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Power series expansions for waves in high-contrast plasmonic crystals

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POWER SERIES EXPANSIONS FOR WAVES
IN HIGH-CONTRAST PLASMONIC CRYSTALS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
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in

The Department of Mathematics

by

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“Any problem which can be understood can also be solved.”

-The Buddha
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To my family.
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Abstract

In this thesis, a method is developed for obtaining convergent power series expansions for dispersion relations in two-dimensional periodic media with frequency dependent constitutive relations. The method is based on high-contrast expansions in the parameter \( \eta = \frac{2\pi d}{\lambda} \), where \( d \) is the period of the crystal cell and \( \lambda \) is the wavelength. The radii of convergence obtained are not too small, on the order of \( \eta \approx 10^{-2} \). That the method applies to frequency dependent media is an important fact, since the majority of the methods available in the literature are restricted to frequency independent constitutive relations. The convergent series for the dispersion relation is used to define an effective property valid for finite cell structure sizes, as opposed to a quasi-static property, valid only in the limit \( \eta \to 0 \).
Chapter 1
Introduction

1.1 Waves in Periodic Media: Differential Equations with a Small Parameter

The electromagnetic properties of a medium with periodic microstructure are described by a rapidly oscillating coefficient $A(x/d)$, periodic with the microstructure period $d$. For a non-magnetic medium, the coefficient $A$ represents the dielectric permittivity, so that the magnetic field, $u$, of a polarized electromagnetic wave in the medium is described by the differential equation

$$-\nabla \cdot (A(x/d)\nabla u) = \frac{\omega^2}{c^2} u,$$

(1.1.1)

where $\omega$ is the frequency of the wave in the medium and $c$ is the speed of light in vacuum. We are interested in two-dimensional media with periodic microstructure. The period cell has an inclusion, as depicted in figure (1.1), with a $C^2$ boundary. This regularity property of the boundary will be assumed throughout the rest of this thesis. The inclusion is allowed to be multiply connected. The coefficient $A$ is then given in the unit period cell by

$$A = \begin{cases} 
\epsilon_p^{-1} & \text{in the inclusion}, \\
\epsilon_{\bar{p}}^{-1} & \text{in the host}.
\end{cases}$$

FIGURE 1.1. Unit cell with plasmonic inclusion.
The change of variables

\[ y = d^{-1}x \Rightarrow \nabla_x = d^{-1}\nabla_y \]

applied to equation (1.4.3) then gives the following boundary value problem in the unit cell

\[
\begin{cases}
\Delta_y u + d^2 \epsilon_p \frac{\omega^2}{c^2} u = 0, \quad y \in \bar{P} \\
\Delta_y u + d^2 \epsilon_p \frac{\omega^2}{c^2} u = 0, \quad y \in P \\
\epsilon_p^{-1} \nabla_y u \bigg|_p \cdot n = \epsilon_p^{-1} \nabla_y u \bigg|_{\bar{P}} \cdot n, \quad y \in \partial P.
\end{cases}
\] (1.1.2)

The above is an example of a *boundary value problem with a parameter*, the parameter in this case being the cell size \(d\) introduced through the change of variables. In effect, we shall be dealing with the Bloch wave version of (1.1.2), which is obtained as follows. Consider a Bloch wave ansatz for the field, that is, assume \(u = e^{i\mathbf{k} \cdot \mathbf{x}} h(x)\), where \(\mathbf{k} \in \mathbb{R}\) is the Bloch wave vector and \(h\) is \(d\)-periodic in \(x_1\) and \(x_2\), where \(\mathbf{x} = (x_1, x_2)\). Inserting the Bloch ansatz into (1.1.2), we obtain

\[
\begin{cases}
(\nabla_y + i\eta \hat{\kappa})(\nabla_y + i\eta \hat{\kappa})h + \epsilon_p \eta^2 \xi_n^2 h = 0, \quad y \in \bar{P} \\
(\nabla_y + i\eta \hat{\kappa})(\nabla_y + i\eta \hat{\kappa})h + \epsilon_p \eta^2 \xi_n^2 h = 0, \quad y \in P \\
\epsilon_p^{-1}(\nabla_y + i\eta \hat{\kappa})h \bigg|_p \cdot n = \epsilon_p^{-1}(\nabla_y + i\eta \hat{\kappa})h \bigg|_{\bar{P}} \cdot n, \quad y \in \partial P.
\end{cases}
\] (1.1.3)

where

\[ \xi_n^2 = \frac{\omega^2}{c^2 \kappa^2} \]

and the parameter is now

\[ \eta = \frac{2\pi k}{d}, \]
where \( k = |k| \). Note that \( \eta \) is dimensionless. In this thesis, we shall focus on a plasmonic crystal, which corresponds to setting

\[
\epsilon_\tilde{p} = 1 \quad \text{and} \quad \epsilon_p = 1 - \frac{\omega_p^2}{\omega^2},
\]

where the constant \( \omega_p \) is specified by the electron density of the plasmon trapped inside the inclusion [19]. The number \( \omega_p \) will not be independent of the cell size, but shall scale according to [36]

\[
\omega_p = \frac{c}{d} = \frac{kc}{\eta}. \tag{1.1.4}
\]

Inserting this scaling into the expression for \( \omega_p \), we obtain

\[
\epsilon_p = 1 - \frac{1}{\eta^2}\xi_\eta^2,
\]

and the limit \( \eta \to 0 \) then gives the boundary value problem

\[
\begin{align*}
\Delta_y h_0 &= 0, \quad y \in \tilde{P} \\
\Delta_y h_0 &= h_0, \quad y \in P \\
\nabla_y h_0 \cdot n &= 0, \quad y \in \partial P
\end{align*} \tag{1.1.5}
\]

This same result is also obtained when \( d \to 0 \) in (1.1.2). Note that the frequency term \( \xi^2 \) is absent from (1.1.5). The main result of this thesis is that (1.1.3) has a solution \( \{h_\eta, \xi_\eta^2\} \) and that the power series representations

\[
h_\eta = h_0 + h_1 \eta + h_2 \eta^2 + \ldots \quad \text{and} \quad \xi_\eta^2 = \xi_0^2 + \xi_1^2 \eta + \xi_2^2 \eta^2 + \ldots \tag{1.1.6}
\]

are convergent for \( \eta \leq R \), where \( R > 0 \). In fact, the issue of existence is settled by directly verifying that the power series are a solution of (1.1.3), after convergence has been proved (see theorem (4.1.2)). The number \( R \) is only a lower bound on the radius of convergence. To have an idea of the smallness of the convergence radius, \( R = 1/340 \) for a circular inclusion with radius \( r = 0.45d \). We remark that the series above represent only the first branch of the dispersion relation.
1.1.1 Other Choices of Scaling for $\epsilon_p$

The fact that the frequency $\omega^2$ does not appear in the boundary value problem (1.1.5) that defines the first term of the series expansion is particular to the scaling (1.1.4). Another choice of scaling that has been investigated recently [5], although we do not treat it in this thesis, is

$$\epsilon_p = 1 \text{ and } \epsilon_p = \epsilon_r/d^2.$$  

The limit $d \to 0$ gives

$$\begin{cases} 
\Delta_y u_0 = 0, \ y \in \bar{P} \\
\Delta_y u_0 + \epsilon_p \frac{\omega^2}{c^2} u_0 = 0, \ y \in P \\
\nabla_y u_0 \big|_{\partial P} \cdot n = 0, \ y \in \partial P 
\end{cases}$$

(1.1.7)

The frequency $\omega^2$ does not disappear and we obtain an eigenvalue problem. There exists a countable infinity of branches for the dispersion relation determined by the eigenvalues of the $u_0$ problem above, as opposed to the single branch passing through the origin obtained for the scaling $\omega_p = c/d$. It would be very interesting to obtain a convergent series expansion for this type of scaling.

The idea of power series expansions in the study of differential equations with a small parameter is very old in mathematics. It was used by Hermann Schwarz in 1885 to give the first rigorous existence proof for the smallest eigenvalue of the Laplacian in a bounded domain. Reference [16] contains an extremely interesting account of the recursive scheme of Schwarz and of its connection to various developments in the history of functional analysis. In this thesis, we obtain such a recursive scheme for the coefficients of the series solution (1.1.6) to the eigenvalue problem $-\nabla \cdot (A(x/d)\nabla u) = \frac{\omega^2}{c^2} u$ in the first branch of the dispersion relation.
Due to the appearance of the series product $\frac{\omega^2}{c^2} u$, the recursion estimates lead to inequalities involving convolution products and certain techniques of combinatorial analysis are required to obtain the exponential bound for the series coefficients.

1.2 Negative Permittivities and Resonance

The so-called “energy equation” [23] for (1.1.1) can be obtained by multiplying both sides of the equation by the complex conjugate $\bar{u}$ of the magnetic field and then using integration by parts. The result is

$$\frac{1}{\epsilon_{\bar{p}}} \int_P |\nabla u|^2 dV + \frac{1}{\epsilon_p} \int_P |\nabla u|^2 dV = \frac{\omega^2}{c^2} \int_Q |u|^2 dV, \quad (1.2.1)$$

where $dV$ denotes the area element. If $\epsilon_p$ and $\epsilon_{\bar{p}}$ have opposite signs, it is conceivable that the terms on the left-hand side could become individually very large, but with cancellations in such a manner that the right-hand side remains small (since $\nabla y h$ is the electric field, except for a rotation and a multiplicative constant, physically this amounts to a large electric field). The convergence for the magnetic field and the frequency established in this thesis show that all three terms in the above equation are small when $\eta$ is small, so that this phenomenon cannot occur. Since the convergence is established only for the first branch of the dispersion relation there is still the possibility of resonances in the higher branches (numerical simulations indicate that this is indeed the case [36]).

1.3 Influence of the Topology of the Host Phase on the Dispersion Relation

In the discussion above, the plasmonic inclusions were isolated from one another by the host phase, which formed a connected region. It turns out that this topological property affects the dispersion relation in a significant way. In chapter 7, we present an analysis of the dispersion relation for a periodic layered medium in which the
permittivity is a step function constant in each layer. For such a structure, the host phase is disconnected. It is shown that when the layer permittivities have opposite signs and the layer thicknesses are sufficiently thin, the first passband does not include the origin \( \omega = 0 \), i.e., as \( k \to 0 \), \( \omega \) approaches some positive number, instead of approaching zero. Since the first term of the power series for \( \xi_n^2 \) is proportional to \( k^2 \) (see (3.2) in the appendix), the first passband must necessarily pass through the origin in those cases (heuristically, it is useful to think that the waves can somehow propagate around the inclusions when the host phase is connected).

### 1.4 The State of the Art in the Study of Dispersion Relations in Plasmonic Crystals

Sub-wavelength plasmonic crystals are a class of *meta-material* that possesses a microstructure consisting of a periodic array of plasmonic inclusions embedded within a dielectric host. The term “sub-wavelength” refers to the regime in which the period of the crystal is smaller than the wavelength of the electromagnetic radiation traveling inside the crystal. Many recent investigations into the behavior of meta-materials focus on phenomena associated with the *quasi-static limit* in which the ratio of the period cell size to wavelength tends to zero. Sub-wavelength micro-structured composites are known to exhibit effective electromagnetic properties that are not available in naturally-occurring materials. Investigations over the past decade have explored a variety of meta-materials, including arrays of micro-resonators, wires, high-contrast dielectrics, and plasmonic components. The first two, especially in combination, have been shown to give rise to unconventional bulk electromagnetic response at microwave frequencies [38, 32, 33] and, more recently, at optical frequencies [35], including negative effective dielectric permittivity and/or negative effective magnetic permittivity. An essential ingredient in creat-
ing this response are local resonances contained within each period due to extreme properties, such as high conductivity and capacitance in split-ring resonators [33].

In the case of plasmonic crystals, the dielectric permittivity \( \epsilon_p \) of the inclusions is frequency dependent and negative for frequencies below the plasma frequency \( \omega_p \),

\[
\epsilon_p(\omega) = 1 - \frac{\omega_p^2}{\omega^2}.
\]  

(1.4.1)

Shvets and Urzhumov [36] have investigated plasmonic crystals in which \( \omega_p \) is inversely proportional to the period of the crystal and for which both inclusion and host materials have unit magnetic permeability. They have proposed that simultaneous negative values for both an effective \( \epsilon \) and \( \mu \) arise at sub-wavelength frequencies that are quite far from the quasi-static limit, that is,

\[
\eta = kd = \frac{2\pi d}{\lambda}
\]  

(1.4.2)

is not very small, where \( d \) is the period of the crystal, \( k \) is the norm of the Bloch wavevector and \( \lambda \) is the wavelength. In this work, we present rigorous analysis of this type of plasmonic crystal by establishing the existence of convergent power series in \( \eta \) for the electromagnetic fields and the first branch of the associated dispersion relation. The effective permittivity and permeability defined according to Pendry [33] are shown to be positive for all \( \eta \) within the radius of convergence \( R \), and, in this regime, the extreme property of the plasma produces no resonance in the effective permittivity or permeability. This regime is well distanced from the resonant regime investigated in Shvets and Urzhumov [36].

The analysis shows that the radii of convergence of the power series is at least \( R_m \), which is not too small, as shown in Table (1.4), which contains values of \( R_m \) for circular inclusions of various radii \( rd \). The number \( R_m \) can be put in physical perspective by fixing the cell size and introducing the lower bound \( \lambda_m \) and the
TABLE 1.1. Lower bounds on the radii of convergence $R$ for circular inclusions of radii $rd$.  

<table>
<thead>
<tr>
<th>$r$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.45</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_m$</td>
<td>1/60</td>
<td>1/68</td>
<td>1/88</td>
<td>1/96</td>
<td>1/340</td>
</tr>
</tbody>
</table>

TABLE 1.2. Values of $\lambda_m$ and $k_M$ for circular inclusions of radii $rd$ when $d = 10^{-7} m$.  

<table>
<thead>
<tr>
<th>$r$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.45</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_m$</td>
<td>38 $\mu m$</td>
<td>43 $\mu m$</td>
<td>56 $\mu m$</td>
<td>73 $\mu m$</td>
<td>214 $\mu m$</td>
</tr>
<tr>
<td>$k_M$</td>
<td>$1.6 \cdot 10^5 m^{-1}$</td>
<td>$1.4 \cdot 10^5 m^{-1}$</td>
<td>$1.1 \cdot 10^6 m^{-1}$</td>
<td>$1.0 \cdot 10^5 m^{-1}$</td>
<td>$3.0 \cdot 10^4 m^{-1}$</td>
</tr>
</tbody>
</table>

upper bound $k_M$. Table 2 contains values of $\lambda_m$ and $k_M$ when $d = 10^{-7} m$. The wavelengths lie in the infrared range and the plasma frequency is $\omega_p = 10^{15} \text{sec}^{-1}$.

We focus on harmonic H-polarized electromagnetic waves in a lossless composite medium consisting of a periodic array of plasmonic rods embedded in a non-magnetic frequency-independent dielectric host material. The rod-host configurations are restricted to those with rectangular symmetry, i.e., configurations invariant under a 180° rotation about the center of the unit cell.

The regime of interest for this investigation is that in which

1. the plasma frequency $\omega_p$ is high

2. the ratio of the cell width to the wavelength is small ($\eta \ll 1$).

From the formula $\epsilon_p = 1 - \omega_p^2 / \omega^2$, it is seen that a high plasma frequency $\omega_p$ gives rise to a large and negative dielectric permittivity $\epsilon_p$ in the plasmonic inclusions. Following [36], the plasma frequency is related to the cell size by

$$\omega_p = \frac{c}{d}.$$  

This results in the relation

$$\epsilon_p = 1 - \frac{c^2}{\omega^2 d^2},$$

where $c$ is the speed of light in vacuum. The governing family of differential equations for the magnetic field is the Helmholtz equation with a rapidly oscillating
coefficient

\[-\nabla \cdot (A(x/d) \nabla u) = \frac{\omega^2}{c^2} u, \tag{1.4.3}\]

in which \( A \) is the matrix defined on the unit period of the crystal by

\[A(y) = \begin{cases} \epsilon_p^{-1} I & \text{in the plasmonic phase,} \\ \epsilon_r^{-1} I & \text{in the host phase,} \end{cases}\]

\( I \) is the identity matrix and \( \epsilon_p = 1 \) and \( \epsilon_r = 1 - \frac{c^2}{\omega^2 d^2} \).

The coefficient \( A \) is not coercive in the regime \( \omega_r \geq \omega \), since \( \epsilon_r \) is negative in this regime.

In Theorem (4.1.2), we obtain the following series expansion for the frequency \( \omega^2 \)

\[\omega^2 = \omega^2_\eta = c^2 k^2 \sum_{m=0}^{\infty} \xi_{2m}^2 \eta^{2m} \tag{1.4.4}\]

in which \( \xi_{2m}^2 \) is a tensor of degree \( 2m + 2 \) in \( \hat{\kappa} \). This gives rise to a convergent power series for an effective index of refraction \( n_{\text{eff}}^2 \) defined through

\[n_{\text{eff}}^2 = \frac{c^2 k^2}{\omega^2_\eta}. \tag{1.4.5}\]

The effective property \( n_{\text{eff}}^2 \) is well defined for all \( \eta \) in the radius of convergence and is not phenomenological in origin but instead follows from first principles using the power series expansion. Interpreting the first term of this series as the quasi-static index of refraction \( n_{\text{qs}}^2 \), the remaining terms then provide the dynamic correctors of all orders. In section 6, we define the effective permeability \( \mu_{\text{eff}} \) and prove that \( n_{\text{eff}}^2 \) and \( \mu_{\text{eff}} \) are both positive for \( \eta \) in some interval \((0, \eta_0]\) and that a mild effective magnetic response emerges for the homogenized composite, even though the component materials are non-magnetic (\( \mu_P = \mu_{\text{P}} = 1 \)). Having defined
\( n_{\text{eff}}^2 \) and \( \mu_{\text{eff}} \), the effective electrical permittivity \( \epsilon_{\text{eff}} \) can be defined through the equation

\[
n_{\text{eff}}^2 = \mu_{\text{eff}}\epsilon_{\text{eff}},
\]

so that \( \epsilon_{\text{eff}} \) is positive whenever both \( n_{\text{eff}}^2 \) and \( \mu_{\text{eff}} \) are positive. Thus, one has a solid basis on which to assert that plasmonic crystals function as materials of positive index of refraction in which both the effective permittivity and permeability are positive. The method developed here can be applied to other types of frequency-dependent dielectric media such as polaratonic crystals. From a physical perspective, this work provides the first explicit description of Bloch wave solutions associated with the first propagation band inside nanoscale plasmonic crystals. In the context of frequency independent dielectric inclusions, the first two terms of \( n_{\text{eff}}^2 \) are identified via Rayleigh sums in [28].

To emphasize the difference between effective properties defined for meta-material structures where the crystal period \( d \) is fixed and effective properties defined in the quasistatic limit, i.e., \( k \) fixed and \( d \to 0 \), we refer to the latter as quasistatic effective properties and denote these with the subscript \( \text{qs} \). The situation considered in this thesis contrasts with the case in which \( \epsilon \approx d^{-2} \) in the inclusion and is large and positive, investigated by Bouchitté and Felbacq [5]. In that case for \( \eta \to 0 \), \( \mu_{\text{qs}}(\omega) \) has poles at Dirichlet eigenvalues of the inclusion and therefore is negative in certain frequency intervals (see also Bouchitté and Felbacq [6, 8, 9]). In fact, what allows us to prove convergence of the power series in the plasmonic case is precisely the absence, due to negative \( \epsilon_p \), of these internal Dirichlet resonances.

In the regime where \( \epsilon_p \) is negative and large, the perturbation methods used for describing Bloch waves in heterogeneous media developed in Odeh and Keller [30, 15, 4] cannot be applied. Our analysis instead makes use of the fact that \( \epsilon_p \) is
negative and large for sub-wavelength crystals and develops high-contrast power series solutions for the nonlinear eigenvalue problem that describes the propagation of Bloch waves in plasmonic crystals. The convergence analysis takes advantage of the iterative structure appearing in the series expansion and is inspired by a technique of Bruno [11] developed for series solutions to quasi-static field problems. We prove that the series converges to a solution of the harmonic Maxwell system for ratios of cell size to wavelength that are not too small. Indeed for typical values of the plasma frequency $\omega_p$ the analysis delivers convergent series solutions for nano scale plasmonic rods at infrared wavelengths.

In section (5) we compute the first two terms of the dispersion relation for circular inclusions Shvets and Urzhumov [36, 37] and provide explicit bounds on the relative error committed upon replacing the full series with its first two terms. The error is seen to be less than 3% for values of $\eta$ up to 20% of the convergence radius, so that the two-term approximation provides a numerically fast and accurate approximation to the dispersion relation.

The high contrast in $\epsilon$ gives rise to effective constants $\epsilon_{\text{eff}}$ and $\mu_{\text{eff}}$. In the bulk relation

$$B_{\text{eff}} = \mu_{\text{eff}} H_{\text{eff}}, \quad (1.4.6)$$

where $B_{\text{eff}}$ is the average over the period cell (a flux), whereas $H_{\text{eff}}$ is the average of $H_3$ over line segments in the matrix parallel to the rods. Taking the ratio of $B_{\text{eff}}/H_{\text{eff}}$ delivers an effective magnetic permeability and one recovers magnetic activity from meta-materials made from non-magnetic materials. This phenomenon was understood by Pendry [33], and has been made rigorous in the quasistatic limit through two-scale analysis in several cases. These include the two-dimensional arrays of inclusions in which $\epsilon$ scales as $d^{-2}$ [5, 6, 8, 9, 33], two dimensional arrays
of ring resonators whose surface conductivity scales as $d^{-1}$ [25], as well as three-dimensional arrays of split-ring wire resonators in which the conductivity scales as $d^{-2}$ [7]. This “non-standard” homogenization has been understood for some decades in problems of porous media and imperfect interface [13, 2, 27, 17, 41] and recently has given rise to interesting effects in composites of both high contrast and high anisotropy [12, 40].

The two-scale analysis in these cases relies on the coercivity of the underlying partial differential equations. The problem of plasmonic inclusions, however, is not coercive because $\varepsilon$ is negative in the plasma—but it is precisely this negative index that underlies the convergence of the power series. As we shall see, the uniqueness of the solution of the Dirichlet problem for $\triangle u - u = 0$ in the plasmonic inclusion gives exponential bounds on the coefficients of the series, which allows us to prove that it converges to a solution of the differential equation (1.4.3). This result presented here shows that by considering a finite number of terms in the series, one has an approximation of the true solution, to any desired algebraic order of convergence. In this context we point out the recent work of [39, 24, 31] that shows that the power series expressed by the formal two scale expansion of Bakhvalov and Panasenko [3] is an asymptotic series in certain cases under the hypothesis that the coefficient $A$ is coercive.
Chapter 2
Mathematical Formulation

2.1 The Boundary Value Problem

We introduce the nonlinear eigenvalue problem describing the propagation of Bloch waves inside a plasmonic crystal and provide the context for the power series approach to its solution.

For points $\mathbf{x} = (x_1, x_2)$ in the $x_1x_2$-plane, the $d$-periodic dielectric coefficient of the crystal is denoted by $\epsilon(\omega, \mathbf{x})$, where

$$
\epsilon(\omega, \mathbf{x}) = \begin{cases} 
\epsilon_p(\omega) & \text{for } \mathbf{x} \in P, \\
\epsilon_{\bar{p}} & \text{for } \mathbf{x} \in \overline{P}.
\end{cases}
$$

Both materials are assumed to have unit magnetic permeability, $\mu_p = \mu_{\bar{p}} = 1$.

We assume a Bloch-wave form of the field, where $\hat{\kappa} = (\kappa_1, \kappa_2)$ is the unit vector along the direction of the traveling wave and $k = 2\pi/\lambda$ is the wave number for a wave of length $\lambda$. The magnetic and electric fields are denoted by $\mathbf{H} = (H_1, H_2, H_3)$ and $\mathbf{E} = (E_1, E_2, E_3)$ respectively. For $H$-polarized time-harmonic waves, the non-vanishing field components are

$$
H_3 = H_3(\mathbf{x})e^{i(k\hat{\kappa}\cdot\mathbf{x} - \omega t)}, \quad E_1 = E_1(\mathbf{x})e^{i(k\hat{\kappa}\cdot\mathbf{x} - \omega t)}, \quad E_2 = E_2(\mathbf{x})e^{i(k\hat{\kappa}\cdot\mathbf{x} - \omega t)} \quad (2.1.1)
$$

in which the fields $H_3(\mathbf{x})$, $E_1(\mathbf{x})$, and $E_2(\mathbf{x})$ are continuous and $d$-periodic in both $x_1$ and $x_2$. The Maxwell equations take the form of the Helmholtz equation (1.4.3), in which substitution of $u = H_3(\mathbf{x})e^{i(k\hat{\kappa}\cdot\mathbf{x} - \omega t)}$ gives

$$
-(\nabla + ik\hat{\kappa})\epsilon_p^{-1}(\nabla + ik\hat{\kappa})H_3 = \frac{\omega^2}{c^2}H_3 \quad \text{in the rods,} \quad (2.1.2)
$$

$$
-(\nabla + ik\hat{\kappa})\epsilon_{\bar{p}}^{-1}(\nabla + ik\hat{\kappa})H_3 = \frac{\omega^2}{c^2}H_3 \quad \text{in the host material,} \quad (2.1.3)
$$
where $H_3$ satisfies the transmission conditions on the interface between the rods and host material given by

$$n \cdot (\epsilon_p^{-1}(\nabla + i\kappa H_3)|_p) = n \cdot (\epsilon_p^{-1}(\nabla + i\kappa H_3)|_b).$$  \hspace{1cm} (2.1.4)

Here, the subscripts indicate the side of the interface where the quantities are evaluated and $\mathbf{n}$ is the unit normal vector to the interface pointing into the host material. We denote the unit vector pointing along the $x_3$ direction by $\mathbf{e}_3$, and the electric field component of the wave is given by $\mathbf{E} = -\frac{ic}{\omega} \mathbf{e}_3 \times \nabla H_3$.

For each value of the wave-vector $\mathbf{k}$, equations ((2.1.2), (2.1.3), (2.1.4)) provide a nonlinear eigenvalue problem for the pair $H_3$ and $\omega^2$. One of the main results of this work is to show that this problem is well posed by explicitly constructing solutions using power series expansions. In order to develop the appropriate expansions, we rewrite ((2.1.2), (2.1.3), (2.1.4)) in terms of $\eta$ and a dimensionless variable $\mathbf{y}$ in $\mathbb{R}^2$ that normalizes a period cell to the unit square $Q = [0,1]^2$, $\mathbf{x} = \mathbf{y}d = \mathbf{y} \eta/k$. We define the $Q$-periodic function

$$h(\mathbf{y}) = H_3(\mathbf{y}d)$$

and for convenience of notation, we redefine

$$\epsilon(\omega, \mathbf{y}) = \begin{cases} 
\epsilon_p(\omega) & \text{for } \mathbf{y} \in P, \\
\epsilon_p & \text{for } \mathbf{y} \in \overline{P},
\end{cases}$$  \hspace{1cm} (2.1.5)

to arrive at the eigenvalue problem that requires the pair $h(\mathbf{y})$ and $\omega^2$ to be a solution of the master system

$$
\begin{aligned}
-(\nabla \mathbf{y} + i\eta \kappa)\epsilon^{-1}(\omega, \mathbf{y})h(\mathbf{y})(\nabla \mathbf{y} + i\eta \kappa) = \eta^2 \frac{\omega^2}{c^2} h(\mathbf{y}) & \text{ for } \mathbf{y} \in P \cup \overline{P}, \\
\mathbf{n} \cdot \epsilon_p^{-1}(\omega)(\nabla \mathbf{y} + i\eta \kappa)h(\mathbf{y})|_p = \mathbf{n} \cdot \epsilon_p^{-1}(\nabla \mathbf{y} + i\eta \kappa)h(\mathbf{y})|_b & \text{ for } \mathbf{y} \in \partial P.
\end{aligned}$$  \hspace{1cm} (2.1.6)
We prove in Theorem (4.1.2) that this eigenvalue problem can be solved by constructing explicit convergent power series solutions.

2.2 Power Series Expansions

We take \( \eta \) to be the expansion parameter for the field \( h(y) \) and the frequency \( \omega^2 \),

\[
\begin{align*}
  h_\eta &= h_0 + \eta h_1 + \eta^2 h_2 + \ldots, \\
  \omega^2_\eta &= \omega^2_0 + \eta \omega^2_1 + \eta^2 \omega^2_2 + \ldots,
\end{align*}
\]

in which the functions \( h_m \) are periodic with period cell \( Q \).

Inserting (2.2.1) into (2.1.6) and identifying coefficients of like powers of \( \eta \) on the right- and left-hand sides yields the equations

\[
\begin{cases}
  \Delta h_m + 2i\hat{\kappa} \cdot \nabla h_{m-1} - h_{m-2} = -\frac{\omega^2}{\varepsilon k^2} h_{m-2-\ell} & \text{in } \bar{P}, \\
  \Delta h_m + 2i\hat{\kappa} \cdot \nabla h_{m-1} - h_{m-2} = h_m - \frac{\omega^2}{\varepsilon k^2} h_{m-2-\ell} & \text{in } P, \\
  \left( \frac{\omega^2}{\varepsilon k^2} \nabla h_{m-2-\ell} - \nabla h_m - ih_{m-1}\hat{\kappa} \right) \big|_{\partial P} \cdot \mathbf{n} = \frac{\omega^2}{\varepsilon k^2} \nabla h_{m-2-\ell} \big|_{\partial P} \cdot \mathbf{n} & \text{on } \partial P,
\end{cases}
\]

for \( m = 0, 1, 2, \ldots \), in which \( h_m \equiv 0 \) and \( \omega^2_m = 0 \) for \( m < 0 \) and the terms involving the subscript \( \ell \) are convolutions written according to the following summation conventions,

\[
\begin{align*}
  a_{\ell b_{n-\ell}} &= \sum_{\ell=0}^{n} a_{\ell b_{n-\ell}}, & a_{\ell b_{n-\ell}^{(\ell < \ell_2)}} &= \sum_{\ell=0}^{\ell_2-1} a_{\ell b_{n-\ell}}, \\
  a_{\ell b_{n-\ell}^{(\ell_1 < \ell < \ell_2)}} &= \sum_{\ell=\ell_1+1}^{\ell_2-1} a_{\ell b_{n-\ell}}, & a_{\ell b_{n-\ell}^{(\ell \text{ even})}} &= \sum_{\ell=0}^{[n/2]} a_{2\ell b_{n-2\ell}},
\end{align*}
\]

where \([n/2]\) denotes the largest integer less than or equal to \( n/2 \). The boundary value problem satisfied by \( h_0 \) in \( \bar{P} \) is

\[
\begin{cases}
  \Delta h_0 = 0 & \text{in } \bar{P}, \\
  \nabla h_0 \big|_{\partial P} \cdot \mathbf{n} = 0 & \text{on } \partial P,
\end{cases}
\]
so that this function is necessarily a constant in $\bar{P}$. We denote this constant value by $\bar{h}_0$. It will be convenient to use the dimensionless parameters $\psi_m$ and $\xi_m^2$ defined through

$$h_m = i^m \bar{h}_0 \psi_m \quad \text{and} \quad \omega_m^2 = c^2 k^2 \xi_m^2. \tag{2.2.2}$$

The equation for $h_m = i^m \bar{h}_0 \psi_m$ corresponds to letting $\eta$ vary along the imaginary axis. In terms of $\psi_m$ and $\xi_m^2$, the above equations for the functions $h_m$ become

$$\begin{align*}
\Delta \psi_m + 2 \hat{\kappa} \cdot \nabla \psi_m - \psi_m - 2 - i \hat{\kappa} \cdot \nabla \psi_m - 1 + \xi_m^2 \psi_m - 2 \ell &= \left(-i \ell \xi_m^2 \psi_m - 2 - i \hat{\kappa} \cdot \nabla \psi_m - 1\right) \text{in } \bar{P}, \\
\Delta \psi_m + 2 \hat{\kappa} \cdot \nabla \psi_m + \psi_m - 2 &= \psi_m - 2 \ell \xi_m^2 \psi_m - 2 \text{in } P,
\end{align*}$$

for $m = 0, 1, 2, \ldots$, in which $\psi_m \equiv 0$ and $\xi_m^2 = 0$ for $m < 0$. We thus have an infinite system which the sequences $\{\psi_m\}_{m=0}^{\infty}$ and $\{\xi_m^2\}_{m=0}^{\infty}$ must satisfy. The system is written in terms of a Poisson equation in $\bar{P}$ with Neumann boundary data and a Helmholtz equation in $P$ with Dirichlet boundary data, namely

$$\begin{align*}
\Delta \psi_m &= G_m \quad \text{in } \bar{P}, \\
\nabla \psi_m \cdot \mathbf{n} &= F_m \quad \text{on } \partial P, \tag{2.2.4}
\end{align*}$$

and

$$\begin{align*}
\Delta \psi_m &= \psi_m + G_m \quad \text{in } P, \\
\psi_m |_\rho &= \psi_m |_{\bar{\rho}} \quad \text{on } \partial P, \tag{2.2.5}
\end{align*}$$

in which

$$\begin{align*}
G_m &= \left(-i \ell \xi_m^2 \psi_{m-2-\ell} - 2 i \hat{\kappa} \cdot \nabla \psi_{m-1} - \psi_{m-2}\right), \\
F_m &= \nabla \left((-i \ell \xi_m^2 \psi_{m-2-\ell}) \cdot \mathbf{n} - \left(\nabla \left((-i \ell \xi_m^2 \psi_{m-2-\ell}) - \psi_{m-1} \hat{\kappa}\right)\right) \cdot \mathbf{n}, \tag{2.2.6}
\end{align*}$$

for $m = 0, 1, 2, \ldots$. 


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by solving (2.2.5) with Dirichlet boundary data $\psi_0 = \psi_0$.

The line integral over the interface $\partial P$ separating the two materials is subject to the standard \textit{solvability condition} given by

$$
\langle G_m \rangle_{\bar{P}} + \langle F_m \rangle_{\partial P} = 0.
$$

Here the area integral over the domains $\bar{P}$ and $P$ are denoted by $\langle \cdot \rangle_{\bar{P}}$ and $\langle \cdot \rangle_P$, while the line integral over the interface $\partial P$ separating the two materials is denoted by $\langle \cdot \rangle_{\partial P}$.

The algorithm for solving the system is as follows. First note from the definition of $\psi_0$ it follows that that $\psi_0 = 1$ for $y$ in $\bar{P}$. The function $\psi_0$ is determined inside $P$ by solving (2.2.5) with Dirichlet boundary data $\psi_0|_{\bar{P}} = \psi_0|_P = 1$ on $\partial P$. Then $\psi_1$ on $\bar{P}$ is the solution of (2.2.4) with Neumann data $\nabla \psi_1|_{\partial P} \cdot \mathbf{n} = -\psi_0|_{\partial P} \hat{k} \cdot \mathbf{n}$ on $\partial P$. The process then continues with the boundary values on $\partial P$ of $\psi_m$ in $\bar{P}$ providing the Dirichlet data for $\psi_m$ in $P$ which, in turn, provides the Neumann data for $\psi_{m+1}$ in $\bar{P}$, up to an additive constant. The term $\xi_{m-2}^2$ is determined by the consistency condition (2.2.7) and an inductive argument can be used to show that it is a monomial of degree $m$ in $\hat{k}$. The equations satisfied by $\psi_0, \ldots, \psi_4$ inside $\bar{P}, P$, and $\partial P$ are listed in Table (2.2) below.

In the next section we identify a large class of shapes for the plasmonic rod cross sections for which the sequences $\{\psi_m\}_{m=0}^{\infty}$ and $\{\xi_m^2\}_{m=0}^{\infty}$ satisfy the infinite system ((2.2.4),

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
$\psi_0 = 1$ & $\Delta \psi_0 = \psi_0$ & $\nabla \psi_0|_{\partial P} \cdot \mathbf{n} = 0$ \\
$\Delta \psi_1 + 2\hat{k} \nabla \psi_0 = 0$ & $\Delta \psi_0 + 2\hat{k} \nabla \psi_0 = \psi_1$ & $(\nabla \psi_0 + \frac{\nabla \psi_0}{\mathbf{n}})|_{\partial P} \cdot \mathbf{n} = 0$ \\
$\Delta \psi_2 + 2\hat{k} \nabla \psi_1 + \psi_0 = \xi_0^2 \psi_0$ & $\Delta \psi_2 + 2\hat{k} \nabla \psi_1 + \psi_0 = \psi_2 + \xi_0^2 \psi_0$ & $(\nabla (\xi_0^2 \psi_0) + \frac{\nabla \psi_2 + \psi_0}{\mathbf{n}})|_{\partial P} \cdot \mathbf{n}$ \\
$\Delta \psi_3 + 2\hat{k} \nabla \psi_2 + \psi_1 = \xi_0^2 \psi_1$ & $\Delta \psi_3 + 2\hat{k} \nabla \psi_2 + \psi_1 = \psi_3 + \xi_0^2 \psi_1$ & $= \nabla (\xi_0^2 \psi_1)|_{\partial P} \cdot \mathbf{n}$ \\
$\Delta \psi_4 + 2\hat{k} \nabla \psi_3 + \psi_2 = \xi_0^2 \psi_2 - \xi_0^2 \psi_0$ & $\Delta \psi_4 + 2\hat{k} \nabla \psi_3 + \psi_2 = (\xi_0^2 \psi_2 - \xi_0^2 \psi_0) + \psi_4$ & $(\nabla (\xi_0^2 \psi_2 - \xi_0^2 \psi_0) + \nabla \psi_4 + \psi_4)|_{\partial P} \cdot \mathbf{n}$ \\
\hline
\end{tabular}
\caption{Table of PDEs for the functions $\psi_m$ obtained from the expansion in $\eta$.}
\end{table}
(2.2.5), (2.2.7)) and \( \langle \psi_m \rangle_P = 0, \ m = 1, 2, \ldots \). The mean zero property of \( \psi_m \) on \( \bar{P} \) provides a tractable scenario for proving the convergence of the resulting power series. This topic is discussed further in section (3).

In what follows we will make use of the equivalent weak form of the infinite system. To introduce the weak form we introduce the space of complex valued square integrable functions with square integrable derivatives \( H^1(Q) \). For \( u \) and \( v \) in \( H^1(Q) \) the inner product is defined by \( (u, v)_{H^1(Q)} = \left( \int_Q u v \, dy + \int_Q \nabla u \cdot \nabla v \, dy \right) \), and the norm is given by \( \|v\|_{H^1(Q)} = (v, v)^{1/2}_{H^1(Q)} \). The \( H^1 \) inner products and norms over \( P \) and \( \bar{P} \) are defined similarly.

The weak form of the infinite system is given in terms of the space \( H^1_{\text{per}}(Q) \) of functions in \( H^1(Q) \) that take the same boundary values on opposite faces of \( Q \).

The weak form of the system ((2.2.4), (2.2.5), (2.2.7)) is given by

\[
\langle \left[ \nabla \sigma_m' - \hat{\kappa} \sigma_m' - 2 \right] \cdot \nabla \bar{v} - \left[ \hat{\kappa} \cdot \nabla \sigma_m' - \sigma_m'' - \sigma_m' + \sigma_m'' \right] \bar{v} \rangle_P + \langle \left[ \nabla \sigma_m' + \hat{\kappa} \psi_m \right] \cdot \nabla \bar{v} - \left[ \hat{\kappa} \cdot \nabla \psi_m + \psi_m + \psi_m' \right] \bar{v} \rangle_P = 0,
\]

for all \( v \in H^1_{\text{per}}(Q) \), where \( \sigma_m' = (-i) \xi \xi^2 \psi_m' \) and \( \sigma_m'' = (-i) \xi \xi^2 \psi_m'' \). The equivalence between ((2.2.4), (2.2.5), (2.2.7)) and the weak form follows from integration by parts and the solvability condition (2.2.7) follows from (2.2.8) on choosing the test function \( v = 1 \) in (2.2.8).
Chapter 3
Solutions of the Infinite System

The goal here is to identify solutions of the infinite system for which one can prove convergence of the associated power series with a minimum of effort. Looking ahead we note that the convergence proof is expedited when one can apply the Poincare inequality to the restriction of $\psi_m$ on $\bar{P}$ for $m$ greater than some fixed value. To this end we seek a solution $\{\psi_m(y)\}_{m=0}^\infty$, $\{\xi_m^2\}_{m=0}^\infty$ such that for $m \geq 1$ one has $\langle \psi_m \rangle_P = 0$ and the sequences satisfy ((2.2.4), (2.2.5), (2.2.7)) or equivalently satisfy (2.2.8). We show that we can find such solutions for the class of plasmonic domains $P$ with rectangular symmetry. Here we suppose that the unit period cell is centered at the origin and the class of rectangular symmetric domains is given by the set of all shapes invariant under $180^\circ$ rotations about the origin. This class includes rectangles and ellipses as well as multiply connected domains. For inclusion within this class and for each $m = 1, 2, 3 \ldots$ it is demonstrated that one can add an arbitrary constant to the restriction of the function $\psi_m$ on $\bar{P}$ with out affecting the solvability condition (2.2.7). Under the assumption of rectangular symmetry for the inclusion $P$, we will show that there exists a pair $\{\psi_m(y)\}_{m=0}^\infty$, $\{\xi_m^2\}_{m=0}^\infty$ satisfying (2.2.8) with the functions $\psi_m$ in the subspace $H^1_\ast(Q) \subseteq H^1_{\text{per}}(Q)$ of real-valued functions with zero average in $\bar{P}$.

3.1 Parity of the Functions $\psi_m$ and Proof of Solvability

We now record the symmetries necessarily satisfied by any solution $\psi_m \in H^1_\ast(Q)$ to ((2.2.4), (2.2.5), (2.2.7)) for plasmonic domains with rectangular symmetry. We denote the dependence of $\psi_m$ on the unit vector $\hat{\kappa}$ by writing $\psi_m^{\hat{\kappa}}$ so that
(i) $\psi_m^{-\hat{\kappa}}(y) = (-1)^m \psi_m^\kappa(y), \ \forall y \in Q$

(ii) $\psi_m^{-\hat{\kappa}}(y) = \psi_m^\kappa(-y), \ \forall y \in Q.$

Statement (i) is true for inclusions of arbitrary shape, while statement (ii) is true only for inclusions with rectangular symmetry. Taken together, these statements imply that $\psi_m^\kappa(-y) = (-1)^m \psi_m^\kappa(y),$ so that $\psi_m^\kappa$ is even or odd in $Q$ according as the index $m$ is even or odd. From its definition, $\psi_0 \equiv 1$ in $\bar{P}$ and trivially satisfies the solvability condition (2.2.7). Below are the graphs of $\psi_0$ and $\psi_1.$ Notice that $\psi_0$ is even and that $\psi_1$ is odd.

![Graph of $\psi_0$. This function is symmetric about the origin.](image1)

![Graph of $\psi_1$ when $\hat{\kappa} = (1,0).$ This function is antisymmetric about the origin.](image2)
The solvability of $\psi_m$ when $m \geq 1$ is proved by induction on $m$ using the weak form (2.2.8). We have the following theorem

**Theorem 3.1.1.** For each $\hat{\kappa}$, there exists a sequence of functions $\lbrace \psi_m \rbrace_{m=1}^\infty$, $\psi_m \in H^1_*(Q)$, and a sequence of real numbers $\lbrace \xi_m^2 \rbrace$, with $\xi_{odd}^2 = 0$, solving the weak form (2.2.8) for each integer $m$.

**Proof.** The proof is divided into the base case ($m = 1$ and $m = 2$) and the inductive step.

**Base case:** The solvability for $\psi_1$ and $\psi_2$ can be established without the need to restrict to rectangular symmetric inclusions. This restriction will be necessary only in the inductive step. Setting $m = 1$ and $v \equiv 1$ in (2.2.8), we see that the left-hand side of (2.2.8) vanishes. This establishes the solvability for $\psi_1$. If we then take $\langle \psi_1 \rangle_P = 0$, we have a solution $\psi_1 \in H^1_*(Q)$. Setting $m = 2$ and $v \equiv 1$ in (2.2.8), we obtain

$$
\langle \sigma'_0 \rangle_P + \langle \sigma'_0 \rangle_P - (\hat{\kappa} \cdot \nabla \psi_1 + \psi_0) = 0.
$$

Since $\langle \psi_0 \rangle_Q > 0$ (see Appendix) and $\langle \sigma'_0 \rangle_P + \langle \sigma'_0 \rangle_P = \xi_0^2 \langle \psi_0 \rangle_Q$, this is one equation in one unknown $\xi_0^2$. Solving for $\xi_0^2$ then gives $\xi_0^2 = \langle \psi_0 \rangle_Q^{-1} \langle \hat{\kappa} \cdot \nabla \psi_1 + \psi_0 \rangle_P$. Choosing this value for $\xi_0^2$ and also taking $\langle \psi_2 \rangle_P = 0$, we have a solution $\psi_2 \in H^1_*(Q)$.

**Inductive step:** Let $2n$ be an even positive integer and assume that (2.2.8) has solutions $\psi_m \in H^1_*(Q)$ for $m = 1, 2, ..., 2n$, with $\xi_{m-2}^2 \in \mathbb{R}$ and $\xi_{odd}^2 = 0$. Then (2.2.8) has solutions $\psi_{2n+1}, \psi_{2n+2} \in H^1_*(Q)$ for $m = 2n + 1, 2n + 2$ with $\xi_{2n-1}^2 = 0$ and $\xi_{2n}^2 \in \mathbb{R}$. 

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The solvability condition for $\psi_{2n+1}$ is obtained by setting $v = 1$ and $m = 2n + 1$ in the weak form, namely

$$\langle [\hat{\kappa} \cdot \nabla \sigma'_{2n-2} - \sigma'_{2n-1} - \sigma''_{2n-3} + \sigma'_{2n-3}] \rangle_P +$$

$$+ \langle [\hat{\kappa} \cdot \nabla \sigma'_{2n-2} - \sigma'_{2n-1} - \sigma''_{2n-3} + \sigma'_{2n-3}] \rangle_{\bar{P}} +$$

$$+ \langle [\hat{\kappa} \cdot \nabla \psi_{2n} + \psi_{2n-1}] \rangle_P = 0.$$ 

The hypothesis $\xi_{odd}^2 = 0$, odd $\leq 2n - 2$, will imply that the convolutions $\sigma_m$, $m \leq 2n - 2$, have the same even/odd property as the functions $\psi_m$. Indeed, writing out $\sigma'_{2n-3}$, we have $\sigma'_{2n-3} = (-1)^l \xi_{\ell}^2 \psi_{2n-3-l}^{(l \ even)}$, and since $2n - 3 - \ell$ is odd when $\ell$ is even, it follows that $\sigma'_{2n-3}$ is a linear combination of odd functions and is, therefore, an odd function. The same reasoning applies to all the other convolutions of index less than or equal to $2n - 2$. Moreover, $\hat{\kappa} \cdot \nabla \sigma'_{2n-2}$ is an odd function, since $\sigma'_{2n-2}$ is even. Thus, all integrals in the consistency condition above vanish (for the integrals in $\bar{P}$ we can also use the fact that all functions belong to $H^1_s(Q)$), except that

$$\langle \sigma'_{2n-1} \rangle_P + \langle \sigma'_{2n-1} \rangle_{\bar{P}} = (-i)^{2n-1} \xi_{2n-1}^2 \langle \psi_0 \rangle_Q.$$ 

Since $\langle \psi_0 \rangle_Q > 0$ (see Appendix), the solvability condition for $\psi_{2n+1}$ is simply $\xi_{2n-1}^2 = 0$. We thus take $\xi_{2n-1}^2 = 0$ to establish the existence of $\psi_{2n+1} = 0$. Moreover, since $\psi_m$ and $\xi_{m-2}$ are real by the induction hypothesis, $0 \leq m \leq 2n$, it follows that $\psi_{2n+1}$ is real-valued. Thus, taking $\langle \psi_{2n+1} \rangle_{\bar{P}} = 0$, we have a solution $\psi_{2n+1} \in H^1_s(Q)$. Also, $\psi_{2n+1}$ is an odd function since its index is odd. We now proceed to the solvability of $\psi_{2n+2}$, namely

$$\langle [\hat{\kappa} \cdot \nabla \sigma'_{2n-1} - \sigma'_{2n-2} - \sigma''_{2n-2} + \sigma'_{2n}] \rangle_P +$$

$$+ \langle [\hat{\kappa} \cdot \nabla \sigma'_{2n-1} - \sigma'_{2n-2} - \sigma''_{2n-2} + \sigma'_{2n}] \rangle_{\bar{P}} +$$

$$+ \langle [\hat{\kappa} \cdot \nabla \psi_{2n+1} + \psi_{2n}] \rangle_P = 0.$$ 

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All terms in the above equation are real numbers, since we assumed $\psi_m$ and $\xi_{m-2}$ real for $0 \leq m \leq 2n$, with $\xi^{\text{odd}}_{2n} = 0$, and we just took $\xi^2_{2n-1} = 0$ and $\psi_{2n+1}$ is real-valued. Thus, this equation contains the only one undetermined term $(-i)^{2n} \xi^2_{2n} \langle \psi_0 \rangle_Q$. Thus, we have one real equation with one real variable, so that taking $\xi^2_{2n}$ to be such as to solve this equation and also taking $\langle \psi_{2n+2} \rangle_P = 0$, we complete the proof of the inductive step.

Once convergence has been established (see theorem (4.1.1)), the fact that the functions $\psi_m$ are real-valued allows us to identify the real an imaginary parts of the series $\sum h_m \eta^m$, as follows

$$\sum h_m \eta^m = \sum (-1)^m \psi_{2m} \eta^2m + i \sum (-1)^m \psi_{2m+1} \eta^{2m+1}.$$  

### 3.2 Explicit Expressions for Tensors

The tensors $\xi^2_0$ and $\xi^2_2$ were calculated using the weak form (2.2.8), as follows. Setting $m = 2$ and $v \equiv 1$ in (2.2.8) and solving for $\xi^2_0$ gives

$$\xi^2_0 = \frac{\langle \hat{\kappa} \cdot \nabla \psi_1 + \psi_0 \rangle_P}{\langle \psi_0 \rangle_Q}. \quad (3.2.1)$$

Setting $m = 4$ and $v \equiv 1$ and solving for $\xi^2_2$ gives

$$\xi^2_2 = \frac{-\xi^2_0 \langle \hat{\kappa} \cdot \nabla \psi_1 \rangle_Q + \xi^2_0 \langle \psi_2 \rangle_P + \xi^2_0 \xi^2_0 \langle \psi_0 \rangle_Q + \xi^2_0 \langle \psi_0 \rangle_Q - \langle \hat{\kappa} \cdot \nabla \psi_3 \rangle_P}{\langle \psi_0 \rangle_Q}. \quad (3.2.2)$$

All integrals appearing in (3.2.1) and (3.2.2) were then computed using the program COMSOL.
Chapter 4

Proof of Convergence

In this section we show that the power series $\sum_{m=0}^{\infty} \bar{p}_m \eta^m$, $\sum_{m=0}^{\infty} p_m \eta^m$ and $\sum_{m=0}^{\infty} \xi^2_m \eta^m$, where $\bar{p}_m = \|\psi_m\|_{H^1(\bar{P})}$ and $p_m = \|\psi_m\|_{H^1(P)}$, converge and provide lower bounds on the radius of convergence. This will then be used to show that the pair $h_\eta = \sum_{m=0}^{\infty} h_m \eta^m$ and $\omega^2_\eta = \sum_{m=0}^{\infty} \omega^2_m \eta^m$ is a solution to (2.1.6). In subsection 4.1, we present the Catalan Bound, which is used to provide a lower bound on the radius of convergence of the power series. In subsection 4.2, we derive inequalities which bound $\bar{p}_m$, $p_m$ and $\xi^2_m$ in terms of lower index terms. In subsection 4.3, we present the properties of the Catalan numbers relevant for bounding convolutions of the kind appearing in (2.2.4) and (2.2.5). In subsection 4.4, we use an inductive argument on the inequalities of subsection 4.2 to prove the Catalan Bound. Finally, in subsection 4.5 we prove that the pair $h_\eta$ and $\xi^2_\eta$ is a solution to the eigenvalue problem (2.1.6).

4.1 The Catalan Bound

The following theorem is one of the central results of this thesis.

**Theorem 4.1.1. (Catalan Bound)**

For every integer $m$, we have that

$$\bar{p}_m, p_m, |\xi^2_m| \leq \beta C_m J^m$$

in which $C_m$ is the $m^{th}$ Catalan number, $\beta = \max\{\bar{p}_0, p_0, |\xi^2_0|\}$ and $J = \max\{J_1, J_2\}$, where the numbers $J_1$ and $J_2$ are determined as follows: $J_1$ is the smallest value of $J$ such that (4.1.1) holds for $m \leq 4$ and $J_2$ is the smallest value of $J$ for which the
following polynomials $Q^*, R^*, S^*$ in the variable $J^{-1}$ are all less than unity

\[ Q^* = \Omega P \{ 2E(4)\beta J^{-2}1/3 + E(4)\beta J^{-3}5/42 + E(4)\beta J^{-4}1/21 + E^2(4)\beta^2 J^{-4}5/42 \} + \]
\[ + 2E(4)\beta J^{-2}1/3 + 2E(4)\beta J^{-3}5/42 + E(4)\beta J^{-4}1/21 + E^2(4)\beta^2 J^{-4}5/42 + J^{-2}5/42 \]
\[ + 2\Omega P \{ 2E(4)\beta J^{-3}5/42 + 2E(4)\beta J^{-4}1/21 + E(4)\beta J^{-5}1/42 + \]
\[ + E^2(4)\beta^2 J^{-5}1/21 + J^{-3}1/21 + 2J^{-2}5/42 \}, \]
\[ R^* = AQ^* + E(4)\beta J^{-2}1/3 + 2J^{-1}1/3 + J^{-2}5/42, \]
\[ S^* = \frac{1}{\langle \psi_0 \rangle Q} \{ 4J\{ \theta_P Q^* + \theta_P E(4)\beta J^{-2}(1/3) + E^2(4)\beta^2 J^{-3}(1/3) + E(4)\beta J^{-3}(5/42) \} \]
\[ + \theta_P E(4)\beta J^{-2}(1/3) + E(4)\theta_P \beta J^{-3}(5/42) + \theta_P J^{-3}(1/21) \} \}
\[ + \theta_P \{ (|\xi_0|^2|R^*| + |\xi_2|^2 J^{-2}(1/7) + p_2 J^{-2}(1/7)) + (0.7976,\beta) \} ] . \]

The constants $A$, $\Omega P$, $\beta$, $\theta_P$, $\theta_P$, $|\xi_0|^2$, $|\xi_2|^2$ and $p_2$ are determined by the particular choice of inclusion, while $E(4) = 16C_2/C_5 \leq 0.7619$.

All bounds obtained here are expressed in terms of the Catalan numbers, area fractions and geometric parameters that appear in the Poincare inequality and in an extension operator inequality. We start by listing these parameters and give the background for their description. It is known [29] that any $H^1(\bar{P})$ function $\phi$ can be extended into $P$ as an $H^1(Q)$ function $E(\phi)$ such that $E(\phi) = \phi$ for $y$ in $\bar{P}$ and
\[
\| E(\phi) \|_{H^1(P)} \leq A \| \phi \|_{H^1(\bar{P})} \quad (4.1.2)
\]
where $A$ is a nonnegative constant and is independent of $\phi$ depending only on $P$. For general shapes $A$ can be calculated via numerical solution of a suitable eigenvalue problem. Constants of this type appear in [11] for high contrast expansions of the
DC fields inside frequency independent dielectric media. The second constant is the Poincaré constant $D^2_P$, given by the reciprocal of the first nonzero Neumann eigenvalue of $\bar{P}$ and we have that $\Omega_P = 1 + D^2_P$. The last two geometric constants appearing in the bounds are the volume fractions $\theta_P$ and $\theta_{\bar{P}}$ of the regions $P$ and $\bar{P}$. Using that $C_m \leq 4^m$ (see section 4.3), theorem (4.1.1) shows that $\sum \bar{p}_m \eta^m$, $\sum \rho_m \eta^m$ and $\sum \xi^2_m \eta^m$ are convergent for $\eta \leq 1/4J$, so that one may prove the following theorem

**Theorem 4.1.2.** *(Solution of the Eigenvalue Problem)* Let $R = 1/4J$, where $J$ is the number prescribed by theorem (4.1.1). Then $\sum_{m=0}^{\infty} \xi^2_m \eta^m$ converges as a series of real numbers and $\sum_{m=0}^{\infty} \psi_m \eta^m$ converges in the $H^1(Q)$ Sobolev norm for $\eta \leq R$. Moreover, $\omega^2_\eta = c^2 k^2 \sum_{m=0}^{\infty} \xi^2_m \eta^m$ and $h_\eta = \bar{h}_0 \sum_{m=0}^{\infty} \psi_m \eta^m$ satisfy the eigenvalue problem given by (1.1.1) (or the master system (2.1.6)).

The table below contains values for $A$, $\Omega_{\bar{P}}$ and $J$ for circular inclusions of radii $rd$. This table corresponds to the tables (1.4) and (1.4) for $R_m$, $\lambda_m$ and $k_M$

| TABLE 4.1. Values of $A$, $\Omega_{\bar{P}}$ and $J$ for circular inclusions of radii $rd$. |
|---|---|---|---|---|---|
| $r$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.45 |
| $A$ | 1.058 | 1.293 | 1.907 | 3.956 | 4.840 |
| $\Omega_{\bar{P}}$ | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| $J$ | 15 | 17 | 22 | 29 | 85 |

### 4.2 The $\bar{p}_m$, $p_m$ and $\xi^2_m$ Inequalities—Stability Estimates

We now derive the inequalities which bound $\bar{p}_m$, $p_m$ and $\xi^2_m$ in terms of lower index terms. These inequalities follow from stability estimates for (2.2.4, 2.2.5, 2.2.6). We first introduce the following notation for the convolutions

$q'_m = |\xi^2_\ell| p_{m-\ell}$, $q''_m = p_{m-\ell} |\xi^2_{\ell-1}| |\xi^2_\ell|$, $q'''_{m-1} = |\xi^2_{\ell}| p_{m-1-\ell}$

$q'_m = |\xi^2_\ell| \bar{p}_{m-\ell}$, $q''_m = \bar{p}_{m-\ell} |\xi^2_{\ell-1}| |\xi^2_\ell|$. 

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Proof.

**Theorem 4.2.1.** Let \( m \geq 0 \) be an integer. Then

\[
\bar{p}_m \leq \Omega \rho [A\{2q'_{m-2} + 2q'_{m-3} + q'_{m-4} + q''_{m-4}\} + 2\bar{q}'_{m-2} + 2\bar{q}'_{m-3} + \bar{q}'_{m-4} \\
+ \bar{q}''_{m-4} + \bar{p}_{m-2} + 2 \Omega \rho (A\{2q'_{m-3} + 2q'_{m-4} + q'_{m-5} + q''_{m-5}\} \\
+ 2\bar{q}'_{m-3} + 2\bar{q}'_{m-4} + \bar{q}'_{m-5} + q''_{m-5} + \bar{p}_{m-3} + 2\bar{p}_{m-2})]
\]

\[
p_m \leq A\bar{p}_m + q'_{m-2} + 2p_{m-1} + p_{m-2} \tag{4.2.1}
\]

\[
|\xi^2_{m-1}| \leq (\psi_0)_{Q}^{-1} \{ \sqrt{\theta \rho} q'^*_{m-1} + \sqrt{\theta \rho} \bar{p}_m + \\
+ \sqrt{\theta \rho} (q'_{m-2} + q''_{m-3} + q'_{m-3}) + \sqrt{\theta \rho} (\bar{q}'_{m-2} + \bar{q}''_{m-3} + \bar{q}'_{m-3}) \},
\]

where the \( \bar{p}_m \) inequality holds for \( m \geq 2 \) only.

We start by proving the \( p_m \) inequality. Recalling that ((2.2.5)) is satisfied by \( \psi_m \) in \( P \) gives

\[
\begin{cases}
\Delta \psi_m = \psi_m + G_m, & \text{in } P \\
\psi_m|_{\partial P} = \psi_m|_{\partial P}, & \text{on } \partial P
\end{cases}
\]

where \( G_m = (-i)^{\ell} \xi^2_{m-2-\ell} - 2\bar{\kappa} \cdot \nabla \psi_{m-1} - \psi_{m-2} \). Write the orthogonal decomposition \( \psi_m = u_m + v_m \), where

\[
\begin{cases}
\Delta u_m = u_m, & \text{in } P \\
u_m = \psi_m, & \text{on } \partial P
\end{cases} \tag{4.2.2}
\]

and

\[
\begin{cases}
\Delta v_m = v_m + G_m, & \text{in } P \\
v_m = 0, & \text{on } \partial P
\end{cases}
\]

We then have that

\[
\|\psi_m\|_{H_1(P)} = \sqrt{\|u_m\|_{H_1(P)}^2 + \|v_m\|_{H_1(P)}^2} \leq \|u_m\|_{H_1(P)} + \|v_m\|_{H_1(P)}. \tag{4.2.3}
\]
Moreover, we have that \( u_m = E(\psi_m) \), where \( E(\cdot) \) is the extension operator of (4.1.2), so that

\[
\|u_m\|_{H_1(P)} \leq A\|\psi_m\|_{H_1(\bar{P})}.
\]

(4.2.4)

The term \( \|v_m\|_{H_1(P)} \) can be bounded using a direct integration by parts on the BVP for \( v_m \)

\[
\|v_m\|_{H_1(P)} \leq \|G_m\|_{L_2(P)}.
\]

(4.2.5)

Now,

\[
\|G_m\|_{L_2(P)} = \|(−i)^{\ell}\xi^2\psi_{m−2−\ell} − 2\hat{\kappa} \cdot \nabla \psi_{m−1} − \psi_{m−2}\|_{L_2(P)}
\]

\[
\leq |\xi^2_{\ell}|\|\psi_{m−2−\ell}\|_{L_2(P)} + 2|\hat{\kappa}|\|\nabla \psi_{m−2}\|^2_{L_2(P)} + \|\psi^2_{m−2}\|_{L_2(P)}
\]

\[
\leq |\xi^2_{\ell}|p_{m−2−\ell} + 2p_{m−1} + p_{m−2},
\]

where \( p_m = \|\psi_m\|_{H_1(\bar{P})} \). Using (4.2.4) and (4.2.5) in (4.2.3) gives

\[
p_m \leq A\bar{p}_m + q'_{m−2} + 2p_{m−1} + p_{m−2},
\]

(4.2.6)

and the \( p_m \) inequality is established. We now prove the \( \bar{p}_m \) inequality. In the weak form ((2.2.8)), set \( v = \psi_m \) in \( \bar{P} \) and \( v = u_m \) in \( P \) to obtain

\[
\langle [\nabla \sigma'_{m−2} + \kappa \sigma'_{m−3}] \cdot \nabla \psi_m − [\hat{\kappa} \cdot \nabla \sigma'_{m−3} − \sigma'_{m−2} − \sigma''_{m−4} + \sigma'_{m−4}] \psi_m \rangle_{\bar{P}} +
\]

\[
+ \langle [\nabla \sigma'_{m−2} + \kappa \sigma'_{m−3}] \cdot \nabla u_m − [\hat{\kappa} \cdot \nabla \sigma'_{m−3} − \sigma'_{m−2} − \sigma''_{m−4} + \sigma'_{m−4}] u_m \rangle_{P} +
\]

\[
+ \langle [\nabla \psi_m + \kappa \psi_{m−1}] \cdot \nabla \psi_m − [\hat{\kappa} \cdot \nabla \psi_{m−1} + \psi_{m−2}] \psi_m \rangle_{\bar{P}} = 0,
\]

We now use the Cauchy-Schwarz inequality on the product of integrals appearing in each individual term. For the convolutions, we obtain

\[
\|\langle \nabla \sigma'_{m−2} \cdot \nabla u_m \rangle_{P} \| = \|\langle (-i)^{\ell}\xi^2\psi_{m−2−\ell} \cdot \nabla u_m \rangle_{P} \|
\]

\[
= |\langle (-i)^{\ell}\xi^2\nabla \psi_{m−2−\ell} \cdot \nabla u_m \rangle_{P} |
\]

\[
\leq |\xi^2_{\ell}|\bar{p}_{m−2−\ell}A\bar{p}_m,
\]

\[
= q'_{m−2}A\bar{p}_m
\]

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where we used that \( \|u_m\|_{H_1(P)} \leq A\bar{p}_m \). For the double-convolutions, we obtain

\[
|\langle \sigma''_{m-4}u_m \rangle_P| \leq q''_{m-2}A\bar{p}_m.
\]

Proceeding similarly with the other terms, we obtain

\[
\langle \nabla \psi_m \cdot \nabla \psi_m \rangle_P \leq \bar{p}_m (A\{2q'_{m-2} + 2q'_{m-3} + q'_{m-4} + q''_{m-4}\} + +2\bar{q}'_{m-2} + 2\bar{q}'_{m-3} + \bar{q}'_{m-4} + \bar{q}''_{m-4} + 2\bar{p}_{m-1} + \bar{p}_{m-2}) \tag{4.2.7}
\]

Since the functions \( \psi_m \) have zero average in \( \bar{P} \), we have the Poincaré inequality

\[
\langle \psi^2_m \rangle_P \leq D_P^2 \langle \nabla \psi_m \cdot \nabla \psi_m \rangle_P, \tag{4.2.8}
\]

where the constant \( D_P \) can be computed from the Rayleigh quotient characterization of the first positive eigenvalue for the free membrane problem in \( \bar{P} \). A simple computation using (4.2.8) then gives \( \bar{p}^2_m \leq \Omega_P \langle \nabla \psi_m \cdot \nabla \psi_m \rangle_P \), where \( \Omega_P = D_P^2 + 1 \).

Using this inequality, ((4.2.7)) gives:

\[
\bar{p}_m \leq \Omega_P (A\{2q'_{m-2} + 2q'_{m-3} + q'_{m-4} + q''_{m-4}\} + +2\bar{q}'_{m-2} + 2\bar{q}'_{m-3} + \bar{q}'_{m-4} + \bar{q}''_{m-4} + \bar{p}_{m-2} + 2\bar{p}_{m-1}). \tag{4.2.9}
\]

It will turn out to be to our advantage to apply (4.2.9) to the last term \( 2\bar{p}_{m-1} \) in (4.2.9) so as to replace it with

\[
\bar{p}_{m-1} \leq \Omega_P (A\{2q'_{m-3} + 2q'_{m-4} + q'_{m-5} + q''_{m-5}\} + +\bar{q}'_{m-3} + 2q'_{m-4} + \bar{q}'_{m-5} + \bar{q}''_{m-5} + \bar{p}_{m-3} + 2\bar{p}_{m-2}). \tag{4.2.10}
\]

Using (4.2.10) in (4.2.9) yields the \( \bar{p}_m \) inequality in (4.2.1), valid for \( m \geq 2 \) (for \( m = 1 \), use (4.2.9)).
Last we establish the $\xi_{m-1}^2$ inequality. Setting $\nu = 1$ in the weak form ((2.2.8)) we obtain

$$
\langle \hat{\kappa} \cdot \nabla \sigma'_{m-3} - \sigma''_{m-2} - \sigma'''_{m-4} + \sigma''_{m-4} \rangle_P + \\
+ \langle \hat{\kappa} \cdot \nabla \sigma'_{m-3} - \sigma''_{m-2} - \sigma'''_{m-4} \rangle_{\bar{P}} + \\
+ \langle \hat{\kappa} \cdot \nabla \psi_{m-1} + \psi_{m-2} \rangle_{\bar{P}} = 0. \quad (4.2.11)
$$

(recall that for $m$ odd, each term on the left-hand side of the above equation vanishes individually). Solving for $\xi_{m-2}^2$ we then obtain

$$
-(\hat{q}^2_{m-2} \xi_{m-2}^2 \langle \psi_0 \rangle_{Q}) = \langle \hat{\kappa} \cdot \nabla \sigma'_{m-3} - \sigma''_{m-2} - \sigma'''_{m-4} + \sigma''_{m-4} \rangle_P + \\
+ \langle \hat{\kappa} \cdot \nabla \sigma'_{m-3} - \sigma''_{m-2} - \sigma'''_{m-4} \rangle_{\bar{P}} + \\
+ \langle \hat{\kappa} \cdot \nabla \psi_{m-1} + \psi_{m-2} \rangle_{\bar{P}}. \quad (4.2.12)
$$

We shall be using this equality for $m \geq 5$ only, so that $\langle \sigma_{m-2}^* \rangle_P = 0$ and $\langle \psi_{m-2} \rangle_P = 0$. Moreover, using that $\langle |\psi_m| \rangle_P \leq \sqrt{\theta_P \langle |\psi_m|^2 \rangle_P}$ and $\langle |\psi_m| \rangle_{\bar{P}} \leq \sqrt{\theta_{\bar{P}} \langle |\psi_m|^2 \rangle_{\bar{P}}}$, where $\theta_P$ and $\theta_{\bar{P}}$ denote the volume fractions of the regions $P$ and $\bar{P}$, we have that $\langle \psi_m \rangle_P \leq \sqrt{\theta_P p_m}$ and $\langle \psi_m \rangle_{\bar{P}} \leq \sqrt{\theta_{\bar{P}} \bar{p}_m}$. Thus, proceeding with (4.2.12) as we did in the previous stability estimates, we obtain

$$
|\xi_{m-2}^2| \leq \langle \psi_0 \rangle_Q^{-1} \{ \sqrt{\theta_P \bar{p}_{m-2-\ell}}|\xi_{\ell}^2|^\ell_{m-2} + \sqrt{\theta_P \bar{p}_{m-1}} + \sqrt{\theta_P \langle q'_{m-3} + q''_{m-4} + q''_{m-4} \rangle} + \\
+ \sqrt{\theta_{\bar{P}} (q'_{m-3} + \bar{q}''_{m-4} + \bar{q}''_{m-4})} \}.
$$

Since the iteration scheme at each step involves $p_m$ and $\bar{p}_m$ and $\xi_{m-1}^2$ we adjust subscripts in the above inequality to obtain the $\xi_{m-1}^2$ inequality

$$
|\xi_{m-1}^2| \leq \langle \psi_0 \rangle_Q^{-1} \{ \sqrt{\theta_P \bar{q}''_{m-1}} + \sqrt{\theta_{\bar{P}} \bar{q}''_{m-2}} + \sqrt{\theta_P \langle q'_{m-2} + q''_{m-3} + q''_{m-3} \rangle} + \\
+ \sqrt{\theta_{\bar{P}} (q'_{m-2} + \bar{q}''_{m-3} + \bar{q}''_{m-3})} \}. \quad (4.2.13)
$$
TABLE 4.2. Values of $\rho_k^m$, where $\rho_m^k = C_{m-k}/C_m$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_k^m$</td>
<td>1</td>
<td>1/3</td>
<td>5/42</td>
<td>1/21</td>
<td>1/42</td>
</tr>
</tbody>
</table>

4.3 The Catalan Numbers

The Catalan numbers $C_m$ are defined algebraically through the recursion

$$C_{m+1} = C_{m-\ell}C_\ell, \quad C_0 = 1.$$  \hfill (4.3.1)

These numbers arise in many combinatorial contexts [21] as well as in the study of fluctuations in coin tossing and random walks [18]. It can be shown that

$$C_m = \frac{1}{m+1} \binom{2m}{m}$$

so that

$$\frac{C_{m+1}}{C_m} = 4 - \frac{6}{m+2} \quad \text{and} \quad \frac{C_m}{C_{m+1}} = \frac{1}{4} + \frac{3}{8m+4}. \hfill (4.3.2)$$

Thus, it is seen that the Catalan numbers form a sequence of the hypergeometric type [1]. The first inequality above provides the exponential bound

$$C_m \leq 4^m. \hfill (4.3.3)$$

It will be convenient to introduce the notation $\rho_m^k = C_{m-k}/C_m$. From (4.3.2), it is clear that $\rho_m^k$ is decreasing in both $m$ and $k$. In section (4.4) we shall make use of Table 4, in which values of $\rho_5^k$ are listed.

4.3.1 The Even Part of the Catalan Convolution

The fact that $\xi_{\text{odd}}^2 = 0$ must be taken into account in order to provide an exponential bound for the incomplete convolution term $q_{m-1}^n$ appearing in the $\xi_{m-1}^2$ inequality in (4.2.1). We thus look for an upper bound on the ratio

$$E(n) = \frac{C_{n-\ell}C_\ell^{(\ell \text{ even})}}{C_{n-\ell}C_\ell}. \hfill (4.3.4)$$

The following lemma gives the estimate $E(n) \leq E(4), \ n \geq 4$. 

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Lemma 4.3.1. The following two statements are true for all \( m \geq 0 \): (i) \( E(2m) \) is a decreasing sequence; and (ii) \( E(2m + 1) = 1/2 \). Thus, for all \( m \geq 4 \), we have that \( E(m) \leq \max\{E(4), 1/2\} = E(4) \leq 0.7619 \).

Proof. Statement (ii) is actually just an observation, as one can see by writing out the sum \( C_n - \ell C_\ell \). Statement (i) can be deduced from the identity \( C_{2m-2\ell} C_\ell = 4^m C_m \) Koshy (2008). Indeed, dividing both sides of this identity by \( C_{2m+1} \), we obtain

\[
E(2m) = \frac{1}{4} \frac{4C_m}{C_{m+1}} \frac{4C_{m+1}}{C_{m+2}} \cdots \frac{4C_{2m}}{C_{2m+1}}.
\]

From (4.3.2), each of the above fractions \( 4C_m/C_{m+\ell} \), \( \ell = 1, 2, \ldots, m+1 \), is a decreasing sequence in \( m \) so that their product is also decreasing in \( m \). This completes the proof.

\[
\square
\]

4.4 Proof of the Catalan Bound

Proof. (Catalan Bound, theorem (4.1.1)) Fix the values of the geometric parameters \( A, \Omega, \theta_P \) and \( \theta_{\bar{P}} \) in (4.2.1). Starting with the initial estimates \( \bar{p}_0 = \theta_P, p_0 = \theta_P \) and \( |\xi_0^P| \leq 1 \), the inequalities (4.2.1) can be used recursively for \( m = 1, 2, 3, 4 \) to determine a number \( J_1 \) such that

\[
\bar{p}_m, p_m, |\xi_m^P| \leq \beta C_m J_1^m, \quad 0 \leq m \leq 4.
\]

We now proceed inductively: assume that

\[
\bar{p}_n, p_n, |\xi_n^P| \leq \beta C_n J^n, \quad n \in \{0, 1, 2, \ldots, m-1\}, \quad (4.4.1)
\]
where \( m \geq 5 \). We then get for the single convolutions

\[
q_{m-k}^\prime = p_{m-k-\ell} |\xi_{\ell}^2|
\]

\[
\leq (\beta C_{m-k-\ell} J^{m-k-\ell}) (\beta C_{\ell} J^\ell)^{(\ell \text{ even})}
\]

\[
= \beta^2 J^{m-k} C_{m-k-\ell} C_{\ell}^{(\ell \text{ even})}
\]

\[
\leq E(4) \beta^2 J^{m-k} C_{m-k-\ell} C_{\ell}
\]

\[
= E(4) \beta J^{-k} \left( \frac{C_{m+1-k}}{C_m} \right) \beta J^m C_m
\]

\[
= E(4) \beta J^{-k} \rho_{m}^{k-1} \beta J^m C_m.
\]

where \( \rho_{m}^k = C_{m-k}/C_m \) and lemma (4.3.1) was used to introduce the factor \( E(4) \).

Similarly, for double convolutions we get

\[
q_{m-k}^{''} \leq E^2(4) \beta^2 J^{-k} \rho_{m}^{k-2} \beta J^m C_m
\]

where the factor \( E(4)^2 \) comes from using lemma (4.3.1) twice. For the non-convolution terms we get \( p_{m-k} \leq J^{-k} \rho_{m}^k \beta J^m C_m \). The same bounds hold for the terms \( \bar{p}_{m-k} \), \( \bar{q}_{m-k}^\prime \) and \( \bar{q}_{m-k}^{''} \), so that we have

\[
\bar{p}_{m-k}, p_{m-k} \leq J^{-k} \rho_{m}^k \beta J^m C_m
\]

\[
\bar{q}_{m-k}^\prime, q_{m-k} \leq E(4) \beta J^{-k} \rho_{m}^{k-1} \beta J^m C_m \tag{4.4.2}
\]

\[
\bar{q}_{m-k}^{''}, q_{m-k}^{''} \leq E^2(4) \beta^2 J^{-k} \rho_{m}^{k-2} \beta J^m C_m.
\]

The proof now essentially consists of applying these bounds to all terms in inequalities (4.2.1). The factor \( J^{-k} \) appearing on the right-hand side of each inequality is the Cat’s Pajamas: by taking \( J \) sufficiently large, it will allow us to close the induction argument. The incomplete convolution \( q_{m-1}^{*} \) presents special difficulties, since attempting a bound of type (4.4.2) for this term does not produce
a factor of $J^{-k}$ (actually, it produces $J^0 = 1$). Lemma (4.3.1) will allow us to deal with this situation.

Recall the $\bar{p}_m$ inequality from (4.2.1)

$$\bar{p}_m \leq \Omega_p[A\{2q'_{m-2} + 2q'_{m-3} + q'_{m-4} + q''_{m-4}\} +$$

$$+ 2\bar{q}'_{m-2} + 2\bar{q}'_{m-3} + \bar{q}'_{m-4} + \bar{q}''_{m-4} + \bar{p}_{m-2} +$$

$$+ 2\Omega_p\{A\{2q'_{m-3} + 2q'_{m-4} + q'_{m-5} + q''_{m-5}\} +$$

$$+ 2\bar{q}'_{m-3} + 2\bar{q}'_{m-4} + \bar{q}'_{m-5} + \bar{q}''_{m-5} + \bar{p}_{m-3} + 2\bar{p}_{m-2}\}]$$

Using (4.4.2) on this inequality gives

$$\bar{p}_m \leq Q_m \beta J^m C_m, \quad (4.4.3)$$

where $Q_m$ is the following polynomial in $J^{-1}$

$$Q_m = \Omega_p [A\{2E(4)\beta J^{-2} 1/3 + E(4)\beta J^{-3} 5/42 + E(4)\beta J^{-4} 1/21 + E^2(4)\beta^2 J^{-4} 5/42\} +$$

$$+ 2E(4)\beta J^{-2} 1/3 + 2E(4)\beta J^{-3} 5/42 + E(4)\beta J^{-4} 1/21 + E^2(4)\beta^2 J^{-4} 5/42 + J^{-2} 5/42 +$$

$$+ 2E(4)\beta J^{-4} 1/21 + E(4)\beta J^{-5} 1/21 + E^2(4)\beta^2 J^{-5} 1/21\}}]\]$$

Since we shall be using this inequality for $m \geq 5$ only, table (4.2) can be used to bound the numbers $\rho_m^b$, so that we may write $Q_m \leq Q^*$, where

$$Q^* = \Omega_p [A\{2E(4)\beta J^{-2} 1/3 + E(4)\beta J^{-3} 5/42 + E(4)\beta J^{-4} 1/21 + E^2(4)\beta^2 J^{-4} 5/42\} +$$

$$+ 2E(4)\beta J^{-2} 1/3 + 2E(4)\beta J^{-3} 5/42 + E(4)\beta J^{-4} 1/21 + E^2(4)\beta^2 J^{-4} 5/42 + J^{-2} 5/42 +$$

$$+ 2E(4)\beta J^{-3} 5/42 + 2E(4)\beta J^{-4} 1/21 + E(4)\beta J^{-5} 1/21 + E^2(4)\beta^2 J^{-5} 1/21\} +$$

$$+ 2E(4)\beta J^{-3} 5/42 + 2E(4)\beta J^{-4} 1/21 + E(4)\beta J^{-5} 1/21 + E^2(4)\beta^2 J^{-5} 1/21 +$$

$$+ E^2(4)\beta^2 J^{-5} 1/21 + J^{-3} 1/21 + 2J^{-2} 5/42\}]. \quad (4.4.4)$$
The strategy now is to determine similar polynomials $R_m$ and $S_{m-1}$ for the other two inequalities, that is $p_m \leq R_m \beta J^m C_m$ and $|\xi_{m-1}^2| \leq S_{m-1} \beta J^{m-1} C_{m-1}$, and then take $J$ large enough that all three polynomials are less than unity, allowing us to complete the induction argument. Having obtained $Q_m$, it is straightforward to obtain $R_m$. Indeed, using (4.4.2) and (4.4.3), the $p_m$ inequality in (4.2.1) yields

$$R_m = AQ_m + E(4)\beta J^{-2}\rho_m^1 + 2J^{-1}\rho_m^1 + J^{-2}\rho_m^2.$$ 

Thus, $R_m \leq R^*$, where

$$R^* = AQ^* + E(4)\beta J^{-2}1/3 + 2J^{-1}1/3 + J^{-2}5/42. \quad (4.4.5)$$

The $\xi_{m-1}^2$ inequality requires a little more care due to the presence of the incomplete convolution term $q_{m-1}''$. For the remaining terms, we proceed as we did with the previous inequalities:

$$\theta\bar{p}\bar{p}_m + \theta p(q'_{m-2} + q''_{m-3} + q'_{m-3}) + \theta\bar{p}(\bar{q}'_{m-2} + \theta p|\xi_{m-3}^2| + \theta p|\xi_{m-3}^2|) \leq$$

$$\{\theta\bar{p}Q_m + \theta p(0.5\beta J^{-2}\rho_m^1 + 0.25\beta^2 J^{-3}\rho_m^1 + 0.5\beta J^{-3}\rho_m^2)$$

$$+ \theta\bar{p}(0.5\beta J^{-2}\rho_m^1 + 0.5\theta\bar{p}\beta J^{-3}\rho_m^2 + J^{-3}\rho_m^3)\} \beta J^m C_m.$$ 

since this is an upper bound on $|\xi_{m-1}^2|$, we must replace the term $\beta J^m C_m$ with $\beta J^{m-1} C_{m-1}$ as follows:

$$\beta J^m C_m = J \left(\frac{C_m}{C_{m-1}}\right) \beta J^{m-1} C_{m-1}$$

$$\leq 4J \beta J^{m-1} C_{m-1}.$$

Using this replacement and the bounds (4.2) on the numbers $\rho_{m}^k$, we obtain the upper bound

$$4J\{\theta\bar{p}Q^* + \theta p(E(4)\beta J^{-2}(1/3) + E^2(4)\beta^2 J^{-3}(1/3) + E(4)\beta J^{-3}(5/42))$$

$$+ \theta\bar{p}(E(4)\beta J^{-2}(1/3) + E(4)\theta\bar{p}\beta J^{-3}(5/42) + \theta p J^{-3}(1/21))\} \beta J^{m-1} C_{m-1} \quad (4.4.6)$$

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It remains to deal with $q_{m-1}^*$. To do this, we first write $q_{m-1}^*$ as follows

$$
q_{m-1}^* = p_{m-1}|\xi_0^2| + p_{m-3}|\xi_2^2| + p_{m-1-\ell}|\xi_\ell^2|^{2<\ell<m-3} + p_2|\xi_{m-3}^2| + .
$$

(recall that we can take $m - 1$ even, since in the case $m - 1$ odd the equality $\xi_{odd} = 0$ leaves nothing to prove) The non-convolution terms then give

$$
p_{m-1}|\xi_0^2| + p_{m-3}|\xi_2^2| + p_2|\xi_{m-3}^2|
\leq \left( |\xi_0^2|R_{m-1} + |\xi_2^2|J^{-2}\frac{C_{m-3}}{C_{m-1}} + p_2J^{-2}\frac{C_{m-3}}{C_{m-1}} \right) \beta J^{m-1} C_{m-1}
\leq \left( |\xi_0^2|R^* + |\xi_2^2|J^{-2}(1/7) + p_2J^{-2}(1/7) \right) \beta J^{m-1} C_{m-1},
$$

since $C_{m-3}/C_{m-1} \leq C_2/C_4 = 1/7$, if $m \geq 5$. The remaining term $p_{m-1-\ell}|\xi_\ell^2|^{2<\ell<m-3}$ is treated in a completely different manner:

$$
p_{m-1-\ell}|\xi_\ell^2|^{2<\ell<m-3} = p_{m-5}|\xi_4^2| + p_{m-7}|\xi_6^2| + \cdots + p_6|\xi_{m-8}^2| + p_4|\xi_{m-5}^2|
\leq (C_{m-5}C_4 + C_{m-7}C_6 + \cdots + C_6C_{m-7} + C_4C_{m-5}) \beta^2 J^{m-1}
\leq (C_{m-1-\ell}C^{(\ell even)} - 2C_2C_{m-3} - 2C_0C_{m-1}) \beta^2 J^{m-1}
\leq (E(4)C_{m-1-\ell}C \ell - 2C_2C_{m-3} - 2C_0C_{m-1}) \beta^2 J^{m-1}
\leq (E(4)C_m - 2C_2C_{m-3} - 2C_0C_{m-1}) \beta^2 J^{m-1}
\leq \left( \frac{E(4)C_m - 2C_2C_{m-3} - 2C_0C_{m-1}}{C_{m-1}} \right) \beta J^{m-1} C_{m-1}
\leq \beta(E(4)4 - 1/4 - 2) \beta J^{m-1} C_{m-1},
$$

Thus, adding (4.4.6), (4.4.8) and (4.4.9), we set

$$
S^* = \langle \psi_0 \rangle^{-1}(4J\{\theta_{p}Q^* + \theta_{p}(E(4)\beta J^{-2}(1/3) + E^2(4)\beta^2 J^{-3}(1/3) + E(4)\beta J^{-3}(5/42)) + \theta_{p}(E(4)\beta J^{-2}(1/3) + E(4)\theta_{p}\beta J^{-3}(5/42) + \theta_{p}J^{-3}(1/21)) \} + \theta_{p}\{(|\xi_0^2|R^* + |\xi_2^2|J^{-2}(1/7) + p_2J^{-2}(1/7) \} + (0.7976\beta) \} \rangle.
$$
Using the fact that \( \langle \psi_0 \rangle_Q \) is bounded from below and that \( \langle \psi_0 \rangle^{-1} 0.7976 \beta \leq 1 \) (see (4.8.3) in the appendix), there exists a number \( J_2 \) such that \( Q^*/(J_2) \), \( R^*/(J_2) \), \( S^*/(J_2) \) \( \leq 1. \) Now, taking \( J = \max\{J_1, J_2\} \), we have shown that the induction hypothesis (4.4.1) implies
\[
\bar{p}_m, p_m, |\xi_m^2| \leq \beta C_m J^m,
\]
so that in fact (4.4.11) holds for every integer \( m. \) \( \Box \)

### 4.5 Proof of Theorem (4.1.2): Solution of the Eigenvalue Problem

**Proof.** The weak form of the master system is
\[
\int_Q \left[ \epsilon^{-1}_\eta (\nabla + i\eta \hat{\kappa}) h_\eta(y) \cdot (\nabla - i\eta \hat{\kappa}) \tilde{v}(y) - \eta^2 \xi^2_\eta h_\eta(y) \tilde{v}(y) \right] = 0 \quad \text{for all } v \in H^1_{\text{per}}(Q).
\]

(4.5.1)

Using that \( \epsilon^{-1}_\eta = \eta^2 \xi^2_\eta / (\eta^2 \xi^2_\eta - 1) \) in \( P \) and \( \epsilon_\eta = 1 \) in \( \bar{P} \), and multiplying by \( (\eta^2 \xi^2_\eta - 1) \), gives the equivalent system
\[
a_\eta(h, \xi^2; v) = 0 \quad \text{for all } v \in H^1_{\text{per}}(Q),
\]
in which
\[
a_\eta(h, \xi^2; v) = -\int_P (\nabla + i\eta \hat{\kappa}) h \cdot (\nabla - i\eta \hat{\kappa}) \tilde{v} + \int_Q \left[ \eta^2 \xi^2 (\nabla + i\eta \hat{\kappa}) h \cdot (\nabla - i\eta \hat{\kappa}) \tilde{v} - (\eta^2 \xi^2 - 1) \eta^2 \xi^2 h \tilde{v} \right].
\]

This form can be expanded in powers of \( \eta \),
\[
a_\eta(h, \xi^2; v) = a_0(h; v) - i\eta a_1(h; v) - \eta^2 a_2(h, \xi^2; v) + i\eta^3 a_3(h, \xi^2; v) + \eta^4 a_4(h, \xi^2; v)
\]
(4.5.2)
in which the $a_m$ are real forms

$$
\begin{align*}
  a_0(h; v) &= -\int_P \nabla h \cdot \nabla \bar{v}, \\
  a_1(h; v) &= \int_P (\hat{h} \cdot \nabla \bar{v} - \nabla h \cdot \hat{\kappa} \bar{v}), \\
  a_2(h, \xi^2; v) &= \int_P h \bar{v} - \xi^2 \int_Q (\nabla h \cdot \nabla \bar{v} + h \bar{v}), \\
  a_3(h, \xi^2; v) &= \xi^2 \int_Q (h \hat{\kappa} \cdot \nabla \bar{v} - \nabla h \cdot \hat{\kappa} \bar{v}), \\
  a_4(h, \xi^2; v) &= \xi^2 \int_Q (1 - \xi^2)h \bar{v}.
\end{align*}
$$

Define the partial sums

$$
\begin{align*}
  \xi_{\eta}^{2,N} &= \sum_{m=0}^{N} \eta^m \xi_m^2 \\
  h_{\eta}^N &= \sum_{m=0}^{N} \eta^m h_m.
\end{align*}
$$

For $\eta < R$, the sequence $\{\xi_{\eta}^{2,N}\}$ converges to a number $\xi_{\eta}^2$ and the sequence $\{h_{\eta}^N\}$ converges in $H_1^1$ to a function $h_{\eta}$; thus

$$
a_j(h_{\eta}^N, \xi_{\eta}^{2,N}; v) \to a_j(h_{\eta}, \xi_{\eta}^2; v) \quad \text{for all } v \in H_1^1(Q) \text{ and } i = 0, \ldots, 4.
$$

Therefore, $a_j(h_{\eta}, \xi_{\eta}^2; v), j = 1, \ldots, 4$, has a convergent series representation in powers of $\eta$, in which the $m^{th}$ coefficient is related to the coefficients $\xi_\ell$ and $h_\ell$ by

$$
\begin{align*}
  (j = 0) & \quad \int_P -\nabla h_m \cdot \nabla \bar{v}, \\
  (j = 1) & \quad \int_P (h_m \hat{\kappa} \cdot \nabla \bar{v} - \nabla h_m \cdot \hat{\kappa} \bar{v}), \\
  (j = 2) & \quad \int_P h_m \bar{v} - \int_Q (\nabla (\xi_\ell^2 h_{\ell - \ell}) \cdot \nabla \bar{v} + (\xi_\ell^2 h_{\ell - \ell}) \bar{v}), \\
  (j = 3) & \quad \int_Q (\xi_\ell^2 h_{\ell - \ell}) \hat{\kappa} \cdot \nabla \bar{v} - \nabla (\xi_\ell^2 h_{\ell - \ell}) \cdot \hat{\kappa} \bar{v}), \\
  (j = 4) & \quad \int_Q (\xi_j^2 h_{\ell - j} - \xi_j^2 \xi_j^2 \xi_{\ell - j} h_{\ell - j}) \bar{v}.
\end{align*}
$$

From these, one obtains the $m^{th}$ coefficient of $a_\eta(h_{\eta}, \xi_{\eta}^2; v)$ (see (4.5.2)), which, by means of the relations $h_m = h_0 i^m \psi_m$, $\xi_\ell^2 h_{\ell - \ell} = h_0 i^m \sigma_m'$ and $\xi_j^2 \xi_j^2 \xi_{\ell - j} h_{\ell - \ell} = h_0 i^m \sigma_m''$,
is seen to be equal to the $-i^m h_0$ times the right-hand side of equation (2.2.8). All these coefficients are therefore equal to zero, and we conclude that $a_n(h_{\eta}, \xi_\eta^2; \nu) = 0$. This proves that the function $h_{\eta}$, together with the frequency $\sqrt{\xi_\eta}$ solve the weak form (4.5.1) of the master system. 

4.6 Computing the Constant A for Circular Inclusions

Given a function $\psi \in H^1_{\text{per}}(\bar{P})$, let $u \in H^1_P$ satisfy

\begin{align*}
\nabla^2 u - u &= 0 \quad \text{in } P, \\
u &= \psi \quad \text{on } \partial P.
\end{align*}

We seek to compute a number $A$ such that $\|u\|_{H^1(P)}^2 \leq A\|\psi\|_{H^1(\bar{P})}^2$ for all $\psi$. Following [11], we will calculate a value of $A$ for circular inclusions $P$ of radius $r_0 < 0.5$ by restricting $\psi$ to the annulus between $P$ and the circle of radius $1/2$. It suffices to consider real-valued functions $\psi$ that minimize the $H^1$ norm in the annulus, that is $\nabla^2 \psi - \psi = 0$, $r_0 < r < 0.5$. A function of this type is given generally by the real part of an expansion $\psi(r, \theta) = \sum_{n=0}^{\infty} \left(c_n I_n(r) + d_n K_n(r)\right) e^{in\theta}$, in which $c_n$ and $d_n$ are complex numbers and $I_n$ and $K_n$ are the “modified” Bessel functions.

The continuous continuation of $\psi$ into the disk with $\nabla^2 u - u = 0$ is given by the real part of $u(r, \theta) = \sum_{n=0}^{\infty} f_n I_n(r) e^{in\theta}$ under the relations

\begin{equation}
f_n = c_n + d_n \frac{K_n(r_0)}{I_n(r_0)}.
\end{equation}

One computes that $\|\text{Re } \psi\|^2 = \frac{1}{2} \|\psi\|^2$ and $\|\text{Re } u\|^2 = \frac{1}{2} \|u\|^2$, so we may work with the complex functions rather than their real parts. The Helmholtz equation in $P$ and integration by parts yield

\begin{equation*}
\|u\|_{H^1(P)}^2 = \int_P (|\nabla u|^2 + |u|^2) \, dA = \int_{\partial P} \bar{u} \partial_n u,
\end{equation*}
and this provides the representation \( \|u\|_{H^1}^2 = 2\pi r_1 \sum_{n=0}^{\infty} \tilde{f}_n I_n(r_1) f_n'(r_1) \). The analogous representation in the annulus is

\[
\|\psi\|_{H^1}^2 = \int_0^{2\pi} (\partial_r \psi(1, \theta) \psi(1, \theta)) d\theta - \int_0^{2\pi} r_0 (\partial_r \psi(r_0, \theta) \overline{\psi(r_0, \theta)}) d\theta
\]

\[
= 2\pi \sum_{n=0}^{\infty} \left( \bar{c}_n I_n(0.5) + \bar{d}_n K_n(0.5) \right) \left( c_n I_n'(0.5) + d_n K_n'(0.5) \right) +
\]

\[
- 2\pi r_0 \sum_{n=0}^{\infty} \left( \bar{c}_n J_n(r_0) + \bar{d}_n K_n(r_0) \right) \left( c_n I_n'(r_0) + d_n K_n'(r_0) \right)
\]

We seek a positive number \( A \) such that, for all choices of complex numbers \( c_n \) and \( d_n \) we have \( 0 \leq A \|\psi\|^2 - \|u\|^2 \). The right-hand-side of this inequality is a quadratic form in all of the coefficients \( (c_n, d_n) \) that depends on \( A \),

\[
A \|\psi\|^2 - \|u\|^2 = m_{11}^n c_n \bar{c}_n + m_{12}^n c_n \bar{d}_n + m_{21}^n d_n \bar{c}_n + m_{22}^n d_n \bar{d}_n,
\]

in which the \( m_{ij}^n \) depend on \( A \) and are conveniently expressed in terms of the functions

\[
\Pi_n(r) = I_n(r) I_n'(r), \quad KK_n(r) = K_n(r) K_n'(r), \quad IK_n(r) = I_n(r) K_n'(r)
\]

\[
KI_n(r) = K_n(r) I_n'(r), \quad JJ_n(r) = \frac{K_n(r)^2}{I_n(r)} I_n'(r).
\]

\[
m_{11}^n = -r_0 \Pi_n(r_0) - Ar_0 \Pi_n(r_0) + A \Pi_n(0.5),
\]

\[
m_{22}^n = -r_0 JJ_n(r_0) - Ar_0 KK_n(r_0) + A KK(0.5),
\]

\[
m_{12}^n = -r_0 KI(r_0) - Ar_0 IK_n(r_0) + A IK_n(0.5),
\]

\[
m_{21}^n = -r_0 KI(r_0) - Ar_0 IK_n(r_0) + A IK_n(0.5).
\]

The form \( m_{ij}^n \) is Hermitian, as one can show that \( m_{12}^n = m_{21}^n \) by using the fact that \( r \text{Wron}[I_n, K_n] \) is constant.
We must find $A > 0$ such that $m_n^{11} \geq 0$ and $m_n^{11}m_n^{22} - m_n^{12}m_n^{21} \geq 0$ for all $n = 0, 1, 2, \ldots$. These quantities are equal to

\[
m_n^{11} = \beta(r_0, 0.5)A - \alpha(r_0),
\]
\[
m_n^{11}m_n^{22} - m_n^{12}m_n^{21} = \epsilon(r_0, 0.5)A^2 - \delta(r_0, 0.5)A,
\]

in which

\[
\alpha_n(r) = r\Pi_n(r),
\]
\[
\beta_n(r, s) = \Pi_n(s) - r\Pi_n(r),
\]
\[
\delta_n(r, s) = rs[\Pi_n(r)KK_n(s) + \Pi_n(s)JJ_n(r) - KI_n(r)IK_n(s) - KI_n(s)KI_n(r)] + r^2[-\Pi_n(r)KK_n(r) - \Pi_n(r)JJ(r) + KI_n(r)IK_n(r) + KI_n(r)KI_n(r)],
\]
\[
\epsilon_n(r, s) = rs[-\Pi_n(r)KK_n(s) - \Pi_n(s)KK_n(r) + KI_n(r)IK_n(s) + KI_n(s)IK_n(r)].
\]

The numbers $\alpha_n(r)$ and $\beta_n(r, s)$ for $r < s$ are positive; the latter because $(rI_nI'_n)' = \frac{1}{r}(r^2 + n^2)I_n^2 + rI_n^2 > 0$. One can show that $\delta$ and $\epsilon$ are positive. Thus, it is sufficient to find $A > 0$ such that, for all $n = 0, 1, 2, \ldots,$

\[
A \geq \max \left\{ \frac{\alpha_n(r_0)}{\beta_n(r_0, 0.5)}, \frac{\delta_n(r_0, 0.5)}{\epsilon_n(r_0, 0.5)} \right\}.
\]

Table 4.1 shows computed values of $A$ for various values of $r_0$.

### 4.7 A Note on the Catalan Bound

This is an illustration of what occurs when one attempts to use induction to prove the existence of an exponential bound for the Catalan numbers. Suppose that we wish to prove inductively that certain positive numbers $\{b_\ell\}_{\ell=0}^\infty$ are bounded exponentially,

\[
b_\ell \leq b_0 r^\ell, \tag{4.7.1}
\]
using the information that they are bounded by the convolution of the previous terms,

\[ b_{m+1} \leq b_\ell b_{m-\ell}. \]  

(4.7.2)

Assuming (4.7.1) for \( 0 \leq \ell \leq m \), inequality (4.7.2) gives

\[ b_{m+1} \leq (b_0 r^\ell) (b_0 r^{m-\ell}) \leq m \frac{b_0^2}{r} r^{m+1} \]

and it is not possible to close the induction. The appearance of the factor \( m \) suggests a factorial bound, which can be established by induction as follows. Assuming

\[ b_\ell \leq b_0 r^\ell \ell!, \]

(4.7.3)

we obtain

\[
\begin{align*}
   b_{m+1} & \leq b_\ell b_{m-\ell} \\
   & \leq b_0^2 r^m \ell!(m - \ell)!.
\end{align*}
\]

(4.7.4)

The convolution \( \ell!(m - \ell)! \) has \( m + 1 \) terms, the largest of which is \( m! \). Thus, inequality (4.7.4) gives

\[
\begin{align*}
   b_{m+1} & \leq \frac{b_0^2}{r} r^{m+1} (m + 1)!,
\end{align*}
\]

(4.7.5)

and the induction is successful for \( r \geq b_0 \).

There exists a special combination of factorials which does give exponential growth, namely the Catalan number \( C_m = \frac{1}{2^m} \binom{2m}{m} \): if we assume that

\[ b_\ell \leq b_0 r^\ell C_\ell, \]

(4.7.6)

then

\[
\begin{align*}
   b_{m+1} & \leq b_\ell b_{m-\ell} \\
   & \leq b_0^2 r^m C_\ell C_{m-\ell} \\
   & = \frac{b_0^2}{r} r^{m+1} C_{m+1},
\end{align*}
\]

and using that \( C_m \leq 4^m \), an exponential bound is obtained.
4.8 A Lower Bound on $\langle \psi_0 \rangle_Q$ and an Upper Bound on $\beta$

Recall the boundary value problem for $\psi_0$ in $P$

$$
\begin{align*}
\Delta \psi_0 &= \psi_0, \quad y \in P \\
\psi_0 &= 1, \quad \partial P.
\end{align*}
$$

(4.8.1)

The inequality $0 \leq \psi_0 \leq 1$ can be established by using in the differential equation for $\psi_0$ the fact that at a minimum (maximum) the second derivative of a function must be non-negative (non-positive). Consider now the case where $P$ is a circular domain of radius $a$. Using polar coordinates, the above boundary value problem becomes

$$
\begin{align*}
(ru')' &= ru, \quad 0 < r < a \\
u(a) &= 1,
\end{align*}
$$

(4.8.2)

where $u = \psi_0$ and $' = \frac{\partial}{\partial r}$, since the the boundary condition $\psi_0 = 1$ eliminates the dependence of $u$ on the angle variable. Integrating the above differential equation from $\delta$ to $r$, $0 < \delta \leq r < a$, and taking the limit $\delta \to 0$, we obtain

$$
u'(r) = \frac{1}{r} \int_0^r \rho u(\rho) \, d\rho.
$$

The inequality $u \leq 1$ then gives $u'(r) \leq r/2$, so that $u(0) \geq u(r) - r^2/4$. Using the boundary condition $u(a) = 1$, we then obtain $u(0) \geq 1 - a^2/4$. Moreover, since $u \geq 0$, the above equality shows that $u$ is increasing, so that its minimum is attained at the origin $r = 0$. Thus, for circular inclusions

$$
\langle \psi_0 \rangle_P \geq \theta_P (1 - \frac{a^2}{4}) = \theta_P (1 - \frac{\theta_P}{4\pi}).
$$

(4.8.3)

That this lower bound holds for all connected inclusions is a consequence of the theorem below.
Theorem 4.8.1. (Sag Theorem) Consider the boundary value problem 4.8.1 for the function $\psi_0$ in a bounded region $P$. Then, of all regions with a given area, the disk minimizes the integral $\langle \psi_0 \rangle_P$. If the region $P$ has multiple components, one should consider the disk whose area equals the sum of the areas of the components.

This theorem has a simple physical interpretation. If one considers an elastic membrane placed over a thin wire frame of constant height (the wire shape corresponds to $\partial P$ and the constant height to the boundary condition $\psi_0 = 1$), then the membrane will sag under the influence of gravity and its shape will be given by the graph of $\psi_0$ (with the correct choice of units for the elasticity constant of the membrane). The above theorem states that a circular wire will produce the greatest sag.

Proof. We have the following identity relating the integral of $\psi_0$ to its $H^1$ norm

$$\langle \psi_0 \rangle_P = \langle \nabla \psi_0 \cdot \mathbf{n} \rangle_{\partial P} = \langle \psi_0 \nabla \psi_0 \cdot \mathbf{n} \rangle_{\partial P} = \langle \nabla \cdot (\psi_0 \nabla \psi_0) \rangle_P = \langle |\nabla \psi_0|^2 + \psi_0^2 \rangle_P = \inf \{ \langle |\nabla \psi|^2 + \psi^2 \rangle_P \},$$

where the infimum is taken over all functions satisfying the boundary condition $\psi = 1$ on $\partial P$. Transforming the domain $P$ by symmetrization [34] into a disk $\hat{P}$ of the same area, each one of the integrals $\langle \psi^2 + |\nabla \psi|^2 \rangle_P$ decreases, so that

$$\inf \{ \langle |\nabla \psi|^2 + \psi^2 \rangle_P \} \geq \inf \{ \langle |\nabla \psi|^2 + \psi^2 \rangle_{\hat{P}} \}.$$

Since $\inf \{ \langle |\nabla \psi|^2 + \psi^2 \rangle_{\hat{P}} \} = \langle |\nabla \psi_0|^2 + \psi_0^2 \rangle_{\hat{P}}$, this completes the proof. \qed
Now that a lower bound on $\langle \psi_0 \rangle_Q$ has been established, it is possible to obtain upper bounds on the quantities $\beta$ and $0.7976 \frac{\sqrt{\theta_P \beta}}{\langle \psi_0 \rangle_Q}$ appearing in the proof of the Catalan bound. Indeed, since $\psi_0 \equiv 1$ in $\bar{P}$, we have $\bar{p}_0 = \theta_P$. Since $\langle \psi_0 \rangle_P = p_0^2$ (see proof of theorem above) and $0 \leq \psi_0 \leq 1$, we have that $p_0 \leq \sqrt{\theta_P}$. Setting $v = \psi_1$ in the weak form for $\psi_1$, we get that $\langle \hat{\kappa} \cdot \nabla \psi_1 \rangle_P = -\langle \nabla \psi_1 \cdot \nabla \psi_1 \rangle_P < 0$. Using this in expression (3.2.1) gives $\xi_0^2 < 1$. Thus, we may take $\beta \leq 1$, which is sufficient for the $\bar{p}_m$ and $p_m$ inequalities. The bound 4.8.3 shows that $0.7976 \frac{\sqrt{\theta_P \beta}}{\langle \psi_0 \rangle_Q} < 1$, which is sufficient for the $|\xi_m^2|$ inequality.
Chapter 5
Effective Properties

In this section we start by identifying an effective property directly from the dispersion relation. We then discuss the relation between effective properties and quasistatic properties. Next we provide explicit error bounds for finite-term approximations to the first branch of the dispersion relation for nonzero values of $\eta$. The error bounds show that numerical computation of the first two terms of the power series delivers an accurate and inexpensive numerical method for calculating dispersion relations for sub-wavelength plasmonic crystals.

5.1 The Effective Index of Refraction

The identification of an effective index of refraction valid for $\eta > 0$ follows directly from the dispersion relation given by the series for $\omega_\eta^2$. Indeed the effective refractive index $n_{\text{eff}}^2$ is defined by expressing the dispersion relation as

$$\omega_\eta^2 = \frac{c^2 k^2}{n_{\text{eff}}^2}$$

and it then follows from the expansion for $\omega_\eta^2$ that the effective refractive index has the convergent power series expansion

$$n_{\text{eff}}^2 = n_{\text{qs}}^2 + \sum_{m=1}^{\infty} \eta^{2m} \xi_{2m}^2.$$ (5.1.2)

We now discuss the relationship between the effective index of refraction and the quasistatic effective properties seen in the $d \to 0$ limit with $k$ fixed. The effective refraction index $n_{\text{eff}}$ can be rewritten in the equivalent form by the equation $n_{\text{eff}}^2 = 1/\xi_\eta^2$. By setting $v = h_\eta$ in the weak form of the master system (4.5.1), it is easily seen that $\xi_\eta^2 > 0$ for all $\eta$ within the radius of convergence, so that $n_{\text{eff}}^2 > 0$ for
those values of η. Following Pendry et al. [33], see also [25], we define the effective permeability by

\[
\mu_{\text{eff}} = \frac{(B_3)_{\text{eff}}}{(H_3)_{\text{eff}}},
\]

and we then define \(\epsilon_{\text{eff}}\) through the equation

\[
n_{\text{eff}}^2 = \epsilon_{\text{eff}}\mu_{\text{eff}}.
\]

The quasi-static effective properties are recovered by passing to the limits

\[
n_{qs}^2 = \lim_{\eta \to 0} n_{\text{eff}}^2, \quad \mu_{qs} = \lim_{\eta \to 0} \mu_{\text{eff}}, \quad \epsilon_{qs} = \lim_{\eta \to 0} \epsilon_{\text{eff}}.
\]

A simple computation shows that \(\mu_{qs} = \langle \psi_0 \rangle_Q > 0\) (see the appendix). Hence, we have that \(\mu_{\text{eff}} > 0\) for η in a neighborhood of the origin, so that \(\epsilon_{\text{eff}} > 0\) for these values of η, since \(n_{\text{eff}}^2 > 0\) for all η in the radius of convergence. Thus, one has a solid basis on which to assert that plasmonic crystals function as materials of positive index of refraction in which both the effective permittivity and permeability are positive.

For circular inclusions we have used the program COMSOL to compute that \(\langle \psi_0 \rangle_Q \approx 0.98\), so that only a mild effective magnetic permeability arises.

Having established that \(h_\eta(y)e^{in\hat{k}\cdot y}\) is the solution to the unit cell problem, we can undo the change of variable \(y = kx/\eta\) to see that the function

\[
\hat{h}_\eta \left( \frac{kx}{\eta} \right) e^{i(k\hat{k}\cdot x - \omega_\eta t)},
\]

where \(\hat{h}_\eta\) is the \(Q\)-periodic extension of \(h_\eta\) to all of \(\mathbb{R}^2\), is a solution of

\[
\nabla_x \cdot (\epsilon_\eta^{-1} \nabla_x \hat{h}_\eta) = \frac{1}{c^2} \partial_{tt} \hat{h}_\eta, \quad x \in \mathbb{R}^2,
\]

for every η in the radius of convergence.
We investigate the quasistatic limit directly using the power series (5.1.5). Here we wish to describe the average field as \( d \to 0 \). To do this we introduce the three-dimensional period cell for the crystal \([0, d]^3 \). The base of the cell in the \( x_1x_2 \) plane is denoted by \( Q_d = [0, d]^2 \) and is the period of the crystal in the plane transverse to the rods. We apply the definition of \( B_{\text{eff}} \) and \( H_{\text{eff}} \) given in Pendry et al. [33] which in our context is

\[
(B_3)_{\text{eff}} = \frac{1}{d^2} \int_{Q_d} \hat{h}_\eta \left( \frac{kx}{\eta} \right) e^{i(k\hat{k} \cdot x - \omega_d t)} dx_1 dx_2 \tag{5.1.7}
\]

and

\[
(H_3)_{\text{eff}} = \frac{1}{d} \int_{(0,0,0)}^{(0,0,d)} \hat{h}_\eta \left( \frac{kx}{\eta} \right) e^{i(k\hat{k} \cdot x - \omega_d t)} dx_3. \tag{5.1.8}
\]

Taking limits for \( k \) fixed and \( d \to 0 \) in (5.1.5) gives

\[
\lim_{d \to 0} (B_3)_{\text{eff}} = \langle \psi_0 \rangle_Q \bar{h}_0 e^{i(k\hat{k} \cdot x - \omega_{qs} t)} \quad \text{and} \quad \lim_{d \to 0} (H_3)_{\text{eff}} = \bar{h}_0 e^{i(k\hat{k} \cdot x - \omega_{qs} t)},
\]

in which \( \omega_{qs}^2 = \frac{c^2 k^2}{n_{qs}^2} \). These are the same average fields that would be seen in a quasistatic magnetically active effective medium with index of refraction \( n_{qs} \) and \( \mu_{qs} \) that supports the plane waves

\[
(B_3)_{qs} = \mu_{qs} \bar{h}_0 e^{i(k\hat{k} \cdot x - \omega_{qs} t)} \quad \text{and} \quad (H_3)_{qs} = \bar{h}_0 e^{i(k\hat{k} \cdot x - \omega_{qs} t)},
\]

where \( \mu_{qs} = \langle \psi_0 \rangle_Q \). It is evident that these fields are solutions of the homogenized equation \( \frac{n_{qs}^2}{c^2} \partial_t u = \Delta u \). This quasistatic interpretation provides further motivation for the definition of \( n_{\text{eff}} \) for nonzero \( \eta \) given by 5.1.2.

Now we apply the definition of effective permeability \( \mu_{\text{eff}} \) given in Pendry et al. [33], together with \( n_{\text{eff}} \) to define an effective permeability \( \epsilon_{\text{eff}} \) for \( \eta > 0 \). The relationships between the effective properties and quasistatic effective properties are used to show that plasmonic crystals function as meta-materials of positive index of refraction in which both the effective permittivity and permeability are positive for \( \eta > 0 \).
5.2 Absolute Error Bounds

The Catalan bound provides simple estimates on the size of the tails for the series $\xi_2^2$ and $h_\eta$,

$$E_{m_0,\xi} = \sum_{m=m_0+1}^\infty \xi_{2m}^2 \eta^{2m} \quad \text{and} \quad E_{m_0,h} = \sum_{m=m_0+1}^\infty h_m \eta^m.$$  

We have established convergence for $\eta \leq 1/4J$, so that we may write $\eta = \alpha/4J$, $0 \leq \alpha \leq 1$. Then, using that $|\xi_{2m}^2| \leq \beta C_2 J^{2m}$, $C_2 \leq 4^{2m}$ and $4J\eta = \alpha$, we have

$$|E_{m_0,\xi}| = \left| \sum_{m=m_0+1}^\infty \xi_{2m}^2 \eta^{2m} \right| \leq \beta \sum_{m=m_0+1}^\infty C_2 J^{2m} \eta^{2m} \leq \beta \sum_{m=m_0+1}^\infty (4J\eta)^{2m} \leq \beta \sum_{m=m_0+1}^\infty \alpha^{2m} = \beta \frac{\alpha^{2m_0+2}}{1-\alpha^2}. \quad (5.2.1)$$

Similarly, for $h_\eta$ we have that

$$\|E_{m_0,h}\|_{H^1(Q)} \leq 2\beta |h_0| \alpha^{m_0+1} \frac{1}{1-\alpha}. \quad (5.2.2)$$

5.3 Relative Error Bounds

In this section, we use the absolute error bound (5.2.2) with $m_0 = 1$ to obtain a relative error bound for the particular case of a circular inclusion of radius $r = 0.45$ [36]. The first term approximation to $h_\eta$ is

$$h_\eta = h_0 \psi_0 + ih_0 \psi_1 \eta + E_{1,h}. \quad (5.3.1)$$

For a circular inclusion of radius $r = 0.45$, we have $J \leq 85$, $\beta \leq 0.79$, $\|\psi_0\| = 0.97$ and $\|\psi_1\| = 0.02$, where $\| \cdot \| = \| \cdot \|_{H^1(Q)}$. Thus, using bound (5.2.2), the
relative error $R_{1,h}$ is bounded by

$$|R_{1,h}| = \frac{|E_{1,h}|}{\|h_0 + ih_0\psi_1\eta\|} \leq \frac{1.58\frac{\alpha^2}{1-\alpha}}{\|h_0\| - \|h_0\psi_1\|\|\eta\|} \leq \frac{1.58\frac{\alpha^2}{1-\alpha}}{0.97 - 0.02\frac{\alpha}{340}},$$

so that for $\alpha \leq 0.2$ the relative error is less than 8.2%. The graphs of $\psi_0$ and $\psi_1$ can be found in the Appendix. The first term approximation to $\xi^2_\eta$ is

$$\xi^2_\eta = \xi^2_0 + \xi^2_2\eta^2 + E_{1,\xi}.$$  \hfill (5.3.2)

In the Appendix we indicate how the tensors $\xi^2_m$ may be computed. For an inclusion of radius $r = 0.45$, we have $\xi^2_0 = 0.36$ and $\xi^2_2 = -0.14$. Thus, using bound (5.2.1), the relative error $R_{1,\xi}$ is bounded by

$$|R_{1,\xi}| = \frac{|E_{1,\xi}|}{|\xi^2_0\psi_0 + \xi^2_2\eta^2|} \leq \frac{\beta\frac{\alpha^4}{1-\alpha^2}}{|\xi^2_0 + \xi^2_2\eta^2|^2} \leq \frac{0.79\frac{\alpha^4}{1-\alpha^2}}{|0.36 - 0.14\frac{\alpha^2}{340}|},$$

so that for $\alpha \leq 0.3$ the relative error is less than 2%. 

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FIGURE 5.1. Solid curve is $R_{1,\xi}$ and dotdash curve is $R_{1,h}$.

FIGURE 5.2. Graph of the first branch of the dispersion relation. Dashed curves represent error bars.
Chapter 6
Stop Bands and Passbands in layered Media

As discussed in the introduction, section (1.3), the topology of the host phase affects the dispersion in a significant way. Here we present an analysis of the dispersion relation for a periodic layered medium which elaborates on this.

6.1 Existence of a Stopband at the Origin for Layered Media

The functional dependence between the wavevector $k$ and the angular frequency $\omega$ for a plane wave $A e^{i(kx - \omega t)}$ propagating in a homogeneous medium is given by $\omega^2 = \frac{1}{c^2} |k|^2$. This is easily obtained by inserting the plane wave ansatz into the wave equation $\Delta u = \frac{1}{c^2} u_{tt}$. In non-homogeneous media, this functional dependence can become much more complicated and many interesting physical phenomena can take place, such as the appearance of stopbands, which occur when complex values of the Bloch wave vector $k$ correspond to the frequency $\omega$. Insertion of $k = \alpha + i\beta$ into the solution $A(x) e^{i(kx - \omega t)}$ produces the factor $e^{-\beta x}$, so that the wave decays in amplitude as it attempts to travel through the medium. When a value of the frequency does not lie in a stopband, it is said to lie in a passband.

The dispersion relation for a layered medium in which $\epsilon$ is a step function constant in each layer was obtained by van der Pol and Strutt in 1920 (see the book by Brillouin [10]). We use this dispersion relation to establish the existence of a stopband in a neighborhood of the origin $\omega = 0$ for layered media with a negative effective permittivity is negative.
The Maxwell equations for a time harmonic, $H$-polarized normally incident wave propagating in a layered medium prescribe that $H$ and $\epsilon^{-1}H'$ must be continuous functions such that

$$(\epsilon^{-1}H')' + \frac{\omega^2}{c^2} H = 0. \quad (6.1.1)$$

where

$$\epsilon(x) = \begin{cases} 
\epsilon_{p}, & -\theta_pd < x < 0 \\
\epsilon_{\bar{p}}, & 0 < x < \theta_{P}d.
\end{cases}$$

and $\mu = 1$.

Using these continuity restrictions, van der Poll and Strutt [10] showed that a solution of the form $e^{ikx}h(x)$, where $h(x + d) = h(x)$, has the following dispersion relation

$$\cos(kd) = \cosh \left( i\theta_{P}d\sqrt{\epsilon_{p} - \omega^2/c^2} \right) \cosh \left( i\theta_{\bar{P}}d\sqrt{\epsilon_{\bar{p}} - \omega^2/c^2} \right) +$$

$$+ \frac{1}{2} \left( \frac{\sqrt{\epsilon_{p}}}{\sqrt{\epsilon_{p}}} + \frac{\sqrt{\epsilon_{\bar{p}}}}{\sqrt{\epsilon_{\bar{p}}}} \right) \sinh \left( i\theta_{P}d\sqrt{\epsilon_{p} - \omega^2/c^2} \right) \sinh \left( i\theta_{\bar{P}}d\sqrt{\epsilon_{\bar{p}} - \omega^2/c^2} \right) \quad (6.1.2)$$

The above is one equation in the seven parameters $\sqrt{\epsilon_{p}}, \sqrt{\epsilon_{\bar{p}}}, \theta_{p}, \theta_{\bar{p}}, d, \omega$ and $k$ (since $c$ is the speed of light in vacuum). We now fix the quantities $\sqrt{\epsilon_{p}}, \sqrt{\epsilon_{\bar{p}}}, \theta_{p}$ and $\theta_{\bar{p}}$ and expand the right-hand and left-hand sides in their Taylor series in $d$. Equating the coefficients of $d^2$ to obtain the following “homogenized” dispersion relation

$$k^2 = (\epsilon_{p}\theta_{p} + \epsilon_{\bar{p}}\theta_{\bar{p}}) \frac{\omega^2}{c^2}. \quad (6.1.3)$$

The quantity $\frac{1}{\epsilon_{p}\theta_{p} + \epsilon_{\bar{p}}\theta_{\bar{p}}}$ is known in the literature as the effective permittivity, $\epsilon_{\text{eff}}$, and from the above equation it follows that, for $\omega^2 > 0$, the wave vector $k$ must have a non-zero imaginary part whenever $\epsilon_{p}\theta_{p} + \epsilon_{\bar{p}}\theta_{\bar{p}} < 0$. This may be restated as follows: when the effective permittivity is negative, the first passband of a layered medium with sufficiently thin layer thicknesses will not include the origin.
This is borne out in the contour plot (6.1) of the function \( \cos(kd) \) below. We set \( \epsilon_p = -y^2 \) and \( \epsilon_\bar{p} = 1 \), where \( y \) is any prescribed real number. The horizontal axis is \( \frac{d\omega}{2\pi c} \) and the vertical axis is \( y \). The volume fractions are taken to be equal \( \theta_p = \theta_\bar{p} = 1/2 \). In the middle region, which contains the origin, the wave vector is real (\( |\cos(kd)| \leq 1 \)). In the lighter (\( \cos(kd) > 1 \)) region and the darker region (\( \cos(kd) < -1 \)), the wave vector has a non-zero imaginary part.

By inserting the plasmonic permittivity \( \epsilon_p = 1 - \frac{c^2}{\omega^2 d^2} \) into (6.1.2) and letting \( d \to 0 \), we arrive at a similar conclusion regarding the shift of the first passband away from the origin. This can be easily visualized in the contour plot. Indeed, setting \( \epsilon_p = -\frac{c^2}{\omega^2 d^2} \) and letting \( d \to 0 \) corresponds to traversing the hyperbola \( y = \frac{2\pi}{x} \) in the sense of decreasing \( x \), since

\[
\epsilon_p = -y^2 \quad \epsilon_\bar{p} = -\frac{c^2}{\omega^2 d^2} \quad x = \frac{\omega d}{2\pi c}.
\]

Now, the graph of the hyperbola has a vertical asymptote at the line \( x = 0 \), so that it will not intersect the pass band region \( |\cos(kd)| \leq 1 \) for \( x \) sufficiently small.

Since the first term of the convergent power series for \( \omega^2 \) established in the previous chapters is proportional to \( k^2 \), the first passband must necessarily pass through the origin in those cases. There, the inclusions of neighboring period cells were isolated from one another by the host material, which does not occur in the topology of a layered medium.
FIGURE 6.1. Contour plots of the dispersion relation for a layered medium.
References


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