid for the macro-level contact of two ideally smooth curved surfaces while for real rough surfaces, the pertinent contact parameters such as the pressure distribution and the contact width deviate from those predicted by the Hertzian approach.
Nearly all tribological components, especially those in relative motion are prone to contact deterioration. In fact, excessive contact failure is still one of the most pervasive failure mechanisms in man-made machine components. The second task in the current research involves the investigation of three of the most important contact degradation processes; adhesive wear, rolling/sliding contact fatigue and fretting fatigue. In tribological studies, wear is often classified into different categories such as adhesive, abrasive, corrosion, and fretting. Among them adhesive wear is believed to be the most common form among all types of wear and because of different parameters involved is hard to model. Contact fatigue wear is the prevailing failure mode in a properly lubricated rolling/sliding element, which is a type of marital degradation commonly experienced in bearings, gears, cams, railways tracks and the like. Material degradation occurs as a result of the accumulation of damage in the material microstructure due to the repeated rolling and sliding. Fretting fatigue is a pervasive type of damage in contacting bodies, subjected to oscillatory motion or vibration. Numerous instances of fretting-induced failure in practical applications ranging from dovetail joints in turbine engines to bolted or riveted structures to orthopedic implants and wire ropes have been reported in the literature.

1.2. Dissertation Outline

This dissertation deals with six sub-topics:

1. Prediction of unlubricated adhesive wear;
2. modeling of rolling/sliding contact fatigue;
3. application of statistical micro-asperity contact models to line contact problem;
4. application of statistical micro-asperity contact models to elliptical point contact problem;
5. deterministic modeling of rough line contact configuration in stick-slip condition; and
6. prediction of mixed-lubricated wear.

The second, third and seventh chapters focus on the contact degradation phenomena while the fourth, fifth and sixth chapters are on the topic of rough surfaces contact. The chapters are written in the form of a journal paper.

Chapter 2 presents a method that applies the principles of continuum damage mechanics (CDM) to predict the appropriate adhesive wear coefficient. Using the CDM approach, we predict the number of cycles before the crack nucleation sets in, evaluate the probability that an asperity forms a wear particle, and use this information to derive an expression for the Archard wear coefficient. This formulation eliminates the empirical nature of wear coefficient, for the approach makes it possible to calculate it using the bulk material properties and surface conditions. In addition, the results of simulation are validated by using a pin-on-disk experimental test results and also by comparing the computed wear coefficient against the available published values.
In chapter 3, the principles of CDM are applied to predict the rolling/sliding contact fatigue crack initiation. The approach involves evaluating the subsurface stresses as well as the state of damage within the contact region. It is shown that the fatigue crack initiation life can be related to the scalar damage parameter $D$. The effect of variable loading on the fatigue behavior of rolling contact with provision for non-linear damage evolution is also investigated.

In chapter 4, different statistical micro-contact models are employed together with the bulk deformation of the bounding solids to predict dry, rough line-contact characteristics such as the apparent pressure profile, contact width and real area of contact. The approach involves solving the micro-contact models and separation formulas simultaneously. Further, based on the results of numerical simulations, useful relationships are provided for the prediction of the maximum contact pressure, contact width, real area of contact and pressure distribution.

In chapter 5, statistical micro-contact models of Greenwood-Williamson (GW) and Kogut-Etsion (KE) are employed along with the bulk deformation of the contacting solids to predict dry rough elliptical point-contact characteristics such as the apparent pressure profile and contact dimensions. The approach involves solving the micro-contact and separation formulas simultaneously. Also presented are convenient formulas that can be readily used for the prediction of the maximum contact pressure, contact dimensions, contact compliance, real area of contact and pressure distribution.

Chapter 6 represents a robust approach for the deterministic prediction of pressure distribution in dry, rough-line contact configuration considering the elastic-fully plastic asperity effects. Adopting the Civarella-Jager approach, a procedure for calculation of the tangential traction distribution for cyclic loading condition in stick-slip regime is also described in detail. The effect of surface roughness on the pressure and tangential traction distribution and sub-surface stress field is evaluated. To illustrate the utility of the approach, the crack initiation risk in a stick-slip (fretting) contact is investigated for different surface roughness values. The methodology is conceptually simple and can be easily implemented in a computer code.

In chapter 7, the CDM approach (developed in chapter 2) in conjunction with the load sharing concept is applied to predict steady state lubricated line-contact wear. Empirical formula for the maximum contact pressure in dry, rough contact (obtained in chapter 4) is employed together with the lubricant film thickness equation to estimate the portions of the load carried by the asperities and the lubricant. The effect of contact temperature on wear is also included using a simplified thermo-elastohydrodynamic analysis in conjunction with fractional film defect concept. In contrast to most wear studies, in this approach the wear coefficient for lubricated contact is obtained based on a purely predictive methodology rather than by experimental measurement. The proposed methodology is quite fast and easy to implement in practical
engineering applications. To show the utility of the approach in real engineering problems, the contact of two spur gear teeth is simulated and their contact and wear performance are evaluated.

Chapter 8 summarizes the main results presented in this dissertation and gives recommendation for future works.
Chapter 2: A Thermodynamic Approach for Prediction of Wear Coefficient Under Unlubricated Sliding Condition*

2.1. Introduction

Wear is a manifestation of degradation between rubbing surfaces in all man-made machine components like hard disk drive, gears and cams, bearing, seals, as well as biomedical transplants. In tribological studies, wear is often classified into different categories such as adhesive, abrasive, fatigue, corrosion, and fretting. Among them adhesive wear is thought to be the most common form [1]. Based on the existing adhesive wear theories (e.g., Asperity Fatigue Theory [2, 3] and Delamination Theory of Wear [4]), adhesive wear involves crack initiation and propagation. In other word, at the final stage of surface degradation, cracks form, propagate and finally wear particles are detached from the surfaces.

A rich volume of existing publications attests to the fact that sliding wear and fatigue are related [5-16]. Noteworthy among these publications are the works of Kragelskii [5] and Rozeanu [6] who pioneered the idea of treating adhesive wear as a fatigue phenomenon. Referring to Figure 2.1, Rozeanu [6] suggested that the wear fragment is produced after “successive steps of the fatigue.” Kragelskii [5] proposed that a loose wear particle is produced by a fatigue process and interpreted the wear coefficient as the inverse of the number of cycles that an asperity submits to stress before it breaks. Additional evidence that corroborate the existence of a relationship between fatigue and sliding wear is presented by Kimura [7]. He stated that having considered “repetitive nature of practical sliding system,” it is a natural consequence that fatigue mechanism is operating. More recently, Arnell et al. [8] provided a plausible explanation on the mechanisms through which asperity fatigue contributes to adhesive wear. According to their work, consideration of wear by fatigue mechanism can describe the observation that a harder material can be worn by a softer one, a phenomenon which cannot be explained by adhesion theory alone. Therefore, in a process involving adhesive wear, the combination of adhesion and fatigue mechanisms are involved. The interested reader is referred to additional body of work by Suh et al. [9], Soda et al. [10] Challen et al. [11], Halling [12], Yamada et al. [13], Finkin [14], Jain and Bahadur [15], Omar and Atkins [16] who describe the relationship between fatigue and adhesive wear.

* Reprinted by permission of Tribology Letters (See Appendix C)
Figure 2.1. Successive steps of the fatigue wear mechanism [6]

The first quantitative law for adhesive wear is attributed to the work of Archard [17] who developed a model based on the contact of asperities. According to Archard [17], the total volume of wear, $V$, when two bodies slide against each other over a total distance $S$ while subject to an applied load $L$ is given by:

$$V = k \frac{L}{p} S$$  \hspace{1cm} (1)

where $p$ is the softer material flow pressure and $k$ is a constant, generally referred to as the wear coefficient. It represents $(1/3)$ of the probability of formation of a wear fragment each time two asperities come into contact with each other. Although the formula is very simple, its validity has been experimentally verified [18]. It is important to note that the constant $k$ in equation (1) is dependent upon the specific pair of materials and can vary widely —over an order of magnitude— depending on the contact conditions and is often evaluated experimentally. The interested reader is referred to Rabinowicz [1] for a detailed discussion.

According to Kragelskii [19] the wear coefficient can be interpreted as the inverse of the number of events $N$ required for formation of wear particle. In other word, $N$ can be interpreted as the number of loading cycles needed for an asperity to produce a wear fragment. The wear coefficient can be considered as the $1/3$ of probability $(P)$ that an asperity forms wear particle from softer surface [20]:

$$k = \frac{P}{3} = \frac{1}{3N}$$  \hspace{1cm} (2)

Based on this relationship, an adhesive wear can be interpreted as the detachment of wear particles from a surface as a result of fatigue. This concept allows one to relate wear to the fatigue properties: for a specific material and the loading condition, given the number of cycles to fatigue failure $N$, one can estimate the wear coefficient.
Efforts made for predicting the number of cycles to failure associated with fatigue are quite voluminous. See an extensive review provided in [21]. Of interest in the present paper is the notion that, in general, material degradation and the associated accumulation of damage can be treated based on the principles of thermodynamics. The validity of this approach to reliability problems has been demonstrated by Feinberg [22] and applications to processes involving surface degradation and wear are recently reported by Bryant et al. [23].

Of particular interest in the present paper is the treatment of damage using the Continuum Damage Mechanics (CDM). Lamaitre [24] and more recently Bhattacharya and Ellingwood [25] have significantly contributed to development of the CDM by treating the “growth of damage” as an irreversible process that obeys the laws of thermodynamics [26]. While application of CDM to fretting wear has already been demonstrated [27], the full potential of the theory to tribology applications remains largely unexplored.

A brief review of CDM is given in section 2.

### 2.2. CDM Theory

The theory of CDM was first proposed by Kachanov as the kinetic equation of damage [28]. Later, Lemaitre [24] proposed a theory based on the thermodynamics potential functions. Recently, Bhattacharya et al. [25, 29, 30] used a thermodynamic framework for damage growth. This approach is attractive because it eliminates the dependency on critical crack sizes and empirical growth parameter, used in previous approaches. Instead, the state of damage in the material is estimated based on “macroscopically obtained material parameter.” In other words, the material degradation is presented as the macroscopic state variable [31-33]. This damage variable, symbolized as ‘$D$’, is assumed to be a continuous variable, represent by a tensor for anisotropic materials [34].

Figure 2.2. Damage at any point in the material
As described by Bhattacharya, different definitions of damage parameter have been proposed [25]. In the present study the definition of damage proposed by Lemaitre [35] has been applied. The damage variable, \( D(\eta) \), is quantified by the surface density of cracks, voids, and cavities (\( \sum dA_D \)), lying on an elemental cross sectional plane (with the area of \( dA \), and normal vector \( \eta \)), which is weighted (\( k_w \)) by the average stress-raising effects of the voids (Figure 2.2). Hence, it can be shown as follows:

\[
D(\eta) = k_w \cdot \frac{\sum dA_D}{dA}
\]

(3)

Damage is said to be isotropic, if it does not depend on the orientation of the normal vector \( \eta \) on this section. In the present analysis we assume that damage is isotropic, so that \( D \) can be represented by a scalar quantity. According to Lamaitre [35], in the case of uni-axial isotropic damage the effective stress \( \sigma' \) acting on the resisting area (\( dA - \sum dA_D \)) of the damaged surface can be related to the calculated normal stress \( \sigma \) neglecting the damage process by the equilibrium of forces:

\[
\sigma' = \frac{\sigma}{1 - D}
\]

(4)

The principle of strain equivalence [36] is the basic hypothesis of CDM. It postulates that a damaged volume of material under the nominal stress \( \sigma \), exhibits the same strain response as a comparable undamaged volume under effective stress \( \sigma' \) with the corresponding modulus of elasticity \( E' \)

\[
E' = E(1 - D)
\]

(5)

Equation (5) describes the relationship between the modulus of elasticity of original undamaged material and damaged material. The elasticity modulus decreases due to the damage. Thus, by measuring \( E' \) at any point, the state of damage at any instant, \( D \), can be computed [36]; the Poisson’s ratio is assumed to be unaffected after damage process [37].

In CDM theory, failure occurs when the damage variable \( D \) reaches it critical value, \( D_c \) (\( D_c \leq 1 \)). CDM hypothesizes that \( D_c \) is an intrinsic material property [38] and can be obtained by performing a simple static tension test. It is worthwhile to note that in CDM theory, failure does not mean fracture. Rather, it indicates a point at which the material has adequately degraded such that “continuity” is lost as a result of micro-cracks and micro-voids formation. A macro-crack can easily develop in a damaged specimen. A limitation of CDM is that it predicts volume wide degradation, implying that the surface degradation phenomena like wear and fretting could not be modeled by CDM. This drawback is overcome by considering that the crack initiates within the asperity, hence degradation occurs just inside the surface. Therefore, wear is treated as the volume wide degradation of the surface layer. The representative volume element (RVE) could be defined in this sense [27].
The CDM model proposed by Bhattacharya and Ellingwood, [25], which was verified experimentally [29], is capable of estimating the number of cycle ($N$) to failure (crack initiation) in terms of only macroscopic material parameter which can be obtained experimentally. Assuming that the system evolves through a set of equilibrium states prior to localization of damage and applying the first and second law of thermodynamics, Bhattacharya and Ellingwood [25, 29, 30] developed the following equation of Isotropic damage growth for a deformable body:

$$T_i + \psi_D(\partial D/\partial \varepsilon_{ij})n_j = 0; \text{ on } \partial R_1$$

In this equation the unknown is $D$ (The damage parameter); $T_i$ is the traction on the boundary $\partial R_1$ of the body ($\partial R_1$ is a part of entire boundary on which the displacement is non zero and hence the work is done on the system by traction forces), $\psi_D$ is the derivative of Helmholtz free energy function with respect to $D$; $\varepsilon_{ij}$ represents the strain and $n_j$ is the normal in $j$ direction.

Applying the condition that in uniaxial loading the far-field stress, $\sigma_\infty$, acts normal to the surface, the above equation yields [29]:

$$\frac{dD}{d\varepsilon} = -\frac{\sigma_\infty}{\psi_D}$$

The solution of this equation by using Ramberg-Osgood type equation for the hysteresis loop gives $D_i$, the damage in $i^{th}$ cycle [29]:

$$D_i = 1 - \left(1 - D_{i-1}\right)\frac{(1 + 1/M)\Delta \varepsilon_{oil} + 1/M \Delta \varepsilon_{oil} + C_i}{(1 + 1/M)\Delta \varepsilon_{pml} + 1/M \Delta \varepsilon_{pml} + C_i} \left\{\begin{array}{ll}
\sigma_{\text{max}} \geq S_e \\
D_i = D_{i-1}
\end{array}\right.$$  

otherwise

$$C_i = \frac{3\sigma_f}{4K} \frac{\Delta \varepsilon_{pml} + 1/M}{1 + 1/M} \Delta \varepsilon_{oil} + \frac{1/M}{1 + 1/M} \Delta \varepsilon_{pml}$$

$S_e$ is the endurance limit, $\sigma_f$ denotes the true failure stress, $M$ represents the cyclic hardening exponent, $D_{i-1}$ is the damage in previous cycle, $K = 2^{1/M} H$, where $H$ is cyclic hardening modulus, and the various $\Delta \varepsilon$ are the strain values from stress-strain curve of loading as shown in Figure 2.3 associated by the related stresses.


The above equation can also be written as:

\[
D_N = 1 - (1 - D_0) \prod_{i=1}^{N} F
\]

where

\[
F = \frac{(1 + 1/M)\Delta\varepsilon_{oi}^{1+1/M} - \Delta\varepsilon_{pli}^{1/M} \Delta\varepsilon_{oi} + C_i}{(1 + 1/M)\Delta\varepsilon_{pmi}^{1+1/M} - \Delta\varepsilon_{pli}^{1/M} \Delta\varepsilon_{pmi} + C_i}
\]

\( F \) represents the loading conditions and material properties. For a virgin material \( D_0 = 0 \).

### 2.3. Wear Rate

We seek to determine a relationship between the numbers of cycles \( (N) \) associated with an asperity breakage and the wear coefficient between two dissimilar materials, with the software one experiencing wear. It is also assumed that crack nucleation is the main cause of asperity breakage and that adhesive wear is assumed to be the predominant cause of wear.

Figure 4 shows a single asperity subjected to a normal force \( L/n \), where \( L \) is the normal load and \( n \) is the number of identical asperities involved in real area of contact, \( A_r \), at a given instant. The Coulomb’s law gives the corresponding shear force \( \mu L/n \), where \( \mu \) is the coefficient of friction. The area of each asperity which resists this shear force is

\[
A = \frac{A_r}{n}
\]

According to [1], the real area of contact \( A_r \) is related to the material flow pressure (hardness) by the following relationship.
where $p$ is the material flow pressure. The force is assumed to be in the form of a square-pulse (Archard [17]) of width $d$, the distance for which asperity contact is made. See Figure 2.4.

![Figure 2.4. Loading conditions of an asperity, (a) Normal and shear force (b) Load-displacement functions of an asperity contact](image)

Under compression loading and the forward shear force, crack initiates and grows by shear [39]. It can be argued that since the shear force acts on the small area of cross section of asperity, so it is a cause of crack initiation and propagation and eventually asperity breakage. Like fretting fatigue, these cracks are attributed to the frictional force [27], [40]. The shear stress sustained by an asperity is given by

$$\tau = \frac{\mu L}{nA}$$

(12)

Considering equation (12) and (13), equation (14) can be simply written as

$$\tau = \mu p$$

(13)

Having computed the corresponding principle stress considering the above shear stress, the related strain values can be then calculated [29]. The solution of this simple case of uniaxial loading is available, Eq. (9), based on the work of Bhattacharya and Ellingwood [29].

### 2.4. Numerical Calculation Procedure

The damage variable for uniaxial case is given by recursive equation (8). We assume that the material is initially virgin, that is $D_0 = 0$. The input parameters are the bulk material properties of the softer contacting material ($E$, $H$, $M$, $S_e$, $\sigma_f$), the coefficient of friction ($\mu$), and the penetration hardness ($p$). The mechanical properties of the simulated materials are shown in Table 2.1.

Employing equation (15) as the asperity loading condition and using the Ramberg-Osgood type equation for the hysteresis curve [29], we compute the different strain limits as shown in
Figure 2.3. $D_i$ is then calculated by equation (8). Next $F'$ is calculated using Equation (9) where we evaluate damage per cycle recursively until the damage reaches a critical value, $D_c$ after $N$ cycles. This represents the number of cycles to failure of an asperity. At that point, the computations are completed and the number of cycles undergone until the failure is recorded. The wear coefficient is then obtained using equation (2).

<table>
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<th>Material</th>
<th>$E$ GPa</th>
<th>$H$ MPa</th>
<th>$M$</th>
<th>$\sigma_f$ MPa</th>
<th>$S_c$ MPa</th>
<th>$D_c$ MPa</th>
<th>$\sigma_p$ MPa</th>
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<td>1180</td>
<td>2340</td>
</tr>
<tr>
<td>Aluminum 6061-T6</td>
<td>68.9</td>
<td>426</td>
<td>16.12</td>
<td>535</td>
<td>96.4</td>
<td>0.3</td>
<td>310</td>
<td>1050</td>
</tr>
<tr>
<td>Aluminum 2024-T4</td>
<td>70.4</td>
<td>856</td>
<td>9.1</td>
<td>683</td>
<td>138</td>
<td>0.1,0.32</td>
<td>304</td>
<td>1340</td>
</tr>
<tr>
<td>Titanium 6Al4V</td>
<td>121.5</td>
<td>1288</td>
<td>10.5</td>
<td>738</td>
<td>510</td>
<td>0.1,0.4</td>
<td>880</td>
<td>3420</td>
</tr>
</tbody>
</table>

### 2.5. Experimental Tests

A series of pin on disk tests was performed using a Tribometer manufactured by CETR, Model UMT-2 with computerized data acquisition system. A schematic of the device is shown in Figure 2.5. The pin was attached to a 2D load sensor which measures the normal and tangential force via a suspension system which maintains the normal force constant during the test. The equipment includes a rotational drive which provides the relative sliding motion between pin and disk at constant speed during each test. The coefficient of friction during the test and also the average friction coefficient are automatically recorded. Figure 2.6 shows 3 different values of friction coefficient obtained for 3 different normal loads.

![Figure 2.5. Schematic of CETR-UMT 2 Tribometer](image)

All tests were conducted in air with 50-60% humidity and at room temperature (25-27 °C). The wearing material was a stationary pin and made of AL 6061-T6 in contact with a disk made of SS 304. The diameter of the pin and the disk were 8 mm and 100 mm, respectively. The range
of sliding velocity tested was 30 mm/s to 120 mm/s, which corresponds to 10 to 40 rpm. The normal load ranged from 3 to 30 N. The speed and load were held constant during each test. The duration of tests ranged from 2 to 5 hours which corresponds to a sliding distance of 250-2500 m. The amount of wear and consequently the wear coefficient were obtained by measuring the weight of the pin before and after the test by means of precise digital scale with the accuracy of 0.0001 gr.

Figure 2.6. Friction Coefficient Values by CETR Machine

2.6. Results and Discussion

The values of wear coefficient for four different materials, Aluminum 2024-T4, Aluminum 6061-T6, Stainless steel 4340 and Titanium 6Al4V are obtained by the procedure described above. We also obtained the wear coefficient of Aluminum 6061-T6 for the case of unlubricated condition experimentally. Numerical and experimental results for Aluminum 6061-T6 are plotted in Figure 2.7. As it can be seen there is reasonably good agreement between the experimental and computed results which shows the validity of the simulation.

Figure 2.8 presents the predicted values of wear coefficients versus coefficient of friction for SAE 4340 and Aluminum 2024-T6. Like previous results, wear coefficient increases with friction. Also shown in Figure 8 are experimental results reported by Rabinowicz [1]. Rabinowicz presented the values of wear coefficient versus coefficient of friction for dissimilar contacting conditions of metals as a unique curve for all different material properties. For more exact results, different experimental curves are needed for different specific material pairs. The importance of all these curves is that they enable a designer to estimate wear coefficient easily by using friction coefficient at an interface, which is quite simple to measure.
Figure 2.7. Predicted and experimental results for wear coefficient of Aluminum 6061-T6 as a function of friction coefficient.

Figure 2.8. Predicted wear coefficient of SAE 4340 and Aluminum 2024 ($D_c=0.1$), Rabinowicz’s [1] results pertain to any dissimilar metal pair.

To study the effect of $D_c$, again the wear coefficient of Aluminum 2024-T4 versus coefficient of friction are depicted in Figure 2.9, but here, the results are shown for two different values of $D_c$ which are reported in the literature for Aluminum [29]. We also employed two different $D_c$ for Titanium (0.1 & 0.4), which are considered to be extreme values for critical damage for metals. Figure 2.10 demonstrates the results for Titanium 6Al4V. As it can be seen, there is not a
significant change in the wear coefficient, showing that the CDM-wear model is not appreciably sensitive on the value of critical damage. Generally, the critical damage, $D_c$, is not a standard property and its values are available for some materials. However, they can be obtained by generating fatigue S-N curves for various $D_c$ values and comparing it with the published experimental S-N curve for fatigue [41].

![Friction Coefficient vs Wear Coefficient](image)

Figure 2.9. Predicted wear coefficient of Aluminum 2024-T4, Rabinowicz [1] results pertain to the wear coefficient of any dissimilar metal pair

Experimental results for Titanium have been reported in the literature (Ming et al. [46] and Neibuhr [47]). Comparison of our predictions and the results published by Ming et al. [46] and Neibuhr [47] is presented in Figure 2.10. As it can be seen there is a relatively good agreement between the experimental and computed results, particularly for the experimental results by Neibuhr and those pertaining to friction coefficient values greater than 0.25. It should be noted that the result reported by Ming et al. are for high velocity sliding tests which the temperature ranges are between 600-1000 °C, while the predicted curve is based on the fatigue properties of the Titanium 6AL-4V at the room temperature. Contact temperature can significantly influences the wear behavior of the Titanium 6AL-4V [46]. Therefore, it can be argued that because of high temperature, the experimental values of wear coefficient are larger than predicted ones for this case.
2.7. Conclusions

The modern continuum damage mechanic approach is applied to predict adhesive wear coefficient by the contribution of asperity fatigue based theory. On the basis of a simple asperity contact model, the loading conditions for a single asperity are obtained. These loadings in conjunction with the damage theory give the strength of the asperity. The asperity strength is the quantity of interest to a tribologist as it is a measure of the wear rate. The continuum damage mechanic model is used for the purpose of calculating asperity strength. This formulation eliminates the empirical nature of wear coefficient, since it is now possible to calculate it using the bulk material properties and surface conditions. The numerical simulation has been then carried out based on current analysis to calculate the wear coefficient as a function of friction coefficient for different type of materials. By carrying out pin-on-disk experiments, wear coefficients for specific material are obtained and compared with predicted values based on the CDM theory. Also the simulated curves are compared with available published experimental work showing that the results are matching within an order of magnitude. This paper is the first step in the application of continuum damage mechanics (CDM) in wear problems. For future studies the underlying assumptions should be investigated and efforts should be directed to the development of a more generalized theory of adhesive wear.

Nomenclature

\( A \) area of an asperity involved, m\(^2\)
\( A_r \) real area of contact, m\(^2\)
\( dA \) elementary area, m\(^2\)
\( D \) damage variable
\( D_i \) damage after \( i \) cycle
\( D_c \) critical damage value
$E$ \quad \text{modulus of elasticity of an undamaged material, GPa}

$E'$ \quad \text{effective modulus of elasticity, GPa}

$H$ \quad \text{cyclic hardening modulus, MPa}

$k$ \quad \text{wear coefficient}

$k_w$ \quad \text{stress-raising effects factor}

$L$ \quad \text{normal force, N}

$M$ \quad \text{cyclic hardening exponent}

$n$ \quad \text{number of asperities}

$n_i$ \quad \text{normal to an elemental cross section}

$N$ \quad \text{number of cycles to failure}

$P$ \quad \text{probability of wear particle formation}

$p$ \quad \text{material flow pressure, Mpa}

$S$ \quad \text{total distance of sliding, m}

$S_e$ \quad \text{endurance limit, MPa}

$T_i$ \quad \text{boundary traction, Mpa}

$V$ \quad \text{wear volume, m$^3$}

$\Delta \varepsilon_{si}$ \quad \text{threshold strain of damage increment in cycle $i$}

$\Delta \varepsilon_{pli}$ \quad \text{initial plastic strain in $i$th cycle}

$\Delta \varepsilon_{pmi}$ \quad \text{final plastic strain in $i$th cycle}

$\Delta \varepsilon_{poli}$ \quad \text{threshold plastic strain of damage increment in cycle $i$}

$\partial R_i$ \quad \text{part of system boundary on which traction is applied}

$\varepsilon$ \quad \text{Strain}

$\mu$ \quad \text{friction coefficient}

$\sigma$ \quad \text{stress, MPa}

$\sigma'$ \quad \text{effective stress, MPa}

$\sigma_f$ \quad \text{true failure stress, MPa}

$\sigma_e$ \quad \text{far field stress, MPa}

$\sigma_{max}$ \quad \text{maximum normal stress, MPa}

$\tau$ \quad \text{shear stress, MPa}

$\psi_D$ \quad \text{partial derivative of Helmholtz free energy with respect to $D$, MPa}

### 2.8. References


Chapter 3: On the Prediction of Fatigue Crack Initiation in Cyclic Rolling/Sliding Contacts with Provision for Loading Sequence Effect

3.1. Introduction

Contact fatigue wear is the prevailing failure mode in a properly lubricated rolling/sliding element, which is a type of material degradation commonly experienced in bearings, gears, cams, railways tracks and the like. Material degradation occurs as a result of the accumulation of damage in the material microstructure due to the repeated rolling and sliding.

Published research efforts dealing with the prediction of the number of cycles to failure associated with fatigue damage are quite voluminous; see, for example, an extensive review provided by Fatemi & Yang [1]. The reader interested in modeling schemes for predicting the contact fatigue life of mechanical elements is referred to Tallian [2], Johnson [3], Harris [4], Tyfoor et al. [5], Franklin et al. [6] and Ciavarella and Monno [7]. These references provide detailed discussions of the subject matter and available prediction methodologies. According to Govindarajan & Gnanamoorthy [8], however, because of the large number of variables that affect the contact fatigue process, none of the available theoretical approaches are recognized as complete and accurate. In fact, several fundamental mechanisms in rolling/sliding contact fatigue are not always clear and vary from one material to another [9].

In general, contact fatigue problems can be divided into two broad categories: low friction coefficient and high friction coefficient. The low friction category deals with the surfaces that are either fully or partially separated by a lubricant, typically referred to as elastohydrodynamic or mixed lubrication regimes. This type of contact fatigue can be further divided into two subcategories that involve either pitting or spalling. According to Ding and Gear [10] many researchers have used the terms pitting and spalling indiscriminately, especially when dealing with problems where the friction coefficient is low. Pitting (or micro pitting) manifests itself in the form of a “shallow crater,” mostly on the rough surfaces [10], while spalling is a macro-scale contact fatigue that usually occurs below the surface as a result of macro-cracks formation, coalescence, and propagation—a failure mechanism that is dominant for smooth surfaces (Figure 3.1(a)). The depth of a spall is typically 2-10 times greater than that of a pit [10]. Fatigue in the high friction category mostly occurs in rail wheels and rail tracks. Similar to the low friction category, cracks are initiated either on the surface or below the surface depending on the surface traction (Figure 3.1(b)). As it can be seen, the cracks direction is the same as the direction of the deformed material microstructure caused by high surface traction. This kind of

* Reprinted by permission of Tribology International (See Appendix C)
rolling contact fatigue crack is initiated by accumulated deformation due to combination of high normal pressure and tangential traction. These can be found in the heads (top) of railways rails and are often called head checks [11].

In the present study, we focus our attention on the general case of the Hertzian contact between two bodies involving macro-scale contact fatigue mechanism. Practical examples include spalling in gears, cams and bearings or surface/subsurface macro-scale fatigue in railway rails. The premise of this paper is that, in general, material degradation and the associated accumulation of damage can be treated using the principles of thermodynamics. The validity of this approach and its application to catastrophic reliability analysis has been documented by Feinberg and Widom [12]. Extension to processes involving degradation and wear (surface degradation) are recently reported in Refs. [13-17]. To tackle problem involving damage growth,
Lamaitre [18] and more recently Bhattacharya and Ellingwood [19] have applied thermodynamically-based continuum damage mechanics (CDM) approach to treat such irreversible processes. While application of CDM to fretting and adhesive wear has already been demonstrated in [14], [20] the full potential of the theory to tribology applications remains largely unexplored. A brief review of CDM is given in section 2; for detail discussion see [14].

In the present study, we compute the stress level change due to the unidirectional travel of Hertzian contact at the edge of a semi-infinite domain and utilize the CDM approach to assess the state of damage. In other words, using the concept of critical damage (to be defined later), we determine the number of cycles to crack initiation and predict the location where the first crack is likely to form. The approach presented utilizes a non-linear damage evolution formula in which the damage status at each cycle depends on the loading condition and damage status of the previous cycle. Thus, the effect of loading sequence is directly taken into account. This is advantageous since available models that use linear damage evolution, such as the Miner’s rule, assume that the number of cycles to crack initiation does not depend on the loading sequence. Hence, they are incapable of accurately considering the effect of variable loading.

3.2. CDM Theory

The theory of CDM was first proposed in 1958 by Kachanov [21]. Later, Lemaitre [18] formulated the theory using the thermodynamics potential functions, and more recently, Bhattacharya and Ellingwood [19],[22] developed a thermodynamic framework for the analysis of the damage growth. This approach is attractive because, unlike other approaches, it eliminates the model dependency on the size of the critical crack and does not require the use of empirical growth parameters. Here, the state of damage in the material is estimated based on the “macroscopically obtained material parameters [22].”

As described by Bhattacharya and Ellingwood [19], different definitions of damage parameter have been proposed. In the present study, we incorporate a damage parameter as defined by Lemaitre [23] which asserts that the damage variable, $D$, is a quantitative measurement of the surface density of micro-cracks and micro-voids, lying on an arbitrary, elemental, cross-sectional plane within the material. We further assume that damage is isotropic and, therefore, independent of the surface orientation. Hence, $D$ can be represented by a scalar quantity. In the case of an isotropic damage, considering equilibrium of forces, the relationship between the effective stress $\sigma'$ acting on the resisting area of the damaged surface and the calculated normal stress $\sigma$ is simply [23]:

$$\sigma' = \frac{\sigma}{1-D}$$

(1)

According to the principle of strain equivalence [24] the damaged volume of the material subjected to the nominal stress, $\sigma$, exhibits the same strain response as a comparable undamaged volume under effective stress, $\sigma'$, with the corresponding modulus of elasticity, $E'$, defined as:
Equation 2 describes the relationship between the modulus of elasticity of the original undamaged material and the damaged material. Thus, by measuring $E'$, the state of damage, $D$, can be predicted [23]. Note that the Poisson’s ratio is assumed to be unaffected by the damage process [25].

In CDM approach, failure occurs when the damage variable, $D$, reaches a critical value, $D_c$ ($D_c \leq 1$). It hypothesizes that $D_c$ is an intrinsic material property [26] and that it can be obtained by performing a simple static tension test. It should also be noted that in CDM theory, failure does not mean fracture. Rather, it indicates that there exists a point at which the material has degraded to the extent that “continuity” is lost due to the formation of a prevailing defect, which, in the context of fatigue process, indicates the initiation of a crack in an initially defect free material.

The CDM model proposed by Bhattacharya and Ellingwood [19], which was verified experimentally [22], is capable of estimating the number of cycle ($N$) to failure (crack initiation) in terms of the macroscopic material parameters that can be obtained experimentally. Assuming that the system evolves through a set of equilibrium states prior to localization of damage and applying the first and the second laws of thermodynamics, Bhattacharya & Ellingwood [19],[22] developed the following equation of isotropic damage growth for a deformable body:

$$0 = \frac{\partial}{\partial t} + \nabla \cdot \mathbf{T} - \mathbf{D}_\psi \nabla D = 0; \text{ on } \partial \mathcal{R}_1$$

In this equation the unknown is $D$ (the damage parameter); $T_i$ is the traction on the boundary $\partial \mathcal{R}_1$ of the body ($\partial \mathcal{R}_1$ is a part of entire boundary on which the displacement is nonzero and hence the work is done on the system by traction forces); $\psi_D$ is the derivative of Helmholtz free energy function with respect to $D$; $\epsilon_{ij}$ represents the strain; and $n_j$ is the normal in the $j$ direction.

Using the Ramberg-Osgood type equation for the hysteresis loop gives $D_i$, the damage in $i^{th}$ cycle. The resulting expression for $D_i$ is [22]:

$$D_i = 1 - (1 - D_{i-1}) F_i \text{ if } \sigma_{\text{max}} \geq S_e$$

where

$$F_i = \frac{(1 + 1/M)^{-1} \Delta \epsilon_{\text{oil}} + 1/M \Delta \epsilon_{\text{ppl}}}{(1 + 1/M)^{-1} \Delta \epsilon_{\text{ppl}} + 1/M \Delta \epsilon_{\text{pml}}}$$

otherwise, if $\sigma_{\text{max}} < S_e$

$$D_i = D_{i-1}$$

where
\[ C_t = \frac{3\sigma_f}{4K} \left( \frac{\Delta\epsilon_{oi}^{1+1/M}}{1+1/M} + \Delta\epsilon_{pli}^{1/M} \right) \Delta\epsilon_{oi} \]

\( S_e \) is the endurance limit, \( \sigma_f \) denotes the true failure stress, \( M \) represents the cyclic hardening exponent, \( K = 2^{1-1/M} H \), where \( H \) is the cyclic hardening modulus, and the various \( \Delta\epsilon \) variable represents the strain values from stress-strain curve of loading associated by the related stresses as described in Ref. [14]. For a virgin material \( D_0 = 0 \).

### 3.3. Rolling-Sliding Contact Stresses

Let us now turn our attention to the stress field generated during a typical rolling-sliding contact. According to the Hertzian contact theory, the pressure distribution in a non-conformal line contact is given by [3]:

\[
p(x) = \frac{2L}{\pi a} \sqrt{1 - \frac{x^2}{a^2}}
\]

where \( a \) is the half-width of Hertzian contact and \( L \) represents the normal load. The frictional force can be simply calculated by the Coulomb friction law:

\[
q(x) = \mu \cdot p(x)
\]

where \( \mu \) is the coefficient of friction between two surfaces.

Referring to Figure 3.2, to simulate repeated rolling/sliding contact, the Hertzian pressure associated with the appropriate tangential traction is moved cyclically in one direction and in a step-wise fashion on a semi-infinite domain. In other words, each cycle consists of a number of static loading steps which are independent of each other, thus neglecting the speed effect. This methodology has been used by other researchers such as Sraml et al. [27], Slack and Sadeghi [28], Taraf et al. [29].

![Figure 3.2. Semi-infinite domain for the Hertzian contact of mechanical elements](image)
For two elastic bodies in line contact, assuming plane strain condition, the different stress components below the contact region are given by the following integrals [3]:

\[
\sigma_x(x,z) = -\frac{2z^2}{\pi} \int_{-a}^{a} \frac{p(s)(x-s)^2}{[(x-s)^2 + z^2]^2} ds - \frac{2}{\pi} \int_{-a}^{a} \frac{q(s)(x-s)^3}{[(x-s)^2 + z^2]^2} ds + \sigma_\theta
\]  \quad (7)

\[
\sigma_z(x,z) = -\frac{2z^3}{\pi} \int_{-a}^{a} \frac{p(s)ds}{[(x-s)^2 + z^2]^2} - \frac{2z^2}{\pi} \int_{-a}^{a} \frac{q(s)(x-s)ds}{[(x-s)^2 + z^2]^2}
\]  \quad (8)

\[
\sigma_y(x,z) = v(\sigma_x(x,z) + \sigma_y(x,z))
\]  \quad (9)

\[
\tau_{xz}(x,z) = -\frac{2z^2}{\pi} \int_{-a}^{a} \frac{p(s)(x-s)ds}{[(x-s)^2 + z^2]^2} - \frac{2z}{\pi} \int_{-a}^{a} \frac{q(s)(x-s)^2 ds}{[(x-s)^2 + z^2]^2}
\]  \quad (10)

where \(x, z\) signify the directions along and normal to the sliding direction, respectively and \(y\) is the normal axis to the plane (Figure 3.2). Note that the last term in equation (7), \(\sigma_\theta\), represents the residual circumferential stress that maybe present due to, for example, tempering and quenching [30]. However, in the present work, it is assumed that there is no residual stress in the material, and therefore, \(\sigma_\theta = 0\). In addition, it should be mentioned that we ignore the effect of material imperfection in the stress calculation. Further, given that typically rolling/sliding contacts results in a high-cycle fatigue, implying that the stresses are below the yield stress, it is reasonable to assume that the material is elastically deformed during the loading cycles [31]. This is particularly relevant to contacts involving low friction coefficient, which is the focus of this paper. This is also in line with the original Lundberg, Palmgren theory [32]. If, on the other hand, one is dealing with an application involving high pressure and high traction, then the effect of plastic deformation would have to be considered [11].

### 3.3.1. Critical Stresses Criterion

A review of the literature reveals that many different stress quantities have been considered as the critical stress for the prediction of a contact fatigue [31], e.g., the maximum orthogonal shear stress, the von Mises equivalent stress, the octahedral shear stress and the maximum shear stress. Experimental evidence reveals that some cracks are first initiated at the depth of the maximum range of orthogonal shear stress, and some are found to be closer to the depth at which maximum shear stress occurs [31]. According to Refs. [33],[34],[35], cracks and especially sub-surface cracks are more frequently found to initiate in the region of maximum shear stress. It has been discussed that since the negative principle stresses in rolling contact fatigue is significantly high (especially when friction coefficient is low), it prevents cracks from opening in the material below contact region where spalls are more likely to form. Thus, it is commonly assumed that the principle normal stresses do not cause damage to the material and hence the maximum shear stress is considered to be the source of crack initiation and propagation ([36],[37],[38],[39]).
Hence, in the present paper, we choose the maximum shear stress as the criterion for initiating fatigue failure. It is defined as:

\[
\tau_{\text{max}} = \max \left\{ \left| \frac{\sigma_1 - \sigma_2}{2} \right|, \left| \frac{\sigma_1 - \sigma_3}{2} \right|, \left| \frac{\sigma_2 - \sigma_3}{2} \right| \right\}
\]

(11)

where \( \sigma_1, \sigma_2, \sigma_3 \) are the principal stresses.

3.4. Numerical Simulation Procedure

In this section we present numerical calculation procedure for evaluating the fatigue behavior of two rolling contact materials: SAE 4340, and AISI 52100. The mechanical properties of those materials are shown in Table 3.1. The half width of contact is specified \( a = 0.25 \) mm and the computational domain is chosen to be \( 12a \) length and \( 5a \) depth with \( 0.01a \) distance between each node horizontally or vertically. This distance yields accurate results for the stress variation at the line of symmetry (Figure 3.2) and assures that stress distribution is not influenced by the boundary effects.

We assume that the material is initially virgin, that is \( D_\theta = 0 \). The input parameters are the bulk material properties of the contacting material \( (E, H, M, S_e, \sigma_f) \) and the loading condition. Next, the equivalent maximum shear stress is computed using Eqs. 7-11 for the entire domain shown in Figure 3.2. We then transform the maximum shear stress into normal stresses using Mohr’s circle. Neglecting the compressive stress components along with using the Ramberg-Osgood type equation for the hysteresis curve [22], we compute different strain limits as described in Ref. [14]. The parameter \( D_t \) is, then, calculated by Eq. 4. Once damage status is updated, new strain values are computed (see Ref. [14]). Next ‘\( F' \) is calculated using Eq. 4 again to evaluate damage per cycle recursively and the process is repeated until the damage reaches the specified critical value of \( D_c \), after \( N \) cycles. Note that using this approach, the state of damage is evaluated in every nodal point below the surface at each time level until one of the points reaches the critical value for the first time. At that stage, the computations are complete and the number of cycles to failure is recorded. The average computational time for solving the damage evolution at different depths on the line of symmetry is around 420 minutes on a computer with 2.7 GHz CPU.

<table>
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<th>( E ) GPa</th>
<th>( H ) MPa</th>
<th>( M )</th>
<th>( \sigma_f ) MPa</th>
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</tbody>
</table>
3.5. Results and Discussion

3.5.1. Stress Distribution

Figure 3.3 shows the results of the dimensionless maximum shear stress, $\tau_{\text{max}}/p_{\text{max}}$, distribution below the contact region plotted as a function of the depth from the surface for five different coefficients of friction ranging from $\mu = 0.0$ to $\mu = 0.4$. This figure shows that the maximum shear stress ($\tau_{\text{max}}$) is strongly influenced by the friction coefficient. As can be seen, when the friction coefficient is low, the maximum shear stress occurs inside the material while for larger coefficients of friction the location of $\tau_{\text{max}}$ approaches the surface. The shift occurs at $\mu = 0.207$. Since the analytical solution for stress distribution does not depend on the material properties, the above statement is general for the elastic plain strain condition in homogenous materials regardless of the value of $a$ and the load. Also, as it can be seen, the maximum shear stress for all friction coefficients tends to an identical value at the depth of about $1.5a$ and beyond, revealing that the friction coefficient effect on stress distribution can be neglected at points deeper than $1.5a$. A similar conclusion is reached by Sraml et al. [43].

Figure 3.3. Maximum shear stress at $x=0$ for different depth($z$)

Figure 3.4(a-e) shows the evolution of the maximum shear stress at the line of symmetry in each cycle due to the contact pressure moving from $-6a$ to $6a$ on the upper edge of the plane. Dimensionless maximum shear stress is depicted at different depths as a function of distance from the line of symmetry, $l$, (see Figure 3.2) for a series of friction coefficient ranging from $\mu = 0.0$ to $\mu = 0.4$. Figure 3.4(f) illustrates how the absolute values of individual components in Eq. 11 vary with the position of pressure for $\mu = 0.4$ and $z=0$. The greatest variation of maximum...
shear stress occurs at the depth of 0.786a for the friction coefficients equal to 0.0, 0.1 and 0.2, while for the friction coefficients of 0.3 and 0.4 its maximum value moves to the surface. These stress variation values are used as an input for the calculation of damage status.

Figure 3.4. (a,b,c,d,e) Maximum shear stress evolution at the $x=0$ for different depths($z$) and coefficients of friction, (f) Evolution of different stress components in Eq.11 at the surface for $\mu=0.4$
3.5.2. Model Validation

To validate the prediction of crack initiation model, we compare the results of the simulations to those obtained experimentally by different researchers, a collection of which are reported by Harris & Barnsby [44] for a cylindrical roller bearing made of carbon vacuum-degassed AISI 52100. These data are the results of bearing tests subjected to different operating conditions. Figures 3.5(a) and 3.5(b) show the maximum pressure plotted as a function of number of cycles for initiation period and total failure, respectively. Also shown are the results of crack initiation and propagation reported by Chen et al. [30] and experimental results provided by Bhattacharyya et al. [45], for the same material. See Table 3.1 for the material’s properties. The results of Ref. [45] are for the high speed contact fatigue with ground and honed surface finish using synthetic oil with additives.

Note that precise determination of the location of the crack initiation at its inception is a difficult task, and therefore, most available experimental results are obtained after complete bearing failure. Fortunately, Chen et al. [30] provided measurement results for the ratio of initiation life to total life measured for GCr15 bearing steel which has the same properties as AISI 52100 [46]. Also the maximum Hertzian pressure of 2450 MPa specified by Chen et al. is close to the experimental load range reported by Harris & Barnsby [44]. Therefore, direct comparison can be made. According to Chen et al. [30], the nucleation period of cracks is about 10% of the total life for AISI 52100 roller bearing steels. The number of cycles in crack initiation period is calculated based on the estimated nucleation life by CDM (Figure 3.5(a)). The portion of total life spent in the initiation is considered to be as reported by Chen et al. and, using this information, the total life is estimated. The results are depicted in Figure 3.5(b) by plotting the maximum Hertzian pressure as a function of cycles. As can be seen, there is a good agreement between the current model and the experimental data for both the crack initiation stage and the total failure. As expected, the number of cycles to failure increases when the maximum Hertzian pressure decreases until reaching the point where \( P_{\text{max}} \) levels off. This corresponds to the location where the stress at the critical point (in this case below the surface) is close to the material endurance limit. Below the endurance limit the total life of the material is theoretically infinite. This is also in line with the findings of Andersson [47] who showed that the fatigue life for some rolling bearing applications can be considered to be infinite, if they are manufactured accurately from “clean” and homogeneous steel and operate under proper lubrication condition.

It should be mentioned that Figure 3.5 is only generated to validate the model for the specific material (AISI 52100) and for specific case of roller bearings. The assumption asserting that the nucleation period is 10% of the total life is not a general statement. Indeed, it depends on different factors [48], [49] including loading condition, material properties, etc. This percentage could be sometimes up to 90% of total life and generally it is an important stage in total failure. It is also important to mention that in the current study, we concern ourselves with the initiation phase of fatigue, not the total failure.
Figure 3.5. Number of cycles to (a) initiation and (b) failure for different maximum Hertzian pressure for AISI 52100

Since the critical damage value for AISI 52100 is not available in the open literature, we employed two different $D_c$ (0.1 & 0.4), which are considered to be extreme values of the critical damage for most of the materials (see Refs. [23],[50]). It should be noted that the maximum and minimum values of the critical damage (0.1 & 0.4) are the average of the theoretical maximum and minimum critical damage values according to different hypotheses. These average values are
obtained based on both the hypothesis of “elastic energy equivalence” and the hypothesis of “elastic strain equivalence”[50]. As can be seen, the CDM-contact fatigue model is not appreciably sensitive on the value of critical damage parameter, especially in the high cycle regime.

3.5.3. Damage Distribution

Figure 3.6(a,b,c,d) shows the evolution of damage below the surface for the first cycle. Here, the rolling element is assumed to be SAE 4340 whose properties are given in Table 1, and it is subjected to the maximum Hertzian pressure of 1800 MPa with an assumed friction coefficient of 0.1. As seen, the center of contact pressure moves from $3.5a$ to $3.5a$ in each cycle. The material is without any damage at the beginning and as the pressure moves along the upper edge, the damage is formed in the material.

Figure 3.6(e) shows the damage evolution at the line of symmetry (see Figure 3.2) based on the maximum shear stress criterion for 175,000 loading cycles. As can be seen, the rate of the damage growth at depth of $z = 0.196$ mm is very sharp, which indicates that a crack is likely to be initiated after around 180,000 cycles. At this stage the surface and near surface remain at very low damage status ($D_c = 0.46$). Figure 3.6(e) also shows that damage evolution is a non-linear process whose growth rate is low at the beginning of the loading cycles and rises rapidly at higher cycles. In other words, the rate of damage evolution increases as time goes on due to the increase in the local stresses that are intensified as damage accumulated in the material (see Refs. [14],[22] for more discussion). This observation is also reported by Lemaitre [24] based on his experimental findings for fatigue damage evolution of AISI 316 stainless steel.

Figure 3.7 shows the number of cycles to crack initiation for different coefficients of friction and at different depth from the surface for SAE 4340. These results correspond to the maximum pressure of 1800 MPa. As can be seen, due to the stress variation effect, the first crack is initiated below the surface at the approximate depth of 0.2 mm when the friction is less than 0.3, which is the case in components that typically operate in mixed lubrication/elastohydrodynamic regime such as gears, cams and rolling element bearings. As the friction increases, the location of the first crack initiation moves toward the surface and for $\mu = 0.3$ and higher it occurs at the surface, which is a common phenomenon in railway rails where the friction coefficient is comparatively high. There is a discontinuity in the graph for $\mu = 0.0$, 0.1 and 0.2; it shows that no crack is initiated there when $P_{\text{max}}=1800$ MPa. In other words, theoretically the life is infinite in this region where the stress level is below the endurance limit.
Figure 3.6. (a,b,c,d) Damage evolution contour for all $x$, at different depth for SAE 4340 and (e) Damage evolution at $x=0$ at different depth
3.5.4. Loading Sequence Effect

Palmgren-Miner’s rule [51] is commonly applied to evaluate cumulative fatigue damage in a component that is subjected to variable amplitude loading. It is written as follows:

\[ D = \sum_{i} \left( \frac{N_x}{N_f} \right)_{i} \]  \hspace{2cm} (12)

where \( N_x \) and \( N_f \) signify the number of applied cycles in each sequence and the number of total cycles to failure for the specific load in that sequence, respectively. The parameter \( N_f \) represents the number of all sequences. Here, \( D=1 \) means failure as opposed to CDM-based interpretation of failure in which the critical damage value (\( D_c \)) is less than unity. In linear incremental damage evolution model such as Palmgren-Miner’s rule, the number of cycles to crack initiation does not depend on the loading sequence. Likewise, for the especial case of bearing contact fatigue, although modification to the original Lundberg-Palmgren [32] relationship make it capable of considering variable loading, it is still not capable of taking into account the loading sequence effect. Moreover, the validity of linear damage increment rule has been questioned by experimental observations [52]. In fact, research shows that cycles of high load followed by low load cause greater damage to the material than when the order is reversed [1], [22]. A unique feature of the present analysis is its capability for taking loading sequence into consideration for the case of rolling/sliding contact fatigue.
Figure 3.8 shows the effect of load change during the unidirectional cyclic loading for total cycles of 40,000 based on the current model. The coefficient of friction is 0.2. Figure 3.8(a) shows the damage status when the rolling/sliding element is subjected to low load for the first 20,000 cycles followed up to high load for the next 20,000 cycles. Figure 3.8(b) demonstrates the damage status in a reverse order. That is, when rolling/sliding element is subjected first to a high load followed by low load for equal number of cycles in each sequence. If each individual loading sequence is considered, it is obvious that the higher the load, the greater is the damage. For example, compare the first loading cycle with 1900 MPa and 2100 MPa illustrated by dotted line in Figs. 8(a) and 8(b). However, when the total damage (solid line) is compared, the outcome is different. The maximum total damage after 40,000 cycles is dropped by around 12.7% when the lower load is applied first. The reason is: when a higher load is applied, more damage (micro cracks) is introduced to the material and this leads to a greater stress concentration for the rest of the material’s life. This behavior can influence the number of cycles to crack initiation as well as the total life of the element.

Figure 3.9 shows the results for the same loading condition but using the Miner’s rule for damage evaluation. Here, the current approach is again used but for the evaluation of each sequence individually (each component in Eq. 12). As mentioned previously, the ultimate value of $D$ is 1 for the Miner’s formation while it is less than unity for CDM approach. In this case, loading sequence does not have any effect on the damage and both Figures 3.9(a) and 3.9(b) show exactly the same total damage values. Clearly, therefore, the loading sequence effect must be taken into consideration for life prediction in rolling/sliding elements that are subjected to variable loading.

Figure 3.9.

Figure 3.8. Effect of Load sequence on the damage status at different depth from the surface-Current model, (a) Low load to high load (b) High load to low load.
Figure 3.9. Effect of Load sequence on the damage status at different depth from the surface-Miner’s rule, (a) Low load to high load (b) High load to low load.

3.6. Conclusions

The modern continuum damage mechanic approach is applied to predict contact fatigue crack initiation. On the basis of a moving Hertzian contact theory, the subsurface stresses for elastic plain strain condition are obtained. These loadings in conjunction with the damage theory enable one to predict the status of damage at the subsurface of the contacting material for given number of cycles. The continuum damage mechanic model is used for the purpose of calculating damage status. This formulation is capable of calculating the damage variable using the bulk material properties and surface conditions. Numerical simulations are carried out to calculate the damage status and predict the location of the first crack and the number of cycles required for its formation. The estimated number of cycles to crack initiation compared to the available experimental results reveals good agreement. The effect of load sequence on the life of the material is also investigated which shows different damage status inside the material due to different loading sequences.

For future studies the underlying assumptions (neglecting material imperfection and residual stresses, elastic deformation and the like) should be investigated and efforts should be directed to the development of a more generalized damage model for contact fatigue problems. Also the existing equations for the prediction of rolling/sliding element life should be modified in a more general form with consideration of variable loading sequence effect.

Nomenclature

- \( a \) half contact width, m
- \( D \) damage variable
- \( D_o \) initial damage
- \( D_i \) damage after \( i^{th} \) cycle
- \( D_c \) critical damage value
- \( E \) modulus of elasticity of an undamaged material, GPa
- \( E' \) effective modulus of elasticity, GPa
\( H \) cyclic hardening modulus, MPa
\( L \) normal force, N
\( l \) distance from the center of Hertzian pressure to the line of symmetry, m
\( M \) cyclic hardening exponent
\( n \) normal to an elemental cross section
\( N \) number of cycles to crack initiation
\( N_i \) number of all sequences
\( N_i \) number of applied cycles in each sequence
\( N_f \) number of total cycles to failure pertinent to the specific load in that sequence
\( p \) normal pressure, MPa
\( q \) tangential traction, MPa
\( S_e \) endurance limit, MPa
\( T_i \) boundary traction, Mpa
\( x, y, z \) coordinates components
\( \nu \) Poisson’s ratio
\( \varepsilon_{ij} \) Strain
\( \psi_D \) partial derivative of Helmholtz free energy with respect to \( D \), MPa
\( \Delta \varepsilon \) different threshold strains of damage increment in cycle \( i \) pertains to stress above \( S_e \) as described in Ref. [22]
\( \partial R_i \) part of system boundary on which traction is applied, m
\( \mu \) friction coefficient
\( \sigma \) stress, MPa
\( \sigma' \) effective stress, MPa
\( \sigma_{max} \) Maximum normal stress, MPa
\( \sigma_f \) true failure stress, MPa
\( \sigma_{\theta} \) residual circumferential stress, MPa
\( \sigma_1, \sigma_2, \sigma_3 \) principle stresses
\( \tau_{max} \) maximum shear stress, MPa

3.7. References


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Chapter 4: Asperity Micro-Contact Models as Applied to the Deformation of Rough Line Contact

4.1. Introduction

Real engineering surfaces are rough at microscopic level and their interactions involve the contact of surface peaks at discrete spots called asperity tips. These interactions play an important role on the tribological performance of bearings, gears, clutches, mechanical seals, hard disk drive, as well as biomedical transplants. Surface roughness affects the contact behavior such as the pressure distribution, contact width, real area of contact and contact resistance. These factors directly affect the load-carrying capacity, friction (traction) force, electro-thermo-mechanical and the wear behavior of tribo-components.

A review of the literature reveals that several approaches are available for predicting rough surface contact characteristics [1-5]. Among them, statistical approach is most convenient. Over the last four decades, different statistical asperity contact models have been proposed. The pioneering work of Greenwood and Williamson [6] treated the problem of nominally-flat rough surfaces by the statistical extension of the elastic Hertzian solution for an individual asperity to an ensemble of asperities with normally distributed heights. Later Chang et al. [7] proposed a model that extended the solution to the elasto-fully plastic deformation of asperities. Treatment of the intermediate regime of deformation was achieved by Zhao et al. [8] who provided an elastic-elasto/plastic-fully plastic model that bridges the elastic and plastic behavior of the asperity using an analytical function. More recent investigations involve the use of the finite element analysis as reported by Kogut and Etsion [9], [10] and Jackson and Green [11], [12]. Their study resulted in the development of convenient empirical elasto-plastic model based on the results of a finite element analysis.

The well-known Hertzian formulas are only valid for the macro-level contact of two ideally smooth curved surfaces while for real rough surfaces, the pertinent contact parameters such as the pressure distribution and the contact width deviate from those predicted by the Hertzian approach. To address this shortcoming, the literature is abundant with studies focusing on the assessment of roughness effect on the contact of spherical bodies [13-23]. In contrast, line-contact problems, i.e. contact of a cylinder and a plate, seem to have received less attention [23-25].

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The first analytical study on the contact of rough curved bodies was performed by Greenwood and Tripp [13] who employed the Greenwood-Williamson asperity contact model together with the bulk surface deformation for circular point contact. In a noteworthy contribution, Gelinck and Schipper [25] extended the approach of Greenwood and Tripp to the deformation of rough line contact using the Greenwood-Williamson model. Gelinck and Schipper [25] also provided convenient empirical formulas for the contact parameters such as maximum pressure and contact width based on the numerical solutions of the rough line contact. These formulations are utilized extensively in the treatment of mixed lubrication problems [26-35] that incorporate the load-sharing concept.

The current paper applies different statistical asperity micro-contact models to the deformation of rough line contact and provides a comparison study among them. The model involves a simultaneous solution of the asperity interaction with the elastic bulk deformation of the surface using the Newton-Raphson technique. It predicts the apparent pressure distribution as well as the contact width and the real area of contact for the line contact configuration. Results presented take into account the elastic or elasto-plastic behavior (hardness or yield stress) of the asperities. In addition, the results of extensive sets of simulations are used to derived expressions for the prediction of the contact characteristics including maximum contact pressure, contact width, real area of contact and pressure distribution function.

4.2. Statistical Asperity Micro-contact Models

In a statistical approach, the treatment of surface roughness requires one to first determine an appropriate relationship for the contact of a single asperity. Subsequently, the influence of an ensemble of asperities must be taken into account to determine the contact behavior of a rough surface. This is achieved by employing the statistical distribution of asperities and specification of surface parameters. The basic underlying assumptions in statistical treatment of micro-contact are [6]: The rough surface is isotropic; asperities are spherical near their summits; all asperities have the same radius of curvature; the asperity heights follow the Gaussian distribution; each individual asperity deforms separately, so there is no interaction among the asperities; and that the bulk surface deformation below the individual asperity is negligible.

We begin by first presenting a brief description of the asperity models followed by the methodology for treating the line-contact problem and the corresponding results.

4.2.1. Greenwood and Williamson (GW)

Greenwood and Williamson [6] proposed a model for the load and real area of contact of rough surfaces based on the elastic Hertzian solution of a single asperity contact and the extension of the results to an ensemble of asperity heights with Gaussian distribution. They provided the following relationships for the contact load and the real area of contact of a flat rough and an ideally smooth flat surface:
\[ P(h) = n \beta \sigma E'A_n \frac{4}{3} \sqrt{\frac{\sigma}{\beta}} \int_{h-y_s}^{\infty} (\overline{\sigma})^{1/2} \phi^*(z^*)dz^* = \xi E'A_n \times \{\Phi_{GW}(h)\} \]

\[ A(h) = \pi \xi A_n \int_{h-y_s}^{\infty} (\overline{\sigma}) \phi^*(z^*)dz^* \]

where

\[ \xi = n \beta \sigma \quad , \quad \phi^*(z^*) = \frac{1}{\sqrt{2\pi}} \left( \frac{\sigma}{\sigma_s} \right) e^{-\left( \frac{\sigma}{\sigma_s} \right)^2 \frac{z^2}{2}} \quad , \quad z^* = \frac{z}{\sigma} \quad , \quad \omega = z^* - \frac{h-y_s}{\sigma} \]

\[ \frac{1}{E'} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2} \]

In these equations, \( n \) and \( \beta \) are the asperity density and radius respectively; \( \sigma \) denotes the standard deviation of the surface heights distribution (henceforth, referred to surface roughness); \( \sigma_s \) represents the standard deviation of summit heights distribution; \( h \) is the separation between two surfaces; \( y_s \) denotes the distance between the mean line of the summit heights and that of the surface heights; and \( z \) represents the asperity (summit) height measured from the mean line of summit heights. See Figure 4.1. \( E_1 \) and \( E_2 \) are the moduli of elasticity and \( \nu_1 \) and \( \nu_2 \) are the Poisson’s ratios of the contacting surfaces. \( A_n \) is the nominal contact area and \( A \) is the real contact area.

![Figure 4.1. contact of a rough surface with an ideally smooth flat surface](image)

Referring to the work of Bush et al. [36] and McCool [37] and by rearranging their formulations for the isotropic rough surfaces with Gaussian distribution of surface heights, the following relationships apply:
\[ \frac{\sigma}{\sigma_s} = \frac{\xi}{\sqrt{\xi^2 - F}} \quad F = 3.71693 \times 10^{-4} \]

\[ \frac{y_s}{\sigma} = G \frac{\xi}{\xi} \quad G = 0.045944 \]

In non-dimensionalized form, the GW relationships can be written as:

\[ p(\bar{h}) = \bar{\beta} \beta \sigma \left\{ \frac{4}{3} \frac{\bar{\sigma}}{\bar{\beta}} \sqrt{\frac{\pi}{W}} \int_0^\infty \frac{G}{\bar{\sigma}} \frac{\xi}{\xi} (\bar{\omega})^{3/2} \phi^*(z^*)dz^* \right\} = \xi \left\{ \phi_{GW}(\bar{h}) \right\} \]

\[ A(\bar{h}) = \frac{A}{A_n} = \pi \xi \left( \frac{3}{2} \frac{\bar{\sigma}}{\bar{\beta}} \frac{G}{\xi} \phi^*(z^*)dz^* = \pi \xi \left( \bar{\lambda}_{GW}(\bar{h}) \right) \right) \]

\[ \bar{\omega} = z^* - \frac{h - y_s}{\sigma} = z^* - \left( \frac{h}{\sigma} - \frac{G}{\xi} \right) \]

with the following dimensionless parameters:

\[ \bar{\sigma} = \frac{\sigma}{R} \quad \bar{\beta} = \frac{\beta}{R} \quad \bar{n} = nR^2 \quad \xi = \bar{\beta} \bar{\sigma} = n\beta\sigma \]

\[ p = \frac{p}{p_H} \quad \bar{h} = \frac{h}{R} \quad \bar{W} = \frac{W}{l E'R} \]

where \( p_H \) is the maximum Hertzian pressure and \( b \) is the Hertzian contact half width:

\[ p_H = \frac{2W}{\pi l b} = \frac{E'h}{2R} = \frac{E'}{l} \sqrt{\frac{W}{\pi}} \]

\[ b = \sqrt{\frac{4WR}{\pi l E'}} = \frac{4WR^2}{\pi} \]

The contact of two cylinders can be treated as a contact of a flat plate and a cylinder with the effective (equivalent) radius. In Eqs. 4 and 5, \( R \) is the effective radius of curvature, \( l \) is the depth of line contact and \( W \) denotes the total contact load. These dimensionless parameters are chosen with regard to the EHL application [38], so that any predictive formulas based on these parameters can be further extended to consider the lubricant effect on the contact characteristics.

The aforementioned asperity formulas are applicable to the contact of two flat surfaces parallel to each other with constant mean separation. The line contact of a curved surface and a flat surface can be treated as a summation of discrete lines each having a different but constant distance from the flat surface. In other words, \( dA_n \) and \( dA \) can be considered as a small element of the nominal contact area and real area of contact, respectively. Hence, for the contact of a cylinder against a flat surface one can write:
\[ A_s = \int dA = \int \left[ \pi n \beta \sigma \int_{h-y_s}^{z^*} \left( z^* - \frac{h-y_s}{\sigma} \right) \phi^* (z^*) dz^* \right] dA_n \]

\[ = \pi n \beta \sigma \int_{-\infty}^{+\infty} \int_{h(x)-y_s}^{z^*} \left( z^* - \frac{h(x)-y_s}{\sigma} \right) \phi^* (z^*) dz^* \]

and in dimensionless form:

\[ \bar{A}_s = \frac{A_s}{bl} = \pi \xi \int_{-\infty}^{+\infty} \left\{ \int_{h}^{\bar{h}} \frac{G}{\bar{G}} \left( z^* - \frac{\bar{h}(X)}{\xi} \right) \phi^* (z^*) dz^* \right\} dX = \pi \xi \int_{-\infty}^{+\infty} \bar{A}_{GW} \left( \bar{h}(X) \right) dX \]  

Similar equations to Eqs. 6 and 7 can be derived for all other statistical micro-contact models presented in the following sections. However, for brevity, the details are not given.

4.2.2. Chang, Etsion and Bogy (CEB)

McCool [1] compared the GW model with other available isotropic and anisotropic models, and concluded that the GW model gives reliable results. Nonetheless, it can be used only when the majority of the contacting asperities deform elastically, which limits its applicability to surfaces with low plasticity index as defined later in this section.

Twenty years after Greenwood and Williamson, Chang et al. [7] incorporated the effect of asperity plastic deformation into the GW model using the concept of volume conservation for plastically deformed asperities. By introducing the critical interface \((\omega_c)\) at the inception of the plastic deformation, they assumed that the asperity deforms elastically below the critical interface, and is fully plastic above this value, and that the volume of the plastically deformed asperity is conserved. They proposed the following integrals for the two regimes of deformation:

\[ P(h) = n \beta \sigma E' A_n \left\{ \frac{4}{3} \sqrt{\frac{\sigma}{\beta}} \int_{h-y_s}^{h-y_s+\omega_c} \left( \frac{\omega}{\omega_c} \right)^{3/2} \phi^* (z^*) dz^* + \pi K \frac{H}{E'} \int_{h-y_s}^{h-y_s+\omega_c} \left( 2\bar{\omega} - \omega_c^* \right) \phi^* (z^*) dz^* \right\} \]

\[ = n \beta \sigma E' A_n \times \{ \Phi_{CEB} (h) \} \]

\[ A(h) = A_s \pi n \beta \sigma \left\{ \int_{h-y_s}^{h-y_s+\omega_c^*} \left( \omega \right) \phi^* (z^*) dz^* + \int_{h-y_s}^{h-y_s+\omega_c^*} \left( 2\bar{\omega} - \omega_c^* \right) \phi^* (z^*) dz^* \right\} \]  

where

\[ \omega_c^* = \frac{\omega_c}{\sigma} = \left( \frac{nKH}{2E} \right)^{1/2} \left( \frac{\beta}{\sigma} \right) \]

In Eq. 8, \(H\) is the hardness of the softer material and is typically taken to be equal to 2.8 times the yield strength \(S_y\) for untreated surfaces. \(K\) is the proportionality factor between the maximum
contact pressure and the hardness at the onset of plastic deformation \( P_{\text{yield}} = KH \). Tabor [39] suggested the constant value of 0.6 for \( K \), while other researchers showed that it depends on the Poisson’s ratio. For example, Chang et al. [40] proposed:

\[
K = 0.454 + 0.41\nu
\]  

(9a)

and Lin [41] suggested the following relationship:

\[
K = 0.4645 + 0.314\nu + 0.1943\nu^2
\]  

(9b)

In the current study the expression provided by Chang et al. [40] is used; nevertheless, all of these relationships yield fairly close values for \( K \).

According to Greenwood and Williamson [6], the plasticity index is expressed as:

\[
\psi = \frac{\sigma_c}{\omega_c}
\]  

(10)

Using Eq. 2 and the definition of the critical interface (Eq. 8), the plasticity index can be written as follows [7]:

\[
\psi = \frac{2E'}{\pi KH} \sqrt{\frac{\sigma}{\beta}} \left( 1 - F \frac{\xi^2}{\bar{\xi}^2} \right)^{1/4}
\]  

(11)

4.2.3. Zhao, Maietta and Chang (ZMC)

The CEB model suffers from a discontinuity in the contact pressure expression at the critical interface. At this point, the average pressure suddenly jumps from \( 2/3KH \) to \( KH \). Moreover, the slope of both the contact load and the real contact area are different at the critical interface. To overcome this limitation, Zhao et al. [8] used a mathematical function that bridges between the elastic and fully plastic segments and maintains the continuity of the load and the contact area expressions as well as their slopes. They provided the following equations:

\[
P(h) = n\beta\sigma E'A_n \left\{ \frac{4}{3} \sqrt{\frac{\sigma}{\theta}} \int_{\frac{h-y_1+\omega_1}{\sigma}}^{h-y_1+\omega_1} \left( \bar{\omega} \sigma^2 \right)^{3/2} \phi^*(z^*)dz^* + 2\pi \frac{H}{E'} \int_{\frac{h-y_1+\omega_1}{\sigma}}^{h-y_1+\omega_1} \left[ 1 - (1 - k) \frac{\ln \omega_2 - \ln \bar{\omega}}{\ln \omega_2 - \ln \omega_1} \right] \right\} = \xi E'A_n \times \{\Phi_{\text{ZMC}}(h)\}
\]

\[
A(h) = A_n \pi n\beta\sigma \left\{ \int_{\frac{h-y_1+\omega_1}{\sigma}}^{h-y_1+\omega_1} \left( \bar{\omega} \sigma^2 \right) \phi^*(z^*)dz^* + 2\int_{\frac{h-y_1+\omega_1}{\sigma}}^{h-y_1+\omega_1} \left( \bar{\omega} \sigma^2 \right) \phi^*(z^*)dz^* \right\}
\]
\[ \int \frac{h - y_s + \omega_c}{\sigma} \left[ 1 - 2 \left( \frac{\omega}{\omega_2 - \theta_1} \right)^{3/2} \phi^*(z^*)dz^* \right] \]

where

\[ \omega_1^* = \omega_c^* = \omega_1 = \left( \frac{3 \pi k H}{4 E^*} \right)^{2} \left( \frac{\beta}{\sigma} \right) ...... \]

Based on the work of Johnson [42], Zhao et. al [8] concluded that fully plastic deformation occurs at the interface which is at least 54 times bigger than the interface at initial plastic deformation (\( \omega_2 \geq 54 \omega_c \)).

4.2.4. Kogut and Etsion (KE)

Although the ZMC approach resolved the continuity problem in CEB model, it was based on the mathematical manipulations not physical considerations. Recently, Kogut and Etsion [9] presented a different approach where they carried out a finite element simulation for the deformation of a single asperity. Subsequently, they proposed convenient empirical expressions which include different asperity deformation regimes. Given the relationships for a single asperity contact, they extended the approach to the contact of rough surfaces using statistical method [10]:

\[ P(h) = n \beta \sigma E^* A_n \left\{ 4 - \int \frac{h - y_s + \omega_c^*}{\sigma} \left( \frac{h - y_s + \omega_c^*}{\sigma} \right)^{3/2} \phi^*(z)dz \right\} + \frac{2}{3} \times 1.03 \pi . K \omega_c^* - 0.425 \left( \frac{H}{E^*} \right) \int \frac{h - y_s + 6 \omega_c^*}{\sigma} \left( \frac{h - y_s + 6 \omega_c^*}{\sigma} \right)^{1.425} \phi^*(z)dz \]

\[ + \frac{2}{3} \times 1.4 \pi K \omega_c^* - 0.263 \left( \frac{H}{E^*} \right) \int \frac{h - y_s + 110 \omega_c^*}{\sigma} \left( \frac{h - y_s + 110 \omega_c^*}{\sigma} \right)^{1.263} \phi^*(z)dz \]

\[ + 2 \pi \frac{H}{E^*} \int_{h - y_s + 110 \omega_c^*}^{\infty} \left( \frac{h - y_s + 110 \omega_c^*}{\sigma} \right)^{1.136} \phi^*(z)dz \]

\[ = n \beta \sigma E^* A_n \times \{ \Phi_{KE}(h) \} \]

\[ A(h) = \pi n \beta \sigma A_n \left\{ \int \frac{h - y_s + \omega_c^*}{\sigma} \left( \frac{h - y_s + \omega_c^*}{\sigma} \right)^{3/2} \phi^*(z^*)dz^* + 0.93 \pi n \beta \sigma : \omega_c^* - 0.136 \int \frac{h - y_s + 6 \omega_c^*}{\sigma} \left( \frac{h - y_s + 6 \omega_c^*}{\sigma} \right)^{1.136} \phi^*(z^*)dz^* \right\} \]

\[ + 0.94 \omega_c^* - 0.146 \int \frac{h - y_s + 110 \omega_c^*}{\sigma} \left( \frac{h - y_s + 110 \omega_c^*}{\sigma} \right)^{1.146} \phi^*(z^*)dz^* + 2 \int_{h - y_s + 110 \omega_c^*}^{\infty} \left( \frac{h - y_s + 110 \omega_c^*}{\sigma} \right)^{1.136} \phi^*(z^*)dz^* \]

Four regimes of deformation can be distinguished in their model. The first and the last ones are the elastic and fully plastic regimes, similar to the ZMC model. However, according to KE,
complete plastic deformation occurs at 110 $\omega_c^*$. The elasto-plastic regime (second and third integrals) is divided into two distinct parts based on the evolution of the plastic core beneath the contact region for an individual asperity contact. It is worth mentioning that the predictions of the KE model yield good agreement as compared to experimental results [43].

4.2.5. Jackson and Green (JG)

Recently, Jackson and Green [11], [12] proposed a new model based on finite element analysis. They used finer meshes compared to Kogut and Etsion and took the effects of material properties and geometry into account during the deformation. Moreover, they extended the contact model of KE to a high asperity deformation up to the value of $a/\beta=0.41$. They showed that the assumption of the elastic deformation for the asperity contact is valid not only within the critical interface limit but also up to 1.9 times the critical interface. However, they did not determine the interface at which fully plastic regime starts. Unlike the KE model which assumes a constant hardness, the JG model considers hardness variation during the deformation. They showed that, in contrast to the KE model, the value of 2.8 for the average pressure to yield strength ratio is not reached even for very high interfaces. The JG model is given as follows:

$$P(h) = n \beta \sigma E' A_n \left\{ \frac{4}{3} \int \frac{\sigma}{\beta} \left( \frac{\sigma}{\omega_{c}} + \frac{1.9 \sigma}{\omega_{c}^*} \right) \phi^*(z) dz + \right.$$

$$\left. \frac{4}{3} \left( \frac{C \pi S_y}{2 E} \right) \int_{\frac{h-y_s}{\omega}}^{\infty} \left( \frac{\sigma}{\omega_{c}^*} \right)^{1/2} \frac{e^{-1} (\sigma)^{1/2}}{4} \frac{\sigma^{5/12} (\sigma)^{5/12}}{} \right\} + \left. \frac{2.84 \times 4}{C} \left( 1 - e^{-0.82} \left( \frac{\sigma}{\beta} \sqrt{\pi} \left( \frac{1}{1.9 \sigma} \right)^{D/2} \right)^{\beta} \phi^*(z^*) dz^* \right) \right\} (14)$$

$$A(h) = n \beta \sigma A_n \left\{ \int_{\frac{h-y_s}{\omega}}^{\infty} \left( \frac{\sigma}{\omega_{c}^*} \right) \phi^*(z^*) dz^* + \left. \left( \frac{1}{1.9 \omega_{c}^*} \right)^D \int_{\frac{h-y_s}{\omega}}^{\infty} \frac{\sigma^{D+1} \phi^*(z^*) dz^*}{1} \right\} \right\}$$

where

$$\omega_{c}^* = \frac{\omega_{c}^*}{\sigma} = \left( \frac{\pi CS_y}{2E} \right)^2 \frac{\beta}{\sigma}$$

$$C = 1.295 \cdot e^{0.736v} \quad D = 0.14 \cdot e^{2.5 \frac{S_y}{E}}$$
4.3. Application to Line Contact

Having described the existing statistical asperity micro-contact models, we now present their application to the line contact problem which involves the bulk deformation of surface. For this purpose, the governing equations for the asperity contact models should be solved simultaneously with the bulk surface deformation of surface. Note that the bulk surface is allowed to deform elastically, while the surface asperities can deform elastically, elasto-plastically or fully plastic depending on the applied load, geometry and surface properties.

4.3.1. Governing equations

The following expression generalizes the asperity pressure according the GW, CEB, ZMC, KE or JG models:

\[ p(h) = n \beta \sigma E' \times \Phi_{GW,CEB,ZMC,KE,JG}(h) \]  

(15)

where \( p(h) \) is the nominal (apparent) pressure for the contact of two flat rough surfaces with constant mean separation. The separation equation for the line contact is [44]:

\[ h(x) = h_0 + x^2 \frac{2}{2R} - \frac{2}{\pi E'} \int_{-\infty}^{+\infty} p(s) \ln|s| ds \]  

(16)

where \( h_0 \) is a constant to be determined and \( x \) is the coordinate along the contact width. In addition, the force balance equation must be satisfied:

\[ W = I \int_{-\infty}^{+\infty} p(x) dx \]  

(17)

In non-dimensionalized form the above equations can be written as:

\[ \bar{p} = \pi \bar{\sigma} \times \Phi_{GW,CEB,ZMC,KE,JG}(h) \]  

(18)

\[ \bar{h}(X) = \bar{h}_0 + X^2 \left( \frac{4W}{\pi} \right) - \frac{1}{\pi} \left( \frac{4W}{\pi} \right) \int_{-\infty}^{+\infty} \bar{p}(S) \ln|X - S| dS \]  

(19)

\[ \frac{\pi}{2} = \int_{-\infty}^{+\infty} \bar{p}(X) dX \]  

(20)

where \( X = x/b \) and \( \Phi(h) \) in Eq. 18 corresponds to each micro-asperity contact model in dimensionless form that are summarized in in Table. 4.1. Introducing two new dimensionless parameters \( \Omega \) (dimensionless hardness) and \( \Upsilon \) (dimensionless yield stress), these expressions represent the dimensionless forms of the GW, CEB, ZMC, KE and JG micro-asperity contact load Eqs 1,8,12,13 and 14.
4.3.2. Numerical Solution Scheme

Equations 18-20 must be solved simultaneously. They can be discretized in a systematic way as follows:

\[
f_i = \bar{p}_i - \bar{p} \beta \sigma \times \left\{ \Phi_{GW, CEB, ZMC, KE, JG}(\tilde{h}_i) \right\}
\]  

(21)
\[ \tilde{h}_i = \tilde{h}_{00} + \frac{X_i^2}{2} \left( \frac{4W}{\pi} \right) - \frac{1}{\pi} \left( \frac{4W}{\pi} \right) \sum_{j=1}^{N} K_{ij} \tilde{p}_j \Delta X \]

\[ f_{N+1} = \Delta X \sum_{j=1}^{N} \tilde{p}_j \frac{\pi}{2} = 0 \]

where \( K_{ij} \) can be written as [45]:

\[ K_{ij} = \left( X_j - X_i + \frac{\Delta X}{2} \right) \left( \ln \left( \frac{X_j - X_i + \Delta X}{2} \right) - 1 \right) - \left( X_j - X_i - \frac{\Delta X}{2} \right) \left( \ln \left( \frac{X_j - X_i - \Delta X}{2} \right) - 1 \right) \]

In these set of expressions (Eq. 21), \( N+1 \) unknown (associated with \( N+1 \) equations) are to be determined. The computations begin by assuming a value of \( \tilde{h}_{00} \) and a pressure distribution. Next, using the Newton-Raphson technique, the equation set (21) are iteratively solved for the pressure distribution as well as the value of \( \tilde{h}_{00} \) until the results converge within a specified error (see Eq. 22).

\[
\begin{bmatrix}
\frac{\partial f_1}{\partial \tilde{p}_1} & \frac{\partial f_1}{\partial \tilde{p}_2} & \cdots & \frac{\partial f_1}{\partial \tilde{p}_N} & \frac{\partial f_1}{\partial \tilde{h}_{00}} \\
\frac{\partial f_2}{\partial \tilde{p}_1} & \frac{\partial f_2}{\partial \tilde{p}_2} & \cdots & \frac{\partial f_2}{\partial \tilde{p}_N} & \frac{\partial f_2}{\partial \tilde{h}_{00}} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{\partial f_N}{\partial \tilde{p}_1} & \frac{\partial f_N}{\partial \tilde{p}_2} & \cdots & \frac{\partial f_N}{\partial \tilde{p}_N} & \frac{\partial f_N}{\partial \tilde{h}_{00}} \\
\Delta X & \Delta X & \cdots & \Delta X & 0
\end{bmatrix}
\begin{bmatrix}
\Delta \tilde{p}_1 \\
\Delta \tilde{p}_2 \\
\vdots \\
\Delta \tilde{p}_N \\
\tilde{h}_{00}
\end{bmatrix}
= 
\begin{bmatrix}
-f_1 \\
-f_2 \\
\vdots \\
-f_N \\
-f_{N+1}
\end{bmatrix}
\]

(22)

4.4. Results and Discussion

Recent models of KE and JG are found to follow the experimental data very closely when the single contact of a sphere and a flat is considered. Nonetheless, for the case of a rough surface with multiple asperity contacts, all of the statistical contact models including KE and JG employ a few surface parameters to conveniently simulate the contact condition instead of a full surface profile. As also noted by Ref. [46], it is unrealistic to anticipate very exact results especially using the real area of contact formula (e.g., Eqs. 3 & 7). Furthermore, there are certain limitations associated with these models that one needs to be aware of. For example, as stated before, all of these models assume that there is no interaction between the asperities and that they deform independently. Therefore, “asperity merging” is not taken into account. Hence, the validity of these models becomes questionable for high loads when results yield large negative separations, which have no physical meaning. In the present study, for the solution of the pressure distribution, we let the calculations continue until the results converge regardless of a large negative separation in order to show that mathematically the model can approach the Hertzian contact under a very high imposed loading condition. This procedure is also adopted by Ref. [25]. In contrast, the calculations of the real area of contact are terminated for high negative
separations. It is worthwhile to mention that in order to achieve convergence, especially at high loads, an under relaxation factor is applied in the simulation of pressure.

In what follows we present a series of contact characteristic prediction results including the apparent pressure distribution, contact width and the real area of contact based on the different micro-contact models presented above and for surfaces with different properties. The surface properties used in the current comparative study are summarized in Table 4.2.

<table>
<thead>
<tr>
<th>Case #</th>
<th>$\sigma$</th>
<th>$\beta$</th>
<th>$n$</th>
<th>$\bar{\sigma}$</th>
<th>$\xi=n\beta\sigma$</th>
<th>$\psi$</th>
<th>$R$(mm)</th>
<th>$l$(mm)</th>
<th>$E^\prime$(GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (smooth)</td>
<td>$0.08 \times 10^{-6}$</td>
<td>$0.32 \times 10^{-4}$</td>
<td>$1.55 \times 10^9$</td>
<td>$4 \times 10^{-6}$</td>
<td>$0.040$</td>
<td>$0.9$</td>
<td>$20$</td>
<td>$45$</td>
<td>$114$</td>
</tr>
<tr>
<td>2 (medium)</td>
<td>$0.3 \times 10^{-6}$</td>
<td>$0.17 \times 10^{-3}$</td>
<td>$1.18 \times 10^9$</td>
<td>$1.5 \times 10^{-5}$</td>
<td>$0.060$</td>
<td>$2.5$</td>
<td>$20$</td>
<td>$45$</td>
<td>$114$</td>
</tr>
<tr>
<td>3 (rough)</td>
<td>$1 \times 10^{-6}$</td>
<td>$0.055 \times 10^{-3}$</td>
<td>$1.15 \times 10^9$</td>
<td>$5 \times 10^{-3}$</td>
<td>$0.064$</td>
<td>$7.5$</td>
<td>$20$</td>
<td>$45$</td>
<td>$114$</td>
</tr>
<tr>
<td>4 (very rough)</td>
<td>$2.95 \times 10^{-6}$</td>
<td>$0.03 \times 10^{-3}$</td>
<td>$0.66 \times 10^9$</td>
<td>$1.47 \times 10^{-4}$</td>
<td>$0.058$</td>
<td>$18.5$</td>
<td>$20$</td>
<td>$45$</td>
<td>$114$</td>
</tr>
</tbody>
</table>

### 4.4.1. Apparent pressure distribution

Four sets of surface parameters are selected and the pressure profiles are determined at two representative high and low loads of $\bar{W} = 0.001$ and $\bar{W} = 0.00001$ (Figures 4.2&4.3). As can be seen, the pressure profiles according to all models are different from the Hertzian pressure, revealing that Hertzian pressure distribution ceases to be valid for rough surfaces. It is evident that at high loads the smooth surface with lowest plasticity index (Figure 4.2a) has a profile close to the Hertzian distribution while the pressure profile for very rough surface significantly deviates from Hertzian solution. The results show the important role of the surface parameters in the pressure distribution. Another important factor is the total load. When the total load increases the pressure profile becomes more similar to the Hertzian type. At the high load of $\bar{W} = 0.001$ and low roughness (Figure 4.2a and b), the pressure profiles nearly coincide with the Hertzian pressure except in the vicinity of $X=1$. This trend persists even for the case of rough surface with $\bar{\sigma} = 5 \times 10^{-5}$ (Figure 4.2c). In contrast, for the low load of $\bar{W}_b=0.00001$, the maximum pressure is significantly less than that of the maximum Hertzian pressure even for smooth surface (Figure 4.3). The maximum pressure is around 30% of the maximum Hertzian pressure for very rough surface and 80% for the smooth one.

All models yield similar results for smooth surface with the plasticity index of lower than 1, even at high loads (Figure 4.2a). However, when the plasticity index and the surface roughness increase, the GW model predicts higher maximum pressure than other models since it assumes that all asperities deform elastically. In almost all cases, the ZMC, KE and JG models predict very close pressure profiles. Although all of these models take into account the transition between the elastic and the fully plastic regimes, the KE and JG models are more accurate since they consider “all states of deformation”.

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It has been shown that when the contact of two flat surfaces is considered, where the separation is constant, the ZMC, KE and JG models predict different values for the contact load when plasticity index is comparatively high \[12\]. Nevertheless, here, for the case of line contact, they show close results. The CEB model predicts a slightly lower maximum pressure compared to other elasto-plastic models at high loads due to the fact that it assumes fully plastic behavior for the asperity at the deformation beyond the critical point \( \omega_c \) while other elasto-plastic models consider the transition between elastic and fully plastic regime as well.

According to Figures 4.2 & 4.3, the pressure distribution extends significantly beyond \( X=1 \) for rough surface. Clearly as the maximum pressure drops, the extension of pressure becomes wider so that the load is satisfied. This is particularly noticeable in situations where the plasticity index is large or the load is low. More discussion on this is presented in section 4.2.
The dimensionless maximum pressure versus the dimensionless load is plotted in Figure 4.4 for two different surface properties. Also shown in this figure is the associated dimensional load. All models predict nearly the same maximum pressure at low loads for both surface types (smooth and rough) which is very small. The maximum pressure is almost equal to Hertzian at very high loads for the smooth surface and comparatively close to that of Hertzian for the rough surface. At high loads, the bulk deformation of the surface dominates the asperity deformation and the surface acts like an ideally smooth one. Again here, the ZMC, KE and JG models predict close results whereas the GW and CEB deviate from them especially for the rough surface (Figure 4.4b). In addition, at very low or very high loads all of the models show close results since at low loads the asperity deformation is not significant and mostly elastic. Further, as
established in the results presented, at high loads all of the models approach the Hertzian solution due to the dominance of the bulk deformation.

Figure 4.4. Normalized maximum apparent pressure for different surface properties at different loads for $\Omega=0.017$

4.4.2. Contact width

The contact width can be estimated according to the pressure distribution. However, unlike the Hertzian distribution in which the contact area has distinct boundaries, for rough surfaces the pressure asymptotically approaches zero at the boundaries. Review of the literature shows that
there is no universal contact width definition for rough surface contact and different researchers have adopted different criteria for its prediction [13],[42],[17]. In the present study, in order to compare these criteria, the following two are adopted:

The first criterion as proposed by Greenwood and Tripp [13] is:

\[
\bar{b}_{\text{eff}} = \frac{b_{\text{eff}}}{b} = \frac{3\pi}{4} \int_0^\infty X \bar{p}(X) dX = \frac{3}{\pi} \int_0^\infty X \bar{p}(X) dX
\]

where \(\bar{b}_{\text{eff}}\) is the dimensionless contact half width, often referred to as the effective contact half width. The factor \(3\pi/4\) is chosen such that the formula yields unity for the Hertzian pressure distribution. It is evident that the denominator equals \(\pi/4\) (see Eq. 20), so only the numerator needs to be calculated.

Another definition — see for example Ref. [17] — considers the contact half width as the distance between the center of the contact and a specific point at which the dimensionless pressure, \(\bar{p}\), is less than an assumed value. This value should be small enough such that the pressure can be assumed negligible beyond that specific point. In the current study, this threshold value is chosen as 0.01 and the associated contact half width is symbolized as \(b_{0.01}\) and referred to as threshold contact half width hereinafter. Should an estimation of the pressure distribution function be desired, the latter definition needs to be used since the former one gives the effective distance, not the actual mathematical distance (see Figure 4.3c).

Figure 4.5 shows the dimensionless contact half width as a function of the load for two different surface properties and based on both definitions. Effective contact half width is demonstrated by solid lines while dashed lines represent the threshold contact half width. As can be seen, the contact width differs significantly from Hertzian width for lower loads while approaches unity for higher loads, showing that Hertzian solution is valid when the load is high. In addition, all of the models show relatively similar results for different loads showing that the contact half width is not appreciably dependent on the type of asperity contact model.

4.4.2.1. Experimental Verification

Precise experimental determination of the contact parameters is a difficult task, and published experimental results are scarce for the case of line contact. Fortunately, Kagami et al. [48] provided measurement results for the contact half width for the case of smooth cylinders and a rough steel plate at four different loads. The pertinent dimensional, mechanical and surface properties of the cylinders and the plate are shown in Table 4.3.
Figure 4.5. Normalized contact half width for different surface properties at different loads for $\Omega=0.017$. 

(a) $\sigma=4 \times 10^{-6}$, $\psi=0.9$ (smooth) 

(b) $\sigma=5 \times 10^{-5}$, $\psi=8$ (rough)
Table 4.3. Dimensional, surface and mechanical properties for the rough plate and the cylinders [48]

<table>
<thead>
<tr>
<th></th>
<th>$R(m)$</th>
<th>$l(m)$</th>
<th>$\sigma(m)$</th>
<th>$\beta(m)$</th>
<th>$n(m^2)$</th>
<th>$E$(GPa)</th>
<th>$H$(MPa)</th>
<th>$\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plate(Steel)</td>
<td>-</td>
<td>-</td>
<td>$1.45 \times 10^{-6}$</td>
<td>$0.28 \times 10^{-4}$</td>
<td>$1.4 \times 10^9$</td>
<td>206</td>
<td>3340</td>
<td>0.29</td>
</tr>
<tr>
<td>Cylinder(Steel)</td>
<td>$1.5 \times 10^{-3}$</td>
<td>$4.49 \times 10^{-3}$</td>
<td>$&lt; 0.1 \times 10^{-6}$</td>
<td>-</td>
<td>-</td>
<td>206</td>
<td>7520</td>
<td>0.29</td>
</tr>
</tbody>
</table>

Demonstrated in Figure 4.6 is the variation of the predicted half of contact width as a function of the total load using KE and JG approaches. Also shown are the experimental results of Kagami et al. [48]. As seen, there is good agreement between the present model and the experimental data for both cylinders using either KE or JG approaches. The half of contact width is obtained based on both of the aforementioned criteria. Note that the experimental data is slightly closer to the threshold contact half width. Nonetheless, two curves are too close to conclude which criterion prevails. However, as stated before, for the estimation of the pressure distribution function, the threshold contact half width needs to be used. In the following sections, only this definition is considered.

![Figure 4.6. Half of the contact width at different loads based on Hertzian, KE and JG approach](image)

4.4.3. Real area of contact

In addition to the contact pressure distribution and the contact width, the real area of contact is another important parameter that influences friction, wear and contact resistance of the materials in contact. Statistical approaches provide estimation of the real area of contact of two flat surfaces. As mentioned before, it can be extended to the contact of curved bodies [13], [25].
After obtaining the apparent pressure distribution as well as the separation as a function of $X$, one can proceed to calculate the dimensionless real contact area using Eqs. 6 & 7 which can be written in the following general form:

$$\overline{A}_* = \frac{A_*}{bl} = \pi \xi \int_{-\infty}^{+\infty} \Lambda_{GW, CEB, ZMC, KE, JG} (X) dX$$  \hfill (24)

where $\Lambda(X)$ corresponds to each micro-asperity model for the real area of contact in dimensionless form. The GW, CEB, ZMC, KE and JG micro-asperity real contact area equations 1, 8, 12, 13 and 14 in their dimensionless form, are summarized in in Table 4.4. It should be noted that since the maximum real area of contact is equal to nominal Hertzian contact area, i.e. $2bl$, the maximum allowable magnitude of $A_*$ in Eq. 24 equals to 2 not 1.

Table 4.4. Real area of contact models in dimensionless form

<table>
<thead>
<tr>
<th>Model</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\overline{\Lambda}_{GW}(\overline{h})$</td>
<td>$\int \frac{\pi}{\xi} \frac{G}{\xi} (\overline{\sigma})^{y^<em>} (\xi^</em>) dz^*$</td>
</tr>
<tr>
<td>$\overline{\Lambda}_{CEB}(\overline{h})$</td>
<td>$\int \frac{\pi}{\xi} \frac{G}{\xi} (\overline{\sigma})^{y^<em>} (\xi^</em>) dz^* + \int \frac{\pi}{\xi} \frac{G}{\xi} (\overline{\sigma} - \overline{\sigma}_1) \frac{G}{\xi} dz^*$</td>
</tr>
<tr>
<td>$\overline{\Lambda}_{ZMC}(\overline{h})$</td>
<td>$\int \frac{\pi}{\xi} \frac{G}{\xi} (\overline{\sigma})^{y^<em>} (\xi^</em>) dz^* + \int \frac{\pi}{\xi} \frac{G}{\xi} (\overline{\sigma} - \overline{\sigma}_1)^2 (\overline{\sigma} - \overline{\sigma}_1) \frac{G}{\xi} dz^*$</td>
</tr>
<tr>
<td>$\overline{\Lambda}_{KE}(\overline{h})$</td>
<td>$\left{ \int \frac{\pi}{\xi} \frac{G}{\xi} (\overline{\sigma})^{y^<em>} (\xi^</em>) dz^* + 0.95 \overline{\sigma}_c \frac{G}{\xi} \frac{G}{\xi} (\overline{\sigma}_c) \frac{G}{\xi} dz^* \right}$</td>
</tr>
<tr>
<td>$\overline{\Lambda}_{JG}(\overline{h})$</td>
<td>$\int \frac{\pi}{\xi} \frac{G}{\xi} (\overline{\sigma})^{y^<em>} (\xi^</em>) dz^* + 2 \int \frac{\pi}{\xi} \frac{G}{\xi} (\overline{\sigma})^{y^<em>} (\xi^</em>) dz^*$</td>
</tr>
</tbody>
</table>

Figure 4.7 illustrates the dimensionless real area of contact as a function of the load for two different surface properties, i.e. smooth and rough. The dimensionless real area of contact increases with the load, as expected. For the smooth surface, all of the models predict close results while for the rough one the CEB model predicts the largest values and the GW model gives lowest ones. The KE and JG models, although different, show comparatively close predictions for the real area of contact.
It is noteworthy to mention that the experimental evidence [39], [43] shows the existence of a linear relationship between the real area of contact and the load ($A_*(W) \propto W^{(a-1)}$). Here, if one plots the real area of contact ($A_*$) instead of $\bar{A}_*$ as a function of the load, the linear relationship between the load and the real area of contact becomes evident for the case of line contact. For
brevity, only the results of curve fitting are presented here for all loads and four different surface properties. For the GW, CEB, ZMC, KE and JG models, the values of $\alpha$ are 0.94, 0.96, 0.96, 0.98 and 1.05 respectively indicating that the KE model best satisfies the proportionality between the load and the real area of contact.

4.4.4. Predictive expressions

Since the calculations of the different micro-asperity contact models along with the bulk deformation of the surface demand considerable time and effort to develop, deriving convenient equations based on the numerical results should be of an interest to the tribology community. Accordingly, in this section, based on the numerical results for the deformation of dry rough line contact, formulas for the estimation of the maximum contact pressure, contact half width and real area of contact are presented. These expressions can be then applied to easily estimate the friction and wear for lubricated contact using the load sharing concept [26, 30, 49] along with adhesive wear of unlubricated contact [50].

It should be noted that for the micro-asperity deformation three models of GW, KE and JG are adopted. Previously, Gelinck and Schipper (GS) [25] provided curve-fit equations for the rough line contact configuration based on GW model. Here, considering the recently developed asperity contact models, i.e. KE and JG, material’s hardness or yield stress are also taken into consideration. In addition, GW model is again curve-fitted primarily because of three reasons. First, the GS model assumes that the product of $\eta\beta\sigma$ is constant and equal to $\zeta = 0.05$, but in this study it is an input parameter since experimental calculations show that it falls within the range of 0.03 and 0.1 [51]. This parameter affects the real area of contact directly (see Eq. 24). Second, the distance between the summit and surface height ($y_s$) was not considered by GS model and finally here due to the nature of elasto-plastic asperity contact models, different non-dimensional parameters are selected so it would be useful to provided simple formulas for GW model in addition to KE and JG. Since KE assumes that hardness is constant, an input for the associated curve-fit formulas is the material’s hardness. In comparison, formulas based on the JG model employ the material’s yield stress. Nonetheless, as mentioned before, these two models predict very close results for the line contact problem when the aforementioned relationship, i.e. $H=2.8S_y$, between hardness and yield stress is assumed.

Table 4.5. Non-dimensionalized parameters and their ranges used in the curve fitting

<table>
<thead>
<tr>
<th>$\frac{\bar{\sigma}}{\beta}$</th>
<th>$\zeta = \bar{n}\beta\bar{\sigma}$</th>
<th>$\bar{\sigma}$</th>
<th>$\bar{\Omega}$</th>
<th>$\Omega$</th>
<th>$\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum</td>
<td>$5 \times 10^{-4}$</td>
<td>$3 \times 10^{-2}$</td>
<td>$1 \times 10^{-7}$</td>
<td>$5 \times 10^{-6}$</td>
<td>$2 \times 10^{-3}$</td>
</tr>
<tr>
<td>Maximum</td>
<td>$2 \times 10^{-1}$</td>
<td>$1 \times 10^{-1}$</td>
<td>$1 \times 10^{-3}$</td>
<td>$4 \times 10^{-3}$</td>
<td>$5 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

Table 4.5 provides the range of the parameters used for the numerical curve fitting. After considerable numerical simulations, the maximum pressure, contact width, and the real area of
contact are curve-fitted based on the GW, KE and JG models and the following formulas are derived. The coefficients for each equation as well as the maximum error due to curve fitting are provided in Table 4.6. Errors associated with these formulas are small, about 5-6% for $\bar{P}_{\text{max}}$ and $\bar{b}_{0.01}$ and slightly greater than 8-10% for the dimensionless real area of contact.

$$\bar{P}_{\text{max}} = \frac{1}{\sqrt{1 + (a\bar{P}^b\beta^c\bar{e}^d\bar{W}^e\Omega^f\gamma^g)^h}}$$

$$\bar{b}_{0.01} = \frac{1}{\sqrt{1 + (a\bar{P}^b\beta^c\bar{e}^d\bar{W}^e\Omega^f\gamma^g)^h}}$$

$$\bar{A}_{GW} = \frac{1}{1 + (a\bar{P}^b\beta^c\bar{e}^d\bar{W}^e\Omega^f\gamma^g)^h} \left[ i + j \log(\bar{P}\beta\sigma) \right]$$

$$\bar{A}_{KE, JG} = \frac{(1 + h \log(\bar{P}\beta\sigma))}{1 + (a\bar{P}^b\beta^c\bar{e}^d\bar{W}^e\Omega^f\gamma^g)^h}$$

### Table 4.6. Coefficients for the curve-fit equations

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$e$</th>
<th>$f$</th>
<th>$g$</th>
<th>$h$</th>
<th>$i$</th>
<th>$j$</th>
<th>$k$</th>
<th>error</th>
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</thead>
<tbody>
<tr>
<td>$\bar{P}_{\text{max}}$</td>
<td>0.8464</td>
<td>0.0837</td>
<td>0.0183</td>
<td>-0.5556</td>
<td>(b-d)</td>
<td>0.0000</td>
<td>0.0000</td>
<td>-1.3782</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>5%</td>
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<tr>
<td>$\bar{b}_{0.01}$</td>
<td>4.1146</td>
<td>-0.0028</td>
<td>0.0439</td>
<td>0.8813</td>
<td>(b-d)</td>
<td>0.0000</td>
<td>0.0000</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>6%</td>
</tr>
<tr>
<td>$\bar{A}_{*}$</td>
<td>0.4694</td>
<td>-0.3463</td>
<td>0.1210</td>
<td>-0.7813</td>
<td>(b-d)</td>
<td>-</td>
<td>-</td>
<td>-1.7522</td>
<td>3.1934</td>
<td>0.6107</td>
<td>-</td>
<td>8%</td>
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<tr>
<td>$\bar{P}_{\text{max}}$</td>
<td>1.1188</td>
<td>-0.1531</td>
<td>-0.1203</td>
<td>0.6304</td>
<td>-0.7161</td>
<td>-0.1423</td>
<td>0.0000</td>
<td>1.1396</td>
<td>-</td>
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<td>-</td>
<td>6%</td>
</tr>
<tr>
<td>$\bar{b}_{0.01}$</td>
<td>3.0826</td>
<td>-0.0310</td>
<td>-0.0175</td>
<td>0.8375</td>
<td>(c-d)</td>
<td>-0.0790</td>
<td>0.0000</td>
<td>-</td>
<td>-</td>
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<td>6%</td>
</tr>
<tr>
<td>$\bar{A}_{*}$</td>
<td>5.9807</td>
<td>-0.3855</td>
<td>-0.3807</td>
<td>-0.3953</td>
<td>-0.4767</td>
<td>1.0169</td>
<td>0.0000</td>
<td>-0.0072</td>
<td>2.3097</td>
<td>1.1966</td>
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<table>
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<th>$h$</th>
<th>$i$</th>
<th>$j$</th>
<th>$k$</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{P}_{\text{max}}$</td>
<td>1.1724</td>
<td>-0.2319</td>
<td>-0.1963</td>
<td>0.8506</td>
<td>-0.9605</td>
<td>0.0000</td>
<td>-0.1979</td>
<td>0.8417</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>6%</td>
</tr>
<tr>
<td>$\bar{b}_{0.01}$</td>
<td>3.1175</td>
<td>-0.0435</td>
<td>-0.0407</td>
<td>0.8791</td>
<td>(c-d)</td>
<td>0.0000</td>
<td>-0.0962</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>6%</td>
</tr>
<tr>
<td>$\bar{A}_{*}$</td>
<td>3.3612</td>
<td>-0.2687</td>
<td>-0.1299</td>
<td>-0.4190</td>
<td>-0.5550</td>
<td>0.0000</td>
<td>1.3520</td>
<td>-0.2570</td>
<td>9.0390</td>
<td>0.0000</td>
<td>1.1033</td>
<td>10%</td>
</tr>
</tbody>
</table>

### 4.4.4.1. Pressure Distribution Function

Having obtained the maximum contact pressure and the contact width expressions, it would be useful if one derives a universal expression for the pressure profile for dry rough line contacts. As can be seen in Figures 4.2&4.3 pressure profiles for medium, rough and very rough surfaces are very similar to Gaussian distribution. In contrast, the profile for smooth surface deviates from
Gaussian distribution, when the maximum pressure approaches unity. The general form of pressure distribution can be written as:

\[
\bar{p}(X) = \bar{p}_{\text{max}} \left( 1 - \left( \frac{X}{\bar{b}_{0.01}} \right)^2 \right)^{\gamma} \quad X \leq \bar{b}_{0.01}
\]

(29)

which simply yields Hertzian solution for \( \gamma = 0.5 \) and \( \bar{b}_{0.01} = 1 \) while for greater values of \( \gamma \), follows the Gaussian distribution. Applying dimensionless force balance equation 20 for the half of the contact configuration – since the pressure profile is symmetric – and ignoring pressure values beyond the point greater than \( \bar{b}_{0.01} (X > \bar{b}_{0.01}) \), one can find the value of \( \gamma \) as follows:

\[
\frac{\pi}{4} = \int_0^{\bar{b}_{0.01}} \bar{p}_{\text{max}} \left( 1 - \left( \frac{X}{\bar{b}_{0.01}} \right)^2 \right)^{\gamma} dX = \frac{\bar{p}_{\text{max}} \bar{b}_{0.01} \sqrt{\pi} \Gamma(\gamma + 1)}{2 \Gamma \left( \gamma + \frac{3}{2} \right)} \quad X \leq \bar{b}_{0.01}
\]

(30)

where \( \Gamma(\gamma) \) is the gamma function. Explicit solution for \( \gamma \) is unavailable; therefore, using numerical curve fitting for different practical ranges of \( \gamma \) and \( \bar{p}_{\text{max}} \times \bar{b}_{0.01} \), the following expression is obtained with the maximum error of less than 2%:

\[
\gamma = -0.7717 + 1.2773 \left( \bar{p}_{\text{max}} \bar{b}_{0.01} \right) \quad 1 \leq \bar{p}_{\text{max}} \times \bar{b}_{0.01} < 2
\]

(31)

4.4.4.2. Example

Consider two cases of the contact of a cylinder and a flat plate reported in Ref. [48]. First case is the contact of a very smooth steel cylinder and rough copper plate whereas the second one is the contact of a very smooth steel cylinder and rough steel plate. The corresponding data are summarized in Table 4.7.

Table 4.7. Load, dimensional, surface and mechanical properties for the rough plate and the cylinders [48]

<table>
<thead>
<tr>
<th>Case #1, W=120 (N)</th>
<th>Case #2, W=81 (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Plate (Cu)</td>
</tr>
<tr>
<td>Plate (St)</td>
<td>-</td>
</tr>
<tr>
<td>Cylinder (#1)</td>
<td>1.5</td>
</tr>
<tr>
<td>Cylinder (#2)</td>
<td>1.5</td>
</tr>
</tbody>
</table>

For brevity only the calculations pertinent to the first case is presented in details. In addition to the necessary parameters for calculating contact characteristics, plasticity index are also computed below. The calculation details are as follows:
\[
\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{1.5 \times 10^{-3}} + \frac{1}{\infty} = R = 1.5 \times 10^{-3}
\]

\[
\frac{1}{E'} = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} = E' = 7.867 \times 10^{10}
\]

\[
\bar{\sigma} = \frac{\sigma}{R} = 0.966 \times 10^{-3} \quad \bar{\beta} = \frac{\beta}{R} = 18.66 \times 10^{-3} \quad \bar{n} = nR^2 = 3.15 \times 10^{3} \quad \bar{\xi} = n\sigma\beta = 0.057
\]

\[
\bar{W} = \frac{W}{lE'R} = 2.26 \times 10^{-4} \quad \Omega = \frac{H}{E'} = 1.61 \times 10^{-2}
\]

\[
\psi = \frac{2}{\pi K \Omega} \sqrt{\frac{\bar{\sigma}}{\bar{\beta}}} \left( 1 - \frac{3.71693 \times 10^{-4}}{\bar{\xi}} \right) = 14.5
\]

\[
\bar{E}_{0.01}^{KE} = \sqrt{1 + 3.0826 \bar{n}^{-0.831} \bar{\beta}^{-0.0175} \bar{\sigma}^{-0.8375} \bar{W}^{-0.855} \Omega^{-0.079}} = 3.86 \quad b = \sqrt{\frac{4WR}{\pi lE'}} = 2.54 \times 10^{-5} \text{ m}
\]

\[
b_{0.01} = b \times \bar{E}_{0.01} = 2.54 \times 10^{-5} \times 3.86 = 9.8 \times 10^{-5} \text{ m}
\]

\[
\bar{p}_{\text{max}} = \frac{1}{\sqrt{1 + \left( \frac{1.1188 \bar{n}^{-0.1531} \bar{\beta}^{-0.1203} \bar{\sigma}^{-0.6304} \bar{W}^{-0.7161} \Omega^{-0.1423} \right)^{0.1396}}} = 0.376
\]

\[
\gamma = -0.7717 + 1.2773 \left( \bar{p}_{\text{max}} \bar{E}_{0.01} \right)^2 = 1.918 \quad \bar{p}(X) = 0.376 \left( 1 - \left( \frac{X}{3.86} \right)^2 \right)^{1.918} \quad X \leq 3.86
\]

The contact half width for the two cases based on the curve-fit equation, numerical calculations and the experimental results are presented in Table 4.8. As seen, the error of curve-fit equation and numerical calculation with respect to experimental result are acceptable.

<table>
<thead>
<tr>
<th>Case</th>
<th>Experimental</th>
<th>Numerical</th>
<th>Error</th>
<th>Curve-fit</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$10.5 \times 10^{-3}$</td>
<td>$10.4 \times 10^{-3}$</td>
<td>1%</td>
<td>$9.8 \times 10^{-5}$</td>
<td>6.6%</td>
</tr>
<tr>
<td>2</td>
<td>$6.4 \times 10^{-3}$</td>
<td>$5.7 \times 10^{-3}$</td>
<td>11%</td>
<td>$5.63 \times 10^{-5}$</td>
<td>12%</td>
</tr>
</tbody>
</table>

Plotted in Figure 4.8 are the pressure profiles for the aforementioned cases. In this figure, pressure profile based on the curve-fit equations 25, 26, 29 and 31 and full numerical calculations are demonstrated. As seen, there is a close agreement between them showing the good accuracy of the approximated function.
4.4.5. Contact of two rough surfaces

All the aforementioned relationships were based on the assumptions in which one of the surfaces is ideally smooth and the other one is rough. It would be useful to discuss the practical situation where both surfaces are rough. As shown by Greenwood and Tripp [13] the contact of two rough surfaces can be treated as a contact of one rough surface with equivalent (effective) parameters and an ideally smooth surface. Based on the relationships provided by McCool [37] and by rearranging them, it is possible to extend all contact formulations and curve-fitted expressions to handle the contact of two rough surfaces.

It has been shown that the surface parameters $\sigma$, $\beta$ and $n$, can be expressed in terms of three quantities, $m_0$, $m_2$ and $m_4$ called “spectral moments” based on the surface profile $\lambda(x)$ [37]:

\[
\begin{align*}
    m_0 &= AVG \left[ \lambda^2(x) \right] = \sigma^2 \tag{33a} \\
    m_2 &= AVG \left[ \left( \frac{d\lambda(x)}{dx} \right)^2 \right] \tag{33b} \\
    m_4 &= AVG \left[ \left( \frac{d^2\lambda(x)}{dx^2} \right)^2 \right] \tag{33c}
\end{align*}
\]

Assuming that the $\lambda(x)$ is a Gaussian random variable, the asperity radius and density is given by [36], [52]:

Figure 4.8. Pressure profile based on the numerical calculation and approximated pressure distribution function
\[ \beta = 0.375 \sqrt[3]{\frac{\pi}{m_4}} \] (34)

\[ n = \frac{m_4}{m_2 6\pi \sqrt{3}} \] (35)

The equivalent spectral moments of two rough surfaces can then be written as follows [37]:

\[ (m_0)_{eq} = (m_0)_1 + (m_0)_2 \] (36a)

\[ (m_2)_{eq} = (m_2)_1 + (m_2)_2 \] (36b)

\[ (m_4)_{eq} = (m_4)_1 + (m_4)_2 \] (36c)

Considering equation set 36 and Eqs. 33a, 34 and 35, the equivalent roughness, asperity radius and asperity density can be written as:

\[ \sigma_{eq} = \sqrt{\sigma_1^2 + \sigma_2^2} \] (37)

\[ \frac{1}{\beta_{eq}} = \sqrt{\frac{1}{\beta_1^2} + \frac{1}{\beta_2^2}} \] (38)

\[ \frac{1}{n_{eq}} = \frac{1}{n_1} \left( \frac{\beta_{eq}}{\beta_1} \right)^2 + \frac{1}{n_2} \left( \frac{\beta_{eq}}{\beta_2} \right)^2 \] (39)

### 4.5. Conclusions

Different statistical micro-contact models including Greenwood-Williamson (GW), Chang-Etsion-Bogy (CEB), Zhou-Maietta-Chang (ZMC), Kogut-Etsion (KE) and Jackson-Green (JG) are employed along with the elastic bulk deformation formula of the line contact to determine the pressure profile, contact width and the real area of contact as a measure of the impact of surface roughness on the contact characteristics.

Results indicates that elastic-plastic micro-contact models that take asperity’s elastic-plastic effects into account predict a lower maximum normal pressure, a greater contact width, and a larger real contact area compared to the predictions of the GW model, which assumes that asperities deform elastically. It is also found that the ZMC, KE and JG models predict closer results as compared to the GW and CEB. The KE model shows the most linear relationship between the load and the real contact area. The KE and JG models are found to be appropriate statistical contact models for analyzing the line contact behavior as opposed to other models since the KE and JG models considers all different regimes of deformation in comparison to GW, CEB and even ZMC. KE and JG models describe the physical behavior of the materials in more detail. Moreover, according to previous studies, the prediction by the KE and JG models are closer to the reported experimental results.

In addition, useful expressions for the prediction of maximum contact pressure, contact width, real area of contact and pressure distribution are proposed based on the GW, KE and JG models.
Using these formulas one can easily estimate the contact characteristics of line contact configuration.

**Nomenclature**

- $A$ : real contact area between two rough flat surfaces, $m^2$
- $A_n$ : nominal contact area between two rough flat surfaces, $m^2$
- $\overline{A}$ : dimensionless real area of contact for the contact of two flat surfaces, $A/A_n$
- $A_*$ : real contact area for the contact of curved surfaces, $m^2$
- $\overline{A}_*$ : dimensionless real contact area for the contact of curved surfaces
- $a$ : contact radius pertinent to the contact of a single asperity and a rigid flat
- $b$ : Hertzian contact half width, m
- $b_{\text{eff}}$ : effective contact half width, m
- $\overline{b}_{\text{eff}}$ : dimensionless effective contact half width
- $b_{0.01}$ : contact half width beyond which pressure is negligible ($\overline{p} \leq 0.01$)
- $\overline{b}_{0.01}$ : dimensionless contact half width
- $d$ : separation based on asperity heights ($h - y_s$), m
- $E_1, E_2, E'$ : moduli of elasticity of first and second surface and the effective modulus of elasticity, GPa
- $H$ : hardness of the softer material, MPa
- $h$ : separation based on surface heights, m
- $h_{00}$ : constant in the separation relationship
- $\overline{h}$ : dimensionless separation
- $\overline{h}_{00}$ : constant in the dimensionless separation relationship
- $K_{ij}$ : influence coefficient
- $K$ : maximum contact pressure factor
- $k$ : mean contact pressure factor
- $l$ : contact depth, m
- $m_0, m_2, m_4$ : spectral moments of a rough surface
- $N$ : number of nodes
- $P$ : contact load for the contact of two flat surfaces with constant mean separation, N
- $p$ : nominal pressure for the contact of two flat surfaces with constant mean separation, $P/A_n$, MPa
- $\overline{p}$ : dimensionless pressure
- $p_H$ : maximum Hertzian pressure, MPa
- $\overline{p}_{\text{max}}$ : maximum dimensionless pressure for the rough surfaces contact
- $r$ : ratio of interface at the onset of fully plastic deformation to the interface at initial yielding
- $R$ : effective radius of curvature, m
- $S_y$ : yield stress, MPa
- $W$ : total normal force for the contact of a cylinder and a flat surface, N
- $\overline{W}$ : dimensionless total normal force
- $x$ : spatial coordinate component along the contact width, m
\( X \)  
dimensionless spatial coordinate component along the contact width, \( x/b \)

\( y \)  
distance between the mean of summit heights and that of the surface heights, m

\( z \)  
asperity height measured from the mean line of summit heights, m

\( z^* \)  
dimensionless asperity height, \( z/\sigma \)

\( \alpha \)  
exponent coefficient for real area of contact and load relationship

\( \nu_1, \nu_2 \)  
Poisson ratios of the first and second surface

\( n \)  
asperity density, \( m^2 \)

\( \bar{n} \)  
dimensionless asperity density

\( \beta \)  
asperity radius, m

\( \bar{\beta} \)  
dimensionless asperity radius

\( \bar{\sigma} \)  
dimensionless standard deviation of surface heights

\( \sigma_s \)  
standard deviation of summit (asperity) heights, m

\( \bar{\sigma} \)  
standard deviation of surface heights, m

\( \sigma \)  
asperity interface, m

\( \bar{\sigma} \)  
dimensionless interface, \( \omega/\sigma \)

\( \omega_c \)  
critical interface according to the CEB, ZMC and KE models, m

\( \omega_c' \)  
critical interface according to the JG model, m

\( \omega_c^* \)  
dimensionless critical interface according to the CEB, ZMC and KE models

\( \omega_c^* \)  
dimensionless critical interface according to the JG model

\( \Omega \)  
ratio of the softer surface hardness to the effective elasticity modulus, \( H/E' \)

\( Y \)  
ratio of the softer material yield stress to the effective elasticity modulus, \( S_y/E' \)

\( \lambda(x) \)  
surface profile

\( \Gamma(\gamma) \)  
gamma function

\( \Phi(h) \)  
function representing different micro-asperity contact models for the contact load

\( \Phi(h) \)  
dimensionless function representing different micro-asperity contact model for the contact load

\( \bar{\Lambda}_{GW}(\bar{h}) \)  
dimensionless function representing different micro-asperity contact model for real contact area

\( \phi^*(z^*) \)  
dimensionless standard normal distribution function

\( \zeta \)  
\( n\beta\sigma \)

\( \psi \)  
plasticity index

\( \gamma \)  
exponential coefficient of universal pressure distribution function

4.6. References


Chapter 5: Elliptical Point Contact of Rough Surfaces: Contact Behaviors and Predictive Formulas

5.1. Introduction

Nearly all engineering surfaces are rough at microscopic level regardless of their method of production. As a result, the contact of two surfaces takes place at discrete micro protrusions called asperity tips. These interactions have long been of keen research interest as they play a fundamental role on the performance of different tribological components ranging from bearings and gears to hard disk drives and biomedical transplants. Surface roughness affects the contact characteristics such as the pressure and sub-surface stress distribution, contact dimensions, real area of contact and contact resistance. These factors directly affect the load-carrying capacity, friction (traction) force, electro-thermo-mechanical, wear and fatigue behavior of tribo-components.

Several approaches have been employed to predict rough surface contact characteristics [1-5]. Broadly, they can be classified into two categories: statistical and deterministic. Depending on the application and the desired results, each of these two approaches has its own advantages and disadvantages. The deterministic approach involves simulation of the real surface profile and hence provides a more detailed description of the pressure, deformation and the sub-surface stress distribution. However, it requires direct measurement of the surface profile and extensive numerical calculations. On the other hand, the statistical approach, although relatively approximate, is more convenient to formulate as it only requires the specification of a few surface parameters and thus lends itself to generalization of predictions for different geometries, surface properties and loading conditions.

Pioneering contribution was made by Greenwood and Williamson (GW) in 1966 who modeled the problem of nominally-flat rough surfaces by the statistical extension of the elastic Hertzian solution for an individual hemispherical asperity to a population of asperities with Gaussian distribution [6]. While many attempts have been done to improve the original GW model for other geometries and asperity distributions, a comparative numerical study by McCool [7] in 1986 revealed that despite many simplifying assumptions in the GW model, it gives promising results. However, more recent research shows that because of the assumption that asperity deformation is purely elastic, the GW model is not amenable for surfaces with high plasticity index or when the load is high. Twenty years after Greenwood and Williamson, Chang et al. [8] developed a model—the so-called CEB model—that extended the solution to the elasto-fully plastic deformation regime using the concept of volume conservation for plastically deformed asperities. By introducing the critical interface at the inception of the plastic...
deformation, they assumed that an asperity deforms elastically below the critical interface, and fully plastically above this value, while ensuring that the volume of the plastically deformed asperity is conserved. A treatment methodology to handle the intermediate regime of deformation was later proposed by Zhao et al. [9] who developed an elastic-elasto/plastic-fully plastic model —commonly referred to as ZMC model— that bridges the elastic and plastic behavior of the asperity using a mathematical function. More recent investigations involve the use of the finite element analysis as reported by Kogut and Etsion [10] (KE model), [11] and Jackson and Green [12], [13] (JG model). The recent study of KE and JG lend themselves to the development of useful empirical elasto-plastic models that consider different deformation regimes.

Let us now turn our attention to the contact of rough curved bodies. While the macro-level contact of two topographically smooth curved bodies can be adequately described by the well-known Hertzian theory, it is far more complex when surfaces are rough and the pertinent contact properties deviate from those predicted by the Hertzian approach. Several studies have been done on the evaluation of roughness effect on the contact of curved bodies, including line-contact problems [14-17], i.e. the contact of a cylinder and a plate, and circular point-contact problem as a special case of elliptical point contact, e.g., contact of a ball and a plate, [18-27]. In contrast, there is still a paucity of information on the general case of elliptical point contact of rough surfaces (e.g. contact of an ellipsoid with a plate). Further, very little attempts have been made to provide useful relationships, akin to Hertzian equations, that can be readily applied to estimate the contact characteristics in the contact of two rough curved bodies.

The first analytical study on the contact of rough curved bodies was performed by Greenwood and Tripp [18] who employed the GW asperity contact model together with the bulk surface deformation for circular point contact. Their model shares the same assumptions as the GW theory. In an interesting attempt, Gelinck and Schipper (GS) extended the approach of Greenwood and Tripp to the deformation of rough line contact using the GW model [15]. Gelinck and Schipper [15] also provided convenient empirical formulas for contact parameters such as the maximum pressure and the contact width based on the numerical solutions of the rough line contact. These formulations are utilized extensively in the treatment of mixed lubrication problems [28-31] that incorporate the load-sharing concept. Recently, Beheshti and Khonsari [17] extended the GS approach to elasto-plastic deformation regime. They provided a comparison among different statistical asperity contact models as applied to the deformation of rough line contact and proposed formulas for analyzing the line-contact based on the elasto-plastic deformation of the asperities that takes into account the surface hardness effect.

This paper applies the statistical asperity micro-contact models of GW and KE to the deformation of elliptical point contact. The GW model is chosen because of its simplicity and popularity. The more recent elasto-plastic KE model shows good agreement with experimental
results [32]. In addition, the previous study by the authors for the line-contact problem revealed that this model is very promising for the contact of curved-surfaces. Here, the numerical simulation involves the simultaneous solution of the asperity interaction with the elastic bulk deformation of the surface using the Jacobi relaxation technique. It predicts the apparent pressure distribution, compliance and real area of contact, as well as the contact dimensions for the elliptical point contact configuration. The analysis presented take into account the elastic or elasto-plastic behavior of the asperities. In addition, the results of extensive sets of simulations are used to derive expressions for the prediction of the contact characteristics including maximum contact pressure, contact dimensions, real area of contact, contact compliance and pressure distribution function. These expressions are useful since they provide fast and easy-to-use predictive relationships for the contact characteristics for the general case of elliptical point contact. These predictions are useful for the development of optimum and sustainable design in applications that require analyses involving tribological performance, thermo-mechanical degradation and electrical resistance.

5.2. Statistical micro-contact models

The modeling methodology in statistical approach is applied in two stages. The first stage is to develop an appropriate relationship for the contact of a single asperity and the second stage is to extend the solution for that individual asperity to an ensemble of asperities in order to determine the contact behavior of a rough surface. This is achieved through employing the statistical distribution of asperities and specification of surface parameters. The basic assumptions in statistical treatment of micro contacts are [6]: The rough surface is isotropic; asperities are spherical, at least near their summits; all asperities have the same radius of curvature; the asperity heights follow the Gaussian distribution; each individual asperity deforms separately and there is no interaction among the neighboring asperities; and that the bulk surface deformation below the individual asperity is negligible.

Before delving into the methodology for treating the point-contact problem and the corresponding results, we begin by first presenting a brief description of the asperity models. For brevity, interested reader is referred to Ref. [17] for detailed review on the development of different statistical asperity contact models. Here, we only present the widely-used GW model and the recent and more KE model. Later, these models are utilized to predict the contact behaviors in elliptical point-contact geometry.

5.2.1. Greenwood and Williamson (GW)

Greenwood and Williamson [6] developed a model for rough surfaces based on the elastic Hertzian solution of a single asperity contact and the extension of the results to a population of asperities with Gaussian distribution of heights. They proposed the following relationships for the contact load and real area of contact of a rough flat and an ideally smooth flat surface:
\[ P(h) = n \beta \sigma E' A_n \frac{4}{3} \sqrt{\pi} \frac{\sigma}{\bar{\sigma}} \left( \frac{h - y_s}{\sigma} \right)^{3/2} \int_{h-y_s}^{\infty} (\bar{\sigma})^{3/2} \phi^*(z^*) dz^* = \Phi_{GW}(h) \]

\[ A(h) = \pi \bar{\sigma} A_n \int_{h-y_s}^{\infty} \frac{(\bar{\sigma})^3}{\sigma} \phi^*(z^*) dz^* = \Lambda_{GW}(h) \]

where

\[ \bar{\sigma} = n \beta \sigma \quad , \quad \phi^*(z^*) = \frac{1}{\sqrt{2\pi} \sigma_s} e^{-\frac{(z^*)^2}{2 \sigma_s^2}} \quad , \quad z^* = \frac{z}{\sigma} \quad , \quad \bar{\sigma} = z^* = \frac{h - y_s}{\sigma} \]

In Eq. (1), \( n \) and \( \beta \) denote the asperity density and radius respectively; \( \sigma \) is the standard deviation of the surface heights distribution (referred to surface roughness hereafter); \( \sigma_s \) is the standard deviation of summit heights distribution; \( A_n \) is the nominal contact area; \( h \) represents the separation between two surfaces; \( y_s \) denotes the distance between the mean line of the summit heights and that of the surface heights; and \( z \) is the asperity (summit) height measured from the mean line of summit heights (see Figure 5.1). \( E_1 \) and \( E_2 \) symbolize the moduli of elasticity, \( \nu_1 \) and \( \nu_2 \) are the Poisson’s ratios of the contacting surfaces. (See, for example, Refs. [8, 17] for more details). Symbols \( \Phi \) and \( \Lambda \) are the general representative of load-separation and real area-separation functions.

For the contact of isotropic surfaces with Gaussian distribution of surface heights, the following relationships exist between \( \sigma, \sigma_s \) and \( y_s \) [33, 34]:

\[ \frac{\sigma}{\sigma_s} = \frac{\bar{\sigma}}{\sqrt{\bar{\sigma}^2 - F}} \quad F = 3.71693 \times 10^{-4} \]

\[ \frac{y_s}{\sigma} = \frac{G}{\bar{\sigma}} \quad G = \frac{1}{\pi \sqrt{48}} \]

5.2.2. Kogut and Etsion (KE)

Recently, Kogut and Etsion [10] presented the results of a comprehensive finite element simulation for the deformation of a single asperity. Subsequently, they proposed empirical expressions which include different asperity deformation regimes from purely elastic to purely plastic. Given the relationships for a single asperity contact, they extended the approach to the contact of rough surfaces using statistical method [11]:

75
\[ P(h) = n \beta E A_n \left( \frac{4}{3} \sqrt{\frac{\sigma}{\beta}} \int_{-\frac{2}{3}}^{h-y_\perp + \omega_\perp^*} \left( \frac{h-y_\perp + \omega_\perp^*}{\sigma} \right)^{3/2} \phi^*(z) \, dz \right) + \frac{2}{3} \times 1.03 \pi K \omega_c^{-0.425} \left( \frac{H}{E'} \right) \frac{h-y_\perp + \omega_\perp^*}{\sigma} \int_{-\frac{2}{3}}^{h-y_\perp + \omega_\perp^*} \left( \frac{h-y_\perp + \omega_\perp^*}{\sigma} \right)^{3/2} \phi^*(z) \, dz \]

\[ + \frac{2}{3} \times 1.4 \pi K \omega_c^{-0.263} \left( \frac{H}{E'} \right) \frac{h-y_\perp + 110 \omega_c^*}{\sigma} \int_{-\frac{2}{3}}^{h-y_\perp + 110 \omega_c^*} \left( \frac{h-y_\perp + 110 \omega_c^*}{\sigma} \right)^{3/2} \phi^*(z) \, dz + 2 \pi \frac{H}{E'} \int_{-\frac{2}{3}}^{h-y_\perp + 110 \omega_c^*} \left( \frac{h-y_\perp + 110 \omega_c^*}{\sigma} \right)^{3/2} \phi^*(z) \, dz = \Phi_{KE}(h) \]

\[ A(h) = \pi n \beta E A_n \left( \frac{h-y_\perp + \omega_\perp^*}{\sigma} \int_{-\frac{2}{3}}^{h-y_\perp + \omega_\perp^*} \left( \frac{h-y_\perp + \omega_\perp^*}{\sigma} \right)^{3/2} \phi^*(z) \, dz + 0.93 \pi n \beta E A_n \omega_c^{-0.136} \int_{-\frac{2}{3}}^{h-y_\perp + 110 \omega_c^*} \left( \frac{h-y_\perp + 110 \omega_c^*}{\sigma} \right)^{3/2} \phi^*(z) \, dz + 0.94 \omega_c^{-0.146} \int_{-\frac{2}{3}}^{h-y_\perp + 110 \omega_c^*} \left( \frac{h-y_\perp + 110 \omega_c^*}{\sigma} \right)^{3/2} \phi^*(z) \, dz + 2 \pi \frac{H}{E'} \int_{-\frac{2}{3}}^{h-y_\perp + 110 \omega_c^*} \left( \frac{h-y_\perp + 110 \omega_c^*}{\sigma} \right)^{3/2} \phi^*(z) \, dz = \Lambda_{KE}(h) \]

where

\[ \omega_c = \frac{\omega_c}{\sigma} = \left( \frac{\pi KH}{2E'} \right)^2 \left( \frac{\beta}{\sigma} \right) \]

and, \( H \) is the hardness of the softer material, which for most metals is typically taken to be equal to 2.8 times the yield strength, \( S_y \), for untreated surfaces. \( K \) is the proportionality factor between the maximum contact pressure and the hardness at the onset of plastic deformation (\( P_{yield} = KH \)). Tabor [35] suggested a constant value of \( K=0.6 \), while other researchers showed that it depends on the Poisson’s ratio. For example, Chang et al. [36] proposed:

\[ K = 0.454 + 0.41\nu \]

In the current study the expression provided by Chang et al. [36] is used; nevertheless, for the typical values of the Poisson’s ratios, all of these relationships yield fairly close numbers for \( K \). The parameter \( \omega_c \) in Eq. (3) is the critical interface at the inception of the plastic deformation, which is originally introduced by Etsion and his coworkers in 1987 (CEB model) [8], where they assumed that below the critical interface the asperity deformation is completely elastic and purely plastic above it. Nevertheless in the recent model of Kogut and Etsion (KE), four regimes of deformation are distinguished. The first and the last ones are the elastic and fully plastic regimes which occur at \( \omega_c^* \) and \( 110 \omega_c^* \), respectively. The elasto–plastic regime –the second and third integrals in Eq. (3)– is divided into two distinct parts based on the evolution of the plastic core beneath the contact region for an individual hemispherical asperity contact. It is worth noting that the predictions of the KE model yield good agreement as compared to experimental results [32].

According to Greenwood and Williamson [6], the plasticity index is expressed as:
Using Eq. (2) and the definition of the critical interface (Eq. (3)), the plasticity index can be written as follows [8]:

\[ \psi = 2 \frac{E'}{\pi KH} \sqrt{\frac{\sigma}{\beta}} \left( 1 - \frac{F}{\bar{z}} \right)^{1/4} \] (6)

5.3. Application to elliptical point contact

In this section, we present the application of the described statistical asperity micro-contact models to the elliptical point contact problem which involves the bulk deformation of surface. In what follows the dimensionless parameters as well as the solution methodology are explained in details.

![Figure 5.1. Elliptical point contact of a rough flat surface and smooth ellipsoid](image)

5.3.1. Dimensionless parameters

5.3.1.1. GW formulation

The GW relationship (Eq. (1)) for elliptical point contact can be non-dimensionalized as follows:

\[
\bar{p}(\bar{h}) = \bar{\beta} \bar{\sigma} \left\{ 4 \frac{2\pi \kappa}{3 W} \frac{2\pi \kappa}{3W} \gamma^2 \int_{-\infty}^{\bar{h}} \bar{G} \left( z - \bar{h} \frac{A}{\bar{z}} \right)^{3/2} \phi^*(z^*) \, dz^* \right\} = \Phi_{GW}(\bar{h})
\] (7)

where

\[ \gamma = \frac{b}{R_x} = \frac{3 D W}{\pi \kappa (D + 1)} \] (8)

\[ \bar{G} = z^* - \frac{\bar{h} - y_s}{\sigma} = z^* - \left( \frac{\bar{h}}{\bar{\sigma}} - \frac{G}{\bar{\zeta}} \right) \]
with the following dimensionless parameters:

\[
\begin{align*}
\sigma &= \frac{\sigma}{R_x} \\
\bar{\sigma} &= \frac{\sigma}{R_x} \\
\bar{\beta} &= \frac{\beta}{R_x} \\
\bar{n} &= nR_x^2 \\
\bar{\xi} &= \bar{n} \bar{\beta} \bar{\sigma} = n\beta\sigma \\
\bar{p} &= \frac{p}{p_H} \\
\bar{h} &= \frac{h}{R_x} \\
\bar{W} &= \frac{W}{E'R_x^2} \\
D &= \frac{R_y}{R_x} \\ R_x &\geq R_y
\end{align*}
\]

where \( W \) denotes the total contact load, \( R_x \) and \( R_y \) represent the radii of curvature in the \( x \) and \( y \) directions; \( p_H \), \( a \) and \( b \) are, respectively, the maximum Hertzian pressure, the Hertzian contact half length (half of the contact dimension in the \( y \) direction) and contact half width (half of contact dimension in the \( x \) direction, see Figure 5.1). They are defined as follows.

\[
\begin{align*}
p_H &= \frac{3W}{2a\pi b} = \frac{3WE'R_x^2}{2\pi b^2} = \frac{3WE'}{2\pi\bar{\beta}^2} \\
b &= \sqrt[3]{\frac{3R_x DW\bar{\sigma}}{\pi E\kappa(D+1)}} = R_x \sqrt[3]{\frac{3DW\bar{\sigma}}{\pi\kappa(D+1)}} \\
\kappa &= \frac{a}{b} \quad a \geq b
\end{align*}
\]

where \( \kappa \) is the so-called ellipticity parameter, which is the ratio of the contact length to the contact width. The ellipticity parameter is only a function of \( D \) defined as:

\[
\kappa = \sqrt{(D+1)\tilde{F}} = \sqrt{(D+1)\tilde{S} - D}
\]

where \( \tilde{F} \) and \( \tilde{S} \) are elliptical integral of the first and second kind and are defined as:

\[
\begin{align*}
\tilde{F}(\kappa) &= \int_0^{\pi/2} \frac{d\tau}{\sqrt{1 - \left(1 - \frac{1}{\kappa^2}\right)\sin^2 \tau}} \\
\tilde{S}(\kappa) &= \int_0^{\pi/2} \sqrt{1 - \left(1 - \frac{1}{\kappa^2}\right)\sin^2 \tau} d\tau
\end{align*}
\]

Note that these integrals are the function of the ellipticity, hence an iterative procedure is needed to calculate \( \kappa \) based on \( D \).

In general, the contact of two ellipsoids can be treated as a contact of a flat plate and an ellipsoid with the effective (equivalent) radii in two perpendicular planes (Figure 5.1). In Eq. (9), \( R_x \) and \( R_y \) can be taken as the effective radii of curvature in \( x \) and \( y \) direction for the contact of two ellipsoids.

The aforementioned asperity formulas are applicable to the contact of two flat surfaces parallel to each other with constant mean separation. The point contact of a curved surface and a flat surface can be treated as a summation of discrete area each having a different but constant
distance from the flat surface. In other words, \( dA_n \) and \( dA \) can be considered as a small element of the nominal contact area and the real area of contact, respectively. Hence, for the contact of an ellipsoid against a flat surface, using the GW formulation, one can write:

\[
A_s = \int dA = \int_{-\infty}^{+\infty} \pi n \beta \sigma \int_{\frac{h-y}{\sigma}}^{\infty} \frac{G}{\sigma} \phi^*(z^*) dz \, dA_n
\]

\[
= \pi n \beta \sigma \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left( \int_{\frac{h(x)-y}{\sigma}}^{\infty} \frac{G}{\sigma} \phi^*(z^*) dz \right) dx \, dy
\]

and in dimensionless form:

\[
\bar{A}_s = \frac{A_s}{\pi ab} = \zeta \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left( \int_{\frac{h(X,Y)}{\sigma}}^{\infty} \frac{G}{\sigma} \phi^*(z^*) dz \right) dx \, dy
\]

5.3.1.2. KE formulation

Analogous expressions can be derived using the KE formulation and by introducing new dimensionless parameter \( \Omega \) (dimensionless hardness, \( \Omega = H/E' \)). The result in dimensionless form reads:

\[
\bar{h} = 2 \pi K \bar{h} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left( \int_{\frac{h(X,Y)}{\sigma}}^{\infty} \frac{G}{\sigma} \phi^*(z^*) dz \right) dx \, dy
\]

Similar to Eq. (14), one can derive the relationship for real area of contact based on KE micro-contact model in dimensionless form:

\[
\bar{A}_s = \zeta \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left( \int_{\frac{h(X,Y)}{\sigma}}^{\infty} \frac{G}{\sigma} \phi^*(z^*) dz \right) dx \, dy
\]
The following expression generalizes the asperity pressure according the GW, and KE models:

\[ p(h(x, y)) = \Phi_{GW, KE}(h(x, y)) \]  

(17)

where \( p(h) \) is the nominal (apparent) pressure for the contact of two flat rough surfaces with constant mean separation. The separation equation for the elliptical point contact is [37]:

\[ h(x, y) = h_0 + \frac{x^2}{2R_x} + \frac{y^2}{2R_y} + \frac{1}{\pi E' \sigma} \int \int_{-\infty}^{\infty} \int \int_{-\infty}^{\infty} \frac{p(s, q) ds dq}{\sqrt{(x - s)^2 + (y - q)^2}} \]  

(18)

where \( h_0 \) is a constant to be determined and \( x \) and \( y \) are the coordinates along the contact width and length. In addition, the force balance equation must be satisfied:

\[ W = \int \int_{-\infty}^{\infty} p(x, y) dx dy \]  

(19)

In non-dimensionalized form the above equations can be written as:

\[ \bar{p}(\bar{h}) = \bar{\pi} \bar{\beta} \bar{\sigma} \times \left[ \Phi_{GW, KE}(\bar{h}) \right] \]  

(20)

\[ \bar{h}(x, y) = \bar{h}_0 + \frac{X^2}{2} - \frac{\gamma^2}{2} + \frac{\kappa^2}{D} - \frac{\kappa^2}{D} + 1 \int \int_{-\infty}^{\infty} \int \int_{-\infty}^{\infty} \frac{\bar{p}(S, Q) dS dQ}{\sqrt{(X - S)^2 + \kappa^2 (Y - Q)^2}} \]  

(21)

\[ \frac{2\pi}{3} = \int \int_{-\infty}^{\infty} \bar{p}(X, Y) dX dY \]  

(22)

where \( X = x/b \), \( Y = y/a \) and \( \Phi(\bar{h}) \) in Eq. (21) corresponds to each micro-asperity contact model in dimensionless form.
To solve equations (20)-(22) simultaneously, we discretize them in a systematic way as follows:

\[
f_{i,j} = \overline{p}_{i,j} - \overline{p}_{i} \sigma \times \left( \Phi_{GW,KE} (\overline{h}_{i,j}) \right) \quad i = 1, \ldots, M
\]

\[
\overline{h}_{i,j} = \overline{h}_{00} + \frac{X_i^2}{2} \gamma^2 + \frac{Y_j^2}{2} \kappa^2 + \frac{2}{\pi^2} \gamma \pi \kappa (D+1) \sum_{k=1}^{M} \sum_{l=1}^{N} K_{ijkl} \overline{p}_{kl}
\]

\[
f_{MN+1} = \Delta X \Delta Y \sum_{i=1}^{M} \sum_{j=1}^{N} \overline{p}_{ij} - \frac{2\pi}{3} = 0
\]

(23)

where \( K_{ijkl} \) is defined as [38]:

\[
K_{ijkl} = \left| X_p \right| \text{arcsinh} \left( \frac{Y_p}{X_p} \right) + \left| Y_p \right| \text{arcsinh} \left( \frac{X_p}{Y_p} \right) - \left| X_m \right| \text{arcsinh} \left( \frac{Y_p}{X_m} \right) - \left| Y_p \right| \text{arcsinh} \left( \frac{X_m}{Y_p} \right)
\]

\[
-\left| X_p \right| \text{arcsinh} \left( \frac{Y_m}{X_p} \right) - \left| Y_m \right| \text{arcsinh} \left( \frac{X_p}{Y_m} \right) + \left| X_m \right| \text{arcsinh} \left( \frac{Y_m}{X_m} \right) + \left| Y_m \right| \text{arcsinh} \left( \frac{X_m}{Y_m} \right)
\]

and

\[
X_p = \frac{X_i - X_k + \Delta x}{2} \quad X_m = \frac{X_i - X_k - \Delta x}{2}
\]

\[
Y_p = Y_j - Y_l + \frac{\Delta y}{2} \quad Y_m = Y_j - Y_l - \frac{\Delta y}{2}
\]

In this set of expressions (Eq.(23)), for \( M \times N \) computational nodes (\( M \) nodes in \( X \) direction and \( N \) nodes in \( Y \) direction) there are \( M \times N + 1 \) unknowns (associated with \( M \times N + 1 \) equations) which should be determined. The unknowns represent the pressure at \( M \times N \) nodes and the constant \( \overline{h}_{00} \). The computational procedure begins by assuming an initial value of \( \overline{h}_{00} \) and a pressure distribution. Next, using the Jacobi relaxation technique, the equation set (23) is solved for the pressure distribution as well as the value of \( \overline{h}_{00} \) until the results between two successive iterations converge within a specified error. Here, the computational domain comprises of 200×200 nodes.

5.4. Results and discussion

5.4.1. Pressure distribution

Two types of surfaces (smooth and rough) are selected and the pressure profiles are determined based on both the GW and KE models. They are depicted in two representative XZ and YZ planes in Figure 5.2. As can be seen, the pressure profiles according to both models are different from the Hertzian pressure, revealing that Hertzian pressure distribution ceases to be valid for rough surfaces. It is evident that the smooth surface (Figures 5.2a&c) has a profile
closer to the Hertzian distribution as compared to the rough surface (Figures 5.2b&d). In fact, the results show the important role of the surface roughness in the pressure distribution.

Figure 5.2. Normalized pressure distribution for smooth and rough surfaces at $W = 1.5 \times 10^{-5}$, $\Omega = 0.015$ and $D = 5$ based on (a)&(b) GW and (c)&(d) KE models

The both the GW and KE models yield relatively similar results for smooth surface with the plasticity index of 1. However, when the plasticity index and the surface roughness increase, the GW model predicts higher maximum pressure than KE model since it assumes that all asperities deform elastically.

Another important factor is the total load. The pressure profiles at two representative high and low loads of $W = 1.5 \times 10^{-5}$ and $W = 1.5 \times 10^{-3}$ are plotted in Figure 5.3. When the total load increases the pressure profile becomes more similar to the Hertzian type. In contrast, for the low load, the maximum pressure is significantly less than that of the maximum Hertzian pressure.
According to Figure 5.2 & 5.3, the pressure distribution extends considerably beyond X=1 and Y=1 for rough surfaces. Clearly as the maximum pressure drops, the extension of pressure becomes wider in both directions, so that the load is satisfied. This is particularly noticeable in situations where the plasticity index is large or the load is low.

5.4.2. Experimental verification

Precise experimental determination of the contact parameters is a difficult task and therefore published experimental results are scarce. Fortunately, Kagami et al. [39] provided measurement results for the contact dimension and the normal compliance for the case of smooth sphere and a rough plates (copper and steel plates) subjected to different loads. The pertinent information on the dimensions as well as the mechanical and surface properties of the sphere and the plates are shown in Table 5.1. In this section, we make use of the results of [39] to verify our theoretical development.
Table 5.1. Dimensional, surface and mechanical properties for the rough plate and the sphere [39]

<table>
<thead>
<tr>
<th></th>
<th>( R ) (m)</th>
<th>( \sigma ) (m)</th>
<th>( \beta ) (m)</th>
<th>( n ) (m²)</th>
<th>( E ) (GPa)</th>
<th>( H ) (MPa)</th>
<th>( \nu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Copper Plate</td>
<td>-</td>
<td>0.457×10⁻⁶</td>
<td>33.3×10⁻⁶</td>
<td>2×10⁹</td>
<td>107</td>
<td>1270</td>
<td>0.34</td>
</tr>
<tr>
<td>Steel Plate A</td>
<td>-</td>
<td>1.45×10⁻⁶</td>
<td>28×10⁻⁶</td>
<td>1.4×10⁹</td>
<td>206</td>
<td>3340</td>
<td>0.29</td>
</tr>
<tr>
<td>Steel Plate B</td>
<td>-</td>
<td>0.457×10⁻⁶</td>
<td>33.3×10⁻⁶</td>
<td>2×10⁹</td>
<td>206</td>
<td>3340</td>
<td>0.29</td>
</tr>
<tr>
<td>Sphere(Steel)</td>
<td>3.18×10⁻³</td>
<td>&lt;0.1×10⁻⁶</td>
<td>-</td>
<td>-</td>
<td>206</td>
<td>7520</td>
<td>0.29</td>
</tr>
</tbody>
</table>

5.4.2.1. Contact radius

The contact length and width can be estimated according to the pressure distribution. However, unlike the Hertzian distribution in which the contact area has distinct boundaries (defined by function \( X^2+Y^2=1 \) in dimensionless form), in rough surfaces the pressure asymptotically approaches zero at the boundaries. A review of the literature shows that there is no universally accepted definition for the contact dimension in rough curved surfaces, and that researchers have adopted different criteria for its prediction [17, 18], [22],[40]. In the present study, in order to compare the numerical results with available experimental measurements, a criterion is adopted based on which the contact half length and width are calculated as the maximum and minimum distance between the center point of the contact and the boundary at which the dimensionless pressure, \( \bar{p} \), is less than an assumed value (see for example Ref. [22]). This value should be small enough such that the pressure can be assumed negligible beyond the specified boundary. In the current study, this threshold value is chosen to be 0.01, and the associated contact half width and half length are symbolized as \( b_{1\%} \) and \( a_{1\%} \), respectively (see Figure 5.2b). These parameters, hereinafter, are referred to as threshold contact half width and threshold contact half length.

The experimental results of Kagami et al. [20] pertain to the case of contact of a smooth steel sphere with rough steel and copper plates. Therefore the contact radii in two perpendicular planes \( (R_x \text{ and } R_y) \) are the same and, as a result, the contact half width and contact half length are equal. We shall, henceforth, refer to this parameter as the contact radius \( (\rho_{1\%}) \). Demonstrated in Figure 5.4 is the variation of the predicted contact radius as a function of the total load based on the Hertzian solution and the predicted behavior based on the current simulation. Also shown are the experimental results of Kagami et al. [20]. As seen, there is good agreement between the present model and the experimental data for both contact types using the KE, which authenticates the approach for the prediction of the contact dimensions. In contrast, the Hertzian solution deviates significantly from the measured results for low to moderate loads. As the load increases, the current simulation results approach to those of the Hertzian solution.
5.4.2.2. Contact compliance

In addition to contact radius, Kagami et al. [20] also reported the experimental results for the contact compliance corresponding to the surfaces listed in Table 1.

The compliance between a ellipsoidal curved body and a plate at the location of \( x \) and \( y \) is the summation of the geometry of the bodies, \( G(x,y) \), the asperity deformation, \( D_a(x,y) \), and the bulk deformation \( D_b(x,y) \). It can be written in the following form:

\[
\delta = G(x,y) + D_a(x,y) + D_b(x,y)
\]  

(24)

where

\[
G(x,y) = \frac{x^2}{2Rx} + \frac{y^2}{2Ry}
\]

and

\[
D_b(x,y) = \frac{1}{\pi E'} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{p(s,q)dsdq}{\sqrt{(x-s)^2 + (y-q)^2}}
\]

For the contact of a sphere (\( R_x = R_y = R \)) and a plate, the asperity deformation is nil at the contact boundaries (e.g., at \( x=b_1% = \rho_1% \) and \( y=0 \)). Applying this condition to Eq. (24), yields:

\[
\delta = \frac{h_{0.01}^2}{2R} + \frac{1}{\pi E'} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{p(s,q)dsdq}{\sqrt{(b_{0.01}-s)^2 + (0-q)^2}}
\]

(25)

After obtaining the solution to the equation set (23), all the values in Eq. (25) are known including the pressure distribution. Hence, one can easily calculate the contact compliance between two bodies.
Figure 5.5 illustrates the contact compliance as a function of total load obtained based on Hertzian solution and the current simulation. Also depicted in Figure 5.5, are the measurement results of Kagami et al. [20] for the contact compliance. As seen, there is very good accordance between the experimental results and numerical predictions which confirms the validity of the approach. Due to the asperity deformation, which is not considered in Hertzian solution, the predicted values of compliance based on Hertzian approach significantly deviate from the experimental results especially at low loads where the asperity deformation is comparable to the bulk deformation of the surfaces.

Figure 5.5. Contact compliance based on Hertzian, current simulation and experimental results of Kagami et al. [20], (a) Smooth Steel Ball on Rough Copper Plate and, (b) Smooth Steel Ball on Rough Steel Plate A

5.4.3. Elliptical point contact vs. line contact

The elliptical point contact solution is the general case for the contact of two curved bodies, and the circular point contact—the case studied in previous section—is a special case of the elliptical point contact, with an equal half-width and half-length contact patch. Another special case is when \( R_y \) approaches infinity, in which case the point contact solution should reduce to that of the line contact. Among the characteristic parameters, compliance is an appropriate physical measure of the effect of ellipticity on the contact behavior in an elliptical point contact configuration. Figure 5.6 shows the compliance as a function of the ellipticity parameter for the contact of two rough surfaces. As seen, the compliance decreases as ellipticity increases until reaching the point where it levels off for relatively high ellipticity values.

The compliance is also computed using the previously developed model by the authors for line- contact configuration [17]. To compare the elliptical point contact solution and those associated with line contact approach, the contact length in line contact configuration \( (l) \) is taken to be equal to the major axes in elliptical point contact \( (2a) \) and all other input are exactly
identical in both simulations. According to the solution of the compliance based on line contact analysis and for the current input values (see Figure 5.6 caption), around the ellipticity of $\kappa = 28$ the two solutions become close, indicating that one can use the line contact formulation to calculate the compliance of the elliptical point contact problem at high ellipticity values of $\kappa \geq 28$.

![Figure 5.6. Effect of ellipticity parameter ($\kappa$) on contact compliance at $W/g_{33} = 5.9 \times 10^{-5}$, $\sigma/g_{33} = 1.5 \times 10^{-4}$](image)

### 5.4.4. Predictive expressions

Since the calculations of different micro-asperity contact models along with the bulk deformation of the surface demand considerable time and effort to develop, deriving convenient equations based on the numerical results is useful for application purposes. Accordingly, in this section, formulas for the estimation of the maximum contact pressure, contact half width and length, real area of contact and contact compliance are presented based on the numerical results for the deformation of dry rough elliptical point contact. These expressions can be then applied to the case of lubricated contact to easily estimate the lubricant film thickness, friction and the wear using the load-sharing concept [28, 41, 42].

Recently, Beheshti and Khonsari carried out extensive numerical simulations for the line-contact configuration based on the statistical micro-asperity contact model of Kogut and Etsion [11] and provided formulas for the prediction of dry contact parameters. In the present study, the approach is extended to the circular or elliptical point contact configuration. In addition, the curve-fitted expressions for the GW model as well as for KE are provided.
Table 5.2. Non-dimensionalized parameters and their ranges used in the curve fitting

<table>
<thead>
<tr>
<th></th>
<th>$\bar{\sigma}/\bar{\beta}$</th>
<th>$\zeta = \bar{n}\bar{\beta}/\bar{\sigma}$</th>
<th>$\bar{\sigma}$</th>
<th>$\bar{W}$</th>
<th>$\Omega$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum</td>
<td>$5 \times 10^{-4}$</td>
<td>$3 \times 10^{-2}$</td>
<td>$5 \times 10^{-6}$</td>
<td>$1 \times 10^{-6}$</td>
<td>$2 \times 10^{-3}$</td>
<td>1</td>
</tr>
<tr>
<td>Maximum</td>
<td>$5 \times 10^{-1}$</td>
<td>$1 \times 10^{-1}$</td>
<td>$1 \times 10^{-3}$</td>
<td>$1 \times 10^{-3}$</td>
<td>$5 \times 10^{-2}$</td>
<td>50</td>
</tr>
</tbody>
</table>

Table 5.2 provides the range of the parameters used for the numerical curve fitting. After considerable numerical simulations (more than 1100 cases), the maximum pressure, contact half width and length, real area of contact and the contact compliance are curve-fitted based on the GW and KE models and the following formulas are derived. The coefficients for each equation as well as the maximum error due to curve fitting are provided in Table 5.3.

$$\bar{P}_{\text{max}} = \frac{1}{1 + \left(\alpha \bar{n} \bar{\beta} \bar{c} \bar{\sigma} \bar{W} \bar{e} \Omega / D \bar{g}\right)^i}$$ \hspace{1cm} (26a)

$$\bar{b}_1 = \left(1 + \left(\alpha \bar{n} \bar{\beta} \bar{c} \bar{\sigma} \bar{W} \bar{e} \Omega / D \bar{g}\right)^i\right)^{1/2}$$ \hspace{1cm} (26b)

$$\bar{a}_1 = \left(1 + \left(\alpha \bar{n} \bar{\beta} \bar{c} \bar{\sigma} \bar{W} \bar{e} \Omega / D \bar{g}\right)^i\right)^{1/2}$$ \hspace{1cm} (26c)

$$\delta = \sqrt[4]{1 + \left(\alpha \bar{n} \bar{\beta} \bar{c} \bar{\sigma} \bar{W} \bar{e} \Omega / D \bar{g}\right)^i}$$ \hspace{1cm} (26d)

$$\mathcal{T}_* = \left(1 + \left(\alpha \bar{n} \bar{\beta} \bar{c} \bar{\sigma} \bar{W} \bar{e} \Omega / D \bar{g}\right)^i\right)^{1/4}$$ \hspace{1cm} (26e)

Table 5.3. Coefficients for the curve-fit equations

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$e$</th>
<th>$f$</th>
<th>$g$</th>
<th>$h$</th>
<th>$i$</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{P}_{\text{max}}$</td>
<td>0.5386</td>
<td>0.1863</td>
<td>0.0312</td>
<td>-0.8563</td>
<td>0.6920</td>
<td>0.0</td>
<td>-0.2643</td>
<td>-0.8106</td>
<td>N/A</td>
<td>6%</td>
</tr>
<tr>
<td>$\bar{b}_1$</td>
<td>3.8427</td>
<td>-0.0043</td>
<td>0.0501</td>
<td>0.7081</td>
<td>-0.4664</td>
<td>0.0</td>
<td>0.2970</td>
<td>0.9914</td>
<td>0.6334</td>
<td>7%</td>
</tr>
<tr>
<td>$\bar{a}_1$</td>
<td>3.0414</td>
<td>-0.0024</td>
<td>0.0456</td>
<td>0.6372</td>
<td>-0.4238</td>
<td>0.0</td>
<td>0.0257</td>
<td>1.2524</td>
<td>0.5925</td>
<td>7%</td>
</tr>
<tr>
<td>$\bar{\delta}$</td>
<td>1.2406</td>
<td>-0.0015</td>
<td>-0.0088</td>
<td>-0.0612</td>
<td>-0.6778</td>
<td>0.0</td>
<td>0.2280</td>
<td>1.7231</td>
<td>N/A</td>
<td>8%</td>
</tr>
<tr>
<td>$\bar{A}_*$</td>
<td>0.6593</td>
<td>-0.7043</td>
<td>-1.8030</td>
<td>0.3028</td>
<td>-0.6563</td>
<td>0.0</td>
<td>0.5345</td>
<td>3.9343</td>
<td>-0.0897</td>
<td>8%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$e$</th>
<th>$f$</th>
<th>$g$</th>
<th>$h$</th>
<th>$i$</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{P}_{\text{max}}$</td>
<td>1.6097</td>
<td>-0.2572</td>
<td>-0.2107</td>
<td>0.6101</td>
<td>-0.4863</td>
<td>-0.2138</td>
<td>0.1701</td>
<td>1.000</td>
<td>N/A</td>
<td>6%</td>
</tr>
<tr>
<td>$\bar{b}_1$</td>
<td>4.8818</td>
<td>-0.0248</td>
<td>0.0021</td>
<td>0.7604</td>
<td>-0.4847</td>
<td>-0.0701</td>
<td>0.3100</td>
<td>0.9788</td>
<td>0.5703</td>
<td>8%</td>
</tr>
<tr>
<td>$\bar{a}_1$</td>
<td>4.7997</td>
<td>-0.0317</td>
<td>-0.0155</td>
<td>0.7712</td>
<td>-0.4917</td>
<td>-0.0025</td>
<td>-0.0900</td>
<td>1.1090</td>
<td>0.500</td>
<td>9%</td>
</tr>
<tr>
<td>$\bar{\delta}$</td>
<td>1.2625</td>
<td>0.01260</td>
<td>0.0107</td>
<td>-0.1344</td>
<td>-0.9612</td>
<td>0.0530</td>
<td>0.3366</td>
<td>1.1729</td>
<td>N/A</td>
<td>8%</td>
</tr>
<tr>
<td>$\bar{A}_*$</td>
<td>0.9530</td>
<td>-0.1241</td>
<td>-0.2407</td>
<td>-0.0143</td>
<td>-0.6087</td>
<td>1.4044</td>
<td>0.4445</td>
<td>9.6490</td>
<td>-0.0533</td>
<td>10%</td>
</tr>
</tbody>
</table>
Similar to the line-contact problem, pressure profiles for medium, rough and very rough surfaces are in general very close to Gaussian distribution. In contrast, the profile for smooth surface deviates from Gaussian distribution when the maximum pressure approaches unity. The general form of pressure distribution can be written as:

$$
\bar{p}(X,Y) = \bar{p}_{\text{max}} \left( 1 - \left( \frac{X}{\bar{b}_{0.01}} \right)^2 - \left( \frac{Y}{\bar{a}_{0.01}} \right)^2 \right)^\gamma, \quad X \leq \bar{b}_{0.01}, \quad Y \leq \bar{a}_{0.01}
$$

which simply yields the Hertzian solution for $\gamma = 0.5$ and $\bar{a}_{0.01}, \bar{b}_{0.01} = 1$ while for greater values of $\gamma$, it follows the Gaussian distribution. Applying the dimensionless force balance Eq. (22) for the entire contact configuration, one can find the value of $\gamma$ as follows:

$$
\frac{2\pi}{3} = \int_{A} \bar{p}_{\text{max}} \left( 1 - \left( \frac{X}{\bar{b}_{0.01}} \right)^2 - \left( \frac{Y}{\bar{a}_{0.01}} \right)^2 \right)^\gamma dXdY = \bar{p}_{\text{max}} \frac{\bar{a}_{0.01} \bar{b}_{0.01} \pi}{\gamma + 1}
$$

Solving for $\gamma$ yields:

$$
\gamma = \frac{3\bar{p}_{\text{max}} \bar{a}_{0.01} \bar{b}_{0.01}}{2} - 1
$$

5.4.5 Example

Consider a case of the contact of a rough copper ellipsoid and a smooth steel ellipsoid. The corresponding data are summarized in Table 5.4. Here, we seek to obtain contact parameters for this case.

<table>
<thead>
<tr>
<th></th>
<th>$W=100$ (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$W$ (N)</td>
</tr>
<tr>
<td>Ellipsoid #1(Cu)</td>
<td>100</td>
</tr>
<tr>
<td>Ellipsoid #2 (St)</td>
<td></td>
</tr>
</tbody>
</table>

The calculations are presented in details in the following; In addition to the necessary parameters for calculating contact characteristics, plasticity index are also computed below:

$$
\frac{1}{R_x} = \frac{1}{R_{x1}} + \frac{1}{R_{x2}} = \frac{1}{20 \times 10^{-3}} + \frac{1}{10 \times 10^{-3}} \quad R_x = 1.4 \times 10^{-3}
$$

$$
\frac{1}{R_y} = \frac{1}{R_{y1}} + \frac{1}{R_{y2}} = \frac{1}{20 \times 10^{-3}} + \frac{1}{30 \times 10^{-3}} \quad R_y = 8.5 \times 10^{-3}
$$

$$
\frac{1}{E'} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2} \quad E' = 7.867 \times 10^{10}
$$

$$
\bar{\sigma} = \frac{\sigma}{R_x} = 5.6 \times 10^{-4} \quad \bar{\beta} = \frac{\beta}{R_x} = 1.4 \times 10^{-2} \quad \bar{n} = nR_x^2 = 1.20 \times 10^4 \quad \xi = n\bar{\sigma}\bar{\beta} = 0.087
$$
\[ D = \frac{R_e}{R_s} = 6 \quad \bar{W} = \frac{W}{E' R_s^2} = 6.2 \times 10^{-4} \quad \Omega = \frac{H}{E'} = 2.54 \times 10^{-2} \]

\[ \psi = \frac{2}{\pi K \xi} \sqrt{\frac{\bar{\sigma}}{\bar{\sigma}} \left( 1 - \frac{3.71693 \times 10^{-4}}{\xi} \right)} = 8.2 \]

\[ \bar{\sigma}_{KE_{\text{max}}} = \frac{1}{\sqrt{1 + 1.6097 \pi^{-0.2572} \bar{\sigma}^{-0.2107} \bar{W}^{-0.4863} \Omega^{-0.2138} D^{-0.1701}}} = 0.743 \]

\[ \bar{\sigma}_{KE_{0.01}} = \left( 1 + \left[ 4.88186 \bar{\sigma}^{-0.0248} \bar{\sigma}^{-0.0021} \bar{\sigma}^{-0.7604} \bar{\sigma}^{-0.4847} \Omega^{-0.0701} D^{-0.031} \right]^{0.9788} \right)^{0.5703} = 1.50 \]

\[ \bar{\sigma}_{KE_{0.01}} = \left( 1 + \left[ 4.7997 \bar{\sigma}^{-0.0317} \bar{\sigma}^{-0.0155} \bar{\sigma}^{-0.7712} \bar{\sigma}^{-0.4917} \Omega^{-0.0025} D^{-0.09} \right]^{0.1.109} \right)^{0.5} = 1.16 \]

\[ \gamma = \frac{3 \bar{\sigma}_{KE_{\text{max}}} \bar{\sigma}_{KE_{0.01}}}{2} - 1 = 0.946 \]

Therefore, the pressure distribution according to the statistical approach can be written as:

\[ \bar{\sigma}(X) = 0.743 \left( 1 - \left( \frac{X}{1.50} \right)^2 - \left( \frac{Y}{1.16} \right)^2 \right)^{0.946} \quad X \leq 1.50 \quad , \quad Y \leq 1.16 \quad (31) \]

Plotted in Figure 5.7 are the pressure profiles for the aforementioned case. In this figure, pressure profile based on the proposed Eq. (31) and full numerical calculations are demonstrated in two XZ and YZ plane. As seen, there is a close agreement between them, revealing the usefulness of the expression developed.

Figure 5.7. Pressure profile based on the numerical calculation and approximated pressure distribution function (a) XZ plane, (b) YZ plane

5.4.6. Contact of two rough surfaces

All the relationships derived so far are based on the assumption that one of the surfaces is ideally smooth and the other one is rough. As shown by Greenwood and Tripp [18] the contact of two rough surfaces can be treated as a contact of one rough surface with equivalent (effective)
parameters and an ideally smooth surface. Based on the relationships provided by McCool [34] and by rearranging them [17], it is possible to extend all contact formulations and curve-fitted expressions to analyze the contact of two rough surfaces. The equivalent roughness, asperity radius and asperity density can be obtained using the following relationships [17]:

\[
\sigma_{eq} = \sqrt{\sigma_1^2 + \sigma_2^2} \tag{32a}
\]

\[
\frac{1}{\beta_{eq}} = \sqrt{\frac{1}{\beta_1^2} + \frac{1}{\beta_2^2}} \tag{32b}
\]

\[
\frac{1}{n_{eq}} = \frac{1}{n_1} \left( \frac{\beta_{eq}}{\beta_1} \right)^2 + \frac{1}{n_2} \left( \frac{\beta_{eq}}{\beta_2} \right)^2 \tag{32c}
\]

5.5. Conclusions

Statistical micro-contact models including Greenwood-Williamson (GW) and Kogut-Etsion (KE) are employed along with the elastic bulk deformation formula of the elliptical point contact to determine the pressure profile, contact width and length, real area of contact and contact compliance as measures of the impact of surface roughness on the contact characteristics.

Results indicates that the consideration of elastic-plastic micro-contact model of KE reduces the magnitude of maximum normal pressure and predicts a greater contact width and length compared to the GW model which assumes that asperities deform elastically. The numerical results for the contact radius and contact compliance are compared with the available experimental measurements and show very good agreement. Thus, the KE model is found to be an appropriate statistical contact model for analyzing the point contact behavior. The model considers all different regimes of deformation in comparison to GW, CEB and even ZMC models and describes the physical behavior of the materials in more detail.

In addition, useful expressions for the prediction of maximum contact pressure, contact width, contact length, real area of contact, contact compliance and pressure distribution are proposed based on the GW and KE models. Using these formulas one can easily estimate the contact characteristics of the elliptical point contact configuration.

**Nomenclature**

- $A$: real contact area between two flat surfaces, m$^2$
- $A_n$: nominal contact area between flat surfaces, m$^2$
- $A_r$: real contact area for the contact of curved surfaces, m$^2$
- $\bar{A}_r$: dimensionless real contact area for the contact of curved surfaces
- $A_{nc}$: nominal contact area between flat and curved surfaces, m$^2$
- $a$: Hertzian contact half length, m
- $b$: Hertzian contact half width, m
contact half length beyond which pressure is negligible (\( \bar{a} \leq 0.01 \))

dimensionless contact half length

contact half width beyond which pressure is negligible (\( \bar{b} \leq 0.01 \))

dimensionless contact half width

to contact radius in \( y \) direction to contact radius in \( x \) direction

separation based on asperity heights (\( h - y_s \)), m

modulus of elasticity of body 1 and 2, GPa

effective modulus of elasticity,

hardness of the softer material, MPa

separation based on surface heights, m

constant in the separation relationship

dimensionless separation

constant in the dimensionless separation relationship

influence coefficient

maximum contact pressure factor

contact length in rectangular conjunction (line contact)

number of nodes

contact load for the contact of two flat surfaces with constant mean separation, N

nominal pressure for the contact of two flat surfaces with constant mean separation, MPa

dimensionless pressure

maximum Hertzian pressure, MPa

maximum dimensionless pressure for the rough surfaces contact

effective radius of curvature in \( x \) and \( y \) directions

total normal force for the contact of a cylinder and a flat surface, N

dimensionless total normal force

spatial coordinate component along and perpendicular to the contact width, m

dimensionless spatial coordinate component along and perpendicular to contact width, \( x/b, y/a \)
distance between the mean of summit heights and that of the surface heights, m

asperity height measured from the mean line of summit heights, m

dimensionless asperity height, \( z/\sigma \)

Poisson ratios of the first and second surface

compliance, m

asperity density, m\(^{-2}\)

dimensionless asperity density

asperity radius, m

dimensionless asperity radius

standard deviation of surface heights, m
\( \bar{\sigma} \)  
- dimensionless standard deviation of surface heights

\( \sigma_s \)  
- standard deviation of summit (asperity) heights, m

\( \rho \)  
- contact radius beyond which pressure is negligible (\( \bar{\rho} \leq 0.01 \))

\( \kappa \)  
- ellipticity parameter, \( a/b \)

\( \omega \)  
- asperity interface, m

\( \bar{\omega} \)  
- dimensionless interface, \( \omega/\sigma \)

\( \omega^* \)  
- critical interface according to the CEB and KE models, m

\( \omega^* \)  
- dimensionless critical interface according to the CEB and KE models

\( \Omega \)  
- ratio of the softer surface hardness to the effective elasticity modulus, \( H/E' \)

\( \lambda(x) \)  
- surface profile

\( \Phi(h) \)  
- function representing different micro-asperity contact models for the contact load

\( \Phi^*(h) \)  
- dimensionless function representing different micro-asperity contact model for the contact load

\( \bar{\Phi}(h), \bar{S}(\kappa) \)  
- elliptical integrals of the first and second kind

\( \phi^*(z^*) \)  
- dimensionless standard normal distribution function

\( \zeta \)  
- \( n\beta\sigma \)

\( \psi \)  
- plasticity index

\( \gamma \)  
- exponential coefficient of universal pressure distribution function

### 5.6. References:


Chapter 6: Deterministic Surface Trawctions in Rough Contact under Stick-Slip Condition: Application to Fretting Fatigue Crack Initiation

6.1. Introduction

Given that all engineering surfaces are rough at microscopic level, the interaction between mating surfaces always begins at the asperity tips. It is therefore, no surprise that the performance of vital mechanical components such as bearings, gears, mechanical seals, hard disk drives, and biomedical transplants is highly affected by the surface roughness. Hence, consideration of surface roughness in modeling is indispensable for realistic characterization of load-carrying capacity, friction and wear as well as the prediction of fatigue life.

While the macro-level contact of two ideally-smooth curved bodies can be adequately described by the well-known Hertzian pressure distribution formula, the situation is far more complex when surfaces are rough to the extent that the pertinent contact properties deviate from those predicted by the Hertzian approach.

A review of the literature reveals that several approaches are available for predicting the contact behavior of bodies with rough surfaces subjected to both normal and tangential loads [1-33]. Broadly, the surface roughness modeling approaches fall in two categories: statistical or deterministic. The statistical approach is typically more convenient to formulate since it only requires the specification of a few surface parameters and lends itself to generalization of predictions for different geometries, surface properties and loading conditions. The classical work of Greenwood and Williamson [1] is, in fact, a statistical approach and many researchers have followed their study for the so-called nominally flat rough surfaces [24-27] as well as curved rough surfaces [34-37]. Deterministic simulation of the surfaces, on the other hand, provides a more detailed description of the pressure, deformation and the sub-surface stress distribution. The deterministic approach is required in problems involving micro-scale phenomena with surface damage—e.g., micro-pitting or surface crack initiation in rough surface fretting fatigue or any near surface failure—in order to determine the sub-surface stress distribution for the contact region with real surface profile. While several studies have been reported on the deterministic prediction of pressure and resulting sub-surface stress field for rough surfaces [8-23], numerical deterministic prediction of the rough surface tangential traction due to stick-slip condition has received only limited attention [38-41]. Stick-slip condition frequently arises in fretting fatigue where typically surface cracks tend to nucleate and grow until

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complete failure occurs. Details on the modeling and experimental analysis of fretting fatigue in smooth contact are abundantly available in literature [42-60]. For the case of rough contacts, however, modeling techniques are very few in number [40, 61, 62] and need to be further developed to predict and describe the fretting crack initiation behavior observed in experiments.

The present work provides a report on the development of a simple approach to predict the deterministic pressure distribution in a rough line contact configuration with the consideration of elastic-perfectly plastic behavior in the asperity tips. The proposed algorithm solves the surface separation and load balance equations simultaneously with an iterative procedure instead of the sequential procedure and, hence, offers a fast convergence rate. In addition, adopting Ciavarella-Jager [63, 64] approach, the calculation methodology for the deterministic tangential traction distribution for cyclic loading condition in stick-slip regime is presented in detail. Results are given for the pressure, tangential distribution and the sub-surfaces stress field, and can be used for determination of surface damage. The described algorithm is conceptually simple and can be conveniently implemented on the computer. The surface tractions obtained by means of the proposed technique are used to examine the fretting fatigue crack initiation phenomenon in rough line-contact configuration.

6.2. Contact Problem Formulation

6.2.1. Pressure Distribution

Presented in this section is the formulation of the normal contact of rough surfaces involving the bulk deformation of the surface in line-contact configuration where plane strain condition applies. Assuming an elastically identical contact pair, the separation equation, \( h(x) \), for the line contact is:

\[
h(x) = h_{00} + \frac{x^2}{2R} - \frac{2}{\pi E'} \int_{-\infty}^{\infty} p(s) \ln(|x - s|) ds + \lambda(x)
\]

(1)

where \( E' \) is the effective modulus of elasticity, \( p(s) \) denotes the pressure, \( h_{00} \) is a constant to be determined, \( x \) is the coordinate along the contact width, \( \lambda(x) \) symbolizes the composite surface profile and \( R \) represents the effective radius of curvature. (see Figure 6.1).
In addition, the force balance equation must be satisfied:

\[ W = l \int_{-\infty}^{+\infty} p(x) dx \]  

(2)

where \( W \) represents the total load and \( l \) is the length of the contact. The above equations can be written in the non-dimensionalized form as:

\[ \bar{h}(X) = \bar{h}_{00} + \frac{X^2}{2} \left( \frac{4\bar{W}}{\pi} \right) - \frac{1}{\pi} \left( \frac{4\bar{W}}{\pi} \right)^{+\infty}_{-\infty} \bar{p}(S) \ln|X - S| dS + \Lambda(X) \]  

(3)

\[ \frac{\pi}{2} = \int_{-\infty}^{+\infty} \bar{p}(X) dX \]  

(4)

with the following dimensionless parameters:

\[ \bar{r}_q = \frac{r_q}{R} \quad \Lambda = \frac{\lambda}{R} \quad X = \frac{x}{b} \]  

(5)

\[ \bar{p} = \frac{p}{p_H} \quad \bar{h} = \frac{h}{R} \quad \bar{W} = \frac{W}{l E' R} \]

where \( p_H \) is the maximum Hertzian pressure and \( b \) is the Hertzian contact half width:

\[ p_H = \frac{2W}{\pi l b} = \frac{E' b}{2R} = E' \sqrt{\frac{W}{\pi}} \]  

(6a)

\[ b = \sqrt{\frac{4WR}{\pi l E'}} = \sqrt{\frac{4WR^2}{\pi}} \]  

(6b)
The reader recognizes that these dimensionless parameters are chosen so that they are identical to those typically used in the formulation of elastohydrodynamic lubrication (EHL) [65], and thus can be further extended to treat lubricated contacts.

It is worth noting that a limit should be considered for the total load if it causes the surface bulk deformation to exceed beyond the elastic limit. However, it is difficult to obtain an exact criterion for the elastic limit of bodies when one deals with rough surfaces. The approach adopted here is based on the consideration of the sub-surface stress field in Hertzian contact problem of smooth surfaces. According to the von Mises yield criterion, the sub-surface region yields when the maximum Hertzian contact pressure is roughly $p_h = 1.8S_y$, where $S_y$ is the yield stress. Using this relationship in Eq. 6a, we arrive at the following limitation for the dimensionless load:

$$
\bar{W} \leq 0.406 \pi \Omega^2
$$

where the hardness is considered to be 2.8 times the yield strength ($H=2.8S_y$) when direct measurement is not available.

Letting $\Omega_c$ represent the contact domain, the deterministic solution must satisfy the following complementarity condition and force balance equation:

$$
\bar{h}(X) = 0 \quad , \quad \bar{p}(X) > 0 \quad , \quad X \subset \Omega_c
$$
$$
\bar{h}(X) > 0 \quad , \quad \bar{p}(X) = 0 \quad , \quad X \subset \Omega_c
$$

$$
\frac{\pi}{2} = \int_{\Omega_c} \bar{p}(X) dX
$$

The first and second relationships in Eq. 8 require the pressure to be positive in the contact domain where separation is nil, while elsewhere pressure is nil where separation is positive. During the computations the local pressure in the contact domain may exceed the hardness ($H$) of the softer material. This, of course, lacks physical meaning and should be appropriately treated. In the current study it is assumed that asperity tips conform to the elastic-perfectly plastic assumption. In other words, they deform elastically up to a certain pressure ($p = H$) and fully plastically thereafter. This is formulated by imposing a ceiling pressure of $p = H$ on the contact pressure. Introducing a new dimensionless parameter $\Omega = H/E'(\text{dimensionless hardness})$, the following restrictions are applied:

$$
\bar{p}(X) = \bar{p}(X) \quad , \quad \text{if} \quad \bar{p}(X) < \Omega \sqrt{\frac{\pi}{W}}
$$
$$
\bar{p}(X) = \Omega \sqrt{\frac{\pi}{W}} \quad , \quad \text{if} \quad \bar{p}(X) \geq \Omega \sqrt{\frac{\pi}{W}}
$$
Consideration of the plastic deformation in this fashion is widely used in the deterministic simulation of the rough surface contact problems. It is worth noting that as the pressure limit is reached, the implicit assumption here is that plastic deformation occurs in the corresponding contact points without further resistant to higher pressure. This is equivalent to initially modifying the surface profile in a fashion that its elastic solution would result in the same pressure distribution as we have obtained by the elastic-perfectly plastic model using the original geometry.

6.2.2. Tangential Traction Distribution

6.2.2.1. Monotonic loading condition

Consider the case that the tangential force changing monotonically from zero to its maximum value \((Q_{\text{max}})\). In “full-slip condition”, the tangential traction distribution, \(q(x)\), is determined based on the Coulomb’s law:

\[
q_{\text{max}}(x) = \mu p(x)
\]

for full slip condition \(11\)

while for “stick-slip condition”, Coulomb’s friction law must be modified by introducing the corrective traction distribution \(q^*(x)\) in the following manner:

\[
q_{\text{max}}(x) = \mu p(x) - q^*(x)
\]

for stick-slip condition \(12\)

which reduces to Eq. 11 in the slip region where \(q^*(x) = 0\). The tangential traction distribution in the stick region in a rough line-contact configuration can be calculated using the approach introduced by Ciavarella and Jager [63, 64]. They showed that the solution for the corrective tangential traction is analogous to that of the normal pressure distribution problem with a corresponding effective load (to be defined later) as an input load. The corrective traction distribution, \(q^*(x)\), can be assumed to be proportional to an imaginary normal pressure \([41]\) with the friction coefficient as the proportionality coefficient \((q^* = \mu p^*)\). Therefore, Eq. 12 can be written as:

\[
q_{\text{max}}(x) = \mu p(x) - \mu p^*(x)
\]

Integrating Eq. 13 over the entire domain and noting that the integration of the left-hand-side of Eq. 13 must yield the total tangential load, one can easily obtain the “effective load” needed to produce the pressure distribution \(p^*(x)\):

\[
W^* = W - \frac{Q_{\text{max}}}{\mu}
\]

(14)
Once the effective load \( W^* \) is known, Eqs. 3, 8 and 9 can be applied to obtain the effective pressure distribution \( p^*(x) \).

### 6.2.2.2 Cyclic loading condition

The equations described in the previous section pertain to the case of a monotonically increasing tangential force. For cyclic tangential loading condition in which \( Q \) varies with time, the history of loading should be taken into consideration [44]. In other words, if the load varies between two extreme values \( (+Q_{\text{max}} \text{ and } -Q_{\text{max}}) \) over a period of time, the steady-state solution is valid only for the two extreme values. Therefore, one needs to consider the full loading history if detailed description of the stress and strain variations through the loading cycle are desired.

Consider an operating condition in which the tangential force \( Q \) increases monotonically to its maximum value \( +Q_{\text{max}} \). Let us assume that we reduce the load from \( +Q_{\text{max}} \) to \( Q \), which is equal to the application of force \( Q_{\Delta} = Q_{\text{max}} - Q \) in the opposite direction. This reduction results in a “reverse” slip near the contact edges where the magnitude of shear in this new slip zone changes from \( \mu p(x) \) to \( -\mu p(x) \) [44]. This is equivalent to the application of the corrective term \(-2\mu p(x)\) in this region. It further leads to the formation of a new stick zone that also needs an extra new corrective traction term [44]. We assume this corrective term to be \( 2q^*(x) \) over the new stick zone. Therefore, the total corrective term related to the application of load \( Q_{\Delta} \) for the unloading becomes:

\[
q_{\Delta}(x) = 2\mu p(x) + 2q^*(x)
\]  

(15)

where the term \( q^*(x) \) is unknown. Analogous to the calculation of \( q^*(x) \), the new corrective term, \( q^*(x) \), can be calculated with the corresponding load of \( 1/2 \) \( Q_{\Delta} \) (Eq. 15 is divided by factor of 2 and integrated). Again the aforementioned new corrective term is assumed to be proportional to the new imaginary normal pressure \( p^*(x) \) with the friction coefficient as the proportionality factor \( (q^* = \mu p^*) \). Thus, the effective load to substitute into the normal pressure formulation to calculate \( p^*(x) \) becomes:

\[
W^* = W - \frac{Q_{\Delta}}{2\mu} = W - \frac{Q_{\text{max}} - Q}{2\mu}
\]

where \( +Q_{\text{max}} \rightarrow Q \rightarrow -Q_{\text{max}} \)  

(16a)

\[
W^* = W - \frac{Q_{\Delta}}{2\mu} = W - \frac{Q_{\text{max}} + Q}{2\mu}
\]

where \( -Q_{\text{max}} \rightarrow Q \rightarrow +Q_{\text{max}} \)  

(16b)

which gives \( p^*(x) \). Finally, the tangential traction distribution during the cyclic unloading (with positive sign) and loading (with negative sign) is:
\[ q(x) = \pm[q_{\text{max}}(x) - q_\Delta(x)] = \pm\mu[-p(x) - p^*(x) + 2p^{**}(x)] \]  

(17)

### 6.2.3. Sub-surface Stress Distribution

The stress components below the contact region of an elastic body in line contact, assuming that plane strain condition prevails, are given by the following integrals [43]:

\[
\sigma_x(x, z) = -\frac{2z}{\pi} \int_{a_1}^{a_2} \frac{p(s)(x-s)^2 ds}{[(x-s)^2 + z^2]^2} - \frac{2}{\pi} \int_{a_1}^{a_2} \frac{q(s)(x-s)^3 ds}{[(x-s)^2 + z^2]^2} 
\]

(18a)

\[
\sigma_z(x, z) = -\frac{2z^2}{\pi} \int_{a_1}^{a_2} \frac{p(s)ds}{[(x-s)^2 + z^2]^2} - \frac{2z^2}{\pi} \int_{a_1}^{a_2} \frac{q(s)(x-s) ds}{[(x-s)^2 + z^2]^2} 
\]

(18b)

\[
\sigma_y(x, z) = \nu_1 (\sigma_x(x, z) + \sigma_z(x, z))
\]

(18c)

\[
\tau_{xz}(x, z) = -\frac{2z}{\pi} \int_{a_1}^{a_2} \frac{p(s)(x-s) ds}{[(x-s)^2 + z^2]^2} - \frac{2z}{\pi} \int_{a_1}^{a_2} \frac{q(s)(x-s)^2 ds}{[(x-s)^2 + z^2]^2} 
\]

(18d)

where \( x \) and \( z \) signify the directions along and normal to the tangential traction direction, respectively, \( y \) is the axis normal to the plane and \( a_1 \) and \( a_2 \) represent the contact boundaries. Once the pressure and tangential traction distributions for the contact region are known, the sub-surface stress field can be determined using Eqs. 18.

### 6.2.4. Numerical Solution Scheme

Equations 3 and 4 are solved simultaneously using the Newton-Raphson’s method. They can be discretized in a systematic way as follows:

\[
f_j = h_j = h_{00} + \frac{X_j^2}{2} - \frac{1}{\pi} \left( \frac{4\pi}{X_j} \right) \sum_{j=1}^{n} K_{ij} p_j \Delta X + \Lambda_j = 0
\]

\[
f_{n+1} = \xi \Delta X \sum_{j=1}^{n} p_j - \frac{\pi}{2} = 0
\]

(19)

where \( \xi \) is the scaling factor—to be defined later—and \( K_{ij} \), assuming constant pressure over each interval \( \Delta x \), can be written as [66]:

\[
K_{ij} = \left( X_j - X_i + \frac{\Delta X}{2} \right) \left( \ln \left( X_j - X_i + \frac{\Delta X}{2} \right) - 1 \right) - \left( X_j - X_i - \frac{\Delta X}{2} \right) \left( \ln \left( X_j - X_i - \frac{\Delta X}{2} \right) - 1 \right)
\]

(20)
Accordingly, we have \( n+1 \) unknowns which are the pressure values at \( n \) nodes (\( \bar{p}_i \)) and the constant \( \bar{h}_{oo} \). These unknowns are associated with \( n+1 \) equations. Typically when dealing with a rough surface, the contact width may exceed the Hertzian contact region, which has a normalized value between -1 and +1 (\( -1 \leq X \leq +1 \)). In the current simulation, the computational domain is chosen to be at least 2-3 times the dimensionless Hertzian contact width.

The computations begin by assuming a pressure distribution \( \bar{p}(x) \), and a constant \( \bar{h}_{oo} \). The Hertzian pressure distribution—for which pressure is only positive over \( -1 \leq X \leq +1 \) and zero everywhere else—can be used for this purpose. However, it is better that the values of all pressure nodes over the entire computation domain are set to a positive value. Here, a Hertzian, like pressure can be assumed but over the entire domain with, for example, the dimensionless maximum pressure of 1/3 and the width equal to the computation domain. In addition, to enforce the contact at all points at the start, \( \bar{h}_{oo} \) is set to an initial negative value. Although the pressure values at \( n \) nodes are initially unknown, the desired solution domain is the nodes that are in contact (\( n_c \)). Therefore, in each time step, when the calculation of the pressure distribution is completed, nodes with negative or zero pressure value are eliminated. In this fashion, the Newton-Raphson algorithm is only applied to the contact region, that is:

\[
\sum_{j=1}^{n_c} \left( \frac{\partial f_j}{\partial \bar{p}_j} \right) (\Delta \bar{p}_j)^{k+1} + \left( \frac{\partial f_j}{\partial \bar{h}_{oo}} \right) (\Delta \bar{h}_{oo})^{k+1} = -f_i^k \quad X_i \subset \Omega_c \quad n_c \leq n
\]

Once the pressure and \( \bar{h}_{oo} \) are determined for the current step, the calculations are repeated again by eliminating non-contacting nodes and the procedure is continued until the results converge. Up until this point, the calculations are performed only by assuming elastic deformation; that is, the restrictive conditions in Eq. 10 are still not satisfied. After obtaining the converged elastic results, the above procedure is repeated, but now the pressure values are limited by the hardness value of the softer body. The pressure values at nodes that are in plastic region are set equal to the hardness (according to Eq. 10 in dimensionless form) and the unknowns are the pressure in the remaining nodes since the pressure at all plastic nodes are already known. After calculating the new pressure distribution, all values are updated and the procedure is repeated until the
results converge within the specified tolerance. The flowchart of the calculation is shown in Figure 6.2.

For the calculation of the \( p^* \) and \( p^{**} \), since the input force is less than \( W \) and all the formulations are non-dimensionalized with respect to \( W \), the resulting distribution should be scaled in both magnitude and extent. To accomplish this, we introduce the following ratio into Eq. 19:

\[
\xi = \frac{W^*}{W} = 1 - \frac{1}{\mu} \left( \frac{Q}{W} \right) \quad \text{for } W^* \tag{22a}
\]

\[
\xi = \frac{W^{**}}{W} = 1 - \frac{1}{2\mu} \left( \frac{Q_{\max} \pm Q}{W} \right) \quad \text{for } W^{**} \tag{22b}
\]

The value of \( \xi \) is one for solving the pressure distribution problem and is calculated using Eq. 22 for the tangential traction distribution.

Figure 6.2. Flowchart of the numerical calculation
6.3. Results and Discussion

6.3.1. Pressure, tangential traction and sub-surface stress distribution for smooth and rough surfaces

In this section we consider two different types of surfaces: randomly generated Gaussian surface and sinusoidal surface. For brevity, details of the numerical procedure to produce Gaussian surface are not presented here; but they can be found elsewhere [67]. Figures 6.3 and 6.4 present the results of prediction of pressure and tangential traction distributions for four different values of standard deviations of surface heights—referred to as the roughness values hereinafter—from very smooth \( (R_q = 2.5 \times 10^{-8}) \) to very rough \( (R_q = 1.5 \times 10^{-5}) \), at the maximum tangential force. The von-Mises stress contour below the surface is also plotted in Figures 6.3 and 6.4. To verify the model, the case of smooth surface is chosen and the analytical solution is compared against the numerical simulation for both the pressure and the tangential tractions in Figure 6.3a. Analytical result, corresponding to the Hertzian solution, is illustrated by discrete marks for both the pressure and tangential traction distributions while solid lines illustrate the numerical solution. As seen, for the case of ideally-smooth surfaces, both approaches show identical results which authenticates the numerical predictions. In addition, the stick zone \( (2\bar{c}) \) can be clearly distinguished in Figure 6.3a. Results are depicted at the assumed condition where the tangential load is at its maximum value (here, \( \bar{Q}_{\text{max}} = 5 \times 10^{-5} \)) and hence, the stick zone is at its minimum value in a full loading cycle. In fact, the extension of the stick zone oscillates between the minimum stick zone and the contact boundaries, as the tangential load cycles between its maximum and minimum values. For the case of smooth contact, the size of the stick zone can be obtained analytically [44]. For the case of rough surface, however, the instantaneous stick zone is equal to the contact zone associated with the first and second corrective pressure terms \( (p^* \text{ and } p^{**}) \), which are calculated numerically. Accordingly, the traction distributions in the stick and slip zones and their variations due to cyclic loading are considered for the full loading cycle in the present study, but are not presented for brevity. It is, nonetheless, noted that variation and the overall traction distributions in the stick zone for rough surfaces are found to be comparable to that of the smooth surface which can be found in details in Ref. [44].

As illustrated, except for the case of ideally smooth surface in Figures 6.3a and 6.4a, the pressure profile and tangential traction are different from that of the Hertzian solution for both randomly generated Gaussian and sinusoidal surfaces, revealing the shortcoming of the Hertzian solution in treating the rough surface contact. The difference between Hertzian and deterministic approaches is more pronounced for the case of very rough surface.
Figure 6.3. Pressure and tangential traction distributions and the associated sub-surface von Mises stress distribution for (a) very smooth (b) smooth (c) rough and (d) very rough numerically generated Gaussian surfaces ($\bar{W} = 1.7 \times 10^{-4}$, $\bar{Q}_{\text{max}} = 5 \times 10^{-5}$, $\Omega = 0.016$, $\mu = 0.4$)
Given the elastic-perfectly plastic assumption, the maximum pressure is not allowed to exceed a certain limit (Figures 6.3d and 6.4d). The imposed limitation for the local pressure in the deterministic model significantly mitigates the local stress values for the rough surfaces. It is worth noting that the present model does not consider micro finite deformation of the individual asperities. In fact, due to the imposition of a local pressure limit, the inclusion of the individual asperity deformation is not expected to considerably modify the stress distribution. Furthermore, simulation of the individual asperity deformation generally requires either detailed finite element analysis or multi-level macro/micro contact analysis. These methods need an assumption for the asperity shape with enormous computational demand which is rather unwarranted compared to the benefits that can be achieved.

The results also show the important role of the surface parameters in the stress distribution. Also shown in Figures 6.3 and 6.4 are the associated von Mises subsurface stresses. As expected, for the smooth surface, the distribution is similar to the Hertzian type where the maximum stress occurs in region either below or at the surface depending on the amount of friction coefficient [68]. However for the case of rough surface, the location of maximum stress is always on the surface and at randomly scattered spots close to the surface peaks.

Since both the load balance and the separation equations are treated simultaneously, results converge rapidly. For a smooth surface, results converge after 8-10 iterations. For the case of very rough surface, typically, 18-20 iterations are needed for convergence. It should be mentioned that to facilitate convergence, especially for the case of rough surface, the “under relaxation” technique is applied to the pressure increments.

Another important factor is the total load. When the total load increases the pressure profile becomes more similar to the Hertzian type. Shown in Figure 6.5 are the pressure profiles and tangential tractions for identical surfaces but with different loads (light and heavy loads). As seen, at the high load of $\tilde{W} = 3.2 \times 10^{-4}$ (Figure 6.5a), the pressure profiles is close to the Hertzian pressure. In contrast, for the low load of $\tilde{W} = 3.2 \times 10^{-5}$, the pressure and tangential traction significantly deviate from the Hertzian solution. In fact, at high loads, results approach the Hertzian solution due to the dominance of the bulk deformation term in the separation formula (See Eq. 1). A similar conclusion has been reported using the statistical approach [37].
Figure 6.4. Pressure and tangential traction distributions and the associated sub-surface von Mises stress distribution for (a) very smooth (b) smooth (c) rough and (d) very rough sinusoidal surfaces, $\Lambda(X) = \sqrt{2R_q} \cos \left(\frac{2\pi X}{0.25}\right)$, $(\bar{W} = 1.7 \times 10^{-4}$, $\bar{Q}_{\text{max}} = 5 \times 10^{-5}$, $\Omega=0.016$, $\mu=0.4)$
Figure 6.5. Pressure and tangential traction (a) heavy load $W = 3.2 \times 10^{-4}$, $Q_{\text{max}} = 1 \times 10^{-4}$ (b) light load ($W = 3.2 \times 10^{-5}$, $Q_{\text{max}} = 1 \times 10^{-5}$ and for $\Omega = 0.016$ and $\mu = 0.4$)

6.3.2. Application to Fretting Fatigue Crack Initiation

6.3.2.1. Crack initiation criterion

One of the most practical criteria for analysis of fatigue crack initiation risk is due to the work of Smith-Watson-Topper (SWT), also known as the critical plane criterion [69]. It is defined by the SWT parameter with the following definition:

$$\Gamma_{\text{SWT}}(X,Z) = \left\{ \frac{\sigma_{n}(X,Z,\theta,T)_{\text{max}} \times \Delta \varepsilon_{n}(X,Z,\theta)}{2} \right\}_{\text{max} \forall \theta}$$

(23)

where $\Gamma_{\text{SWT}}(X,Z)$ represents the SWT parameter on and below the surface, with a higher SWT parameter corresponding to a higher cracking risk. The parameter $\sigma_{n}$ is the dimensionless stress at time $T$, and $\Delta \varepsilon_{n}$ is the maximum strain range, both normal to the plane at the angle of $\theta$. The values of normal stress and strain are calculated for all points on and below the surface, in all directions and for a complete loading cycle over time, $T$. At each point, the value of SWT parameter is equal to the largest value of the product of the maximum normal stress and the maximum normal strain range experienced during the loading cycle considering all of the directions. It is worth mentioning that there are infinite numbers of possible planes (directions) to be considered. Here, we calculate the values at every $1^\circ$. The plane with the maximum value of the SWT parameter is called the critical plane.
The SWT parameter is calculated for a full loading cycle and presented in the following at different surface roughness values. Illustrated in Figure 6.6, is the distribution of the SWT parameter on and below the contact surface for the case of a smooth surface. As seen, the maximum value of SWT parameter occurs at the surface. Below the surface the value of the SWT parameter decreases rapidly.

Figure 6.7 shows the SWT parameter plotted at the surface \((Z = 0)\) for four different surface roughness values. As stated before, the SWT parameter represents the fatigue crack initiation risk. According to Figures 6.7a & 6.7b, for the “very smooth” as well as “smooth” surface, the highest crack initiation risk occurs at the contact edges \((X = \pm 1)\), which is also verified numerically by others using either the SWT [70] or other crack initiation criteria [58, 71, 72]. Experimental studies involving optical observation of the fretting region also confirm this finding [70, 72-74]. Furthermore, for the very smooth and the smooth surfaces, the SWT value falls to zero near the center of the contact zone. As the surface roughness increases, the risk of crack initiation becomes greater while the highest SWT value does not necessarily occur exactly at the contact edges (See Figures 6.7c & 6.7d). For instance, the value of SWT parameter for the very rough surface is almost 2.5 times that of the very smooth surface but the highest crack initiation risk still occurs close to the contact edges (approximately at \(X = -0.9\)). However, in contrast to the smooth surface, the SWT parameter is not zero near the central region of contact.

![Figure 6.6. The SWT parameter for the entire domain for a smooth surface \((\bar{W} = 1.7 \times 10^{-4}, \bar{Q}_{max} = 5 \times 10^{-5}, \Omega = 0.016 \text{ and } \mu = 0.4)\)]
Figure 6.7. Distribution of the SWT parameter at the surface for (a) very smooth, (b) smooth, (c) rough and (d) very rough surface ($\bar{W} = 1.7 \times 10^{-4}$, $\bar{Q}_{\text{max}} = 5 \times 10^{-5}$, $\Omega = 0.016$, $\mu = 0.4$)

6.3.2.2. Crack initiation risk

Perusal of the available literature on the effect of surface roughness on the fretting crack initiation reveals a scarcity of experimental data. To the best of the authors’ knowledge, the only quantitative experimental study on the surface roughness effect is due to the work of Proudhon et al [75]. The experimental results obtained by Proudhon et al. [75], to a large extent, confirm the findings in this study. These results confirm that a higher crack initiation risk occurs at higher roughness values with its location close to the contact edges [75]. In what follows, the variation of crack initiation risk versus the tangential load amplitude is numerically predicted for three different surface roughness values. The predictions are then compared with the experimental results obtained by Proudhon et al. [75].

Using the SWT parameter, defined earlier, it is possible to define and calculate the crack initiation risk $d_{\text{SWT}}$ at a specific number of cycles ($N$) [75]:

$$d_{\text{SWT}} = \ldots$$
\[ d_{\text{SWT}} = \frac{\Gamma_{\text{SWT}} \times p_H}{\frac{\sigma'_f}{E} (2N)^{2b'} + \sigma'_f \varepsilon'_f (2N)^{b'+c'}} \]  

(24)

where \( \sigma'_f \) is the fatigue strength coefficient, \( b' \) and \( c' \) denote the fatigue strain and fatigue ductility exponents respectively, and \( \varepsilon'_f \) represents the fatigue ductility coefficient. It is worth noting that the SWT parameter is dimensionless with respect to \( p_H \) and therefore it should be multiplied by \( p_H \) in Eq. 24. For a given number of cycles \( (N) \), if the value of \( d_{\text{SWT}} \) is greater than one, a crack is likely to be initiated, whereas values lower than one indicate no cracking risk. It is obvious that when the value of \( d_{\text{SWT}} \) is one, Eq. 24 reduces to the well-known SWT fatigue life prediction formula [69].

6.3.2.3. Averaging technique

In order to predict the crack initiation risk for a given set of material properties and at specific number of cycles, one needs to calculate the SWT parameter over the contact surface as explained earlier. However, as shown in Figures 6.6 and 6.7, the resulting distributions of the SWT parameter exhibit a very high gradient near the contact edges. These intense variations of the SWT parameter occur as a corollary of the intense stress and strain gradients near the contact edges, which often occur over a length scale comparable to the materials microstructure. As a consequence, any life prediction methodology that uses the maximum local values is bound to produce unrealistic results. As a remedy, many studies suggest that an appropriate life prediction parameter must necessarily be an averaged quantity in order to resolve this issue [49, 58, 70, 71, 75-82]. However, the available body of evidence in literature does not establish a unique definition of either the size or the shape of the averaging zone [58]. Nonetheless, in view of the recent studies [58, 80], the slip width appears to be a pertinent parameter for defining the size of the averaging zone in line-contact fretting configuration. Therefore, following the definition proposed in [58], the averaging region is centered on the point where the value of the SWT parameter is maximum, and has a length equal to the slip width \( (b-c) \). The size of the averaging zone defined in this fashion, is variable and, in general, depends on the loading conditions as well as the contact width. It is, however, worthwhile to note that, customarily, experimental observations are used to determine the averaging zone which best fits the experiments. In the present study, in contrast, it is attempted to use a definition for the averaging zone which is previously found to produce consistent predictions [58], independent of the experimental results. Employing the averaging region described above, the average value of the SWT parameter is obtained and the cracking risk is calculated using Eq. 24 for a specific number of cycles, \( N \). The results are presented in the following section.
6.3.2.4. Prediction of the crack initiation risk

The $d_{SWT}$ parameter, as a function of the maximum tangential force per unit length, $Q_{\text{max}}/l$, is calculated and plotted in Figure 6.8 for three different randomly generated surfaces with different mean roughness values in line-contact configuration (e.g. contact of a cylinder and a flat surface). The graphs in Figure 6.8 are obtained for a set of loading conditions including $W/l = 280$ N/mm, $R=0.049$ m, $l=0.0044$ m, $E_1, E_2=72$ GPa and $N=50,000$. The associated dimensionless parameters are specified in the figure caption and the material properties are given in Table 1. As expected, the value of $d_{SWT}$ increases continuously with the $Q_{\text{max}}$, however at different rates for each different roughness value. A critical value can be defined for the amplitude of the tangential force ($Q_c$) that results in initiation of a crack at the given number of cycles. When the value of $d_{SWT}$ becomes equal to one (marked by a star on each graph), $Q_{\text{max}}$ is equal to its critical value ($Q_c$) and a crack is very likely to be initiated at $N=50,000$ cycles. Examination of Figure 6.8 clearly indicates that a rougher surface tends to reduce the value of critical tangential force amplitude. In other words, for a given tangential force amplitude, the crack initiation risk is higher for higher surface roughness.

![Figure 6.8. The crack initiation risk as a function of $Q_{\text{max}}$ for different surface roughness values ($W = 1.4 \times 10^{-4}$, $\bar{Q}_{\text{max}} = 0 \sim 1.5 \times 10^{-4}, \Omega = 0.02, \mu = 1.1$)](image)

The determination of the critical tangential force amplitude can alternatively be achieved using a trial-and-error procedure, in which the value of $Q_{\text{max}}$ (as an input) is updated recursively until $d_{SWT}$ becomes close enough to one ($d_{SWT} = 1 \pm 0.05$). The error margin is defined as a criterion for stopping the calculations. Using this procedure, the critical tangential force amplitude is predicted for different surface conditions corresponding to the experimental work of Proudhon et al. [75], in which the fretting crack initiation in an aluminum alloy (2024-T351) was
investigated. The experimental work of Proudhon et al. [75] investigates a cylinder-versus-flat configuration (line-contact configuration) involving flat specimens made of Al 2024-T351 and cylindrical counter bodies made of Al 7075-T6 [75]. The surfaces of the flat specimens were polished to achieve a smooth surface ($R_a \approx 0.05 \mu m$). Cylindrical counter bodies were produced with three different mean roughness values of 0.11, 0.6 and 0.75 µm. For each surface roughness value, the investigation involved determination of the critical tangential force amplitude leading to initiation of a crack at $N=50,000$ for a given normal force value (280 N/mm). This was accomplished by varying the tangential force amplitude in an incremental fashion until a crack is observed in the specimens [75]. Material properties of the specimens used in the experiments are shown in Table 6.1. The radius of the cylindrical counter body was 49 mm, the contact length was 4.4 mm and the friction coefficient was reported as $\mu=1.1\pm0.1$.

<table>
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<th>Material</th>
<th>$E$ (GPa)</th>
<th>$\nu$</th>
<th>$\sigma_f'$ (MPa)</th>
<th>$\epsilon_f'$</th>
<th>$b'$</th>
<th>$c'$</th>
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<td>Al2024T351</td>
<td>72.4</td>
<td>0.33</td>
<td>714</td>
<td>0.166</td>
<td>-0.078</td>
<td>-0.538</td>
</tr>
</tbody>
</table>

Shown in Figure 6.9 are the predictions of the critical tangential force amplitude as a function of the mean surface roughness. It is noted that the roughness values are reported as the mean roughness values instead of dimensionless values for easier comparison with the experiments in Ref. [75]. The simulations are carried out for randomly generated surfaces. A constant normal force value of 280 N/mm is used in the simulations. A large number of simulations had to be carried out so as to account for the randomness in the nature of the generated surface profiles. In this fashion, a prediction band is obtained for the critical tangential force amplitude. As seen, the prediction band converges to a line as the surface roughness decreases. This behavior is, in fact, anticipated in view of the diminishing effect of randomness at lower roughness values.

Also shown in Figure 6.9 are the experimentally obtained critical tangential force amplitudes according to the work of Proudhon et al. [75]. Regardless of the scatter in the simulation results, they reasonably agree with the experiments. The simulations generally predict lower critical tangential force amplitudes for higher roughness values. However, discrepancies are observed between the predictions and experimental measurements which can be ascribed to a number of factors including, but not limited to, the definition of the averaging technique, the exact value of the friction coefficient, and other possible differences that inevitably exist between the experimental conditions and simulations, such as the exact surface properties in terms of the distribution and orientation of the surface asperities. In this regard, further experimental data for other material types, friction coefficients and surface properties, are needed in order to conclusively identify a possible source of discrepancy. Nevertheless, the overall examination of
the results in Figure 6.9 indicates a reasonable agreement between the predictions of the critical tangential force amplitude and the experimental measurements.

![Figure 6.9](image)

**Figure 6.9.** The critical tangential force amplitudes as reported experimentally [75] and obtained numerically for different mean roughness values using randomly generated surfaces

This case study, in fact, further supports the authenticity of the proposed methodology for calculation of the deterministic surface tractions in rough line contacts. The contact stresses calculated as a result of these tractions, can be used to predict surface failure phenomena in a variety of contact configurations.

### 6.4. Conclusions

Consideration of the surface roughness is essential to gain a thorough understanding of the load-carrying capacity, friction, wear, thermal and electrical properties as well as the life of the components in contact under dry or heavy lubricated condition. For the treatment of micro-scale damage—e.g. micro-pitting or surface crack initiation in rough surface fretting fatigue or any near surface failure—one must follow a deterministic approach to determine the sub-surface stress distribution for the contact region with real surface profile.

In the current work, a robust and comprehensive method is presented in detail to predict the deterministic pressure distribution for rough line-contact problems assuming elastic-perfectly plastic behavior of the asperity tips. The described algorithm is conceptually simple and can be readily implemented on the computer. The algorithm is capable of solving the surface separation and load balance equations together with an iterative procedure and has a fast convergence rate. In addition, adopting the Ciavarella-Jager approach, the calculation of the deterministic
tangential traction distribution for cyclic loading condition in stick-slip regime is explained in
detail. The sub-surface stress field due to the application of normal and tangential forces is also
determined. The results show the important role of the surface roughness and normal load in the
deterministic pressure and tangential traction distributions. High surface roughness values and
low normal loads result in pressure and tangential traction distributions that significantly deviate
from the Hertzian solution. The approach can be easily utilized to evaluate the contact damage
evolution due to stick-slip (fretting) condition.

To illustrate the utility of the technique, the fretting crack initiation risk for surfaces with
different roughness values subjected to cyclic stick-slip condition was evaluated. To resolve the
stress gradient effect, a special averaging technique was adopted that yields consistent
predictions by automatically adjusting the averaging zone size based on the slip width. Results
show that higher surface roughness values increase the risk of crack initiation which agrees with
the available experimental observation. Both the simulation and experimental results indicate the
contact edges or their vicinity to be the most probable sites for crack initiation. In addition, the
overall examination of the numerical simulation results indicates a reasonable agreement
between the predictions of the critical tangential force amplitude and the experimental
measurements. This case study authenticates the prediction results of the proposed methodology
for calculation of the deterministic surface tractions in rough line contacts that can be used to
predict surface failure phenomena in a variety of contact configurations.

Nomenclature

\(a_1, a_2\) contact boundaries, (m)

\(b\) Hertzian contact half width, (m)

\(c\) stick zone half width, (m)

\(\bar{c}\) dimensionless stick zone half width, \(c/b\)

\(E_1, E_2\) modulus of elasticity of body 1 and 2, (GPa)

\(E'\) effective modulus of elasticity, \(\frac{1}{E'} = \frac{1-v_1^2}{E_1} + \frac{1-v_2^2}{E_2}\), (GPa)

\(H\) hardness of the softer material, (MPa)

\(h\) separation gap based on surface heights, (m)

\(h_{00}\) constant in the separation relationship

\(\bar{h}\) dimensionless separation

\(\bar{h}_{00}\) constant in the dimensionless separation relationship

\(K_{ij}\) influence coefficient

\(l\) contact length, (m)

\(N\) number of total nodes
number of nodes in contact
\[ n_c \]
number of cycles to crack initiation
\[ N \]
pressure, (MPa)
\[ p(x) \]
dimensionless pressure
\[ \tilde{p}(X) \]
maximum Hertzian pressure, (MPa)
\[ p_{H} \]
imaginary pressures to calculate corrective traction terms, (MPa)
\[ p^{*}(x), p^{**}(x) \]
tangential traction, (MPa)
\[ q(x) \]
dimensionless tangential traction
\[ \tilde{q}(X) \]
tangential traction at the maximum tangential force, (MPa)
\[ q_{\text{max}}(x) \]
corrective traction terms, (MPa)
\[ q^{*}(x), q^{**}(x) \]
ratio of the effective load to the normal load
\[ \zeta \]
tangential force, (N)
\[ Q \]
maximum force for the contact of a cylinder and a flat surface, (N)
\[ Q_{\text{max}} \]
radii of the contacting bodies 1 and 2 and their effective radius of curvature,
\[ R_{1}, R_{2}, \frac{1}{R} = \frac{1}{R_{1}} + \frac{1}{R_{2}}, (m) \]
mean roughness value, (m)
\[ R_{a} \]
standard deviation of surface heights, (m)
\[ R_{q} \]
dimensionless standard deviation of surface heights, \( \bar{R}_{q}/R \)
\[ S_{y} \]
yield stress, (MPa)
\[ \bar{S}_{\text{vm}} \]
dimensionless von Mises stress
\[ W \]
total normal force for the contact of a cylinder and a flat surface, (N)
\[ W^{*}, W^{**} \]
effective loads for calculation of the corrective term for stick region, (N)
\[ \bar{W} \]
dimensionless total normal force
\[ x, y, z \]
spatial coordinates, (m)
\[ X, Z \]
dimensionless spatial coordinates
\[ \bar{\sigma}_{n} \]
normal stress non-dimensionalized with respect to \( p_{H} \)
\[ \mu \]
friction coefficient
\[ \nu_{1}, \nu_{2} \]
Poisson’s ratios for bodies 1 and 2
\[ \Omega \]
ratio of the softer surface hardness to the effective elasticity modulus, \( H/E' \)
\[ \Omega_{c} \]
contact region
\[ \Gamma_{\text{SWT}} \]
SWT parameter
\[ \lambda(x) \]
surface profile, (m)
\[ \Lambda(X) \]
dimensionless surface profile
6.5. References


Chapter 7: An Engineering Approach for the Prediction of Wear in Mixed Lubricated Contacts*

7.1. Introduction

Almost all tribological components ranging from bearings and gears to artificial hip and knee joints are susceptible to wear. While, comparatively, the severity of wear in a properly lubricated contact is less pronounced than in a dry contact, it, nevertheless, does occur. In fact, excessive lubricated sliding wear is still one of the most pervasive surface failure mechanisms in man-made machine components. In general, wear can also influence many of the operational characteristics of a mechanical system such as vibration and noise level, load-carrying capacity and friction.

Given its significance, it is not at all surprising that the literature is abundant with different methodologies for sliding wear evaluation. Since the pioneering work on dry adhesive wear by Holm [1] and Archard [2], numerous attempts on quantitative predictions of wear in both dry and lubricated contacts have been reported [3-18]. Yet a widely accepted model of wear remains elusive, and notwithstanding its remarkably complicated nature, a predictive methodology for estimating lubricated wear is of considerable interest in engineering practice.

Let us now direct our attention to a special regime of lubrication known as mixed elastohydrodynamic lubrication (EHL), in which the hydrodynamic film may not be adequately large to completely separate the two interacting surfaces. As a result, the surface asperities come into intimate contact. Interestingly, in conjunction with the fluid film, these asperities do provide some load-carrying capacity. Nevertheless, this direct solid contact between lubricated surfaces leads to sliding wear in tribological components operating under mixed EHL condition.

The current study proposes an engineering approach for the prediction of steady-state lubricated wear in mixed EHL regime. Based on the load-sharing concept, empirical formula for the elasto-plastic asperity contact of two dry rough cylinders previously developed by the authors [19] are extended to the case of lubricated contacts. Also, a steady state thermal analysis is performed to account for the average temperature rise inside the contact. Moreover, the continuum damage mechanic (CDM) approach is utilized to estimate the Archard wear coefficient for dry contact as an alternative to experimental measurements. This coefficient is then modified using the average temperature rise in conjunction with the fractional film defect concept and considering the load-carrying shared between the asperities and the fluid. Results are presented for the steady-state sliding wear in mixed EHL regime in line-contact configuration.

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is worth emphasizing that in contrast to most wear studies, in this approach the wear coefficient for lubricated contact is obtained based on a purely predictive methodology rather than by experimental measurement.

7.2. Wear Modeling Methodology

Presented in this section is the approach to modify the original Archard wear equation for the treatment of mixed lubrication. The methodologies to obtain each of these parameters are also presented.

7.2.1. Modification of Archard wear equation for the mixed lubrication problem

According to Archard theory [2], the steady state wear volume in dry sliding condition can be predicted by:

$$ V = k \cdot \frac{W \cdot S}{H} $$

where $V$ is the wear volume, $k$ represents the wear coefficient for dry sliding condition, $W$ denotes the total load, $S$ is the sliding distance, and $H$ symbolizes the hardness. The same equation can be used in the case of mixed-lubricated configuration, however, with two modifications [5]. They are: the consideration of the fractional film defect and implementation of the load-sharing concept. These concepts are described as follows.

7.2.1.1. Fractional film defect

Our goal is to ultimately calculate wear for mixed-lubrication regime in which only a part of the load is carried by the asperities, i.e. solid-to-solid contacts. Although there is no hydrodynamic pressure in these contact spots, still the existence of oil molecules results in the reduction of metal on metal contact, lowering the wear coefficient. This is taken into account by multiplying the dry sliding wear coefficient by a so-called fractional film defect coefficient $\psi$ ($k_b = k \times \psi$) [5]. At this stage of the modification, the resulting parameter $k_b$ can be interpreted as the Archard wear coefficient for boundary lubrication wherein no hydrodynamic fluid pressure exists to detach the contacting surfaces. Physically, the oil molecules trapped between the surfaces reduce the direct metal on metal contacts, and thus $\psi$ can be considered to be the ratio of the direct metal-on-metal contact area to the real area of contact. In other words, $\psi$ can be viewed as the probability that an asperity comes into contact directly with another asperity while it passes over the mating surface in a region that is not occupied by “adsorbed lubricant molecules” [5].

The $\psi$ parameter is calculated based on thermal desorption theory, and using the following relationship suggested by Kingsbury [20] and Rowe [21]:

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\[ \psi = 1 - \exp \left\{ - \left[ \frac{a_s}{u_s t_0} \exp \left( - \frac{E_a}{R_g T_s} \right) \right] \right\} \tag{2} \]

where \( u_s \) is the sliding velocity, \( a_s \) denotes the diameter of the area associated with an adsorb molecule, \( t_0 \) is the fundamental time of vibration of the molecule in the adsorbed state, \( E_a \) represents the heat of adsorption of the lubricant on the surface, \( R_g \) is the gas constant and \( T_s \) symbolizes the absolute temperature of the surface. The only unknown in Eq. (2) is the surface temperature. In order to estimate the surface temperature, a simplified thermal model developed by Akbarzadeh and Khonsari [22] is used here. Since only the average surface temperature rise is required, there is no need for complicated and time-consuming simulation involving full thermal EHL analysis. Here, a simplified energy equation is solved. The boundary conditions are the temperature of the surrounding surfaces, estimated based on the approach developed by Blok [23] and Jaeger [24]. In addition, the effect of asperities surface traction is considered in the model as a heat generation term in boundary condition equations. The interested reader is referred to [22] for details.

7.2.1.2. Fluid and asperities load sharing

In mixed lubrication, a part of the load is carried by the fluid and a part by the asperities. The load carried by the fluid is associated with the elastohydrodynamic regime and does not contribute to wear whereas the load directly carried by the asperity contact is directly responsible for wear. In order to treat this, the total load needs to be replaced by the asperity load in Archard wear relationship. We propose to evaluate the portion of the load carried by asperities in mixed lubricating condition using the load-sharing methodology wherein the total load \( (W) \) can be written as the sum of two portions [25]:

\[ W = W_f + W_a \tag{3} \]

where \( W_f \) is the load carried by the fluid and \( W_a \) is the load carried through direct surface contact, i.e. the asperity contact. Likewise, the total interface pressure \( (p) \) is comprised of two parts, namely asperity pressure \( (p_a) \) and fluid pressure \( (p_f) \):

\[ p = p_f + p_a \tag{4} \]

Accordingly, two load-sharing ratios (scaling factors) are defined as follows.

Scaling factor for the fluid part is:

\[ \gamma_1 = \frac{W}{W_f} = \frac{p}{p_f} \tag{5a} \]

and the scaling factor for the asperity portion is:
The following relationship for the sum holds:

\[ \frac{1}{\gamma_1} + \frac{1}{\gamma_2} = 1 \]  

(5c)

Substituting (5b) into Eq. (1) and introducing the fractional film defect coefficient, Eq. (1) yields:

\[ V = k \psi \frac{W_a S}{H} = k \psi \frac{W S}{\gamma_2 H} = k_1 \frac{W S}{H} \]

(6)

where \( k_1 \) can be considered as the wear coefficient for mixed lubrication regime. Typically, this parameter is obtained experimentally for the desired condition. In the current study, in contrast, this coefficient is obtained by simulation rather than experimental measurements. In Eq. (6), there are three unknowns in the modified Archard equation; \( k, \gamma_2 \) and \( \psi \). In what follows, a procedure to calculate each of these parameters is presented.

### 7.2.2. Determination of \( k \)

#### 7.2.2.1. Experimentally measured values

Similar to \( k_1 \), \( k \) is typically obtained experimentally by performing, for example, a pin-on-disk dry sliding test. Values for \( k \) for a variety of contacting couples can be found in the literature (see Ref. [26], for example) and when available can be directly used in Eq. (6). Please see, for instance, the example provided in Section 3.3.

#### 7.2.2.2. Numerically predicted values by continuum damage mechanics

Using the methodology previously developed by the authors [27]—as an alternative to experimental measurements—we proceed to determine the dry wear coefficient based on the fatigue theory of adhesive wear. A brief description of the approach is presented in this section.

The idea of treating adhesive wear as a fatigue phenomenon is pioneered by Kragelskii [28] and Rozeanu [29]. They suggested that a wear fragment is produced after “successive steps of the fatigue”. The wear coefficient, \( k \), can be considered as \( 1/3 \) of the probability of formation of a wear fragment from softer material each time two asperities come into contact with each other. Accordingly, Kragelskii postulated that a loose wear particle is formed by fatigue process and interpreted the probability of formation of wear debris as the inverse of the number of cycles that an asperity experiences stress before it breaks. Published evidence that corroborates the existence of a relationship between fatigue and sliding wear can be found in Refs. [30-38]. Using the fatigue theory of adhesive wear, \( k \) can be simply expressed by [27], [39]:

\[ k = \frac{1}{3N} \]

(7)
where $N$ is the total number of cycles needed for an asperity to break. This concept allows us to relate wear to the fatigue properties. Of particular interest here is the treatment of fatigue damage using the thermodynamically-based continuum damage mechanics (CDM). As demonstrated in recent studies [40-44], the damage growth in different types of materials and the associated irreversible processes can be effectively modeled within the thermodynamic framework. Although application of CDM has been recently demonstrated in several tribological problems [27, 45-52], it is still in embryonic stage and remains largely unexplored. In the present study, the CDM theory is utilized, as an auspicious alternative to conventional empirical fatigue theories, to calculate the number of cycles to failure for an asperity.

Recently, Bhattacharya and Ellingwood developed a thermodynamic framework for the damage growth analysis [53]. The CDM model proposed by Bhattacharya and Ellingwood is capable of predicting the number of cycle ($N$) to failure in terms of only the macroscopic material parameters. Assuming that the system evolves through a set of equilibrium states prior to localization of damage and applying the first and the second laws of thermodynamics along with the Ramberg-Osgood type equation for the hysteresis loop, Bhattacharya and Ellingwood derived the following equation of isotropic damage growth which gives $D_i$, the damage in $i^{th}$ cycle [53]:

$$D_i = 1 - (1 - D_{i-1}) F_i \quad \text{if } S_{max} \geq S_e$$

where

$$F_i = \frac{(1 + 1/M)^{-1} \Delta \varepsilon_{ad}^{1+1/M} - \Delta \varepsilon_{pl}^{1/M} \Delta \varepsilon_{ad} + C_i}{(1 + 1/M)^{-1} \Delta \varepsilon_{pmi}^{1+1/M} - \Delta \varepsilon_{pl}^{1/M} \Delta \varepsilon_{pmi} + C_i}$$

and

$$C_i = \frac{3S_f}{4K} \frac{\Delta \varepsilon_{ad}^{1+1/M}}{1 + 1/M} + \Delta \varepsilon_{pl}^{1/M} \Delta \varepsilon_{pl}$$

otherwise, if $S_{max} < S_e$

$$D_i = D_{i-1}$$

$S_e$ is the endurance limit, $S_f$ denotes the true failure stress, $S_{max}$ is the applied stress at its maximum value, $M$ represents the cyclic hardening exponent, $K = 2^{1-1/M} H_m$, where $H_m$ is the cyclic hardening modulus, and the various $\Delta \varepsilon$ variable represents the strain values from stress-strain curve of loading associated by the related stresses as described in Refs. [27, 53]. For a virgin material the damage variable is assumed to be zero. The state of the damage, i.e., the damage variable, is calculated using Eq. (8) where damage per cycle is evaluated recursively until after $N$ cycles it reaches a critical value, $D_c$. This represents the number of cycles to “failure of an asperity”. At that point, the computations are completed and the number of cycles to failure is recorded. Subsequently, wear coefficient is estimated based on Eq. (7). Interested reader is referred to Ref. [27] for further details.
7.2.3. Calculating $\gamma_2$

As established before, in the present study the load-sharing concept [25] is adopted to address the mixed lubrication problem. This powerful technique was applied by Gelinck and Schipper [54] who formulated an approach specifically for the line-contact problem. They developed a curve-fitted equation based on the Greenwood-Williamson (GW) statistical approach [55] with the Gaussian asperity heights distribution to predict the contact properties for dry rough line-contact configuration wherein the deformation of asperities was assumed to be completely elastic based on the original GW model [56]. In a separate study [54], they used Moes’s equation for the central film thickness of the smooth contact [57] in conjunction with their curve-fitted formula for dry rough line contact [56] to treat the mixed lubricated contact problem. This approach has been further developed and extensively put to use in the treatment of mixed lubrication problems in various applications with promising results [22, 58-61].

In the present study, curve-fitted formula provided recently by the authors [19] that accounts for elastic-elasto/plastic-fully plastic deformation of the asperities is utilized. In addition, Pan and Hamrock’s film thickness equation [62] is used instead of Moes’ equation. In late 80s, Pan and Hamrock—as compared to their famous preceding studies of Ertel-Grubin (1949) and Dowsen-Higgins (1962)—modeled the EHL line-contact problem with more precise numerical simulation (due to the advent of new computes). The developed model has also no limitation for the normal load. Besides, recent comprehensive and precise numerical simulation of EHL line-contact problem shows very good accordance with Pan and Hamrock’s film thickness equation [63]. In what follows we present the methodology to calculate $\gamma_2$. By the virtue of the load-sharing concept and scaling factors, the problem is divided into two sections: fluid part and asperity contact.

7.2.3.1. Fluid part

Pan and Hamrock’s central film thickness equation for the smooth line-contact configuration is given by [62]:

$$\bar{h}_c = 2.922 \left( \frac{\bar{W}}{2} \right)^{-0.166} \left( \frac{\bar{U}}{2} \right)^{0.692} (2G)^{0.47} \quad (9)$$

with the dimensionless parameters defined as:

$$\bar{h}_c = \frac{h_c}{R}, \quad \bar{W} = \frac{W}{lE'R}, \quad \bar{U} = \frac{\mu_0 u_r}{E'R}, \quad G = \alpha E'$$

(10)

where $h_c$ is the central film thickness, $R$ denotes the effective radius of curvature, $l$ is the length of the contact, $E'$ represents the effective modulus of elasticity, $\mu_0$ is the lubricant viscosity at the ambient pressure, $u_r$ represents the effective rolling velocity of two contacting surfaces $(u_1 + u_2)/2$, and $\alpha$ is the pressure-viscosity coefficient which is given as [62]:

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where $Z$ is the pressure-viscosity index. Note that the factor of two is introduced in Eq. (9) to account for the difference between the definition of effective modulus of elasticity in the current study and that of the Pan and Hamrock’s definition.

For mixed lubrication regime, in which the effect of roughness cannot be neglected, Eq. (9) can be modified considering the fact that only a part of the total load, i.e. $1/\gamma_1$, is supported by the fluid (see Appendix A for details). The resultant modified equation, accounting for the roughness effect, is given as:

\[
\bar{h}_c = 2.922 \left( \frac{W}{2} \right)^{-0.166} \left( \frac{U}{2} \right)^{0.602} (2G)^{0.47} \gamma_1^{0.222}
\]

\[ \text{(12)} \]

At high slide-to-roll ratios, the temperature rises considerably inside the contact due to viscous shear heating. It is worth mentioning that the above film thickness equation is based on the isothermal solution of the EHL problem in which the thermal effect is not considered. According to [64, 65], at the typical rolling speeds the temperature has a modest influence on the shape of the pressure and film profile and in general very little effect on the magnitude of the film thickness [64]. However, later studies show that the minimum film thickness experiences a considerable reduction [66, 67] due to the relative sliding in contact region and its consequent temperature rise. In fact, several correction factors are available in the open literature for the purpose of modifying the minimum film thickness equation (e.g. Ref. [67]). Nonetheless, examination of the film profiles presented in the literature [64, 66, 68] reveals that unlike the minimum film thickness, the central film thickness and the associated central pressure are not appreciably affected by the sliding inside the contact. At high rolling speeds, however, the inlet shear heating can affect the film profile and consequently the central film thickness [62, 69]. Thermal correction factors considering this effect are available in the literature (e.g., Ref. [69]) for the central film thickness. For the range of input parameters in this study, this thermal correction factor reduces the film thickness by less than 2%. Hence, this correction factor is not considered here and, accordingly, Eq. (12) is used as the final solution for the central film thickness value.

7.2.3.2. Asperity contact

According to Kogut and Etsion [70] for the contact of two flat surfaces with elastic-elasto/plastic-fully plastic asperity behavior and Gaussian distribution (the peaks and the valleys are distributed statistically even), the contact pressure as a function of the average separation ($\bar{h}$), in non-dimensionalized form, is given by:
where

\[ \phi^*(z^*) = \frac{1}{\sqrt{2\pi}} \frac{\sigma}{\sigma_s} e^{-\left(\frac{\sigma_z^2}{2\sigma_s^2}\right)} \]

\[ \bar{d} = \frac{d}{\sigma} = \frac{h - y_s}{\sigma} \]

\[ \xi = n\beta\sigma = \bar{n}\beta\sigma \]

\[ \overline{\omega} = \bar{z} - d \]

\[ \omega^* = \left(\frac{\pi K \Omega}{2}\right)^2 \left(\frac{\beta}{\sigma}\right) \]

with the following dimensionless parameters:

\[ \bar{\sigma} = \frac{\sigma}{R} \]

\[ \bar{\beta} = \frac{\beta}{R} \]

\[ \bar{n} = nR^2 \]

\[ \bar{z} = \frac{z}{\sigma} \]

\[ \Omega = \frac{h}{R} \]

\[ \frac{\sigma}{\sigma_s} = \frac{\xi}{\sqrt{\xi^2 - 3.71693 \times 10^{-4}}} \]  \hspace{1cm} (15a)

\[ \frac{y_s}{\sigma} = \frac{0.045944}{\xi} \]  \hspace{1cm} (15b)

In these equations, \( p_H \) is the maximum Hertzian pressure; \( h \) is the separation between two surfaces which can also be considered as a lubricant film thickness in lubrication regime; \( n \) and \( \beta \) denote the asperity density and radius respectively; \( \sigma \) is the standard deviation of the surface heights distribution (henceforth, referred to as the surface roughness); \( \sigma_s \) represents the standard deviation of summit heights distribution and \( y_s \) denotes the distance between the mean line of the summit heights and that of the surface heights, both of which can be calculated based on \( \sigma \) for the Gaussian surface (see Eq. 15). The parameter \( z \) represents the asperity (summit) height measured from the mean line of summit heights as depicted in Figure 7.1. Referring to the work of Bush et al. [71] and McCool [72] for isotropic rough surfaces with Gaussian distribution of surface heights, the following relationships apply [19]:

\[ \frac{\sigma}{\sigma_s} = \frac{\xi}{\sqrt{\xi^2 - 3.71693 \times 10^{-4}}} \]  \hspace{1cm} (15a)

\[ \frac{y_s}{\sigma} = \frac{0.045944}{\xi} \]  \hspace{1cm} (15b)
Recently, Beheshti and Khonsari [19] have carried out extensive numerical simulations for the line-contact configuration (Figure 7.1) based on the statistical micro-asperity contact model of Kogut and Etsion [70] and reported convenient formulas for the prediction of dry contact parameters such as central pressure, contact half-width, real area of contact and pressure distribution function. According to [19], the central (maximum) contact pressure at dry rough line-contact configuration is given by:

$$
\bar{p}_{\text{central}} = \frac{1}{\sqrt{1 + \left(1.1188 \sigma^{-0.1531} \beta^{-0.1203} \sigma^{-0.6304} \Omega^{-0.7161} \omega^{-0.1423} \right)^{0.1396}}} \tag{16}
$$

Analogous to the fluid part discussed in section 2.3.1, only $1/\gamma_2$ portion of the total load is carried by the asperities and, hence Eq. (16) should be modified accordingly (see Appendix B). The resultant modified central pressure equation is given as:

$$
\left(\bar{p}_a\right)_{\text{central}} = \frac{1}{\gamma_2 \sqrt{1 + \left(1.1188 \sigma^{-0.1531} \beta^{-0.1203} \sigma^{-0.6304} \Omega^{-0.7161} \Omega^{-0.1423} \gamma_2^{-0.2954} \right)^{0.1396}}} \tag{17}
$$

7.2.3.3. Combining the smooth EHL and dry rough contact models to calculate $\gamma_2$

Our objective here is to find $\gamma_2$ (or alternatively $\gamma_1$) to be used in the modified Archard equation. The modified central film thickness equation, Eq. (12), has two unknowns to be determined: $\gamma_1$ and $\overline{h}_c$. Eq. (17) gives the pressure at the center of the contact. Alternatively, Eq. (13) can also be used to calculate the central pressure if the central film thickness is known. Thus, another relationship is obtained by equating Eq. (13) to Eq. (17) for the center of the contact as follows:
Substituting the central film thickness from Eq. (12) into Eq. (18) and considering load-sharing concept, Eq. (5c), Eq. (18), in expanded form yields:

\[
\left( \bar{p}_a \right)_{central} = \xi \sqrt{ \frac{\pi}{R} \Phi_{KE} (\bar{h}_c) }
\]  

(18)

Eq. (19), with one unknown, is the only equation to be solved in order to determine the load sharing ratio, \( \gamma_2 \), in line-contact configuration. This equation can be easily solved for \( \gamma_2 \), for example, using the bisection technique.

In many published studies on the contact of rough surfaces, roughness values (\( \sigma \)) are reported while the values for \( n \) and \( \beta \) are not. It is worth mentioning that the effect of \( n \) and \( \beta \) on asperity load share is less appreciable compared to \( \sigma \) in their typical range [63]. In cases that \( \sigma \) is known and \( n \) and \( \beta \) are not directly reported, one may resort to an estimate considering the fact that \( \xi = n \beta \sigma \) typically varies between 0.02 and 0.1, and that \( \sigma / \beta \) ranges between 0.0001 and 0.1 [19, 25, 56, 73]. The sensitivity of the model on the statistical parameter choice is discussed in the example presented in Section 3.3.

After applying all the aforementioned modifications and calculating the unknowns (\( k, \gamma_2 \) and \( \psi \)), wear volume is estimated using Eq. (6). Figure 7.2 demonstrates the procedure and the steps required for the calculation of the wear volume.

7.3. Results and Discussion

7.3.1 Model Validation

Published experimental results that comprehensively report the essential input parameters to be used in the present wear model are really scarce. Wu and Cheng [7] provided wear measurement results for mixed elastohydrodynamic condition and, fortunately, their report also includes input values required to simulate the associated wear using the current model. To validate the model, the experimental results reported by Wu and Cheng [7] are compared with the simulation results for the steady-state wear.
They measured wear in a two-disk machine configuration where the diameters of the specimens (upper disk) and the supporting lower disk were 1.27 cm and 7.62 cm and both had a width of 0.635 cm. Initially, both disks had a hardness of Rc60–62, but in order to induce wear, the specimens were tempered to Rc56 (equivalent to 6013 MPa to be used in Archard wear volume equation and CDM simulations). The composite surface roughness was approximately 0.42 µm. According to [7], the experiment was carried out at the rolling velocity and the maximum Hertzian pressure of 1.83 m/s and 2.0 GPa, respectively, and the inlet temperature was 42 °C. Experiments were performed over a wide range of slip-to-roll ratios (SRR) from approximately pure rolling (SRR=0.001) to pure sliding (SRR=2.0). The specimen (AISI 52100) properties used in the CDM simulation in order to estimate $k$ are listed in Table 7.1.

<table>
<thead>
<tr>
<th>$E_1$ (GPa)</th>
<th>$H_m$ (MPa)</th>
<th>$S_f$ (MPa)</th>
<th>$S_e$ (MPa)</th>
<th>$M$</th>
<th>$D_e$</th>
<th>$f_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>206.9</td>
<td>3443</td>
<td>2586</td>
<td>768</td>
<td>6.22</td>
<td>0.25</td>
<td>0.25</td>
</tr>
</tbody>
</table>
Using the CDM simulation, the number of cycles to failure is calculated and subsequently the dry wear coefficient is estimated. Note that, alternatively, $k$ can be found in the literature. Its published value obtained experimentally is reported to be $5 \times 10^{-4}$ \cite{7, 26} while the CDM calculation estimates it as $4.43 \times 10^{-4}$ and the later value is used in the current study.

Table 7.2. Properties of the specimen and the oil for the EHL and thermal analysis \cite{7, 22}

<table>
<thead>
<tr>
<th>$\mu_0$ (Pa·s)</th>
<th>Z</th>
<th>$\rho_f$ (Kg/m$^3$)</th>
<th>$K_f$ (N/s·K)</th>
<th>$\rho_r$</th>
<th>$K_r$ (N/s·K)</th>
<th>$f_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.032</td>
<td>0.6</td>
<td>840</td>
<td>0.14</td>
<td>7850</td>
<td>47</td>
<td>0.13</td>
</tr>
</tbody>
</table>

Following the approach outlined in \cite{22}, an average temperature rise was estimated and substituted into Eq. (2) to determine the fractional film defect coefficient $\psi$. Tables 7.2 and 7.3 show the parameters that have been used in thermal analysis and the wear parameters associated with Kingsbury \cite{20} and Rowe \cite{21} approach, respectively.

Table 7.3. Values for wear parameters [7]

<table>
<thead>
<tr>
<th>$a$ (m)</th>
<th>$E_a$ (kJ/mole)</th>
<th>$R_g$ (J/mole·K)</th>
<th>$t_0$ (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3 \times 10^{-10}$</td>
<td>49</td>
<td>8.31</td>
<td>$3 \times 10^{-12}$</td>
</tr>
</tbody>
</table>

Table 7.4 illustrates the surface properties used in the rough EHL analysis. Wu and Cheng reported the surface roughness of 0.42 µm for unworn surfaces at the beginning of the tests. For lower SRR values (0.001-0.1) the surface roughness was 0.41 µm while they reported the minimum roughness of 0.34 µm after the test for larger SRR values. Therefore, in the current study the average of the roughness values before and after the test, which also change linearly with SRR values, are considered.

Table 7.4. Surface properties \cite{7, 19}

<table>
<thead>
<tr>
<th>$\sigma$ (m)</th>
<th>$\sigma/\beta$</th>
<th>$\xi=\beta/\sigma$</th>
<th>$H$ (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.38~0.41)$\times 10^{-6}$</td>
<td>(3.1~3.4)$\times 10^{-2}$</td>
<td>0.0571~0.0601</td>
<td>6013</td>
</tr>
</tbody>
</table>

Figure 7.3 shows the variation of the predicted volumetric wear per sliding distance—termed simply as wear henceforth—as a function of the slide-to-roll ratio for the geometry and loading conditions pertinent to the experimental results of Wu and Cheng and also their experimental wear measurement results. As seen, the wear results from the current simulation are in a good accord with the experimentally measured wear for different SRR values showing the capability of the approach to predict steady-state wear in mixed EHL problems.
Generally, for a given running period, one expects to see a larger accumulated wear volume when SRR increases. However, as seen in the measured wear results, in terms of the wear (wear for the same sliding distance), for the SRR varying between 0.001 and approximately 0.8, larger SRR values yield lower wear. There are two possible reasons for this observation as explained by Wu and Cheng. First reason is the fact that during the tests, the rolling velocity was kept constant for all different SRR values and as a result, to obtain a specific sliding distant, the specimens underwent more rolling in smaller SRR. This resulted in more asperity interaction that might have contributed to increasing wear. The second reason, as also documented by Wu and Cheng, is that according to the thermal desorption theory, a higher sliding velocity allows adsorbed molecules less time to detach from the surface. Hence, as seen in Figure 7.3, a decrease of wear with the increase in SRR is observed. However, this is only observed for small SRR values. In fact, there exist a transition point where for higher SRR values (here, approximately more than 0.8) a drastic rise in wear can be observed. Interestingly, the numerical results also show the same trend which can be attributed to the second reason (thermal desorption theory, Eq. (2)).

For high SRR values it seems that the model under predicts the wear. It can be attributed to the following reason: as mentioned before, current model is based on the statistical asperity contact model in conjunction with Johnson’s load sharing concept. Based on this theory, it is implicitly assumed that the film thickness for the entire contact area is constant and equal to the

![Figure 7.3. Comparison of the wear results vs. different slide-to-roll ratios based on the current approach with those reported by Wu and Cheng [7]](image)
central film thickness [54]. In fact, for typical EHL problem, it is a valid assumption, since the examination of complete film profile shows that except near the outlet, where the fluid film thickness abruptly drops to its minimum value, it is almost constant and close to central film thickness. As stated in section 7.2.3.1, unlike central film thickness, minimum film thickness reduces significantly at high SRRs. Consequently, higher wear happens at the outlet than that one would obtain with the model using central film thickness. The discrepancy between the experimental and the numerical simulation results at high SRRs can be related to this fact.

### 7.3.2. Load and Roughness Effects

Figure 7.4 shows the predicted wear for four different roughness values from relatively smooth surface ($\bar{\sigma} = 7.5 \times 10^{-6}$) to medium ($\bar{\sigma} = 1.5 \times 10^{-5}$) to rough ($\bar{\sigma} = 4 \times 10^{-5}$) and very rough ($\bar{\sigma} = 9 \times 10^{-5}$) surfaces. The predicted percentage of the load carried by the asperities associated with each case is also shown in Figure 7.4. For the smooth and medium surfaces only a very small portion of load is sustained by the asperities while for the case of very rough surface ($\bar{\sigma} = 9 \times 10^{-5}$) 61% of the load is carried by the asperities. Due to this fact, wear increases significantly in rougher surfaces. For all of these cases, a reduction in wear is observed with increasing SRR similar to Figure 7.3. However, a minimum wear (transition point) can be identified only for the rough and very rough surfaces. As the surface roughness increases, the optimum point occurs at slightly smaller SRR value.

![Figure 7.4. Wear for different surface roughness values and vs. different slide-to-roll ratios ($W = 4.4 \times 10^{-4}$)](image-url)
Depicted in Figure 7.5 is the wear prediction as a function of dimensionless load for different roughness values and at SRR of 1. As expected, wear increases as the load becomes larger. The increase is more pronounced for smoother surfaces as compared to surfaces with larger roughness values. This observation may be related to the viscous dissipation in the fluid. In the smooth surface, the major part of the load is carried by the fluid. Increasing the load causes a substantial increase in the fluid viscosity and the shear rate which consequently leads to greater amount of thermal dissipation and surface temperature rise. In contrast, for rough surfaces, significant part of the load is carried by the asperities, so the thermal viscous dissipation—which is highly dependent on the load—is less influential.

In order to show the effect of slide-to-roll ratio at different loads and different roughness values, sixteen sets of simulations are plotted in Figure 7.6. As seen, similar to Figure 7.4, for the smooth and medium surfaces wear decreases as the SRR increases. In addition, the wear trends for all SRR values are identical. However, for rough and very rough surfaces a transition in wear behavior is observed; for the small slide-to-roll ratios (below 1) the same behavior is observed similar to smooth and medium surfaces, whereas at high SRR value of 1.8 wear drastically rises with increasing load. This transition in the wear behavior (minimum wear point) can also be seen in Figures 7.3 and 7.4 for relatively rough surfaces.
Figure 7.6. Wear as a function of dimensionless load at different slide-to-roll ratios for (a) smooth, (b) medium, (c) rough and (d) very rough surfaces

Note that in using the model, careful attention should be given to the magnitude of the imposed load. At larger loads, mainly due to the small values of the film thickness, the assumption of mixed-lubrication regime may not be valid and lubrication regime may fall into boundary lubrication which is not within the scope of the current work.

7.3.3. Example

Consider the case of the contact between two cylinders rolling and sliding against each other. Practically, this can also be considered as a single point on the line of action in a gear. The objective is to estimate the wear rate assuming mixed lubrication regime prevails. Properties of the two rollers and the operating condition are summarized in Tables 7.5 and 7.6 respectively. The oil and wear properties are assumed based on the data in Tables 7.2 and 7.3.

| Table 7.5. Dimensional, surface and mechanical properties for two cylinders |
|------------------|------------|---------|----------|---------|--------|--------|--------|
|                  | $R$(mm)   | $l$(mm) | $\sigma$(um) | $\beta$(um) | $n$(m$^2$) | $E$(GPa) | $H$(MPa) | $v$     |
| Roller (1)       | 20        | 10      | 0.3        | 28      | $7\times10^5$ | 209    | 5500   | 0.29   |
| Roller (2)       | 25        | 10      | 0.4        | 28      | $7\times10^5$ | 209    | 6000   | 0.29   |

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The effective radius of curvature and modulus of elasticity are calculated as follows:

\[
\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{2 \times 10^{-2}} + \frac{1}{2.5 \times 10^{-2}} \quad R = 1.11 \times 10^{-2} \text{ m}
\]

\[
\frac{1}{E'} = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \quad E' = 114.1 \text{ GPa}
\]

Since both surfaces are rough, equivalent values for the surface parameters should be calculated. The relationships for the calculation of composite roughness, asperity radius of curvature and asperity density can be found in Ref. [19]:

\[
\sigma = \frac{\sigma_{eq}}{R} = \sqrt{\frac{\sigma_1^2 + \sigma_2^2}{R}} = 4.5 \times 10^{-5} \quad \bar{\beta} = \frac{\beta_{eq}}{R} = \sqrt{\frac{\beta_1^2 + \beta_2^2}{R}} = 1.8 \times 10^{-3} \quad \bar{n} = n_1 R^2 = 8.64 \times 10^5
\]

\[
\bar{\xi} = \bar{n} \bar{\beta} \bar{\sigma} = 0.069 \quad \Omega = \frac{H}{E'} = 4.82 \times 10^{-2} \quad \bar{W} = \frac{W}{l E' R} = 1.97 \times 10^{-4}
\]

\[
u_r = \frac{R_1 \omega_1 + R_2 \omega_2}{2} = 1.70 \text{ m/s} \quad \nu_s = R_1 \omega_1 - R_2 \omega_2 = 0.78 \text{ m/s}
\]

\[
\overline{U} = \frac{\mu_0 \nu_r}{E' R} = 4.29 \times 10^{-11} \quad \text{SRR} = \frac{\nu_s}{\nu_r} = 0.46 \quad \alpha = Z \left( 5.1 \times 10^{-9} \left( \ln \left( \nu_0 \right) + 9.67 \right) \right) = 1.9 \times 10^{-8} \quad G = a E' = 2.17 \times 10^3
\]

Having calculated the dimensionless parameters, they are substituted in Eq. (19) to estimate \( \gamma_2 \):

\[
\frac{1}{\gamma_2} \sqrt{1 + \left[ 0.1188 \pi \sigma - 0.1531 \bar{\beta} - 0.1203 \bar{\sigma} - 0.6304 \bar{W} - 0.7161 \Omega - 0.1423 \gamma_2 - 0.2954 \right]} = 1.396
\]

\[
\bar{\xi} \sqrt{\frac{\pi}{\bar{W}}} \left\{ \Phi_{KE} \left( 2.922 \left( \frac{\bar{W}}{2} \right) - 0.166 \overline{U} - 0.692 (2G)^{0.47} \left( \frac{\gamma_2}{\gamma_2 - 1} \right)^{0.222} \right) \right\}
\]

Solving Eq. (22) for \( \gamma_2 \) yields: \( \gamma_2 = 1.87 \). The average temperature rise must be estimated and subsequently used to calculate \( \psi \). Here, the thermal analysis method of Ref. [22] is applied and the result is 322 K. In this example, instead of calculating Archard wear coefficient \( k \) based on CDM analysis, \( k \) is chosen based on available experimental data for dry contact condition. Assuming general steel on steel contact, a pertinent value for \( k (=126 \times 10^{-4}) \) can be found in [26]. Using these specifications, wear volume per sliding distance is determined:
\[
\psi = 1 - \exp\left(- \frac{A_g}{u_s t_0} \exp\left(- \frac{E_a}{R g T_s}\right) \right) = 1.43 \times 10^{-6}
\]

\[
\frac{V}{S} = \frac{k \psi W}{\gamma_2} = 4.37 \times 10^{-9} \text{ mm}^3/\text{mm}
\]

It is worth to evaluate the sensitivity of the model on the choice of statistical parameter \((n, \beta\) and \(\sigma\)). The effect of roughness values, \(\sigma\), are already examined and is illustrated in Figures 7.4 and 7.5. Here, to investigate the effect of \(n\) and \(\beta\), we assume that all input parameters including roughness value is exactly the same as in the previous example, except that, we change the values of \(\beta_{eq}\) to 10\(\mu\)m (half of its previous value) and \(n_{1,2}\) to \(1.4 \times 10^{10}\) (twice previous values) in order to keep the \(\xi\) value constant and within the typical range. Using these numbers in the models yields the following results: \(\gamma_2 = 1.63, \psi = 1.5 \times 10^{-6}\) and \(V/S = 5.29 \times 10^9\). As noted the amount of wear volume \((V)\) per unit sliding distance \((S)\) changes 13% by considerably changing \(n\) and \(\beta\). Nonetheless, changing \(\sigma\) values results in substantial amount of wear change as can be seen in Figures 7.4 and 7.5.

### 7.4. Application of the mixed lubrication model to the spur gear teeth contact

In this section we seek to provide an example to demonstrate how the developments in this dissertation can be applied to predict the performance of a mechanical component operating in the mixed-EHL regime. For this purpose, we consider the contact of two spur gear teeth and predict their contact behavior and wear characteristics. For modeling purposes, at each point along the line of action (LoA), the contact of pinion and gear is replaced by that of two cylinders having the radii of \(R_1\) and \(R_2\). The points of contact in spur gears are always along this line (Figure 7.7).

Figure 7.7. Contact of a pinion and a gear
As the point of contact moves on the pinion or the gear tooth, the transmitted force, the speed and the radius of the curvature of the cylindrical rollers vary along the LoA. The pitch radius of the pinion \( r_p \) and the gear \( r_g \) are calculated using the following formulas [77]:

\[
r_p = \frac{N_p \times m}{2} \tag{24a}
\]
\[
r_g = \eta \times r_p \tag{24b}
\]

where \( m \) represents the pinion-gear module, \( \eta \) is the gear ratio (speed ratio) and \( N_p \) is the number of pinion teeth. The radius of curvature for the contact points on the pinion and gear teeth are given as [78]:

\[
R_1 = r_p \times \sin \varphi + \varepsilon \tag{25a}
\]
\[
R_2 = r_g \times \sin \varphi - \varepsilon \tag{25b}
\]

where \( \varepsilon \) is the coordinate along the LoA, i.e. the coordinates of the points on the pinion tooth that come into contact, and \( \varphi \) is the so-called pressure angle defined as the angle between the LoA and the line tangent to both pinion and gear pitch circles (Figure 7.7). Having calculated the radii of two rollers, the equivalent radius of contact \( R \) can also be calculated. The variation of the radii of curvature of the pinion and gear and the equivalent radius are illustrated in Figure 7.8a. The negative part of the horizontal axis refers to pinion dedendum and the positive part refers to pinion addendum. The smallest value of the coordinate along the LoA \( \varepsilon \) pertains to the beginning of the contact mesh and the largest point refers to the end of the mesh, i.e. the end of the contact. The coordinate with zero value pertains to the pitch point (see Figure 7.8a).

The linear speeds of the points on the pinion and gear teeth are defined using the following equations:

\[
u_1 = 2\pi \omega_p \left( r_p \sin \varphi + \varepsilon \right) \tag{26a}
\]
\[
u_2 = 2\pi \omega_g \left( r_g \sin \varphi - \varepsilon \right) \tag{26b}
\]

where \( \omega_p \) and \( \omega_g \) are the pinion and gear rotational speeds, respectively.

For the simulation purposes, a typical pinion and gear set is chosen. Shown in Table 7.7 are the selected properties of the pinion and gear for the current analysis. The mechanical properties of contacting bodies and the oil and wear properties are assumed based on the data given in Tables 7.1, 7.2 and 7.3.
Table 7.7. Pinion and gear properties

<table>
<thead>
<tr>
<th>Number of pinion teeth ((N_p))</th>
<th>Gear ratio ((\eta))</th>
<th>Module ((m))</th>
<th>Pinion rotational speed ((\omega_p))</th>
<th>Pinion width ((l))</th>
<th>Transmitted load ((F_t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>26</td>
<td>2.5</td>
<td>0.003175 m</td>
<td>1500 rpm</td>
<td>0.1 m</td>
<td>70 kN</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Pressure angel ((\phi))</th>
<th>Effective roughness ((\sigma))</th>
<th>Effective asperity radius ((\beta))</th>
<th>Effective asperity density ((n))</th>
<th>Pinion hardness</th>
<th>Gear hardness</th>
</tr>
</thead>
<tbody>
<tr>
<td>20°</td>
<td>0.4 µm</td>
<td>8 µm</td>
<td>2.5\times10^{-10} (m^{-2})</td>
<td>6400 MPa</td>
<td>6800 MPa</td>
</tr>
</tbody>
</table>

The variation of the rolling speed, \(u_r = (u_1 + u_2) / 2\), the sliding speed \(\left( |u_2 - u_1| \right)\) and the associated slide-to-roll ratio (SRR) are depicted in Figure 7.8b. The rolling speed monotonically increases through the end of the contact. In contrast, the sliding speed decreases starting from the initial point of contact to the pitch point where it becomes zero (pure rolling condition). It again increases after the pitch point as the contact point moves toward the end of the contact.

In spur gears, the load carried by each tooth \((W)\) varies along the LoA since the number of teeth in contact changes. At the initial point of the pinion-gear engagement, there are two pairs of teeth in contact where the amount of contact load changes linearly from 1/3 to 2/3 of the total transmitted load \((F_T)\). However, the contact load jumps to the total load when only one pair of teeth is in contact and then drops to 2/3 of the total load again when another pair comes into contact. The contact load then reduces linearly to 1/3 of the total load at the end of the contact. The variation of the load along the LoA for one tooth is shown in Figure 7.8c and the associated maximum Hertzian pressure is illustrated in Figure 7.8d.

Having calculated all the required parameters for the mixed-lubricated wear model, the steady state wear analysis is performed for all of the contact points on the pinion tooth. Another important output of the model includes the film thickness, the load-sharing ratios and the surface temperature. All the output is calculated using the methodology described in Section 7.2. The load-sharing ratios are determined by solving Eq. 19 for each point along the LoA. Subsequently, the film thickness value is calculated using the modified central film thickness formula (Eq. 12) which considers the roughness effect. The Archard wear coefficient is estimated using the proposed methodology (CDM approach) described in detail in Chapter 2. Similar to the previous section, the surface temperature along the LoA is estimated based on the thermal model developed by Akbarzadeh and Khonsari [22] and, subsequently, the fractional film defect coefficient is calculated using Eq. 2. Having estimated all the necessary parameters for the modified Archard wear equation (Eq. 6), the wear rate (wear volume per second) for each point along the LoA is eventually calculated.
Figure 7.8. Variation of (a) pinion, gear and equivalent radii, (b) rolling speed, sliding speed and slide-to-roll ratio, (c) contact load and (d) maximum Hertzian pressure along the line of action.

Figure 7.9a shows the variation of the central lubricant film thickness ($h_c$) along the LoA. As the contact point moves on the pinion tooth, the film thickness increases monotonically except for the parts where a sudden increase of the contact load is experienced (see Figure 7.8c). The film thickness is highly dependent on the rolling velocity, and therefore, as the rolling velocity increases along the LoA (Figure 7.8b), the film thickness becomes larger.

Figure 7.9b shows the portions of the load carried by the asperities and the fluid film. At the first point of contact where the rolling speed is minimum, the asperities take the largest amount of the load as compared to all the other points on the LoA. As the contact point moves along the LoA, the fluid film gets thicker and, subsequently, the fluid part becomes larger while the asperity part decreases. The asperity load share has its lowest value at the end of contact where the rolling speed is maximum.

Figure 7.9c demonstrates the variation of the surface temperature along the LoA. The temperature of the lubricant and that of the surface are highly dependent on the amount of SRR.
As a result, the first point of contact has the maximum temperature. The surface temperature reduces to its minimum value at the pitch point where the SRR is nil and again starts to rise as the contact point moves toward the end of contact.

Finally, Figure 7.9d shows the variation of the volumetric wear rate along the LoA. As seen, the highest amount of wear is experienced at the beginning of the contact. This finding can be justified in light of two facts. First, the sliding speed is maximum at the beginning of the contact, and second, because of the low rolling speed, the asperity load ratio has its highest value (Figure 7.9b). As the contact point passes along the LoA, the wear decreases to its minimum value of zero exactly at the pitch point.

![Figure 7.9. Variation of (a) central film thickness, (b) asperity and fluid load ratios, (c) surface temperature and (d) steady-state volumetric wear along the line of action](image)

As mentioned, the current model uses the elastic-plastic asperity model of KE to estimate the load-sharing ratios between the asperities and the fluid. Demonstrated in Figure 7.10 is the variation of the wear rate (along the LoA) predicted by the current model as well as the wear model proposed by Akbarzadeh and Khonsari [3] considering fully elastic behavior for the asperities deformation (GW model). As seen, the trends of wear rate according to both models
are identical. Nonetheless, the current model estimates wear slightly less than the one developed by Akbarzadeh and Khonsari. There are two possible explanations for this observation. The first reason is that the dry Archard wear coefficient predicted by the CDM approach is slightly smaller than that used by Akbarzadeh and Khonsari based on the experimentally obtained value (see Section 7.3.1). Second, the elastic-plastic asperity contact model predicts lower contact pressure compared to the elastic model of GW which results in lower asperity load ratio and consequently thicker fluid film thickness. This directly leads to a lower amount of wear prediction in modified Archard wear formula (Eq. 6). As the point of contact moves along the LoA and the rolling speed increases, the fluid film becomes thicker and the asperity load ratio reduces. As a result, for high rolling speeds, the difference between the current model and that of Akbarzadeh and Khonsari becomes negligible.

Figure 7.10. Comparison of the variation of volumetric wear along the LoA based on the current simulation and Akbarzadeh and Khonsari results [3] ($F_t = 0.125 \times 10^5 \text{N}$, $l = 0.1 \text{m}$, $\omega_p = 600 \text{rpm}$, $\sigma = 0.4 \mu\text{m}$, $\eta = 3$)

Figure 7.11 shows the effect of pinion’s rotational speed on the volumetric wear rate. As the speed increases, both the sliding and rolling speeds increase. This results in two opposing effects. As the rolling speed increases, the lubricant film thickness becomes larger. A thicker fluid film reduces the asperity load share and, hence, results in the reduction of the wear rate. On the other hand, a larger sliding speed leads to more wear rate. As a result, for higher pinion speeds, wear rate has its largest values at the beginning of the contact in which the sliding speed is high, whereas wear drastically decreases toward the end of contact as the rolling speed increases and the sliding speed decreases.
Figure 7.11. Effect of pinion speed on the variation of volumetric wear along the LoA

Figure 7.12, shows the effect of surface roughness on the volumetric wear. An increase in the surface roughness leads to a higher asperity load share and therefore higher wear rate. This condition is consistent for all contact points along the LoA. As expected, the surface roughness has a significant effect on the amount of wear.

Figure 7.12. Effect of surface roughness on the variation of volumetric wear along the LoA

To illustrate the significant influence of lubrication in gears, let us simply assume that the pinion and the gear operate under the dry condition. The volumetric wear rate for the points along the LoA is calculated using the Archard dry wear coefficient predicted by the CDM
approach and is illustrated in Fig 7.13a. The variation of the associated lubricated wear is also plotted in Figure 7.13b. As seen, the amount of wear in dry condition is significantly larger within orders of magnitude compared to the lubricated case.

Figure 7.13. Variation of the steady-state volumetric wear along the line of action (a) dry condition (b) lubricated condition ($F_t = 0.5 \times 10^5$ N, $\omega_p=1000$ rpm)

7.5. Conclusions

An engineering approach for the prediction of lubricated wear in mixed EHL line contact is developed. It utilizes the load-sharing concept, and the empirical formulas for the contact of two dry rough cylinders in combination with EHL central film thickness formula to predict the percentage of the load sustained by the asperities. The empirical formula is based on Kogut-Etsion (KE) asperity micro contact models taking elasto-plastic deformation of asperities into consideration. The Archard wear coefficient for the dry sliding condition is approximated using the continuum damage mechanics (CDM) based on the fatigue theory of adhesive wear. Finally, by means of a simple thermal analysis for the EHL line-contact configuration the average temperature rise at the contact is calculated and considering the fractional film defect concept, the Archard wear coefficient is modified for the steady-state sliding wear in the mixed EHL regime for the line-contact configuration.

Comparison of the predicted wear volume per sliding distance with those measured experimentally shows good agreement indicating the usefulness of the approach to predict wear in mixed lubricated contacts. Simulation results indicate that wear is highly dependent on the roughness values, the total load and the slide-to-roll ratio. However, the effect of load is more pronounced in lower roughness values. Interestingly, for larger roughness there exists a SRR value at which the wear volume per sliding distance is minimum while such a minimum cannot be identified in surfaces with low roughness value. In addition, wear drastically rises with the increase in slide-to-roll ratio in rougher surfaces at higher slide-to-roll ratios.
The utility of the approach in a real engineering problem, e.g. the contact of two spur gear teeth, is also demonstrated. The model is able to predict the film thickness, asperity and fluid load ratios, surface temperature and steady-state wear rate along the line of action on the pinion or gear for different geometries, operating conditions and surface properties.

Nomenclature

\(a_x\) diameter of area associated with an adsorb molecule, (m)
\(b\) Hertzian contact half width, (m)
\(d\) separation based on summit (asperity) heights, \(h - y_s\), (m)
\(\bar{d}\) dimensionless separation based on summits heights
\(D\) damage variable
\(D_c\) critical damage value
\(E_{1,2}\) modulus of elasticity of body 1 and 2, (GPa)
\(E'\) effective modulus of elasticity, \(\frac{1}{E'} = \frac{1-v_1^2}{E_1} + \frac{1-v_2^2}{E_2}\), (GPa)
\(E_a\) heat of adsorption of the lubricant on surface, (kJ/mole)
\(f_b\) coefficient of friction for the boundary lubricated contact
\(f_d\) coefficient of friction for the dry contact
\(G\) material number
\(H\) hardness of the softer material, (MPa)
\(H_m\) cyclic hardening modulus, (MPa)
\(h\) separation based on surface heights, (m)
\(h_{00}\) constant in the separation relationship, (m)
\(h_c\) central separation or film thickness in line-contact configuration, (m)
\(\bar{h}_c\) dimensionless separation
\(\bar{h}_{c}'\) dimensionless central separation or film thickness in line-contact configuration
\(K\) maximum contact pressure factor
\(K_f\) fluid thermal conductivity (N/s·K)
\(K_r\) roller thermal conductivity (N/s·K)
\(k\) original Archard wear coefficient
\(k_b\) boundary lubricated wear coefficient
\(k_l\) mixed lubricated wear coefficient
\(l\) contact length, (m)
\(M\) cyclic hardening exponent
\(N\) number of cycles for an asperity to break off
\(p\) pressure, (MPa)
\(p_a\) asperity pressure, (MPa)
\(p_f\) fluid pressure, (MPa)
\(\bar{p}\) dimensionless pressure
\( p_H \)  
maximum Hertzian pressure, (MPa)

\( \bar{p}_{max} \)  
maximum dimensionless pressure for the rough surfaces contact

\( R \)  
effective radius of curvature, (m)

\( R_g \)  
gas constant, (J/mole·K)

\( S \)  
sliding distance, (m)

\( S_{RR} \)  
slide-to-roll ratio

\( S_{max} \)  
maximum stress

\( S_e \)  
endurance limit, (MPa)

\( S_f \)  
true failure stress, (MPa)

\( S_y \)  
yield stress, (MPa)

\( T_s \)  
absolute temperature of the surface

\( t_0 \)  
fundamental time of vibration of the molecule in the adsorb state, (s)

\( u_1, u_2 \)  
linear velocity of the first and second rollers

\( u_r \)  
effective rolling velocity, (m/s)

\( u_s \)  
sliding velocity, (m/s)

\( U \)  
dimensionless velocity

\( V \)  
wear volume, (m³)

\( W \)  
total normal force, (N)

\( W_a \)  
asperity load, (N)

\( W_f \)  
fluid load, (N)

\( \bar{W} \)  
dimensionless total normal force

\( x \)  
spatial coordinate component along the contact width, (m)

\( y_s \)  
distance between the mean of summit heights and that of the surface heights, (m)

\( z \)  
average asperity height measured from the mean line of summit heights, (m)

\( \bar{z} \)  
dimensionless asperity height, (z/σ)

\( Z \)  
pressure viscosity index

\( \alpha \)  
pressure viscosity coefficient

\( \psi \)  
fractional film defect coefficient

\( \nu_1, \nu_2 \)  
Poisson ratios of the first and second surfaces

\( n \)  
average density of asperities, (m⁻²)

\( \bar{n} \)  
dimensionless average density of asperities

\( \beta \)  
average asperity radius, (m)

\( \bar{\beta} \)  
dimensionless average asperity radius

\( \gamma_1, \gamma_2 \)  
scaling factors for the load-sharing concept

\( \sigma \)  
surface roughness (standard deviation of surface heights), (m)

\( \bar{\sigma} \)  
dimensionless surface roughness

\( \sigma_s \)  
standard deviation of summit heights, (m)

\( \rho_f \)  
fluid density (kg/m³)
\[ \rho_r \] roller density (kg/m³)

\[ \mu_0 \] fluid viscosity at inlet, (Pa·s)

\[ \omega \] asperity interface, (m)

\[ \overline{\omega} \] dimensionless interface, \( \omega/\sigma \)

\[ \omega^*_c \] dimensionless critical interface according to the KE model

\[ \omega_r \] rotational speed, (rpm)

\[ \Omega \] ratio of the softer surface hardness to the effective elasticity modulus, \( H/E' \)

\[ \Phi_{KE}(\overline{h}) \] dimensionless function representing KE micro-asperity contact model for the contact load

\[ \phi^*(z^*) \] dimensionless standard normal distribution function

\[ \zeta \] \( n \beta \sigma \)

### 7.6. References


[64] H.S. Cheng, A Refined Solution to Thermal-Elastohydrodynamic Lubrication of Rolling and Sliding Cylinders, Asle Trans, 8 (1965) 397-408.


Chapter 8: Summary and Future Works

8.1 Summary and Conclusions

This dissertation is comprised of two major interrelated foci. The first focus is to investigate the effect of surface roughness on the behavior of contacting bodies through both deterministic and statistical approaches. The second objective involves the assessment of three of the most observed contact degradation processes including rolling/sliding contact fatigue, adhesive wear in unlubricated and mixed lubricated contacts and fretting fatigue. In order to investigate the last two degradation phenomena, the results obtained from the first objective are directly utilized. A summary and conclusions of the main results is as follows:

- The modern continuum damage mechanics (CDM) approach is applied to predict adhesive wear coefficient by the contribution of asperity fatigue based theory. This approach eliminates the empirical nature of wear coefficient and makes it possible to calculate the wear coefficient using the bulk material properties and surface conditions. Numerical simulations have been carried out to calculate the wear coefficient as a function of friction coefficient for different type of materials. By carrying out pin-on-disk experiments, wear coefficients for Aluminum 6061-T6 are obtained and compared with predicted values based on the CDM theory. Also the simulated curves are compared with available published experimental work showing good agreement.

- The CDM approach is again applied to predict rolling/sliding contact fatigue crack initiation. On the basis of a moving Hertzian contact theory, the subsurface stresses for elastic plain strain condition are obtained. These loadings in conjunction with the damage theory enable one to predict the status of damage at the subsurface of the contacting material for given number of cycles. Numerical simulations are carried out to calculate the damage status and predict the location of the first crack and the number of cycles required for its formation. The estimated number of cycles to crack initiation compared to the available experimental results reveals good agreement. The effect of load sequence on the life of the material is also investigated which shows different damage status inside the material due to different loading sequences.

- Different statistical micro-contact models including Greenwood-Williamson (GW), Chang-Etsion-Bogy (CEB), Zhou-Maietta-Chang (ZMC), Kogut-Etsion (KE) and Jackson-Green (JG) are employed along with the elastic bulk deformation formula of the line contact to determine the pressure profile, contact width and the real area of contact as a measure of the
impact of surface roughness on the contact characteristics. Results indicate that elastic-plastic micro-contact models that take asperity’s elastic-plastic effects into account predict a lower maximum normal pressure, a greater contact width, and a larger real contact area compared to the predictions of the GW model, which assumes that asperities deform elastically. The KE and JG models are found to be most realistic statistical contact models for analyzing the line contact behavior as opposed to other models since they fully consider the details of regimes of deformation. In addition, useful expressions for the prediction of maximum contact pressure, contact width, real area of contact and pressure distribution are proposed based on the GW, KE and JG models.

- Statistical micro-contact models including Greenwood-Williamson (GW) and Kogut-Etsion (KE) are again employed, however along with the elastic bulk deformation formula of the elliptical point contact to determine the pressure profile, contact width and length, real area of contact and contact compliance. The numerical results for the contact radius and contact compliance are compared with the available experimental measurements and show good agreement. In addition, useful expressions for the prediction of maximum contact pressure, contact width, contact length, real area of contact, contact compliance and pressure distribution are developed based on the GW and the KE models. Using these formulas one can easily estimate the contact characteristics of the elliptical point contact configuration.

- A robust and comprehensive method is presented in detail to predict the deterministic pressure distribution for rough line-contact problems assuming elastic-perfectly plastic behavior of the asperity tips. The described algorithm is conceptually simple and can be readily implemented in a computer code. In addition, adopting the Ciavarella-Jager approach, the calculation of the deterministic tangential traction distribution for cyclic loading condition in stick-slip regime is explained in detail. The sub-surface stress field due to the application of normal and tangential forces is also determined. The approach can be easily utilized to evaluate the contact damage evolution due to stick-slip (fretting) condition. To illustrate the utility of the technique, the fretting crack initiation risk for surfaces with different roughness values subjected to cyclic stick-slip condition was evaluated. High surface roughness values and low normal loads result in pressure and tangential traction distributions that significantly deviate from the Hertzian solution. The Results also show that higher surface roughness values increase the risk of crack initiation which agrees with the available experimental observations. The numerical predictions of the current simulations, in agreement with the published experimental results, indicate that the contact edges or their vicinity are the most probable sites for crack initiation.

- An engineering approach for the prediction of lubricated wear in mixed EHL line contact is developed. It utilizes the load-sharing concept, and the empirical formulas for the contact of two dry rough cylinders in combination with EHL central film thickness formula to predict
the percentage of the load sustained by the asperities. The empirical formula is based on Kogut-Etsion (KE) asperity micro contact models taking elasto-plastic deformation of asperities into consideration (Chapter 4). The Archard wear coefficient for the dry sliding condition is approximated using the continuum damage mechanics (CDM) based on the fatigue theory of adhesive wear (Chapter 2). Finally, by means of a simple thermal analysis for the EHL line-contact configuration the average temperature rise at the contact is calculated. Further, by considering the fractional film defect concept, the Archard wear coefficient is modified for the steady-state sliding wear in the mixed EHL regime for the line-contact configuration. Comparison of the predicted wear volume per sliding distance with those measured experimentally by others shows good agreement indicating the usefulness of the approach to predict wear in mixed lubricated contacts. Simulation results indicate that wear is highly dependent on the roughness values, the total load and the slide-to-roll ratio. The utility of the approach in a real engineering problem, e.g. the contact of two spur gear teeth, is also illustrated. The model is capable of predicting the film thickness, asperity and fluid load ratios, surface temperature and steady-state wear rate along the line of action on the pinion or gear for different geometries, operating conditions and surface properties.

8.2 Recommendations for Future Works

The following recommendations are made for possible future research:

- Regarding the modeling of rolling/sliding contact fatigue, the underlying assumptions (neglecting material imperfection and residual stresses, elastic deformation and the like) can be investigated and efforts can be directed to the development of a more generalized damage model for contact fatigue problems.

- In addition to the simulation of crack initiation phase, crack propagation can be also modeled with the use of CDM for the rolling/sliding contact fatigue problem. However, due to the complex stress filed after cracking, sophisticated modeling technique such as the finite element analysis should be utilized in order to calculate the stress filed.

- The statistical modeling for the curved bodies contact can be extended into the contact of bodies with bilayer material types. In this fashion, the contact of rough surfaces with coating material can be considered.

- Engineering mixed-lubricated wear modeled developed in chapter 7 for the line-contact problem, can be easily extended to the elliptical point contact problem using the results of Chapter 5 and also the available EHL film thickness formulas for lubricated smooth elliptical point contact.
The results of chapter 5 for the dry rough elliptical point contact can be extended to lubricated contact condition in virtue of the load sharing concept and by using the available EHL film thickness formulas to simulate the complex contact in metal-on-metal artificial hip and knee joints and to calculate the film thickness and wear considering the roughness effects.
Appendix A: Load Sharing Concept, Fluid Part

The formulation of the EHL problem for the smooth surface consists of the Reynolds equation, the separation-pressure expression (deformation equation) and the load balance equation:

$$\frac{\partial}{\partial x} \left( \frac{\rho h^3}{\mu} \frac{\partial p_f}{\partial x} \right) = 12 u_r \frac{\partial (\rho h)}{\partial x} \quad (A1)$$

$$h(x) = h_{00} + \frac{x^2}{2R} - \frac{2}{\pi E'} \int_{-\infty}^{\infty} p_f(s) \ln|x-s| ds \quad (A2)$$

$$W = l \int_{-\infty}^{\infty} p_f(x) dx \quad (A3)$$

The EHL central film thickness relationship (Chapter 7), Eq. (9), is based on the simultaneous numerical solution of these three formulas for the smooth contact [62] in which the total load is sustained by the interface pressure produced by the fluid only.

In mixed lubrication regime, the pressure in Reynolds equation is still the fluid pressure while the deformation is produced by the total pressure and, of course, the force balance equation must be satisfied for the total pressure [25, 54]. Alternatively, the separation equation and load balance equation can be written in terms of the fluid pressure considering the load-sharing concept (Eq. (5a)):

$$\frac{\partial}{\partial x} \left( \frac{\rho h^3}{\mu} \frac{\partial p_f}{\partial x} \right) = 12 u_r \frac{\partial (\rho h)}{\partial x} \quad (A4)$$

$$h(x) = h_{00} + \frac{X^2}{2R} - \frac{2}{\pi E'} \int_{-\infty}^{\infty} p_f(s) \ln|x-s| ds \quad \rightarrow \quad h(x) = h_{00} + \frac{X^2}{2R} - \frac{2}{\pi \left( \frac{E'}{\gamma_1} \right)} \int_{-\infty}^{\infty} p_f(s) \ln|x-s| ds \quad (A5)$$

$$W = l \int_{-\infty}^{\infty} p_f(x) dx \quad \rightarrow \quad \left( \frac{W}{\gamma_1} \right) = l \int_{-\infty}^{\infty} p_f(x) dx \quad (A6)$$
It is noted that Eqs. A1-A3 (for the smooth surface) convert to A4-A6 (for the mixed lubricated condition) by replacing the equivalent modulus of elasticity $E'$ by $E'/\gamma_1$ in the deformation equation and the total load $W$ by $W/\gamma_1$ in the load balance equation. Likewise, applying the same substitutions, the film thickness equation for the smooth case, Eq. (9), converts to Eq. (12) considering the fact that only a portion of the load is carried by the fluid.
Appendix B: Load Sharing Concept, Asperity Part

The formulation of the dry rough line-contact problem includes the contact pressure equation (KE asperity contact equation), the separation-pressure expression (deformation equation) and the load balance equation:

\[ p_a(h) = n\beta\sigma E^{1/2} \Phi_{KE}(h) \] \hspace{1cm} \text{(B1)}

\[ h(x) = h_0 + \frac{x^2}{2R} - \frac{2}{\pi E'} \int_{-\infty}^{\infty} p_a(s) \ln|x-s| ds \] \hspace{1cm} \text{(B2)}

\[ W = l \int_{-\infty}^{\infty} p_a(x) dx \] \hspace{1cm} \text{(B3)}

The central (maximum) contact pressure equation, Eq. (16) in Chapter 7, is obtained by a regression analysis based on the simultaneous numerical solution of these three formulas for dry rough line-contact configuration [19] in which the total load is sustained by the interface pressure produced by the asperity contact only.

In mixed lubrication regime, analogous to the fluid part discussed in Appendix A, the pressure in asperity contact equation is the asperity pressure whereas the deformation is produced by the total pressure \( p \) and, obviously, the force balance equation must be satisfied for the total pressure. According to Eq. (5a), only \( 1/\gamma_2 \) portion of the total load is carried by the asperities and, hence the separation equation and the load balance equation can be alternatively written in terms of asperity pressure \( p_a \):

\[ p_a(h) = n\beta\sigma E^{1/2} \Phi_{KE}(h) \] \hspace{1cm} \rightarrow \hspace{1cm} p_a(h) = (\gamma_2 n\beta\sigma \left( \frac{E'}{\gamma_2} \right)^{1/2} \Phi_{KE}(h) \] \hspace{1cm} \text{(B4)}

\[ h(x) = h_0 + \frac{x^2}{2R} - \frac{2}{\pi E'} \int_{-\infty}^{\infty} p_a(s) \ln|x-s| ds \] \hspace{1cm} \rightarrow \hspace{1cm} h(x) = h_0 + \frac{x^2}{2R} - \frac{2}{\pi \left( \frac{E'}{\gamma_2} \right)} \int_{-\infty}^{\infty} p_a(s) \ln|x-s| ds \] \hspace{1cm} \text{(B5)}

\[ W = l \int_{-\infty}^{\infty} p(x) dx \] \hspace{1cm} \rightarrow \hspace{1cm} \frac{W}{\gamma_2} = l \int_{-\infty}^{\infty} p_a(x) dx \] \hspace{1cm} \text{(B6)}
It is noted that Eqs. B1-B3 (for dry rough surface) give same results as B4-B6 (for mixed lubricated condition) by replacing the modulus of elasticity by $E'/\gamma_2$, the total applied load by $W/\gamma_2$ and $n$ by $n\cdot\gamma_2$. Accordingly, applying these modifications, the central pressure for dry rough surface, Eq. (16) yields Eq. (17) for the mixed lubricated condition considering the fact that only a portion of the load is carried by the asperities.
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Vita

Ali Beheshti was born in Esfahan, Iran in 1982. He received his BS and MS in Mechanical Engineering from Isfahan University of Technology (Esfahan) in 2004 and 2007, respectively where he graduated with honor (top 2%). He worked as a researcher and consultant at the center for metal forming, Isfahan University of Technology, Esfahan, Iran from 2004 to 2008. He also worked as the manager for quality control and design consultant at Behineh Saz Foulad Company in Isfahan, Iran from 2003 to 2008. He then joined Center for Rotating Machinery, Louisiana State University, Baton Rouge, LA, USA where he started his Ph.D. research under the supervision of Prof. Michael M. Khonsari, Dow Chemical Endowed Chair and Professor, in 2008.

Ali Beheshti has made technical contributions in several mechanical engineering areas, including industrial sheet metal forming, computational solid mechanics, tribology and damage mechanics and he has authored 10 scientific articles in internationally peer-reviewed journals. He has received the best research award of 2013 from Mechanical Engineering Department, Louisiana State University for his research outcome related to his Ph.D. work. He is a member of American Society of Mechanical Engineers (AMSE), Society of Tribologist and Lubrication Engineers (STLE), Phi Kappa Phi honor society and Gold Key International honor society.

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