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INVARIANTS OF LEGENDRIAN PRODUCTS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
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Table of Contents

Abstract

This thesis investigates a construction in contact topology of Legendrian submanifolds called the Legendrian product. We investigate and compute invariants for these Legendrian submanifolds, including the Thurston-Bennequin invariant and Maslov class; Legendrian contact homology for the product of two Legendrian knots; and generating family homology.

Chapter 1
Introduction

A *contact structure* $\xi = \ker(\alpha)$ on a smooth manifold Y^{2n+1} is a hyperplane field that satisfies the non-integrability condition $\alpha \wedge (d\alpha)^n > 0$ for a global 1-form α. The basic example of a contact manifold is $\mathbb{R}^{2n+1}$ with contact form $\alpha = dz - \sum y_i dx_i$. More generally, the 1-jet space $J^1(M) = T^*M \times \mathbb{R}$ of a smooth manifold M and the spherical cotangent bundle ST^*M admit contact structures $\xi_{\text{1-jet}} = \ker(dz - \lambda)$ and $\xi_{ST^*M} = \ker(\lambda)$, respectively, where λ is the canonical Liouville 1-form on T^*M.

One way to understand a contact manifold Y is by studying its *Legendrian* submanifolds: submanifolds of dimension n that are everywhere tangent to the contact structure ξ. The motivation for this paper is exploring the contact geometry of higher dimensional (≥ 5) contact manifolds, by constructing many examples of Legendrian submanifolds. The Legendrian submanifolds of $\mathbb{R}^3$, Legendrian knots, are well-explored and have found importance in low-dimensional topology. However, much less is known about higher dimensional contact manifolds and their Legendrian submanifolds. Several authors have built examples, including the frontspinning construction of Ekholm, Etnyre and Sullivan [EES05b], a generalization to spheres of arbitrary dimensions by Golovko [Gol], hypercube tori defined by Baldridge and McCarty [BM] and tori given as the trace of Legendrian isotopies by Ekholm and Kalman [EK08]. In fact, the former three constructions should naturally be understood as products in the sense of this thesis.

Some well-worn ideas in contact geometry in dimension 3, such as the dichotomy between tight and overtwisted contact structures [1] and convex surface theory, have not yet been satisfactorily extended to higher dimensions. One of particular interest here is the connection, for Legendrian knots, between generating families and augmentations of the Legendrian contact homology dg-algebra.

The main topic of this thesis is a product operation on Legendrian submanifolds and computing contact-geometric invariants. The construction of Legendrian products is straightforward. For a detailed explanation of the terminology, see Chapter 2. Let $P \times \mathbb{R}$, and $Q \times \mathbb{R}$ be contact manifolds such that $(P, d\lambda), (Q, d\eta)$ are exact symplectic manifolds and with contact forms $\alpha = dz - \lambda$, $\beta = dz - \eta$, respectively. Take Legendrian submanifolds $L_1 \in P \times \mathbb{R}$ and $L_2 \in Q \times \mathbb{R}$ with Reeb chords $\{a_i\}, \{b_j\}$ and let $\bar{L}_1, \bar{L}_2$ denote their Lagrangian projections in P, Q. Then $\bar{L}_1 \times \bar{L}_2$ is an exact Lagrangian submanifold of $P \times Q$.

[1] After this thesis was submitted and defended, Borman, Eliashberg and Murphy released a preprint [BEM] generalizing the notion of overtwisted contact structures to higher-dimensions and proved the analogue of Eliashberg's original, foundational result in dimension 3: the inclusion of of the space of overtwisted contact structures on a manifold M^{2n+1} into the space of almost-contact structures on M is a weak homotopy equivalence.

Definition 1. The *Legendrian product* $L_1 \times L_2$ is the Legendrian submanifold in $P \times Q \times \mathbb{R}$ given by the lift of $\bar{L}_1 \times \bar{L}_2$.

The product $L_1 \times L_2$ is immersed in general and is embedded if the sets of Reeb chord actions $\{\mathcal{Z}(a_i)\}, \{\mathcal{Z}(b_j)\}$ are disjoint.

As pointed out to the author by Lenhard Ng, the simplest example is given by taking L_1 to be a collection of k points in $\mathbb{R} = J^1(\mathbb{R}^0)$. Then $L_1 \times L_2$ is simply k parallel copies of L_2, displaced vertically in the z-direction. To see that the product construction is not well defined, as stated in the above remark, suppose that L_1 consists of two points $\{0, \epsilon\}$. For large ϵ, then $L_1 \times L_2$ consists of two, unlinked copies of L_2. For small ϵ, the product $L_1 \times L_2$ consists of L_2 and a pushoff of L_2 in the z-direction, which are linked if the Thurston-Bennequin invariant $tb(L_2)$ is nonzero.

An obvious question to ask is whether a particular contact-geometric invariant of $L_1 \times L_2$ admits a nice, Kunneth formula-type description in terms of the invariants of L_1 and L_2. Interestingly, however, the answer is no for many invariants. This is because these invariants depend upon the relative lengths of Reeb chords in L_1 and L_2, a geometric property that is not preserved under Legendrian isotopy.

As a consequence, the product construction is not well-defined up to Legendrian isotopy of the factors, that is, a Legendrian isotopy of L_1 or L_2 does not necessarily extend to a Legendrian isotopy of $L_1 \times L_2$. In fact, by varying the Legendrian embedding of a factor within its Legendrian isotopy class, one can obtain infinitely many, non-Legendrian isotopic products. This is the first main theorem of the thesis.

Theorem A. *Let $L_1 \subset \mathbb{R}^{2n+1}, L_2 \subset \mathbb{R}^{2m+1}$ be chord generic Legendrians such that n, m have different parity. Then there exists an infinite family of Legendrians $\{L_1^\alpha\}$ all Legendrian isotopic to L_1 such that the family of Legendrian products $\{L_1^\alpha \times L_2\}$ are pairwise non-Legendrian isotopic.*

The proof of this theorem relies on the formula in Theorem B of the Thurston-Bennequin invariant.

Bennequin and Thurston [Ben83] introduced an invariant of Legendrian knots by associating to a Legendrian knot L a particular framing prescribed by the contact structure. If L is nullhomologous, the difference between the nullhomologous framing and the contact framing is an integer-valued invariant, called the Thurston-Bennequin number. Tabachnikov [Tab88] introduced a higher-dimensional generalization and for nullhomologous Legendrians, this invariant can be computed [EES05b] in a manner similar to the writhe of knots.

The second main theorem of this thesis is a formula for computing the Thurston-Bennequin invariant of products.

Theorem B. *The Thurston-Bennequin number of $K \times L$ is given by:*

$$tb(K \times L) = (-1)^{mn} tb(K)\chi(T^*L) + \chi(T^*K)tb(L) + tb(K)tb(L) + \sum_{i,j} \tau(a_i, b_j)\sigma(a_i)\sigma(b_j)$$

where $\chi(T^*L)$ denotes the pairing $\langle e(T^*L), [L]\rangle$ of the Euler class of T^*L with the fundamental class of L and

$$\tau(a_i, b_j) = \begin{cases} (-1)^n & \text{if } \mathcal{Z}(a_i) < \mathcal{Z}(b_j) \\ (-1)^m & \text{if } \mathcal{Z}(a_i) > \mathcal{Z}(b_j) \end{cases}$$

The last term in the tb formula requires some elaboration. Locally, each transverse double point is the intersection of two open $(m+n)$-disks in a single point. Just as computing the sign of a knot crossing requires knowing which strand passes over the other, computing the sign of a Reeb chord requires knowing which disk passes "over" the other (i.e. greater z-coordinate). In the product $L_1 \times L_2$, after a suitable perturbation, there is a Reeb chord $c_{i,j}$ for each pair (a_i, b_j) of Reeb chords of L_1 and L_2. However, determining which disk is "over" and which is "under" depends upon the relative lengths of the Reeb chords a_i, b_j.

Generating Family Homology

In some cases, a Legendrian submanifold $L \subset J^*(M)$ can be obtained from a generating family of functions $F : M \times \mathbb{R}^N \to \mathbb{R}$. Traynor [Tra01] first studied nonclassical invariants of Legendrian knots in terms of generating families. Since then, others have extended this approach [Hen11, HR13, JT06]. For more details on generating families and generating family homology, see Chapter 4.

Of particular interest is the connection between generating families and augmentations of the Chekanov-Eliashberg DGA (see below for background on this invariant and its generalization to Legendrian contact homology).

Theorem 1.1 (Fuchs [Fuc03], Fuchs-Ishkhanov [FI04], Sabloff [Sab06], Chekanov–Pushkar [CP05])**.** *A Legendrian knot L admits a generating family if and only if the Chekanov-Eliashberg DGA of L admits an augmentation.*

For example, the stabilization operation on Legendrian knots destroys the potential for a Legendrian knot to admit a generating family or an augmentation of its DGA.

The proof of this theorem relies on equating generating families and augmentations with normal rulings. In particular, Chekanov and Pushkar [CP05] showed that a Legendrian knot admits a generating family if and only if it admits a graded normal ruling of its front diagram. Fuchs [Fuc03] showed that if a Legendrian admits a graded normal ruling, then its Chekanov-Eliashberg DGA admits an augmention, although the correspondence is not 1-to-1. Fuchs and Ishkhanov [FI04] and independently Sabloff [Sab05] proved the converse.

A generating family $F : M \times \mathbb{R}^N \to \mathbb{R}$ determines a difference function $\delta : M \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and the generating family homology (either total or relative) is the relative singular homology of a suitable pair of sublevel sets of this function.

On the other side, an augmentation ϵ determines a linearized version $H_*^\epsilon(L)$ of the Chekanov-Eliashberg DGA of a Legendrian knot. Fuchs and Rutherford [FR11] found that the correspondence between generating family homology and augmentations can be strengthened.

Theorem 1.2 (Fuchs-Rutherford [FR11])**.** *If L admits a generating family F, there exists an augmentation ϵ of the Chekanov-Eliashberg DGA such that*

$$\mathcal{G}H_*(L, [F]) \simeq H_*^\epsilon(L)$$

In Chapter 4 we show that if L_1 and L_2 admit generating families, then so does $L_1 \times L_2$. In addition, under the assumption that $L_1 \gg L_2$, meaning that all the Reeb chord actions of L_1 are much greater than all the Reeb chord actions of L_2, then

Theorem C. *For $L_1 \gg L_2$, then the total and relative generating family homology over a field $\mathbb{F}$ satisfies a Kunneth-type formula*

$$\mathcal{G}H_*(L_1 \times L_2, [F_1 \oplus F_2]) \simeq \mathcal{G}H_*(L_1, [F_1]) \otimes_{\mathbb{F}} H_*(L_2; \mathbb{F})$$
$$\widetilde{\mathcal{G}H}_*(L_1 \times L_2, [F_1 \oplus F_2]) \simeq \widetilde{\mathcal{G}H}_*(L_1, [F_1]) \otimes_{\mathbb{F}} H_*(L_2; \mathbb{F})$$

The generating family homology can still be computed if we relax the assumption that $L_1 \gg L_2$, however it would require more concrete information about L_1, L_2 and the level sets of the difference functions. Crucially, the topology of the sublevel sets depends on the relative lengths of Reeb chord actions in L_1 and L_2.

Theorem C can be considered as a generalization of a result obtained in [EES09]. The frontspinning operation, introduced by Ekholm, Etnyre and Sullivan [EES05b], is a suspension procedure that for any Legendrian $L \subset \mathbb{R}^{2n-1}$ produces a Legendrian $\Sigma L \subset \mathbb{R}^{2n+1}$ of topological type $L \times S^1$. In fact, we will show in Theorem 2.3 that frontspinning is Legendrian isotopic to the product of $L \times U$, where U is the unique $tb = -1$ Legendrian unknot in $\mathbb{R}^3$.

Sabloff [Sab06] found that linearized contact homology of Legendrian knots satisfies a form of duality, up to a term representing the fundamental class of the underlying S^1 of the Legendrian knot. Ekholm, Etnyre and Sabloff [EES09] later generalized this to vertically-displaceable Legendrians and found an exact triangle relating the linearized contact homology to the singular homology of the underlying manifold of the Legendrian. In [EES05a], Ekholm, Etnyre and Sullivan found that if the Legendrian contact homology of L has an augmentation, then so does the LCH of ΣL. Using this duality, the linearized contact homology of ΣL with respect to this augmentation was calculated in [EES09]. This calculation of linearized contact homology agrees with the calculation of generating family homology in Theorem C, offering partial evidence for a continued relationship between generating families and linearized contact homology in higher dimensions.

Legendrian Contact Homology

In [EGH00], Eliashberg, Givental and Hofer sketched a unified framework for Floer-type invariants in contact and symplectic topology called *Symplectic Field Theory* (SFT). It conjectures several levels of increasingly powerful invariants of Legendrian submanifolds. The lowest level, Legendrian Contact Homology (LCH), was constructed for $\mathbb{R}^3$ by Chekanov [Che02] (see also [Eli98]) and for many higher-dimensional contact manifolds by Ekholm, Etnyre and Sullivan [EES05a], [EES07].

Chekanov [Che02] defined the first nonclassical invariant of Legendrian knots, called the *Chekanov-Eliashberg DGA*. To the Lagrangian projection of a Legendrian knot, it associates a dg-algebra $(\mathcal{A}, \partial)$, where $\mathcal{A}$ is a tensor algebra generated over some field $\mathbb{F}$ by the intersections of the knot projection and the differential ∂ counts immersed disks with boundary on the knot projection and corners at the knot crossings. This invariant distinguishes two Legendrian 5_2-knots with the same Thurston-Bennequin invariant and rotation number and these were the first examples of Legendrian nonsimple knots.

Ekholm, Etnyre and Sullivan constructed Legendrian Contact Homology, an invariant of higher-dimensional Legendrian submanifolds that generalizes the Chekanov-Eliashberg DGA. The invariant associated to a Legendrian submanifold L is a dg-algebra $(\mathcal{A}, \partial)$, with the underlying algebra $\mathcal{A}$ generated by the *Reeb chords* of L. When the ambient contact manifold is a 1-jet space $J^1(M)$, the differential ∂ is determined by counts of rigid, punctured, pseudoholomorphic disks in T^*M with boundary on the Lagrangian projection $\overline{L}$. For a more complete definition of Legendrian Contact Homology, augmentations and linearized contact homology, see Chapter 2.

A key technical detail in the definition of Legendrian contact homology is that the differential counts a restricted collection of pseudoholomorphic curves. These curves are genus 0, with one boundary component on $\overline{L}$, of formal dimension 0 (rigid disks) and containing exactly 1 positive puncture on $\overline{L}$. Such restrictions are necessary to construct a well-defined theory and ensure that $\partial^2 = 0$.

Consider the Lagrangian projection $\overline{L_1 \times L_2} \subset T^*(M_1 \times M_2)$ and give $T^*(M_1 \times M_2)$ the product almost-complex structure from T^*M_1 and T^*M_2. If $u : (\mathbb{D}, \partial\mathbb{D}) \to (T^*(M_1 \times M_2), \overline{L_1} \times \overline{L_2})$ is a holomorphic disk, it determines a pair of holomorphic disks $u_i : (\mathbb{D}, \partial\mathbb{D}) \to (T^*M_i, \overline{L_i})$ by projection. In the opposite direction, a pair (u_1, u_2) of holomorphic disks lifts to a disk u if and only if the conformal structures on the domain agree.

This is a simplification, however, because even if L_1 and L_2 are sufficiently generic to define Legendrian contact homology, the product, as constructed, is highly degenerate. This Morse-Bott degeneracy must be perturbed by some Morse function in order to calculate LCH. Reversing the degeneration, some 'thin' pieces of holomorphic disks may collapse into constant disks or gradient flow lines of the Morse function used to perturb $L_1 \times L_2$. Thus, it is necessary to consider generalized disks (see [EES09] for a simpler version of this in the context of LCH).

After the perturbation, a sequence $\mathbf{s} = s_1 s_2 \ldots s_m$ of punctures on $\overline{L_1} \times \overline{L_2}$ determines a sequence of punctures $\mathbf{s}^1 = s_1^1 s_2^1 \ldots s_m^1$ on $\overline{L}_1$ and a sequence of punctures $\mathbf{s}^2 = s_1^2 s_2^2 \ldots s_m^2$ on $\overline{L}_2$.

Thus, conjecturally, the moduli of disks with boundary on $\overline{L_1 \times L_2}$ and punctures at $\mathbf{s} = s_1 s_2 \ldots s_m$ is

$$\mathcal{M}_{L_1 \times L_2}(\mathbf{s}) \simeq \widetilde{\mathcal{M}}_{L_1}(\mathbf{s}^1) \times_{\Delta_m} \widetilde{\mathcal{M}}_{L_2}(\mathbf{s}^2) \tag{1.1}$$

where Δ_m is the space of conformal structures on the disk $\mathbb{D}$ with m marked points on its boundary. That is, there is a correspondence between holomorphic disks u in $J^1(M_1 \times M_2)$ with boundary on $\overline{L_1 \times L_2}$ and punctures at $\mathbf{s}$ and pairs of generalized disks (u_1, u_2), where u_i is a disk in $J^1(M_i)$, boundary on $\overline{L_1}$ and punctures at $\mathbf{s}^i$, such that the conformal structures on the domains of u_1, u_2 are equivalent.

Some important considerations include:

- the formal dimension of u_1, u_2 may be strictly greater than 0 even if the formal dimension of u is 0;
- the puncture of u_i at s_k^i on L_i may be positive even if the corresponding puncture u at s_k on $L_1 \times L_2$ is negative.

Thus, in order to determine $\mathcal{M}_{L_1 \times L_2}(\mathbf{w})$ it is necessary to consider moduli of disks that are *not* counted by the LCH differential. So, a priori, one cannot expect to determine $LCH(L_1 \times L_2)$ algebraically from $LCH(L_1)$ and $LCH(L_2)$, just as the Thurston-Bennequin invariant and generating family homology on $L_1 \times L_2$ cannot be determined by the invariants of the factors.

A key tool to compute LCH are *gradient flow trees*. For the Chekanov-Eliashberg DGA, the differential can be computed directly from the Lagrangian projection of a knot. However, this is not true in higher dimensions. When the ambient contact manifold is a 1-jet space $J^1(M)$, Ekholm [Ekh07], building on work of Fukaya and Oh [FO97], has shown that the differential can be computed in terms of these objects instead of pseudoholomorphic disks. Specifically, Ekholm established a 1 to 1 identification between rigid, punctured pseudoholomorphic disks and rigid gradient flow trees. Using flow trees has significant computational advantages since the analytic difficulty of working with pseudoholomorphic disks is much greater than that of working with flow trees. An application of this technique was used in [EENS13] to compute knot contact homology, an invariant of knots in S^3, as the Legendrian contact homology of a Legendrian torus $\Lambda_K \subset ST^*\mathbb{R}^3 \simeq J^1(S^2)$.

In Chapter 5, we prove a version of the conjecture in Equation 1.1.

Theorem D. *For L_1, L_2 Legendrian knots in $\mathbb{R}^3$, there exists a perturbation $\widetilde{L}_\epsilon$ of $L_1 \times L_2$ such that the moduli of rigid gradient flow trees on $\widehat{L}_\epsilon$ is the fiber product*

$$\mathcal{M}^{\nabla}_{\widetilde{L}_\epsilon}(\mathbf{s}) \simeq \widetilde{\mathcal{M}}^{\nabla}_{L_1}(\mathbf{s}^1) \times_{\mathcal{T}_m} \widetilde{\mathcal{M}}^{\nabla}_{L_2}(\mathbf{s}^2)$$

where $\mathcal{T}$ is the moduli of metric trees.

After the computation of knot contact homology [EENS13], this is only the second major computation of Legendrian contact homology in dimension 5.

Chapter 2
Preliminaries

A *symplectic* manifold (X, w) is a manifold X with a closed, nondegenerate 2-form ω. That is, ω^n is a volume form on X, and, as a consequence, the dimension of X must be even. A submanifold $L \subset X$ is *isotropic* if the restriction $\omega|_L$ is identically 0 and *Lagrangian* if it is isotropic and has dimension n. A Lagrangian submanifold is *exact* if the restriction λ_L is exact. For any (immersed) Lagrangian L, there is an oriented diffeomorphism between the normal bundle NL of L in X and the cotangent bundle T^*L. Moreover, there is a symplectomorphism between some neighborhood of L in X and a neighborhood of the 0-section in T^*L. A symplectic manifold (X, ω) is *exact symplectic* if the symplectic form is exact, i.e. $\omega = d\lambda$ for some primitive λ.

Some basic examples of symplectic manifold include $(\mathbb{R}^{2n}, \omega_{std})$ with symplectic form $\omega_{std} = \sum_i dx_i \wedge dy_i$ and the cotangent bundle $(T^*M, d\lambda)$ of an arbitrary manifold M, where λ is the canonical Liouville 1-form on the cotangent bundle.

A *contact form* α on an odd-dimensional manifold Y^{2n+1} is a 1-form such that $\alpha \wedge (d\alpha)^n$. A *contact structure* $\xi = \ker(\alpha)$ is a maximally non-integrable hyperplane field given as the kernel of some contact 1-form. The pair (Y, ξ) of a manifold with a contact structure is a *contact manifold*. A submanifold L of some contact manifold (Y, ξ) is *isotropic* if it is everywhere tangent to the hyperplane field ξ and is *Legendrian* if it is isotropic and has dimension n. A continuous, one-parameter family of Legendrian submanifolds is a *Legendrian isotopy*.

Some basic examples of contact manifolds include $(\mathbb{R}^{2n+1}, \xi_{std})$ where $\xi_{std} = \ker(dz - \sum_i y_i dx_i)$, and the 1-jet space $J^1(M)$ of an arbitrary manifold. The 1-jet space is the bundle whose sections are 1st-order Taylor approximations of functions on M and is diffeomorphic to $T^*M \times \mathbb{R}$. It has a canonical contact structure $\xi = \ker(dz - \lambda)$ where λ is the Liouville 1-form on T^*M. The standard contact structure on $\mathbb{R}^{2n+1}$ can be interpreted as arising in this way, since $\mathbb{R}^{2n+1} \simeq T^*\mathbb{R}^n \times \mathbb{R}$ and the Liouville form on $T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$ is $\lambda = \sum_i y_i dx_i$.

This is an example of a more general phenomenon. If $(X, d\lambda)$ is exact symplectic, then the 1-form $\alpha = dz + \lambda$ is a contact form on the product manifold $X \times \mathbb{R}$ (z is the coordinate on the second factor) as $\alpha \wedge (d\alpha)^n = \omega^n \wedge dz \neq 0$. This contact form induces a contact structure $\xi = \ker(\alpha)$ and the contact manifold $(X \times \mathbb{R}, \xi)$ is called the *contactization* of X. These have a distinguished vector field, the *Reeb vector field* $R_\alpha = \partial_z$, which satisfies $\iota_{R_\alpha} d\alpha = 0$ and $\alpha(R_\alpha) = 1$.

From now on, we restrict to contact manifolds that are 1-jet spaces. On a 1-jet space $J^1(M)$, there are two important projection maps. The *Lagrangian projection* $\Pi : J^1(M) \to T^*M$ onto the cotangent bundle of M, the *front projection* $\Pi_F : J^1(M) \to M \times \mathbb{R}$ extending the fiber bundle projection of the cotangent bundle, and the *base projection* $\Pi_M : J^1(M) \to M$.

Throughout this paper, we distinguish points and sets $x, U \subset X \times \mathbb{R}$ from their images under the Lagrangian projection Π through bar notation, i.e $\Pi(x) = \bar{x}, \Pi(U) = \bar{U}$.

The projection $\overline{L}$ of a Legendrian submanifold is exact Lagrangian. Let $Z : L \to \mathbb{R}$ be the restriction of the projection $J^1(M) \to \mathbb{R}$ to L. Then the Legendrian condition $\alpha|_L = 0$ implies that $\lambda_{\overline{L}} = dZ$. Furthermore, given an exact Lagrangian submanifold $\overline{L}$ in T^*M, there exists a *lift* $L \subset J^1(M)$ of $\overline{L}$, well-defined up translation in the z-direction, such that L is an immersed Legendrian submanifold. For any pair of points, $x, y \in \overline{L}$, the difference in z-coordinates of a lift must satisfy

$$Z(y) - Z(x) = \int_\gamma dz = \int_\gamma \lambda = f(y) - f(x),$$

where f is a primitive for λ_L. Thus, $\overline{L}$ can be lifted to some L by first choosing a point x and assigning it some z-coordinate and then extending this to all points $y \in \overline{L}$.

A *Reeb chord* c for some Legendrian submanifold L is an integral curve of the Reeb vector field that begins and ends on L. Since the Reeb vector field flows in the z-direction, the endpoints c^+, c^- of the chord project to the same point $\bar{c}$ and every multiple point of the projection $\bar{L}$ lifts to at least one Reeb chord. A Legendrian submanifold is *chord generic* if its Lagrangian projection has a finite number of transverse double points. Double points of $\bar{L}$ correspond to unique Reeb chords since $\mathbb{R}$ is not compact and this implies that L has a finite number of Reeb chords. The *action* $\mathcal{Z}(c)$ of a Reeb chord is its length, which is equal to the difference of z-coordinates $Z(c^+) - Z(c^-)$.

When L is orientable, each Reeb chord has a sign $\sigma(c)$ defined as follows. Choose neighborhoods U_+, U_- around c^+, c^- and call these the *upper sheet* and *lower sheet.* Let $V_+ := \Pi_*(T_{c^+}L)$ and $V_- := \Pi_*(T_{c^-}L)$. Since c corresponds to a transverse double point in the Lagrangian projection, $V_+ \oplus V_-$ span $T_{\bar{c}}T^*M$. An orientation on L induces orientations on $V_\pm$. If the orientation of $V_+ \oplus V_-$ agrees with the orientation of T^*M, then $\sigma(c) = 1$; otherwise $\sigma(c) = -1$. The sign is independent of the choice of orientation on L.

Let (Y, α) be an oriented contact manifold with contact structure $\xi = \ker(\alpha)$ and recall that the 2-form $d\alpha$ restricts to a symplectic form on ξ. If a submanifold L is isotropic, then $TL \subset \xi|_L$ and define $TL^\perp$ to be the symplectic subbundle of $\xi|_L$ whose fibers are the symplectic orthogonal complements to the fibers of TL. The *conformal symplectic normal bundle* of L in M is the quotient bundle,

$$CSN_Y(L) = TL^\perp / TL.$$

If dim $Y = 2n+1$ and dim $L = m$ then $CSN_Y(L)$ has rank $2n - 2m$. If we choose an almost complex structure J on ξ compatible with $d\alpha$, then the normal bundle of L in Y splits as

$$NL = \langle R_\alpha \rangle \oplus J(TL) \oplus CSN_Y(L)$$

The contact form α restricts to a contact form on the fibers of $\langle R_\alpha \rangle \oplus CSN_Y(L)$ and so it is a contact subbundle of NL. Furthermore, there exists a contactomorphism between suitable neighborhoods of $L \subset Y$ and the 0-section of $J^1(L) \oplus CSN_Y(L)$. Similarly, given a contact embedding $(Y, \xi) \hookrightarrow (Y', \xi')$, the *conformal symplectic normal* bundle $CSN_{Y'}(Y)$ is the symplectic subbundle $(\xi)^\perp \subset \xi'|_Y$ given by taking the symplectic orthogonal complement to ξ in $\xi'|_Y$. This bundle can be identified with the normal bundle NY of Y in Y'. There also exists a contactomorphism between suitable neighborhoods of $Y \subset Y'$ and the 0-section of $CSN_{Y'}(Y)$.

2.1 Invariants

In this section, we describe several invariants of Legendrian submanifolds up to Legendrian isotopy.

2.1.1 Thurston-Bennequin

Suppose that $\bar{L}$, the projection of L to T^*M, has a finite number of transverse double points. Then these double points are in one-to-one correspondance with the Reeb chords and L is called chord generic. Each chord c can then be assigned a sign $\sigma(c) = \pm 1$ as describe above and in [EES05b], it is shown that the Thurston-Bennequin invariant is given by

$$tb(L) = \sum_c \sigma(c).$$

Thus, since the Thurston-Bennequin number can be computed as a signed count of intersection points in $\overline{L}$, we will interpret it as the 'writhe' of L with respect to the Lagrangian projection.

2.1.2 Maslov Class

Let Λ_n be the Grassmann manifold of Lagrangian subspaces in the standard affine symplectic space $(\mathbb{R}^{2n}, \omega)$. Fix some Lagrangian subspace $\Lambda \in \Lambda_n$ and let $\Sigma_k \subset \Lambda_n$ be the set of all Lagrangian planes in $\mathbb{R}^{2n}$ that intersect Λ along a subspace of dimension k. Then the *Maslov cycle* is the algebraic subvariety,

$$\Sigma = \overline{\Sigma_1} = \Sigma_1 \cup \cdots \cup \Sigma_n,$$

which has codimension 1 in Λ_n. For a path $\Gamma : [0,1] \to \Lambda_n$, we can define an intersection number of Γ and Σ as follows. Fix a Lagrangian complement W to Λ and suppose that $\Gamma(t')$ intersects Σ. For t near t', there exists a family $w(t) \in W$ of vectors such that for all $v \in \Gamma(t') \cap \Sigma$ the vector $v + w(t) \in \Gamma(t)$. Then there is a quadratic form $Q = \frac{d}{dt}|_{t'} \omega(v, w(t))$ on $\Gamma(t') \cap \Sigma$ and the signature of this quadratic form is the intersection number of Γ at t'.

If Γ is a loop, then the *Maslov index* $\mu(\Gamma)$ is the total intersection number of Γ with Σ. The map μ defines an isomorphism $H_1(\Lambda_n) \simeq \pi_1(\Lambda_n) \simeq \mathbb{Z}$.

Let $\overline{L} \subset \mathbb{R}^{2n}$ be an immersed Lagrangian. A global trivialization of $T\mathbb{R}^{2n}$ induces a map $f : L \to \Lambda_n$, where each point is sent to its Lagrangian tangent plane. The *Maslov class* is the pullback of a generator m of $H^1(\Lambda_n; \mathbb{Z})$,

$$\mu_L := f^*(m).$$

The *Maslov number* of L is the minimal nonzero Maslov index of some first homology class of L,

$$m(L) := \min_{\alpha \in H_1(L;\mathbb{Z})} \{|\mu_L(\alpha)| \mid \mu_L(\alpha) \neq 0\}.$$

If $\mu(\alpha) = 0$ for all $\alpha \in H_1(L;\mathbb{Z})$, then define $m(L) = 0$.

2.1.3 Legendrian Contact Homology

In the following description of Legendrian contact homology, we assume that the Legendrian submanifold L is connected, oriented and admits a spin structure. In addition, we restrict to when the ambient contact manifold is $\mathbb{R}^{2N+1} = \mathbb{C}^N \times \mathbb{R}$ and the submanifold L is chord generic.

The Legendrian contact homology $(\mathcal{A}, \partial)$ of a chord-generic Legendrian $L \subset \mathbb{R}^{2N+1}$ is a differential-graded algebra associated to L. Two different versions of LCH have been defined in the literature. The original, *homology-commutative* version and a *fully-noncommutative* or *homology-noncommutative* version used in [EENS13]. Since the homology-commutative version can be obtained from the non-commutative version, we will only define the latter.

Let R be a unital, commutative ring. In practice, this ring will often by $\mathbb{F}_2$, the field with two elements, or $\mathbb{Z}$. Since L is chord-generic, there is a finite set of Reeb chords $\{c_1, \dots, c_n\}$, which are in bijective correspondence with the transverse double points of $\overline{L}$. Let $Q = \{q_1, \dots, q_n\}$ be a set of variables, where the Reeb chord c_i corresponds to the letter q_i.

In the homology-commutative version, the underlying algebra $\mathcal{A}_{comm}$

$$\mathcal{A}_{comm} := R[H_1(L;\mathbb{Z}) \times \mathrm{Fr}(Q)]$$

is the group ring of the direct product of $H_1(L)$ with the free group generated by the set Q. In the noncommutative version, the underlying algebra $\mathcal{A}_{non-comm}$

$$\mathcal{A}_{non-comm} := R[H_1(L;\mathbb{Z}) * \mathrm{Fr}(Q)]$$

is the group ring of the free product of $H_1(L)$ with $\mathrm{Fr}(Q)$.

Thus, a monomial term in $\mathcal{A}_{non-comm}$ has the form

$$r\alpha_1 s_1 \alpha_2 s_2 \dots s_m \alpha_{m+1}$$

where $r \in R; \alpha_1, \dots, \alpha_{m+1} \in H_1(L)$ and $s_1, \dots, s_m \in Q$.

The commutative version can be recovered by quotienting out the commutator $[H_1(L), \mathrm{Fr}(q)]$ in $\mathcal{A}_{non-comm}$. From now on, we will use $\mathcal{A}$ to denote $\mathcal{A}_{non-comm}$.

Before defining the grading and differential, we first need to make a few choices. First, choose a point $x \in L$ and a collection of paths $\gamma_i^{\pm}$ in L from $c_i^{\pm}$ to x, called *endpoint paths*. The unions $\gamma_i = \gamma_i^+ \cup -\gamma_i^-$ is a path from c_i^+ to c_i^-, called a *capping path* for the Reeb chord c_i.

Since $\overline{L}$ is Lagrangian, the bundle $\overline{\gamma}_i^*(T\overline{L})$ is a path of Lagrangian planes in $\Lambda(N)$. The path can be closed to a loop in $\Lambda(n)$ by a positive rotation (see [EES05a] for details). The *Conley-Zehnder index* $CZ(\gamma_i)$ of the capping path γ_i is the Maslov index of this closed loop. Using the Conley-Zehnder index, we can define a map $|\,| : Q \to \mathbb{Z}$ by

$$|q_i| := CZ(\gamma_i) - 1$$

and extend this to a group homomorphism $\mathrm{Fr}(Q) \to \mathbb{Z}_a$, where $\mathbb{Z}_a$ is the underlying additive group of the integers. While this homomorphism depends on the choices of baspeoint x and endpoints paths $\gamma_i^{\pm}$, it is well defined modulo the Maslov number $m(L)$ of L since if γ_i' is another capping path, then the Maslov index of the closed path $\gamma_i * -\gamma_i'$ is a multiple of $m(L)$.

Extending the map $|\,|$ to $H_1(L;\mathbb{Z})$ by setting $|\alpha| = 0$ for all $\alpha \in H_1(L;\mathbb{Z})$, the map $|\,|$ becomes a grading

$$|\,| : \mathcal{A} \to \mathbb{Z}/m(L)\mathbb{Z}$$

on the group ring $\mathcal{A} = R[H_1(L;\mathbb{Z}) * \mathrm{Fr}(Q)]$ that is well-defined modulo $m(L)$.

Differential

Let $\mathbb{D}$ denote the unit disk in $\mathbb{C}$, let $\mathbf{z} = \{z_1, \ldots, z_m\}$ be a collection of m points in $\partial\mathbb{D}$ and define $\mathbb{D}_{\mathbf{z}} := \mathbb{D} \setminus \mathbf{z}$ and $\partial\mathbb{D}_{\mathbf{z}} := \partial\mathbb{D} \setminus \mathbf{z}$. Furthermore, let $\beta_1, \ldots, \beta_m$ be the components of $\partial\mathbb{D}_{\mathbf{z}}$, so that $\partial\beta_i = z_i - z_{i-1}$.

Consider smooth maps

$$u : (\mathbb{D}_{\mathbf{z}}, \partial\mathbb{D}_{\mathbf{z}}) \to (\mathbb{C}^N, \overline{L})$$

such that u lifts to a continuous map $\widetilde{u} : \partial\mathbb{D}_{\mathbf{z}} \to L \subset \mathbb{R}^{2N+1}$.

The map u has a *puncture* at a Reeb chord c if, for $z \in \mathbb{D}$ and some $k = 1, \ldots, m$, $u(z) \to \overline{c}$ as $z \to z_k$ but u does not lift continuously to a map

$$\widetilde{u} : \partial\mathbb{D}_{\mathbf{z}} \cup z_k \to L$$

The puncture is *positive* if

$$\lim_{z \to z_k} \widetilde{u}(z) = \begin{cases} c_- & \text{for } z \in \beta_k \\ c_+ & \text{for } z \in \beta_{k+1} \end{cases}$$

Conversely, the puncture is *negative* if

$$\lim_{z \to z_k} \widetilde{u}(z) = \begin{cases} c_+ & \text{for } z \in \beta_k \\ c_- & \text{for } z \in \beta_{k+1} \end{cases}$$

Let $B = (b_0, \ldots, b_m) \in H_1(L;\mathbb{Z})^{m+1}$ be an $(m+1)$-tuple of 1^{st} homology classes in L, let s_0 be some Reeb chord of L, and let $\mathbf{s} = (s_1, \ldots, s_m)$ be an m-tuple of Reeb chords of L.

Define $\mathcal{M}_B(s_0; \mathbf{s})$ to be the moduli space of holomorphic maps $u : (\mathbb{D}_{\mathbf{z}}, \partial\mathbb{D}_{\mathbf{z}}) \to (\mathbb{C}^N, \overline{L})$ for some collection $\mathbf{z}$ of $m+1$ points, modulo reparametrization of the domain, such that

1. u has a positive puncture at s_0, with $u(z) \to s_0$ as $z \to z_{m+1}$

2. u has negative punctures at $s_1, \dots, s_m$, in cyclic order, with $u(z) \to s_i$ as $z \to z_i$
3. the closed loop formed by the union of $\widetilde{u}(\beta_i)$ and the appropriate endpoint paths at s_{i-1}, s_i represents the homology class b_0.

For generic L and almost complex structures J on $\mathbb{C}^N$, the moduli space $\mathcal{M}_B(s_0; \mathbf{s})$ is a manifold of dimension

$$\dim \mathcal{M}_B(s_0; \mathbf{s}) = |s_0| - \sum_{k=1}^{m} |s_k| - 1 \tag{2.1}$$

The moduli space is transversely cutout as the 0-section of the differential operator $\overline{\partial}_J$. In[EES05a], it is shown that transversality can be achieved by fixing J to be the standard complex structure on $\mathbb{C}^N$ and only perturbing the embedding $L \hookrightarrow \mathbb{R}^{2N+1}$.

Moreover, via Gromov compactness, the moduli space $\mathcal{M}_B(s_0; \mathbf{s})$ can be compactified by broken disks. As a consequence, if $\dim \mathcal{M}_B(s_0; \mathbf{s}) = 0$ then the moduli space is compact and $|\mathcal{M}_B(s_0; \mathbf{s})|$ is a nonnegative integer.

Since L admits a spin structure, the moduli spaces $\mathcal{M}_B(s_0; \mathbf{s})$ can be coherently oriented [ENS02, EES05c] and we can define $|\mathcal{M}_B(s_0; \mathbf{s})|$ to be a signed count of points in the moduli space, with signs determined by the coherent orientation scheme.

Using the holomorphic curve date can define a differential $\partial : A \to \mathcal{A}$. First, set

$$\begin{aligned}
\partial \alpha &:= 0 \\
\partial s_0 &:= \sum_{\substack{B=(\beta_0,\dots,\beta_m); \\ |s_0| - \sum |s_k| = 1}} |\mathcal{M}_B(s_0; \mathbf{s})| \beta_0 s_1 \beta_1 s_1 \dots \beta_{m-1} s_m \beta_m
\end{aligned}$$

then extend this to a differential on $\mathcal{A}$ by the graded Leibnitz formula

$$\partial(ab) = \partial(a)b + (-1)^{|a|} a \partial(b)$$

The algebra $(\mathcal{A}, \partial)$ depends upon the explicit embedding of L and is not invariant under Legendrian isotopy. To obtain an isotopy invariant, it is necessary to consider $(\mathcal{A}, \partial)$ up to stable tame isomorphism.

Let $\mathcal{A} = R[G * \mathrm{Fr}(n)]$ be an algebra and let $a_1, \dots, a_n$ be the generators of $\mathrm{Fr}(n)$. Define $\mathcal{A}_i$ to be the subalgebra $R[G * \mathrm{Fr}(a_1, \dots, \widehat{a_i}, \dots, a_n)]$, where $\widehat{a_i}$ signifies that the i^{th} letter a_i has been removed from the generating set. An R-algebra automorphism ϕ of $\mathcal{A}$ is *elementary* if there exists some $i = 1, \dots, n$ such that ϕ has the form

$$\begin{array}{lll}
\phi(g) & = g & \text{for some } g \in G \\
\phi(a_j) & = g_j a_j h_j & \text{for some } g_j, h_j \in G \\
\phi(a_i) & = g_i a_i h_i + u & \text{for some } g_i, h_j \in G \text{ and } u \in \mathcal{A}_i
\end{array}$$

A *tame* isomorphism is a composition of elementary isomorphisms. An automorphism of dg-algbras is elementary or tame if the underlying map on algebras is elementary or tame.

Let (S_k, d) be a dg-algebra, where $S_k := R[\mathrm{Fr}(e_k, e_{k-1})]$ is an R-algebra with $|e_k| = |e_{k-1}| + 1 = k$ and differential $\partial(e_k) = e_{k-1}$ and $\partial(e_{k-1}) = 0$. A degree k *stabilization* $(S_k\mathcal{A}, \partial)$ of a dg-algebra $(\mathcal{A}, \partial)$ is the free product of $(\mathcal{A}, \partial)$ with (S_k, d).

A *stable tame isomorphism* of dg-algebras is a composition of stabilizations and tame isomorphisms. Note that all stable tame isomorphisms are homotopy equivalences.

Theorem 2.1 ([EES05a, EES07])**.** *The map $\partial : \mathcal{A} \to \mathcal{A}$ is a differential on $\mathcal{A}$. In particular, ∂ has degree -1 and $\partial^2 = 0$. Moreover, the stable tame isomorphism class of $(\mathcal{A}, \partial)$ is invariant under Legendrian isotopies of L.*

An *augmentation* ϵ on $(\mathcal{A}, \partial)$ is a graded morphism $\epsilon : (\mathcal{A}, \partial) \to (\mathbb{F}, 0)$ of dg-algebras over $\mathbb{F}$, where $\mathbb{F}$ is a 1-dimensional $\mathbb{F}$-algebra supported in grading 0 and the differential on $\mathbb{F}$ is the 0-map. In particular, $\epsilon(1) = 1$ ($\mathbb{F}$-algebra morphism); $\epsilon \circ \partial = 0$ (chain map); ϵ is supported on the 0-graded piece of $\mathcal{A}$ (graded morphism).

In general, the homology of $(\mathcal{A}, \partial)$ is infinite-dimensional over $\mathbb{F}$. However, given an augmentation ϵ, it is possible to extract a it linearized version of the homology that is finite-dimensional over $\mathbb{F}$.

Recall that each element in $\mathcal{A}$ has the form

$$a = \sum_{i=0} \sum_{j=1}^{n_i} \alpha_{j,0} s_{j,1} \alpha_{j,1} \cdots s_{j,i} \alpha_{j,i}$$

Thus, the differential splits

$$\partial = \partial^0 + \partial^1 + \partial^2 + \dots$$

where $\partial_i(a)$ is the sum of n_i monomials in $\partial(a)$ with i Reeb chord elements. The condition $\partial^2 = 0$ implies that $(\partial^0)^2 = 0$, but it is not always true that $(\partial^1)^2) = 0$.

For an augmentation, define an isomorphism $\phi_\epsilon : \mathcal{A} \to \mathcal{A}$ by $\phi_\epsilon(q) = q + \epsilon(q)$. Conjugating by this isomorphism induces a new differential $\partial_\epsilon = \phi_\epsilon^{-1} \partial \phi_\epsilon$. For this new differential, $(\partial_\epsilon^1)^2 = 0$ and so there is a chain complex $(\mathbb{F}\langle q_1, \dots, q_n \rangle, \partial_\epsilon^1)$ whose homology is a finite-dimensional vector space over $\mathbb{F}$. The *linearized homology $H_*^\epsilon(L)$* of $(\mathcal{A}, \partial)$ with respect to the augmentation ϵ is the homology of this chain complex.

Note that the linearized homology itself is not an invariant of L; only the sets of augmentations $\{\epsilon\}$ and sets of linearized homologies $\{H_*^\epsilon\}$ with respect to these homologies are invariant under Legendrian isotopy.

2.2 Spinning

In [EES05b], Etnyre, Ekholm and Sullivan defined a construction, called frontspinning, which takes a Legendrian $L \subset \mathbb{R}^{2n-1}$ and produces a Legendrian $\Sigma L \subset \mathbb{R}^{2n+1}$ of topological type $L \times S^1$.

Suppose that a Legendrian $L \subset \mathbb{R}^{2n+1}$ is given by the embedding $f : L \to \mathbb{R}^{2n+1}$ and parametrized such that

$$f(L) = (x_1(L), \dots, x_n(L), y_1(L), \dots, y_n(L), z(L))$$

Let $\Pi_F : \mathbb{R}^{2n+1} \to \mathbb{R}^{n+1}$ be the projection onto the x and z coordinates. The *front projection* of L is the subsvariety

$$\Pi_F(L) = (x_1(L), \ldots, x_n(L), z(L))$$

If S is a subvariety of $\mathbb{R}^{n+1}$ such that $\frac{\partial}{\partial z} \notin T_x S$ for any $x \in S$, then S lifts to an isotropic subvariety of $\mathbb{R}^{2n+1}$ as the y-coordinates can be chosen at each point to satisfy the contact condition. The *frontspinning* of L is the lift to $\mathbb{R}^{2n+3}$ of the following subvariety of $\mathbb{R}^{n+2}$:

$$S = (\cos\theta x_1(L), \sin\theta x_1(L), x_2(L), \ldots, x_n(L), z(L))$$

for $\theta \in [0, 2\pi]$. This can be thought of as spinning the front projection of L around the plane $x_0 = x_1 = 0$ in $\mathbb{R}^{n+2}$ and the intersections of S with the planes $(\cos\theta x_1, \sin\theta x_1, x_2, \ldots, x_n, z)$ give a family of front projections of L as θ varies.

In fact, frontspinning should more naturally be seen as the product of L with a standard Legendrian unknot U with $tb = -1, r = 0$. Consider S^{2k-1} as the unit sphere in $\mathbb{R}^{2k}$ and take the obvious contact embedding $S^{2k-1} \hookrightarrow \mathbb{R}^{2n+1} \simeq \mathbb{R}^{2k} \times \mathbb{R}^{2n-2k+1}$.

Before proving this, we first give an alternate interpretation of this construction that generalizes to arbitrary contact manifolds. Choose some Legendrian $K \subset M$, a contact embedding $M \hookrightarrow Y'$ of codimension $2m$ with trivial $CSN_{Y'}(Y)$ and a Legendrian $L \subset \mathbb{R}^{2m+1}$. By scaling and translation, we can assume that L lies in a suitably small neighborhood of the origin. It follows that the conformal symplectic normal bundle of K in Y' splits as

$$CSN_{Y'}(K) = CSN_Y(K) \oplus CSN_{Y'}(Y)$$

and by assumption, since K is Legendrian, this bundle is trivial of rank $2m$.

Definition 2.2. The *spinning* of L by K, denoted $K \times_Y L$, is the Legendrian submanifold of Y' obtained as the image of $K \times L$ under the contactomorphism that identifies neighborhoods of K in $J^1(K) \times \mathbb{R}^{2m}$ and Y'.

It is clear from the construction that ambient contact isotopies of K, Y and L extend to Legendrian isotopies of $K \times_Y L$, provided that L is contained in a suitably small neighborhood of 0.

We now prove the following theorem and corollary.

Theorem 2.3. *Let L be a Legendrian submanifold of S^{2k-1} (equiv $\mathbb{R}^{2k-1}$) and K a Legendrian submanifold of $\mathbb{R}^{2n-2k+1}$, chosen so that all Reeb chord actions of K are much less than all Reeb chord actions of L. Then $K \times_{S^{2k-1}} L$ and $K \times L$ are Legendrian isotopic in $\mathbb{R}^{2n+1}$*

Let U denote the Whitney embedding of the standard Legendrian unknot of S^3 with exactly one Reeb chord of length 1.

Corollary 2.4. *Let $L \subset \mathbb{R}^{2n-1}$ be Legendrian all of whose Reeb chords have action $\mathcal{Z}(c) \ll 1$. Then ΣL and $U \times L$ are Legendrian isotopic in $\mathbb{R}^{2n+1}$.*

Consider (S^3, ξ_{std}) as the unit sphere in $\mathbb{R}^4$. Then for the embedding $(S^3, \xi_{std}) \hookrightarrow \mathbb{R}^{2n+1} \simeq \mathbb{R}^4 \times \mathbb{R}^{2n-3}$, the conformal symplectic normal bundle $CSN_{\mathbb{R}^{2n+1}}(S^3)$ is trivial. Let U be the submanifold

$$U = (\cos\theta, 0, \sin\theta, 0),\ \theta \in S^1$$

It is easy to verify that U is Legendrian and isotopic to the standard Legendrian unknot with $tb = -1, r = 0$.

Lemma 2.5. *The Legendrian submanifolds ΣL and $U \times_{S^3} L$ are identical.*

Proof. Let J be the standard complex structure on $\mathbb{C}^2$. Then we can trivialize $TU, J(TU)$ as

$$T_\theta U = \langle -\sin\theta\partial_{x_1} + \cos\theta\partial_{x_2}\rangle$$
$$J(T_\theta U) = \langle -\sin\theta\partial_{y_1} + \cos\theta\partial_{y_2}\rangle$$

and trivialize $CSN_{\mathbb{R}^{2n+1}}(U)$ as

$$CSN_{\mathbb{R}^{2n+1}}(W) = \langle \cos\theta\partial_{x_1} + \sin\theta\partial_{x_2}, \partial_{x_3}, \partial_{y_3}, \dots, \partial_{x_n}, \partial_{y_n}, \partial_z\rangle$$

The Reeb vector field in S^3 along U is given by $R = \cos\theta\partial_{y_1} + \sin\theta\partial_{y_2}$. It follows from above that the frontspun ΣL is obtained by restricting to L in each fiber of the bundle $\langle R\rangle \oplus CSN_{\mathbb{R}^{2n+1}}(W)$ □

We can now prove Theorem 2.3

Proof of Theorem 2.3. Choose a Darboux ball around the point $(1, 0, \dots, 0) \in \mathbb{R}^{2n+1}$, given by some map $f : B^{2n+1} \to \mathbb{R}^{2n+1}$ of the unit ball, that restricts to a Darboux ball $f' : B^{2k-1} \to S^{2k-1}$ on S^{2k-1} as well. Thus, $f(K \times L)$ is exactly $f'(K) \times_{S^{2k-1}} L$. Choose some Legendrian isotopy $K(t)$ in S^{2k-1} so that $K(0) = K$ and $K(1) = f'(K)$ in this Darboux ball on S^{2k-1}. The isotopy extends to some suitable neighborhood of K and isotope L so that $K \times_{S^{2k-1}} L$ lies in this neighborhood. The Legendrian isotopy of K thus extends to a Legendrian isotopy from $K \times_{S^{2k-1}} L$ to $f(K \times L)$.

Recall that the contact disk theorem states that for any two contact embeddings $g, h : B^{2n+1} \to M^{2n+1}$ of the unit ball into a contact manifold, there is a contact isotopy $i : M \to M$ such that $i \circ h = g$. Thus, there exists some isotopy $i : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ such that $i \circ f = id$ and this isotopy sends $f(K \times L)$ to $K \times L$. Furthermore, since L must live in a suitably small neighborhood of the origin, whose diameter must be less than the length of the Reeb chords of K, it follows that all its Reeb chord actions must be less than this diameter. □

Corollary 2.4 now follows directly from Lemma 2.5 and Theorem 2.3.

Chapter 3
Classical Invariants

The goal of this section is to prove Theorems A and B.

Theorem A. *Let $L_1 \subset \mathbb{R}^{2n+1}, L_2 \subset \mathbb{R}^{2m+1}$ be chord generic Legendrians such that n, m have different parity. Then there exists an infinite family of Legendrians $\{L_1^\alpha\}$ all Legendrian isotopic to L_1 such that the family of Legendrian products $\{L_1^\alpha \times L_2\}$ are pairwise non-Legendrian isotopic.*

To prove this, we calculate the Thurston-Bennequin invariant of these Legendrian products in terms of the embeddings of L_1, L_2. There are two well-known, classical invariants of Legendrian knots: the Thurston-Bennequin number and the rotation number. These have been generalized to higher dimensions by Tabachnikov and by Ekholm, Etnyre and Sullivan. The latter described the Thurston-Bennequin invariant for homologically-trivial Legendrians L as the linking number of L with a pushoff L' of itself along the Reeb vector field

$$tb(L) = lk(L, L')$$

In our setting, when the ambient contact manifold is $\mathbb{R}^{2n+1}$, this can be computed in a manner similar to the writhe of knots. The Maslov class is a cohomology class $\mu \in H^1(L;\mathbb{Z})$ that assigns to each 1-dimensional homology class the Maslov index of a path representing that class. In dimension 3, the rotation number is $\frac{1}{2}\mu(\gamma)$ where γ is a generator of $H_1(S^1;\mathbb{Z})$. Suppose that $L_1 \in \mathbb{R}^{2n+1}$ and $L_2 \in \mathbb{R}^{2m+1}$ are chord generic Legendrians with Reeb chords $\{a_i\}, \{b_j\}$. Then we can obtain the following formula for the classical invariants of their product.

Theorem B. *The Thurston-Bennequin number of $L_1 \times L_2$ is given by:*

$$\begin{aligned} tb(L_1 \times L_2) = (-1)^{mn} tb(L_1)\chi(T^*L_2) + \chi(T^*L_1) tb(L_2) \\ + tb(L_1)tb(L_2) + \sum_{i,j} \tau(a_i, b_j)\sigma(a_i)\sigma(b_j) \end{aligned}$$

where $\chi(T^*L)$ denotes the pairing $\langle e(T^*L), [L]\rangle$ of the Euler class of T^*L with the fundamental class of L and

$$\tau(a_i, b_j) = \begin{cases} (-1)^n & \text{if } \mathcal{Z}(a_i) < \mathcal{Z}(b_j) \\ (-1)^m & \text{if } \mathcal{Z}(a_i) > \mathcal{Z}(b_j) \end{cases}$$

The Maslov class of $L_1 \times L_2$ is given by

$$\mu_{L_1 \times L_2} = \mu_{L_1} \oplus \mu_{L_2} \in H^1(L_1 \times L_2; \mathbb{Z}) \simeq H^1(L_1;\mathbb{Z}) \oplus H^1(L_2;\mathbb{Z})$$

The last term in the tb formula requires some elaboration. Locally, each transverse double point is the intersection of two open $(m+n)$-disks in a single point. Just as computing the sign of a knot crossing requires knowing which strand passes over the other, computing the sign of a Reeb chord requires knowing which disk passes "over" the other (i.e. greater z-coordinate). In the product $L_1 \times L_2$, after a suitable perturbation, there is a Reeb chord $c_{i,j}$ for each pair (a_i, b_j) of Reeb chords of L_1 and L_2. However, determining which disk is "over" and which is "under" depends upon the relative lengths of the Reeb chords a_i, b_j.

3.1 Thurston-Bennequin

One problem with this product is that while each factor is chord generic, the product is not. In fact, each Reeb chord is part of some family of Reeb chords given by either $a_i \times L_2$ or $L_1 \times b_j$. In order to make $L_1 \times L_2$ chord generic, we must perturb it slightly. Let f, g be C^1-small Morse functions on L_1, L_2 whose critical points are away from the endpoints of the Reeb chords and such that the endpoints lie in different level sets (i.e. $f(a_i^+) \neq f(a_i^-), g(b_j^+) \neq g(b_j^-)$). Thus, there exist neighborhoods U_i^+, U_i^- of a_i^+, a_i^- such that $f(U_i^+)$ and $f(U_i^-)$ are disjoint and similarly there exist such neighborhoods W_j^+, W_j^- of each b_j^+, b_j^-. Denote the critical points of f by m_k^1 and the critical points of g by m_l^2. We can identify a small neighborhood of $L_1 \times L_2$ with a neighborhood of the 0-section in its 1-jet space $J^1(L_1 \times L_2)$ and perturb it by a Legendrian isotopy to the graph of fg in the 1-jet space $J^1(L_1 \times L_2)$.

Lemma 3.1. *The perturbed Legendrian is chord generic and has the following Reeb chords:*

- ***Reeb/Morse****: one for each pair (a_i, m_l^2) of Reeb chord for L_1, Morse critical point of L_2, denoted $a_i \otimes m_l^2$*
- ***Morse/Reeb****: one for each pair (m_k^1, b_j) of Morse critical point of L_1, Reeb chord for L_2, denoted $m_k^1 \otimes b_j$*
- ***Reeb/Reeb****: two for each pair (a_i, b_j) of Reeb chord for L_1, Reeb chord for L_2, denoted $c_{i,j}$ and $d_{i,j}$*

We will refer to these as A-chords, B-chords, C-chords and D-chords, respectively.

Proof. In the Lagrangian projection, a neighborhood of $\overline{L_1 \times L_2}$ is symplectomorphic to a neighborhood $\eta(0)$ of the 0-section of the cotangent bundle. Moreover, we can assume that the map $v|_{0-\text{section}}$ is injective away from $T^*(U_i^\pm \times W_j^\pm)|_\eta$. The perturbation pushes $L_1 \times L_2$ off to the graph of $d(fg) = gdf + fdg$. Let x, y denote points in L_1, L_2 and $\bar{x}, \bar{y}$ denote their projections. Then the perturbation maps $(\bar{x}, \bar{y})$ to $(\bar{x} + g(y)df(x), \bar{y} + f(x)dg(y))$.

Now, suppose that $(\bar{x}+g(y)df(x), \bar{y}+f(x)dg(y)) = (\bar{x'}+g(y')df(x'), \bar{y'}+f(x')dg(y'))$ for some $x, x' \in L_1, y, y' \in L_2$. If $x = x'$, then either x is a Morse critical point or y, y' lie in the same level set. In the first case, we get intersection points coming from the intersection points of $f(x)dg$, the pushoff, whose intersection points are in 1-1 correspondance with the intersection points of L. This gives the A-chords. The second is impossible, since if y, y' are distinct then we must have (up to relabeling) that $y \in W_j^+$ and $y' \in W_j^-$ and so y, y' cannot lie in the same level set. Repeating this for $y = y'$ will yield the B-chords.

Finally, consider when both pairs x, x' and y, y' are distinct. Prior to the perturbation, there was a unique transverse intersection point of $\overline{U_i^+} \times \overline{W_j^+}$ and $\overline{U_i^-} \times \overline{W_j^-}$. Similarly, there was a unique transverse intersection point of $\overline{U_i^+} \times \overline{W_j^-}$ and $\overline{U_i^-} \times \overline{W_j^+}$. Since the perturbation is C^1-small, we can assume that after the perturbation, there remains a unique intersection point in each case. Thus, the intersection point $(\bar{x}, \bar{y}) = (\bar{x'}, \bar{y'})$ under consideration must be one of these two; the first case we label $d_{i,j}$ and the second we label $c_{i,j}$. □

We can compute the signs of the intersection points as well:

Lemma 3.2. *The Reeb chords of the perturbation have the following actions*

- $\mathcal{Z}(a_i \otimes m_l^2) \approx \mathcal{Z}(a_i)$
- $\mathcal{Z}(m_k^1 \otimes b_j) \approx \mathcal{Z}(b_j)$
- $\mathcal{Z}(c_{i,j}) \approx |\mathcal{Z}(a_i) - \mathcal{Z}(b_j)|$
- $\mathcal{Z}(d_{i,j}) \approx \mathcal{Z}(a_i) + \mathcal{Z}(b_j)$

and signs

- $\sigma(a_i \otimes m_l^1) = (-1)^{mn}\sigma(a_i)\sigma(m_l^2)$
- $\sigma(m_k^1 \otimes b_j) = (-1)^{mn}\sigma(m_k^1)\sigma(b_j)$
- $\sigma(c_{i,j}) = (-1)^{mn}\sigma(a_i)\sigma(b_j)\tau(i,j)$
- $\sigma(d_{i,j}) = (-1)^{mn}\sigma(a_i)\sigma(b_j)$

where

$$\tau(i,j) = \begin{cases} (-1)^n & \text{if } \mathcal{Z}(a_i) < \mathcal{Z}(b_j) \\ (-1)^m & \text{if } \mathcal{Z}(a_i) > \mathcal{Z}(b_j) \end{cases}$$

Proof. As above, let $V_+ := (\Pi_P)_*(T_{a^+}K)$ and $V_- := (\Pi_P)_*(T_{a^-}K)$, which we think of as the tangent planes to the upper and lower sheets $\bar{U_+}, \bar{U_-}$ in P at $\bar{a}$, and define W_+, W_- similarly for L. Then for the Reeb/Morse chords, the tangent planes of the upper and lower sheets at $\overline{a_i \otimes m_l^1}$ are given by $V_+ \oplus W_+$ and $V_- \oplus W_-$. The sign of the Reeb chord is given by the orientation of $V_+ \oplus W_+ \oplus V_- \oplus W_-$, which is $(-1)^{mn}$ times the orientation given by $V_1 + \oplus V_- \oplus W_+ \oplus W_-$, whose sign is given by $\sigma(a_i)\sigma(m_k^2)$. The situation is similar for the Morse/Reeb chords and each $d_{i,j}$.

However, for the $c_{i,j}$, the calculation is different. If $\mathcal{Z}(a_i) < \mathcal{Z}(b_j)$, the tangent plane to the upper sheet at $\bar{c_i, j}$ is given by $V_+ \oplus W_-$ and to the lower sheet by

$V_- \oplus W_+$. Thus, the orientation on $V_+ \oplus W_- \oplus V_- \oplus W_+$ is $(-1)^{mn}(-1)^{m^2}$ times that of $V_+ \oplus V_- \oplus W_+ \oplus W_-$, which is given by $\sigma(a_i)\sigma(b_j)$. But if $\mathcal{Z}(a_i) > \mathcal{Z}(b_j)$, then the sign of the Reeb chord is given by the orientation of $V_- \oplus W_+ \oplus V_+ \oplus W_-$, which differs from $\sigma(a_i)\sigma(b_j)$ by $(-1)^{mn}(-1)^{n^2}$. □

We can now prove Theorem B. Suppose that $\bar{L}$ has a finite number of transverse double points. Then these double points are in one-to-one correspondance with the Reeb chords and L is called chord generic. Each chord c can then be assigned a sign $\sigma(c) = \pm 1$ and in [EES05b], it is shown that the Thurston-Bennequin invariant is given by

$$tb(L) = \sum_c \sigma(c)$$

Proof of Theorem B. The Thurston-Bennequin calculation follows directly from Lemma 3.2 by summing over all indices i, j, k, l. □

Remark 3.3. Notice that this formula is consistent with the result in [EES05b] that for L of even dimension, the Thurston-Bennequin number is $-\frac{1}{2}\chi(\nu)$, where ν is the oriented normal bundle to L in the Lagrangian projection. Suppose that the dimensions of K and L have the same parity. Then the dimension of their product is even and so this result applies. If n, m are odd, then the Euler characteristic of both manifolds vanish by Poincare duality, the Euler characteristic of their product vanishes, and $\tau_{i,j}$ is always negative. Thus, we get that:

$$tb(L_1 \times L_2) = -(0 + 0 + tb(L_1)tb(L_2) - tb(L_1)tb(L_2)) = 0 = \chi(L_1 \times L_2)$$

If n, m are even, then $\tau_{i,j}$ is always 1 and we can use the immersed version of the Lagrangian Neighborhood Theorem to identify T^*L_1 with ν_{L_1} and T^*L_2 with ν_{L_2} and obtain

$$\begin{aligned} tb(L_1 \times L_2) &= \left(-\frac{1}{2}\right)\chi(\nu_{L_1})\chi(\nu_{L_2}) + \chi(\nu_{L_1})\left(-\frac{1}{2}\right)\chi(\nu_{L_2}) \\ &+ \left(-\frac{1}{2}\right)\chi(\nu_{L_1})\left(-\frac{1}{2}\right)\chi(\nu_{L_1}) + \left(-\frac{1}{2}\right)\chi(\nu_{L_1})\left(-\frac{1}{2}\right)\chi(\nu_{L_2}) \\ &= -\frac{1}{2}\chi(\nu_{L_1})\chi(\nu_{L_2}) \\ &= -\frac{1}{2}\chi(\nu_{L_1 \times L_2}) \end{aligned}$$

since it is clear from the construction that the normal bundle of the product is the product of the normal bundles.

3.2 Maslov class

In order to calclate the Maslov class, we can make the necessary choices so that each condition splits. Specifically, take $\Lambda' \in \Lambda_n$ and $\Lambda'' \in \Lambda_m$ and associated Maslov cycles $\Sigma(\Lambda'), \Sigma(\Lambda'')$. Then $\Lambda = \Lambda' \oplus \Lambda''$ is Lagrangian in $\mathbb{R}^{2n+2m}$ and so defines a Maslov cycle $\Sigma(\Lambda)$ that splits as

$$\Sigma(\Lambda)_k = \sum_{i+j=k} \Sigma(\Lambda')_i \oplus \Sigma(\Lambda'')_j$$

Moreover, we can choose Lagrangian complements such that $W = W' \oplus W''$.

Take a path $\Gamma \in \Lambda_{n+m}$ and its projections $\Gamma' \in \Lambda_n, \Gamma'' \in \Lambda_m$. At each intersection point $\Gamma(t') \subset \Sigma(\Lambda')$, the signature of the associated quadratic form is the sum of the signatures of the associated quadratic forms for the intersections of $\Gamma'(t'), \Gamma''(t')$ and $\Sigma(\Lambda'), \Sigma(\Lambda'')$.

Thus, the Maslov index splits as $\mu(\Gamma) = \mu(\Gamma') + \mu(\Gamma'')$. Therefore, it follows that the Maslov class $\mu_{L_1 \times L_2}$ splits as well.

3.3 Proof of Theorem A

Choose some Darboux ball of radius 3ϵ and isotope L_1 so that its intersection with the Darboux ball is two disjoint disks given by two parallel Lagrangian planes of distance ϵ apart. By a Hamiltonian isotopy supported in the Darboux ball, we can add two canceling transverse double points corresponding to two Reeb chords b, a, labeled so that $\mathcal{Z}(a) > \mathcal{Z}(b)$. Now, by scaling either L_1 or L_2 and a perturbation, we can assume that there is exactly one Reeb chord e of L_2 such that $\mathcal{Z}(a) > \mathcal{Z}(e) > \mathcal{Z}(b)$. For all other Reeb chords e' of L_2, the terms $\tau(a, e')\sigma(a)\sigma(e')$ and $\tau(b, e')\sigma(b)\sigma(e')$ cancel. However, $\tau(a, e)\sigma(a)\sigma(e)$ and $\tau(b, e)\sigma(b)\sigma(e)$ have the same sign since the pairs $\tau(a, e), \tau(b, e)$ and $\sigma(a), \sigma(b)$ are distinct. Moreover, we can add arbitrarily many pairs of Reeb chord pairs $\{(a_i, b_i)\}$ so that $\mathcal{Z}(a_i) = \mathcal{Z}(a_j)$ and $\mathcal{Z}(b_i) = \mathcal{Z}(b_j)$ for all i, j. Thus, we can add $2n * (\tau(a, e)\sigma(a)\sigma(e))$ to the Thurston-Bennequin invariant for arbitrary nonnegative integer n.

Chapter 4
Generating Family Homology

One of the motivation of this thesis is the close relationship between generating families and holomorphic curve invariants of Legendrian knots in dimension 3.

Theorem 1.1 (Fuchs [Fuc03], Fuchs-Ishkhanov [FI04], Sabloff [Sab06], Chekanov–Pushkar [CP05]). *A Legendrian knot L admits a generating family if and only if the Chekanov-Eliashberg DGA of L admits an augmentation.*

Theorem 4.1 (Fuchs-Rutherford [FR11]). *If L admits a generating family F, there exists an augmentation ϵ of the Chekanov-Eliashberg DGA such that*

$$\mathcal{G}H_*(L, [F]) \simeq H_*^{\epsilon}(L)$$

It would be interesting to know whether similar results, connecting generating families and augmentations or generating family homology and linearized contact homology, hold in higher dimensions.

If L_1, L_2 admit generating families F_1, F_2, we will show that $L_1 \times L_2$ admits a generating familiy $F_1 \oplus F_2$ that is well-defined on equivalence classes of generating families (Lemma 4.3). When $L_1 \gg L_2$, the generating family homology of $L_1 \times L_2$ with respect to this generating family has a simple description. The main theorem we prove in this chapter is the following:

Theorem C. *For $L_1 \gg L_2$, the total and relative generating family homology over a field $\mathbb{F}$ satisfies a Kunneth-type formula*

$$\mathcal{G}H_*(L_1 \times L_2, [F_1 \oplus F_2]) \simeq \mathcal{G}H_*(L_2, [F_2]) \otimes_{\mathbb{F}} H_*(L_1; \mathbb{F})$$
$$\widetilde{\mathcal{G}H}_*(L_1 \times L_2, [F_1 \oplus F_2]) \simeq \widetilde{\mathcal{G}H}_*(L_2, [F_2]) \otimes_{\mathbb{F}} H_*(L_1; \mathbb{F})$$

Recall the frontspinning construction described in Chapter [?]. Given some $L \subset \mathbb{R}^{2n-1}$, there is a Legendrian $\Sigma L \subset \mathbb{R}^{2n+1}$ whose underlying topological type is $L \times S^1$. Let U denote the unique $tb = -1$ Legendrian knot in $\mathbb{R}^3$. For any Legendrian $L \subset \mathbb{R}^{2n+1}$, we can choose a representative of U such that $U \gg L$. In Chapter [?], we showed that in fact ΣL and $U \times L$ are Legendrian isotopic.

If the Legendrian contact homology DGA of L admits an augmentation ϵ, then the Legendrian contact homology DGA of ΣL admits an augmentation ϵ_Σ [EES05b]. In addition, the linearized contact homology with respect to this augmentation satisfies a Kunneth-like theorem.

Theorem 4.2 (Ekholm-Etnyre-Sabloff [EES09]).

$$H_*^{\epsilon_\Sigma}(\Sigma L) \simeq H_*^{\epsilon}(L) \otimes H_*(S^1)$$

Given the identification of ΣL and $U \times L$ in the formal similarity between Theorem C and Theorem 4.2 is some evidence that a generalization of Fuchs and Rutherford's result may hold in higher dimensions. In addition, Sabloff and Sullivan obtained similar results when L_1 is the 1-jet graph of a function on M_1 [SS].

Without the assumption that $L_1 \gg L_2$, the above theorem does not hold. Reeb chord actions determine the critical values of the difference function δ_F and thus modifying the geometry of Reeb chords may affect the relative homology of a pair of level sets of the difference.

Organization. In Section 4.1 we review background material on generating families for Legendrian submanifolds and generating family homology, as well as how generating families interact with the product construction. Subsequently, in Section 4.2, we complete the proof of Theorem C.

4.1 Background

Throughout this paper, if $f : M \to \mathbb{R}$ and $g : N \to \mathbb{R}$ are functions, we will use $f \oplus g$ to denote the function

$$f \oplus g(x, w) = f(x) + g(w)$$

for $x \in M$ and $w \in N$.

4.1.1 Generating Families

Let M be an n-dimensional manifold, $f : M \times \mathbb{R}^N \to \mathbb{R}$ a smooth function and (x, y) denote coordinates on $M \times \mathbb{R}^N$. The differential df of f is a section of the cotangent bundle $T^*(M \times \mathbb{R}^N) \simeq T^*M \times T^*\mathbb{R}^N$ and composing with projections we have two maps

$$\begin{aligned} df_x :& M \times \mathbb{R}^N \to T^*M \\ df_y :& M \times \mathbb{R}^N \to T^*\mathbb{R}^N \simeq \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \end{aligned}$$

where the final projection is onto the fibers of $T^*\mathbb{R}^N$.

If the origin $\mathbf{0}$ is a regular value of df_y then the preimage

$$\Sigma_f = df_y^{-1}(\mathbf{0})$$

is a smooth, n-dimensional submanifold of $M \times \mathbb{R}^N$ called the *fiber critical set*. There is a Lagrangian immersion $\partial_f : \Sigma_f \to T^*M$ defined by

$$\partial_f(x, y) = (x, df_x(x, y)) \quad \text{for } (x, y) \in \Sigma_f \subset M \times \mathbb{R}^N$$

and a Legendrian immersion $j^1_f : \Sigma_f \to J^1M$ defined by

$$j^1_f(x, y) = (x, df_x(x, y), f(x, y)) \quad \text{for } (x, y) \in \Sigma_f \subset M \times \mathbb{R}^N$$

Let Λ denote the image of ∂_f and L the image of j^1_f. The function f is a *generating family* of functions for Λ and L and we say that L, Λ admit the generating family f.

If a given Legendrian submanifold L admits a generating family, then in fact it admits infinitely many generating families since each generating family can by modified by one of the following two operations:

Stabilization. Let Q be a nondegenerate quadratic form on $\mathbb{R}^k$ and let $f \oplus Q : M \times \mathbb{R}^N \times \mathbb{R}^k \to \mathbb{R}$ denote the function defined by

$$f \oplus Q(x, y, y') = f(x, y) + Q(y')$$

Then $\mathbf{0}$ is still a regular value of $d(f \oplus Q)$ and the fiber critical set is

$$\Sigma_{f \oplus Q} = \Sigma_f \times 0 \subset \left(M \times \mathbb{R}^N\right) \times \mathbb{R}^k$$

Thus, $f \oplus Q$ is also a generating family for L.

Fiber-preserving diffeomorphism. Let $\Phi : M \times \mathbb{R}^N \to M \times \mathbb{R}^N$ be a fiber-preserving diffeomorphism, i.e there is a commutative triangle

$$\pi \circ \Phi = \pi$$

where $\pi : M \times \mathbb{R}^N \to M$ is the projection of the trivial bundle onto its base. Let $\Phi_x : \mathbb{R}^N \to \mathbb{R}^N$ be the restriction of Φ to the $\mathbb{R}^N$-fiber over x. Then by the chain rule and the fact that Φ preserves fibers we have

$$\begin{aligned} d(f \circ \Phi)_y(x, y) &= df_y(\Phi(x, y)) \circ d\Phi(x, y) \\ d(f \circ \Phi)_x(x, y) &= df_x(\Phi(x, y)) \circ d\Phi(x, y) \qquad &= df_x(x, \Phi_x(y)) \end{aligned}$$

Since each Φ is a diffeomorphism, $\mathbf{0}$ is still a regular value with fiber critical set

$$\Sigma_{f \circ \Phi} = \Phi^{-1}(\Sigma_f)$$

But since $\partial_{f \circ \Phi} = \partial_f \circ \Phi$ and $j^1_{f \circ \Phi} = j^1_f \circ \Phi$, the image of $j^1_{f \circ \Phi}$ is still L.

There is an equivalence relation on generating families obtained by declaring that $f_1 \sim f_2$ if f_1 and f_2 are related by some sequence of stabilizations and fiber-preserving diffeomorphisms. Let $[f]$ denote the equivalence class of a generating family f under this equivalence relation.

A function $f : M \times \mathbb{R}^n \to \mathbb{R}$ is *linear at infinity* if there exists a compactly supported function f_c and linear function A on $\mathbb{R}^n$ such that

$$f(x, y) = f^c(x, y) + A(y)$$

Note that a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ extends to a fiber-preserving diffeomorphism Φ_T on $M \times \mathbb{R}^n$ by applying T fiberwise. Moreover, for any linear function A there exists some T such that $A(T(y)) = y_1$. Thus, every linear-at-infinity generating family is equivalent to a linear-at-infinity generating function where $A(y) = y_1$.

4.1.2 Legendrian product

Suppose that L_1 and L_2 admit generating families $f_1 : M_1 \times \mathbb{R}^{n_1} \to \mathbb{R}$ and $f_2 : M_2 \times \mathbb{R}^{N_2} \to \mathbb{R}$. Define the function $f_1 \oplus f_2 : M_1 \times M_2 \times \mathbb{R}^{N_1+N_2} \to \mathbb{R}$ by

$$f_1 \oplus f_2(x, w, y, z) = f_1(x, y) + f_2(w, z)$$

Lemma 4.3. *The function $f_1 \oplus f_2 : M_1 \times M_2 \times \mathbb{R}^{n_1+n_2} \to \mathbb{R}$ is a generating family for $L_1 \times L_2$. If $f_1 \sim f_1'$ and $f_2 \sim f_2'$ then $f_1 \oplus f_2 \sim f_1' \oplus f_2'$. Finally, if f_1, f_2 are linear-at-infinity then $f_1 \oplus f_2$ is equivalent to a linear-at-infinity function.*

Proof. If $\mathbf{0}$ is a regular value of $d(f_1)_y$ and $d(f_2)_z$, then it is a regular value of $d(f_1 \oplus f_2)_{y,z}$ as well and the fiber critical set is

$$\begin{aligned}\Sigma_{f_1 \oplus f_2} &= d(f_1 \oplus f_2)_{y,z}^{-1}(\mathbf{0}) = d(f_1)_y^{-1}(\mathbf{0}) \times d(f_2)_z^{-1}(\mathbf{0}) \\ &= \Sigma_{f_1} \times \Sigma_{f_2}\end{aligned}$$

Thus, the image of $j^1_{f_1 \oplus f_2}$ is $L_1 \times L_2$.

Secondly, it is clear that stabilizations and fiber-preserving diffeomorphisms of the factors can be extended to the product.

Now assume f_1, f_2 are linear-at-infinity, so

$$\begin{aligned}f_1(x, y) &= f_1^c(x, y) + A(y) \\ f_2(w, z) &= f_2^c(w, z) + B(z)\end{aligned}$$

for some compactly supported functions f_1^c, f_2^c and linear functions $A(y) = \sum a_i y_1$ and $B(z) = \sum b_j z_j$. After applying a fiber-preserving diffeomorphism, we can assume that $A(y) = y_1$ and $B(z) = z_1$.

Let U, V be sets containing the supports of f_1^c, f_2^c, respectively and let Ψ be a smooth bump function on $M_1 \times M_1 \times \mathbb{R}^{n_1+n_2}$ that equals 0 on $U \times V$ and 1 on the complement of some compact neighborhood W of $U \times V$. Define a fiber-preserving diffeomorphism Φ

$$\Phi(x, w, y, z) := (x, w, y_1 - \Psi \cdot f_2^c, y_2, \ldots, y_{n_1}, z_1 - \Psi \cdot f_1^c, z_2, \ldots, z_{n_1})$$

Then

$$\begin{aligned}f_1 \oplus f_2 \circ \Phi &= f_1^c(x, y_1 - \Psi \cdot f_2^c(w, z), y_2, \ldots, y_{n_1}) + f_2^c(w, z_1 - \Psi \cdot f_1^c(x, y), z_2, \ldots, z_{n_2}) \\ &\quad + y_1 - \Psi \cdot f_2^c(w, z) + z_1 - \Psi \cdot f_1^c(x, y) \\ &= f_1^c(x, y_1 - \Psi \cdot f_2^c(w, z), y_2, \ldots, y_{n_1}) - \Psi \cdot f_1^c(x, y) \\ &\quad + f_2^c(w, z_1 - \Psi \cdot f_1^c(x, y), z_2, \ldots, z_{n_2}) - \Psi \cdot f_2^c(w, z) + y_1 + z_1 \\ &= g_1 + g_2 + C(y, z)\end{aligned}$$

where

$$\begin{aligned}g_1 &:= f_1^c(x, y_1 - \Psi \cdot f_2^c(w, z), y_2, \ldots, y_{n_1}) - \Psi \cdot f_1^c(x, y) \\ g_2 &:= f_2^c(w, z_1 - \Psi \cdot f_1^c(x, y), z_2, \ldots, z_{n_2}) - \Psi \cdot f_2^c(w, z) \\ C(y, z) &:= y_1 + z_1\end{aligned}$$

Since C is linear, it just remains to show that $g_1 + g_2$ is compactly supported. The support of g_1 lies in $U \times M_2 \times \mathbb{R}^{n_2}$ and since f_2^c is supported in V,

$$g_1 = f_1^c(x, y_1 - \Psi \cdot f_2^c(w, z), y_2, \dots, y_{n_1}) - \Psi \cdot f_1^c(x, y) = (1 - \Psi) f_1^c(x, y)$$

on $M_1 \times \mathbb{R}^{n_1} \times (M_2 \times \mathbb{R}^{n_2} \setminus V)$. But the support of $1 - \Psi$ lies in W so g_1 is in fact supported in W. A similar argument shows that g_2 is supported in W so the sum $g_1 + g_2$ is compactly supported. $\square$

4.1.3 Generating family homology

Define $\underline{l} = \min_c\{\mathcal{Z}(c)\}$ and $\bar{l} = \max_c\{\mathcal{Z}(c)\}$ to be the minimum and maximum actions of a Reeb chord of L. Furthermore, suppose that ϵ, ω are positive, real constants such that

$$\epsilon < \underline{l} \leq \bar{l} < \omega \tag{4.1}$$

Let $f : M \times \mathbb{R}^N \to \mathbb{R}$ be a generating family for L. The *difference function* $\delta_f : M \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ for f is defined by

$$\delta_f(x, y, y') = f(x, y') - f(x, y)$$

The critical points of this difference function encode information about the Reeb chords of the Legendrian L and the relative homology of level sets of the difference function δ_f can be used to define invariants of L.

Proposition 4.4 ([FR11, ST])**.** *The critical points $Crit(\delta_f)$ of δ_f come in two families*

- *Two critical points (x, y_1, y_2) and (x, y_2, y_1) with critical values $\pm\mathcal{Z}(c)$ for each Reeb chord c, where c has endpoints at $j^1_f(x, y_1)$ and $j^1(x, y_2)$.*
- *A critical submanifold $\{(x, y, y) | (x, y) \in \Sigma_f\}$ with critical value 0.*

The total and relative generating family homologies of $(L, [F])$ are the relative singular homologies of two sublevel sets of the difference function δ_F

Definition 4.5. Let $F : M \times \mathbb{R}^N \to \mathbb{R}$ be a generating family of functions for $L \subset J^*M$. The **total generating family homology** is defined to be

$$\widetilde{\mathcal{G}H}_*(L, F) := H_{*+N+1}(\delta^\omega, \delta^{-\epsilon}; \mathbb{F})$$

and the **relative generating family homology** is defined to be

$$\mathcal{G}H_*(L, F) := H_{*+N+1}(\delta^\omega, \delta^\epsilon; \mathbb{F})$$

The generating family homology does not depend on the choices made in the definition.

Theorem 4.6 (Sabloff-Traynor[ST]). *The total and relative generating family homology is well-defined up to the equivalence relation on generating families. In addition, if F is a linear-at-infinity generating family for L, then $\widetilde{\mathcal{G}H}_*(L,[F])$ and $\mathcal{G}H_*(L,[F])$ is independent of the choices of ϵ, ω satisfying Inequality 4.1.*

In addition, generating families persist through Legendrian isotopies [Che96a, JT06] and isotopies induce isomorphisms on the generating family homology.

Theorem 4.7 ([ST]). *If L is a compact Legendrian submanifold in $J^1(M)$, the set of generating family homologies is invariant under Legendrian isotopy.*

Thus, the set of generating family homologies gives an invariant of the Legendrian submanifold.

The following key lemma will be useful in proving Theorem C.

Lemma 4.8 ([ST]). *The pair $(\delta^{\omega}, \delta^{-\omega})$ is acyclic for all ω sufficiently large.*

4.2 Proof of Theorem C

In this section, we prove the main theorem, Theorem C. Generating family homology $\mathcal{G}H_*(L,[F])$ is the relative homology of sublevels sets of the difference function δ_F for F. In order to establish a Kunneth formula for generating family homology, we first give a criterion to relate the relative homologies of sublevel sets $(\delta^b_{F_1}, \delta^a_{F_1}), (\delta^d_{F_2}, \delta^c_{F_2})$ on the factors and the relative homology of sublevel sets $(\delta^{b+d}_{F_1 \oplus F_2}, \delta^{a+c}_{F_1 \oplus F_2})$ for the product. Using this fact, we apply the exact triangle in homology associated to a triple to complete the proof.

First, the following important lemma describes a criterion for when the relative homology of a particular pair of sublevel sets on the product can be computed from the relative homology of corresponding pairs of sublevel sets on the factors.

Lemma 4.9. *Let $f : M \times \mathbb{R}^{n_1} \to \mathbb{R}$ and $h : N \times \mathbb{R}^{n_2} \to \mathbb{R}$ be linear-at-infinity, smooth functions and let $a < b$ and $c < d$ be constants. Suppose that for all critical points $(x, y) \in Crit(f \oplus h)$ with critical value in the interval $(a + c, b + d)$ that $a < f(x) < b$ and $c < h(y) < d$. Then there is an isomorphism of relative homology of sublevel sets*

$$H_*((f \oplus h)^{b+d}, (f \oplus h)^{a+c}; \mathbb{F}) \simeq H_*(f^b, f^a; \mathbb{F}) \otimes_{\mathbb{F}} H_*(h^d, h^c; \mathbb{F})$$

Proof. As a preliminary observation, note that the critical points $\mathrm{Crit}(f \oplus h)$ of the function $f \oplus h$ are of the form $x \times y$, where $x \in \mathrm{Crit}(f)$ and $y \in \mathrm{Crit}(h)$, and the associated critical value is $f \oplus h(x, y) = f(x) + h(y)$.

Define two subsets of $M \times \mathbb{R}^{n_1} \times N \times \mathbb{R}^{n_2}$

$$A := \big((f \oplus h)^{a+d} \cap f^a \times N \times \mathbb{R}^{n_2}\big) \cup \big((f \oplus h)^{b+c} \cap M \times \mathbb{R}^{n_1} \times h^c\big)$$
$$B := f^{-1}([a, b]) \times h^{-1}([c, d])$$

Thus, by assumption, all the critical points of $f \oplus h$ with critical values between $a + c$ and $b + d$ lie in B.

The following claim and its proof are similar to a standard fact in Morse theory (Theorem 3.1 in [Mil63], Lemma 2.3 in [ST]) that if there are no critical points between a pair of level sets of a Morse function, then the corresponding sublevel sets are homotopy equivalent.

Claim. A deformation retracts onto $(f \oplus h)^{a+c}$ and $(f \oplus h)^{b+d}$ deformation retracts onto $A \cup B$.

By the Claim, we have that $H_*((f \oplus h)^{b+d}, (f \oplus h)^{a+c}; \mathbb{F})$ is isomorphic to the relative homology $H_*(A \cup B, A; \mathbb{F})$. Furthermore, by excision

$$H_*(A \cup B, A; \mathbb{F}) \simeq H_*(f^b \times h^d, f^b \times h^c \cup f^a \times h^d; \mathbb{F})$$

and the lemma follows by the Kunneth formula for relative homology over a field.

To prove the Claim, we construct the appropriate deformation retractions using the negative gradient flow of $\nabla(f \oplus h)$.

We can assume that $f = f^c + y_1$ and $h = h^c + z_1$, where f^c, h^c are compactly supported and $\mathbf{y} = (y_1, \ldots, y_{n_1})$ and $\mathbf{z} = (z_1, \ldots, z_{n_2})$ are coordinates on $\mathbb{R}^{n_1}, \mathbb{R}^{n_2}$, respectively. Choose metrics g_1, g_2 on $M \times \mathbb{R}^{n_1}, N \times \mathbb{R}^{n_2}$ that restrict to the standard Euclidean metric on the affine factors. Thus, $|\nabla f|$ and $|\nabla h|$ equal 1 outside of compact sets.

Let $\nabla(f \oplus h) = (\nabla f, \nabla h)$ be the gradient of $(f \oplus h)$ with respect to the split metric $g = g_1 \oplus g_2$. Then $|\nabla(f \oplus h)| \geq 1$ outside of a compact set.

Thus, since all critical points in $(f \oplus h)^{-1}([a+c, b+d])$ lie in B, the vector field $\nabla(f \oplus h)$ is bounded away from $\mathbf{0}$ on $(f \oplus h)^{-1}([a+c, b+d]) \setminus B$.

Clearly, each point in $(f \oplus h)^{-1}(a+c)$ lies on a unique gradient trajectory. In addition, each point in ∂A lies on a unique gradient trajectory. If $(x, y), (x', y')$ are distinct points on the same trajectory γ, with $f \oplus h(x, y) < f \oplus h(x', y')$ then

$$f(x) < f(x') \quad \text{and} \quad h(y) < h(y')$$

But there are no pairs of points $(x, y), (x', y')$ in ∂A that satisfy these inequalities.

Since $\nabla(f \oplus h)$ is bounded away from $\mathbf{0}$, we can normalize the vector field so that for each gradient trajectory γ through $A \setminus (f \oplus h)^{a+c}$, then $\gamma(0) \in \partial A$ and $\gamma(1) \in (f \oplus h)^{-1}(a+c)$. Moreover, each point in $(x, y) \in A \setminus (f \oplus h)^{a+c}$ lies on some such trajectory $\gamma_{x,y}$.

Thus, we can define a deformation retract $H : A \to (f \oplus h)^{a+c}$

$$H(x, y) := \begin{cases} (x, y) & \text{if } f \oplus h(x, y) \leq a + c \\ \\ \gamma_{x,y}(1) & \text{if } (x, y) \in A \setminus (f \oplus h)^{a+c} \end{cases}$$

A similar argument shows that $(f \oplus h)^{b+d}$ deformation retracts onto $A \cup B$. $\square$

We can now prove the main theorem.

Proof of Theorem C. For the sake of notation, let δ denote the difference function $\delta_{F_1 \oplus F_2}$.

Since $L_1 \gg L_2$, we can choose constants ϵ_1, ω_1 and ϵ_2, ω_2 to simultaneously satisfy Inequality 4.1 and

$$\epsilon_2 < \omega_2 \ll \epsilon_1 < \omega_1 \tag{4.2}$$

and so that $(\delta_{F_2}^{\omega_2}, \delta_{F_2}^{-\omega_2})$ is acylic as in Lemma 4.8. Furthermore, choose some $0 < \epsilon_3 \ll \epsilon_2$.

Recall that $\mathrm{Crit}(\delta) = \mathrm{Crit}(\delta_{F_1}) \times \mathrm{Crit}(\delta_{F_2})$. The constant ω_2 was chosen so that $-\omega_2 < \delta_{F_2}(y) < \omega_2$ for all $y \in \mathrm{Crit}(\delta_{F_2})$. In addition, $\omega_2 \ll \epsilon_1$ so we can apply Lemma 4.9 to obtain

$$H_*(\delta^{\omega_1+\omega_2}, \delta^{\epsilon_1-\omega_2}; \mathbb{F}) \simeq H_*(\delta_{F_1}^{\omega_1}, \delta_{F_1}^{\epsilon_1}; \mathbb{F}) \otimes H_*(\delta_{F_2}^{\omega_2}, \delta_{F_2}^{-\omega_2}; \mathbb{F}) \simeq 0$$

since $(\delta_{F_2}^{\omega_2}, \delta_{F_2}^{-\omega_2})$ is acylic. Similarly, we can again apply Lemma 4.9 to obtain

$$H_*(\delta^{\epsilon_1-\omega_2}, \delta^{\epsilon_2}; \mathbb{F}) \simeq H_*(\delta_{F_1}^{\epsilon_1-2\omega_2}, \delta_{F_1}^{-\epsilon_3}; \mathbb{F}) \otimes H_*(\delta_{F_2}^{\omega_2}, \delta_{F_2}^{\epsilon_2+\epsilon_3}; \mathbb{F})$$

According to Proposition 4.4, the only portion of the critical locus of δ_{F_1} between $\epsilon_1 - 2\omega_2$ and $-\epsilon_3$ is a critical submanifold diffeomorphic to L_1, and Theorem 4.6 states that the relative generating family homology is invariant under small perturbations of ϵ_2. Thus

$$H_*(\delta^{\epsilon_1-\omega_2}, \delta^{\epsilon_2}; \mathbb{F}) \simeq H_*(L_1; \mathbb{F}) \otimes \mathcal{G}H_*(L_2; [f_2])$$

The triple $(\delta^{\omega_1+\omega_2}, \delta^{\epsilon_1-\omega_2}, \delta^{\epsilon_2})$ induces an exact triangle

$$\begin{tikzcd}
H_*(\delta^{\epsilon_1-\omega_2}, \delta^{\epsilon_2}) \arrow[rr, "i_*"] & & H_*(\delta^{\omega_1+\omega_2}, \delta^{\epsilon_2}) \arrow[dl, "j_*"] \\
& H_*(\delta^{\omega_1+\omega_2}, \delta^{\epsilon_1-\omega_2}) \arrow[ul, "{[-1]}" near end, "d_*"'] &
\end{tikzcd}$$

However, the third term vanishes and so the map i_* is an isomorphism, which proves the theorem for relative generating family homology.

A similar argument can be applied to the triple $(\delta^{\omega_1+\omega_2}, \delta^{\epsilon_1-\omega_2}, \delta^{-\epsilon_2})$ to prove the theorem for total generating family homology. □

Remark 4.10. Theorem C can be improved to apply to homology over non-field coefficients. In particular, the fact that i_* is an isomorphism still holds over any coefficients. The only modification is the appropriate modification of the Kunneth theorem in the second application.

Chapter 5
Legendrian contact homology for the product of two Legendrian knots

The goal of this chapter is to describe a criterion for computing the Legendrian contact homology of a product of two Legendrian knots.

Recall from Chapter 2 that the LCH of a Legendrian L is a dg-algebra $(\mathcal{A}, \partial)$, where A is an R-algebra generated by $H_1(L)$ and formal variables $q_1, \ldots, q_n$ corresponding to the Reeb chords of L, and the differential counts rigid, pseudoholomorphic curves.

Let $\mathbf{s} = (s_1, \ldots, s_k)$ be a k-tuple of Reeb chords on a perturbation $\widetilde{L}_\epsilon$ of $L_1 \times L_2$. By Lemma 5.18, this determines two k-tuples $\mathbf{s}^1, \mathbf{s}^2$ of Reeb chords and Morse critical points on L_1 and L_2, respectively. For $\mathbf{s}_i = (s^i_1, \ldots, s^i_k)$, let $\widetilde{\mathcal{M}}^\nabla_{L_i}(\mathbf{s}^i)$ be the moduli of generalized flow trees on L_i with punctures at $s^i_1, \ldots, s^i_k$. Let $\widetilde{\mathcal{M}}^\nabla_{L_1}(\mathbf{s}^1) \times_{\mathcal{T}_m} \widetilde{\mathcal{M}}^\nabla_{L_2}(\mathbf{s}^2)$ be the moduli space of compatible pairs (Γ_1, Γ_2), where $\Gamma_i \in \widetilde{\mathcal{M}}^\nabla_{L_i}(\mathbf{s}^i)$, as defined in Section 5.6.

The main theorem of this chapter is the following:

Theorem D. *For L_1, L_2 Legendrian knots in $\mathbb{R}^3$, there exists a perturbation $\widetilde{L}_\epsilon$ of $L_1 \times L_2$ such that the moduli of rigid gradient flow trees on $\widehat{L}_\epsilon$ is the fiber product*

$$\mathcal{M}^\nabla_{\widetilde{L}_\epsilon}(\mathbf{s}) \simeq \widetilde{\mathcal{M}}^\nabla_{L_1}(\mathbf{s}^1) \times_{\mathcal{T}_m} \widetilde{\mathcal{M}}^\nabla_{L_2}(\mathbf{s}^2)$$

where $\mathcal{T}$ is the moduli of metric trees.

Since Ekholm has established a 1-1 correspondence between rigid holomorphic disks and rigid gradient flow trees, this is sufficient to compute the Legendrian contact homology of $L_1 \times L_2$.

Organization. In Section 5.1, we describe background on metric trees and gradient flow trees on Legendrian submanifolds. In Section 5.2, we specifically discuss gradient flow trees for Legendrian knots, the correspondence with holomorphic disks, and generalized flow trees. In Section 5.3, we outline a strategy to compute gradient flow trees when L is a Legendrian surface by decomposing rigid trees into minimal partial trees. In Sections 5.5 to 5.6, we construct an explicit perturbation of $L_1 \times L_2$ that allows us to compute the gradient flow trees, then describe the miminal partial trees and local descriptions of gradient flow trees on the perturbed Legendrian. Finally, in Section 5.7, we compute the proof of Theorem D.

5.1 Gradient Flow Trees

In this section, we review background on gradient flow trees and prove some basic results.

First, we review some basic notation for Legendrian submanifolds in 1-jet spaces.

Let $L \subset J^1(M)$ be a Legendrian submanifold. The *front projection* is the map $\Pi_F : J^1(M) \to J^0(M)$, the *base projection* is the map $\Pi_B : J^1(M) \to M$ and the *Lagrangian projection* is the map $\Pi_P : J^1(M) \to T^*M$. The cotangent bundle T^*M has a natural, exact symplectic form $\omega = d\lambda$, where the Liouville form λ is locally defined by

$$\lambda = \sum_i p_i dq_i$$

where p are coordinates on M and q are the conjugate coordinates to p in the fibers of T^*M. The 1-jet space $J^1(M)$ of any smooth manifold has a natural contact structure, the hyperplane field ξ given as the kernel of the 1-form $dz - \lambda$. The 1-jet graph Γ_f in $J^1(M)$ of any smooth function on M is Legendrian since $\frac{\partial \Gamma_f}{\partial p_i} = 1, \frac{\partial \Gamma_f}{\partial z} = \frac{\partial f}{\partial p_i}$ and $\Gamma_f(q_i) = \frac{\partial f}{\partial p_i}$. However, in general, Legendrian submanifolds of $J^1(M)$ are only locally the graphs of some function.

Cusps

The *caustic* or *cusp set* Σ of L is the subset of L where the base projection Π_B fails to be an immersion. When dim L ≤ 2, there are two generic local models for a point Σ. Let x_1, x_2, y_1, y_2, z be coordinates on $J^1(M)$ and u_1, u_2 coordinates on L. A standard *cusp edge*

$$x_1 = \frac{1}{2}u_1^2 \qquad x_2 = u_2$$
$$z = \frac{1}{3}u_1^3 + \frac{\beta}{2}u_1^2 + \alpha u_2$$

for some constants α, β, and a *swallow-tail*

$$x_1 = u_2 \qquad\qquad y_1 = \alpha + \frac{1}{2}u_1^2 + \beta u_1$$
$$x_2 = \frac{1}{3}u_1^3 + \frac{1}{2}u_1 u_2 \qquad\qquad y_2 = u_1 + \beta$$
$$z = \frac{1}{4}u_1^4 + \frac{1}{2}u_1^2 u_2 + \frac{\beta}{3}u_1^3 + \beta u_1 u_2 + \alpha u_2$$

for some constants α, β.

Let Σ_s denote the set of swallow-tail singularities of L.

A Legendrian $L \subset J^1(M)$ is *front generic* if

- Σ is a codimension 1 subset of L and each point in $\Sigma \setminus \Sigma_s$ is a standard cusp edge.
- Σ_s is a compact, codimension 2 subset of L
- $\Pi_B : \Sigma \setminus \Sigma_s \to M$ is a self-transverse immersion with $\Pi_B(\Sigma \setminus \Sigma_s)$ and $\Pi_B(\Sigma_s)$ disjoint

Points $x \in M$ can be partitioned according to the number of cusp-points in $\Pi_B^{-1}(x)$. If L is front-generic, then there are either 0,1 or 2 such points and if there is 1 point, it can either be a standard cusp edge or a swallow-tail singularity. Let $c_x = \Pi_B^{-1}(x) \cap \Sigma$ and $s_x = \Pi_B^{-1} \cap \Sigma_s$ and set $n_x = |\Pi_B^{-1}(x)|$.

Case 1. $c_x = s_x = 0$. Then there exists a neighborhood U of x in M, disjoint open sets $V_1, \ldots, V_{n_x}$ in L, and functions $f_1, \ldots, f_{n_x} : U \to \mathbb{R}$ such that $V_i = \Gamma_{f_i}$ is the 1-jet graph of f_i. In particular, each open set V_i is a neighborhood in L of a unique point in the fiber $\Pi_B^{-1}(x)$.

Case 2. $c_x = 1$ and $s_x = 0$. Then there exists a neighborhood U of x with open sets $V_2, \ldots, V_{n_x}$ disjoint from Σ and functions $f_2, \ldots, f_{n_x}$ as above. Let $y \in \Sigma \cap \Pi_B^{-1}(x)$ be the unique cusp-point above x. Then for a sufficiently small neighborhood V of y, the set $V \setminus \Sigma$ has two disjoint connected components V^+, V^- and functions $f^\pm : W^\pm := \Pi_B(V^\pm) \to \mathbb{R}$ such that $V^\pm = \Gamma_{f^\pm}$ is the 1-jet graph. Extend $f^\pm$ to the closures $\overline{W}^\pm$. Then $f^+ = f^-$ on $\partial \overline{W}^+ \cap \Pi_B(\Sigma) = \partial \overline{W}^- \cap \Pi_B(\Sigma)$ and the limit of df^+ and df^- as x approaches $\Pi_B(\Sigma)$ from the interior of $W = W^+ \cap W^-$.

Case 3. $c_x = 2$ and $s_x = 0$. There exists an open U in M and open sets $V_3, \ldots, V_{n_x}$ and functions $f_3, \ldots, f_{n_x}$ as above. In addition there exist open sets V_1, V_2 of the two cusp points $y_1, y_2 \in \Sigma \cap \Pi_B^{-1}(x)$, with $V_1^\pm, V_2^\pm$ and functions $f_1^\pm, f_2^\pm$ as above.

Case 4. $c_x = 1$ and $s_x = 1$. There there exists a neighborhood U of x in M with open sets $V_2, \ldots, V_{n_x}$ and functions $f_2, \ldots, f_{n_x}$ as above. In coordinates near x as in the local model of the swallow-tail singularity above, L is the graph of a unique function f over the set $W^1 = \{(x_1, x_2)|\}$ and 3 functions f^1, f^2, f^3 over $W^3 = \{?\}$.

In all cases, we refer to the functions *local functions for L over U, W*.

Take some f_i and f^+ as in cases 2,3 or 4. Then $f_i - f^+$ is defined on $U \cap W^+$ and we can choose a chart on M that identifies x with 0 in $\mathbb{R}^2$, $\Pi_B(\Sigma)$ with $\mathbb{R} \times 0$ and W_+ with the upper-half plane $\mathbb{R} \times \mathbb{R}_{\geq 0}$. In these coordinates, the gradient vector field of the local difference function $f_i - f^+$ along $\Pi_B(\Sigma)$ splits into components

$$\nabla(f_j - f^+) = a_1(x_1)\partial_{x_1} + a_2(x_1)\partial_{x_2}$$

for some functions a_1, a_2 on $\Pi_B(\Sigma)$. The tangency locus $T(f_i, f^+)$ for the pair (f_i, f^+) is the 0-set of a_2. Note that since $\nabla f^+ = \nabla f^-$ the tangency locus is independent of the choice of f^+ or f^-. Generically, this a transversely cut-out submanifold consisting of a collection of points, is disjoint from swallow-tail points and double points of $\Pi_B(\Sigma)$ and $T(f_i, f^+)$ is disjoint from $T(f_i, f^+)$ if $i \neq j$.

The *tangency index* $\tau(v)$ of a vertex v is 1 if $\phi(v)$ lies in the tangency locus for some pair of functions f_i, f^+ and 0 otherwise.

Flow lines

Fix a metric g on M and let ∇ be the g-gradient.

Let $\gamma : (a, b) \to M$ be a path. Suppose there exists an open cover $\{U_\alpha\}$ of the image of γ such that over each U_α there are functions f_1^α, f_2^α are local functions for L over U_α with $f_i^\alpha = f_i^\beta$ on $U_\alpha \cap U_\beta$ for $i = 1, 2$. By abuse of notation in the following discussion, we refer to functions f_1, f_2 over all of γ, even though if γ is merely immersed the analytic continuation of f_1 along γ may become f_2, for example. (This is justified because the Legendrians we consider in this paper satisfy the following *global function property*: There exists a neighborhood $\nu(\Sigma)$ of the cusp set Σ such that Π_B, restricted to a single connected component of $L \setminus \nu(\Sigma)$ is injective.

The path γ is an *flow line* for L if there exists functions f_1, f_2 over γ such that the path satisfies the negative gradient flow equation

$$\gamma'(t) = -\nabla(f_1 - f_2)(\gamma(t)) \qquad \text{for all } t \in (a,b)$$

A flow line determines a pair of paths $\widetilde{\gamma}_1 = f_1(\gamma(t)) : (a,b) \to L$ and $\widetilde{\gamma}_2 = f_2(\gamma(t)) : (b,a) \to L$ in L called *1-jet lifts*. Note that the lifts $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ are oriented in opposite directions in the sense that

$$d\Pi_B(\widetilde{\gamma}_1') = -d\Pi_B(\widetilde{\gamma}_2') = -\nabla(f_1 - f_2)$$

The *cotangent lifts* $\overline{\gamma}_1, \overline{\gamma}_2$ of the flow line γ are the images of the 1-jet lifts in the Lagrangian projection $\overline{L}$ of L.

Lemma 5.1 ([Ekh07]). *Each flow line $\gamma(t)$ can be extended to a maximal domain of definition of the form $[a,b], [a,\infty), (-\infty,b]$ or $(-\infty,\infty)$. If γ has a compact end at $l = a, b$ then the limit of at least one of the 1-jet lifts at the compact end lies in the cusp set Σ. That is, for some $i = 1, 2$*

$$\lim_{t \to l} \widetilde{\gamma}_i(t) \in \Sigma$$

If γ has a noncompact end at $\pm\infty$, then the limit of γ at this end is a critical point of $f_1 - f_2$

$$\lim_{t \to \pm\infty} \gamma(t) \in \mathit{Crit}(f_1 - f_2)$$

Metric Trees

A *ribbon tree* Γ is a finite, connected, acyclic graph with a fixed cyclic ordering of the edges incident to each vertex. Contrary to some conventions, we do not require that Γ have no vertices of valence 2. An *external vertex* of Γ is a valence-1 vertex and an *external edge* is an edge incident to an external vertex. All other vertices and edges are *internal*. Let $V_E(\Gamma)$ and $V_I(\Gamma)$ be the sets of external and internal vertices and $E_E(\Gamma)$ and $E_I(\Gamma)$ the sets of external and internal edges.

Let $T_{j,k}$ denote the set of equivalence classes of ribbon trees with k external vertices, j valence-2 vertices and a distinguish external vertex v_1. Two trees Γ, Γ' are equivalent if there is a ribbon graph isomorphism $i : \Gamma \to \Gamma'$ that sends the distinguish vertex v_1 of Γ to the distinguished vertex v_1' of Γ'.

Let $V_2(\Gamma)$ be the valence-2 vertices. For each $v \in V_2$, contracting the vertex induces a projection $\pi_v : T_{j,k} \to T_{j-1,k}$.

Let $C \subset \{1, \ldots, k\}$ be a subset and $P = C^c$ its complement in $\{1, \ldots, k\}$. A choice of C gives a partition of the set of external vertices $V_{Ext} = V_C \cup V_P$ and of the set of external edges $E_{Ext} = E_C \cup E_P$. Let $T_{j,k,C}$ denote pairs (Γ, C) of a ribbon tree $\Gamma \in T_{j,k}$ and a subset C.

A *metric* on a ribbon graph $\Gamma \in T_{j,k,C}$ is a function $m : E_I(\Gamma) \cup E_C(\Gamma) \to \mathbb{R}_{\geq 0}$ that assigns a finite, nonnegative length to each internal edge and the external edges indexed by C. The remaining external edges are considered to have infinite length. A *metric tree* is a ribbon tree in $T_{j,k,C}$ equipped with a metric m.

For a given pair (Γ, C), let $\mathcal{T}(\Gamma, C)$ be the set of of metric trees whose underlying ribbon tree is Γ. It is clear that this space is diffeomorphic to $\mathbb{R}_{\geq 0}^{|E_{Int} \cup E_C|}$.

Let $\mathcal{T}_{j,k,C} = \cup_{j' \leq j} \cup_{\Gamma \in T_{j',k}} \mathcal{T}(\Gamma, C)$ be the union of all metric trees with k external vertices and at most j valence-2 vertices.

Given the triple (Γ, C, m) we choose specific parametrizations of each edge of $\Gamma \setminus V_P$ as follows. First, fix an orientation on each edge in (Γ, C).

- If $k = 2$ and $j = |C| = 0$, then $E_P = \{e\}$ and identify the unique edge with the real line
$$e \simeq (-\infty, \infty)$$
- Otherwise, for each $e \in E_P$ identify e in an orientation-preserving manner with the half-open interval
$$e \simeq [0, \infty) \qquad \text{or} \qquad e \simeq (-\infty, 0]$$
- For $e \in E_I \cup E_C$, identify e via an orientation-preserving isometry with the closed interal
$$e \simeq [0, m(e)]$$

5.1.1 Flow Trees

Let $(\Gamma, C) \in \mathcal{T}_{j,k,C}$ be a metric tree and $\phi : \Gamma \to M$ be a continuous map. Fix an arbitrary orientation on each edge of Γ, identify each edge e with an interval I_e as described above and let $\phi^e : I_e \to M$ be the restriction of ϕ to this parametrized edge.

Definition 5.2. A (closed) *gradient flow tree* (Γ, ϕ) for L consists of a metric tree Γ in $\mathcal{T}_{j,k,C}$ and a continuous map $\phi : \Gamma \to M$ such that:

1. (*Flow lines*) For each oriented edge e, there are distinct local functions f_1^e, f_2^e for L such that ϕ^e is a flow line of the difference function $f_1^e - f_2^e$.

2. (*Nonconstant*) The flow line ϕ^e is nonconstant if $e \in E_I$.

3. (*Punctures*) For each $e \in E_P$ and corresponding $v \in V_P$,
$$\lim_{t \to \pm\infty} \phi^e(t) \in \mathrm{Crit}(f_1^e - f_2^e)$$

4. (*Cusp ends*) For $e \in E_C$ and corresponding $v \in V_C$,
$$\lim_{t \to v} \widetilde{\phi}_1^e(t) = \lim_{t \to v} \widetilde{\phi}_2^e(t) \in \Sigma$$

5. (*1-jet lifts*) For each $v \in E_I \cup E_C$ with incident edges $e_1, \ldots, e_l$ is cyclic order then
$$\widetilde{\phi}^{e_i}_{\leftarrow}(v) = \widetilde{\phi}^{e_{i+1}}_{\rightarrow}(v)$$
for all $i = 1, \ldots, l$ (setting $l+1 = 1$), where $\widetilde{\phi}^{e}_{\leftarrow}$ is the 1-jet lift of ϕ^e oriented towards v and $\widetilde{\phi}^{e}_{\rightarrow}$ the 1-jet lift oriented away from v.

6. (*2-valent vertices*) For 2-valent vertices v
$$\widetilde{\phi}^{e_1}_{\leftarrow}(v) = \widetilde{\phi}^{e_2}_{\rightarrow}(v) \in \Sigma$$
for at least one ordering e_1, e_2 of the edges incident to v.

7. (*Cotangent lifts*). The union of the cotangent lifts $\bigcup_{e \in E(\Gamma)}(\overline{\phi}^e_1 \cup \overline{\phi}^e_2)$ is a closed, orieted curve in the Lagrangian projection $\overline{L}$.

A *open (or partial) flow tree* (Γ, ϕ) only satisfies conditions (1),(2),(3),(5),(6),(7). A vertex $v \in E_C$ which fails criterion (4) is a *special puncture*.

The vertices E_P are the *punctures* of the gradient flow tree.

Remark 5.3. Note that switching the orientation on an edge in Γ does not affect whether (Γ, ϕ) is a gradient flow tree. Switching the orientation on e interchanges f^e_1 and f^e_2 in (1),(2),(3),(4) and since the orientations on the 1-jet lifts do not depend on the orientation of e, criteria (5),(6),(7) are unaffected.

Remark 5.4. In contrast to [Ekh07], we require that all punctures occur at the external vertices of the tree Γ. Such a condition is necessary to appropriately describe and prove Theorem D. However, this yields equivalent notions of gradient flow trees.

Let v be a nonspecial puncture of a flow tree (Γ, ϕ), e the edge incident to v, u the other vertex on e and $\{e_1, \ldots, e_k\}$ the other edges incident to u. Then v is *1-valent* if ϕ^e paramterizes a nonconstant flow line of $f^e_1 - f^e_2$ and is *k-valent* if ϕ^e parametrizes a constant flow line of $f^e_1 - f^3_2$.

Let Γ be a flow tree, possibly with special punctures, and x a point in Γ. If x is not a vertex, then $\Gamma \setminus x$ consists of two ribbon trees Γ_1, Γ_2. If x is a k-valent vertex, then $\Gamma \setminus x$ consists of k ribbon trees $\Gamma_1, \ldots, \Gamma_k$. In both cases, the flow tree structure on Γ induces a partial flow tree structure on each Γ_i, with a new special puncture at x. The process is called *cutting* the tree Γ at x.

A vertex v is *splittable* if there exist nonadjacent edges e_i, e_j such that $\widetilde{\phi}^{e_i}_{\leftarrow}(v) = \widetilde{\phi}^{e_j}_{\rightarrow}(v)$. In this case, Γ is obtained from two trees Γ_1, Γ_2 be identifying $v_1 \sim v_2$ two vertices $v_1 \in \Gamma_1$ and $v_2 \in \Gamma_2$, where the edges incident to v_1 are $e_j, \ldots, e_i$ and the edges incident to v_2 are $e_{i+1}, \ldots, e_{j-1}$. After possibly contracting the vertices v_1 or v_2, the pairs $(\Gamma_1, \phi|_{\Gamma_1})$ and $(\Gamma_2, \phi|_{\Gamma_2})$ are flow trees for L.

An example of a splitable vertex is a *multiply covered* vertex. A vertex is multiply-covered if it is kp-valent, with $k \geq 2$, and for all $i = 1, \ldots, kp$ the local functions satisfy $f^{e_i}_j = f^{e_{i+k}}_j$ for $j = 1, 2$ and $(\phi^{e_i})' = (\phi^{e_{i+k}})'$ near v. In other words, $e_i, e_{i+k}, e_{i+2k}, \ldots, e_{i+(p-1)k}$ parametrize the same flow line for all $i = 1, \ldots, p$.

All punctures, special or nonspecial, have signs defined as follows. Let e be the external edge incident to the puncture, oriented away from v, with ϕ_e a flow line of $f_1^e - f_2^e$. The puncture is *positive* if $f_1^e < f_2^e$ and is *negative* if $f_2^e > f_1^e$ in a neighborhood of $\phi_e(v)$ in M.

Labeling and orientation of flow trees

We will use the following conventions to label and orient diagrams of flow trees. Suppose that $\phi(\Gamma) \subset U$ and $\{f_1, \ldots, f_n\}$ are local functions for L defined over U. An oriented edge e of Γ is labeled by the ordered pair (i, j) if $f_1^e = f_i$ and $f_2^e = f_j$.

For example, given three local functions f_i, f_j, f_k, Figure 5.1 describes the four possible ways (up to reindexing the three functions f_i, f_j, f_k) of labeling and orienting edges of a flow tree incident to a 3-valent vertex to satisfy criterion (5) of Definition 5.2.

5.1.2 Dimension formulae

Each puncture v of Γ has an *index* $I(v)$ defined as follows. Let e be the edge incident to v; if the puncture at v is positive, orient e away from v, and if the puncture at v is negative, orient e towards v. This ensures that $f_1^e > f_2^e$ near v. Then if v is a nonspecial puncture, $I(v)$ is the Morse index of $f_1^e - f_2^e$ at v. If v is a special puncture, the $I(v) = n + 1$ if p is positive and $I(v) = -1$ if p is negative.

Let v be a nonpuncture of Γ of valence k. Then by criterion (5) in Definition 5.2, for each edge e incident to v there is a pair of 1-jet lifts $(\widetilde{\phi}^{e_i}_{\leftarrow}, \widetilde{\phi}^{e_{i+1}}_{\rightarrow})$ whose union forms a path through some $y \in \Pi_B^{-1}(\phi(v))$. Let $f_\leftarrow, f_\rightarrow$ be the local functions defining the sheets containing these two points.

If $y \in \Sigma$, then $\mu(e, v) = 1$ if $f_\rightarrow > f_\leftarrow$ near $\Pi_B(\Sigma)$ and $\mu(e, v) = -1$ if $f_\rightarrow > f_\leftarrow$ near $\Pi_B(\Sigma)$.

If dim $L = 1$, define $\operatorname{sgn}(y) = -1$ if y is a left-cusp and $\operatorname{sgn}(y) = 1$ if y is a right-cusp. If dim $L = 2$ define $\operatorname{sgn}(y) = 1$.

If $y \notin \Sigma$, define $\mu(e, v) = 0$.

The *Maslov content* $\mu(v)$ is the sum

$$\mu(v) = \sum_e \operatorname{sgn}(y)\mu(e, v)$$

From now on, we suppress the map ϕ and use only Γ to denote a flow tree.

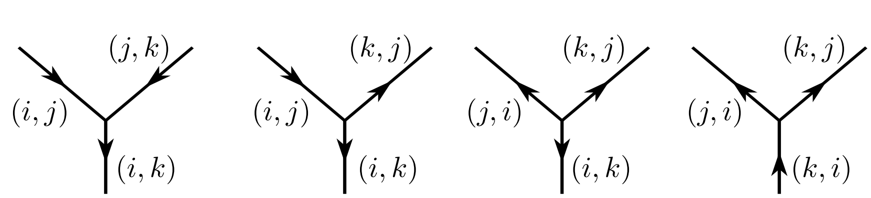

FIGURE 5.1: Allowable orientations at 3-valent vertices

Let p denote a positive puncture, q a negative puncture, and r a non-puncture of Γ and $P(\Gamma), N(\Gamma), R(\Gamma)$ the sets of positive, negative and non-punctures of Γ.

Definition 5.5. The *formal dimension* of a flow tree Γ is

$$\dim(\Gamma) = (n-3) + \sum_{p \in P(\Gamma)} (I(p) - (n-1)) - \sum_{q \in Q(\Gamma)} (I(q) - 1) + \sum_{r \in R(\Gamma)} \mu(r) \quad (5.1)$$

Also define the *partial flow tree dimension*

$$\operatorname{pdim}(\Gamma) = \dim(\Gamma) - \#\text{special punctures}$$

A Legendrian $L \subset J^1(M)$ satisfies the *preliminary transversality condition* if all partial flow trees have pdim ≥ 0.

Let $P_k(\Gamma), Q_k(\Gamma), R_k(\Gamma)$ denote the sets of k-valent positive, negative and non-punctures of Γ. Let $i(\Gamma) = |E_I|$ be the number of internal edges of Γ.

Definition 5.6. The *geometric dimension* of Γ is

$$\begin{aligned} \operatorname{gdim} \Gamma := i(\Gamma)(n+1) &+ \sum_{p \in P_1(\Gamma)} I(p) + \sum_{q \in Q_k(\Gamma)} (n - I(q)) \\ &+ \sum_{r \in R_1(\Gamma)} (n + 1 - \sigma(r)) - \sum_{k \geq 2} k(|P_k(\Gamma)| + |Q_k(\Gamma)|) \\ &- \sum_{k \geq 2} \sum_{r \in R_k(\Gamma)} (n(k-1) + \sigma(r) + \tau(r)) \end{aligned}$$

The *geometric codimension* of Γ is

$$\operatorname{gcodim} \Gamma := \dim \Gamma - \operatorname{gdim} \Gamma$$

Apart from k-valent punctures, there are several important types of vertices in a gradient flow tree.

Definition 5.7. An end, switch, Y_0 or Y_1-vertex is defined to be:

- *End.* A vertex $r \in E_C(\Gamma)$ that satisfies criterion (4) in Definition 5.2. In particular, $\mu(r) = 1$
- *Switch.* A 2-valent vertex with $\phi(v)$ in the tangency locus $T(f_i, f^+)$ for some local functions, $\phi(e_1)$ a flowline for $f_i - f^+$ and $\phi(e_2)$ a flowline for $f_i - f^-$, for some ordering of the incident edges e_1, e_2. In addition, we require that $\mu(v) = -1$.
- *Y_0-vertex.* A 3-valent vertex with $\phi(v)$ disjoint from $\Pi_B(\Sigma)$
- *Y_1-vertex.* A 3-valent vertex with $\phi(v)$ in $\Pi_B(\Sigma)$ and $\mu(v) = 1$.

Ekholm [Ekh07] gave a criterion for rigid trees and showed that these are the only possible vertices that can appear in a rigid flow tree.

Lemma 5.8 ([Ekh07]). *If Γ is a partial flow tree, then $0 \leq$ gcodim $\Gamma \leq$ dim Γ. Furthermore, if gcodim $\Gamma = 0$, then the vertices of Γ can only be (1) 1-valent puncture; (2) 2-valent puncture; (3) End; (4) Switch; (5) Y_0-vertex; (6) Y_1-vertex. In particular, if Γ is a rigid partial tree, then all of its vertices must be one of the above 6 types.*

Let $s_1, \ldots, s_l \in \{p_1, q_1, \ldots, p_n, q_n\}$. The moduli of (closed) gradient flow trees on L with punctures at $s_1, \ldots, s_l$ is the set

$$\mathcal{M}_L^{\nabla}(s_1, \ldots, s_l) := \left\{ (\Gamma, \phi) \middle| \begin{array}{ll} v_i \text{ is a negative puncture at } c_j & \text{if } s_i = q_j \\ v_i \text{ is a positive puncture at } c_j & \text{if } s_i = p_j \end{array} \right\}$$

where $v_1, \ldots, v_l$ are the cyclically-ordered vertices in $V_P(\Gamma)$ and v_1 it the distinguished vertex of Γ.

5.1.3 Generalized Flow Trees .

We broaden the notion of gradient flow tree for L by including a second type of edge. Let $F : L \to \mathbb{R}$ be a fixed Morse function on L with l critical points $m_1, \ldots, m_l$. For a metric g on M, let $\widetilde{g} := \frac{1}{K}\Pi^*(g)$ be the induced metric on L given by pulling back g by the base projection and scaled by some large positive constant K.

Let $E = E_{SFT} \cup E_{Morse}$ be a partition of the set of edges, the *SFT edges* and *Morse edges*. Let $\overline{E_{SFT}}$ and $\overline{E_{Morse}}$ be the ribbon graphs obtained as the closure of the union of the edges in E_{SFT} and E_{Morse}.

Analogously to Definition 5.2, we define generalized flow trees.

Definition 5.9. A *generalized flow tree* $\widetilde{\Gamma}$ consists of a metric tree $\Gamma \in \mathcal{T}_{j,k,C}$, a partition $E = E_{SFT} \cup E_{Morse}$ of the edges into two sets and continuous maps $\phi : \overline{E_{SFT}} \to M$ and $\psi : \overline{E_{Morse}} \to L$ such that

1. (*Flowlines*)

   (a) If $e \in E_{SFT}$, then ϕ^e is the flowline of some difference function $f_1^e - f_2^e$

   (b) If $e \in E_{Morse}$, then ψ^e is a gradient flowline of $\nabla^{\widetilde{g}} F$

2. (*(Non)constant*) If $e \in E_{SFT}$ is an interior edge, then the flowline ϕ^e is nonconstant. If $e \in E_{SFT}$ is an exterior edge or $e \in E_{Morse}$ then ϕ^e or ψ^e may be constant.

3. (*Punctures*) For each $e \in E_P$ and corresponding $v \in V_P$,

   (a) (SFT puncture) If $e \in E_{SFT}$, then

$$\lim_{t \to \pm\infty} \phi^e(t) \in \mathrm{Crit}(f_1^e - f_2^e)$$

(b) (Morse puncture) If $e \in E_{Morse}$, then

$$\lim_{t \to \pm\infty} \psi^e(t) \in \mathrm{Crit}(F)$$

4. (*Cusp ends*) If $e \in E_C$, then

$$\lim_{t \to v} \widetilde{\phi}_1^e(t) = \lim_{t \to v} \widetilde{\phi}_2^e(t)$$

   If $e \in E_C \cap E_{SFT}$, this is an *SFT cusp* and if $e \in E_C \cap E_{Morse}$ it is a *Morse cusp*

5. (*1-jet lifts*) The 1-jet lifts satisfy Condition (5) in Definition 5.2, where the 1-jet lifts of a Morse edge are two copies of the image of ψ^e, oriented in opposite directions.

6. (*2-valent vertices*) If v is 2-valent, then both incident edges $e_1, e_2 \in E_{SFT}$ and satify Condition (6) in Definition 5.2

7. (*Cotangent lifts*) The union of the cotangent lifts is a closed, oriented curve in the Lagrangian projection $\overline{L}$, where the cotangent lifts of a Morse edge are the images of the 1-jet lifts in the Lagrangian projection.

Let $s_1, \ldots, s_l \in \{p_1, q_1, \ldots, p_n, q_n, m_1^+, m_1^-, \ldots, m_l^+, m_l^-\}$. The moduli of (closed) gradient flow trees on L with punctures at $s_1, \ldots, s_l$ is the set

$$\widetilde{\mathcal{M}}_L^{\nabla}(s_1, \ldots, s_l) := \left\{ (\Gamma, \phi) \;\middle|\; \begin{array}{ll} v_i \text{ is a positive puncture at } c_j & \text{if } s_i = p_j \\ v_i \text{ is a negative puncture at } c_j & \text{if } s_i = q_j \\ v_i \text{ is a positive Morse puncture at } m_k & \text{if } s_i = m_k \\ v_i \text{ is a negative Morse puncture at } m_k & \text{if } s_i = m_k \end{array} \right\}$$

where $v_1, \ldots, v_l$ are the cyclically-ordered vertices in $V_P(\Gamma)$ and v_1 it the distinguished vertex of Γ.

Definition 5.10. Some important types of vertices in a generalized flow tree are:

1. *SFT 1-valent puncture.* A 1-valent puncture at the external vertex of some edge $e \in E_P \cap E_{SFT}$.

2. *Morse 1-valent puncture.* A 1-valent puncture at the external vertex of some edge $e \in E_P \cap E_{Morse}$.

3. *SFT 2-valent puncture.* A 2-valent puncture at the external vertex of some edge $e \in E_P \cap E_{SFT}$, as in Remark 5.4, where the remaining edges e_1, e_2 incident to the corresponding internal vertex u are in E_{SFT}.

4. *Mixed 2-valent puncture* A 2-valent puncture at the external vertex of some edge $e \in E_P \cap E_{SFT}$, as in Remark 5.4, such that $e_1 \in E_{SFT}$ and $e_2 \in E_{Morse}$, where e_1, e_2 are the remaining edges incident to the corresponding internal vertex u.

5. *SFT end.* A vertex $r \in E_C \cap E_{SFT}$ that satisfies criterion (4) in Definition 5.9.

6. *Morse end.* A vertex $r \in E_C \cap E_{Morse}$ that satisfies criterion (4) in Definition 5.9.

7. *SFT* Y_0 A 3-valent internal vertex as in Definition 5.7, where each incident edge lies in E_{SFT}

8. *Morse* Y_0 A 3-valent internal vertex as in Definition 5.7, where each incident edge lies in E_{Morse}

9. *Mixed* Y_0 A 3-valent internal vertex as in Definition 5.7, where two edges lie in E_{SFT} and the third is in E_{Morse}.

Generically, a vertex of a generalized tree will be one of the aforementioned types or a (SFT) switch or Y_1-vertex.

5.2 Flow Trees in Dimension 1

Let $L \subset J^1(\mathbb{R}) = \mathbb{R}^3$ be a Legendrian knot. Let Σ_l be the set of left-cusp-points in L and Σ_r the set of right-cusp points in L. Then $L \setminus \Sigma = \Sigma_l \cup \Sigma_r$ consists of $2n$ connected components $S_1, \ldots, S_{2n}$ where $n = |\Sigma_l| = |\Sigma_r|$. Each connected component S_i is the graph of a function $f_i : \Omega_i \to \mathbb{R}$ on the open set $\Omega_i = \Pi(S_i)$. Define $C_{i,j} = \text{Crit}(f_i - f_j)$ and $C = \cup_{1 \le i < j \le 2n} C_{i,j}$.

The Legendrian knot L is *generic* if

- the difference function $f_i - f_j$ on $\Omega_i \cap \Omega_j$ is Morse for all $1 \le i < j \le 2n$
- the sets $\text{Crit}(f_i - f_j)$ and $\text{Crit}(f_{i'} - f_{j'})$ are disjoint for $(i, j) \neq (i', j')$
- $\Pi_B(\Sigma)$ consists of $2n$ disjoint points and is disjoint from C

Recall that the Lagrangian projection $\overline{L}$ is an immersed submanifold of $T^*\mathbb{R} = \mathbb{R}^2$. The differentials df_i of the local functions are sections $df_i : \Omega_i \to T^*\mathbb{R}$ of the cotangent bundle and since L is Legendrian, $\overline{L}$ is the closure of the union of the images of these sections. Transverse double points of $\overline{L}$ - the knot crossings in the Lagrangian projection of the knot - are therefore given by the critical points of the functions $\{f_i - f_j\}$. If L is generic then the Lagrangian projection is a generic knot projection of L.

When dim $L = 1$, the dimension formula (Equation 5.1) becomes

$$\dim(\Gamma) = -2 + \sum_{p \in P(\Gamma)} I(p) - \sum_{q \in Q(\Gamma)} (I(q) - 1) + \sum_{r \in R(\Gamma)} \mu(r) \tag{5.2}$$

There are no switches when dim $L = 1$ since the tangency locus has codimension 1 in Σ.

5.2.1 Correspondence with holomorphic disks

For the study of Legendrian knots [Che02, Eli98, Ng10], it is standard to define invariants in terms of holomorphic immersions $u : (\mathbb{D}^2, \partial\mathbb{D}^2) \to (\mathbb{C}, \overline{L})$, where $T^*\mathbb{R}$ is identified with $\mathbb{C}$. However, these invariants could equivalently be defined in terms of gradient flow trees. We will review the correspondence between holomorphic disks and gradient flow trees.

Proposition 5.11. *There is a bijective correspondence between holomorphic maps $u : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{C}, \overline{L})$ modulo reparametrization and gradient flow trees on L.*

Let u be such a holomorphic map. Let $W = \{w_1, \ldots, w_k\}$ be the set of points in $w \in \partial\mathbb{D}$ such that $u(w)$ is a transverse double point of $\overline{L}$ but w is not a critical point of $\Pi \circ u$. For each w_i, let e_i be a copy of the interval $[0, 1]$, and define

$$D := \mathbb{D} \cup \{e_1, \ldots, e_k\}/\{w_i \sim e_1(0)\}$$

where the point w_i is identified with the endpoint 0 in the edge e_i.

For such a holomorphic map u, define

$$\Gamma_u := D/\left\{x \sim y \begin{array}{l} \text{if } x, y \in \mathbb{D} \text{ lie in the same connected component} \\ \text{of a level set of } \Pi \circ u \end{array}\right\}$$

After extending $\Pi \circ u$ to all of D by setting $\Pi \circ u(e_i) = \Pi \circ u(w_i)$, let $\widehat{u} : \Gamma_u \to \mathbb{R}$ be the induced map on the quotient Γ_u.

Lemma 5.12. *The pair $(\Gamma_u, \widehat{u})$ is a ribbon tree. Furthermore, if $u' = u \circ P$ where $P : \mathbb{D}^2 \to \mathbb{D}^2$ is biholomorphic, then $(\Gamma_u, \widehat{u})$ and $(\Gamma_{u'}, \widehat{u'})$ are isometric.*

Proof. First, define two subsets of $\mathbb{D}$

$$\begin{aligned} V &:= \text{critical level sets of } \Pi \circ u \cup \{e_1(1), \ldots, e_k(1)\} \\ &\quad\; \cup \{\text{connected components of regular level sets containing some } w_i\} \\ E &:= \mathbb{D} \setminus V \end{aligned}$$

The connected components of a regular level set of $\Pi \circ u$ are diffeomorphic to the interval $I = [0, 1]$. Thus, each connected component E' of E is diffeomorphic to $I \times (0, 1)$. In addition, the closure $\overline{E'}$ intersects exactly two connected components of V.

As a result, connected components of $E/\sim$ and $V/\sim$ are open intervals and points, respectively, and if e is a connected component of $E/\sim$ then ∂e consists of two points in $V/\sim$. Thus Γ_u is a graph. In addition, there is a cyclic ordering on edges at each v induced by the cyclic incidence of the connected components of E with the preimage of v in $\mathbb{D}^2$.

Finally, since P is a diffeomorphism, it descends to a ribbon graph isomorphism $\widehat{P} : \Gamma_u \to \Gamma_{u'}$ with $\widehat{u'} = \widehat{u} \circ \widehat{P}$, hence $\widehat{P}$ is an isometry. $\square$

Proof of Proposition 5.11. Let $u : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{C}, \overline{L})$ be a holomorphic map and Γ_u the induced metric tree. By Lemma 5.12, the metric tree Γ_u is well-defined up to conformal reparametrization of u.

In the terminology from the proof of Lemma 5.12, if E' is some connected component of E then $E' \cap \partial\mathbb{D}$ has two connected components γ_1, γ_2 with $u(\gamma_1), u(\gamma_2)$ in the graphs of df_i, df_j, respectively for some i, j. If e' is the edge in Γ_u corresponding to E', then define $\phi^{e'} : e' \to \mathbb{R}$ to parametrize the unique flow line of $f_i - f_j$ along the interval $\Pi \circ u(\overline{E'})$. To accomplish this, it will be necessary to modify the metric on e' to satisfy the gradient equation. However, this modification in well-defined up to conformal reparametrization of u by Lemma 5.12. If e_i is an external edge corresponding to some w_i, then define $\phi^{e_i} : e_i \to \Pi \circ u(w_i)$.

The 1-jet lifts of $\phi^{e'}$ are $u(\gamma_1), u(\gamma_2)$ and so since u is continuous, the 1-jet lifts satisfy criteria (4)-(7). Hence (Γ_u, ϕ) is a gradient flow tree on L.

Conversely, suppose (Γ, ϕ) is a gradient flow tree on L. Choose an embedding $i : (\Gamma, E_e) \to (\mathbb{D}, \partial\mathbb{D})$, an identification $\nu(i(\Gamma)) \simeq \mathbb{D}$ of some neighborhood of the image of i with the disk, and a quotient map $q : \nu(i(\Gamma)) \to \Gamma$ such that

- $q^{-1}(x) \simeq [0, 1]$ if x is in the interior of an edge of Γ
- $q^{-1}(v) \simeq C_k$ if v is a k-valent vertex, where C_k is the k-corolla

$$C_k := \left\{ (r, \zeta_k) \in \mathbb{C} \,\middle|\, 0 \le r \le 1 \text{ and } \zeta_k \text{ is a } k^{\text{th}} \text{ root of unity} \right\}$$

We can define a continuous map $u^* : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{C}, \overline{L})$ as follows. For each edge e of Γ, the map ϕ^e parametrizes a flow line of some local difference function $f_i - f_j$. Define u^* on the fibers of q over the interior of the edges as

$$\begin{aligned} u^* : q^{-1}(x) \simeq [0, 1] \to T^*\mathbb{R} \\ u^*(t) := (\phi^e(x), df_j(\phi^e(x)) + t(df_i(\phi^e(x)) - df_j(\phi^e(x)))) \end{aligned}$$

and then extend this continuously to a map on $\mathbb{D}$. By the Riemann mapping theorem, u^* can be replaced by a holomorphic immersion u with the same image. $\square$

5.3 Flow Trees in Dimension 2

The main goal of this subsection is to describe a strategy to calculate all rigid gradient flow trees when $\dim L = 2$. The underlying idea is to decompose a rigid, closed tree by repeatedly cutting it into several partial trees until we have decomposed it into minimal partial trees of dimension 0 or 1. Reversing this process, we can obtain rigid, closed trees by gluing partial trees together. This strategy will be useful in Section 5.7 for computing the gradient flow trees on the product $L_1 \times L_2$.

This strategy is a formalization of the underlying approach in [EENS13] to compute gradient flow trees in dimension 2.

The following is an overview of this process.

1. Start with the collection Φ_0 of rigid partial trees with 0 or 1 nonspecial punctures. These are neighborhoods of a *switch* or a *1-valent* or *2-valent puncture*, respectively (Corollary 5.16).

2. For some $\Gamma \in \Phi_0$, each edge of Γ ending with a special puncture has a maximal domain of definition. If that domain is noncompact, then the special puncture becomes a nonspecial puncture. If that domain is compact, the special puncture may become an *end*; may become a Y_1*-vertex* and form two new special punctures; or neither.

3. Alternatively, special punctures from two different partial trees may agree and if the 1-jet lifts glue up appropriately, can form a new Y_0*-vertex* with a new special puncture.

4. Repeat steps (2) and (3). Assuming transversality, there are a finite number of initial pieces and the iterated process ends after a finite number of steps. If all external vertices of the resulting tree Γ are either nonspecial punctures or ends, then Γ is a closed, rigid flow tree on L

We now elaborate on the steps above.

When $\dim L = 2$, the dimension formula (Equation 5.1) becomes

$$\dim(\Gamma) = -1 + \sum_{p \in P(\Gamma)} (I(p) - 1) - \sum_{q \in Q(\Gamma)} (I(q) - 1) + \sum_{r \in R(\Gamma)} \mu(r) \qquad (5.3)$$

Extending special punctures

Suppose that Γ has a special puncture at v and let e be the corresponding external edge. Then $\phi^e : [0, m(e)] \to M$ is a flow line of the negative gradient flow of some local difference function $f_1 - f_2$ for some f_1, f_2. By Lemma (?), either ϕ^e can be extended to a flow line $\phi^e_{max} : [0, \infty) \to M$ or there is some constant $b > m(e)$ such that ϕ^e can be extended to a flow line $\phi^e_{max} : [0, b] \to M$ and b is the maximal possible value for such an extension.

There is a one parameter family of trees Γ_t modeled on the same underlying ribbon tree as Γ but with the external edge parametrizing an interval of length t, with either $t \in (0, \infty)$ or $t \in (0, b)$. In other words, we can assume that the special puncture occurs at any point in the image of ϕ^e_{max}.

If ϕ^e_{max} is defined on $[0, \infty)$, then the vertex v satisfies criteria (3) of Definition (1.1) and the special puncture becomes a nonspecial puncture.

If ϕ^e_{max} is defined on $[0, b]$, it may be true that f_1 and f_2 form a cusp edge above $\phi^e_{max}(b)$. In this case, the edge e satisfies criterion (4) of Definition (1.1) and is a cusp end.

If f_1 and f_2 do not form a cusp edge, then it may be possible to extend Γ by adding a Y_1-vertex.

Forming Y_1-vertices

Suppose Γ has an external edge e parametrizing a flow line ϕ_e of some $f_1 - f_2$ whose maximal domain of definition has a compact end. Thus, either

$$\lim_{t \to v} \widehat{\phi}_1^e \in \Sigma \qquad \text{or} \qquad \lim_{t \to v} \widehat{\phi}_2^e \in \Sigma$$

where v is the external vertex on the boundary of e.

In the first case, suppose that f_1 and some f_3 form a cusp edge in Σ with f_1 the upper sheet and f_3 the lower sheet. Then for every f_4 defined in a neighborhood of $\phi^e(v)$ we can extend Γ by adding a Y_1-vertex at $\phi^e(v)$. Add two external edges e_1, e_2 to Γ at v in the cyclic order $\{e, e_1, e_2\}$ and orient these edges away from v. Let ϕ^{e_1} parametrize the downward gradient flow line of $f_3 - f_4$ that passes through $\phi^e(v)$ and let ϕ^{e_2} parametrize the downward gradient flow line of $f_4 - f_2$ that passes through $\phi^e(v)$. Then the 1-jet lifts agree and we have a new flow tree Γ' with one extra special punture extending Γ.

Note that

$$\text{pdim } \Gamma' = \text{pdim } \Gamma + \mu(v) + 1 = \text{pdim } \Gamma$$

as $\mu(v) = -1$. If f_1 and f_3 had formed a cusp with f_3 the upper sheet instead, then the new 3-valent vertex v would have $\mu(v) = 1$ instead so pdim Γ' would equal pdim $\Gamma + 2$. So if Γ had been a rigid special tree, then Γ' would not be rigid.

In the second case, suppose that f_2 and some f_2 form a cusp edge in Σ with f_3 the upper sheet and f_1 the lower sheet. Again, for every f_4 defined in a neighborhood of $\phi^e(v)$ we can extend Γ by adding a Y_1-vertex at $\phi^e(v)$. Add two external edges e_1, e_2 to Γ at v in the cyclic order $\{e, e_1, e_2\}$ and orient these edges away from v. Let ϕ^{e_1} parametrize the downward gradient flow line of $f_1 - f_4$ that passes through $\phi^e(v)$ and let ϕ^{e_2} parametrize the downward gradient flow line of $f_4 - f_3$ that passes through $\phi^e(v)$.

Forming Y_0-vertices

Let Γ_1 be a partial tree with special vertex at some v_1 and external edge e_1 oriented towards v_1 and Γ_2 a partial tree with special vertex at some v_2 and external edge e_2 oriented towards v_2. Suppose that ϕ^{e_1} parametrizes a flow line of $f_1 - f_2$ and ϕ^{e_2} parametrizes a flow line of $f_2 - f_3$. Then at every point $x \in \phi^{e_1}_{max} \cap \phi^{e_2}_{max}$ we can form a Y_0-vertex. Form a single tree $\Gamma = \Gamma_1 \cup_{v_1 \sim v_2} \Gamma_2$ and add an external edge e so that the cyclic ordering of edges at $v = v_1 \sim v_2$ is $\{e_1, e_2, e\}$. Let ϕ^{e_i} be the restriction of $\phi^{e_i}_{max}$ to $[0, (\phi^{e_i}_{max})^{-1}(x)]$. Let ϕ^e parametrize the negative gradient flow line of $f_1 - f_3$ that passes through x.

Lemma 5.13. *Let Γ be a rigid flow tree, possibly with special punctures, and $x \in \Gamma$ some point that is not a vertex. Then cutting Γ at x yields two partial flow trees Γ_0, Γ_1 with*

$$pdim\ \Gamma_0 = 0 \qquad and \qquad pdim\ \Gamma_1 = 1$$

Proof. After cutting Γ into two trees Γ_a, Γ_b we have from the dimension formula (Equation 5.1)

$$\begin{aligned}\dim\,\Gamma_a &= -1 + \sum_{p\in P(\Gamma_a)} (I(p)-1) - \sum_{q\in Q(\Gamma_a)} + \sum_{r\in R(\Gamma_a)} \mu(r)\\ \dim\,\Gamma_b &= -1 + \sum_{p\in P(\Gamma_b)} (I(p)-1) - \sum_{q\in Q(\Gamma_b)} + \sum_{r\in R(\Gamma_b)} \mu(r)\end{aligned}$$

Note that special punctures contribute +2 to the dimension formula regardless of sign, so

$$\begin{aligned}\dim\,\Gamma_a + \dim\,\Gamma_b &= -2+2+2+\sum_{p\in P(\Gamma)} (I(p)-1) - \sum_{q\in Q(\Gamma)} + \sum_{r\in R(\Gamma)} \mu(r)\\ &= \dim\,\Gamma + 3\end{aligned}$$

Since Γ_a, Γ_b each have a new special puncture, this implies that

$$\text{pdim}\Gamma_a + \text{pdim}\Gamma_b = \text{pdim}\Gamma + 1$$

but since pdim Γ_a, pdim $\Gamma_b \geq 0$, one partial tree must have dimension 0 and the other dimension 1. □

Lemma 5.14. *Let Γ be a partial tree with pdim $\Gamma = i$ for $i = 0, 1$. Let p be a special puncture, e the edge incident to p and v the other vertex of e.*

If v is a Y_0 -vertex, then cutting Γ at v yields 3 partial flow trees $\Gamma_1, \Gamma_2, \Gamma_3$ such that $\Gamma_3 = e$ and

$$\textit{pdim}\ \Gamma_1 = 0 \qquad \textit{pdim}\ \Gamma_2 = i$$

If v is a Y_1-vertex, then cutting Γ at v yields 3 partial flow trees $\Gamma_1, \Gamma_2, \Gamma_3$ such that $\Gamma_3 = e$ and

$$\textit{pdim}\ \Gamma_1 = 1 \qquad \textit{pdim}\ \Gamma_2 = i$$

Proof. The partial flow tree $\Gamma_3 = e$ has special punctures at v, p. This implies that $\text{dim}\Gamma_3 = 3$ and $\text{pdim}\Gamma_3 = 1$. The rest follows by dimension formula since

$$\begin{aligned}\text{pdim}\Gamma &= \text{pdim}\Gamma_1 + \text{pdim}\Gamma_2 + \text{pdim}\Gamma_3 - 2\\ &= \text{pdim}\Gamma_1 + \text{pdim}\Gamma_2 - 1\end{aligned}$$

and $\text{pdim}\Gamma_i \geq 0$. □

A rigid partial tree Γ is *minimal* if it contains no rigid subtree Γ'.

Lemma 5.15. *Each rigid tree Γ can be cut at some collection of distinct points $x_1, \ldots, x_k, v_1, \ldots, v_l$, where $x_1, \ldots, x_k$ are interior points of Γ and $v_1, \ldots, v_l$ are Y_0 or Y_1 vertices of Γ, into minimal partial trees $\Gamma_1, \ldots, \Gamma_{k+2l+1}$ of dimension 0 or 1.*

Proof. If Γ has no interior edges, it has 1 edge either connected two punctures or connecting a puncture and an end. Cut along this edge. By Lemma 5.13, one partial tree is rigid and the other has pdim $= 1$. Clearly, the rigid partial tree has no subtrees so is minimal.

If Γ has interior edges, cut along some interior edge. Iterate the following two steps until there are no more 3-valent vertices, 2-valent punctures or switches connected to a special puncture:

1. Cut at each Y_0 and Y_1 vertex incident to a special puncture. By Lemma 5.14, the remaining partial trees have pdim ≤ 1.

2. If a 2-valent puncture or switch has an edge connecting it to a special puncture, then cut on the other edge incident to the vertex.

The resulting partial trees must be a neighborhood of one of the types 1- or 2-valent vertices in Lemma 5.8. Moreover, by Lemma 5.14 and Lemma **??** all resulting partial trees have pdim ≤ 1. Neighborhoods of 1-valent vertices are clearly minimal. Cutting at a switch or a 2-valent puncture results in 2 partial trees consisting of an edge and 2 special punctures, which each have pdim $= 1$. Hence, neighborhoods of switches and 2-valent punctures are minimal as well. □

Corollary 5.16. *The minimal rigid partial trees of L are neighborhoods of the 3 types of vertices: 1-valent puncture at some v with $I(v) = 1$; 2-valent puncture at some v with $I(v) = 0, 2$; or switch. The minimal 1-dimensional partial trees of L are maximal flow lines and 1-valent punctures at some v with $I(v) = 0, 2$.*

5.4 A perturbation of $L_1 \times L_2$

In order to compute rigid flow trees on a Legendrian L, it must be front-generic and satisfy the preliminary transversality condition. Even if L_1, L_2 satisfy these two conditions, their product $L_1 \times L_2$, will not. In this section, we construct an explicit perturbation of $L = L_1 \times L_2$ to $\widetilde{L}_\epsilon$ for some arbitrarily small constant ϵ that is both front generic.

As constructed in Chapter 2, the Legendrian $L_1 \times L_2$ is not even chord-generic: there are families of Reeb chords $a \times L_2$ and $L_1 \times b$ for Reeb chords a of L_1 and b of L_2. From the gradient tree perspective, this is a Morse-Bott degeneration. Importantly, there are two, 'perpendicular' Morse-Bott degenerations, one along L_1 and the other along L_2, that must be perturbed simultaneously.

In Subsection 5.4.1, we describe an explicit Morse-Bott perturbation of $L = L_1 \times L_2$ to some L_ϵ, and in Subsection 5.4.2, we describe how to perturb L_ϵ near the cusp set to achieve front-genericity.

The following lemma is a straightforward consequence of the definition of the product operation (Definiton 1; compare with Lemma 4.3).

Lemma 5.17. *If $\{f_i^1 : \Omega_i^1 \to \mathbb{R}\}$ are local defining functions for L_1 and $\{f_k^2 : \Omega_k^2 \to \mathbb{R}\}$ are local defining functions for L_2 then $\{H_{i,k} := f_i^1 + f_k^2 : \Omega_i^1 \times \Omega_k^2 \to \mathbb{R}\}$ are local defining functions for $L_1 \times L_2$.*

Recall the L_1, L_2 satisfy the outside cusps conditions. Let $X \in \Sigma_l$ denote the left-cusp-point such that $\Pi_B(X) = a$. Index the components S_i of $L \setminus \Sigma$ as follows. S_1 is the component whose negatively-oriented boundary point is X. Then following L in the direction of the oriention on L index the components in order $S_2, \dots, S_{2n}$ so that X is the positively-oriented boundary point of S_{2n}.

In addition, we can perturb f_i, f_j, without introducting any new Reeb chords, so that for some $\zeta > 0$, they equal

$$f_i = (-1)^{I(p)}\frac{C}{2}x^2 + ax + c + d$$
$$f_j = ax + dj$$

for some constants C, a, c, d, on the interval $(p - \zeta, p + \zeta)$ for each $p \in \mathrm{Crit}(f_i - f_j)$ and thus

$$\nabla(f_i - f_j) = (-1)^{I(p)} C x \partial_x$$

on this interval.

5.4.1 Morse-Bott perturbation

Throughout this section, we use x, y to denote coordinates in L_1, L_2, respectively, and s, t to denote coordinates on $\mathbb{R}^2$.

For $\mu = 1, 2$ let $\Sigma^\mu = \Sigma^\mu_l \cup \Sigma^\mu_r$ be the cusp-points of L_μ, $\{S^\mu_1, \dots, S^\mu_{2n_\mu}\}$ the connected components of $L_\mu \setminus \Sigma^\mu$ which are the graphs of the functions $\{f^\mu_i : \Omega^\mu_i \to \mathbb{R}\}$.

Let $A^\mu = \Pi{-}1([z_\mu, z_\mu + \lambda_\mu])$ be an interval in S^μ_1. For $\mu = 1, 2$ let $\mu^* = 2, 1$.

Pick points $\alpha^- < \alpha^+ \in (z_1, z_1 + \lambda_1)$ and $\beta^- < \beta^+ \in (z_2, z_2 + \lambda_2)$.

Let $\eta_1, \eta_2 > 0$ and $0 < \zeta \ll \lambda_1, \lambda_2$ be small constants. Let F^1 be a Morse function on L_1 with 2 critical points, an index 1 critical point at $\Pi^{-1}(\alpha^+)$ and an index 0 critical point at $\Pi^{-1}(\alpha^-)$ such that $|\nabla F^1| = \eta_1$ on the complement of $\Pi^{-1}((\alpha^- - \zeta, \alpha^+ + \zeta))$. Similarly, let F^2 be a Morse function on L_2 with 2 critical points, an index 1 critical point at $\Pi^{-1}(\beta^+)$ and an index 0 critical point at $\Pi^{-1}(\beta^-)$ suc that $|\nabla F^2| = \eta_2$ on the complement of $\Pi^{-1}((\beta^- - \zeta, \beta^+ + \zeta))$. Furthermore, let $F^1_i = \Pi^{-1} \circ F^1 : \Omega^1_i \to \mathbb{R}$ and $F^2_k : \Pi^{-1} \circ F^2 : \Omega^2_k \to \mathbb{R}$.

Define

$$H^\epsilon_{i,k} := f^1_i + f^2_k + \epsilon F^1_i F^2_k$$

Let L_ϵ be the Legendrian submanifold defined by these local functions, so that $L_0 = L = L_1 \times L_2$. Moreover, for ϵ sufficiently small, this defines a Legendrian isotopy of L.

Lemma 5.18. *For ϵ sufficiently small, there are four families $a^\pm_q, b^\pm_p, c_{p,q}, d_{p,q}$ of Reeb chords on L_ϵ, where $p \in Crit(f^1_i - f^1_j)$ and $q \in Crit(f^2_k - f^2_l)$, for some $1 \le i < j \le n_1$ and $1 \le k < l \le n_2$, with the following coordinates in $\mathbb{R}^2$*

$$a^\pm_q = (\alpha^\pm, q) \qquad c_{p,q} = (p + (-1)^{I(p)+1}\epsilon\eta_1\mu^1_{i,j,q}, q + (-1)^{I(q)+1}\epsilon\eta_2\mu^2_{k,l,p})$$
$$b^\pm_p = (p, \beta^\pm) \qquad d_{p,q} = (p + (-1)^{I(p)+1}\epsilon\eta_1\mu^1_{i,j,q}, q + (-1)^{I(q)}\epsilon\eta_2\mu^2_{k,l,p})$$

for some constants $\mu^1_{i,j,q}$ and $\mu^2_{k,l,p}$. These critical points have the following Morse indices:

$$I(a_q^+) = I(q) + 1 \qquad I(a_q^-) = I(q)$$
$$I(b_p^+) = I(p) + 1 \qquad I(b_p^-) = I(p)$$
$$I(c_{p,q}) = I(p) + I(q) \qquad I(d_{p,q}) = I(p) - I(q)$$

Proof. In the following discussion, we will always assume that $i \leq j$.

The gradient of $H^\epsilon_{i,k} - H^\epsilon_{j,l}$ is

$$\nabla(H^\epsilon_{i,k} - H^\epsilon_{j,l}) = \nabla(f_i^1 - f_j^1) + \nabla(f_k^2 - f_l^2) + \epsilon\nabla(F_i^1F_k^2 - F_j^1F_l^2)$$

In the first case, let $i \neq j$ and $k \neq l$. For small ϵ, all critical points must lie in some small neighborhood of some point $p \times q$, where $p \in \mathrm{Crit}(f_i^1 - f_j^1)$ and $q \in \mathrm{Crit}(f_k^2 - f_l^2)$. By assumption, in such a neighborhood

$$\nabla_{\partial_t}F_i^1 = \nabla_{\partial_t}F_j^1 = \nabla_{\partial_s}F_k^2 = \nabla_{\partial_s}F_l^2 = 0$$

and

$$|\nabla_{\partial_s}F_i^1| = |\nabla_{\partial_s}F_j^1| = \eta_1 \qquad |\nabla_{\partial_t}F_k^2| = |\nabla_{\partial_t}F_l^2| = \eta_2$$

and

$$f_i^1 - f_j^1 = (-1)^{I(p)}\frac{1}{2}(s-p)^2 \qquad f_k^2 - f_l^2 = (-1)^{I(q)}\frac{1}{2}(t-q)^2$$

and so the gradient becomes

$$\begin{aligned}\nabla(H^\epsilon_{i,k} - H^\epsilon_{j,l}) = \big(&(-1)^{I(p)}(s-p) + \epsilon\eta_1\left((-1)^{i+1}F_k^2 - (-1)^{j+1}F_l^2\right), \\ &(-1)^{I(q)}(t-q) + \epsilon\eta_2\left((-1)^{k+1}F_i^1 - (-1)^{l+1}F_j^1\right)\big)\end{aligned}$$

In addition, near $p \times q$, $F_i^1 > F_j^1$ if and only if $i < j$ and $F_k^2 > F_l^2$ if and only if $k < l$.

This gives two critical points: $c_{p,q}$ when $k < l$, and $d_{p,q}$ when $k > l$. Note that there is a sign change in the second case. Setting $\mu^1_{i,j,q} := ((-1)^{i+1}F_k^2 - (-1)^{j+1}F_l^2)$ and $\mu^2_{k,l,p} := \left((-1)^{k+1}F_i^1 - (-1)^{l+1}F_j^1\right)$, the statement about the coordinates of these critical points follows.

To compute the Morse index, note that for ϵ small, the Hessian $\nabla^2(H^\epsilon_{i,k} - H^\epsilon_{j,l})$ is arbitrarily close to $\nabla^2(f_i^1 - f_j^1) \oplus \nabla^2(f_k^2 - f_l^2)$.

Now, let $i = j$. For small ϵ, a critical point must lie in a neighborhood of $\mathbb{R} \times q$, for some $q \in \mathrm{Crit}(f_k^2 - f_l^2)$. In this neighborhood, the gradient becomes

$$\nabla(H^\epsilon_{i,k} - H^\epsilon_{i,l}) = \left(\epsilon(F_k^2 - F_l^2)\nabla F_i^1, (-1)^{I(q)}(t-q) + \epsilon\eta_2F_i^1\left((-1)^{k+1} - (-1)^{l+1}\right)\right)$$

This gives two critical points a_q^+ and a_q^-, corresponding to the index 1 and index 0 critical points of F^1.

The statement about Morse indices is similar. For small ϵ, the Hessian $\nabla^2(H^\epsilon_{i,k} - H^\epsilon_{i,l})$ is arbitrarily close to $\nabla^2(F^1_i) \oplus \nabla^2(f^2_k - f^2_l)$.

The case of $k = l$ is similar and gives two critical points b^+_p and b^-_p for each $p \in \mathrm{Crit}(f^1_i - f^1_j)$ and the two critical points of F^2.

□

The vertical rigid flow lines of $H^\epsilon_{1,k} - H^\epsilon_{1,l}$ and $H^\epsilon_{1,k'} - H^\epsilon_{1,l'}$ coincide for all pairs $(k,l),(k',l')$; and similarly for the horizontal rigid flow lines of $H^\epsilon_{i,1} - H^\epsilon_{j,1}$ and $H^\epsilon_{i',1} - H^\epsilon_{j',1}$ for all pairs $(i,j),(i',j')$. Hence, to achieve preliminary transversality, it is necessary to perturb L_ϵ.

Lemma 5.19. *For arbitrarily small $\zeta > 0$, there exists a perturbation of L_ϵ, supported in $A^1 \times L_2 \cup L_1 \times A^2$, and functions $\alpha^\pm_{k,l} : S^2_1 \to (0,\zeta)$ for $1 \le k < l \le n_2$, supported in A^2, and $\beta^\pm_{i,j} : A^1 \to (0,\zeta)$ for $1 \le i < j \le n_1$, supported in A^1, such that*

- *The collections $\{\alpha^+_{k,l}\}, \{\alpha^-_{k,l}\}, \{\beta^+_{i,j}\}, \{\beta^-_{i,j}\}$ are lexicographically ordered. That is, $\alpha^\pm_{k,l} < \alpha^\pm_{k',l'}$ if $k < k'$ or if $k = k'$ and $l < l'$; similarly, $\beta^\pm_{i,j} < \beta^\pm_{i',j'}$ if $i < i'$ or $i = i'$ and $j < j'$.*
- *There is a 1-1 correspondence of Reeb chords before and after the perturbation. Further, if $a^\pm_q, b^\pm_p$ denote the perturbed Reeb chords, then the coordinates of the Reeb chords in $\mathbb{R}^2$ are*
$$\begin{aligned} a^+_q &= (\alpha^+ + \alpha^+_{k,l}(q), q) & a^-_q &= (\alpha^- + \alpha^-_{k,l}(q), q) & &\text{if } q \in \mathit{Crit}(f^2_k - f^2_l) \\ b^+_p &= (p, \beta^+ + \beta^+_{i,j}(p)) & b^-_p &= (p, \beta^- + \beta^-_{i,j}(p)) & &\text{if } p \in \mathit{Crit}(f^1_i - f^1_k) \end{aligned}$$
- *If $a^\pm_q$ has Morse index 1, then the stable and unstable manifolds are contained in the lines $\{t = q\}$ and $\{s = \alpha^\pm + \alpha^\pm_{k,l}(t)\}$. If $b^\pm_p$ has Morse index 1, then the stable and unstable manifolds are contained in the lines $\{s = p\}$ and $\{t = \beta^\pm + \beta^\pm_{i,j}(s)\}$.*

Proof. The proof is similar to the proof of Lemma 3.9 in [EENS13]. □

For $s_0, t_0 \in \mathbb{R}$ and $D > 0$, a *D-corridor* of s_0 (respectively t_0) is the open neighborhood $(s_0 - D, s_0 + D) \times \mathbb{R}$ (respectively $\mathbb{R} \times (t_0 - D, t_0 + D)$).

Lemma 5.20. *For all constants $C, D > 0$, there exists an $\epsilon > 0$ such that, with respect to the standard metric on $\mathbb{R}^2$,*

- *$\left|\nabla_{\partial_s}(H^\epsilon_{1,k} - H^\epsilon_{1,l})\right| / \left|\nabla_{\partial_t}(H^\epsilon_{1,k} - H^\epsilon_{1,l})\right| < C$ on the complement of D-corridors of all $q \in \mathit{Crit}(f^2_k - f^2_l)$.*
- *$\left|\nabla_{\partial_t}(H^\epsilon_{i,1} - H^\epsilon_{j,1})\right| / \left|\nabla_{\partial_s}(H^\epsilon_{i,1} - H^\epsilon_{j,1})\right| < C$ on the complement of D-corridors of all $p \in \mathit{Crit}(f^1 - i - f^1_j)$.*

Proof. This is clear from the description of the gradients of $H^\epsilon_{1,k} - H^\epsilon_{1,l}$ and $H^\epsilon_{i,1} - H^\epsilon_{j,1}$ in the proof of Lemma 5.18. □

This lemma implies that a general flowline of $\nabla(H^\epsilon_{i,k} - H^\epsilon_{i,l})$ or $\nabla(H^\epsilon_{i,k} - H^\epsilon_{j,k})$ flows essentially perpendicular to a D-corridor outside that D-corridor.

5.4.2 Perturbation along caustic

Even after the Morse-Bott perturbation described above, the Legendrian L_ϵ is not front-generic. However, we can choose an isotopy of L_ϵ to some $\widetilde{L}_\epsilon$, supported in an arbitrarily small neighborhood of the caustic Σ, so that $\widetilde{L}_\epsilon$ is front-generic.

The caustic Σ of $L_1 \times L_2$ is the union of the submanifolds $\Sigma^1 \times L_2$ and $L_1 \times \Sigma^2$. These submanifolds intersect transversely at $x \times y$ for each $x \in \Sigma^1$ and $y \in \Sigma^2$, where there is a double cusp as in Figure 5.2. Locally, there are coordinates

$$x_1 = \frac{u_1^2}{2} \qquad x_2 = \frac{u_2^2}{2}$$

$$z = \frac{1}{3}(u_1^3 + u_2^3)$$

near this double cusp. Each double cusp is formed by four sheets $S_{i,j}$, $S_{i,j+1}$, $S_{i+1,j}$, $S_{i+1,j+1}$ meeting at a point, with four cusp edges formed by pairs of sheets $(S_{i,j}, S_{i+1,j})$, $(S_{i,j}, S_{i,j+1})$, $(S_{i+1,j}, S_{i+1,j+1})$, $(S_{i,j+1}, S_{i+1,j+1})$ emanating from the double cusp. For a generic choice of $\Phi = (\Phi_1, \Phi_2) : L_1 \times L_2 \to \mathbb{R}^2$ supported in a small neighborhood of $x \times y$, there is a Legendrian isotopy defined for $\lambda \in [0, \epsilon']$ for some $\epsilon' > 0$

$$x_1 = \frac{(u_1 - \lambda\Phi_1)^2}{2} \qquad x_2 = \frac{(u_2 - \lambda\Phi_2)^2}{2}$$

$$z = \frac{1}{3}(u_1^3 + u_2^3)$$

such that the double cusp is perturbed into a swallow-tail singularity formed by the sheets $S_{i,j}, S_{i,j+1}, S_{i+1,j}$ and the cusp edges formed by the pairs $(S_{i+1,j}, S_{i+1,j+1})$ and $(S_{i,j+1}, S_{i+1,j+1})$ become a single cusp edge. See Figure 5.2. We choose the neighborhood on which Φ is defined to be disjoint from the region affected by the Morse-Bott perturbation above.

For $x \in \Sigma^1$ define $T_{x,k} = x \times S_k^2$. Then even after the perturbation at the double cusp, the projections $\Pi_B(T_{x,k})$ and $\Pi_B(T_{x,l})$ for $k \neq l$ are tangent along much of their interior. Similarly, for $y \in \Sigma^2$ define $T_{y,i} = S_i^1 \times y$, the projections $\Pi_B(T_{y,i})$ and $\Pi_B(T_{y,j})$ for $i \neq j$ are tangent along much of their interior. In standard cusp-edge coordinates along $T_{x,k}$ (resp. $T_{y,i}$)

$$x_1 = \frac{1}{2}u_1^2 - \epsilon' \phi_{x,k}$$

where $\phi_{x,k}$ is a bump function on a neighborhood of $T_{x,k}$ that agrees with the choices of functions Φ near the double cusps. For small $\epsilon' > 0$ and generic choices of $\phi_{x,k}$ as k varies, this defines a Legendrian isotopy.

In particular, we can also assume the tangency locus along $T_{x,k}$ is transversely cut-out as well. Thus, we have proved the following lemma.

Lemma 5.21. *There exists a Legendrian isotopy of L_ϵ to some $\widetilde{L}_\epsilon$, supported in an arbitrarily small neighborhood of Σ, such that $\widetilde{L}_\epsilon$ is front-generic and satisfies the transversality condition. In addition, the isotopy introduces no new critical points* *and the gradient vector field is unchanged in neighborhoods of $p \in Crit(f_i^1 - f_j^1)$ and $p \in Crit(f_k^2 - f_l^2)$ for all i, j, k, l*

5.5 Local description of rigid trees on $L_1 \times L_2$

In this section, we give local descriptions of all vertices of rigid trees on $L_1 \times L_2$, after it has been isotoped according to the perturbation in the previous section. First, in Subsection 5.5, we describe the minimal partial trees on $L_1 \times L_2$. In the successive subsections, we describe Y_1-vertices (Subsection 5.5) and Y_0-vertices (Subsection 5.5), of which there are four types.

Minimal Partial Trees

In this subsection, we describe all minimal partial trees. Recall from Corollary 5.16 that rigid minimal partial trees can be either 1-valent punctures, 2-valent punctures, or switches.

Lemma 5.22. *The minimal partial trees are*

1. *For each critical point c of Morse index 1, there are four rigid partial flow trees with a 1-valent puncture at c*

2. *For each critical point $c \in Crit(H_{i,j} - H_{k,l})$ of index 0 or 2, and pair (r, s) with either $r \neq i, j$ or $s \neq k, l$ there is a rigid partial flow tree with a 2-valent puncture at c.*

In particular, there are no switches.

Proof. If $I(c) = 1$, then there are two rigid flowlines in the stable manifold $S_c(H_{i,j} - H_{k,l})$ and two rigid flowlines in the unstable manifold $S_c(H_{i,j} - H_{k,l})$. Any partial flowline determines a minimal partial tree and by the dimension formula each has pdim $= 0$. A partial tree with one 2-valent puncture at c and 2 special punctures would have pdim $= 2$ and so is not rigid.

For (r, s) satisfying the assumptions, there is a flowline of $H_{i,r} - H_{k,s}$ and a flowline of $H_{j,r} - H_{l,s}$ that pass through c. There is a unique way to glue up the 1-jet lifts to satisfy criteria (5) of Definition 5.2 to obtain a partial flow tree with

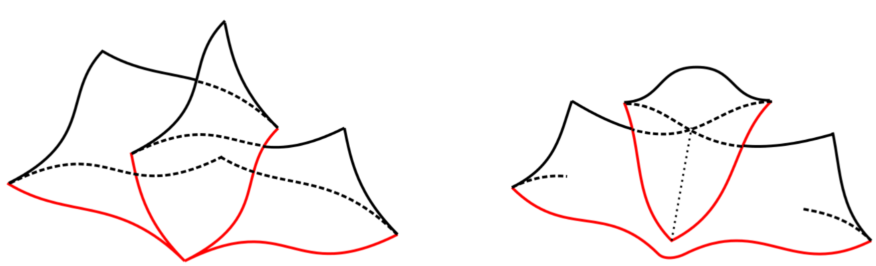

FIGURE 5.2: Local picture of a double cusp before (left) and after (right) a perturbation. The caustic is in red.

a 2-valent puncture at c. Applying the dimension formula shows that pdim $= 0$ and so the partial tree is rigid.

To rule out switches, note the following. A switch has a vertex at a tangency point z of the local function difference gradient along the projection of the caustic. However, we can assume by our choice of perturbation along the caustic that there is a boundary-parallel flowline γ of $\nabla(H^{\epsilon}_{i,k} - H^{\epsilon}_{j,l})$ in $\Omega := \Omega^1_i \times \Omega^2_k \cap \Omega^1_j \times \Omega^2_l$ such that there are no critical points of $H^{\epsilon}_{i,k} - H^{\epsilon}_{j,l}$ in the disk bounded by γ and $\partial\Omega$ and that a unique tangency point z lies in the $\partial\Omega$ component of the boundary of this disk. This point divides this boundary component into two regions, one where $\nabla(H^{\epsilon}_{i,k} - H^{\epsilon}_{j,l})$ points outward and one where it points inward.

Each flow line that begins on the inward-pointing part has a maximal domain of definition,but it cannot cross γ or limit to a critical point, so it must end on the cusp set. In particular, it must end on the outward-pointing part. In other words, flow lines must connect the inward-pointing component of $\partial\Omega$ to the outward-pointing component. Take a sequence of such flowlines with initial points limiting to the tangency point. The endpoints of these flowlines must limit to the tangency point as well. Otherwise, there is a region whose boundary on the cusp set is all outward-pointing. But this is impossible, since we can work back from any outward-pointing and it has to end on the inward-pointing component. Thus, there is a constant flow line at the tangency point and there can be no flowline leaving from the tangency point. Hence it cannot be part of any switch.

By Corollary 5.16 these are the only possible rigid minimal trees. □

Y_1-vertices

Recall that not every partial flow tree Γ becomes a closed flow tree after extending all the external edges to their maximal domains of definitions. If this occurs, we can add a Y_1-vertex at the cusp to enlarge the tree, at the expense of increasing the number of special punctures of Γ by 1.

In the current setting, the utility of adding Y_1-vertices depends upon the pair of local difference functions at the special puncture. If ϕ_e is a flow line of $(i,i),(k,l)$ for $k \neq l$ or of $(i,j)(k,k)$ for $i \neq j$, then we have the following lemma.

Lemma 5.23. *Let Γ be a partial flow tree and e a special edge parametrizing a flowline of $(i,i);(k,l)$ in a D-corridor of some $q \in Crit(f^2_k - f^2_l)$. Then Γ can be extended to a partial flow tree Γ' with pdim $\Gamma' =$ pdim. Γ, the same number of special punctures, and a new special edge parametrizing a flowline of $(i+1,i+1);(k,l)$, if $(\phi^e)'$ points outward at its special puncture, or a flowline of $(i-1,i-1);(k,l)$, if $(\phi^e)'$ points inward at its special puncture.*

A similar statement holds if ϕ_e parametrizes a flowline of $(i,j);(k,k)$.

Proof. This is described in Figure 5.3 for the first case; the other cases follow similarly. Flow lines are in black and the projection of the caustic is in red. The perturbation of the caustic preserves the outward-pointing gradient of the local difference function in the specified interval and can be chosen so that the domain of $H^{\epsilon}_{i,l}$ extends slightly farther then the domain of $H^{\epsilon}_{i,k}$. □

The outside cusp condition ensures that all Y_1-vertices must occur near the boundary of this domain but for orientation reasons the above Y_1-vertices are the only ones possible.

Y_0-vertices

In this subsection, we describe possible Y_0-vertices.

An edge e of a gradient flow tree on $\widehat{L}_\epsilon$ is *fat* if it parametrizes a flow line of $\nabla(H^\epsilon_{i,k} - H^\epsilon_{j,l})$ for $i \neq j$ and $k \neq l$. The edge is *thin* if it parametrizes a flow line of $\nabla(H^\epsilon_{i,k} - H^\epsilon_{i,l})$ or $\nabla(H^\epsilon_{i,k} - H^\epsilon_{j,k})$. Y_0 vertices can be classified into 4 types, according to how many incident edges are fat and thin: Type I has 3 fat edges; Type II has 2 fat edges and 1 thin edge; Type III has 1 fat edge and 2 thin edges; and Type IV has 3 thin edges.

Type I. The first type of Y_0 vertices are formed by joining two partial trees, one with an external edge parametrizing some $(i, j); (k, l)$ and the other parametrizing some $(j, r); (l, s)$ for i, j, r and k, l, s distinct. The new external edge parametrizes $(i, r); (k, s)$. See Figure 5.4.

Type II. The second type of Y_0-vertices are formed either by joining two partial trees with fat special edges, to obtain a new partial tree with thin special edge, or by joining two partial trees, one with a fat special edge, the other with a thin special edge, to obtain a new partial tree with fat special edge. See Figure 5.5.

Type III. The possible indices and orientations of the third type of Y_0-vertices are described in Figure 5.6.

Type IV. Finally, there are two instances of Type IV Y_0-vertices. The first happens near in D-corridors of critical points of the perturbing Morse functions F^1, F^2. The second occur in D-corridor surrounding a Type II or Type III vertex.

5.6 Pairs of flow trees on $L_1 \times L_2$

In this section, we define the space $\widetilde{\mathcal{M}}^{\nabla}_{L_1}(\mathbf{s}^1) \times_{\mathcal{T}_m} \widetilde{\mathcal{M}}^{\nabla}_{L_2}(\mathbf{s}^2)$ of compatible pairs of generalized flow trees and that this space is transversely cut-out. We then describe several types of vertices for a generic pair of generalized flow trees on L_1 and L_2. and show that each vertex for a pair of trees in $\widetilde{\mathcal{M}}^{\nabla}_{L_1}(\mathbf{s}^1) \times_{\mathcal{T}_m} \widetilde{\mathcal{M}}^{\nabla}_{L_2}(\mathbf{s}^2)$ is one of these types.

Let $\mathbf{s} = (s_1, \ldots, s_k)$ be a k-tuple of Reeb chords on $L_1 \times L_2$. Recall that by Lemma 5.18, this determines two k-tuples $\mathbf{s}^1, \mathbf{s}^2$ of Reeb chords and Morse critical points on L_1 and L_2, respectively.

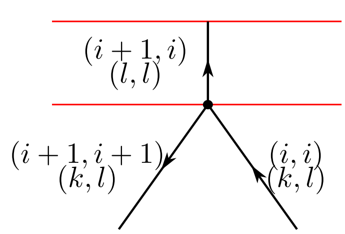

FIGURE 5.3: Adding a Y_1-vertex to extend a thin edge past a cusp

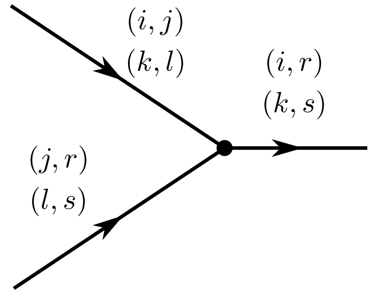

FIGURE 5.4: Type I

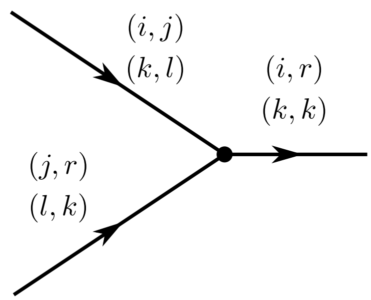

FIGURE 5.5: Type II

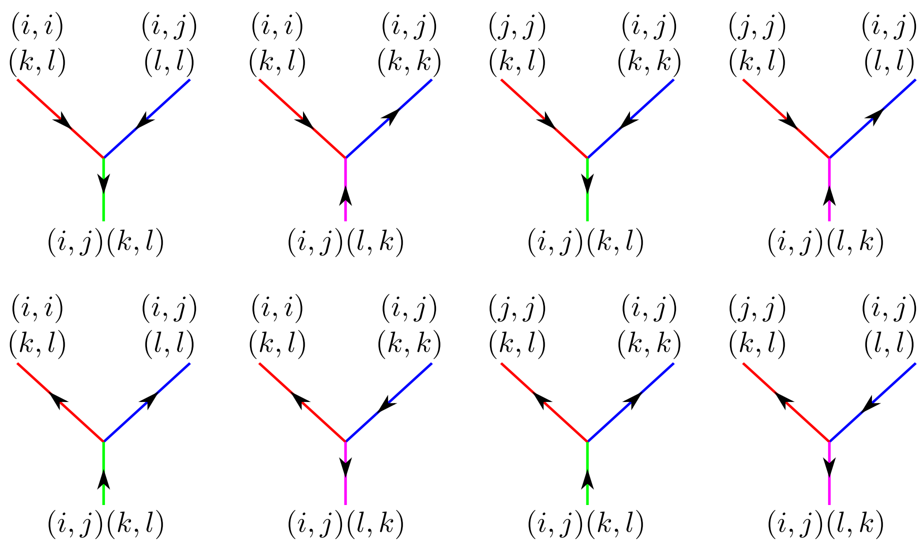

FIGURE 5.6: Possible orientations at a Type III Y_0-vertex

For $\mathbf{s}_i = (s^i_1, \ldots, s^i_k)$, let $\widetilde{\mathcal{M}}^{\nabla}_{L_i}(\mathbf{s}^i)$ be the moduli of generalized flow trees on L_i with punctures at $s^i_1, \ldots, s^i_k$.

Definition 5.24. The moduli space $\widetilde{\mathcal{M}}^{\nabla}_{L_1}(\mathbf{s}^1) \times_{\mathcal{T}_m} \widetilde{\mathcal{M}}^{\nabla}_{L_2}(\mathbf{s}^2)$ is the set of pairs (Γ_1, Γ_2), where $\Gamma_i \in \widetilde{\mathcal{M}}^{\nabla}_{L_i}(\mathbf{s}^i)$, such that

- There is an isometry $\phi : \Gamma_1 \to \Gamma_2$ of the underlying metric trees
- If e is an edge of Γ_1 (respectively Γ_2) parametrizing a Morse edge, then $\phi(e)$ (respectively $\phi^{-1}(e)$) parametrizes an SFT edge.
- If e is an interior edge of Γ_1, then either e or $\phi(e)$ parametrizes a nonconstant flowline.

Lemma 5.25. *After an arbitrarily small perturbation of L_1 and L_2, the space $\widetilde{\mathcal{M}}^{\nabla}_{L_1}(\mathbf{s}^1) \times_{\mathcal{T}_m} \widetilde{\mathcal{M}}^{\nabla}_{L_2}(\mathbf{s}^2)$ is a transversely cut-out.*

Proof. This follows by a modification of the proof of transversality for standard flow trees in Proposition 3.14 in [Ekh07]. □

Definition 5.10 described 9 types of generic vertices in a generalized flow tree. The following is a complete list of potential pairings of these vertex-types for a pair of trees in $(\widetilde{\Gamma}_1, \widetilde{\Gamma}_2) \in \widetilde{\mathcal{M}}^{\nabla}_{L_1}(\mathbf{s}^1) \times_{\mathcal{T}_m} \widetilde{\mathcal{M}}^{\nabla}_{L_2}(\mathbf{s}^2)$:

1	SFT 1-valent puncture	SFT 1-valent puncture
2	SFT 1-valent puncture	Morse 1-valent puncture
3	SFT 2-valent puncture	SFT 2-valent puncture
4	SFT 2-valent puncture	Morse 1-valent puncture $\cup$ Y_0-vertex (either Morse/mixed/SFT)
5	SFT 2-valent puncture	Mixed puncture
6	Mixed puncture	Mixed puncture
7	SFT end	SFT end
8	SFT end	Morse end
9	SFT Y_0	SFT Y_0
10	SFT Y_0	Mixed Y_0
11	SFT Y_0	Morse Y_0
12	Mixed Y_0	Mixed Y_0

Finally, analogously to Lemma 5.8, all pairs of vertices in a generic pair of generalized flow trees must be one of these types.

Lemma 5.26. *For a pair $(\Gamma_1, \Gamma_2) \subset \widetilde{\mathcal{M}}^{\nabla}_{L_1}(\mathbf{s}^1) \times_{\mathcal{T}_m} \widetilde{\mathcal{M}}^{\nabla}_{L_2}(\mathbf{s}^2)$, each pair of vertices must be one of the above 12 types.*

Proof. The proof of Lemma 5.8 in [Ekh07] and transversality established in Lemma 5.25 imply that each vertex in the underlying ribbon tree must be either 1-valent or 3-valent. Since the 9 types of vertices in Definition 5.10 are the only possibilites for 1- and 3-valent vertices in a generalized flow tree, the above list is exhaustive of all compatible pairs. □

5.7 Proof of Theorem D

After Lemmas 5.19 and 5.21, the perturbed $\widetilde{L}_\epsilon$ is front-generic and satisfies the preliminary transversality conditions. Hence, we can utilize the algorithm from Subsection 5.3 to compute the rigid gradient flow trees on $\widetilde{L}_\epsilon$.

On the other side, Lemma 5.25 guarantees that $\widetilde{\mathcal{M}}^{\nabla}_{L_1}(\mathbf{s}^1) \times_{\mathcal{T}_m} \widetilde{\mathcal{M}}^{\nabla}_{L_2}(\mathbf{s}^2)$ is transversely cut-out and Lemma 5.26 ensures that each vertex for a pair of trees in $\widetilde{\mathcal{M}}^{\nabla}_{L_1}(\mathbf{s}^1) \times_{\mathcal{T}_m} \widetilde{\mathcal{M}}^{\nabla}_{L_2}(\mathbf{s}^2)$ is one of the 12 types in Lemma 5.26.

Reversing the degeneration, as $\epsilon \to 0$, the images of the edges of a gradient flow tree converge to cascades, in the sense of [BH13]. That is, for $\epsilon > \epsilon' > 0$ sufficiently small, there is a 1-1 correspondence between gradient flow trees on $\widetilde{L}_\epsilon$ and gradient flow trees on $\widetilde{L}_{\epsilon'}$ and for each flow tree Γ on $\widetilde{L}_\epsilon$ there is a corresponding flow tree $\Gamma_{\epsilon'}$ on $\widetilde{L}_{\epsilon'}$. In addition, there is a sequence $\{\epsilon_\nu\}$ that converges to 0 such that for each tree Γ on $\widetilde{L}_\epsilon$, the images of the sequence $\{\Gamma_{\epsilon_\nu}\}$ converges to some $\widetilde{\Gamma}$ whose edges are *cascades*, described as follows.

If ϕ_e parametrizes a flowline of $H^\epsilon_{i,k} - H^\epsilon_{j,l}$ with $i \neq j$ and $k \neq l$, then ϕ^e converges to a flowline $\tilde{\phi}^e$ of $H_{i,k} - H_{j,l}$. As a consequence, its projection onto the factors is an SFT flowline of $f^1_i - f^1_j$ and an SFT flowline of $f^2_k - f^2_l$.

If ϕ_e parametrizes a flowline of $H^\epsilon_{i,k} - H^\epsilon_{i,l}$, then ϕ^e converges to a piecewise-smooth path $\tilde{\phi}^e = \tilde{\phi}^e_1 \cup \tilde{\phi}^e_2$, where $\tilde{\phi}^e_1$ parametrizes a flowline of $H_{i,k} - H_{i,l}$ and $\tilde{\phi}^e_2$ is a path in the critical submanifold $\mathbb{R} \times \{x\}$ for some critical point $x \in \mathrm{Crit}(f^2_k - f^2 l)$. After projecting, the image of $\tilde{\phi}^e$ is a Morse flowline on L_1 and an SFT flowline on L_2. The case of ϕ^e parametrizing a flowline of $H^\epsilon_{i,k} - H^\epsilon_{j,k}$ is analogous.

Thus, each Γ determines a unique pair of generalized gradient flow trees on L_1 and L_2. It is clear that this pair satisfies the criteria of Definition 5.9 and that there is an isometry of the underlying metric trees.

To prove the converse, we can build up flow trees on $\widetilde{L}_\epsilon$ from according to the procedure in Subsection 5.3. The strategy of the proof is as follows: as we build up the rigid trees on $\widehat{L}_\epsilon$ from minimal partial trees according to the algorithm of Subsection 5.3, we will show that there is a 1-1 correspondence between the partial trees we construct and pairs of generalized partial trees on the factors L_1, L_2. That is, minimal partial trees on $\widehat{L}_\epsilon$ correspond to pairs of minimal generalized partial trees on L_1, L_2, extending special punctures correspond to extending the special punctures on the corresponding pair of partial generalized trees, forming Y_1-vertices corresponds to extending a Morse edge through a cusp on L_1 or L_2, and forming Y_0-vertices correspond to forming a pair of Y_0 vertices and that such Y_0-vertices exist if and only if the metric length on the pairs of edges are the same.

Extending edges and Y_1-vertices. Y_1-vertices only occur by trying to extend a thin flowline through a cusp. Thus, it corresponds to extending a Morse edge on one tree and an SFT edge on the other. However, by Lemma **??**, it occurs in a D-neighborhood of a critical point of the difference function that the SFT edge is parametrizing. Thus, the SFT edge is essentially constant and extending through the Y_1-vertex corresponds to extending the Morse edge through a cusp.

Minimal partial trees. Now, we describe the correspondence between minimal partial trees and pairs of minimal generalized partial trees. In fact, to properly describe the correspondence, we need to relate pairs of minimal generalized partial trees to partial trees on $\widehat{L}_\epsilon$ that are not minimal. This is because the ribbon tree underlying a 2-valent puncture actually has 3 external vertices, which become 1 normal and 2 special punctures, a 3-valent vertex and the edge connecting this vertex to the nonspecial puncture parametrizes a constant flowline. However, in $\widehat{L}_\epsilon$, this edge may parametrize a nonconstant edge.

By Lemma 5.22, the only minimal partial trees we need to consider are 1-valent and 2-valent punctures. The following table describes the correspondence between the minimal partial trees of Lemma 5.22 and the generic pairs of vertices of generalized trees in Lemma 5.26.

Type 1 (SFT 1-valent puncture/ SFT 1-valent puncture)	1-valent puncture at critical point of $H^\epsilon_{i,k} - H^\epsilon_{j,l}$ with $i \neq j$ and $k \neq l$
Type 2 (SFT 1-valent puncture/ Morse puncture)	1-valent puncture at critical point of $H^\epsilon_{i,k} - H^\epsilon_{j,k}$ or $H^\epsilon_{i,k} - H^\epsilon_{i,l}$
Type 3 (SFT 2-valent puncture/ SFT 2-valent puncture)	2-valent puncture at critical point of $H^\epsilon_{i,k} - H^\epsilon_{j,l}$ with i, j, r distinct and k, l, s distinct
Type 4 (SFT 2-valent puncture/Morse 1-valent puncture $\cup$ Y_0-vertex (SFT/Mixed/Morse))	Type I, II, IV Y_0 vertex (respectively) $\cup$1-valent puncture at critical point of $H^\epsilon_{i,k} - H^\epsilon_{j,k}$ or $H^\epsilon_{i,k} - H^\epsilon_{i,l}$
Type 5 (SFT 2-valent puncture/ Mixed puncture)	2-valent puncture at critical point of $H^\epsilon_{i,k} - H^\epsilon_{j,l}$ with either $r = i$, $r = j$, $s = k$ or $s = l$.

Ends. If a special edge of a partial flow tree parametrizes a thin edge and its maximal domain of definition is compact, i.e. it can be extended to the caustic of $\widetilde{L}_\epsilon$, then in fact it can be closed off to form an end. Since this edge is thin, it corresponds to pair of SFT and Morse edges, which each must have an end. Thus, ends in $\widetilde{L}_\epsilon$ correspond to Type 8 in Lemma 5.26. However, Type 7 cannot occur, since the sheets defined by $H^\epsilon_{i,k}$ and $H^\epsilon_{j,l}$ do not meet at the caustic if $i \neq j$ and $k \neq l$.

Y_0**-vertices**. Finally, the remaining types of generic pairs of vertices in generalized trees correspond to Y_0-vertices as follows:

Type 6 (Mixed puncture/ Mixed puncture)	Type III Y_0 in neighborhood of $p \times q$ for $p \in \mathrm{Crit}(f_i^1 - f_j^1), q \in \mathrm{Crit}(f_k^2 - f_l^2)$
Type 9 (SFT Y_0/SFT Y_0)	Type I Y_0
Type 10 (SFT Y_0/Mixed Y_0)	Type II Y_0
Type 11 (SFT Y_0/Morse Y_0)	Type IV Y_0
Type 12 (Mixed Y_0/Mixed Y_0)	Type III Y_0

This completes the proof of Theorem D.

Chapter 6
Examples

6.1 Whitney spheres

A Whitney sphere W_c^n is the Legendrian sphere $S^n \in \mathbb{R}^{2n+1}$ given as the lift of the following exact Lagrangian immersion in $\mathbb{C}^n \simeq \mathbb{R}^{2n}$:

$$w : S^n = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : |x|^2 + y^2 = 1\} \mapsto c(1 + iy)x$$

where c is some positive real constant. This immersion has exactly 1 transverse double point at $w(0, 1) = w(0, -1)$ and so the Legendrian W_c^n has exactly one Reeb chord, of length determined by c.

Example 6.1. Consider the product $W_a^1 \times W_b^2$ of a single 1-sphere and a single 2-sphere. Since the parities of the dimensions are different, the dimension of the product is odd and the Thurston-Bennequin invariant will be useful. We have

$$\begin{array}{ll} tb(W_a^1) = -1 & \chi(T^*S^1) = 0 \\ tb(W_b^2) = 1 & \chi(T^*S^2) = -2 \end{array}$$

Denote the Reeb chords a and b. The τ factor for this pair is

$$\tau(a, b) = \begin{cases} (-1)^1 = -1 & \text{if } a < b \\ (-1)^2 = 1 & \text{if } a > b \end{cases}$$

Thus

$$tb(W_a^1 \times W_b^2) = \begin{cases} 2 & \text{if } a < b \\ 0 & \text{if } a > b \end{cases}$$

Now, consider the product $W_a^1 \times W_b^4$ of a single 1-sphere and a single 4-sphere. We have

$$\begin{array}{ll} tb(W_a^1) = -1 & \chi(T^*S^1) = 0 \\ tb(W_b^4) = 1 & \chi(T^*S^4) = 2 \end{array}$$

The τ factor for this pair is

$$\tau(a, b) = \begin{cases} (-1)^1 = -1 & \text{if } a < b \\ (-1)^4 = 1 & \text{if } a > b \end{cases}$$

Thus

$$tb(W_a^1 \times W_b^4) = \begin{cases} -2 & \text{if } a < b \\ 0 & \text{if } a > b \end{cases}$$

Note that for both examples, the case $a > b$ was already calculated in [EES05b].

Example 6.2. Let W_a^1, W_b^1, W_e^1 be three Whitney unknots. To calculate the Thurston-Bennequin invariant of $W_a^1 \times W_b^1 \times W_e^1$ apply theorem B twice. The torus $W_a^1 \times W_b^1$ has $tb = 0$ and $\chi(T^*T^2) = 0$ and, after perturbing, has six Reeb chords with actions

$$\mathcal{Z}(a \otimes m_i) \approx \mathcal{Z}(a)$$

$$\mathcal{Z}(m_i \otimes b) \approx \mathcal{Z}(b)$$

$$\mathcal{Z}(c) \approx |\mathcal{Z}(a) - \mathcal{Z}(b)|$$

$$\mathcal{Z}(d) \approx \mathcal{Z}(a) + \mathcal{Z}(b)$$

and signs

$$\sigma(a \otimes m_i) = (-1)\sigma(a)(-1)^i \qquad \sigma(m_i \otimes b) = (-1)\sigma(b)(-1)^i$$

$$\sigma(c) = (-1)^2\sigma(a)\sigma(b) \qquad \sigma(d) = (-1)\sigma(a)\sigma(b)$$

where m_i is a unique Morse critical point of index i. Since $tb(W_a^1 \times W_b^1) = \chi(T^*T^2) = \chi(T^*S^1) = 0$ the only potential nonzero term in theorem B is the τ-term when computing tb for the triple product. Note that for the A chords, $\tau(a \otimes m_0, e) = \tau(a \otimes m_1, e)$, so that

$$\tau(a \otimes m_0, e)\sigma(a \otimes m_0)\sigma(e) + \tau(a \otimes m_1, e)\sigma(a \otimes m_1)\sigma(e) = 0$$

since the signs of the Morse critical points m_0, m_1 cancel. Similarly, the τ terms involving the B chords cancel. Now,

$$\tau(c, e)\sigma(c)\sigma(e) = \begin{cases} 1 & \text{if } |a - b| < e \\ -1 & \text{if } |a - b| > e \end{cases}$$

$$\tau(d, e)\sigma(d)\sigma(e) = \begin{cases} -1 & \text{if } a + b < e \\ 1 & \text{if } a + b > e \end{cases}$$

since $\tau(a, b) = -1$ and $\sigma(a) = \sigma(b) = \sigma(c) = -1$. Thus,

$$tb(W_a^1 \times W_b^1 \times W_e^1) = \begin{cases} 2 & \text{if } a + b > e \text{ and } |a - b| < e \\ 0 & \text{otherwise} \end{cases}$$

In other words, $tb = 2$ if (a, b, c) satisfy the triangle inequality and $tb = 0$ otherwise.

6.2 Knots

The product of two Legendrian knots is a torus, whose Euler characteristic and Thurston-Bennequin invariant are 0. However, there are interesting tb calculations for products of three Legendrian knots.

Example 6.3. Let K_1, K_2, K_3 be a collection of three Legendrian knots with Reeb chords $\{a_i\}, \{b_j\}, \{c_k\}$. Define

$$\tau(a,b,c) = \begin{cases} 2 & \text{if } a+b > c \text{ and } |a-b| < c \\ 0 & \text{otherwise} \end{cases}$$

Then the arguments in example 5.2 can be repeated for each triple of Reeb chords (a,b,c) and the Thurston-Bennequin invariant is

$$tb(K_1 \times K_2 \times K_3) = \sum_{i,j,k} \tau(a_i, b_j, c_k)\sigma(a_i)\sigma(b_j)\sigma(c_k)$$

Take K_1 to be a once-stabilized unknot, K_2 a standard unknot after applying a Reidemeister-1 move, and K_3 a right-handed trefoil. These can be chosen to have the front projections and Lagrangian projections depicted in figure 1.

K_1 has two Reeb chords a_1, a_2; K_2 has three Reeb chords b_1, b_2, b_3; and K_3 has five Reeb chords c_1, c_2, c_3, c_4, c_5. By abuse of notation, a_i, b_j, c_k will refer to both the chord and its action. These chords have signs

$$\begin{aligned} &\sigma(a_1) = \sigma(a_2) = -1 \\ &\sigma(b_1) = \sigma(b_2) = -1 &&& \sigma(b_3) = 1 \\ &\sigma(c_1) = \sigma(c_2) = -1 && \sigma(c_3) = \sigma(c_4) = &\sigma(c_5) = 1 \end{aligned}$$

so $tb(K_1) = -2$, $tb(K_2) = -1$ and $tb(K_3) = 1$. The Reeb chord actions cannot be chosen completely arbitrarily as each face of the knot diagram in the Lagrangian projection must satisfy the area identity determined by Stokes's Theorem. For a face F of K, the boundary ∂F lies in $\overline{K}$ and the double points of the knot projection split the boundary into segments $\{\overline{\gamma_l}\}$, indexed counter-clockwise. A corner of the face is *positive* if near the Reeb chord, the z-coordinate of γ_{l+1} is greater than the z-coordinate of γ_l and *negative* otherwise.

Lemma 6.4. *(Area Identity) Let A be a face of the knot diagram in the Lagrangian projection and $\gamma = \partial A$ be its boundary. Then*

$$\int_A \omega = \sum_p \mathcal{Z}(p) - \sum_n \mathcal{Z}(n)$$

where p, n are the Reeb chords corresponding to the positive and negative corners of A

Since each face of a knot diagram must have positive area, it follows that

$$\begin{gathered} b_1, b_2 > b_3 \\ c_1, c_2 > c_3, c_4, c_5 \end{gathered}$$

and that this is the only restriction on the actions. With this in mind and without loss of generality, let a, b_+, b_-, c_+, c_- refer to some chord in the sets $\{a_1, a_2\}$, $\{b_1, b_2\}$, $\{b_3\}$, $\{c_1, c_2\}$, $\{c_3, c_4, c_5\}$, respectively.

The minimum tb is achieved if all triples $(a, b_+, c_+), (a, b_-, c_-)$ satisfy the triangle inequality and none of the triples $(a, b_+, c_-), (a, b_-, c_+)$ satisfy it, since $\sigma(a)\sigma(b_+)\sigma(c_+) = \sigma(a)\sigma(b_-)\sigma(c_-) = -1$ and $\sigma(a)\sigma(b_+)\sigma(c_-) = \sigma(a)\sigma(b_-)\sigma(c_+) = 1$. Then

$$tb = (8)(2)(-1) + (6)(2)(-1) + 12(2)(0) + 4(2)(0) = -28$$

For instance, set $a = 5$, $b_+ = c_+ = 10$ and $b_- = c_- = 3$.

Now, for a 5-tuple (a, b_+, b_-, c_+, c_-), if both triples $(a, b_+, c_-), (a, b_-, c_+)$ satisfy the triangle inequality, then (a, b_+, c_+) must as well since $b_+ > b_-$ and $c_+ > c_-$. However, this is not true if only one such triple does. Thus, since there are more c_- chords than b_- chords, the maximum tb is achieved if all possible triples (a, b_+, c_-) satifies the triangle inequality but no other triple does. For instance, set $a = 5$, $b_+ = 6$, $c_+ = 12$ and $b_- = c_- = 2$. Then

$$tb = (8)(2)(0) + (6)(2)(0) + 12(2)(1) + 4(2)(0) = 24$$

By switching the values assigned to the b's and c's, it's possible to achieve many tb values between -28 and 24.

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