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Limit theorems for weighted stochastic systems of interacting particles

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LIMIT THEOREMS FOR WEIGHTED STOCHASTIC SYSTEMS OF INTERACTING PARTICLES

A Dissertation

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The Department of Mathematics

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Abstract

The goal of this dissertation is to (a) establish the weak convergence of empirical measures formed by a system of stochastic differential equations, and (b) prove a comparison result and compactness of support property for the limit measure.

The stochastic system of size $n$ has coefficients that depend on the empirical measure determined by the system. The weights for the empirical measure are determined by a further $n$-system of stochastic equations. There is a random choice among $N$ types of weights. The existence and uniqueness of solutions of the interacting system, weak convergence of the empirical measures, and the identification of the limit form the first part of this work. The second part deals with particular cases of interacting systems for which qualitative properties of the limit can be proved. The properties established are (i) pathwise comparison of solutions, and (ii) compactness of support for the weak limit of the empirical measures.
Chapter 1
Introduction

In this thesis we study the asymptotic behavior of some randomly interacting diffusion processes on $\mathbb{R}^d$ with time-varying weights of various types as the number of processes tends to infinity. The theory of interacting diffusion systems has been studied here by a general interaction that includes mean field interaction. The motivations to our study come from the following research and applications of stochastic interacting systems:

(a) The Propagation of Chaos Problem

In the neurophysiological literature, it is well known that a neuron cell is spatially extended and neuronal activities are realized through the synaptic inputs that occur randomly in time and at different locations on the neuron’s surface (see [15]). The asymptotic behavior of voltage potentials of large assemblages of interacting neurons is investigated in infinite-dimensional spaces such as the dual of a nuclear space. The model is established by starting from an $n$-particle interacting system, then as $n$ goes to infinity, the limit (in probability) of the sequence of empirical measures determined by such an $n$-particle system system is identified to be the law of the unique solution of the McKean-Vlasov equation. See [6] [7] and [8].

Let $X^j_n$ be $\mathbb{R}^d$-valued processes, $1 \leq j \leq n$, governed by the system of equations:

$$X^j_n(t) = X^j_n(0) + \int_0^t \left( a(s, X^j_n(s)) + \frac{1}{n} \sum_{i=1}^n b(s, X^i_n(s), X^j_n(s)) \right) ds$$

$$+ \int_0^t \left( \sigma(s, X^j_n(s)) + \frac{1}{n} \sum_{i=1}^n c(s, X^i_n(s), X^j_n(s)) \right) dW_j(s)$$

(1.1)
with \( X_j^n(0), 1 \leq j \leq n \), being either independent and identically distributed \( \mathbb{R}^d \)-valued random variables or exchangeable random vectors, \( \{W_j\}, 1 \leq j \leq n \) being independent copies of a standard \( \mathbb{R}^d \)-valued Brownian motion. The empirical average that appears in the coefficients is known as mean field interaction.

The empirical measure derived from the above interacting system is given by:

\[
\mu_n(\omega, B) = \frac{1}{n} \sum_{j=1}^{n} \delta_{X_j^n(\cdot, \omega)}(B), \quad \forall B \in \mathcal{B}(C([0, T]; \mathbb{R}^d)).
\]  

(1.2)

Under suitable conditions, such as the \emph{boundedness} of the initial random variable \( X_j^n(0) \), \emph{linear growth} and \emph{joint continuity} for the coefficients \( a, b, \sigma, c \) and a \emph{Lipschitz condition} on the state variables, the sequence of empirical measures \( \{\mu_n\}_{n=1}^{\infty} \) converges in distribution to a limit (denoted by \( \mu \)), and the limit \( \mu \) can be identified as the distribution of process \( Y_t \) given by:

\[
Y_t = Y_0 + \int_0^t A(s, Y_s, \mu_s)ds + \int_0^t B(s, Y_s, \mu_s)dW_s
\]  

(1.3)

where \( \mu_s \) is the distribution of \( Y_s \), and

\[
A(s, u, \lambda) = a(s, u) + \int_{\mathbb{R}^d} b(s, u, y)\lambda(dy),
\]

\[
B(s, u, \lambda) = \sigma(s, u) + \int_{\mathbb{R}^d} c(s, u, y)\lambda(dy).
\]

The stochastic equation (1.3) is known as the McKean-Vlasov equation for the interacting system as \( n \to \infty \).

(b) \textit{Spin Glass Models}

This is a problem that arises from physics in which the mean field theory can be used to solve many interesting problems. A typical spin glass model is usually described by the following \( n \)-particle random \textit{Hamiltonian}:

\[
H^{\theta, i,j}_n(x_1, \cdots, x_n) = -\frac{1}{2} \sum_{i,j=1}^{n} \theta_{i,j} x_i x_j
\]  

(1.4)
where $x_1, \cdots, x_n$ denote $n$ continuous spins in $\mathbb{R}$ and $\theta_{i,j}$ is a set of real valued random variables defined on some probability space. It is well known that the corresponding Gibbs distribution for the Hamiltonian at inverse temperature $\beta = \frac{1}{T}$ is given by:

$$P_{n,\beta}^{\theta_{i,j}}(dx_1, \cdots, dx_n) = \frac{1}{Z_{n,\beta}} e^{-\beta H_{n,\beta}^{\theta_{i,j}}(x)} \prod_{i=1}^{n} \mu(dx_i),$$

where $\mu$ is a probability measure on $\mathbb{R}$ which represents the distribution of a single spin without the interactions, and $Z_{n,\beta}^{\theta_{i,j}}$ is a normalization factor known as the partition function and is given by:

$$Z_{n,\beta}^{\theta_{i,j}} = \int_{\mathbb{R}^n} e^{-\beta H_{n,\beta}^{\theta_{i,j}}(x)} \prod_{i=1}^{n} \mu(dx_i) \quad (1.5)$$

The central problem in equilibrium statistical physics is to determine the free energy function for each particle $f(\beta)$ in the thermodynamic limit as $n \to \infty$

$$f(\beta) = \lim_{n \to \infty} \frac{1}{n} f_n^{\theta_{i,j}}(\beta) \quad (1.6)$$

if this limit exists in some sense, where $f_n^{\theta_{i,j}}(\beta)$ is the free energy of $n$-particle system defined by: $-\beta \cdot f_n^{\theta_{i,j}}(\beta) = \log Z_{n,\beta}^{\theta_{i,j}}$.

The determination of the random interaction $\theta_{i,j}$ between the $i$th and $j$th spins becomes very important in studying the existence of the thermodynamic limit as $n \to \infty$ in (1.6) to obtain the free energy per spin. One example is the Hopfield neural network model in which the random coupling $\{\theta_{i,j}\}$ is given by:

$$\theta_{i,j} = \frac{1}{n} \sum_{k=1}^{q} \xi_i^k \xi_j^k = \xi_i \cdot \xi_j \quad (1.7)$$

where $\cdot$ denotes the inner product between $\xi_i$ and $\xi_j$ in the vector space $\mathbb{R}^d$. One of the applications of the Hopfield neural network is to act as an associative memory model in which the $p$ vectors $\theta_1, \ldots, \theta_p$ in $\mathbb{R}^p$ correspond to $p$ stored patterns and under some
appropriate dynamics the network with the random interaction \( \{\theta_{i,j}\} \) defined by (1.7) will evolve toward one of these patterns asymptotically (see [9]).

From the stochastic differential equation point of view, the dynamical mean field approach to the spin glass problem suggests one to consider the following SDEs in \( \mathbb{R}^d \):

\[
dx^n_j(t) = b(x^n_j(t))dt + \frac{1}{n} \sum_{i=1}^n c(\xi_j, \xi_i)\Phi(x^n_j(t), x^n_i(t))dt + \sigma(x^n_j(t))dB_j(t) \quad (1.8)
\]

where \( x^n_j(0), j = 1, ... n \) are independent and identically distributed random variables. \( B_j (j = 1, ... n) \) are the copies of a standard Brownian motion. \( \xi_j \) and \( \xi_i \) are the \( \mathbb{R}^d \)-valued i.i.d. random vectors defined on some probability space. \( \xi_j \) takes \( N \) distinct values from a set of \( \mathcal{A} = \{h_1, ..., h_N\} \) with the probability \( p_k = P\{\xi_j = h_k\} \) with \( \sum_{k=1}^N p_k = 1 \).

Besides, \( b : \mathbb{R}^d \rightarrow \mathbb{R}^d, \sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d \) and \( \Phi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) are measurable functions.

(c) **Spread of AIDS Among Interactive Transmission Groups**

One of the most important questions in the spread of an epidemic is the prediction of its future behavior at time \( t \), given its initial state at \( t = 0 \). There is a rich mathematical literature on this subject where both deterministic and stochastic models are considered (see [22]).

Suppose four transmission groups in the AIDS epidemic are given by:

1. Homosexual persons, infected by homosexual contact with HIV infective.

2. Blood transfusion recipients, infected by donors with HIV.

3. Intravenous drug users (IVDUs) sharing HIV-infected needles.

4. Persons infected by heterosexual contact with HIV infective.

Then in the model of the spread of AIDS, \( N \) communities and these four transmission groups are considered. Each community has at least one nonempty transmission group.
Let $X^N(i, k, t)$ represent the size of the HIV+ population at time $t$ in the $i$th community ($1 \leq i \leq N$) and transmission group $k$ for $1 \leq k \leq 4$. The change of the size of HIV+ in the transmission group $(i, k)$ may be caused by the following reasons: (a) individuals newly infected by the HIV+ members in the transmission group $(j, l)$ for $1 \leq j \leq N, 1 \leq l \leq 4$, (b) immigrants from the transmission group $(i, k)$, and (c) removals from the transmission group $(i, k)$ due to death, immigration, etc. Hence this spread of AIDS model can be described by a system of equations:

$$dX^N(i, k, t) = dW(i, k, t) + \left\{ \frac{1}{N} \sum_{j=1}^{N} \left( \sum_{l=1}^{4} b_{k,l}(X^N(i, k, l), X^N(j, l, t)) \right) \right\}^{\frac{1}{2}} dt$$

$$X^N(i, k, 0) = X(i, k), \quad i = 1, \ldots, N. \quad k = 1, \ldots, 4.$$

(1.9)

It has been shown that each of the $N$ interacting diffusions $\{X^N(i, k, t); 1 \leq k \leq 4, t \in [0, T]\}$ in (1.9) has a natural limit $\{\hat{X}_{i,k}(t); 1 \leq k \leq 4, t \in [0, T]\}$ as $N \to \infty$ and they are the solutions of Liouville type nonlinear stochastic differential equation:

$$dX_k(t) = dW_k(t) + \left\{ \int \left[ \sum_{l=1}^{4} b(X_k(t), y) \right]^{\beta} u_t(dy) \right\}^{\frac{1}{2}} dt$$

$$X_k(0) = X_k$$

(1.10)

with $W_k = W_{1,k}, X_k = X_{1,k},$ $u_t = P^{X(t)}$ is the measure generated by $X(t) := \{X_l(t); 1 \leq l \leq 4\}$.

More recently stochastic weights are introduced into the the stochastic systems of interacting particles and the studies are correspondingly conducted by the weighted empirical measure. In a paper by Kurtz and Xiong [19], the model of an interacting
particle system is presented as:

\[
X^n_i(t) = X^n_i(0) + \int_0^t c(X^n_i(s), V(s)) ds + \int_0^t \sigma(X^n_i(s), V(s)) dB_i(s) \\
+ \int_{U \times [0, t]} \alpha(X^n_i(s), V(s), u) W(du ds), \quad i = 1, 2, \ldots \tag{1.11}
\]

\[
A^n_i(t) = A^n_i(0) + \int_0^t A^n_i(s) d(X^n_i(s), V(s)) ds + \int_0^t A^n_i(s) \gamma(X^n_i(s), V(s)) dB_i(s) \\
+ \int_{U \times [0, t]} A^n_i(s) \beta(X^n_i(s), V(s), u) W(du ds), \quad i = 1, 2, \ldots \tag{1.12}
\]

\[
V(t) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n A^n_i(t) \delta_{X^n_i(t)}. \tag{1.13}
\]

where \(X^n_i \in \mathbb{R}^d\) and \(A^n_i \in \mathbb{R}\) are the location and weight of the \(i\)th particle respectively. \(\delta_x\) is the Dirac measure at \(x\) and the limit exists in the weak topology on \(\mathcal{M}(\mathbb{R}^d)\), the collection of all finite signed Borel measures on \(\mathbb{R}^d\).

The weighted empirical measure is defined by: \(U_n = \frac{1}{n} \sum_{i=1}^n A^n_i \delta_{X^n_i}\). Since it is not natural to use the limit (say \(V\)) of \(\{U_n\}_{n \geq 1}\) in an interacting particle system before the existence of solutions \((X^n_i, A^n_i)\) has been shown. In this thesis, the limit measure \(V(t) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n A^n_i(t) \delta_{X^n_i(t)}\) is replaced by \(U_n(t) = \frac{1}{n} \sum_{i=1}^n A^n_i(t) \delta_{X^n_i(t)}\), so that a finite system is considered before the limit \(V\) is taken.

The weights of particles in a system are governed not only by interactions between particles, but also by the interactions between groups. From the model of spread of AIDS among different interaction transmission groups, it is necessary to introduce random grouping factors into such a system whenever the members of various types form several different groups. Namely particles whose physical or chemical features are somehow similar are more likely to bond together and form a group with randomness that is independent of the probability space on which \(X^n_j, A^n_j, B_j\) and \(W\) are defined. Therefore the introduction of an index random variable \(L_j\) as a random grouping factor for our
particle system becomes one of the novelties in this thesis. Another novelty is that the boundedness of the coefficients for $X^n_j$ has been relaxed to linear growth.

In Chapter 2 some preliminary theories, rules, formulas and inequalities are reviewed and then in Chapter 3 we present the model of the stochastic systems of interacting particles and the assumptions that are necessary to support the proofs.

In Chapter 4 the existence of solutions $(X^n_j, A^n_j, U_n)$ for an $n$-particle system will be studied by the Euler scheme, tightness criterions, weak convergence of probability measures and the techniques of martingale problems.

In Chapter 5 we first show the uniqueness of solutions $(X^n_j, A^n_j, U_n)$ for the $n$-particle system by the Itô formula and the technique of stopping times, and then prove the convergence of weighted empirical measures $\{U_n\}$ to the limit $\theta$ in distribution. Then $\theta$ is the solution of the following SPDE:

$$
\langle \theta(t), \phi \rangle = \langle \theta(0), \phi \rangle + \int_0^t \langle \theta(s), d\langle \cdot, \theta \rangle \phi + \mathcal{L} \theta(s) \phi \rangle \cdot ds \\
+ \int_0^t \int G \langle \theta(s), \beta(\cdot, \theta(s), z) \phi + \alpha(\cdot, \theta(s), z) \nabla \phi \rangle W(dz ds)
$$

(1.14)

In Chapter 6 we first investigate the conditions for which the comparison between two interacting particle systems with the same location but different time-varying weights. The comparison is studied for two cases: (i) two weight processes with different drift but the same diffusion and space-time noise, (ii) two weight processes with drifts, diffusion and white noise all different. Lastly the compactness of support property for the solutions to an interacting particle system is shown by the Feller Test for explosion.
Chapter 2
Preliminaries

In this chapter we briefly review the preliminaries from probability theory and stochastic analysis that are used in this thesis.

2.1 Weak Convergence of Probability Measures

Theorem 2.1. Let $X_1, ..., X_n$ and $X$ be a family of real-valued random variables defined on the probability space $(\Omega, \mathcal{F}, P)$. Then,

1. $X_n \to X$ almost surely implies that $X_n \to X$ in probability,
2. $X_n \to X$ in probability implies that $X_n \to X$ in distribution,
3. $X_n \to X$ in $L^p$ ($p \geq 1$) implies that $X_n \to X$ in probability,
4. $X_n \to X$ in probability implies that there exists a subsequence $\{X_{n_k}\}$ such that $X_{n_k} \to X$ almost surely.

Theorem 2.2. (Skorokhod Representation Theorem) Let $X_n$ and $X$ be real-valued random variables such that $X_n \to X$ in distribution. There exists a probability space $(\Omega, \mathcal{F}, P)$ and measurable mappings $\xi_n$ and $\xi$ from $(\Omega, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

(a) $\mathcal{L}(\xi_n) = \mathcal{L}(X_n)$, $\mathcal{L}(\xi) = \mathcal{L}(X)$ where $\mathcal{L}$ denotes the law of a random variable.
(b) $\xi_n \to \xi$ almost surely as $n \to \infty$.

Definition 2.3. Let $S$ be a metric space, $\mathcal{B}(S)$ be the totality of all Borel sets in $S$. A sequence of probability measures $\{P_n\}_{n \geq 1}$ is said to be weakly convergent to a probability measure $P$ if

$$\lim_{n \to \infty} \int_S f(x) \cdot P_n(dx) = \int_S f(x) \cdot P(dx)$$
for every bounded, continuous function $f$ defined on $S$.

**Definition 2.4.** A sequence of probability measures $\{P_n\}_{n \geq 1}$ is said to be tight if for any given $\epsilon > 0$, there exists a compact set $K_\epsilon$ in $S$ such that $P_n(K_\epsilon) > 1 - \epsilon$, $\forall n$.

**Theorem 2.5.** Let $S$ be a metric space, $\mathcal{B}(S)$ be the totality of all Borel sets in $S$. Let $P_n, P$ be probability measures defined on the space $(S, \mathcal{B}(S))$. The following are equivalent:

1. $P_n \rightarrow P$ in distribution.
2. $\lim_{n \to \infty} \int_S f(x)P_n(dx) = \int_S f(x)P(dx)$ for all bounded, uniformly continuous functions $f$.
3. $\limsup_n P_n(F) \leq P(F)$ for all closed set $F$ in $S$.
4. $\liminf_n P_n(G) \geq P(G)$ for all open set $G$ in $S$.
5. $\lim_n P_n(A) = P(A)$ for all $P$-continuity sets $A$ in $S$.

Note that a set $H$ is called a $P$-continuity set if its boundary $\partial H$ satisfies $P(\partial H) = 0$.

**Theorem 2.6.** (Prohorov’s Theorem). Suppose that $\{P_n\}_{n \geq 1}$ is a sequence of tight probability measures defined on a complete and separable metric space. Then $\{P_n\}_{n \geq 1}$ has a weakly convergent subsequence $\{P_{n_k}\}$.

**Definition 2.7.** Let $\Pi$ be a family of probability measures on $(S, \mathcal{B}(S))$. We call $\Pi$ relatively compact if every sequence of elements of $\Pi$ contains a weakly convergent subsequence.

**Theorem 2.8.** (Prohorov’s Theorem). Let $\Pi$ be a family of probability measures on $(S, \mathcal{B}(S))$. If $S$ is a metric space, then $\Pi$ is tight implies its relative compactness. If $S$ is a complete and separable metric space, then tightness is equivalent relative compactness.
Definition 2.9. Let \( \{X_n\}_{n \geq 1} \) is a sequence of random variables of \( S \). Let \( P_n \) be the distribution of \( X_n \). Then \( \{X_n\}_{n \geq 1} \) is tight if \( \{P_n\}_{n \geq 1} \) is tight.

Theorem 2.10. (Tightness Criterion). A sequence of probability measures \( \{P_n\}_{n \geq 1} \) is tight on \( \mathbb{R}^d \) if these two conditions hold:

1. For any \( \epsilon > 0 \), there exists a positive number \( K \) such that
   \[
   \sup_{n \geq 1} P_n \{ y(t) \in \mathbb{R}^d; \sup_{t \in [0,T]} |y(t)| \geq K \} < \epsilon
   \]
2. For any \( \epsilon > 0, a > 0 \), there exists a positive number \( \delta \) such that
   \[
   \sup_{n \geq 1} P_n \{ y(t) \in \mathbb{R}^d; \sup_{t_1, t_2 \in [0,T], |t_1 - t_2| < \delta} |y(t_1) - y(t_2)| \geq a \} < \epsilon
   \]
   for any \( t_1, t_2 \) in \( [0, T] \) satisfying \( |t_1 - t_2| < \delta \).

Theorem 2.11. (Tightness Criterion). A sequence of \( \mathbb{R}^d \)-valued stochastic processes \( \{X_n\}_{n \geq 1} \) defined on the probability space \( (\Omega, \mathcal{F}, \mathbf{P}) \) is tight if these two conditions hold:

1. For any \( \epsilon > 0 \), there exists a positive number \( K \) such that
   \[
   \mathbf{P} \{ \omega; \sup_{t \in [0,T]} |X_n(t)| \geq K \} < \epsilon
   \]
2. For any \( \epsilon > 0, a > 0 \), there exists a positive number \( \delta \) such that for all \( n \geq 1 \)
   \[
   \mathbf{P} \{ \omega; \sup_{t_1, t_2 \in [0,T], |t_1 - t_2| < \delta} |X_n(t_1) - X_n(t_2)| \geq a \} < \epsilon
   \]

Theorem 2.12. (Kolmogorov’s Tightness Criterion) A sequence of \( \mathbb{R}^d \)-valued random processes \( \{X_n(t)\}_{n=1}^{\infty} \) is tight if and only if:

1. \( \{X_n(0)\}_{n=1}^{\infty} \) is tight,
2. there exist constants \( \gamma \geq 0, \alpha > 1 \) and \( K > 0 \) such that
   \[
   \mathbf{E}|X_n(t_2) - X_n(t_1)|^\gamma \leq K |t_2 - t_1|^\alpha.
   \]
Definition 2.13. A stochastic process $B(t, \omega)$ is called a \textit{Brownian motion} if

1. $P\{\omega; B(0, \omega) = 0\} = 1$.

2. For any $0 \leq s < t$, the random variable $B(t) - B(s)$ is normally distributed with mean zero and variance $t - s$, i.e., for any $a < b$,

$$P\{a \leq B(t) - B(s) \leq b\} = \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{-\frac{x^2}{2(t-s)}} \, dx.$$ 

3. $B(t, \omega)$ has independent increments, i.e., for any $0 \leq t_1 < t_2 < \cdots < t_n$, the random variables $B(t_1), B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1})$ are independent.

4. Almost all sample paths of $B(t, \omega)$ are continuous functions, i.e.,

$$P\{\omega; B(\cdot, \omega) \text{ is a continuous function of } t\} = 1.$$ 

Definition 2.14. A random variable $\tau: \Omega \to [a, b]$ is a \textit{stopping time} with respect to the filtration $\{\mathcal{F}_t; a \leq t \leq b\}$ if $\{\omega; \tau(\omega) \leq t\} \in \mathcal{F}_t$ for all $t \in [a, b]$, i.e., the set $\{\tau \leq t\}$ is $\mathcal{F}_t$-measurable.

In the above definition $b$ is allowed to be $\infty$. With the filtration $\mathcal{F}_t$ assumed to be right continuous, we have the following characterization of a stopping time.

Definition 2.15. An $\mathcal{F}_t$-adapted stochastic process $X_t$, $a \leq t \leq b$ is called a \textit{local martingale} with respect to $\{\mathcal{F}_t\}$ if there exists a sequence of stopping times $\tau_n, n = 1, 2, \ldots$, such that

1. $\tau_n$ increases monotonically to $b$ almost surely as $n \to \infty$;

2. For each $n$, $X_{t \wedge \tau_n}$ is a martingale with respect to $\{\mathcal{F}_t; a \leq t \leq b\}$.

A cornerstone result in martingale theory is the theorem called the \textit{Doob-Meyer decomposition}, which states that under certain conditions, a submartingale $X(t)$ with respect
to a right continuous filtration \( \{ \mathcal{F}_t \} \) can be decomposed as a sum of a martingale \( M(t) \) and an increasing process \( A(t) \), i.e.,

\[
X(t) = M(t) + A(t).
\] (2.1)

### 2.2 Stochastic Integrals and the Itô Formulas

**Definition 2.16.** A stochastic process \( X(t, \omega) \) is said to be adapted with respect to the filtration \( \{ \mathcal{F}_t \} \) if \( X(t, \omega) \) is \( \mathcal{F}_t \)-measurable for each \( t \).

**Theorem 2.17.** Let \( B(t) \) be Brownian motion. Let \( L^2([a, b] \times \Omega) \) denote the space of all adapted stochastic processes \( X(t, \omega), a \leq t \leq b, \omega \in \Omega \) such that

\[
\int_a^b E|f(t, \omega)|^2 dt < \infty.
\]

For any \( f \in L^2([a, b] \times \Omega) \), the integral \( I(f) = \int_a^b f(t)dB(t) \) is a random variable with \( EI(f) = 0 \) and

\[
E|I(f)|^2 = \int_a^b E|f(t)|^2 dt.
\]

**Theorem 2.18.** (One-Dimensional Itô formula). Let \( B(t) \) be a Brownian motion. Let \( X(t) \) be a \( S \)-valued stochastic process given by

\[
X(t) = X(0) + \int_0^t a(s, X(s))dB(s) + \int_0^t b(s, X(s))ds
\]

where \( a \in L^2_{n.a.}([0, T] \times S) \) and where \( b \in L^1_{n.a.}([0, T] \times S) \).

Suppose \( f(t, x) \) is a continuous function with continuous second partial derivatives. Then \( f(t, X(t)) \) is also a stochastic process and

\[
f(t, X(t)) = f(0, X(0)) + \int_0^t \frac{\partial f}{\partial x}(s, X(s))a(s, X(s))dB(s)
\]

\[
+ \int_0^t \left[ \frac{\partial f}{\partial s}(s, X(s)) + \frac{\partial f}{\partial x}(s, X(s))b(s, X(s)) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s, X(s))a(s, X(s))^2 \right] ds
\]
Theorem 2.19. (Multi-Dimensional Itô formula). Let $B_1(t), ..., B_m(t)$ be $m$ independent Brownian motions. Let $X_1(t), ..., X_n(t)$ be $n$ stochastic processes given by

$$X_i(t) = X_i(0) + \sum_{j=1}^{m} \int_0^t a_{ij}(s) dB_j(s) + \int_0^t b_i(s) ds, \quad 1 \leq i \leq n,$$

where $a_{i,j} \in L^2_{n.a.}([0,T] \times \Omega)$ and $b_i \in L^1_{n.a.}([0,T] \times \Omega)$. Suppose $f(t, x)$ is a continuous function with continuous second partial derivatives. Then $f(t, X(t))$ is also a stochastic process and

$$f(t, X_1(t), ..., X_n(t)) = f(0, X_1(0), ..., X_n(0)) + \int_0^t \frac{\partial f}{\partial s}(s, X_1(s), ..., X_n(s)) ds$$

$$+ \sum_{i=1}^{n} \int_0^t \frac{\partial f}{\partial x_i}(s, X_1(s), ..., X_n(s)) dX_i(t) + \frac{1}{2} \sum_{i,l=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_l}(s, X_1(s), ..., X_n(s)) dX_i(t) dX_l(t)$$

2.3 Stochastic Differential Equations

2.3.1 Strong and Weak Solutions

Let $b_i(t, x), \sigma_{ij}(t, x); 1 \leq i \leq d, 1 \leq j \leq r$ be Borel measurable functions from $[0, \infty) \times \mathbb{R}^d$ into $\mathbb{R}$. Let $b(t, x) = \{b_i(t, x)\}_{1 \leq i \leq d}$ be a $d \times 1$ drift vector and $\sigma(t, x) = \{\sigma_{ij}(t, x)\}_{1 \leq j \leq r}$ be a $d \times r$ dispersion matrix. Consider the stochastic differential equation (SDE): 

$$dX(t) = b(t, X(t)) dt + \sigma(t, X(t)) dB(t)$$

where $B(t)$ is $r$-dimensional Brownian motion and $X(t) = (X_1(t), ..., X_d(t))$ is a suitable stochastic processes with continuous sample paths and values in $\mathbb{R}^d$, the solution of the equation.

Definition 2.20. A strong solution of the above SDE on the given probability space $(\Omega, \mathcal{F}, P)$ and with respect to the fixed Brownian motion $B$ and initial condition $\xi$, is a process $X = \{X(t); 0 \leq t < \infty\}$ with continuous sample paths and with the properties:
1. $X$ is adapted to the filtration $\{\mathcal{F}_t\}$,

2. $\mathbb{P}[X(0) = \xi] = 1$,

3. $\mathbb{P}[\int_0^t \{|b_i(s, X(s))| + \sigma_{i,j}^2(s, X(s))\} ds < \infty] = 1$ holds for every $1 \leq i \leq d$, $1 \leq j \leq r$ and $0 \leq t < \infty$,

4. the integral version of the above equation

$$X_i(t) = X_i(0) + \int_0^t b_i(s, X(s)) ds + \sum_{j=1}^r \int_0^t \sigma_{i,j}(s, X(s)) dB_j(s); \ 1 \leq i \leq d.$$ holds almost surely.

**Definition 2.21.** A weak solution of the above stochastic differential equation is a triple $(X, B(t)), (\Omega, \mathcal{F}, \mathbb{P}), \{\mathcal{F}_t\}$, where

1. $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, and $\{\mathcal{F}_t\}$ is a filtration of sub-$\sigma$-fields of $\mathcal{F}$,

2. $X = \{X(t), \mathcal{F}_t; t \in [0, \infty)\}$ is a continuous, adapted $\mathbb{R}^d$-valued process, $B = \{B(t), \mathcal{F}_t; t \in [0, \infty)\}$ is an $r$-dimensional Brownian motion.

3. $\mathbb{P}[\int_0^t \{|b_i(s, X(s))| + \sigma_{i,j}^2(s, X(s))\} ds < \infty] = 1$ holds for every $1 \leq i \leq d$, $1 \leq j \leq r$ and $0 \leq t < \infty$,

4. the integral version of the above equation

$$X_i(t) = X_i(0) + \int_0^t b_i(s, X(s)) ds + \sum_{j=1}^r \int_0^t \sigma_{i,j}(s, X(s)) dB_j(s); \ 1 \leq i \leq d.$$ holds almost surely.

**Definition 2.22.** Suppose $X$ and $\tilde{X}$ are two solutions of the SDE:

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB(t).$$

The pathwise uniqueness of solutions for this SDE holds if $X(t) = \tilde{X}(t)$ whenever
1. \(X\) and \(\tilde{X}\) are defined on the same probability space \((\Omega, \mathcal{F}, P)\) with the same \(\sigma\)-field \(\mathcal{F}_t\) and the same Brownian motion \(B(t)\),

2. \(X(0) = \tilde{X}(0)\) a.s.

**Theorem 2.23.** *(Yamada and Watanabe (1971)).* Pathwise uniqueness implies uniqueness in the sense of the probability law.

Note that this theorem yields a remarkable corollary that existence of a weak solution and pathwise uniqueness imply the existence of a strong solution.

### 2.3.2 Martingale Problems

Let \(X(t)\) be a \(\mathbb{R}^d\)-valued process governed by the following SDE:

\[
X(t) = X(0) + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dB(s) \quad (*)
\]

with \(X(0) = \xi\) with the distribution \(\mu_0\) satisfying \(\mu_0 = P \cdot \xi^{-1}\), \(B(t)\) being a Brownian motion, \(X\) a stochastic process, all defined on the same probability space \((\Omega, \mathcal{F}, P)\).

For any \(f \in C_2^b(\mathbb{R}^d)\), define the infinitesimal generator:

\[
\mathcal{L}^f(X(t)) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(X(t)) \cdot \frac{\partial^2 f}{\partial x_i \partial x_j} X(t) + \sum_{i=1}^d b_i(X(t)) \cdot \frac{\partial f}{\partial x_i} X(t)
\]

with \(a_{ij}(x) = \sum_{k=1}^d \sigma_{ik}(x)\sigma_{kj}(x)\). Then \(\tilde{P}\) solves the \((\mathcal{L}, \mu_0)\)-martingale problem posed by the SDE (*) if

\[
\tilde{P}\{\omega; x(0, \omega) \in B\} = \mu_0(B), \quad \forall B \in \mathcal{B}(\mathbb{R}^d).
\]

and \(M_f(t) = f(X(t)) - \int_0^t \mathcal{L}^f(X(s))ds\) is a \(\tilde{P}\)-martingale for any \(f \in C_2^b(\mathbb{R}^d)\).

**Theorem 2.24.** A solution \(\tilde{P}\) of a martingale problem posed by the above SDE (*) is a weak solution of the SDE (*).

**Theorem 2.25.** Suppose \(\mu_0, b(t, u), \sigma(t, u)\) satisfy the following conditions:
1. Initial Condition: \( \int_{\mathbb{R}^d} |u|^2 \mu_0(du) < \infty \),

2. Coercivity Condition: \( 2 < b(t, u), u + \sum_{i,j}^d |\sigma_{ij}(t, u)|^2 \leq \theta (1 + |u|^2) \),

3. Linear Growth Condition: \( |b(t, u)|^2 + \sum_{i,j}^d |\sigma_{ij}(t, u)|^2 \leq \theta (1 + |u|^2) \),

4. \( b_i : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d, \sigma_{ij} : [0, T] \times \mathbb{R}^d \rightarrow L(\mathbb{R}^d, \mathbb{R}^d) \) are both jointly continuous.

Then for a fixed \( T \in [0, \infty) \), there exists a probability measure \( \mu^* \) on \( \left( C_{[0, T]}^{\mathbb{R}^d}, B(C_{[0, T]}^{\mathbb{R}^d}) \right) \) such that

1. \( P^* = \mu^* \cdot X(t, \cdot), \mu_0 = P^* \cdot \xi^{-1} \),

2. \( \mu^* \) solves the martingale problem w.r.t. the infinitesimal generator \( \mathcal{L}_s \).

Namely, \( \forall f \in [0, T] \times C_{[0, T]}^{\mathbb{R}^d} \) and \( X \in C_{[0, T]}^{\mathbb{R}^d} \), the real-valued stochastic process

\[
M^f_t(X) := f(t, X(t)) - f(0, X(0)) - \int_0^t \left( \frac{\partial f}{\partial s}(s, X(s)) + \mathcal{L}_s f(s, X(s)) \right) ds
\]

is a \( P^* \)-martingale on the space \( \left( C_{[0, T]}^{\mathbb{R}^d}, B(C_{[0, T]}^{\mathbb{R}^d}), \mu^* \right) \).

### 2.4 Useful Inequalities

**Theorem 2.26. (Itô Isometry).** Let \( B(t) \) be a Brownian motion, \( X \) be a \( \mathcal{F}_t \)-adapted stochastic process both defined on a probability space \( (\Omega, \mathcal{F}, P) \) with \( \mathbb{E} \left\{ \int_0^T X(t)^2 dt \right\} < \infty \), then

\[
\mathbb{E} \left\{ \int_0^T X(t) dB(t) \right\}^2 = \mathbb{E} \left\{ \int_0^T X(t)^2 dt \right\}.
\]

**Theorem 2.27. (Doob's Maximal Inequality).** Let \( X \) be a martingale on \([0, \infty)\). If \( \mathbb{E}|X_T|^p < \infty \) for any \( p > 1 \) and any \( 0 \leq T \leq \infty \), then

\[
\mathbb{E} \left\{ \sup_{0 \leq t \leq T} |X_t| \right\}^p \leq \left( \frac{p}{p-1} \right)^p \mathbb{E}|X_T|^p.
\]
In particular, if $X$ is a martingale with $M_T \in L^2$ (namely, $p = 2$), then

$$\mathbb{E}\left\{ \sup_{0 \leq t \leq T} |X_t| \right\}^2 \leq 4\mathbb{E}|X_T|^2.$$

**Theorem 2.28.** (Gronwall’s Inequality). Let $\alpha : [0, T] \to \mathbb{R}$ be an integrable function. Suppose $\varphi(t)$ is continuous for all $t \in [0, T]$ and satisfies

$$0 \leq \varphi(t) \leq \alpha(t) + \beta \int_0^t \varphi(s) ds; \quad 0 \leq t \leq T. \quad (2.2)$$

with $\beta \geq 0$. Then

$$\varphi(t) \leq \alpha(t) + \beta \int_0^t \alpha(s) e^{\beta(t-s)} ds; \quad 0 \leq t \leq T.$$

In particular, if $\alpha(t) = A$ (a constant) in (2.2), namely if

$$0 \leq \varphi(t) \leq A + \beta \int_0^t \varphi(s) ds; \quad 0 \leq t \leq T$$

with $\beta \geq 0$, then

$$\varphi(t) \leq A \cdot e^{\beta t}; \quad 0 \leq t \leq T.$$
Chapter 3
Stochastic Systems of Interacting Particles

3.1 The Model for Interacting Particle Systems

Consider a stochastic system with $n$ particles. For any $1 \leq j \leq n$, let $X^n_j$ and $A^n_j$ ($1 \leq j \leq n$) be the location and weight of the $j$-th particle respectively that are both defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Let the index set $\{L_j\}_{j \geq 1}$ be a sequence of independent and identically distributed random variables defined on $(\Omega, \mathcal{F}, P)$ that are independent of $\{X^n_j(0), A^n_j(0), B^n_j\}_{1 \leq j \leq n}$ and $W$. For each $j$, $L_j$ takes values from a finite index set $\mathcal{A} = \{1, \ldots, k, \ldots, N\}$ with the probability $p_k = P_{\omega; L_j(\omega) = k}$ and $\sum_{k=1}^N p_k = 1$.

The position and weight of the $j$th particle at time $t$ are governed by the following stochastic differential equations:

$$X^n_j(t) = X^n_j(0) + \int_0^t c(X^n_j(s), U^n_j(s))ds + \int_0^t \sigma(X^n_j(s), U^n_j(s))dB^n_j(s)$$
$$+ \int_0^t \int_G \alpha(X^n_j(s), U^n_j(s), v)W(dvds)$$

(3.1)

$$A^n_j(t) = A^n_j(0) + \int_0^t A^n_j(s)d_{k_j}(X^n_j(s), U^n_j(s))ds + \int_0^t A^n_j(s)\gamma_{k_j}(X^n_j(s), U^n_j(s))dB^n_j(s)$$
$$+ \int_0^t \int_G A^n_j(s)\beta_{k_j}(X^n_j(s), U^n_j(s), v)W(dvds)$$

(3.2)

$$U^n(t) = \frac{1}{n} \sum_{j=1}^n A^n_j(t)\delta_{X^n_j(t)}$$

(3.3)

where $X^n_j(0), 1 \leq j \leq n$ are independent and identically distributed or exchangeable $\mathbb{R}^d$-valued random variables or exchangeable random vectors, $A^n_j(0), 1 \leq j \leq n$ are independent and identically distributed $\mathbb{R}$-valued random variables, or exchangeable random vectors. $\{B_j\}_{j=1}^n$ are copies of a $m$-dimensional standard Brownian motion.
which is independent of $W$. $W$ is a space-time Gaussian white noise defined on $G \times [0, \infty)$ with
\[\mathbb{E}[W(A, t)W(B, t)] = \mu(A \cap B)t\]
for any $A, B \in \mathcal{B}(G)$, where $G$ is a complete and separable metric space and $\mu$ is a $\sigma$-finite Borel measure defined on $G$.

Let $\delta_x$ denote the Dirac delta measure at $x$. The random signed measure $U^n$ is a weighted sum of Dirac delta measures at $X^n_j$ and it is known as a \textit{weighted empirical measure}. For any $E \in \mathcal{B}(\mathcal{C}(\mathbb{R}^d))$, and $\omega \in \tilde{\Omega}$, we have:
\[U^n(\cdot, \omega, E) = \frac{1}{n} \sum_{i=1}^{n} A^n_i(\cdot, \omega) \delta_{X^n_i}(\cdot, \omega)(E)\]

### 3.2 Assumptions

For each $k = 1, \ldots, N$, define:
\[
\gamma(x, u, k) = \gamma_k(x, u), \quad d(x, u, k) = d_k(x, u), \quad \beta(x, u, v, k) = \beta_k(x, u, v).
\]

Then the equation (3.2) is rewritten as:
\[
A^n_j(t) = A^n_j(0) + \int_0^t A^n_j(s)d(X^n_j(s), U^n(s), L_j)ds + \int_0^t A^n_j(s)\gamma(X^n_j(s), U^n(s), L_j)dB^n_j(s)
+ \int_0^t \int_G A^n_j(s)\beta(X^n_j(s), U^n(s), \nu, L_j)W(dvds)
\]

Let $\mathcal{M}(\mathcal{C}([0, T]; \mathbb{R}^d))$ be the space of all signed Borel measures defined on $\mathcal{C}([0, T]; \mathbb{R}^d)$, $\mathcal{M}_+(\mathcal{C}([0, T]; \mathbb{R}^d))$ be the space of all Borel measures defined on $\mathcal{C}([0, T]; \mathbb{R}^d)$, both equipped with the topology of weak convergence of measures. The following assumptions are needed to prove the existence and uniqueness of the solution to the interacting particle system (3.1)-(3.3)-(3.4).

- (S1) Moment Boundedness of Initial Values
\{X_j^n(0), A_j^n(0)\}_{j=1}^n \text{ are exchangeable (or i.i.d.) } \mathbb{R}^d\text{-valued and } \mathbb{R}\text{-valued random variables which are independent of the Brownian motions } \{B_j\}_{j=1}^n \text{ and Gaussian white noise } W \text{ and satisfy}

$$
\mathbb{E}\left\{|X_j^n(0)|^2 + A_j^n(0)^2\right\} < \infty.
$$

(S2) Boundedness Condition

There exists a finite constant $K > 0$ for all $k = 1,...,N$ such that

\[
\begin{align*}
\mathcal{d} : \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \times \mathbb{N} \to \mathbb{R}, \\
\gamma : \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \times \mathbb{N} \to \mathbb{R}, \\
\mathcal{\beta} : \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \times G \times \mathbb{N} \to \mathbb{R}
\end{align*}
\]

satisfy the condition:

\[
\mathcal{d}(x, u, k)^2 + |\gamma(x, u, k)|^2 + \int_G |\mathcal{\beta}(x, u, v, k)|^2 \mu(dv) \leq K^2.
\]

(S3) Linear Growth Condition

There exists a finite constant $K > 0$ for all $j = 1,...,n$ such that

\[
\begin{align*}
c : \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \to \mathbb{R}^d, \\
\sigma : \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \to \mathbb{R}^{d \times m}, \\
\alpha : \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \times G \to \mathbb{R}^d
\end{align*}
\]

satisfy the condition:

\[
|c(x, u)|^2 + |\sigma \sigma^T(x, u)| + \int_G |\alpha(x_j, u, v)|^2 \mu(dv) \leq K^2(1 + |x|^2 + \|u\|^2).
\]

where the norm $u$ is given by its total variation: $\|u\| := \|u^+_n\| + \|u^-_n\|$.

If $u$ is replaced by the weighted empirical measure $u_n = \frac{1}{n} \sum_{j=1}^n \phi_j \delta_{\xi_j}$, then the norm of $u_n$ will be $\|u_n\| = \frac{1}{n} \sum_{j=1}^n |\phi_j|$ for any $\phi_j \in \mathbb{R}$ and $\xi_j \in \mathbb{R}^d$. 

20
(S4) **Lipschitz Condition**

For any \( x_1, x_2 \in \mathbb{R}^d, u_1, u_2 \in \mathcal{M}(\mathbb{R}^d) \), any decomposition of \( u_1, u_2 \):

\[
  u_1 = u_1^+ - u_2^-, \quad u_2 = u_2^+ - u_2^-
\]

and \( u_i^+, u_i^- \in \mathcal{M}_+(\mathbb{R}^d) \) \((i = 1, 2)\) there exists a finite constant \( K > 0 \) for \( k = 1, 2, \ldots, N \) such that

\[
  \left| c(x_1, u_1) - c(x_2, u_2) \right|^2 + \left| \sigma(x_1, u_1) - \sigma(x_2, u_2) \right|^2 \\
  + \int_{G} |\alpha(x_1, u_1, v) - \alpha(x_2, u_2, v)|^2 \mu(dv) \\
  + |d(x_1, u_1, k) - d(x_2, u_2, k)|^2 + |\gamma(x_1, u_1, k) - \gamma(x_2, u_2, k)|^2 \\
  + \int_{G} |\beta(x_1, u_1, v, k) - \beta(x_2, u_2, v, k)|^2 \mu(dv) \\
  \leq K^2 \left( |x_1 - x_2|^2 + \rho(u_1^+, u_2^+)^2 + \rho(u_1^-, u_2^-)^2 \right)
\]

where the distance function \( \rho \) defined by

\[
  \rho(\lambda_1, \lambda_2) = \sup_{\phi \in \mathbb{B}_1} |< \phi, \lambda_1 > - < \phi, \lambda_2 > |
\]

for \( \lambda_1, \lambda_2 \in \mathcal{M}_+(\mathbb{R}^d) \) and \( \mathbb{B}_1 = \{ \phi; \ |\phi(x) \leq 1, \ |\phi(x) - \phi(y)| \leq |x-y|, \ \forall x, y \in \mathbb{R}^d \} \). Note that \( \rho \) is called the Wasserstein metric, which determines a topology of weak convergence on the space \( \mathcal{M}_+(\mathbb{R}^d) \) of finite positive measures.
Chapter 4
Existence of the Solutions

In this chapter we start with an interacting system with \( n \) particles and investigate the existence of the solutions to such a system under the assumptions given in Chapter 3.

4.1 Euler Scheme

For the existence of the solutions to the system (3.1)-(3.3)-(3.4), the Euler scheme is applied to produce a sequence of approximations \( \{X^n_{jm}, A^n_{jm}, U^m_{j}\}_{m \geq 1} \) for the position and weight of the \( j \)th particle and the weighted empirical measure.

Suppose initial values \( \{X^n_j(0), A^n_j(0)\}_{j=1}^n \), a family of copies of a standard Brownian motion \( \{B^n_j\}_{j=1}^n \), a space-time noise \( W \) are given. For each positive integer \( m \geq 1 \), the interval \([0, T]\) is divided into \( m \) equal parts with the step size \( \frac{T}{m} \).

For \( t = 0 \), define

\[
X^n_{jm}(0) = X^n_j(0), \quad A^n_{jm}(0) = A^n_j(0), \quad U^m_{m}(0) = \frac{1}{n} \sum_{j=1}^n A^n_{jm}(0) \delta_{X^n_j(0)} = U^n(0).
\]

For \( t \in (0, \frac{T}{m}] \), define

\[
X^n_{jm}(t) = X^n_{jm}(0) + \int_0^t c(X^n_{jm}(0), U^m_{m}(0)) ds + \int_0^t \sigma(X^n_{jm}(0), U^m_{m}(0)) dB_j(s)
\]

\[
+ \int_0^t \int_G \alpha(X^n_{jm}(0), U^m_{m}(0), v) W(dvds)
\]

\[
A^n_{jm}(t) = A^n_{jm}(0) + \int_0^t A^n_{jm}(0) d(A^n_{jm}(0), U^m_{m}(0), L_j) ds
\]

\[
+ \int_0^t A^n_{jm}(0) \gamma(A^n_{jm}(0), U^m_{m}(0), L_j) dB_j(s)
\]

\[
+ \int_0^t \int_G A^n_{jm}(0) \beta(A^n_{jm}(0), U^m_{m}(0), v, L_j) W(dvds)
\]
For any \( t \in \left( \frac{iT}{m}, \frac{(i+1)T}{m} \right] \) and \( i = 1, \ldots, m - 1 \), define:

\[
X_{jm}^n(t) = X_{jm}^n\left(\frac{iT}{m}\right) + \int_{\frac{iT}{m}}^{t} \left(\begin{array}{c}
  c(X_{jm}^n(\frac{iT}{m}), U_m^i(\frac{iT}{m})) \\
  \sigma(X_{jm}^n(\frac{iT}{m}), U_m^i(\frac{iT}{m}))
\end{array}\right) ds \\
+ \int_{\frac{iT}{m}}^{t} \left(\begin{array}{c}
  \alpha(X_{jm}^n(\frac{iT}{m}), U_m^i(\frac{iT}{m}), v) \\
  \beta(X_{jm}^n(\frac{iT}{m}), U_m^i(\frac{iT}{m}), v, L_j)
\end{array}\right) W(dvds)
\]

\[\text{(4.1)}\]

\[
A_{jm}^n(t) = A_{jm}^n\left(\frac{iT}{m}\right) + \int_{\frac{iT}{m}}^{t} A_{jm}^n(\frac{iT}{m})d(A_{jm}^n(\frac{iT}{m}), U_m^i(\frac{iT}{m}), L_j)ds \\
+ \int_{\frac{iT}{m}}^{t} A_{jm}^n(\frac{iT}{m})\gamma(A_{jm}^n(\frac{iT}{m}), U_m^i(\frac{iT}{m}), L_j)dB_j(s) \\
+ \int_{\frac{iT}{m}}^{t} \int_G A_{jm}^n(\frac{iT}{m})\beta(A_{jm}^n(\frac{iT}{m}), U_m^i(\frac{iT}{m}), v, L_j)W(dvds)
\]

\[\text{(4.2)}\]

\[
U_m^n(t) = \frac{1}{n} \sum_{j=1}^{n} A_{jm}^n(t)\delta X_{jm}^n(t).
\]

\[\text{(4.3)}\]

Let

\[
Y_{jm}^n(t) = X_{jm}^n\left(\frac{iT}{m}\right), \quad Z_{jm}^n(t) = A_{jm}^n\left(\frac{iT}{m}\right), \quad \forall t \in \left(\frac{iT}{m}, \frac{(i+1)T}{m}\right]
\]

Then the above system (4.1)-(4.2)-(4.3) can be written as:

\[
X_{jm}^n(t) = X_{jm}^n(0) + \int_{0}^{t} c(Y_{jm}^n(s), V_m^i(s))ds \\
+ \int_{0}^{t} \sigma(Y_{jm}^n(s), V_m^i(s))dB_j(s) \\
+ \int_{0}^{t} \int_G \alpha(Y_{jm}^n(s), V_m^i(s), v)W(dvds)
\]

\[\text{(4.4)}\]

\[
A_{jm}^n(t) = A_{jm}^n(0) + \int_{0}^{t} Z_{jm}^n(s)d(Y_{jm}^n(s), V_m^i(s), L_j)ds \\
+ \int_{0}^{t} Z_{jm}^n(s)\gamma(Y_{jm}^n(s), V_m^i(s), L_j)dB_j(s) \\
+ \int_{0}^{t} \int_G Z_{jm}^n(s)\beta(Y_{jm}^n(s), V_m^i(s), v, L_j)W(dvds)
\]

\[\text{(4.5)}\]
\[ V^n_m(t) = \frac{1}{n} \sum_{j=1}^{n} Z^n_{jm}(t) \delta_{Y^n_{jm}(t)} \]  

(4.6)

To achieve the existence of a solution for such a system, we will show the weak convergence of the sequence of the approximations \( \{X^n_{jm}, A^n_{jm}, U^n_m\}_{m \geq 1} \) as \( m \to \infty \).

### 4.2 Tightness

**Lemma 4.1.** Under the assumptions (S1), (S2) and (S3), there exists a finite constant \( K > 0 \) such that

1. \( \mathbb{E} \sup_{t \in [0,T]} \left\{ |X^n_{jm}(t)|^2 + A^n_{jm}(t)^2 \right\} < \infty \),
2. \( \mathbb{E} \left\{ |X^n_{jm}(t_2) - X^n_{jm}(t_1)|^2 + |A^n_{jm}(t_2) - A^n_{jm}(t_1)|^2 \right\} \leq K|t_2 - t_1|, \quad \forall t_1, t_2 \in [0,T]. \)

**Proof.**

1. By the definition of the norm of weighted empirical measure \( u \) in (S3),
   \[ \|V^n_m(t)\|^2 \leq \left( \frac{1}{n} \sum_{j=1}^{n} |Z^n_{jm}(t)| \right)^2 \leq \frac{1}{n} \sum_{j=1}^{n} Z^n_{jm}(t)^2. \]

By the exchangeability of \( \{Z^n_{jm}\}_{j=1}^{n} \),
\[ \mathbb{E}\|V^n_m(t)\|^2 \leq \mathbb{E}\left( \frac{1}{n} \sum_{j=1}^{n} Z^n_{jm}(t) \right)^2 \leq \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}|Z^n_{jm}(t)|^2 = \mathbb{E}|Z^n_{1m}(t)|^2 \]  

(4.7)

Squaring the both sides of equation (4.5), we have:
\[ A^n_{jm}(t)^2 \leq 4A^n_{jm}(0)^2 + 4 \left| \int_0^t Z^n_{jm}(s) d(Y^n_{jm}(s), V^n_m(s), L_j) ds \right|^2 \]
\[ + 4 \left| \int_0^t Z^n_{jm}(s) \gamma(Y^n_{jm}(s), V^n_m(s), L_j) dB_j(s) \right|^2 \]
\[ + 4 \left| \int_0^t \int_G Z^n_{jm}(s) \beta(Y^n_{jm}(s), V^n_m(s), v, L_j) W(dvds) \right|^2 \]

Since \( \{L^n_j\} \) is conditionally independent of \( \{X^n_{jm}, A^n_{jm}\}_{m \geq 1} \), it is also conditionally independent of \( \{Y^n_{jm}, Z^n_{jm}\}_{m \geq 1} \). Therefore,
\[ \mathbb{E}^P \left\{ Z^n_{jm}(s)^2 \cdot |\gamma(Y^n_{jm}(s), V^n_m(s), L_j)|^2 \right\} = \sum_{k=1}^{N} p_k \cdot \mathbb{E} \left\{ Z^n_{jm}(s)^2 \cdot |\gamma(Y^n_{jm}(s), V^n_m(s), k)|^2 \right\}. \]
By Doob’s inequality,
\[
\mathbb{E} \sup_{t \in [0,T]} A_{jm}^n(t)^2 \leq 4 \mathbb{E} A_{jm}^n(0)^2 + 4T \cdot \mathbb{E} \int_0^T Z_{jm}^n(s)^2 \cdot d(Y_{jm}^n(s), V_m^n(s), L_j)^2 ds
\]
\[+ 16 \mathbb{E} \int_0^T Z_{jm}^n(s)^2 \cdot |\gamma(Y_{jm}^n(s), V_m^n(s), L_j)|^2 ds
\]
\[+ 16 \cdot \mathbb{E} \int_0^T \int_G Z_{jm}^n(s)^2 \cdot \beta(Y_{jm}^n(s), V_m^n(s), v, L_j)^2 \mu(dv) ds
\]
\[\leq 4 \mathbb{E} A_{jm}^n(0)^2 + 4T \sum_{k=1}^N p_k \cdot \mathbb{E}^P \int_0^T Z_{jm}^n(s)^2 \cdot |d(Y_{jm}^n(s), V_m^n(s), k)|^2 ds
\]
\[+ 16 \sum_{k=1}^N p_k \cdot \mathbb{E} \int_0^T Z_{jm}^n(s)^2 \cdot |\gamma(Y_{jm}^n(s), V_m^n(s), k)|^2 ds
\]
\[+ 16 \cdot \sum_{k=1}^N p_k \cdot \mathbb{E} \int_0^T \int_G Z_{jm}^n(s)^2 \cdot \beta(Y_{jm}^n(s), V_m^n(s), v, k)^2 \mu(dv) ds
\]

By the assumption (S2), Fubini’s theorem and the fact \(\sum_{k=1}^N p_k = 1\),
\[
\mathbb{E} \sup_{t \in [0,T]} A_{jm}^n(t)^2 \leq 4 \mathbb{E} A_{jm}^n(0)^2 + (32 + 4T) K^2 \cdot \sum_{k=1}^N p_k \cdot \mathbb{E} \int_0^T Z_{jm}^n(s)^2 ds
\]
\[= 4 \mathbb{E} A_{jm}^n(0)^2 + (32 + 4T) K^2 \cdot \mathbb{E} \int_0^T Z_{jm}^n(s)^2 ds
\]
\[= K_2 + K_1 \int_0^T \mathbb{E}\{Z_{jm}^n(s)^2\} ds \quad (4.8)
\]

where \(K_1 = (32 + 4T) K^2\), \(K_2 = 4 \mathbb{E} A_{jm}^n(0)^2\).

By squaring the both sides of the equation (4.4),
\[
|X_{jm}^n(t)|^2 \leq 4|X_{jm}^n(0)|^2 + 4 \left| \int_0^t c(Y_{jm}^n(s), V_m^n(s)) ds \right|^2 + 4 \left| \int_0^t \sigma(Y_{jm}^n(s), V_m^n(s)) dB_j(s) \right|^2
\]
\[+ 4 \left| \int_0^t \int_G \alpha(Y_{jm}^n(s), V_m^n(s), v) W(dv ds) \right|^2 .
\]
By Doob’s inequality,
\[
E \sup_{t \in [0,T]} |X_{jm}^n(t)|^2 \leq 4E|X_{jm}^n(0)|^2 + 4TE^P \int_0^T |c(Y_{jm}^n(s), V_{jm}^n(s))|^2 ds \\
+ 16E \int_0^T |\sigma \sigma^T(Y_{jm}^n(s), V_{jm}^n(s))| ds \\
+ 16E \int_0^T \int_G |\alpha(Y_{jm}^n(s), V_{jm}^n(s), v)|^2 \mu(dv) ds.
\]

By linear growth condition (S3) and the result in (4.7),
\[
E \sup_{t \in [0,T]} |X_{jm}^n(t)|^2 \leq 4E|X_{jm}^n(0)|^2 + (32 + 4T)K^2E \int_0^T (1 + |Y_{jm}^n(s)|^2 + |V_{jm}^n(s)|^2) ds \\
\leq K_4 + K_3 \int_0^T E(|Y_{jm}^n(s)|^2 + Z_{1m}^n(s)^2) ds \quad (4.9)
\]
where \( K_3 = (32 + 4T)K^2, K_4 = 4E|X_{jm}^n(0)|^2 + (32 + 4T)K^2T. \)

For any \( t \in (\frac{iT}{m}, \frac{(i+1)T}{m}] \), \( Z_{jm}^n(t) = A_{jm}^n(\frac{iT}{m}), Y_{jm}^n(t) = X_{jm}^n(\frac{iT}{m}) \) with \( i = 0, ..., m-1 \),
the inequalities (4.8) and (4.9) can be rewritten as:
\[
E \sup_{t \in [0,T]} A_{jm}^n(t)^2 \leq K_2 + K_1 \cdot \sum_{i=0}^{m-1} \int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} E A_{jm}^n(\frac{iT}{m})^2 ds \\
\leq K_2 + \frac{K_1 T}{m} \sum_{i=0}^{m-1} E A_{jm}^n(\frac{iT}{m})^2 \quad (4.10)
\]
\[
E \sup_{t \in [0,T]} |X_{jm}^n(t)|^2 \leq K_4 + K_3 \cdot \sum_{i=0}^{m-1} \int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} E \left\{ |X_{jm}^n(\frac{iT}{m})|^2 + A_{jm}^n(\frac{iT}{m})^2 \right\} ds \\
\leq K_4 + \frac{K_3 T}{m} \sum_{i=0}^{m-1} E \left\{ |X_{jm}^n(\frac{iT}{m})|^2 + A_{jm}^n(\frac{iT}{m})^2 \right\} \quad (4.11)
\]

When \( t \in \left(0, \frac{T}{m}\right] \), we have \( i = 0 \). Therefore,
\[
E \sup_{t \in (0, \frac{T}{m}]} A_{jm}^n(t)^2 \leq K_2 + \frac{K_1 T}{m} \cdot E A_{jm}^n(0)^2 < \infty
\]
\[
E \sup_{t \in (0, \frac{T}{m}]} |X_{jm}^n(t)|^2 \leq K_4 + \frac{K_3 T}{m} \cdot E |X_{jm}^n(0)|^2 + \frac{K_3 T}{m} \cdot E A_{jm}^n(0)^2 < \infty.
\]
For \( t \in \left( \frac{iT}{m}, \frac{(i+1)T}{m} \right] \), assume that
\[
\mathbb{E} \sup_{t \in \left( \frac{iT}{m}, \frac{(i+1)T}{m} \right]} \{ A^n_{jm}(t)^2 + |X^n_{jm}(t)|^2 \} < \infty.
\]
By the induction on the index \( i \), we can show:
\[
\mathbb{E} \sup_{t \in \left( \frac{(i+1)T}{m}, \frac{(i+2)T}{m} \right]} \{ A^n_{jm}(t)^2 + |X^n_{jm}(t)|^2 \} < \infty.
\]
In fact, by using the arguments leading up to (4.10) and (4.11), we have:
\[
\mathbb{E} \sup_{t \in \left( \frac{(i+1)T}{m}, \frac{(i+2)T}{m} \right]} A^n_{jm}(t)^2 \leq K_2 + K_1 \int_{\frac{(i+1)T}{m}}^{\frac{(i+2)T}{m}} \mathbb{E} A^n_{jm} \left( \frac{(i+1)T}{m} \right)^2 ds \\
\leq K_2 + \frac{K_1 T}{m} \int_{\frac{(i+1)T}{m}}^{\frac{(i+2)T}{m}} \mathbb{E} A^n_{jm} \left( \frac{(i+1)T}{m} \right)^2 < \infty. \tag{4.12}
\]
\[
\mathbb{E} \sup_{\frac{(i+1)T}{m} \leq t \leq \frac{(i+2)T}{m}} |X^n_{jm}(t)|^2 \\
\leq K_4 + K_3 \int_{\frac{(i+1)T}{m}}^{\frac{(i+2)T}{m}} \mathbb{E} \left\{ |X^n_{jm} \left( \frac{(i+1)T}{m} \right)|^2 + A^n_{jm} \left( \frac{(i+1)T}{m} \right)^2 \right\} ds \\
\leq K_4 + \frac{K_3 T}{m} \mathbb{E} |X^n_{jm} \left( \frac{(i+1)T}{m} \right)|^2 + \frac{K_3 T}{m} \mathbb{E} A^n_{jm} \left( \frac{(i+1)T}{m} \right)^2 < \infty. \tag{4.13}
\]
Thus the part 1 of the lemma is proved.

2. From (4.4), for any \( t_1, t_2 \in [0, T] \) such that \( t_1 \leq t_2 \),
\[
X^n_{jm}(t_2) - X^n_{jm}(t_1) = \int_{t_1}^{t_2} c(Y^n_{jm}(s), V^n_m(s)) ds + \int_{t_1}^{t_2} \sigma(Y^n_{jm}(s), V^n_m(s)) dB_j(s) \\
+ \int_{t_1}^{t_2} \int_G \sigma(Y^n_{jm}(s), V^n_m(s), v) W(dvds)
\]
By the Itô isometry, we have:
\[
\mathbb{E} |X^n_{jm}(t_2) - X^n_{jm}(t_1)|^2 \leq 3(t_2 - t_1) \cdot \mathbb{E} \int_{t_1}^{t_2} |c(Y^n_{jm}(s), V^n_m(s))|^2 ds \\
+ 3 \cdot \mathbb{E} \int_{t_1}^{t_2} |\sigma(Y^n_{jm}(s), V^n_m(s))|^2 ds \\
+ 3 \cdot \mathbb{E} \int_{t_1}^{t_2} \int_G |\alpha(Y^n_{jm}(s), V^n_m(s), v)|^2 \mu(dv) ds.
\]
By linear growth condition (S3),

\[ E|X_{jm}(t_2) - X_{jm}(t_1)|^2 \leq (3T + 6)K_1^2 \int_{t_1}^{t_2} E(1 + |Y_{jm}(s)|^2 + \|V_m^n(s)\|^2)ds \]

\[ \leq (3T + 6)K_1^2 \int_{t_1}^{t_2} E \left( 1 + \sup_{0 \leq i \leq m-1} |X_{jm}^{n}\left(\frac{iT}{m}\right)|^2 + \sup_{0 \leq i \leq m-1} A_{jm}^{n}\left(\frac{iT}{m}\right)^2 \right) ds \]

\[ \leq (3T + 6)K_1^2 \cdot \left( 1 + E \sup_{t \in [0, T]} |X_{jm}^{n}(t)|^2 + E \sup_{t \in [0, T]} A_{jm}^{n}(t)^2 \right) \cdot (t_2 - t_1) \]

\[ \leq K(t_2 - t_1) \]

where

\[ K = (3T + 6)K_1^2 \cdot \left( 1 + E \sup_{t \in [0, T]} |X_{jm}^{n}(t)|^2 + E \sup_{t \in [0, T]} A_{jm}^{n}(t)^2 \right) < \infty. \]

From (4.5), for any \( t_1, t_2 \in [0, T] \) such that \( t_1 \leq t_2 \),

\[ E|A_{jm}^{n}(t_2) - A_{jm}^{n}(t_1)|^2 \leq 3(t_2 - t_1) \cdot \sum_{k=1}^{N} p_k \cdot E \int_{t_1}^{t_2} Z_{jm}^{n}(s)^2 \cdot |d(Y_{jm}^{n}(s), V_m^{n}(s), k)|^2 ds \]

\[ + 3 \sum_{k=1}^{N} p_k \cdot E \int_{t_1}^{t_2} Z_{jm}^{n}(s)^2 \cdot |\gamma(Y_{jm}^{n}(s), V_m^{n}(s), k)|^2 ds \]

\[ + 3 \sum_{k=1}^{N} p_k \cdot E \int_{t_1}^{t_2} \int_{G} Z_{jm}^{n}(s)^2 \beta(Y_{jm}^{n}(s), V_m^{n}(s), v, k)^2 \mu(dv) ds. \]

Then by the assumption (S2),

\[ E|A_{jm}^{n}(t_2) - A_{jm}^{n}(t_1)|^2 \leq (3T + 6)K_2^2 \cdot \sum_{k=1}^{N} p_k \cdot E \int_{t_1}^{t_2} Z_{jm}^{n}(s)^2 ds \]

\[ \leq (3T + 6)K_2^2 \cdot \int_{t_1}^{t_2} E \sup_{1 \leq i \leq m-1} A_{jm}^{n}(\frac{iT}{m})^2 ds \]

\[ \leq (3T + 6)K_2^2 \cdot \int_{t_1}^{t_2} E \sup_{s \in [0, T]} A_{jm}^{n}(s)^2 ds \]

\[ \leq K'(t_2 - t_1) \]

where \( K' = (3T + 6)K_2^2 \cdot E \sup_{t \in [0, T]} A_{jm}^{n}(t)^2 < \infty \) by part 1 of this Lemma. \( \square \)
Theorem 4.2. Under the assumptions (S1), (S2) and (S3), for each $j = 1, \ldots, n$,

(a) the sequence $\{X_{jm}^n\}_{m=1}^\infty$ is tight in $C([0, T]; \mathbb{R}^d)$,

(b) the sequence $\{A_{jm}^n\}_{m=1}^\infty$ is tight in $C([0, T]; \mathbb{R})$.

Proof.

(a) Let $K = \mathbf{E} \sup_{t \in [0, T]} |X_{jm}^n(t)|^2$. By Chebyshev’s inequality and Lemma 4.1, for any $\epsilon > 0$, there exists a number $a$ such that when $a \geq \sqrt{K/\epsilon}$,

$$
\mathbf{P}\{ \sup_{t \in [0, T]} |X_{jm}^n(t)| \geq a \} \leq \frac{1}{a^2} \cdot \mathbf{E} \sup_{t \in [0, T]} |X_{jm}^n(t)|^2 < \frac{K^2}{a^2} \leq \epsilon;
$$

For any $\epsilon > 0$, $a > 0$, there exists a number $\delta$ such that when $\delta \leq \frac{a^2}{K^2} \cdot \epsilon$,

$$
\mathbf{P}\{ \sup_{|t_2 - t_1| < \delta} |X_{jm}^n(t_2) - X_{jm}^n(t_1)| \leq \frac{1}{a^2} \cdot \mathbf{E} \sup_{|t_2 - t_1| < \delta} |X_{jm}^n(t_2) - X_{jm}^n(t_1)|^2
$$

$$
\leq \frac{1}{a^2} K^2 |t_2 - t_1| < \frac{K^2}{a^2} \cdot \delta \leq \epsilon
$$

$\forall t_1, t_2 \in [0, T]$ with $|t_2 - t_1| < \delta$. Therefore $\{X_{jm}^n\}_{m=1}^\infty$ is tight in $C(\mathbb{R}^d \times [0, T])$ by the tightness criterion.

(b) The proof of tightness of $\{A_{jm}^n\}_{m=1}^\infty$ in $C(\mathbb{R} \times [0, T])$ is quite similar.

□

By Prokhorov’s Theorem, the tightness of a family of probability measures on a complete and separable metric space is equivalent to the relative compactness of the family of probability measures.

Since $C([0, T]; \mathbb{R}^d)$ and $C([0, T]; \mathbb{R})$ both are complete and separable metric spaces equipped with the topology of uniform convergence, for each $j = 1, \ldots, n$, the family $\{X_{jm}^n, A_{jm}^n\}_{m=1}^\infty$ are relatively compact in $C([0, T]; \mathbb{R}^d) \times C([0, T]; \mathbb{R})$. We are allowed to choose a subsequence of the above sequence $\{X_{jm}^n, A_{jm}^n\}_{m=1}^\infty$ if necessary. Let $\{\hat{X}_j^n, \hat{A}_j^n\}$ be the limit of the chosen subsequence which is denoted by the same notation.
\( \{X^m_n, A^m_{jm}\}_{m=1}^\infty \) (in order to avoid the complexity in notation), i.e. \( X^m_{jm} \to \hat{X}^n_j, A^m_{jm} \to \hat{A}^n_j \) in distribution.

**Theorem 4.3.**

\[
U^n_m = \frac{1}{n} \sum_{j=1}^{n} A^n_{jm} \delta X^n_{jm} \Rightarrow \frac{1}{n} \sum_{j=1}^{n} \hat{A}^n_{jm} \delta \hat{X}^n_j = \hat{U}^n \text{ as } m \to \infty.
\]

**Proof.** Let \( h(x,a) = a \cdot g(x) \) where \( x \in \mathbb{R}^d \) and \( g \) is a bounded and continuous function on \( \mathbb{R}^d \). Let \( \lambda^m_n \) be the law of \( U^n_m = \frac{1}{n} \sum_{j=1}^{n} A^n_{jm} \delta X^n_{jm} \), \( \hat{\lambda}^n \) be the law of \( \hat{U}^n = \frac{1}{n} \sum_{j=1}^{n} \hat{A}^n_{jm} \delta \hat{X}^n_j \). Then for a fixed \( n \) and as \( m \to \infty \), we have:

\[
\int_{C([0,T];\mathbb{R}^d) \times C([0,T];\mathbb{R})} h(x,a) \lambda^n_m (dx, da) = \frac{1}{n} \mathbb{E} \sum_{j=1}^{n} A^n_{jm} g(X^n_{jm})
\]

\[
\to \frac{1}{n} \mathbb{E} \sum_{j=1}^{n} \hat{A}^n_{jm} g(\hat{X}^n_j) = \int_{C([0,T];\mathbb{R}^d) \times C([0,T];\mathbb{R})} h(x,a) \lambda^n (dx, da)
\]

because it has been shown from above that \( X^m_{jm} \to \hat{X}^n_j, A^m_{jm} \to \hat{A}^n_j \) in distribution. So for a fixed \( n \) and as \( m \to \infty \),

\[
\mathbb{E} \left< g, U^n_m \right> \to \mathbb{E} \left< g, \hat{U}^n \right>, \quad \forall g \in C_b(\mathbb{R}^d).
\]

that is,

\[
\left< g, \lambda^n_m \right> \to \left< g, \hat{\lambda}^n \right>, \quad \forall g \in C_b(\mathbb{R}^d)
\]

Therefore \( U^n_m \to \hat{U}^n \) in distribution.

\( \square \)

4.3 Martingale Problems

In this section it will be shown the limit \((\hat{X}^n_j, \hat{A}^n_j, \hat{U}^n)\) obtained in the last section solves the system (3.1)-(3.3)-(3.4).
Let
\[ X^n(t) = (X^n_1(t), \ldots, X^n_j(t), \ldots, X^n_n(t)), \quad A^n(t) = (A^n_1(t), \ldots, A^n_j(t), \ldots, A^n_n(t)), \]
\[ L^n = (L_1, \ldots, L_j, \ldots, L_n). \]

Define
\[ \sigma_j(X^n(t), A^n(t)) := \sigma(X^n(t), U^n(t)), \quad \gamma_j(X^n(t), A^n(t), L^n) := \gamma(X^n_j(t), U^n(t), L_j) \]
and similarly for \( c_j, \alpha_j, d_j \) and \( \beta_j \).

Then the system equations (3.1) and (3.4) can be written as:
\[ X^n_j(t) = X^n_j(0) + \int_0^t c_j(X^n(s), A^n(s))ds + \int_0^t \sigma_j(X^n(s), A^n(s))dB_j(s) \]
\[ + \int_0^t \int_G \alpha_j(X^n(s), A^n(s), v)W(dvds) \] (4.14)
\[ A^n_j(t) = A^n_j(0) + \int_0^t A^n_j(s)ds + \int_0^t A^n_j(s)\gamma_j(X^n(s), A^n(s), L^n)dB_j(s) \]
\[ + \int_0^t \int_G A^n_j(s)\beta_j(X^n(s), A^n(s), v, L^n)W(dvds) \] (4.15)

To show the limit \( \{\hat{X}^n_j, \hat{A}^n_j, \hat{U}^n\} \) (obtained from theorem 4.2 and 4.3) is a solution of the system (4.14) and (4.15), the martingale problem posed by the system is considered.

For any function \( f \in C^2_b(\mathbb{R}^{n \times d} \times \mathbb{R}^n) \), by the Itô formula, we have:
\[ df(X^n(t), A^n(t)) = \sum_{j=1}^n f'_{x_j}(X^n(t), A^n(t))dX^n_j(t) + \sum_{j=1}^n f'_{a_j}(X^n(t), A^n(t))dA^n_j(t) \]
\[ + \frac{1}{2} \sum_{j=1}^n \sum_{i=1}^n f''_{x_jx_i}(X^n(t), A^n(t))d\langle X^n_j(t), X^n_i(t) \rangle \]
\[ + \frac{1}{2} \sum_{j=1}^n \sum_{i=1}^n f''_{a_ja_i}(X^n(t), A^n(t))d\langle A^n_j(t), A^n_i(t) \rangle \]
\[ + \sum_{j=1}^n \sum_{i=1}^n f''_{x_ja_i}(X^n(t), A^n(t))d\langle X^n_j(t), A^n_i(t) \rangle \] (4.16)
Notice that

\[
d\langle X^n_j(t), X^n_i(t) \rangle = \begin{cases} 
\sigma_j \sigma_j^T(X^n(t), A^n(t))dt \\
+ \int_G |\alpha_j(X^n(t), A^n(t), v)|^2 \mu(dv)dt, & \text{if } i = j \\
\int_G \alpha_j \alpha_i^T(X^n(t), A^n(t), v) \mu(dv)dt, & \text{if } i \neq j 
\end{cases}
\]

\[
d\langle A^n_j(t), A^n_i(t) \rangle = \begin{cases} 
A^n_j(t)^2 \gamma_j \gamma_j^T(X^n(t), A^n(t), L^n)dt \\
+ \int_G A^n_j(t)^2 |\beta_j(X^n(t), A^n(t), v, L^n)|^2 \mu(dv)dt, & \text{if } i = j \\
\int_G A^n_j(t)A^n_i(t)\beta_j \beta_i(X^n(t), A^n(t), v, L^n) \mu(dv)dt, & \text{if } i \neq j 
\end{cases}
\]

\[
d\langle X^n_j(t), A^n_i(t) \rangle = \begin{cases} 
A^n_i(t) \sigma_j(X^n(t), A^n(t)) \gamma_j^T(X^n(t), A^n(t), L^n)dt \\
+ \int_G A^n_i(t) \alpha_j(X^n(t), A^n(t), v) \beta_j(X^n(t), A^n(t), v, L^n) \\
\int_G A^n_i(t) \alpha_j(X^n(t), A^n(t), v) \beta_i(X^n(t), A^n(t), v, L^n) \\
\cdot \mu(dv)dt, & \text{if } i = j \\
\cdot \mu(dv)dt, & \text{if } i \neq j 
\end{cases}
\]

Therefore, the full expression for \(df(X^n(t), A^n(t))\) is:

\[
df(X^n(t), A^n(t)) = \sum_{j=1}^n f'_{x_j}(X^n(t), A^n(t))c_j(X^n(t), A^n(t))dt \\
+ \sum_{j=1}^n f'_{x_j}(X^n(t), A^n(t))\sigma_j(X^n(t), A^n(t))dB_j(t) \\
+ \sum_{j=1}^n f'_{x_j}(X^n(t), A^n(t))\int_G \alpha_j(X^n(t), A^n(t), v)W(dvdt) \\
+ \sum_{j=1}^n f'_{a_j}(X^n(t), A^n(t))A^n_j(t)d_j(X^n(t), A^n(t), L^n)dt \\
+ \sum_{j=1}^n f'_{a_j}(X^n(t), A^n(t))A^n_j(t)\gamma_j(X^n(t), A^n(t), A^n(t), L^n)dB_j(t)
\]
+ \sum_{j=1}^{n} f'_{a_j}(X^n(t), A^n(t))A^n_j(t)\beta_j(X^n(t), A^n(t), v, L^n)W(dvdt)

+ \frac{1}{2} \sum_{j=1}^{n} f''_{x_jx_j}(X^n(t), A^n(t))\sigma^T_j\sigma_j(X^n(t), A^n(t))dt

+ \frac{1}{2} \sum_{j=1}^{n} f''_{x_jx_j}(X^n(t), A^n(t)) \int_G \alpha^T_j \alpha_j(X^n(t), A^n(t), v)\mu(dv)dt

+ \sum_{i,j=1, i\neq j}^{n} \sum f''_{x_jx_j}(X^n(t), A^n(t)) \int_G \alpha^T_j \alpha_j(X^n(t), A^n(t), v)\mu(dv)dt

+ \frac{1}{2} \sum_{j=1}^{n} f''_{a_ja_j}(X^n(t), A^n(t))A^n_j(t)^2\gamma_j(X^n(t), A^n(t), L^n)dt

+ \frac{1}{2} \sum_{j=1}^{n} f''_{a_ja_j}(X^n(t), A^n(t)) \int_G A^n_j(t)^2\beta_j(X^n(t), A^n(t), v, L^n)^2\mu(dv)dt

+ \sum_{i,j=1, i\neq j}^{n} \sum f''_{a_ja_i}(X^n(t), A^n(t)) \int_G A^n_j(t)A^n_i(t)\beta_j(X^n(t), A^n(t), v, L^n)\mu(dv)dt

+ \sum_{j=1}^{n} f''_{x_ja_j}(X^n(t), A^n(t)) \int_G A^n_j(t)\alpha^T_j(X^n(t), A^n(t), v)\beta_j(X^n(t), A^n(t), v, L^n)\mu(dv)dt

+ \sum_{j=1}^{n} f''_{x_ja_j}(X^n(t), A^n(t)) \int_G A^n_j(t)\alpha^T_j(X^n(t), A^n(t), v)\beta_j(X^n(t), A^n(t), v, L^n)\mu(dv)dt

+ \sum_{i,j=1, i\neq j}^{n} \sum f''_{x_ja_i}(X^n(t), A^n(t)) \int_G A^n_i(t)\alpha^T_j(X^n(t), A^n(t), v)\beta_i(X^n(t), A^n(t), v, L^n)\mu(dv)dt

Let

\mathbf{x} = (x_1, ..., x_j, ..., x_n), \quad \mathbf{a} = (a_1, ..., a_j, ..., a_n), \quad \mathbf{1} = (l_1, ..., l_j, ..., l_n).
Define an infinitesimal operator $\mathcal{L}_1$ by:

$$
\mathcal{L}_1 f(x, a) = \sum_{j=1}^{n} f'_{x_j}(x, a)c_j(x, a) + \sum_{j=1}^{n} f'_{a_j}(x, a)d_j(x, a, l) + \frac{1}{2} \sum_{j=1}^{n} f''_{x_jx_j}(x, a)\sigma\sigma^{T}(x, a)$$
$$+ \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} f''_{x_jx_i}(x, a) \int_{G} \alpha_j\alpha_i^{T}(x, a, v)\mu(dv)$$
$$+ \frac{1}{2} \sum_{j=1}^{n} f''_{a_ja_j}(x, a)a_ja_j\gamma\gamma^{T}(x, a, l)$$
$$+ \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} f''_{a_ja_i}(x, a)a_ja_i \int_{G} \beta_j\beta_i(x, a, v, l)\mu(dv)$$
$$+ \sum_{j=1}^{n} f''_{x_ja_j}(x, a)a_j\sigma\sigma^{T}(x, a)\gamma_j(x, a, l)$$
$$+ \sum_{j=1}^{n} \sum_{i=1}^{n} f''_{x_ja_i}(x, a)a_j \int_{G} \alpha_j(x, a, v)\beta_j(x, a, v, l)\mu(dv)$$

Using the Euler approximations and their weak limit, we recall the following:

$$U^n_m = \frac{1}{n} \sum_{j=1}^{m} A^n_{jm}\delta X^n_{jm}, \quad \hat{U}^n_m = \frac{1}{n} \sum_{j=1}^{m} \hat{A}^n_{jm}\delta \hat{X}^n_{jm}, \quad V^n_m = \frac{1}{n} \sum_{j=1}^{m} Z^n_{jm}\delta Y^n_{jm}.$$ 

$$Y^n_{jm}(t) = X^n_{jm}(\frac{iT}{m}), \quad Z^n_{jm}(t) = A^n_{jm}(\frac{iT}{m}), \quad t \in (\frac{iT}{m}, \frac{(i+1)T}{m}].$$

Let $\lambda^n_m$ be the joint distribution of $(X^n_m, A^n_m, L^n)$. Let $\hat{\lambda}^n$ be the joint distribution of $(\hat{X}^n, \hat{A}^n, L^n)$.

For any $f \in C^2_b(\mathbb{R}^{n \times d} \times \mathbb{R}^n)$, it is clear that

$$\mathcal{M}^n_t(X^n_m, A^n_m, L^n) = f(X^n_m(t), A^n_m(t)) - f(X^n_m(0), A^n_m(0)) - \int_0^t \mathcal{L}f(X^n_m(s), A^n_m(s))ds$$
In this section, for a fixed \( n \) is a \( \hat{X} \) is a \( \lambda \) \( \{ \}
\begin{align*}
&= f(X^n_m(t), A^n_m(t)) - f(X^n_m(0), A^n_m(0)) \\
- & \left\{ \sum_{j=1}^{n} \int_{0}^{t} f'_{x_j}(X^n_m(s), A^n_m(s))c_j(Y^n_m(s), Z^n_m(s))ds \right. \\
& \left. + \sum_{j=1}^{n} \int_{0}^{t} f''_{x_{j},x_j}(X^n_m(s), A^n_m(s))d_j(Y^n_m(s), Z^n_m(s), L^n)ds \right. \\
& \left. + \frac{1}{2} \sum_{j=1}^{n} \int_{0}^{t} f''_{x_{j},x_j}(X^n_m(s), A^n_m(s))\sigma_j^T \sigma_j(Y^n_m(s), Z^n_m(s))ds \right. \\
& \left. + \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} \int_{0}^{t} f''_{x_{j},x_i}(X^n_m(s), A^n_m(s)) \int_{G} \alpha_{j}^T \alpha_{i}(Y^n_m(s), Z^n_m(s), v)\mu(dv)ds \right. \\
& \left. + \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} \int_{0}^{t} f''_{x_{j},a_j}(X^n_m(s), A^n_m(s))Z^n_{jm}(s)\gamma_j^T \gamma_j(Y^n_m(s), Z^n_m(s), L^n)ds \right. \\
& \left. + \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} \int_{0}^{t} f''_{x_{j},a_i}(X^n_m(s), A^n_m(s))Z^n_{jm}(s) \int_{G} \beta_j^T \beta_i(Y^n_m(s), Z^n_m(s), v, L^n)\mu(dv)ds \right. \\
& \left. + \sum_{j=1}^{n} \int_{0}^{t} f''_{x_{j},a_j}(X^n_m(s), A^n_m(s))Z^n_{jm}(s)\sigma_j^T (Y^n_m(s), Z^n_m(s)) \gamma_j(Y^n_m(s), Z^n_m(s), L^n)ds \right. \\
& \left. + \sum_{j=1}^{n} \sum_{i=1}^{n} \int_{0}^{t} f''_{x_{j},a_i}(X^n_m(s), A^n_m(s))Z^n_{jm}(s) \cdot \int_{G} \alpha_{j}(Y^n_m(s), Z^n_m(s), v) \beta_i(Y^n_m(s), Z^n_m(s), v, L^n)\mu(dv)ds \right\}.
\end{align*}

is a \( \lambda^n_m \)-martingale. Then it will be shown by the method of martingale problem that

\[
\mathcal{N}_t^f(\hat{X}^n, \hat{A}^n, L^n) = f(\hat{X}^n(t), \hat{A}^n(t)) - f(\hat{X}^n(0), \hat{A}^n(0)) - \int_{0}^{t} \mathcal{L}f(\hat{X}^n(s), \hat{A}^n(s))ds
\]

is a \( \hat{\lambda}^n \)-martingale.

### 4.4 Existence of the Solutions to the Interacting Particle System

In this section, for a fixed \( n \) we will show the convergence of the sequence of martingales

\[
\{ \mathcal{M}_t^n(X^n_m, A^n_m, L^n) \}_{m=1}^\infty
\]

to the limiting process \( \mathcal{N}_t^f(\hat{X}^n, \hat{A}^n, L^n) \).
Lemma 4.4. Let \( \lambda \) be the joint distribution of \((x, a)\). Under the linear growth (S3),

\[
\mathbb{E}^\lambda \left\{ |c_j(x, a)|^2 + |\sigma_j^T \sigma_j(x, a)| + \int_G |\alpha_j(x, a, v)|^2 \mu(dv) \right\} < \infty.
\]

Proof. By Lemma 4.1 and the linear growth condition (S3),

\[
\mathbb{E}^\lambda \left\{ |c_j(x, a)|^2 + |\sigma_j^T \sigma_j(x, a)| + \int_G |\alpha_j(x, a, v)|^2 \mu(dv) \right\} = \mathbb{E}^\lambda \left\{ |c(x, u)|^2 + |\sigma \sigma^T(x, u)| + \int G |\alpha(x, u, v)|^2 \mu(dv) \right\}
\]

\[
\leq K^2 \cdot \mathbb{E}^\lambda \left\{ 1 + |x_j|^2 + \frac{1}{n} \sum_{j=1}^n a_j^2 \right\}
\]

\[
\leq K^2 \cdot \mathbb{E}^\lambda \left\{ 1 + |x_j|^2 + a_j^2 \right\} < \infty
\]

where \( u \) is the weighted measure defined by \( u = \frac{1}{n} \sum_j a_j \delta_{x_j} \).

For any positive number \( R \) (sufficiently large), we define for \( 1 \leq i \leq d, \ 1 \leq l \leq m \),

1. the \( i \)-th component of \( c_j \):

\[
c^{i,R}_j(x, a) = \begin{cases} 
    c^j_i(x, a), & \text{if } |c^j_i(x, a)| < R, \\
    R, & \text{if } |c^j_i(x, a)| \geq R
\end{cases}
\]

2. the \( i, l \)-th entry of \( \sigma_j \):

\[
\sigma^{i,l,R}_j(x, a) = \begin{cases} 
    \sigma^{i,l}_j(x, a), & \text{if } |\sigma^{i,l}_j(x, a)| < R, \\
    R, & \text{if } |\sigma^{i,l}_j(x, a)| \geq R
\end{cases}
\]

3. the \( i \)-th component of \( \alpha_j \):

\[
\alpha^{i,R}_j(x, a) = \begin{cases} 
    \alpha^j_i(x, a), & \text{if } |\alpha^j_i(x, a)| < R, \\
    R, & \text{if } |\alpha^j_i(x, a)| \geq R
\end{cases}
\]
Lemma 4.5. Let $c_j^R = \{c_j^{i,R}\}_{i=1}^d$, $\sigma_j^R = \{\sigma_j^{i,l,R}\}_{i=1,l=1}^{d,m}$, $\alpha_j^R = \{\alpha_j^{i,R}\}_{i=1}^d$. Let $\lambda$ be the joint distribution of $(x, a)$. If $R \to \infty$, then

\[
\mathbb{E}^{\lambda}\left\{ |c_j(x, a) - c_j^R(x, a)| + |\sigma_j^T \sigma_j(x, a) - \sigma_j^R(\sigma_j^R)^T(x, a)| \right. \\
+ \int_G |\alpha_j^T \alpha_j(x, a, v) - (\alpha_j^R)^T \alpha_j^R(x, a, v)|\mu(dv) \} \to 0.
\]

Proof. By the Hölder’s inequality, Chebyshev’s inequality and Lemma 4.4, we have:

\[
\mathbb{E}^{\lambda}|\sigma_j^T \sigma_j(x, a) - \sigma_j^R(\sigma_j^R)^T(x, a)| \\
= \sum_{l=1}^m \mathbb{E}^{\lambda}|\sigma_j^{i,l} \sigma_j^{l,k}(x, a) - \sigma_j^{i,l,R} \sigma_j^{l,k,R}(x, a)| \\
= \sum_{l=1}^m \int_H |\sigma_j^{i,l} \sigma_j^{l,k}(x, a) - R^2| \cdot 1_{\{(x,a);|\sigma_j^{i,l} \sigma_j^{l,k}(x,a)| \geq R^2\}} \lambda(dx da) \\
\leq \sum_{l=1}^m \left( \int_H |\sigma_j^{i,l} \sigma_j^{l,k}(x, a) - R^2|^2 \lambda(dx da) \right)^{1/2} \left( \int_H 1_{\{(x,a);|\sigma_j^{i,l} \sigma_j^{l,k}(x,a)| \geq R^2\}} \lambda(dx da) \right)^{1/2} \\
= \sum_{l=1}^m \left( \mathbb{E}^{\lambda}|\sigma_j^{i,l} \sigma_j^{l,k}(x, a) - R^2|^2 \right)^{1/2} \cdot \left( P\{\omega; |\sigma_j^{i,l} \sigma_j^{l,k}(x, a)| \geq R^2\} \right)^{1/2} \\
\leq \sum_{l=1}^m \left( \mathbb{E}^{\lambda}(|\sigma_j^{i,l}(x, a) - R|^2 + |\sigma_j^{l,k}(x, a) + R|^2) \right)^{1/2} \left( \frac{1}{R^2} \cdot \mathbb{E}^{\lambda}|\sigma_j^{i,l}(x, a)|^2 \right)^{1/2} \\
\to 0 \text{ as } R \to \infty
\]

which means,

\[
\mathbb{E}^{\lambda}\int_G |\alpha_j^T \alpha_j(x, a, v) - (\alpha_j^R)^T \alpha_j^R(x, a, v)|\mu(dv) \to 0
\]

Along similar arguments, the convergence of the other terms to 0 can be established.

Lemma 4.6. Assume that $(X^n, A^n)$ is a solution of the system. Let $\lambda$ be the joint distribution of $(X^n, A^n, L^n)$. Then

\[
\mathbb{E}^{\lambda}\left\{ |c_j(X^n(t), A^n(t)) - c_j(Y^n(t), Z^n(t))|^2 + |\sigma_j(X^n(t), A^n(t)) - \sigma_j(Y^n(t), Z^n(t))|^2 \right. \\
+ |\alpha_j(X^n(t), A^n(t), v) - \alpha_j(Y^n(t), Z^n(t), v)|^2 \mu(dv) \} \to 0
\]
\[ + \int_{G} |\alpha_j(X^n(t), A^n(t), v) - \alpha_j(Y^n(t), Z^n(t), v)|^2 \mu(dv) \]
\[ + |d_j(X^n(t), A^n(t), L^n) - d_j(Y^n(t), Z^n(t), L^n)|^2 \]
\[ + |\gamma_j(X^n(t), A^n(t), L^n) - \gamma_j(Y^n(t), Z^n(t), L^n)|^2 \]
\[ + \int_{G} |\beta_j(X^n(t), A^n(t), v, L^n) - \beta_j(Y^n(t), Z^n(t), v, L^n)|^2 \mu(dv) \}
\[ \longrightarrow 0 \text{ as } m \to \infty \ \forall t \in [0, \infty) \]

**Proof.**

**Step 1.** By lemma 4.1, for any \( t \in \left( \frac{m+1}{m} \right) \),

\[
|c_j(X^n(t), A^n(t)) - c_j(Y^n(t), Z^n(t))|^2 = |c_j(X^n(t), A^n(t)) - c_j(X^n(\frac{m+1}{m}), A^n(\frac{m+1}{m})))|^2 \\
= |c(X^n_j(t), U^n(t)) - c(X^n_\frac{m+1}{m}(t), U^n(\frac{m+1}{m})))|^2 \\
\leq K^2 \left\{ |X^n_j(t) - X^n_\frac{m+1}{m}(\frac{m+1}{m})|^2 + \rho(U^n_+ (t), U^n_+ (\frac{m+1}{m}))^2 + \rho(U^n_- (t), U^n_- (\frac{m+1}{m}))^2 \right\} \\
\leq K^2 \left\{ |X^n_j(t) - X^n_\frac{m+1}{m}(\frac{m+1}{m})|^2 + 2 \left( \frac{1}{n} \sum_{j=1}^{n} \frac{1}{n} \sum_{j=1}^{n} |X^n_j(t) - X^n_\frac{m+1}{m}(\frac{m+1}{m})|^2 \right) \\
+ 2 \left( \frac{1}{n} \sum_{j=1}^{n} |A^n_j(t) - A^n_\frac{m+1}{m}(\frac{m+1}{m})|^2 \right) \right\} \quad (4.18) \]

and for any \( t_1, t_2 \in [0, T] \), there exists a finite constant \( K > 0 \) such that

\[
\mathbb{E}\{ |X^n_j(t_2) - X^n_j(t_1)|^2 + |A^n_j(t_2) - A^n_j(t_1)|^2 \leq K|t_2 - t_1|. \]

This result actually implies the continuity of \( X^n_j(t) \) and \( A^n_j(t) \) in \( t \). That is, if \( t \to \frac{m T}{m} \) from the right, then

\[
\mathbb{E}\left| X^n_j(t) - X^n_j(\frac{m T}{m}) \right|^2 \to 0, \quad \mathbb{E}\left| A^n_j(t) - A^n_j(\frac{m T}{m}) \right|^2 \to 0. \quad (4.19) \]
Therefore on the right-hand side of (4.18), we only need to show the convergence of the second term:

\[ 2 \left( \frac{1}{n} \sum_{j=1}^{n} A_j^n(t)^2 \right) \cdot \left( \frac{1}{n} \sum_{j=1}^{n} |X_j^n(t) - X_j^n(iT/m)|^2 \right). \]

Particularly, it is crucial to show that the factor \( \frac{1}{n} \sum_{j=1}^{n} A_j^n(t)^2 \) is finite. Since \( \mathbb{E}A_j^n(t)^2 < \infty \), it is reasonable to introduce

\[ \tau_M = \inf \{ t; \frac{1}{n} \sum_{j=1}^{n} A_j^n(t)^2 > M^2 \} \]

Then

1. \( \forall t \leq \tau_M, \frac{1}{n} \sum_{j=1}^{n} A_j^n(t)^2 \leq M^2 < \infty. \)

2. By the Chebyshev’s inequality and for a fixed \( n \),

\[ \mathbb{P}\{ t > \tau_M \} = \mathbb{P}\{ \omega; \frac{1}{n} \sum_{j=1}^{n} A_j^n(t)^2 > M^2 \} \leq \frac{1}{M^2} \cdot \mathbb{E}\{ \frac{1}{n} \sum_{j=1}^{n} A_j^n(t)^2 \} \]

\[ = \frac{1}{n \cdot M^2} \sum_{j=1}^{n} \mathbb{E}A_j^n(t)^2 \longrightarrow 0 \text{ as } M \to \infty. \]

Therefore, \( \tau_M \to T \) as \( M \to \infty \) almost sure. By the dominated convergence theorem and (4.19), and if \( F_M = \{ t \leq \tau_M \} \) for any \( t \leq \tau_M \), then

\[ \mathbb{E}^\lambda |c_j(X^n(t), A^n(t)) - c_j(Y^n(t), Z^n(t))|^2 \cdot 1_{F_M} \]

\[ \leq K^2 \cdot \{ \mathbb{E}|X_j^n(t) - X_j^n(iT/m)|^2 \cdot 1_{F_M} + 2 \cdot \mathbb{E}(\frac{1}{n} \sum_{j=1}^{n} |A_j^n(t) - A_j^n(iT/m)|^2 \cdot 1_{F_M}) \]

\[ + 2 \cdot \mathbb{E}(\frac{1}{n} \sum_{j=1}^{n} A_j^n(t)^2)(\frac{1}{n} \sum_{j=1}^{n} |X_j^n(t) - X_j^n(iT/m)|^2 \cdot 1_{F_M}) \} \]

\[ \leq K^2 \cdot \{ \mathbb{E}|X_j^n(t) - X_j^n(iT/m)|^2 + \frac{2}{n} \sum_{j=1}^{n} \mathbb{E}|A_j^n(t) - A_j^n(iT/m)|^2 \]

\[ + \frac{2M^2}{n} \sum_{j=1}^{n} \mathbb{E}|X_j^n(t) - X_j^n(iT/m)|^2 \} \]
\( \rightarrow 0 \) as \( m \to \infty \), if \( n \) with \( M \) are fixed.

Therefore, for any \( t \leq T \),

\[
E^{\lambda} |c_j(X^n(t), A^n(t)) - c_j(Y^n(t), Z^n(t))|^2 \to 0 \text{ as } m \to \infty.
\]

Similarly

\[
E^{\lambda} \left| \left( \sigma_j(X^n(t), A^n(t)) - \sigma_j(Y^n(t), Z^n(t)) \right) \cdot \left( \sigma_j(X^n(t), A^n(t)) - \sigma_j(Y^n(t), Z^n(t)) \right)^T \right| \to 0,
\]

\[
E^{\lambda} \int_G |\alpha_j(X^n(t), A^n(t), v) - \alpha_j(Y^n(t), Z^n(t), v)|^2 \mu(dv) \to 0, \quad \text{as } m \to \infty.
\]

**Step 2.** By Lemma 4.1,

\[
E^{\lambda}|d_j(X^n(t), A^n(t), L^n) - d_j(Y^n(t), Z^n(t), L^n)|^2
\]

\[
= E^{\lambda}|d_j(X^n(t), A^n(t), L^n) - d_j(X^n(t), \frac{iT}{m}, A^n(t), L^n)|^2
\]

\[
= E^{\lambda}|d_j(X^n(t), U^n(t), L_j) - d(X^n(t), \frac{iT}{m}, U^n(t), L_j)|^2
\]

\[
= \sum_{k=1}^{N} p_k \cdot E^{\lambda}|d_j(X^n(t), U^n(t), k) - d(X^n(t), \frac{iT}{m}, U^n(t), k)|^2
\]

\[
\leq \sum_{k=1}^{N} p_k \cdot K^2 \cdot E^{\lambda}\left\{ |X^n_j(t) - X^n_j(\frac{iT}{m})|^2 + \rho \left( U^n_+(t), U^n_+(\frac{iT}{m}) \right)^2 \\
+ \rho \left( U^n_-(t), U^n_-(\frac{iT}{m}) \right)^2 \right\}
\]

\[
\leq K^2 \sum_{k=1}^{N} p_k \cdot \left\{ E^{\lambda}|X^n_j(t) - X^n_j(\frac{iT}{m})|^2 \\
+ E^{\lambda}\left( \frac{2}{n} \sum_{j=1}^{n} A^n_j(t)^2 \right) \left( \frac{1}{n} \sum_{j=1}^{n} |X^n_j(t) - X^n_j(\frac{iT}{m})|^2 \right) \\
+ E^{\lambda}\left( \frac{2}{n} \sum_{j=1}^{n} |A^n_j(t) - A^n_j(\frac{iT}{m})|^2 \right) \right\}
\]

\[(4.20)\]
By the similar techniques of stopping times that we applied in Step 1, we can control
the upper bound of $\frac{1}{n} \sum_{j=1}^{n} A^n_j(t)^2$ and therefore,

$$E^\lambda |d_j(X^n(t), A^n(t), L^n) - d_j(Y^n(t), Z^n(t), L^n)|^2 \rightarrow 0,$$

$$E^\lambda |\gamma_j(X^n(t), A^n(t), L^n) - \gamma_j(Y^n(t), Z^n(t), L^n)|^2 \rightarrow 0,$$

$$E^\lambda \int_G |\beta_j(X^n(t), A^n(t), v, L^n) - \beta_j(Y^n(t), Z^n(t), v, L^n)|^2 \mu(dv) \rightarrow 0, \quad \text{as } m \rightarrow \infty.$$  

□

Next we move on to the most important theorem in this chapter.

**Theorem 4.7.** Under the conditions (S1), (S2), (S3) and (S4), for a fixed $n$,

$$\int_H M^T_f(x, a, l) \cdot \lambda^n_m(dx, da, dl) \rightarrow \int_H N^T_f(x, a, l) \cdot \hat{\lambda}^n(dx, da, dl)$$

where $H = C(\mathbb{R}^{n \times d}) \times C(\mathbb{R}^n) \times \mathbb{N}^n$.

**Proof.** The convergence is established by showing the convergence term by term between $M^T_f$ and $N^T_f$.

(i) It has been proved that $\lambda^n_m \rightarrow \hat{\lambda}^n$ and $f \in C^2_b(\mathbb{R}^{n \times d} \times \mathbb{R}^n)$, therefore

$$\int_H f(x, a) \lambda^n_m(dx, da) \rightarrow \int_H f(x, a) \hat{\lambda}^n(dx, da)$$

Then we have

$$E f(X^n_m(t), A^n_m(t)) \rightarrow E f(\hat{X}^n_m(t), \hat{A}^n_m(t)), \quad E f(X^n_m(0), A^n_m(0)) \rightarrow E f(\hat{X}^n_m(0), \hat{A}^n_m(0)).$$

(ii) To show

$$E \int_0^t f_x^T f_y^T(A^n_m(s), Z^n_m(s))ds$$

$$\rightarrow E \int_0^t f_x^T(\hat{X}^n_m(s), \hat{A}^n_m(s))c_j(\hat{Y}_m(s), \hat{Z}_m(s))ds,$$
we start with

\[ \left| \int_H \int_0^t f'_{x_j}(x(s), a(s)) c_j(y(s), z(s)) ds \cdot \lambda^n_m(dx da) \right| \]

\[ - \int_H \int_0^t f'_{x_j}(x(s), a(s)) c_j(x(s), a(s)) ds \cdot \hat{\lambda}^n(dx da) \]

\[ \leq \int_0^t \left| \int_H f'_{x_j}(x(s), a(s)) c_j(y(s), z(s)) \cdot \lambda^n_m(dx da) \right| ds \]

\[ - \int_H f'_{x_j}(x(s), a(s)) c_j(x(s), a(s)) \cdot \hat{\lambda}^n(dx da) \right| ds \]

\[ \leq \int_0^t \left\{ \left| \int_H f'_{x_j}(x(s), a(s)) c_j(y(s), z(s)) \cdot \lambda^n_m(dx da) \right| - \int_H f'_{x_j}(x(s), a(s)) c_j(x(s), a(s)) \cdot \lambda^n_m(dx da) \right\} ds \]

\[ \leq \int_0^t \int_H \left| f'_{x_j}(x(s), a(s)) \right| \cdot \left| c_j(y(s), z(s)) - c_j(x(s), a(s)) \right| \cdot \lambda^n_m(dx da) ds \]

\[ + \int_0^t \left\{ \left| \int_H f'_{x_j}(x(s), a(s)) c_j(x(s), a(s)) \lambda^n_m(dx da) \right| \right. \]

\[ - \int_H f'_{x_j}(x(s), a(s)) c_j^R(x(s), a(s)) \lambda^n_m(dx, da) \right| \]

\[ + \int_H f'_{x_j}(x(s), a(s)) c_j^R(x(s), a(s)) \lambda^n_m(dx da) \]

\[ - \int_H f'_{x_j}(x(s), a(s)) c_j^R(x(s), a(s)) \hat{\lambda}^n(dx da) \right| \}

\[ ds \]

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\[
\begin{align*}
&\leq \int_0^t E^{\lambda_n} |f_{x_j} (X^n_m(s), A^n_m(s))| \cdot |c_j(Y^n_m(s), Z^n_m(s)) - c_j(X^n_m(s), A^n_m(s))| ds \\
&\quad + \int_0^t E^{\lambda_n} |f_{x_j} (X^n_m(s), A^n_m(s))| \cdot |c_j(X^n_m(s), A^n_m(s)) - c_j^R(X^n_m(s), A^n_m(s))| ds \\
&\quad + \int_0^t \left| \int_H f_{x_j}(x(s), a(s)) c_j^R(x(s), a(s)) \lambda^n_m (dx da) \right| ds \\
&\quad - \int_H f_{x_j}(x(s), a(s)) c_j^R(x(s), a(s)) \hat{\lambda}^n_m (dx da) \bigg| ds \\
&\quad + \int_0^t E^{\lambda^n} |f_{x_j}(\hat{X}^n(s), \hat{A}^n(s))| \cdot |c_j^R(\hat{X}^n(s), \hat{A}^n(s)) - c_j(\hat{X}^n(s), \hat{A}^n(s))| ds
\end{align*}
\tag{4.21}
\]

By the Schwarz inequality and Lemma 4.6 and for any \(f'_{x_j} \in C^2_b(\mathbb{R}^{n \times d} \times \mathbb{R}^n)\), the first term in (4.21) approaches zero as \(m \to \infty\), that is,

\[
\int_0^t E^{\lambda_n} \left( |f_{x_j} (X^n_m(s), A^n_m(s))| \cdot |c_j(Y^n_m(s), Z^n_m(s)) - c_j(X^n_m(s), A^n_m(s))| \right) ds \\
\leq \int_0^t \left\{ E^{\lambda_n} |f_{x_j} (X^n_m(s), A^n_m(s))|^2 \right\}^{1/2} \cdot \left\{ E^{\lambda_n} |c_j(Y^n_m(s), Z^n_m(s)) - c_j(X^n_m(s), A^n_m(s))|^2 \right\}^{1/2} ds \\
\to 0 \text{ as } m \to \infty.
\]

Let

\[
\sup_{1 \leq j \leq n} |f'_{x_j}| \leq K \text{ for some } K > 0.
\]

By the Schwarz inequality and Lemma 4.5, the second and fourth terms in (4.21) both approach to zero. For instance,

\[
\int_0^t E^{\lambda_n} |f_{x_j} (X^n_m(s), A^n_m(s))| \cdot |c_j(X^n_m(s), A^n_m(s)) - c_j^R(X^n_m(s), A^n_m(s))| ds \\
\leq \int_0^t K \cdot E^{\lambda_n} \cdot |c_j(X^n_m(s), A^n_m(s)) - c_j^R(X^n_m(s), A^n_m(s))| ds \\
\to 0 \text{ as } R \to \infty.
\]

Since \(f_{x_j}\) and \(c_j^R\) are bounded, the third term in (4.21) tends to zero because of the weak convergence of \(\lambda^n_m\) to \(\hat{\lambda}^n\), i.e., for any \(j\) and a positive number \(R\),

\[
\left| \int_H f_{x_j}(x(s), a(s)) c_j^R(x(s), a(s)) \lambda^n_m (dx da) \right|
\]
and the second term in (4.22) will be:

$$\leq \sum_{k=1}^{N} p_k \int_{0}^{t} K_1 K_2 \cdot E^{\lambda_m}|Z_{jm}(s) - A_{jm}(s)| ds$$
\[
\leq \sum_{k=1}^N \sum_{i=0}^{m-1} \int_{t}^{t_i} K_1 K_2 \cdot \mathbf{E}^{\lambda_m} \left| A_{jm}^n \left( \frac{iT}{m} \right) - A_{jm}^n(s) \right| ds
\]
\[
\leq K_1 K_2 K_3 \sum_{i=0}^{m-1} \int_{t}^{t_i} \left( s - \frac{iT}{m} \right) ds
\]
\[
= K_1 K_2 K_3 \sum_{i=0}^{m-1} \frac{1}{2} \left( \frac{(i+1)T}{m} - \frac{iT}{m} \right)^2
\]
\[
= \frac{1}{2} K_1 K_2 K_3 \cdot m \cdot \left( \frac{T}{m} \right)^2
\]
\[
= \frac{K_1 K_2 K_3 T^2}{2m} \to 0 \quad \text{as } m \to \infty
\]

To consider the convergence of the third term in (4.22), we introduce a stopping time \( \tau_M \) defined by:
\[
\tau_M = \inf \left\{ t; |A_{jm}^n(t)| \vee |\hat{A}_j^n(t)| \geq M \right\} \vee T
\]
for a sufficiently large number \( 0 < M < \infty \). Then
\[
|A_{jm}^n(t)| \cdot 1_{\{0 \leq t \leq \tau_M\}} \leq M < \infty, \quad |\hat{A}_j^n(t)| \cdot 1_{\{0 \leq t \leq \tau_M\}} \leq M < \infty,
\]
\[
\mathbf{P}\{t > \tau_M\} = \mathbf{P}\{\omega; \sup_{s \leq t} |A_{jm}^n(s)| \vee \sup_{s \leq t} |\hat{A}_j^n(s)| \geq M\}
\]
\[
\leq \frac{1}{M^2} \cdot \left\{ \mathbf{E} \sup_{s \leq t} |A_{jm}^n(s)|^2 + \mathbf{E} \sup_{s \leq t} |\hat{A}_j^n(s)|^2 \right\} \to 0 \quad \text{as } M \to \infty.
\]

Then for any \( t \geq 0 \), we have:
\[
\lambda_m^n \{ a; \sup_{0 \leq s \leq t} |a_s| \geq M \} = \mathbf{P}\{\omega; \sup_{s \leq t} |a(s)| \geq M\}
\]
\[
\leq \frac{1}{M^2} \int \sup_{0 \leq s \leq t} |a(s)|^2 \lambda_m^n(da) \to 0 \quad \text{as } M \to \infty.
\]

For any \( \epsilon > 0 \), the third term in (4.22)
\[
\sum_{k=1}^N p_k \int_0^t \left| \int_H f_{a_j}(x(s), a(s)) a_j(s) d_j(x(s), a(s), k) \cdot \lambda^n_m(dx \cdot da) \right. \left. - \int_H f_{a_j}(x(s), a(s)) a_j(s) d_j(x(s), a(s), k) \cdot \hat{\lambda}^n(dx \cdot da) \right| ds
\]
\[
\leq \sum_{k=1}^{N} p_k \int_0^t \left| \int_H f_{a_j}(x(s), a(s)) d_j(x(s), a(s), k) \cdot 1_{0 \leq s \leq \tau_M} \cdot \lambda_m^n (dx \cdot da) \right| ds + \epsilon t
\]

Let \( m \to \infty \). Then by the Lebesgue dominated convergence theorem and the weak convergence of \( \lambda_m^n \) to \( \hat{\lambda}^n \) and letting \( m \to \infty \), we have:

\[
\lim_{m \to \infty} \sum_{k=1}^{N} p_k \int_0^t \left| \int_H f_{a_j}(x(s), a(s)) d_j(x(s), a(s), k) \cdot 1_{0 \leq s \leq \tau_M} \cdot \lambda_m^n (dx \cdot da) \right| ds + \epsilon t
\]

\[
= \sum_{k=1}^{N} p_k \int_0^t \lim_{m \to \infty} \left| \int_H f_{a_j}(x(s), a(s)) d_j(x(s), a(s), k) \cdot 1_{0 \leq s \leq \tau_M} \cdot \lambda_m^n (dx \cdot da) \right| ds + \epsilon t
\]

\[
= \epsilon t \to 0
\]

because \( f_{a_j}, d_j \) and \( a_j(s) \cdot 1_{0 \leq s \leq \tau_M} \) are all bounded and \( \epsilon \) can be arbitrarily small.

Furthermore, \( P\{t \geq \tau_M\} \to 0 \) implies that \( P\{t < \tau_M\} = 1 \). Hence \( \lim_{M \to \infty} P\{t < \tau_M\} = P\{t < \infty\} = 1 \). Therefore for any \( t \in [0, \infty) \), the third term in (4.22) approaches to zero as \( M \to \infty \) and \( m \to \infty \). Therefore (iii) is proved.

(iv) We need to show

\[
E \int_0^t f_{x_jx_j}''(X_m^n(s), A_m^n(s))\sigma_j^T \sigma(Y_m^n(s), Z_m^n(s))ds
\]

\[
E \to \int_0^t f_{x_jx_j}''(\hat{X}_m^n(s), \hat{A}_m^n(s))\sigma_j^T \sigma(\hat{X}_m^n(s), \hat{A}_m^n(s))ds.
\]

Let \( H \) be defined the same as before.

\[
\left| \int_H \int_0^t f_{x_jx_j}''(x(s), a(s))\sigma_j^T \sigma_j(y(s), z(s))ds \cdot \lambda_m^n (dxdadl) \right|
\]

\[
- \int_H \int_0^t f_{x_jx_j}''(x(s), a(s))\sigma_j^T \sigma_j(x(s), a(s))ds \cdot \hat{\lambda}^n (dxdadl) \right|
\]
By Lemma 4.4, 4.5 and 4.6, each of the terms converges to 0 as \( m \to \infty \).

(v) To show

\[
\mathbb{E} \int_0^t \int_{H} f_{x_i}''(x(s), a(s)) \sigma_j^R(\sigma_j^R)^T(x(s), a(s)) \cdot \lambda_m(dx da) + f_{x_i}''(x(s), a(s)) \sigma_j^R(\sigma_j^R)^T(x(s), a(s)) \cdot \hat{\lambda}_m(dx da) \bigg| ds
\]

This part can be proved by the same procedure as in part (iv). The proof is skipped here.

(vi) Aim to show:

\[
\mathbb{E} \int_0^t f''_{a_j}(X_m^n(s), A_m^n(s)) \cdot Z''_{jm}(s) \cdot \sigma_j^R(\sigma_j^R)^T(Y_m^n(s), Z_m^n(s), L^n) ds
\]

Let

\[
\sup_{1 \leq j \leq n} |f''_{a_j}| \leq K_1, \quad \sup_{1 \leq j \leq n} |\sigma_j^T \sigma_j| \leq K_2
\]

Then

\[
\int_H \int_0^t f''_{a_j}(x(s), a(s)) \cdot z_j(s) \cdot \gamma_j^T(y(s), z(s), l) ds \cdot \lambda_m(dx da dl)
\]
Consider the convergence of the first and third terms in (4.24), we introduce a stopping time to control the upper bounds of $A^n_{jm}$ and $\hat{A}^n_{jm}$. For a suitable large number $M$, we
define

\[ \tau_M = \inf\{ t; \sup_{s \leq t} |A_j^n(s)| \vee \sup_{s \leq t} |\dot{A}_j^n(s)| \geq M \} \bigvee T. \]

Then we have:

\[ P\{ t > \tau_M \} = P\{ \omega; \sup_{s \leq t} |A_j^n(t)| \vee \sup_{s \leq t} |\dot{A}_j^n(t)| \geq M \} \leq \frac{1}{M^2} \cdot E \sup_{s \leq t} |A_j^n(t)|^2 + E \sup_{s \leq t} |\dot{A}_j^n(t)|^2 \rightarrow 0 \text{ as } M \rightarrow \infty. \]

Let

\[ H_M = \left\{ (x, a, l) \in H; \sup_{0 \leq s \leq t} |a_j(s)| \leq M \right\} \]

As \( M \rightarrow \infty, H_M \rightarrow H \). For any \( t \leq \tau_M \), the first term in (4.24)

\[
\sum_{k=1}^{N} p_k \int_0^t \int_H K_1 \cdot |z_j(s)| \cdot |\gamma_j \gamma_j^T(y(s), z(s), k) - \gamma_j \gamma_j^T(x(s), a(s), k)| \lambda_m^n(dx da) ds
\]

\[ = \lim_{M \rightarrow \infty} \sum_{k=1}^{N} p_k K_1 \int_0^t \int_{H_M} |z_j(s)| \{ 1_{\{ 0 \leq t \leq \tau_M \}} \cdot |\gamma_j \gamma_j^T(y(s), z(s), k) - \gamma_j \gamma_j^T(x(s), a(s), k)| \lambda_m^n(dx da) ds
\]

\[ = \lim_{M \rightarrow \infty} \sum_{k=1}^{N} p_k K_1 \cdot \int_0^t \int_{H_M} \left\{ |z_j(s)|^2 1_{\{ 0 \leq t \leq \tau_M \}} \cdot |\gamma_j(y(s), z(s), k) + \gamma_j(x(s), a(s), k)|^2 \right\}^{\frac{1}{2}} \lambda_m^n(dx da)
\]

\[ \cdot \int_{H_M} \left\{ |\gamma_j(y(s), z(s), k) - \gamma_j(x(s), a(s), k)|^2 \right\}^{\frac{1}{2}} \lambda_m^n(dx da) ds \rightarrow 0 \text{ as } m \rightarrow \infty \]

by Lemma 4.6, the boundedness of \( \gamma_j \), the Schwarz inequality and dominated convergence theorem.

For any \( \epsilon > 0 \), the third term in (4.24)

\[
\sum_{k=1}^{N} p_k \int_0^t \left| \int_H f_{\gamma_j a_j}^m(x(s), a(s)) a_j(s) \gamma_j \gamma_j^T(x(s), a(s), k) \cdot \lambda_m^n(dx da)ight| ds
\]

\[ \quad - \int_H f_{\gamma_j a_j}^m(x(s), a(s)) a_j(s) \gamma_j \gamma_j^T(x(s), a(s), k) \cdot \tilde{\lambda}^n(dx \cdot da) \right| ds
\]

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By letting $m \to \infty$, and Dominated Convergence Theorem, we have:

\[
\lim_{M \to \infty} \sum_{k=1}^{N} p_k \int_0^t \left| \int_{H_M} f''_{a_{j}, a_j}(x(s), a(s)) a_j(s) \gamma_j^T \gamma_j (x(s), a(s), k) \cdot 1_{\{0 \leq s \leq \tau_M\}} \cdot \lambda_m^n (dxda) \right| ds + \epsilon t
\]

\[
= \sum_{k=1}^{N} p_k \int_0^t \left| \int_{H_M} f''_{a_{j}, a_j}(x(s), a(s)) a_j(s) \gamma_j^T (x(s), a(s), k) \cdot 1_{\{0 \leq s \leq \tau_M\}} \cdot \lambda_m^n (dxda) \right| ds + \epsilon t
\]

\[
= \epsilon t \to 0
\]

because $f''_{a_{j}, a_j}, \gamma_j$ and $a_j(s) \cdot 1_{\{0 \leq s \leq \tau_M\}}$ are all bounded and $\epsilon$ can be arbitrarily small and approach to zero. Furthermore, $P\{t \geq \tau_M\} \to 0$ implies that $P\{t < \tau_M\} = 1$. Hence $\lim_{M \to \infty} P\{t < \tau_M\} = P\{t < \infty\} = 1$. Therefore for any $t \in [0, \infty)$, the third term in (4.24) approaches to zero as $M \to \infty$ and $m \to \infty$. Therefore (vi) is proved.

(vii) To show the convergence of the rest three terms,

\[
E \int_0^t f''_{x_{j}, x_j}(X^n_{m}(s), A^n_{m}(s))Z^n_{jm}(s)Z^n_{im}(s) \int_G \beta_j \beta_i (Y^n_{m}(s), Z^n_{im}(s), v, L^n) \mu (dv) ds
\]

\[
- E \int_0^t f''_{x_{j}, x_j}(\hat{X}^n(s), \hat{A}^n(s)) \hat{A}^n_i(s) \hat{A}^n_j(s) \int_G \beta_j \beta_i (\hat{X}^n(s), \hat{A}^n(s), v, L^n) \mu (dv) ds,
\]

\[
E \int_0^t f''_{x_{j}, x_j}(X^n_{m}(s), A^n_{m}(s))Z^n_{jm}(s) \cdot \sigma_j (Y^n_{m}(s), Z^n_{m}(s)) \gamma_i^T (Y^n_{m}(s), Z^n_{m}(s), v, L^n) ds
\]

\[
- E \int_0^t f''_{x_{j}, x_j}(\hat{X}^n(s), \hat{A}^n(s)) \hat{A}^n_i(s) \cdot \sigma_j (\hat{X}^n(s), \hat{A}^n(s)) \gamma_i^T (\hat{X}^n(s), \hat{A}^n(s), v, L^n) ds,
\]

\[
E \int_0^t f''_{x_{j}, x_j}(X^n_{m}(s), A^n_{m}(s))Z^n_{im}(s) \int_G \alpha_j (Y^n_{m}(s), Z^n_{m}(s), v) \beta_i (Y^n_{m}(s), Z^n_{m}(s), v, L^n) \mu (dv) ds
\]
\[ \int_0^t f''_{x,a}(\hat{X}^n(s), \hat{A}^n(s)) \int_G \alpha_j(\hat{X}^n(s), \hat{A}^n(s), v) \beta_i(\hat{X}^n(s), \hat{A}^n(s), v, L^n) \mu(dv) ds \]

the proof follows along the same method that are applied in the previous steps.

Combine all the parts (i)-(vii), the convergence of \( M^f_t(x^n_m, a^n_m, L^n) \) to the limiting process \( N^f_t(\hat{X}^n, \hat{A}^n, L^n) \) is established.

□

Recall that \( \lambda^n_m \) is the joint distribution of \((X^n_m, A^n_m, L^n)\), and \( \lambda^n \) is the joint distribution of \((\hat{X}^n, \hat{A}^n, L^n)\). The next theorem will lead us to the existence of solution for the interacting system.

**Theorem 4.8.** Under the assumptions (S1)-(S4), the limit process \( N^f_t(\hat{X}^n, \hat{A}^n, L^n) \) is a \( \lambda^n \)-martingale, where \( \lambda^n \) is the weak limit of \( \{\lambda^n_m\}_{m \geq 1} \).

**Proof.** Let \( g_1, g_2, \ldots, g_k \) be a family of bounded and continuous functions in the space of \( C_b(\mathbb{R}^d \times \mathbb{R}) \). Let for a fixed \( t \in [0, T] \), suppose the times \( r_1, r_2, \ldots, r_k, s \) satisfy \( 0 \leq r_1 \leq r_2 \ldots \leq r_k \leq s \leq t \leq T \).

Since \( M^f_t(x^n_m, a^n_m, L^n) \) is \( \lambda^n_m \)-martingale, then

\[
\int_H \left( M^f_t - M^f_s \right) \cdot g_1(x(r_1), a(r_1)) \cdot g_2(x(r_2), a(r_2)) \cdots g_k(x(r_k), a(r_k)) \cdot \lambda^n_m(dxdadl) = 0.
\]

Exactly as the same as in the proof of Theorem 4.7,

\[
0 = \int_H \left( M^f_t - M^f_s \right) \cdot g_1(x(r_1), a(r_1)) \cdot g_2(x(r_2), a(r_2)) \cdots g_k(x(r_k), a(r_k)) \cdot \lambda^n_m(dxdadl)
\]

\[
\rightarrow \int_H \left( N^f_t - N^f_s \right) \cdot g_1(x(r_1), a(r_1)) \cdot g_2(x(r_2), a(r_2)) \cdots g_k(x(r_k), a(r_k)) \cdot \lambda^n(dxdadl)
\]

as \( m \to \infty \). Therefore \( N^f_t \) is an \( \lambda^n \)-martingale.

\[ \square \]
Chapter 5

Weak Convergence of the Weighted Empirical Measures

In this chapter we first study the conditions under which the solution to the interacting particle system is unique. Consider the stochastic system (3.1)-(3.3)-(3.4) defined on the probability space \( \{\Omega, \mathcal{F}, P\} \):

\[
X^n_j(t) = X^n_j(0) + \int_0^t c(X^n_j(s), U^n(s)) + \int_0^t \sigma(X^n_j(s), U^n(s))dB^n_j(s) \\
   + \int_0^t \int \alpha(X^n_j(s), U^n(s), v) W(dvds) \\
A^n_j(t) = A^n_j(0) + \int_0^t d(X^n_j(s), U^n(s), L_j) + \int_0^t A^n_j(s)\gamma(X^n_j(s), U^n(s), L_j)dB^n_j(s) \\
   + \int_0^t \int G^{A^n_j(s)}(X^n_j(s), U^n(s), v, L_j) W(dvds) \\
U^n(t) = \frac{1}{n} \sum_{j=1}^n A^n_j(t) \delta_{X^n_j(t)}
\]

with everything defined as the same as in Chapter 3.

5.1 Uniqueness of the Solution to the Interacting Particle System

Lemma 5.1. Suppose \( (X^n_j, A^n_j, U^n) \) is a solution of the system (3.1)-(3.3)-(3.4). Then under the assumptions (S1)-(S3), for any \( t \in [0, T] \),

1. \( E\|U^n(t)\|^2 \leq E|A^n_1(t)|^2 \),

2. \( E\sup_{t \in [0, T]}(|X^n_j(t)|^2 + A^n_j(t)^2) < \infty \).

Proof. By the definition of \( \|u\| \) in (S3),

\[
\|U^n(t)\|^2 = \left( \frac{1}{n} \sum_{j=1}^n |A^n_j(t)| \right)^2 \leq \frac{1}{n} \sum_{j=1}^n A^n_j(t)^2
\]

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and by the exchangeability of \( \{A^n_j\}_{j=1}^n \),

\[
E\|U^n(t)\|^2 \leq E\left( \frac{1}{n} \sum_{j=1}^{n} |A^n_j(t)| \right)^2 \leq \frac{1}{n} \sum_{j=1}^{n} E|A^n_j(t)|^2 = E|A^n_1(t)|^2.
\]

By the inequality \((a + b + c + d)^2 \leq 4(a^2 + b^2 + c^2 + d^2)\),

\[
|X^n_j(t)|^2 \leq 4|X^n_j(0)|^2 + 4 \int_0^t \sigma(X^n_j(s), U^n(s))dB_j(s)\right|^2
\]

\[+ 4 \int_0^t c(X^n_j(s), U^n(s))ds \right|^2 + 4 \int_0^t \int_G \alpha(X^n_j(s), U^n(s), v)W(dvd\sigma)^2.
\]

By Doob’s inequality,

\[
E \sup_{t \in [0,T]} |X^n_j(t)|^2 \leq 4E|X^n_j(0)|^2 + 16E \int_0^T |\sigma\sigma^T (X^n_j(s), U^n(s))|ds
\]

\[+ 4T \int_0^T |c(X^n_j(s), U^n(s))|^2 ds + 16E \int_0^T \int_G |\alpha(X^n_j(s), U^n(s), v)|^2 \mu(dv)ds.
\]

By the linear growth (S3),

\[
E \sup_{t \in [0,T]} |X^n_j(t)|^2 \leq 4E|X^n_j(0)|^2 + (32 + 4T)K_1^2 \cdot E \int_0^T (1 + |X^n_j(s)|^2 + \|U^n(s)\|^2)ds
\]

\[\leq 4E|X^n_j(0)|^2 + (32 + 4T)K_1^2 T + K_1^2 \int_0^T E(|X^n_j(s)|^2 + A^n_1(s)^2)ds
\]

\[\leq K_2 + K_1^2 \int_0^T E \sup_{s \in [0,T]} \left(|X^n_j(s)|^2 + A^n_1(s)^2\right)ds \quad (5.1)
\]

where \( K_2 = 4E|X^n_j(0)|^2 + (32 + 4T)K_1^2 T \).

By the inequality \((a + b + c + d)^2 \leq 4(a^2 + b^2 + c^2 + d^2)\),

\[
A^n_j(t)^2 \leq 4A^n_j(0)^2 + 4 \left| \int_0^t A^n_j(s)\gamma(X^n_j(s), U^n(s), L_j)dB_j(s) \right|^2
\]

\[+ 4 \left| \int_0^t A^n_j(s)d(X^n_j(s), U^n(s), L_j)ds \right|^2
\]

\[+ 4 \left| \int_0^t \int_G A^n_j(s)\beta(X^n_j(s), U^n(s), v, L_j)W(dvd\sigma) \right|^2
\]
Since $L_j$ is independent of $X^n_j$ and $A^n_j$, then

$$\mathbb{E}\{A^n_j(s)^2 \cdot |\gamma(X^n_j(s), U^n(s), L_j)|^2\} = \sum_{k=1}^{N} p_k \cdot \mathbb{E}\{A^n_j(s)^2 \cdot |\gamma(X^n_j(s), U^n(s), k)|^2\},$$

$$\mathbb{E}\{A^n_j(s)^2 \cdot |d(X^n_j(s), U^n(s), L_j)|^2\} = \sum_{k=1}^{N} p_k \cdot \mathbb{E}\{A^n_j(s)^2 \cdot |d(X^n_j(s), U^n(s), k)|^2\},$$

$$\mathbb{E}\{A^n_j(s)^2 \cdot \int_G \beta(X^n_j(s), U^n(s), v, L_j)^2 \mu(dv)\} = \sum_{k=1}^{N} p_k \cdot \mathbb{E}\{A^n_j(s)^2 \cdot \int_G \beta(X^n_j(s), U^n(s), v, k)^2 \mu(dv)\}.$$ 

By Doob’s inequality,

$$\mathbb{E} \sup_{t \in [0,T]} A^n_j(t)^2 \leq 4\mathbb{E}A^n_j(0)^2 + 16\mathbb{E} \int_0^T A^n_j(s)^2 \cdot |\gamma(X^n_j(s), U^n(s), L_j)|^2 ds$$

$$+4T \cdot \mathbb{E} \int_0^T A^n_j(s)^2 \cdot d(X^n_j(s), U^n(s), L_j)^2 ds$$

$$+16 \cdot \mathbb{E} \int_0^T \int_G A^n_j(s)^2 \cdot \beta(X^n_j(s), U^n(s), v, L_j)^2 \mu(dv) ds$$

$$\leq 4\mathbb{E}A^n_j(0)^2 + 16 \sum_{k=1}^{N} p_k \cdot \mathbb{E} \int_0^T A^n_j(s)^2 \cdot |\gamma(X^n_j(s), U^n(s), k)|^2 ds$$

$$+4T \cdot \sum_{k=1}^{N} p_k \cdot \mathbb{E} \int_0^T A^n_j(s)^2 \cdot |d(X^n_j(s), U^n(s), k)|^2 ds$$

$$+16 \cdot \sum_{k=1}^{N} p_k \cdot \mathbb{E} \int_0^T \int_G A^n_j(s)^2 \cdot \beta(X^n_j(s), U^n(s), v, k)^2 \mu(dv) ds$$

Then by the boundedness assumption (S2) and the fact $\sum_{k=1}^{N} p_k = 1$,

$$\mathbb{E} \sup_{t \in [0,T]} A^n_j(t)^2 \leq 4\mathbb{E}A^n_j(0)^2 + (32 + 4T) K_3^2 \cdot \sum_{k=1}^{N} p_k \cdot \mathbb{E} \int_0^T A^n_j(s)^2 ds$$

$$= 4\mathbb{E}A^n_j(0)^2 + (32 + 4T) K_3^2 \cdot \mathbb{E} \int_0^T A^n_j(s)^2 ds$$

$$= 4\mathbb{E}A^n_j(0)^2 + (32 + 4T) K_3^2 \cdot \mathbb{E}\{A^n_j(s)^2\} ds.$$ 

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By Gronwall’s inequality,

\[ E \sup_{t \in [0, T]} A^n_j(t)^2 \leq K_4 \cdot e^{K_5 T} < \infty, \quad (5.2) \]

where \( K_4 = 4E A^n_j(0)^2 \), \( K_5 = (32 + 4T)K_3^2 \) are finite.

With (5.1) and (5.2), we conclude:

\[ E \sup_{t \in [0, T]} |X^n_j(t)|^2 \leq K_2 + K_1^2 \cdot \int_0^T E \sup_{t \in [0, T]} |X^n_j(s)|^2 ds + K_1^2 \int_0^T K_4 \cdot e^{K_5 T} ds \]

By Gronwall’s inequality,

\[ E \sup_{t \in [0, T]} |X^n_j(t)|^2 \leq K_6 \cdot e^{K_7 T} < \infty. \]

where \( K_6 = K_2 + K_1^2 K_4 \cdot e^{K_5 T} \), \( K_7 = K_1^2 \).

\[ \square \]

**Corollary 5.2.** Under the assumptions (S1), (S2) and (S3), for a fixed \( n \) and any \( T > 0 \),

\[ E \sup_{t \in [0, T]} \|U^n(t)\|^2 < \infty. \]

**Proof.** By Lemma 5.1, it immediately follows that

\[ E \sup_{t \in [0, T]} \|U^n(t)\|^2 \leq E \sup_{t \in [0, T]} \|A^n_1(t)\|^2 < \infty. \]

\[ \square \]

Suppose \((X^n_j, A^n_j, U^n)\) and \((\tilde{X}^n_j, \tilde{A}^n_j, \tilde{U}^n)\) are two solutions for the interacting particle system (3.1), (3.3) and (3.4) with the same initial values, that is,

\[ X^n_j(0) = \tilde{X}^n_j(0), \quad A^n_j(0) = \tilde{A}^n_j(0), \quad U^n(0) = \tilde{U}^n(0). \]

Let

\[ A^{n+}_j(t) = \begin{cases} A^n_j(t), & \text{if } A^n_j(t) \geq 0 \\ 0, & \text{if } A^n_j(t) < 0 \end{cases}, \quad A^{n-}_j(t) = \begin{cases} 0, & \text{if } A^n_j(t) \geq 0 \\ -A^n_j(t), & \text{if } A^n_j(t) < 0 \end{cases} \]
\[ \hat{A}_j^n(t) = \begin{cases} \hat{A}_j^n(t), & \text{if } \hat{A}_j^n(t) \geq 0 \\ 0, & \text{if } \hat{A}_j^n(t) < 0 \end{cases} \]
\[
\hat{A}^{-n}_j(t) = \begin{cases} 0, & \text{if } \hat{A}_j^n(t) \geq 0 \\ -\hat{A}_j^n(t), & \text{if } \hat{A}_j^n(t) < 0 \end{cases}
\]

Define
\[
U^{n+}(t) = \frac{1}{n} \sum_{j=1}^{n} A_j^{n+}(t) \delta X_j^n(t), \quad U^{n-}(t) = \frac{1}{n} \sum_{j=1}^{n} A_j^{n-}(t) \delta X_j^n(t)
\]
\[
\hat{U}^{n+}(t) = \frac{1}{n} \sum_{j=1}^{n} \hat{A}_j^{n+}(t) \delta \hat{X}_j^n(t), \quad \hat{U}^{n-}(t) = \frac{1}{n} \sum_{j=1}^{n} \hat{A}_j^{n-}(t) \delta \hat{X}_j^n(t)
\]

**Lemma 5.3.**

\[
\rho(U^{n+}(t), \hat{U}^{n+}(t))^2 + \rho(U^{n-}(t), \hat{U}^{n-}(t))^2 \leq 2 \left( \frac{1}{n} \sum_{j=1}^{n} A_j^{n}(t)^2 \right) \cdot \left( \frac{1}{n} \sum_{j=1}^{n} |X_j^n(t) - \hat{X}_j^n(t)|^2 \right) + 2 \left( \frac{1}{n} \sum_{j=1}^{n} |A_j^{n}(t) - \hat{A}_j^n(t)| \right)^2
\]

*Proof.* By the Wasserstein metric defined in (S4),

\[
\rho(U^{n+}(t), \hat{U}^{n+}(t)) = \sup_{\phi \in B_1} \left| \frac{1}{n} \sum_{j=1}^{n} A_j^{n+}(t) \phi(X_j^n(t)) - \frac{1}{n} \sum_{j=1}^{n} \hat{A}_j^{n+}(t) \phi(\hat{X}_j^n(t)) \right|
\]

\[
\leq \frac{1}{n} \sum_{j=1}^{n} \sup_{\phi \in B_1} \left| A_j^{n+}(t) \phi(X_j^n(t)) - \hat{A}_j^{n+}(t) \phi(\hat{X}_j^n(t)) \right|
\]

\[
\leq \frac{1}{n} \sum_{j=1}^{n} \sup_{\phi \in B_1} \left| A_j^{n+}(t) \phi(X_j^n(t)) - A_j^{n+}(t) \phi(\hat{X}_j^n(t)) \right|
\]

\[
+ \left| A_j^{n+}(t) \phi(\hat{X}_j^n(t)) - \hat{A}_j^{n+}(t) \phi(\hat{X}_j^n(t)) \right|
\]

\[
\leq \frac{1}{n} \sum_{j=1}^{n} \sup_{\phi \in B_1} |A_j^{n+}(t)| \cdot |\phi(X_j^n(t)) - \phi(\hat{X}_j^n(t))|
\]

\[
+ \leq \frac{1}{n} \sum_{j=1}^{n} \sup_{\phi \in B_1} |A_j^{n+}(t) - \hat{A}_j^{n+}(t)| \cdot |\phi(\hat{X}_j^n(t))|
\]

Using the property $|\phi(x)| \leq 1$, we have

\[
\rho(U^{n+}(t), \hat{U}^{n+}(t)) \leq \frac{1}{n} \sum_{j=1}^{n} |A_j^{n+}(t)||X_j^n(t) - \hat{X}_j^n(t)| + \frac{1}{n} \sum_{j=1}^{n} |A_j^{n+}(t) - \hat{A}_j^{n+}(t)|
\]

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Similarly we have:

\[ \rho(U^n(t), \widehat{U}^n(t)) \leq \frac{1}{n} \sum_{j=1}^{n} |A^n_j(t)||X^n_j(t) - \widehat{X}^n_j(t)| + \frac{1}{n} \sum_{j=1}^{n} |A^n_j(t) - \widehat{A}^n_j(t)| \]

Adding (5.3) and (5.4) together,

\[ \rho(U^n(t), \widehat{U}^n(t)) + \rho(U^n(t), \widehat{U}^n(t)) \]

\[ \leq \frac{1}{n} \sum_{j=1}^{n} |A^n_j(t)||X^n_j(t) - \widehat{X}^n_j(t)| + \frac{1}{n} \sum_{j=1}^{n} |A^n_j(t) - \widehat{A}^n_j(t)| \]

By the Schwarz inequality and the fact \( \left( \frac{1}{n} \sum_{j=1}^{n} x_j \right)^2 \leq \frac{1}{n} \sum_{j=1}^{n} x_j^2 \),

\[ \rho(U^n(t), \widehat{U}^n(t))^2 + \rho(U^n(t), \widehat{U}^n(t))^2 \]

\[ \leq \left( \rho(U^n(t), \widehat{U}^n(t)) + \rho(U^n(t), \widehat{U}^n(t)) \right)^2 \]

\[ \leq 2 \left( \frac{1}{n} \sum_{j=1}^{n} |A^n_j(t)||X^n_j(t) - \widehat{X}^n_j(t)| \right)^2 + 2 \left( \frac{1}{n} \sum_{j=1}^{n} |A^n_j(t) - \widehat{A}^n_j(t)| \right)^2 \]

\[ \leq 2 \left( \frac{1}{n} \sum_{j=1}^{n} A^n_j(t)^2 \right) \left( \frac{1}{n} \sum_{j=1}^{n} |X^n_j(t) - \widehat{X}^n_j(t)|^2 \right) + 2 \left( \frac{1}{n} \sum_{j=1}^{n} |A^n_j(t) - \widehat{A}^n_j(t)| \right)^2 \]

Next we will show the uniqueness of the solution to the system (3.1)-(3.3)-(3.4). Applying the Itô formula to \( f(z) = \ln z \) with \( z = A^n_j(t) \),

\[ A^n_j(t) = A^n_j(0) \cdot \exp \left\{ \int_0^t \gamma(X^n_j(s), U^n(s), L_j)dB_j(s) + \int_0^t d(X^n_j(s), U^n(s), L_j)ds \right. \]

\[ + \int_0^t \int_G \beta(X^n_j(s), U^n(s), v, L_j)W(dvds) - \frac{1}{2} \int_0^t |\gamma(X^n_j(s), U^n(s), L_j)|^2ds \]

\[ \left. - \frac{1}{2} \int_0^t \int_G \beta(X^n_j(s), U^n(s), v, L_j)^2\mu(dv)ds \right\} \]

(5.5)
\[
\hat{A}_j^n(t) = \hat{A}_j^n(0) \cdot \exp \left\{ \int_0^t \gamma(\hat{X}_j^n(s), \hat{U}^n(s), L_j)dB_j(s) + \int_0^t d(\hat{X}_j^n(s), \hat{U}^n(s), L_j)ds \right. \\
\left. + \int_0^t \int_G \beta(\hat{X}_j^n(s), \hat{U}^n(s), v, L_j) W(dvds) - \frac{1}{2} \int_0^t |\gamma(\hat{X}_j^n(s), \hat{U}^n(s), L_j)|^2 ds \right. \\
\left. - \frac{1}{2} \int_0^t \int_G \beta(\hat{X}_j^n(s), \hat{U}^n(s), v, L_j)^2 \mu(dv)ds \right\} 
\]

(5.6)

By the exchangeability of \( \{A_j^n\} \) and \( \{\hat{A}_j^n\} \), the boundedness of coefficients \( \gamma, d, \beta \) for the weight processes \( A_j^n \) and \( \hat{A}_j^n \), \( \frac{1}{n} \sum_{j=1}^n A_j^n(t)^2 \) and \( \frac{1}{n} \sum_{j=1}^n \hat{A}_j^n(t)^2 \) both exist. Namely, they are finite.

**Theorem 5.4.** Under the assumptions (S1), (S2), (S3) and (S4),

\[
(X_j^n, A_j^n, U^n) = (\hat{X}_j^n, \hat{A}_j^n, \hat{U}^n)
\]

almost surely and for any \( t \in [0, \infty) \).

**Proof.** First define a set of stopping times by

\[
\tau_m = \inf \left\{ t; \frac{1}{n} \sum_{j=1}^n A_j^n(t)^2 > m \right\}, \quad \hat{\tau}_m = \inf \left\{ t; \frac{1}{n} \sum_{j=1}^n \hat{A}_j^n(t)^2 > m \right\}
\]

Let \( \eta_m = \tau_m \wedge \hat{\tau}_m \). By (5.5) and (5.6) and the fact \( |e^x - e^y| \leq (e^x \vee e^y) \cdot |x - y| \),

\[
|A_j^n(t) - \hat{A}_j^n(t)|
\]

\[
\leq (|A_j^n(t) \vee \hat{A}_j^n(t)|) \left| \int_0^t \left( \gamma(X_j^n(s), U^n(s), L_j) - \gamma(\hat{X}_j^n(s), \hat{U}^n(s), L_j) \right) dB_j(s) \\
+ \int_0^t \left( d(X_j^n(s), U^n(s), L_j) - d(\hat{X}_j^n(s), \hat{U}^n(s), L_j) \right) ds \\
+ \int_0^t \int_G \left( \beta(X_j^n(s), U^n(s), v, L_j) - \beta(\hat{X}_j^n(s), \hat{U}^n(s), v, L_j) \right) W(dvds) \\
- \frac{1}{2} \int_0^t \left( |\gamma(X_j^n(s), U^n(s), L_j)|^2 - |\gamma(\hat{X}_j^n(s), \hat{U}^n(s), L_j)|^2 \right) ds \\
- \frac{1}{2} \int_0^t \int_G \left( \beta(X_j^n(s), U^n(s), v, L_j)^2 - \beta(\hat{X}_j^n(s), \hat{U}^n(s), v, L_j)^2 \right) \mu(dv)ds \right|
\]
By the Schwarz inequality, for any \( t \in [0, \eta_m \wedge T] \),

\[
\left( \frac{1}{n} \sum_{j=1}^{n} |A^n_j(t) - \hat{A}^n_j(t)| \right)^2 \leq \frac{1}{n} \sum_{j=1}^{n} \left( A^n_j(t)^2 \vee \hat{A}^n_j(t)^2 \right) \times \\
\times \frac{1}{n} \sum_{j=1}^{n} \left| \int_{0}^{t} \left( \gamma(X^n_j(s), U^n(s), L_j) - \gamma(\hat{X}^n_j(s), \hat{U}^n(s), L_j) \right) dB_j(s) \right|
\]

\[
+ \left| \int_{0}^{t} \left( d(X^n_j(s), U^n(s), L_j) - d(\hat{X}^n_j(s), \hat{U}^n(s), L_j) \right) ds \right|^2
\]

\[
+ \left| \int_{0}^{t} \int_{G} \left( \beta(X^n_j(s), U^n(s), v, L_j) - \beta(\hat{X}^n_j(s), \hat{U}^n(s), v, L_j) \right) W(dvds) \right|^2
\]

\[
- \frac{1}{2} \int_{0}^{t} \left( |\gamma(X^n_j(s), U^n(s), L_j)|^2 - |\gamma(\hat{X}^n_j(s), \hat{U}^n(s), L_j)|^2 \right) ds
\]

\[
- \frac{1}{2} \int_{0}^{t} \int_{G} \left( \beta(X^n_j(s), U^n(s), v, L_j)^2 - \beta(\hat{X}^n_j(s), \hat{U}^n(s), v, L_j)^2 \right) \mu(dv) ds \right|^2
\]

then

\[
\left( \frac{1}{n} \sum_{j=1}^{n} |A^n_j(t) - \hat{A}^n_j(t)| \right)^2 \leq \frac{1}{n} \sum_{j=1}^{n} \left( A^n_j(t)^2 + \hat{A}^n_j(t)^2 \right) \times \\
\times \frac{5}{n} \sum_{j=1}^{n} \left| \int_{0}^{t} \left( \gamma(X^n_j(s), U^n(s), L_j) - \gamma(\hat{X}^n_j(s), \hat{U}^n(s), L_j) \right) dB_j(s) \right|^2
\]

\[
+ \left| \int_{0}^{t} \left( d(X^n_j(s), U^n(s), L_j) - d(\hat{X}^n_j(s), \hat{U}^n(s), L_j) \right) ds \right|^2
\]

\[
+ \left| \int_{0}^{t} \int_{G} \left( \beta(X^n_j(s), U^n(s), v, L_j) - \beta(\hat{X}^n_j(s), \hat{U}^n(s), v, L_j) \right) W(dvds) \right|^2
\]

\[
+ \frac{1}{4} \cdot \left| \int_{0}^{t} \left( |\gamma(X^n_j(s), U^n(s), L_j)|^2 - |\gamma(\hat{X}^n_j(s), \hat{U}^n(s), L_j)|^2 \right) ds \right|^2
\]

\[
+ \frac{1}{4} \cdot \left| \int_{0}^{t} \int_{G} \left( \beta(X^n_j(s), U^n(s), v, L_j)^2 - \beta(\hat{X}^n_j(s), \hat{U}^n(s), v, L_j)^2 \right) \mu(dv) ds \right|^2
\]

and then

\[
\left( \frac{1}{n} \sum_{j=1}^{n} |A^n_j(t) - \hat{A}^n_j(t)| \right)^2
\]
By Doob’s inequality,

\[
\frac{10m^2}{n} \sum_{j=1}^{n} \left\{ \int_{0}^{t} \left| \gamma(X_j^n(s), U^n(s), L_j) - \gamma(\hat{X}_j^n(s), \hat{U}^n(s), L_j) \right| dB_j(s)^2 \\
+ t \int_{0}^{t} \left| d(X_j^n(s), U^n(s), L_j) - d(\hat{X}_j^n(s), \hat{U}^n(s), L_j) \right|^2 ds \\
+ \left| \int_{0}^{t} \int_{G} \left( \beta(X_j^n(s), U^n(s), v, L_j) - \beta(\hat{X}_j^n(s), \hat{U}^n(s), v, L_j) \right) W(dv ds) \right|^2 \\
+ \frac{t}{4} \int_{0}^{t} \left| \gamma(X_j^n(s), U^n(s), L_j) + \gamma(\hat{X}_j^n(s), \hat{U}^n(s), L_j) \right|^2 ds \\
+ \frac{t}{4} \int_{0}^{t} \int_{G} \left| \beta(X_j^n(s), U^n(s), v, L_j) - \beta(\hat{X}_j^n(s), \hat{U}^n(s), v, L_j) \right|^2 \mu(dv) ds \right\}
\]

By Doob’s inequality,

\[
E \left( \frac{1}{n} \sum_{j=1}^{n} \left| A_j^n(t \wedge \eta_m) - \hat{A}_j^n(t \wedge \eta_m) \right| \right)^2 \\
\leq \frac{10m^2}{n} \sum_{j=1}^{n} E \left\{ 4 \int_{0}^{t} \left| \gamma(X_j^n(s), U^n(s), L_j) - \gamma(\hat{X}_j^n(s), \hat{U}^n(s), L_j) \right|^2 \cdot 1_{\{s \leq \eta_m\}} ds \\
+ t \int_{0}^{t} \left| d(X_j^n(s), U^n(s), L_j) - d(\hat{X}_j^n(s), \hat{U}^n(s), L_j) \right|^2 \cdot 1_{\{s \leq \eta_m\}} ds \\
+ 4 \int_{0}^{t} \int_{G} \left| \beta(X_j^n(s), U^n(s), v, L_j) - \beta(\hat{X}_j^n(s), \hat{U}^n(s), v, L_j) \right|^2 \mu(dv) \cdot 1_{\{s \leq \eta_m\}} ds \\
+ \frac{t}{4} \int_{0}^{t} \left| \gamma(X_j^n(s), U^n(s), L_j) + \gamma(\hat{X}_j^n(s), \hat{U}^n(s), L_j) \right|^2 ds \\
+ \frac{t}{4} \int_{0}^{t} \int_{G} \left| \beta(X_j^n(s), U^n(s), v, L_j) - \beta(\hat{X}_j^n(s), \hat{U}^n(s), v, L_j) \right|^2 \mu(dv) \cdot 1_{\{s \leq \eta_m\}} ds \right\}
\]

By assigning a value \( k \) to the grouping factor \( L_j \), and by the assumptions (S2) and (S4),
\[ \mathbb{E}\left( \frac{1}{n} \sum_{j=1}^{n} |A_j^n(t \wedge \eta_m) - \hat{A}_j^n(t \wedge \eta_m)| \right)^2 \]

\[ \leq \frac{10m^2}{n} \sum_{j=1}^{n} \sum_{k=1}^{N} p_k \cdot \mathbb{E}\left\{ 4 \int_0^t |\gamma(X_j^n(s), U^n(s), k) - \gamma(X_j^n(s), \hat{U}^n(s), k)|^2 \cdot 1_{\{s \leq \eta_m\}} ds \right. \]

\[ + \frac{t}{4} \int_0^t \left| \gamma(X_j^n(s), U^n(s), k) - \gamma(X_j^n(s), \hat{U}^n(s), k) \right|^2 \cdot 1_{\{s \leq \eta_m\}} ds \]

\[ + \frac{t}{4} \int_0^t \left| \beta(X_j^n(s), U^n(s), v, k) - \beta(X_j^n(s), \hat{U}^n(s), v, k) \right|^2 \cdot 1_{\{s \leq \eta_m\}} ds \]

\[ \leq \frac{10m^2}{n} \sum_{j=1}^{n} \sum_{k=1}^{N} p_k \cdot (8K_2^2 + K_1^2 K_2 T) \cdot \mathbb{E}\left\{ |X_j^n(s) - \hat{X}_j^n(s)|^2 \right. \]

\[ + \rho(U^n(s), \hat{U}^n(s))^2 + \rho(U^n(s), \hat{U}^n(s))^2 \} \cdot 1_{\{s \leq \eta_m\}} ds \]

Let \( K_3 = 8K_2^2 + K_1^2 K_2 T \). By Lemma 5.3 and the fact \( \sum_{j=1}^{N} p_k = 1 \),

\[ \mathbb{E}\left( \frac{1}{n} \sum_{j=1}^{n} |A_j^n(t \wedge \eta_m) - \hat{A}_j^n(t \wedge \eta_m)| \right)^2 \]

\[ \leq 10m^2 K_3 \cdot \frac{1}{n} \sum_{j=1}^{n} \int_0^t \mathbb{E}\left\{ |X_j^n(s) - \hat{X}_j^n(s)|^2 \right. \]

\[ + 2 \left( \frac{1}{n} \sum_{j=1}^{n} A_j^n(s)^2 \right) \cdot \left( \frac{1}{n} \sum_{j=1}^{n} |X_j^n(s) - \hat{X}_j^n(s)|^2 \right) \]

\[ + 2 \left( \frac{1}{n} \sum_{j=1}^{n} A_j^n(s)^2 \right) \cdot \left( \frac{1}{n} \sum_{j=1}^{n} A_j^n(s)^2 \right) \cdot \left| A_j^n(s) - \hat{A}_j^n(s) \right|^2 \cdot 1_{\{s \leq \eta_m\}} ds. \]

Let \( K_4 = 10m^2 K_3 \) and recall that

\[ \tau_m = \inf \left\{ t; \frac{1}{n} \sum_{j=1}^{n} A_j^n(t)^2 > m \right\}, \quad \bar{\tau}_m = \inf \left\{ t; \frac{1}{n} \sum_{j=1}^{n} \hat{A}_j^n(t)^2 > m \right\} \]
and \( \eta_m = \tau_m \wedge \hat{\tau}_m \), so we have:

\[
\mathbb{E}\left( \frac{1}{n} \sum_{j=1}^{n} \left| A^n_j(t \wedge \eta_m) - \tilde{A}^n_j(t \wedge \eta_m) \right|^2 \right)
\]

\[
\leq K_4 \cdot \left\{ \frac{1}{n} \sum_{j=1}^{n} \int_0^{t \wedge \eta_m} \mathbb{E}|X^n_j(s) - \tilde{X}^n_j(s)|^2 ds + \frac{2m^2}{n} \sum_{j=1}^{n} \int_0^{t \wedge \eta_m} \mathbb{E}|X^n_j(s) - \tilde{X}^n_j(s)|^2 ds \right. 
\]

\[
+ 2 \int_0^{t \wedge \eta_m} \mathbb{E}\left( \frac{1}{n} \sum_{j=1}^{n} |A^n_j(s) - \tilde{A}^n_j(s)| \right)^2 ds \right\}
\]

\[
\leq K_5 \left\{ \frac{1}{n} \sum_{j=1}^{n} \int_0^{t \wedge \eta_m} \mathbb{E}|X^n_j(s) - \tilde{X}^n_j(s)|^2 ds + \int_0^{t \wedge \eta_m} \mathbb{E}\left( \frac{1}{n} \sum_{j=1}^{n} |A^n_j(s) - \tilde{A}^n_j(s)| \right)^2 ds \right\}
\]

(\text{where } K_5 = K_4(1 + 2m^2) \lor 2K_4).

\[
= K_5 \left\{ \int_0^{t \wedge \eta_m} \mathbb{E}\left( \frac{1}{n} \sum_{j=1}^{n} |X^n_j(s) - \tilde{X}^n_j(s)|^2 \right) ds + \int_0^{t \wedge \eta_m} \mathbb{E}\left( \frac{1}{n} \sum_{j=1}^{n} |A^n_j(s) - \tilde{A}^n_j(s)| \right)^2 ds \right\}
\]

(5.7)

On the other hand, for any \( t \leq T \),

\[
|X^n_j(t \wedge \eta_m) - \tilde{X}^n_j(t \wedge \eta_m)|^2
\]

\[
\leq 3 \left| \int_0^t \left( \sigma(X^n_j(s), U^n(s)) - \sigma(\tilde{X}^n_j(s), U^n(s)) \right) \cdot 1_{\{s \leq \eta_m\}} dB_j(s) \right|^2 
\]

\[
+ 3 \left| \int_0^t \left( c(X^n_j(s), U^n(s)) - c(\tilde{X}^n_j(s), \tilde{U}^n(s)) \right) \cdot 1_{\{s \leq \eta_m\}} ds \right|^2 
\]

\[
+ 3 \left| \int_0^t \int_G \left( \alpha(X^n_j(s), U^n(s)) - \alpha(\tilde{X}^n_j(s), \tilde{U}^n(s)) \right) \cdot 1_{\{s \leq \eta_m\}} W(dvds) \right|^2
\]

By Doob’s inequality, for any \( t \leq T \),

\[
\mathbb{E}|X^n_j(t \wedge \eta_m) - \tilde{X}^n_j(t \wedge \eta_m)|^2
\]

\[
\leq 12 \mathbb{E} \int_0^t |\sigma(X^n_j(s), U^n(s)) - \sigma(\tilde{X}^n_j(s), U^n(s))|^2 \cdot 1_{\{s \leq \eta_m\}} ds 
\]

\[
+ 3t \cdot \mathbb{E} \int_0^t |(c(X^n_j(s), U^n(s)) - c(\tilde{X}^n_j(s), \tilde{U}^n(s))|^2 \cdot 1_{\{s \leq \eta_m\}} ds
\]

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\[ + 12E \int_0^t \int_G |\alpha(X^n_j(s), U^n(s)) - \alpha(\tilde{X}^n_j(s), \tilde{U}^n(s))|^2 \cdot 1_{\{s \leq \eta_m\}} \mu(dv)ds. \]

By the Lipschitz condition (S4),

\[
E|X^n_j(t \land \eta_m) - \tilde{X}^n_j(t \land \eta_m)|^2 \\
\leq (24K^2 + 3K^2T) \cdot E \int_0^{t \land \eta_m} \left\{ |X^n_j(s) - \tilde{X}^n_j(s)|^2 + \rho(U^{n+}(s), \tilde{U}^{n+}(s))^2 \\
+ \rho(U^{n-}(s), \tilde{U}^{n-}(s))^2 \right\} ds
\]

Let \( K(T) = 24K^2 + 3K^2T \), then by Lemma 5.3,

\[
E|X^n_j(t \land \eta_m) - \tilde{X}^n_j(t \land \eta_m)|^2 \\
\leq K(T) \cdot E \int_0^{t \land \eta_m} \left\{ |X^n_j(s) - \tilde{X}^n_j(s)|^2 + 2 \left( \frac{1}{n} \sum_{j=1}^n A^n_j(s)^2 \right) \cdot \left( \frac{1}{n} \sum_{j=1}^n |X^n_j(s) - \tilde{X}^n_j(s)|^2 \right) \\
+ 2 \left( \frac{1}{n} \sum_{j=1}^n |A^n_j(s) - \tilde{A}^n_j(s)|^2 \right) \right\} ds
\]

\[
\leq K(T) \cdot E \int_0^{t \land \eta_m} \left\{ |X^n_j(s) - \tilde{X}^n_j(s)|^2 + 2m^2 \left( \frac{1}{n} \sum_{j=1}^n |X^n_j(s) - \tilde{X}^n_j(s)|^2 \right) \\
+ 2 \left( \frac{1}{n} \sum_{j=1}^n |A^n_j(s) - \tilde{A}^n_j(s)|^2 \right) \right\} ds
\]

Let \( K_6 = K(T)(2m^2 + 1) \lor 2K(T) \). Averaging both sides of the above inequality,

\[
E \left( \frac{1}{n} \sum_{j=1}^n |X^n_j(t \land \eta_m) - \tilde{X}^n_j(t \land \eta_m)|^2 \right)
\leq K_6 \cdot \left\{ \int_0^{t \land \eta_m} E \left( \frac{1}{n} \sum_{j=1}^n |X^n_j(s) - \tilde{X}^n_j(s)|^2 \right) ds + \int_0^{t \land \eta_m} E \left( \frac{1}{n} \sum_{j=1}^n |A^n_j(s) - \tilde{A}^n_j(s)|^2 \right) \right\}
\]

(5.8)

Let \( K_7 = K_5 \lor K_6 \). Adding (5.7) and (5.8), we have:

\[
E \left( \frac{1}{n} \sum_{j=1}^n |X^n_j(t \land \eta_m) - \tilde{X}^n_j(t \land \eta_m)|^2 \right) + E \left( \frac{1}{n} \sum_{j=1}^n |A^n_j(t \land \eta_m) - \tilde{A}^n_j(t \land \eta_m)|^2 \right)
\]
\[
\leq 2K_T \cdot \int_0^t \left\{ \mathbb{E} \left( \frac{1}{n} \sum_{j=1}^{n} |A^n_j(t \wedge \eta_m) - \hat{A}^n_j(t \wedge \eta_m)| \right)^2 \\
+ \mathbb{E} \left( \frac{1}{n} \sum_{j=1}^{n} |X^n_j(t \wedge \eta_m) - \hat{X}^n_j(t \wedge \eta_m)|^2 \right) \right\} ds
\]

By Gronwall's inequality,
\[
\mathbb{E} \left( \frac{1}{n} \sum_{j=1}^{n} |A^n_j(t \wedge \eta_m) - \hat{A}^n_j(t \wedge \eta_m)|^2 \right) + \mathbb{E} \left( \frac{1}{n} \sum_{j=1}^{n} |X^n_j(t \wedge \eta_m) - \hat{X}^n_j(t \wedge \eta_m)|^2 \right) \leq 0
\]
therefore,
\[
\mathbb{E} |A^n_j(t \wedge \eta_m) - \hat{A}^n_j(t \wedge \eta_m)| = 0, \quad \mathbb{E} |X^n_j(t \wedge \eta_m) - \hat{X}^n_j(t \wedge \eta_m)| = 0.
\]
which means, for each \(m\), each \(j\) and any \(t \in [0, T]\),
\[
X^n_j(t \wedge \eta_m) = \hat{X}^n_j(t \wedge \eta_m), \quad A^n_j(t \wedge \eta_m) = \hat{A}^n_j(t \wedge \eta_m) \quad \text{almost surely.}
\]
and also \(U^n(t \wedge \eta_m) = \hat{U}^n(t \wedge \eta_m)\) almost surely. Hence \((X, A, U) = (\hat{X}, \hat{A}, \hat{U})\) almost surely and for any \(t \in [0, t \wedge \eta_m]\).

Furthermore, from the definition of stopping time \(\eta_m\), Chebyshev's inequality and the exchangeability of \(\{A^n_j\}_{j=1}^n\), it follows that
\[
P \{ t \geq \eta_m \} \leq \left\{ \sup_{s \in [0, t]} \frac{1}{n} \sum_{j=1}^{n} A^n_j(s)^2 \geq m^2 \right\} \leq \frac{1}{m^2} \cdot \mathbb{E} \left\{ \sup_{s \in [0, t]} \frac{1}{n} \sum_{j=1}^{n} A^n_j(s)^2 \right\}
\]
\[
\leq \frac{1}{m^2} \cdot \frac{1}{n} \cdot \sum_{j=1}^{n} \mathbb{E} \left\{ \sup_{s \in [0, t]} A^n_j(s)^2 \right\} \leq \frac{1}{m^2} \cdot \mathbb{E} \left\{ \sup_{s \in [0, t]} A^n_1(s)^2 \right\}
\]
From Lemma 5.1, we have: \(\mathbb{E} \{ \sup_{s \in [0, t]} A^n_1(s)^2 \}\) is finite and therefore,
\[
P \{ t \geq \eta_\infty \} = \lim_{m \to \infty} P \{ t \geq \eta_m \} = 0
\]
Since \(\eta_\infty = \infty\), by letting \(t \to \infty\), then for any \(t \in [0, \infty)\), we have:
\[
(X^n_j, A^n_j, U^n) = (\hat{X}^n_j, \hat{A}^n_j, \hat{U}^n), \quad a.s.
\]
\[
\square
\]

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5.2 Weak Convergence of Weighted Empirical Measure

Let $\mathcal{B}(C([0,T];\mathbb{R}^d))$ be the Borel $\sigma$-algebra of $C([0,T];\mathbb{R}^d)$. Suppose $\{X^n, A^n, U^n\}$ is the unique solution for the above system. Notice that for any $\omega \in \Omega$ and any Borel set $B \in \mathcal{B}(C([0,T];\mathbb{R}^d))$,

$$U^n(\omega, B) = \frac{1}{n} \sum_{j=1}^{n} A^n_j(\cdot, \omega) \cdot \delta_{X^n_j(\cdot, \omega)}(B).$$

$U^n$ is the weighted empirical measure, also a signed and random measure defined on the measure space $\mathcal{M}(C([0,T];\mathbb{R}^d))$. In this section, we will show the weak convergence of $\{U^n\}_{n=1}^\infty$ under the assumptions (S1) (S2) (S3) and (S4).

**Lemma 5.5.** Under the assumptions (S1) (S2) and (S3) introduced in chapter 3, there exists a constant $K > 0$ such that for any $t_1, t_2 \in [0, T]$,

1. $\mathbb{E}\sup_{t \in [0,T]} \left\{ |X^n_j(t)|^2 + A^n_j(t)^2 \right\} < \infty$,

2. $\mathbb{E}\left[ |X^n_j(t_2) - X^n_j(t_1)|^2 + |A^n_j(t_2) - A^n_j(t_1)|^2 \right] \leq K \cdot |t_2 - t_1|$, 

3. $\{X^n_j\}_{n \geq 1}$ is tight in $C([0,T];\mathbb{R}^d)$ and $\{A^n_j\}_{n \geq 1}$ is tight in $C([0,T];\mathbb{R})$.

**Proof.** We will skip the proof because it is essentially the same as the method applied in Lemma 4.1 and Theorem 4.2.

For any Borel set $B \in \mathcal{B}(C([0,T];\mathbb{R}^d))$, consider the characteristic function $1_B(x)$:

$$1_B(x) = \begin{cases} 
1, & \text{if } x \in B \\
0, & \text{if } x \notin B 
\end{cases}$$

and the function $J_B$:

$$J_B(x, a) = \frac{1}{n} \sum_{j=1}^{n} a_j \cdot 1_B(x_j)$$
where \( \mathbf{x} = (x_1, \ldots, x_j, \ldots, x_n) \), \( \mathbf{a} = (a_1, \ldots, a_j, \ldots, a_n) \).

Since for any \( B \in \mathcal{B}((0, T]; \mathbb{R}^d) \),

\[
J_B(\mathbf{X}^n(t), \mathbf{A}^n(t)) = \frac{1}{n} \sum_{j=1}^{n} A^n_j(t) \cdot 1_B(X^n_j(t)) = U^n(t, B) \quad (5.9)
\]

For any Borel set \( B \in \mathcal{B}((0, T]; \mathbb{R}^d) \), we define:

1. \( \tilde{U}^n(B) = \mathbb{E}\{U^n(B)\} = \text{the mean of } U^n \),

2. \( \eta^n = \mathcal{L}(U^n) = \text{the law of } U^n \).

**Theorem 5.6.** Under the assumptions (S1) (S2) and (S3), (a) the sequence \( \{\tilde{U}^n\}_{n=1}^\infty \) is tight in \( \mathcal{M}(\mathcal{C}((0, T]; \mathbb{R}^d)) \), (b) the sequence \( \{\eta^n\}_{n=1}^\infty \) is tight in \( \pi(\mathcal{M}(\mathcal{C}((0, T]; \mathbb{R}^d))) \).

**Proof.**

(a) By (5.9) and the Schwarz inequality, for any \( B \in \mathcal{B}((0, T]; \mathbb{R}^d) \),

\[
\tilde{U}^n(t, B) = \mathbb{E}\{U^n(B)\} = \mathbb{E}\{I_B(\mathbf{X}^n(t))\}
\]

\[
= \mathbb{E}\left\{ \frac{1}{n} \sum_{j=1}^{n} A^n_j(t) \cdot 1_B(X^n_j(t)) \right\}
\]

\[
= \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}\left\{ A^n_j(t) \cdot 1_B(X^n_j(t)) \right\}
\]

\[
= \frac{1}{n} \sum_{j=1}^{n} \left\{ \mathbb{E}|A^n_j(t)|^2 \right\}^{\frac{1}{2}} \cdot \left\{ \mathbb{E}|1_B(X^n_j(t))|^2 \right\}^{\frac{1}{2}}
\]

\[
= \frac{1}{n} \sum_{j=1}^{n} \left\{ \mathbb{E}|A^n_j(t)|^2 \right\}^{\frac{1}{2}} \cdot \left( \mathbb{P}\{X^n_j(t) \in B\} \right)^{\frac{1}{2}} \quad (5.10)
\]

From Lemma 5.1, there is a finite constant \( K > 0 \) such that

\[
\mathbb{E}A^n_j(t)^2 \leq \mathbb{E}\left\{ \sup_{t \in [0, T]} A^n_j(t)^2 \right\} \leq K < \infty,
\]

then from (5.10) we have: \( \tilde{U}^n(t, B) \leq \left( K \cdot \mathbb{P}\{X^n_j(t) \in B\} \right)^{\frac{1}{2}} \).
Since \( \{X^n_t\}_{n=1}^\infty \) is tight in \( C([0, T]; \mathbb{R}^d) \), for any \( \epsilon > 0 \), there exists a compact set \( D \in C([0, T]; \mathbb{R}^d) \) such that \( \mathbf{P}\{X^n_t(t) \in D^c\} < \epsilon \). Hence \( \tilde{U}^n(t, D^c) \leq \left( K \cdot \mathbf{P}\{X^n_t(t) \in D^c\} \right)^{1/2} < \sqrt{K \cdot \epsilon} \).

(b) Using part (a), there exists a compact set \( D_i \in C([0, T]; \mathbb{R}^d) \) such that

\[
\tilde{U}^n(D_i^c) = \int_{\mathcal{M}([0, T]; \mathbb{R}^d))} \lambda(D_i^c) \cdot \eta^n(d\lambda) \leq \frac{\epsilon}{i^3}, \quad \forall \epsilon > 0, i \in \mathbb{N}
\]

For each \( i \in \mathbb{N} \), define a set \( E_i \in \mathcal{M}(C([0, T]; \mathbb{R}^d)) \) by:

\[
E_i = \lambda; \lambda(D_i) \geq 1 - \frac{1}{i}
\]

Let

\[
E = \bigcap_{i=1}^\infty E_i = \bigcap_{i=1}^\infty \{\lambda; \lambda(D_i) \geq 1 - \frac{1}{i}\}.
\]

Since each \( E_i \) is a closed and tight subset of probability measures defined on a complete and separable metric space \( C([0, T]; \mathbb{R}^d)) \), then \( E_i \) is also compact subset (see Chapter II in Parthasarathy 1967). Notice that the intersection of a family of compact sets is also compact, then \( E = \bigcap_{i=1}^\infty E_i \) is compact in \( \mathcal{M}(C([0, T]; \mathbb{R}^d)) \).

Then by the DeMorgan’s Law,

\[
\eta^n(E^c) = \eta^n\left(\bigcup_{i=1}^\infty \{\lambda; \lambda(D_i^c) \geq 1 - \frac{1}{i}\}\right) \leq \sum_{i=1}^\infty \eta^n\{\lambda; \lambda(D_i^c) \geq 1 - \frac{1}{i}\}
\]

\[
\leq \sum_{i=1}^\infty \frac{\mathbb{E}\eta^n\{U^n(D_i^c)\}}{1/i} = \sum_{i=1}^\infty \frac{\int_{\mathcal{M}(C([0, T]; \mathbb{R}^d))} \lambda(D_i^c) \cdot \eta^n(d\lambda)}{1/i}
\]

\[
\leq \sum_{i=1}^\infty i \cdot \left(\frac{\epsilon}{i^3}\right) = \left(\sum_{i=1}^\infty \frac{1}{i^2}\right) \cdot \epsilon < 2\epsilon
\]

Hence the tightness of \( \{\eta^n\}_{n=1}^\infty \) is proved.
We can pick a subsequence \( \{\eta^n_k\}_{k=1}^\infty \) of the relatively compact set of probability measures \( \{\eta^n\}_{n=1}^\infty \) obtained from Theorem 5.6, such that \( \eta^n_k \rightarrow \eta \) (say). Let \( \theta \) denotes the \( M(C([0,T];\mathbb{R}^d)) \)-valued random variable whose law is given by the limit \( \eta \). Then the following two statements are equivalent:

\[
\eta^n_k \rightarrow \eta \text{ weakly} \iff U^n_k \rightarrow \theta \text{ in distribution.}
\]

5.3 The Limit of Weighted Empirical Measure

In the last section, it has been proved that the probability law \( \eta^n \) of a subsequence of weighted empirical measures \( \{U^n_k\}_{k=1}^\infty \) converges weakly to \( \eta \). In other words, there exists a \( M(C([0,T];\mathbb{R}^d)) \)-valued random variable \( \theta \) whose law is given by the limit \( \eta \) and \( U^n_k \rightarrow \theta \) in distribution as \( n \rightarrow \infty \). In this section we are going to verify that the probability law \( \eta \) of the limit \( \theta \) solves some type of stochastic partial differential equation (SPDE).

Consider the interacting particle system again:

\[
X^n_j(t) = X^n_j(0) + \int_0^t c(X^n_j(s), U^n(s))ds + \int_0^t \sigma(X^n_j(s), U^n(s))dB^n_j(s) + \int_0^t \int_G \alpha(X^n_j(s), U^n(s), v)W(dvds)
\]

\[
A^n_j(t) = A^n_j(0) + \int_0^t d(X^n_j(s), U^n(s), L_j)ds + \int_0^t A^n_j(s)\gamma(X^n_j(s), U^n(s), L_j)dB^n_j(s) + \int_0^t \int_G A^n_j(s)\beta(X^n_j(s), U^n(s), v, L_j)W(dvds)
\]

\[
U^n(t) = \frac{1}{n} \sum_{j=1}^n A^n_j(t)\delta_{X^n_j(t)}
\]

In order to study the limit \( \theta \), for any \( \phi \in C^2_b(\mathbb{R}^d) \), apply the 2-dimensional Itô formula to the process \( A^n_j(t)\phi(X^n_j(t)) \),

\[
A^n_j(t)\phi(X^n_j(t)) = A^n_j(0)\phi(X^n_j(0)) + \int_0^t A^n_j(s)\phi(X^n_j(s)) \cdot \gamma(X^n_j(s), U^n(s), L_j)dB^n_j(s)
\]
\[A^n_j(s)\phi(X^n_j(s)) = A^n_i(0)\phi(X^n_i(0)) + \int_0^t A^n_j(s)\phi(X^n_j(s)) \cdot \gamma(X^n_j(s), U^n(s), L_j)dB_j(s)
+ \int_0^t A^n_j(s)\phi(X^n_j(s)) \cdot d(X^n_j(s), U^n(s), L_j)ds
+ \int_0^t \int_G A^n_j(s)\phi(X^n_j(s)) \cdot \beta(X^n_j(s), U^n(s), v, L_j)W(dvds)
+ \int_0^t A^n_j(s)\phi'(X^n_j(s)) \cdot \left\{ c(X^n_j(s), U^n(s)) + \sigma^T(X^n_j(s), U^n(s))\gamma(X^n_j(s), U^n(s), L_j)
\right\}ds
+ \frac{1}{2} \int_0^t A^n_j(s)\phi''(X^n_j(s)) \cdot \left\{ \sigma\sigma^T(X^n_j(s), U^n(s))
\right\}ds
+ \int_0^t A^n_j(s)\phi'(X^n_j(s)) \cdot \sigma(X^n_j(s), U^n(s))dB_j(s)
+ \int_0^t A^n_j(s)\phi'(X^n_j(s)) \cdot \int_G \alpha(X^n_j(s), U^n(s), v)W(dvds) \quad (5.11)\]

Let
\[g(x, u, l) = c(x, u) + \sigma(x, u)\gamma^T(x, u, l) + \int_G \alpha(x, u, v)\beta(x, u, v, l)\mu(dv)\]
\[h(x, u) = \sigma\sigma^T(x, u) + \int_G \alpha^T(x, u, v)\mu(dv)\]

The equation (5.11) becomes

\[A^n_j(t)\phi(X^n_j(t)) = A^n_i(0)\phi(X^n_i(0)) + \int_0^t A^n_j(s)\phi(X^n_j(s)) \cdot \gamma(X^n_j(s), U^n(s), L_j)dB_j(s)
+ \int_0^t A^n_j(s)\phi(X^n_j(s)) \cdot d(X^n_j(s), U^n(s), L_j)ds
+ \int_0^t \int_G A^n_j(s)\phi(X^n_j(s)) \cdot \beta(X^n_j(s), U^n(s), v, L_j)W(dvds)
+ \int_0^t A^n_j(s)\phi'(X^n_j(s)) \cdot g(X^n_j(s), U^n(s), L_j)ds
+ \frac{1}{2} \int_0^t A^n_j(s)\phi''(X^n_j(s)) \cdot h(X^n_j(s), U^n(s))ds
+ \int_0^t A^n_j(s)\phi'(X^n_j(s)) \cdot \sigma(X^n_j(s), U^n(s))dB_j(s)\]
The equation (5.12) is rewritten as:

\[
+ \int_0^t A_j^n(s) \phi'(X_j^n(s)) \int_G \alpha(X_j^n(s), U^n(s), v) W(dvds) \tag{5.12}
\]

Define an operator \( \mathcal{L}_i(u) \) by

\[
\mathcal{L}_i(u) \phi(x) = \sum_{i=1}^d g_i(x, u, l) \cdot \frac{\partial \phi(x)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d h_{i,j}(x, u) \cdot \frac{\partial^2 \phi(x)}{\partial x_i \partial x_j} \tag{5.13}
\]

The equation (5.12) is rewritten as:

\[
A_j^n(t) \phi(X_j^n(t)) = A_j^n(0) \phi(X_j^n(0))
\]

\[
+ \int_0^t A_j^n(s) \cdot \left\{ \phi(X_j^n(s))d(X_j^n(s), U^n(s), L_j) + \mathcal{L}_{L_j}(U^n(s)) \phi(X_j^n(s)) \right\} ds
\]

\[
+ \int_0^t A_j^n(s) \cdot \left\{ \phi(X_j^n(s)) \gamma(X_j^n(s), U^n(s), L_j) + \phi'(X_j^n(s)) \gamma(X_j^n(s), U^n(s)) \right\} dB_j(s)
\]

\[
+ \int_0^t \int_G A_j^n(s) \left\{ \phi(X_j^n(s)) \beta(X_j^n(s), U^n(s), v, L_j) + \phi'(X_j^n(s)) \alpha(X_j^n(s), U^n(s), v) \right\} W(dvds)
\]

By averaging both sides of the above equation, we have:

\[
\frac{1}{n} \sum_{j=1}^n A_j^n(t) \phi(X_j^n(t)) = \frac{1}{n} \sum_{j=1}^n A_j^n(0) \phi(X_j^n(0))
\]

\[
+ \int_0^t \frac{1}{n} \sum_{j=1}^n A_j^n(s) \cdot \left\{ \phi(X_j^n(s))d(X_j^n(s), U^n(s), L_j) + \mathcal{L}_{L_j}(U^n(s)) \phi(X_j^n(s)) \right\} ds
\]

\[
+ \frac{1}{n} \sum_{j=1}^n \int_0^t A_j^n(s) \cdot \left\{ \phi(X_j^n(s)) \gamma(X_j^n(s), U^n(s), L_j) + \phi'(X_j^n(s)) \sigma(X_j^n(s), U^n(s)) \right\} dB_j(s)
\]

\[
+ \int_0^t \int_G \frac{1}{n} \sum_{j=1}^n A_j^n(s) \cdot \left\{ \phi(X_j^n(s)) \beta(X_j^n(s), U^n(s), v, L_j) + \phi'(X_j^n(s), U^n(s), v) \right\} W(dvds)
\]

Then we have:

\[
\langle U^n(t), \phi \rangle = \langle U^n(0), \phi \rangle
\]

\[
+ \int_0^t \langle U^n(s), d(\cdot, U^n(s), L_j) \phi + \mathcal{L}_{L_j}(U^n(s)) \phi \rangle
\]

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\[ + \int_0^t \langle U^n(s), \beta(\cdot, U^n(s), \nu, L_j) \phi + \alpha(\cdot, U^n(s), \nu) \nabla \phi \rangle W(d\nu ds) \]
\[ + \frac{1}{n} \sum_{j=1}^n \int_0^t A^n_j(s) \cdot F(X^n_j(s), U^n(s), L_j) dB_j(s) \]  

(5.14)

where \( F(x, u, l) = \phi(x) \gamma(x, u, l) + \phi'(x) \sigma(x, u) \).

Let
\[ V^n = \frac{1}{n} \sum_{j=1}^n A^n_j \delta(X^n_j, L_j). \]

Suppose \( \mathcal{A} = \{1, 2, \ldots, N\}. \) Then for any \( B \in \mathcal{B} \left(C([0, T]; \mathbb{R}^d)\right) \),
\[ V^n(B \times \mathcal{A}) = \frac{1}{n} \sum_{j} A^n_j \delta(X^n_j, L_j)(B \times \mathcal{A}) = \frac{1}{n} \sum_{j} A^n_j \delta X^n_j(B) = U^n(B) \]

Then the above equation can be written as
\[ \langle V^n(t), \phi \rangle = \langle V^n(0), \phi \rangle \]
\[ + \int_0^t \langle V^n(s), d(\cdot, V^n(s)) \phi + \mathcal{L}(V^n(s)) \phi \rangle ds \]
\[ + \int_0^t \int_G \langle V^n(s), \beta(\cdot, V^n(s), z) \phi + \alpha(\cdot, V^n(s), z) \nabla \phi \rangle W(dz ds) \]
\[ + \frac{1}{n} \sum_{j=1}^n \int_0^t A^n_j(s) F(X^n_j(s), V^n(s)) dB_j(s) \]

(5.15)

where \( \mathcal{L}(V^n) = \mathcal{L}_{L_j}(U^n) \) and is given by
\[ \mathcal{L}(v) \phi(x) = \sum_{i=1}^d g_i(x, v) \cdot \frac{\partial \phi(x)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d h_{i,j}(x, v) \cdot \frac{\partial^2 \phi(x)}{\partial x_i \partial x_j}. \]

Then it will be shown that under some extra condition, the last term in (5.15) driven by a Brownian motion \( B_j \) actually goes zero as \( n \to \infty \), and using the sequence of new empirical measures \( \{V^n\}_{n=1}^\infty \), we can identify the limit measure as the solution (see [19], [24]) of an SPDE.

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Lemma 5.7. Suppose $E\{X^n_j(0)^4 + A^n_j(0)^4\} < \infty$. Under assumptions (S2) and (S3),

$$E \sup_{t \in [0, T]} \left\{ |X_j(t)|^4 + A^n_j(t)^4 \right\} < \infty.$$ 

Proof.

Step 1. By the same method that is used in Lemma 5.1, we have:

$$E \sup_{t \in [0, T]} A^n_j(t)^2 \leq N_1 \cdot E \left\{ A^n_j(0)^2 \right\} \quad (5.16)$$

where $N_1 > 0$ is a finite constant.

$$E \sup_{t \in [0, T]} |X^n_j(t)|^2 \leq N_2 \cdot \left( N_3 + 4E \left\{ |X^n_j(0)|^2 \right\} + N_4 \cdot E \left\{ A^n_j(0)^2 \right\} \right) \quad (5.17)$$

where $N_2, N_3, N_4$ are all finite positive constants.

By the 4th moment bounded condition and Jensen inequality,

$$\left\{ E|X^n_j(0)|^2 \right\}^2 \leq E|X^n_j(0)|^4 < \infty, \quad \left\{ EA^n_j(0)^2 \right\}^2 \leq EA^n_j(0)^4 < \infty.$$ 

then by (5.16) and (5.17),

$$E \left\{ \sup_{t \in [0, T]} |X^n_j(t)|^2 \right\} < \infty, \quad E \left\{ \sup_{t \in [0, T]} A^n_j(t)^2 \right\} < \infty \quad (5.18)$$

Step 2. Applying the Itô formula to $f(z) = z^2$ with $z = A^n_j(t)$, we have:

$$A^n_j(t)^2 = A^n_j(0)^2 + \int_0^t 2A^n_j(s)\gamma(X^n_j(s), U^n(s), L_j)dB_j(s)$$

$$+ \int_0^t 2A^n_j(s)d(X^n_j(s), U^n(s), L_j)ds$$

$$+ \int_0^t \int_G 2A^n_j(s)\beta(X^n_j(s), U^n(s), v, L_j)W(dvds)$$

$$+ \int_0^t A^n_j(s)^2 \cdot |\gamma(X^n_j(s), U^n(s), L_j)|^2 ds$$

$$+ \int_0^t \int_G A^n_j(s)^2\beta(X^n_j(s), U^n(s), v, L_j)^2 \mu(dv)ds$$

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Squaring the both sides of the above inequality,

\[
A_j^n(t)^4 = 6A_j^n(0)^4 + 24 \int_0^t A_j^n(s) \gamma(X_j^n(s), U^n(s), L_j) dB_j(s)^2 \\
+ 24 \left| \int_0^t A_j^n(s) d(X_j^n(s), U^n(s), L_j) ds \right|^2 \\
+ 24 \left| \int_0^t \int_G A_j^n(s) \beta(X_j^n(s), U^n(s), v, L_j) W(dvds) \right|^2 \\
+ 6 \left| \int_0^t A_j^n(s)^2 \cdot |\gamma(X_j^n(s), U^n(s), L_j)|^2 ds \right|^2 \\
+ 6 \left| \int_0^t \int_G A_j^n(s)^2 \beta(X_j^n(s), U^n(s), v, L_j)^2 \mu(dv) ds \right|^2
\]

By Doob’s inequality,

\[
\mathbb{E} \sup_{t \in [0,T]} A_j^n(t)^4 \leq 6\mathbb{E} A_j^n(0)^4 + 96\mathbb{E}^P \int_0^t A_j^n(s)^2 \cdot |\gamma(X_j^n(s), U^n(s), L_j)|^2 ds \\
+ 24T \cdot \mathbb{E} \int_0^t A_j^n(s)^2 d(X_j^n(s), U^n(s), L_j)^2 ds \\
+ 96 \cdot \mathbb{E} \int_0^t \int_G A_j^n(s)^2 \beta(X_j^n(s), U^n(s), v, L_j)^2 \mu(dv) ds \\
+ 6T \cdot \mathbb{E} \int_0^t A_j^n(s)^4 \cdot |\gamma(X_j^n(s), U^n(s), L_j)|^4 ds \\
+ 6T \cdot \mathbb{E} \int_0^t \int_G A_j^n(s)^4 \beta(X_j^n(s), U^n(s), v, L_j)^4 \mu(dv) ds \\
\leq 6\mathbb{E} A_j^n(0)^4 + 96 \sum_{k=1}^N p_k \cdot \mathbb{E}^P \int_0^t A_j^n(s)^2 \cdot |\gamma(X_j^n(s), U^n(s), k)|^2 ds \\
+ 24T \sum_{k=1}^N p_k \cdot \mathbb{E} \int_0^t A_j^n(s)^2 d(X_j^n(s), U^n(s), k)^2 ds \\
+ 96 \sum_{k=1}^N p_k \cdot \mathbb{E} \int_0^t \int_G A_j^n(s)^2 \beta(X_j^n(s), U^n(s), v, k)^2 \mu(dv) ds \\
+ 6T \sum_{k=1}^N p_k \cdot \mathbb{E} \int_0^t A_j^n(s)^4 \cdot |\gamma(X_j^n(s), U^n(s), k)|^4 ds
\]
By the assumption (S2),

\[ +6T \sum_{k=1}^{N} p_k \cdot E \int_0^t \int_G A_j^n(s)^4 \beta(X_j^n(s), U^n(s), v, k)^4 \mu(dv)ds \]

By the application of Itô formula to \( f(z) = z^2 \) with \( z = X_j^n(t) \),

\[ X_j^n(t)^2 = X_j^n(0)^2 + \int_0^t 2X_j^n(s)\sigma(X_j^n(s), U^n(s))dB_j(s) + \int_0^t 2X_j^n(s)c(X_j^n(s), U^n(s))ds + \int_0^t \int_G 2X_j^n(s)\alpha(X_j^n(s), U^n(s), v)W(dvds) + \int_0^t \sigma\sigma^T(X_j^n(s), U^n(s))ds + \int_0^t \int_G \alpha\alpha^T(X_j^n(s), U^n(s), v)\mu(dv)ds \]

Squaring both sides of the above equation, we have:

\[ |X_j^n(t)|^4 = 6|X_j^n(0)|^4 + 24 \left| \int_0^t X_j^n(s)\sigma(X_j^n(s), U^n(s))dB_j(s) \right|^2 + 24 \left| \int_0^t X_j^n(s)c(X_j^n(s), U^n(s))ds \right|^2 \]
By the assumption (S3), Lemma 5.1 and the Schwarz inequality

\[ +24 \left| \int_0^t \int_G X^n_j(s) \alpha(X^n_j(s), U^n(s), v) W(dv ds) \right|^2 \]
\[ +6 \left| \int_0^t \sigma \sigma^T (X^n_j(s), U^n(s)) ds \right|^2 \]
\[ +6 \left| \int_0^t \int_G \alpha \alpha^T (X^n_j(s), U^n(s), v) \mu(dv) ds \right|^2 \]

By Doob’s inequality,

\[ E \sup_{t \in [0,T]} |X^n_j(t)|^4 = 6 \cdot E|X^n_j(0)|^4 + 96 \cdot E \int_0^T |X^n_j(s)|^2 \cdot |\sigma \sigma^T (X^n_j(s), U^n(s))| ds \]
\[ +24T \cdot E \int_0^T |X^n_j(s)|^2 \cdot |c(X^n_j(s), U^n(s))|^2 ds \]
\[ +96 \cdot E \int_0^T \int_G |X^n_j(s)|^2 \cdot |\alpha \alpha^T (X^n_j(s), U^n(s), v)| \mu(dv) ds \]
\[ +6T \cdot E \int_0^T |\sigma \sigma^T (X^n_j(s), U^n(s))|^2 ds \]
\[ +6T \cdot E \int_0^T \int_G |\alpha \alpha^T (X^n_j(s), U^n(s), v)|^2 \mu(dv) ds \]

By the assumption (S3), Lemma 5.1 and the Schwarz inequality

\[ E \sup_{t \in [0,T]} |X^n_j(t)|^4 \leq 6 \cdot E|X^n_j(0)|^4 \]
\[ + (192 + 24T) K_1^2 \mathbb{E} \int_0^T |X^n_j(s)|^2 \left( 1 + |X^n_j(s)|^2 + \|U^n(s)\|^2 \right) ds \]
\[ +12T K_1^4 \mathbb{E} \int_0^T \left( 1 + |X^n_j(s)|^2 + \|U^n(s)\|^2 \right)^2 ds \]
\[ \leq 6 \cdot E|X^n_j(0)|^4 + (192 + 24T) K_1^2 \int_0^T \mathbb{E} \left( |X^n_j(s)|^2 + |X^n_j(s)|^4 + |X^n_j(s)|^2 |A^n_1(s)|^2 \right) ds \]
\[ +12T K_1^4 \int_0^T \mathbb{E} \left( 1 + |X^n_j(s)|^2 + |A^n_1(s)|^2 \right)^2 ds \]
\[ \leq 6 \cdot E|X^n_j(0)|^4 + K_2^2 \int_0^T \left( E|X^n_j(s)|^2 + E|X^n_j(s)|^4 + \frac{1}{2} \{ E|X^n_j(s)|^4 + E|A^n_1(s)|^4 \} \right) ds \]
\[ +36T K_1^4 \int_0^T \left( 1 + E|X^n_j(s)|^4 + E|A^n_1(s)|^4 \right) ds \]
Therefore,

\[
\mathbb{E} \sup_{t \in [0,T]} |X_j^n(t)|^4 \leq 6 \cdot \mathbb{E}|X_j^n(0)|^4 + K_3 + K_4 \int_0^T \left( \mathbb{E}|X_j^n(s)|^2 + \mathbb{E}|X_j^n(s)|^4 + \mathbb{E}|A_1^n(s)|^4 \right) ds
\]

\[
\leq 6 \cdot \mathbb{E}|X_j^n(0)|^4 + K_3 + K_4 \int_0^T \left( \mathbb{E} \sup_{s \in [0,T]} |X_j^n(s)|^2 + \mathbb{E} \sup_{s \in [0,T]} |X_j^n(s)|^4 + \mathbb{E} \sup_{s \in [0,T]} |A_1^n(s)|^4 \right) ds
\]

(5.20)

where \( K_3 = 36T^2K_1^4, \ K_4 = (3/2)K_2^2 \lor 36TK_1^4. \)

By (5.17), (5.18), (5.20) and the Gronwall’s inequality, we have:

\[
\mathbb{E} \sup_{s \in [0,T]} \left\{ |A_j^n(s)|^4 + |X_j^n(t)|^4 \right\} < \infty.
\]

\[
\square
\]

**Lemma 5.8.** Suppose \( \mathbb{E}\{|X_j^n(0)|^4 + A_1^n(0)|^4\} < \infty. \) Under assumptions (S1),(S2),(S3),

\[
\lim_{n \to \infty} \mathbb{E} \sup_{t \in [0,T]} \left| \frac{1}{n} \sum_{j=1}^n \int_0^t A_j^n(s) F(X_j^n(s), V^n(s)) \cdot dB_j(s) \right|^2 = 0
\]

**Proof.** By Doob’s inequality and Schwartz inequality,

\[
\mathbb{E} \sup_{t \in [0,T]} \left| \frac{1}{n} \sum_{j=1}^n \int_0^t A_j^n(s) F(X_j^n(s), V^n(s)) \cdot dB_j(s) \right|^2
\]

\[
\leq \frac{4N}{n^2} \sum_{j=1}^n \mathbb{E} \int_0^T A_j^n(s)^2 \cdot |F(X_j^n(s), V^n(s))|^2 \cdot ds
\]

\[
= \frac{4N}{n^2} \sum_{j=1}^n \mathbb{E} \int_0^T A_j^n(s)^2 \cdot \phi(X_j^n(s))\gamma(X_j^n(s), V^n(s)) + \phi'(X_j^n(s))\sigma(X_j^n(s), V^n(s))|^2 \cdot ds
\]

(recall: \( F(x,v) = \phi(x)\gamma(x,v) + \phi'(x)\sigma(x,v) \))

\[
\leq \frac{8N}{n^2} \sum_{j=1}^n \mathbb{E} \int_0^T A_j^n(s)^2 \phi(X_j^n(s))^2 |\gamma(X_j^n(s), V^n(s))|^2 ds
\]

\[
+ \frac{8N}{n^2} \sum_{j=1}^n \mathbb{E} \int_0^T A_j^n(s)^2 \phi'(X_j^n(s))^2 |\sigma\sigma^T(X_j^n(s), V^n(s))| ds
\]

(5.21)
For any \( \phi \in C_b^2(\mathbb{R}^d) \), there exist some constants \( K_1, K_2 \) and \( K_3 \) such that

\[
\sup_{t \in [0,T]} \left( |\phi(X_j^n(t))|^2 + |\phi'(X_j^n(t))|^2 \right) \leq K_1^2 < \infty
\]

\[
\sup_{1 \leq k \leq N} |\gamma(x, v)|^2 \leq K_2^2 < \infty
\]

\[
|\sigma \sigma^T(x, v)| \leq K_3(1 + |x|^2 + \|v\|^2).
\]

Then from (5.21), we have:

\[
\mathbb{E} \sup_{t \in [0,T]} \left| \frac{1}{n} \sum_{j=1}^n \int_0^T A_{j}^n(s) F(X_j^n(s), V^n(s))dB_j(s) \right|^2
\]

\[
\leq \frac{8N}{n^2} \sum_{j=1}^n \mathbb{E} \int_0^T A_{j}^n(s)^2 \phi(X_j^n(s))^2 |\gamma(X_j^n(s), V^n(s))|^2 ds
\]

\[
+ \frac{8N}{n^2} \sum_{j=1}^n \mathbb{E} \int_0^T A_{j}^n(s)^2 \phi'(X_j^n(s))^2 |\sigma \sigma^T(X_j^n(s), V^n(s))| ds
\]

\[
\leq \frac{8N}{n^2} \sum_{j=1}^n K_1^2 K_2 \cdot \mathbb{E} \int_0^T A_{j}^n(s)^2 ds
\]

\[
+ \frac{8N}{n^2} \sum_{j=1}^n K_1^2 K_3 \cdot \mathbb{E} \int_0^T A_{j}^n(s)^2 \cdot (1 + |X_j^n(s)|^2 + \|V^n(s)\|^2) ds
\]

\[
\leq \frac{8NK_1^2 K_2}{n^2} \sum_{j=1}^n \int_0^T \mathbb{E} \sup_{s \in [0,T]} A_{j}^n(s)^2 ds
\]

\[
+ \frac{8NK_1^2 K_3}{n^2} \sum_{j=1}^n \int_0^T \mathbb{E} \left( A_{j}^n(s)^2 + A_{j}^n(s)^2 |X_j^n(s)|^2 + A_{j}^n(s)^2 \|U^n(s)\|^2 \right) ds
\]

\[
\leq (8NK_1^2 K_2^2 + 8NK_1^2 K_3^2) \cdot \frac{1}{n^2} \sum_{j=1}^n \int_0^T \mathbb{E} \sup_{s \in [0,T]} \{A_{j}^n(s)^2\} ds
\]

\[
+ \frac{8NK_1^2 K_3^2}{n^2} \sum_{j=1}^n \int_0^T \mathbb{E} \left( A_{j}^n(s)^2 |X_j^n(s)|^2 + A_{j}^n(s)^2 \left( \frac{1}{n} \sum_{i=1}^n A_{i}^n(s)^2 \right) \right) ds
\]

\[
\text{(recall: } \|v\| = \|u\| = \frac{1}{n} \sum_{i=1}^n |a_i|)\]

\[
\leq (8NK_1^2 K_2^2 + 8NK_1^2 K_3^2) \cdot \frac{1}{n^2} \left( \sum_{j=1}^n K_4T \right)
\]
\[ +8NK_1^3K_3^2 \cdot \frac{1}{n^2} \sum_{j=1}^{n} \int_{0}^{T} \left( \frac{1}{2} \mathbb{E} A_j^n(s)^4 \right) \left( \int_{0}^{T} \mathbb{E} A_j^n(s)^4 + \mathbb{E} A_j^n(s)^4 \right) ds \]

(where \( K_4 = \mathbb{E} \sup_{t \in [0,T]} \{ A_j^n(s)^2 \} \))

By Lemma 5.7, there exist finite constants \( K_5 \) and \( K_6 \) such that

\[ \mathbb{E} \sup_{s \in [0,T]} |X_j^n(s)|^4 \leq K_5, \quad \mathbb{E} \sup_{s \in [0,T]} A_j^n(s)^4 \leq K_6 \]

Let \( K_7 = (8NK_1^2K_2^2 + 8NK_1^2K_3^2) \), \( K_8 = 8NK_1^3K_2^2 \), then

\[ \mathbb{E} \sup_{t \in [0,T]} \left| \frac{1}{n} \sum_{j=1}^{n} \int_{0}^{T} A_j^n(s) F(X_j^n(s), V^n(s)) dB_j(s) \right|^2 \]

\[ \leq \frac{1}{n} \cdot K_7 K_4 T + K_8 \cdot \frac{1}{n^2} \sum_{j=1}^{n} \int_{0}^{T} \left( \frac{1}{2} K_6 + \frac{1}{2} K_5 + \frac{1}{2n} \cdot 2nK_6 \right) ds \]

\[ = \frac{1}{n} \cdot \left( K_7 K_4 + \frac{3}{2} K_8 K_6 + \frac{1}{2} K_8 K_5 \right) T \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \]

\[ \Box \]

Since \( U^n \rightarrow \theta \) in distribution, by the above Lemmas along with the arguments in [19] and [24] we obtain:

**Theorem 5.9.** Suppose \( \mathbb{E} \sup_{t \in [0,T]} \left( |X_j^n(0)|^4 + A_j^n(0)^4 \right) < \infty \). Then under the assumptions (S1),(S2) and (S3), the distributional limit \( \theta \) of \( V^n \) is a solution of the following partial differential equation (written in a weak form):

\[ \langle \theta(t), \phi \rangle = \langle \theta(0), \phi \rangle + \int_{0}^{t} \langle \theta(s), d\left( \cdot, \theta(s) \right) \phi + \mathcal{L} \left( \theta(s) \right) \phi \rangle ds \]

\[ + \int_{0}^{t} \int_{G} \langle \theta(s), \beta \left( \cdot, \theta(s), z \right) \phi + \alpha \left( \cdot, \theta(s), z \right) \nabla \phi \rangle W(dzds) \]

where \( \mathcal{L}(V^n) = \mathcal{L}_{L_j}(U^n) \) and is given by:

\[ \mathcal{L}(v)\phi(x) = \sum_{i=1}^{d} g_i(x, v) \cdot \frac{\partial \phi(x)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} h_{i,j}(x, v) \cdot \frac{\partial^2 \phi(x)}{\partial x_i \partial x_j}. \]
It is worthwhile to note that given any subsequence \( \{U_{n k}\}_{k=1}^{\infty} \) of the weighted empirical measures \( \{U_n\}_{n=1}^{\infty} \), there exists a further subsequence \( \{U_{nk_m}\}_{m=1}^{\infty} \) which converges in distribution. By the above Lemma and Theorem, we obtain that the limit of any such weakly convergent subsubsequence has the same limit \( \theta \). Therefore the sequence \( \{U_n\}_{n=1}^{\infty} \) itself converges to \( \theta \).

**Remark.** Notice that the limit measure \( \theta \) is *absolutely continuous* with respect to the Lebesgue measure space.

**Remark.** Let the density \( \varphi \) of the limit \( \theta \) defined by \( \theta(t, B) = \int_B \varphi(t, x)dx \). In fact, \( \varphi \) is the Radon-Nikodym density of \( \theta \) and solves the following SPDE:

\[
d\varphi(t, x) = \left\{ \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i x_j} \left( h_{i,j}(x, \varphi(t, \cdot)) \varphi(t, x) \right) \\
+ d(x, \varphi(t, \cdot)) \varphi(t, x) - \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left( g_i(x, \varphi(t, \cdot)) \varphi(t, x) \right) \right\} dt \\
- \int_G \left\{ \beta(x, \varphi(t, \cdot), z) \varphi(t, x) - \frac{\partial}{\partial x_i} \left( \alpha(x, \varphi(t, \cdot), z) \varphi(t, x) \right) \right\} W(dzdt).
\]

where

\[
g(x, v) = c(x, v) + \sigma^T(x, v) \gamma(x, v) + \int_G \alpha(x, v, z) \beta(x, v, z) \mu(dz) \\
h(x, v) = \sigma \sigma^T(x, v) + \int_G \alpha \alpha^T(x, v, z) \mu(dz).
\]

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Chapter 6
Comparison and Compact Support for the Interacting Particle Systems

In this section we will first study the comparison between two weight processes which are governed by two stochastic differential equations. Then the compact support property for the solution of an interacting particle system will be shown by the method of Feller Test for explosion.

6.1 Two Systems with the Different Drift and the Same Diffusion Terms

Consider two particle systems (6.1)-(6.2)-(6.3) and (6.1)-(6.4)-(6.5) given by

\[ X^n_j(t) = X^n_j(0) + \int_0^t c(X^n_j(s))ds + \int_0^t \sigma(X^n_j(s))dB_j(s) \]
\[ \quad + \int_0^t \int_G \alpha(X^n_j(s), z)W(dzds) \]  
(6.1)

\[ A^n_j(t) = A^n_j(0) + \int_0^t A^n_j(s)b(X^n_j(s), U^n(s), L_j)ds + \int_0^t A^n_j(s)\gamma(X^n_j(s), U^n(s), L_j)dB_j(s) \]
\[ \quad + \int_0^t \int_G A^n_j(s)\beta(X^n_j(s), U^n(s), L_j, z)W(dzds) \]  
(6.2)

\[ U^n(t) = \frac{1}{n} \sum_{j=1}^n A^n_j(t)\delta X^n_j(t) \]  
(6.3)

\[ C^n_j(t) = C^n_j(0) + \int_0^t C^n_j(s)d(X^n_j(s), V^n(s), L_j)ds + \int_0^t C^n_j(s)\gamma(X^n_j(s), V^n(s), L_j)dB_j(s) \]
\[ \quad + \int_0^t \int_G C^n_j(s)\beta(X^n_j(s), V^n(s), L_j, z)W(dzds) \]  
(6.4)

\[ V^n(t) = \frac{1}{n} \sum_{j=1}^n C^n_j(t)\delta X^n_j(t) \]  
(6.5)
where \( A_j^n, B_j, W \) are defined as the same as in Chapter 3, \( X_j^n \) is defined to be the location process without the dependence on the empirical measure either \( U^n \) or \( V^n \).

The coefficients \( c, \sigma, \alpha \) for the location process \( X_j^n \):

\[
c : \mathbb{R}^d \to \mathbb{R}^d, \quad \sigma : \mathbb{R}^d \to \mathbb{R}^{d \times m}, \quad \alpha : \mathbb{R}^d \times G \to \mathbb{R}^d.
\]

satisfy the linear growth condition as usual:

\[
|c(x)|^2 + |\sigma^T \sigma(x)| + \int_G |\alpha(x, v)|^2 \mu(dv) \leq K^2(1 + |x|^2).
\]

The two weight processes \( A_j^n(t) \) and \( C_j^n(t) \) depend on the empirical measures, and are different for their drift coefficients, i.e. \( b(x, u) \) is not necessary to be equal to \( d(x, u) \) for all \( x, u \).

Let

\[
A^n(t) = (A_1^n(t), ..., A_j^n(t), ..., A_n^n(t)), \quad C^n(t) = (C_1^n(t), ..., C_j^n(t), ..., C_n^n(t))
\]

\[
X^n(t) = (X_1^n(t), ..., X_j^n(t), ..., X_n^n(t)), \quad L^n = (L_1, ..., L_j, ..., L_n).
\]

Define

\[
b_j(X^n(t), A^n(t), L^n) = b(X_j^n(t), U^n(t), L_j), \quad d_j(X^n(t), A^n(t), L^n) = d(X_j^n(t), U^n(t), L_j).
\]

\[
\gamma_j(X^n(t), A^n(t), L^n) = \gamma(X_j^n(t), U^n(t), L_j), \quad \gamma_j(X^n(t), C^n(t), L^n) = \gamma(X_j^n(t), V^n(t), L_j).
\]

\[
\beta_j(X^n(t), A^n(t)) = \beta(X_j^n(t), U^n(t), z), \quad \beta_j(X^n(t), C^n(t)) = \beta(X_j^n(t), V^n(t), z).
\]

Then the two equations (6.2) and (6.4) are rewritten as:

\[
A_j^n(t) = A_j^n(0) + \int_0^t A_j^n(s) b_j(X^n(s), A^n(s), L^n) ds + \int_0^t A_j^n(s) \gamma_j(X^n(s), A^n(s), L^n) dB_j(s)
\]

\[
+ \int_0^t \int_G A_j^n(s) \beta_j(X^n(s), A^n(s), L^n, z) W(dz ds)
\]

\[
(6.6)
\]
\[ C^n_j(t) = C^n_j(0) + \int_0^t C^n_j(s) d_j(X^n(s), C^n(s), L^n) ds + \int_0^t C^n_j(s) \gamma_j(X^n(s), C^n(s), L^n) dB_j(s) + \int_0^t \int_G C^n_j(s) \beta_j(X^n(s), C^n(s), L^n, z) W(dz ds) \] (6.7)

The following conditions are necessary for the comparison between (6.6) and (6.7).

Let
\[ x = (x_1, ..., x_j, ..., x_n), \quad l = (l_1, ..., l_j, ..., l_n), \]
\[ a = (a_1, ..., a_j, ..., a_n), \quad c = (c_1, ..., c_j, ..., c_n). \]

(A1) \[ b_j(x, a, l) \geq d_j(x, c, l) \quad \text{if} \quad a_i \geq c_i \text{ and } a_j = c_j \text{ for } j \neq i. \]

(A2) \[ b_j \text{ and } d_j : \mathbb{R}^{n \times d} \times \mathcal{M}(\mathbb{R}^{n \times d}) \times \mathbb{N} \to \mathbb{R} \text{ are both jointly continuous and satisfy the Lipschitz condition on the second component uniformly with respect to the first and the third components,} \]
\[ |b_j(x, a, l) - b_j(x, c, l)| \leq K \left( \frac{1}{n} \sum_{i=1}^n |a_i - c_i| \right), \quad |d_j(x, a) - d_j(x, c)| \leq K \left( \frac{1}{n} \sum_{i=1}^n |a_i - c_i| \right) \]
for some finite and positive constant \( K \).

(A3) \( \gamma_j : \mathbb{R}^{n \times d} \times \mathcal{M}(\mathbb{R}^{n \times d}) \times \mathbb{N} \to \mathbb{R} \) and \( \beta_j : \mathbb{R}^{n \times d} \times \mathcal{M}(\mathbb{R}^{n \times d}) \times \mathbb{N} \times G \to \mathbb{R} \) are both jointly continuous and satisfy:
\[ |\gamma_j(x, a, l) - \gamma_j(x, c, l)| \leq K |a_j - c_j|, \quad \left| \int_G |\beta_j(x, a, l, z) - \beta_j(x, c, l, z)| \mu(dz) \right| \leq K |a_j - c_j| \]
for some finite and positive constant \( K \).

(A4) The bounded condition for the coefficients \( b_j, d_j, \alpha_j \) and \( \beta_j \):
\[ |b_j(x, a, l)|^2 + |d_j(x, a, l)|^2 + |\gamma_j(x, a, l)|^2 + \int_G |\beta_j(x, a, l, z)|^2 \mu(dz) \leq K^2 \]
for some finite and positive constant \( K \).

Notice that for any \( t \in [0, \infty) \), \( A^n_j(t) \) and \( A^n_j(0) \) have the same signs, \( C^n_j(t) \) and \( C^n_j(0) \) have the same signs because:

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1. applying the Itô formula to the function \( f(z) = \ln z \) with \( z = A^n_j(t) \), we have:

\[
A^n_j(t) = A^n_j(0) \cdot \exp \left\{ \int_0^t b_j(X^n(s), A^n(s), L^n)ds + M_j(t) - \frac{1}{2} \langle M \rangle_t \right\}
\] (6.8)

2. applying the Itô formula to the function \( f(z) = \ln z \) with \( z = C^n_j(t) \), we have:

\[
C^n_j(t) = C^n_j(0) \cdot \exp \left\{ \int_0^t d_j(X^n(s), C^n(s), L^n)ds + M_j(t) - \frac{1}{2} \langle M \rangle_t \right\}
\] (6.9)

where

\[
M_j(t) = \int_0^t \gamma_j(X^n(s), A^n(s), L^n)dB_j(s) + \int_0^t \int_G \beta_j(X^n(s), A^n(s), L^n)W(dzds)
\] (6.10)

and the quadratic variation of \( M_j(t) \) is

\[
\langle M \rangle_t = \int_0^t \gamma^2_j(X^n(s), A^n(s), L^n)ds + \int_0^t \int_G \beta^2_j(X^n(s), A^n(s), L^n, z)\mu(dz)ds.
\] (6.11)

**Theorem 6.1.** Under the assumptions (A1)-(A4),

\[
A^n(0) \geq C^n(0), \quad \text{P - a.s.} \quad \Rightarrow \quad \text{P}\left\{ A^n(t) \geq C^n(t), \forall t \in [0, \infty) \right\} = 1.
\]

**Proof.** Since the stochastic weights \( A^n_j(t) \) and \( C^n_j(t) \) can be positive and negative, without loss of generality we proceed with our proof for the four different cases.

**Case (1).** \( A^n_j(0) > C^n_j(0) > 0 \) for each \( j = 1, \ldots, n \).

For each \( j = 1, \ldots, n \), define a new process \( Z^n_j(t) \) by: \( Z^n_j(t) = A^n_j(t) - C^n_j(t) \) and a set of stopping times:

\[
T_j = \inf \left\{ t; Z^n_j(t) < 0 \right\}, \quad T = \bigwedge_j T_j
\]

\[
S_j = \inf \left\{ t; |A^n_j(t)| > M \right\}, \quad S = \bigwedge_j S_j
\]

\[
R_j = \inf \left\{ t; |C^n_j(t)| > M \right\}, \quad R = \bigwedge_j R_j
\]
Let $\tau = T \wedge S \wedge R$. Then we have for any $t \in [0, \tau)$,

$$Z^n_j(t) = A^n_j(t) - C^n_j(t) \geq 0, \ |A^n_j(t)| \leq M, \ |C^n_j(t)| \leq M$$

and $A^n_j(t) > 0, C^n_j(t) > 0, \forall t \in [0, \infty)$ by (6.8) and (6.9).

Applying the Itô formula to the function $f(z) = \frac{1}{z+\epsilon}$ with $z = Z^n_j(t)$ and for any $\epsilon > 0$, we have:

$$df(Z^n_j(t)) = \frac{(-1)}{(Z^n_j(t) + \epsilon)^2} \cdot dZ^n_j(t) + \frac{1}{2} \cdot \frac{2}{(Z^n_j(t) + \epsilon)^3} \cdot \left( dZ^n_j(t) \right)^2$$

$$= - \frac{1}{(Z^n_j(t) + \epsilon)^2} (dA^n_j(t) - dC^n_j(t)) + \frac{1}{(Z^n_j(t) + \epsilon)^3} (dA^n_j(t) - dC^n_j(t))^2$$

$$= - \frac{1}{(Z^n_j(t) + \epsilon)^2} \left\{ A^n_j(t) b_j(X^n(t), A^n(t), L^n) - C^n_j(t) d_j(X^n(t), C^n(t), L^n) \right\} dt$$

$$- \frac{1}{(Z^n_j(t) + \epsilon)^2} \left\{ A^n_j(t) \gamma_j(X^n(t), A^n(t), L^n) - C^n_j(t) \gamma_j(X^n(t), C^n(t), L^n) \right\} dB_j(t)$$

$$- \frac{1}{(Z^n_j(t) + \epsilon)^2} \int_G \left\{ A^n_j(t) \beta_j(X^n(t), A^n(t), L^n, z) - C^n_j(t) \beta_j(X^n(t), C^n(t), L^n, z) \right\} W(dz dt)$$

$$+ \frac{1}{(Z^n_j(t) + \epsilon)^3} \left\{ A^n_j(t) \gamma_j(X^n(t), A^n(t), L^n) - C^n_j(t) \gamma_j(X^n(t), C^n(t), L^n) \right\}^2 dt$$

$$+ \frac{1}{(Z^n_j(t) + \epsilon)^3} \int_G \left\{ A^n_j(t) \beta_j(X^n(t), A^n(t), L^n, z) - C^n_j(t) \beta_j(X^n(t), C^n(t), L^n, z) \right\}^2 \mu(dz) dt$$

Integrating both sides of the above equations from 0 to $t \wedge \tau$, we have:

$$f(Z^n_j(t \wedge \tau)) = f(Z^n_j(0))$$

$$- \int_0^{t \wedge \tau} \frac{1}{(Z^n_j(s) + \epsilon)^2} \left\{ A^n_j(s) b_j(X^n(s), A^n(s), L^n) - C^n_j(s) d_j(X^n(s), C^n(s), L^n) \right\} ds$$

$$- \int_0^{t \wedge \tau} \frac{1}{(Z^n_j(s) + \epsilon)^2} \left\{ A^n_j(s) \gamma_j(X^n(s), A^n(s), L^n) - C^n_j(s) \gamma_j(X^n(s), C^n(s), L^n) \right\} dB_j(s)$$

$$- C^n_j(s) \gamma_j(X^n(s), C^n(s), L^n) \right\} dB_j(s)$$

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The first term in (6.12) satisfies:

\[ f(Z_j^n(0)) = \frac{1}{A_j(0) - C_j^n(0) + \epsilon} > 0 \]

because \( A_j^n(0) > C_j^n(0) > 0 \), P-a.s. for \( j=1,...,n \).

To consider the second term in (6.12), we introduce a new stochastic vector \( \mathbf{F} \):

\[ \mathbf{F}^n(s) = \left( C_{1}^n(s), ..., C_{j-1}^n(s), A_j^n(s), C_{j+1}^n(s), ..., C_{n}^n(s) \right). \]

\( \mathbf{F}^n \) is created by replacing the \( j \)th entry of \( \mathbf{C}^n \) by \( A_j^n \) and leaving other entries the same.

Breaking up the second term in (6.12), we have:

\[- \int_0^{t \wedge \tau} \frac{1}{(Z_j^n(s) + \epsilon)^2} \cdot \left\{ A_j^n(s) b_j(X^n(s), A^n(s), L^n) - C_j^n(s) d_j(X^n(s), C^n(s), L^n) \right\} ds \]

\[ = \int_0^{t \wedge \tau} \frac{1}{(Z_j^n(s) + \epsilon)^2} \cdot \left\{ A_j^n(s) b_j(X^n(s), A^n(s), L^n) - A_j^n(s) d_j(X^n(s), \mathbf{F}^n(s), L^n) \right\} ds \]

\[ - \int_0^{t \wedge \tau} \frac{1}{(Z_j^n(s) + \epsilon)^2} \cdot \left\{ A_j^n(s) d_j(X^n(s), \mathbf{F}^n(s), L^n) - A_j^n(s) d_j(X^n(s), C^n(s), L^n) \right\} ds \]

\[ - \int_0^{t \wedge \tau} \frac{1}{(Z_j^n(s) + \epsilon)^2} \cdot \left\{ A_j^n(s) d_j(X^n(s), C^n(s), L^n) - C_j^n(s) d_j(X^n(s), C^n(s), L^n) \right\} ds \]

\[ = \int_0^{t \wedge \tau} \frac{1}{(Z_j^n(s) + \epsilon)^2} \cdot \left\{ b_j(X^n(s), A^n(s), L^n) - d_j(X^n(s), \mathbf{F}^n(s), L^n) \right\} ds \]

\[ - \int_0^{t \wedge \tau} \frac{1}{(Z_j^n(s) + \epsilon)^2} \cdot \left\{ d_j(X^n(s), \mathbf{F}^n(s), L^n) - d_j(X^n(s), C^n(s), L^n) \right\} ds \]
By the condition (A1), for any $s \in [0, t \land \tau)$,

$$b_j(X^n(s), A^n(s), L^n) \geq d_j(X^n(s), F^n(s), L^n)$$

Comparing $A^n$ and $F^n$, we have for any $s \in [0, t \land \tau)$,

$$A^n_j(s) = A^n_j(s), \quad \& \quad A^n_i(s) \geq C^n_i(s) \quad (i \neq j).$$

We can see that first term in (6.13)

$$- \int_0^{t \wedge \tau} \frac{1}{(Z^n_j(s) + \epsilon)^2} \cdot \left\{ A^n_j(s) - C^n_j(s) \right\} d_j(X^n(s), C^n(s), L^n) ds \quad (6.13)$$

By the condition (A2), the second term in (6.13) can be rewritten as:

$$- \int_0^{t \wedge \tau} \frac{1}{(Z^n_j(s) + \epsilon)^2} \cdot A^n_j(s) \cdot \left\{ b_j(X^n(s), A^n(s), L^n) - d_j(X^n(s), F^n(s), L^n) \right\} ds \leq 0 \quad (6.14)$$

By the condition (A4), the third term in (6.13) becomes

$$- \int_0^{t \wedge \tau} \frac{1}{(Z^n_j(s) + \epsilon)^2} \cdot \left\{ A^n_j(s) - C^n_j(s) \right\} d_j(X^n(s), C^n(s), L^n) ds$$

$$\leq \int_0^{t \wedge \tau} \frac{Z^n_j(s)}{(Z^n_j(s) + \epsilon)^2} \cdot |d_j(X^n(s), C^n(s), L^n)| ds$$

$$\leq \int_0^{t \wedge \tau} \frac{K_2 \cdot Z^n_j(s) \cdot |d_j(X^n(s), C^n(s), L^n)|, ds}{(Z^n_j(s) + \epsilon)^2}$$

$$K_2 \leq \int_0^{t \wedge \tau} \frac{1}{Z^n_j(s) + \epsilon} \cdot ds \quad (6.16)$$
Combining (6.14), (6.15) and (6.16), the second term in (6.13) becomes:

\[-\int_0^{t\wedge \tau} \frac{1}{(Z^n_j(s) + \epsilon)^2} \left\{ A^n_j(s)b_j(X^n(s), A^n(s), L^n) - C^n_j(s)d_j(X^n(s), C^n(s), L^n) \right\} ds \]

\[\leq \left( K_2 + \frac{K_1 M}{n} \right) \int_0^{t\wedge \tau} \frac{1}{Z^n_j(s) + \epsilon} \cdot ds \quad (6.17)\]

The third and fourth terms in (6.12) are martingales. We can denote them by \( MG_1 \) and \( MG_2 \) respectively. Consider the fifth term in (6.12). By the condition (A3) and (A4), we have:

\[\int_0^{t\wedge \tau} \frac{1}{(Z^n_j(s) + \epsilon)^3} \cdot \left\{ A^n_j(s)\gamma_j(X^n(s), A^n(s), L^n) - C^n_j(s)\gamma_j(X^n(s), C^n(s), L^n) \right\}^2 ds \]

\[\leq 2 \int_0^{t\wedge \tau} \frac{1}{(Z^n_j(s) + \epsilon)^3} \cdot \left\{ A^n_j(s)\gamma_j(X^n(s), A^n(s), L^n) - C^n_j(s)\gamma_j(X^n(s), A^n(s), L^n) \right\}^2 ds \]

\[+ 2 \int_0^{t\wedge \tau} \frac{1}{(Z^n_j(s) + \epsilon)^3} \cdot \left\{ C^n_j(s)\gamma_j(X^n(s), A^n(s), L^n) - C^n_j(s)\gamma_j(X^n(s), C^n(s), L^n) \right\}^2 ds \]

\[\leq 2 \int_0^{t\wedge \tau} \frac{1}{(Z^n_j(s) + \epsilon)^3} \cdot |A^n_j(s) - C^n_j(s)|^2 \cdot |\gamma_j(X^n(s), A^n(s), L^n)|^2 \cdot ds \]

\[+ 2 \int_0^{t\wedge \tau} \frac{1}{(Z^n_j(s) + \epsilon)^3} \cdot C^n_j(s)^2 \cdot |\gamma_j(X^n(s), A^n(s), L^n) - \gamma_j(X^n(s), C^n(s), L^n)|^2 \cdot ds \]

\[\leq 2 \int_0^{t\wedge \tau} \frac{1}{(Z^n_j(s) + \epsilon)^3} \cdot Z^n_j(s)^2 \cdot K_3^2 \cdot ds \]

\[+ 2 \int_0^{t\wedge \tau} \frac{1}{(Z^n_j(s) + \epsilon)^3} \cdot C^n_j(s)^2 \cdot K_4^2 \cdot |A^n_j(s) - C^n_j(s)|^2 \cdot ds \]

\[\leq \left( 2K_3^2 + 2M^2K_4^2 \right) \int_0^{t\wedge \tau} \frac{1}{Z^n_j(s) + \epsilon} \cdot ds \quad (6.18)\]

To study the sixth term in (6.12). By the conditions (A3) and (A4),

\[\int_0^{t\wedge \tau} \frac{1}{(Z^n_j(s) + \epsilon)^3} \cdot \int_G \left\{ A^n_j(s)b_j(X^n(s), A^n(s), L^n, z) - C^n_j(s)b_j(X^n(s), C^n(s), z) \right\}^2 \mu(dz) ds \]

\[\leq 2 \int_0^{t\wedge \tau} \int_G \frac{1}{(Z^n_j(s) + \epsilon)^3} \cdot \left\{ A^n_j(s)b_j(X^n(s), A^n(s), L^n, z) \right\}^2 \mu(dz) ds \]

\[\leq 2 \int_0^{t\wedge \tau} \int_G \frac{1}{(Z^n_j(s) + \epsilon)^3} \cdot \left\{ A^n_j(s)b_j(X^n(s), A^n(s), L^n, z) \right\}^2 \mu(dz) ds \]

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\[-C^n_j(s)\beta_j(X^n(s), A^n(s), L^n, z)\bigg\}^2\mu(dz)ds

\[+2\int_0^{t\land \tau} \int_G \frac{1}{(Z^n_j(s) + \epsilon)^3} \cdot \left\{ C^n_j(s)\beta_j(X^n(s), A^n(s), L^n, z) - C^n_j(s)\beta_j(X^n(s), C^n(s), L^n, z)\right\}^2\mu(dz)ds\]

\[\leq 2\int_0^{t\land \tau} \int_G \frac{1}{(Z^n_j(s) + \epsilon)^3} \cdot |A^n_j(s) - C^n_j(s)|^2 \cdot |\beta_j(X^n(s), A^n(s), L^n, z)|^2\mu(dz)ds\]

\[+2\int_0^{t\land \tau} \int_G \frac{1}{(Z^n_j(s) + \epsilon)^3} \cdot C^n_j(s)^2 \cdot |\beta_j(X^n(s), A^n(s), L^n, z) - \beta_j(X^n(s), C^n(s), L^n, z)|^2\mu(dz)ds\]

\[\leq 2\int_0^{t\land \tau} \frac{1}{(Z^n_j(s) + \epsilon)^3} \cdot Z^n_j(s)^2 \cdot K_5^2 \cdot ds\]

\[+2\int_0^{t\land \tau} \frac{1}{(Z^n_j(s) + \epsilon)^3} \cdot C^n_j(s)^2 \cdot K_5^2 \cdot |A^n_j(s) - C^n_j(s)|^2 \cdot ds\]

\[\leq \left(2K_5^2 + 2M^2K_6^2\right) \int_0^{t\land \tau} \frac{1}{Z^n_j(s) + \epsilon} \cdot ds \quad (6.19)\]

Notice that \(\mu(G) = 1\) has been used here. Then by (6.12), (6.17), (6.18) and (6.19) altogether,

\[f(Z^n_j(t \land \tau)) \leq f(Z^n_j(0)) + MG_1 + MG_2 +\]

\[+ \left(\frac{K_1M}{n} + K_2 + 2K_3^2 + 2M^2K_4^2 + 2K_5^2 + 2M^2K_6^2\right) \int_0^{t\land \tau} \frac{1}{Z^n_j(s) + \epsilon} \cdot ds\]

Let \(\tilde{K} = \frac{K_1M}{n} + K_2 + 2K_3^2 + 2M^2K_4^2 + 2K_5^2 + 2M^2K_6^2\). By taking the expectation on both sides of the above inequality, we have:

\[\mathbb{E}f(Z^n_j(t \land \tau)) \leq \mathbb{E}f(Z^n_j(0)) + \tilde{K} \cdot \mathbb{E} \int_0^{t\land \tau} \frac{1}{Z^n_j(s) + \epsilon} \cdot ds\]

\[\leq \mathbb{E}f(Z^n_j(0)) + \tilde{K} \cdot \mathbb{E} \int_0^t \mathbb{E}f(Z^n_j(s \land \tau)) \cdot ds\]

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Finally by Gronwall’s inequality, \( E_f(Z^n_j(t \wedge \tau)) \leq E_f(Z^n_j(0)) \cdot e^{\tilde{K}t} \) for each \( j = 1, 2,..., n \).

In other words, this inequality can be rewritten as:

\[
E\left(\frac{1}{A^n_j(t \wedge \tau) - C^n_j(t \wedge \tau) + \epsilon}\right) \leq E\left(\frac{1}{A^n_j(0) - C^n_j(0) + \epsilon}\right) \cdot e^{\tilde{K}t}.
\]

Note that \( \epsilon > 0 \) can be arbitrarily small and in this case \( A^n_j(0) > C^n_j(0) \). So allowing \( \epsilon \to 0 \) to get

\[
0 < E\left(\frac{1}{A^n_j(t \wedge \tau) - C^n_j(t \wedge \tau)}\right) \leq E\left(\frac{1}{A^n_j(0) - C^n_j(0)}\right) \cdot e^{\tilde{K}t} < \infty
\]

which means \( P\{t < \tau\} = 1 \) for any \( t > 0 \).

**Case (2).** \( A^n_j(0) \geq C^n_j(0) > 0 \) for \( j = 1, 2,..., n \).

First define a new process for each \( j \) by \( D^n_j(0) = A^n_j(0) + \epsilon \) for \( \epsilon > 0 \). Then we are free to apply the same method as that in case (1) because \( D^n_j(0) > C^n_j(0) \). That is, for each \( j = 1,..., n \)

\[
D^n_j(t) = D^n_j(0) + \int_0^t D^n_j(s)\beta_j(X^n(s), D^n(s), L^n)ds + \int_0^t A^n_j(s)\gamma_j(X^n(s), D^n(s), L^n)dB_j(s)
\]

\[
+ \int_0^t \int_G A^n_j(s)\beta_j(X^n(s), D^n(s), L^n, z)W(dzds)
\]

where \( D^n(s) = (D^n_1(s), D^n_2(s), ..., D^n_n(s)) \). Recall that \( C^n_j(t) \) is governed by:

\[
C^n_j(t) = C^n_j(0) + \int_0^t C^n_j(s)d_j(X^n(s), C^n(s), L^n)ds + \int_0^t C^n_j(s)\gamma_j(X^n(s), C^n(s), L^n)dB_j(s)
\]

\[
+ \int_0^t \int_G C^n_j(s)\beta_j(X^n(s), C^n(s), L^n, z)W(dzds)
\]

where \( C^n(s) = (C^n_1(s), ..., C^n_j(s), ..., C^n_n(s)) \).

Under the conditions (A1)-(A4), the initial condition \( D^n_j(0) > C^n_j(0) \) implies that \( P\{D^n_j(t) \geq C^n_j(t), \ \forall \ j, \ \forall t\} = 1 \). By the continuity of the solutions with respect to the initial data implies that \( D^n_j(t) \to A^n_j(t) \), a.s. when \( \epsilon \to 0 \). This allows us to conclude that \( P\{A^n_j(t) \leq C^n_j(t), \ \forall \ j, \ \forall t\} = 1 \).
Case (3). \( A^n_j(0) \geq 0 \geq C^n_j(0) \) for \( j = 1, ..., n \).

Since for any \( t \in [0, \infty) \), \( A^n_j(t) \) and \( A^n_j(0) \) have the same signs, and \( C^n_j(t) \) and \( C^n_j(0) \) have the same signs. So immediately we have \( A^n_j(t) \geq C^n_j(t) \) a.s.

Case (4). \( 0 \geq A^n_j(0) \geq C^n_j(0) \) for \( j = 1, ..., n \).

To compare the two solutions both of which have negative initial values, we can define:

\[
I^n_j(0) = -A^n_j(0), \quad J^n_j(0) = -C^n_j(0)
\]

so that \( J^n_j(0) \geq I^n_j(0) \geq 0 \). By such a transformation, we can turn it the same as the case (2). Hence \( J^n_j(t) \geq I^n_j(t) \) a.s., that is \( A^n_j(t) \geq C^n_j(t) \) a.s.

\[
□
\]

Let

\[
g(x, u, l) = c(x) + \sigma(x)\gamma^T(x, u, l) + \int_G \alpha(x, z)\beta(x, u, z, l)\mu(dz)
\]

\[
h(x) = \sigma\sigma^T(x) + \int_G \alpha^T(x, z)\alpha(x, z)\mu(dz)
\]

For any function \( \phi \in C^2_b(\mathbb{R}^d) \), define an operator \( \mathcal{L}_t(u) \) by

\[
\mathcal{L}_t(u)\phi(x) = \sum_{i=1}^d g_i(x, u, l) \cdot \frac{\partial \phi(x)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d h_{ij}(x) \cdot \frac{\partial^2 \phi(x)}{\partial x_i \partial x_j}.
\]

Consider the following two SPDEs:

\[
\left\langle \theta(t), \phi \right\rangle = \left\langle \theta(0), \phi \right\rangle + \int_0^t \left\langle \theta(s), \ b(\cdot, \theta(s))\phi + \mathcal{L}(\theta(s))\phi \right\rangle ds
\]

\[
+ \int_0^t \int_G \left\langle \theta(s), \beta(\cdot, \theta(s), z)\phi + \alpha(\cdot, z) \nabla \phi \right\rangle W(dz ds)
\]

(6.21)

where \( \theta \) is the distributional limit of \( U^n(t) = \frac{1}{n} \sum_{j=1}^n A^n_j(t)\delta_{X^n_j(t)} \) with \( (X^n_j, A^n_j, U^n) \) governed by the system (6.1)-(6.2)-(6.3), and \( \mathcal{L}(\theta) = \mathcal{L}_t(\theta) \).

\[
\left\langle \vartheta(t), \phi \right\rangle = \left\langle \vartheta(0), \phi \right\rangle + \int_0^t \left\langle \vartheta(s), \ d(\cdot, \vartheta(s))\phi + \mathcal{L}(\vartheta(s))\phi \right\rangle ds
\]

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where $\vartheta$ is the distributional limit of $V^n(t) = \frac{1}{n} \sum_{j=1}^{n} C^n_j(t) \delta_{X^n_j(t)}$ with $(X^n_j, C^n_j, V^n)$ governed by the system (6.1)-(6.4)-(6.5), and $\mathcal{L}(\vartheta) = \mathcal{L}_l(\vartheta)$.

We have:

**Theorem 6.2.** Under the assumptions (A1)-(A4), $\theta(0) \geq \vartheta(0)$ a.s. implies that $\theta(t) \geq \vartheta(t)$ a.s. for any $t \in [0, \infty)$.

### 6.2 Two Systems with the Drift, Diffusion and Gaussian Noise Terms All Different

In this section we will investigate the comparison between two interacting particle systems with different drifts, different diffusions and space-time noise terms which are given by the following system of equations:

\[
X^n_j(t) = X^n_j(0) + \int_0^t c(X^n_j(s)) ds + \int_0^t \sigma(X^n_j(s)) dB_j(s) + \int_0^t \int_G \alpha(X^n_j(s), z) W(dzds) (6.23)
\]

\[
A^n_j(t) = A^n_j(0) + \int_0^t A^n_j(s) b(X^n_j(s), U^n(s), L^n_j) ds + \int_0^t A^n_j(s) \gamma(X^n_j(s), U^n(s), L^n_j) dB_j(s) + \int_0^t \int_G A^n_j(s) \beta(X^n_j(s), U^n(s), L^n_j, z) W(dzds) (6.24)
\]

\[
U^n(t) = \frac{1}{n} \sum_{j=1}^{n} A^n_j(t) \delta_{X^n_j(t)} (6.25)
\]

\[
C^n_j(t) = C^n_j(0) + \int_0^t C^n_j(s) d(X^n_j(s), V^n(s), L^n_j) ds + \int_0^t C^n_j(s) \eta(X^n_j(s), V^n(s), L^n_j) dB_j(s) + \int_0^t \int_G C^n_j(s) \varphi(X^n_j(s), V^n(s), L^n_j, z) W(dzds) (6.26)
\]
\[ V^n(t) = \frac{1}{n} \sum_{j=1}^{n} C_j^n(t) \delta X_j^n(t) \]  

(6.27)

where the drift coefficients \( b \) and \( d \) are random functions with the dependence on \( \omega \), diffusion coefficients \( \sigma \) and \( \eta \) are deterministic functions without dependence on \( \omega \), and so are the space-time noise coefficients \( \beta \) and \( \varrho \).

Let

\[ A^n(t) = (A^n_1(t), ..., A^n_j(t), ..., A^n_n(t)), \quad C^n(t) = (C^n_1(t), ..., C^n_j(t), ..., C^n_n(t)) \]

\[ L^n = (L^n_1, ..., L^n_j, ..., L^n_n) \]

Define

\[ b_j(x^n(t), A^n(t), L^n) = b(X^n_j(t), U^n(t), L^n_j), \quad d_j(x^n(t), A^n(t), L^n) = d(X^n_j(t), U^n(t), L^n_j) \]

\[ \gamma_j(x^n(t), A^n(t), L^n) = \gamma(X^n_j(t), U^n(t), L^n_j), \quad \eta_j(x^n(t), C^n(t), L^n) = \eta(X^n_j(t), V^n(t), L^n_j) \]

\[ \beta_j(x^n(t), A^n(t), L^n) = \beta(X^n_j(t), U^n(t), L^n_j, z), \]

\[ \varrho_j(x^n(t), C^n(t), L^n) = \varrho(X^n_j(t), V^n(t), L^n_j, z) \]

Then the equations (6.24) and (6.26) are rewritten as:

\[ A^n_j(t) = A^n_j(0) + \int_0^t A^n_j(s)b_j(x^n(s), A^n(s), L^n)ds + \int_0^t A^n_j(s)\gamma_j(x^n(s), A^n(s), L^n)dB_j(s) \]

\[ + \int_0^t \int_G A^n_j(s)\beta_j(x^n(s), A^n(s), L^n, z)W(dzds) \]  

(6.28)

\[ C^n_j(t) = C^n_j(0) + \int_0^t C^n_j(s)d_j(x^n(s), C^n(s), L^n)ds + \int_0^t C^n_j(s)\eta_j(x^n(s), C^n(s), L^n)dB_j(s) \]

\[ + \int_0^t \int_G C^n_j(s)\varrho_j(x^n(s), C^n(s), L^n, z)W(dzds) \]  

(6.29)

Let

\[ x = (x_1, ..., x_j, ..., x_n), \quad a = (a_1, ..., a_j, ..., a_n), \]

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\[ \mathbf{c} = (c_1, ..., c_j, ..., c_n), \quad \mathbf{l} = (l_1, ..., l_j, ..., l_n). \]

Let \( \xi_j = A_j^n(0), \ \zeta_j = C_j^n(0). \) Define

\[ \hat{\gamma}_j(x, a, l) = (a_j \gamma_j(x, a, l))^{-1}, \quad \hat{\eta}_j(x, c, l) = (c_j \gamma_j(x, c, l))^{-1} \]

\[ I_j(x, a, l) = a_j^2 \gamma_j^2(x, a, l) + \int_G a_j^2 \beta_j^2(x, a, l, z) \mu(dz) \]

\[ J_j(x, c, l) = c_j^2 \eta_j^2(x, c, l) + \int_G c_j^2 \rho_j^2(x, c, l, z) \mu(dz) \]

The following conditions are necessary for the comparison between the two different weights governed by two different stochastic differential systems (6.23)-(6.28)-(6.25) and (6.23)-(6.29)-(6.27), with different drifts, different diffusion terms and space-time noise terms for \( j = 1, ..., n. \)

**Conditions.** The random functions \( b_j \) and \( d_j: \mathbb{R}^n \times \mathcal{M}(\mathbb{R}^n) \times \mathbb{N} \times \Omega \to \mathbb{R} \) are jointly continuous in the first two components. The deterministic functions \( \gamma \) and \( \eta: \mathbb{R}^n \times \mathcal{M}(\mathbb{R}^n) \times \mathbb{N} \to \mathbb{R}, \beta \) and \( \rho: \mathbb{R}^n \times \mathcal{M}(\mathbb{R}^n) \times \mathbb{N} \times G \to \mathbb{R} \) are jointly continuous and positive \( C^1 \) functions. \( \gamma \) and \( \eta \) both have the first partial derivatives and satisfy the conditions:

\[ (C1). \quad \frac{b_j(x, a, l)}{\gamma_j(x, a, l)} + \frac{1}{2} \frac{\partial \hat{\gamma}_j(x, a, l)}{\partial a_j} \cdot I_j(x, a, l) \geq \frac{d_j(x, c, l)}{\eta_j(x, c, l)} + \frac{1}{2} \frac{\partial \hat{\eta}_j(x, c, l)}{\partial c_j} \cdot J_j(x, c, l) \]

for any \( a, c \in \mathbb{R}^n \) and for almost all \( \omega \in \Omega. \)

\[ (C2). \quad \int_G \frac{\beta_j(x, a, l, z) \mu(dz)}{\gamma_j(x, a, l)} \geq \int_G \frac{\rho_j(x, c, l, z) \mu(dz)}{\eta_j(x, c, l)} \]

\[ (C3). \quad -\infty < \int_{\xi}^{y} \frac{du_j}{u_j \cdot \gamma_j(x, u, l)} \leq \int_{\xi}^{y} \frac{du_j}{u_j \cdot \eta_j(x, u, l)} < \infty, \forall y \in \mathbb{R} \]

for any \( y \in \mathbb{R}, \mathbf{u} = (u_1, ..., u_j, ..., u_n) \in \mathbb{R}^n. \)
Theorem 6.3. Under conditions (C1), (C2) and (C3), for any \( j \) from 1 to \( n \), we have

\[
P\left\{ A^n_j(t) \geq C^n_j(t), \; \forall t \in [0, \infty) \right\} = 1.
\]

Proof. For any \( y \in \mathbb{R} \), define:

\[
S(y) = \int_{\xi}^{y} \frac{du_j}{u_j \cdot \gamma_j(x,u,l)}, \quad T(y) = \int_{\xi}^{y} \frac{du_j}{u_j \cdot \eta_j(x,u,l)} \quad (6.30)
\]

Applying the Itô formula to \( S(y) \) with \( y = A^n_j(t) \), we have:

\[
dS(A^n_j(t)) = \frac{1}{A^n_j(t) \gamma_j(X^n(t), A^n(t), L^n)} \cdot dA^n_j(t) + \frac{\partial}{2\partial A^n_j(t)} \left( \frac{1}{A^n_j(t) \gamma_j(X^n(t), A^n(t), L^n)} \right) \cdot (dA^n_j(t))^2
\]

\[=
\frac{1}{A^n_j(t) \gamma_j(X^n(t), A^n(t), L^n)} \cdot \left( A^n_j(t) b_j(X^n(t), A^n(t), L^n) dt + A^n_j(t) \gamma_j(X^n(t), A^n(t), L^n) dB_j(t) + \int_G A^n_j(t) \beta_j(X^n(t), A^n(t), L^n, z) W(dzdt) \right)
\]

\[+ \frac{\partial \gamma_j(X^n(t), A^n(t), L^n)}{2\partial A^n_j(t)} \cdot I_j(X^n(t), A^n(t), L^n) dt
\]

Integrating both sides from 0 to \( t \), we have:

\[
S(A^n_j(t)) = \int_0^t dB_j(s) + \int_0^t \int_G \frac{\beta_j(X^n(s), A^n(s), L^n, z) \mu(dz)}{\gamma_j(X^n(s), A^n(s), L^n)} W(dzds)
\]

\[+ \int_0^t \frac{b_j(X^n(s), A^n(s), L^n)}{\gamma_j(X^n(s), A^n(s), L^n)} ds + \frac{1}{2} \int_0^t \frac{\partial \gamma_j(X^n(s), A^n(s), L^n)}{\partial A^n_j(s)} \cdot I_j(X^n(s), A^n(s), L^n) ds \quad (6.31)
\]

Applying the Itô formula to \( T(y) \) with \( y = C^n_j(t) \), we have:

\[
dT(C^n_j(t)) = \frac{1}{C^n_j(t) \eta_j(X^n(t), C^n(t), L^n)} \cdot dC^n_j(t) + \frac{\partial}{2\partial C^n_j(t)} \left( \frac{1}{C^n_j(t) \eta_j(X^n(t), C^n(t), L^n)} \right) \cdot (dC^n_j(t))^2
\]

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Since Combining (6.33) and (6.34) together, we have:

By the condition (C3) and by letting \( y \)

For almost all \( t \), we have:

Integrating both sides from 0 to \( t \), we have:

With conditions (C1) and (C2), between (6.31) and (6.32) we have:

For almost all \( \omega \in \Omega \).

Then by (6.30), this implies:

By the condition (C3) and by letting \( y = C^n_j(t) \), we have:

Combining (6.33) and (6.34) together, we have:

Since \( \gamma_j \) is positive, from (6.35) we have: \( A^n_j(t) \geq C^n_j(t) \) for almost all \( \omega \in \Omega \). That is, \( P\{A^n_j(t) \geq C^n_j(t), \forall t \in [0, \infty)\} = 1 \).

Let

\[
g(x, u, l) = c(x) + \sigma(x)\gamma^T(x, u, l) + \int_G \alpha(x, z)\beta(x, u, z, l)\mu(dz)
\]

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\[ h(x) = \sigma \sigma^T(x) + \int_G \alpha \alpha^T(x, z) \mu(dz) \]

For any function \( \phi \in C^2_b(\mathbb{R}^d) \), define an operator \( \mathcal{L}_t(u) \) by

\[ \mathcal{L}_t(u) \phi(x) = \sum_{i=1}^d g_i(x, u, l) \cdot \frac{\partial \phi(x)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d h_{ij}(x) \cdot \frac{\partial^2 \phi(x)}{\partial x_i \partial x_j}. \]

Let

\[ \tilde{g}(x, u, l) = c(x) + \sigma^T(x) \eta(x, u, l) + \int_G \alpha(x, z) \phi(x, u, z, l) \mu(dz) \]
\[ \tilde{h}(x) = \eta \eta^T(x) + \int_G \alpha^T(x, z) \alpha(x, z) \mu(dz) \]

For any function \( \phi \in C^2_b(\mathbb{R}^d) \), define an operator \( \tilde{\mathcal{L}}_t(u) \) by

\[ \tilde{\mathcal{L}}_t(u) \phi(x) = \sum_{i=1}^d \tilde{g}_i(x, u, l) \cdot \frac{\partial \phi(x)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d \tilde{h}_{ij}(x) \cdot \frac{\partial^2 \phi(x)}{\partial x_i \partial x_j}. \]

Consider the following two SPDEs:

\[ \langle \theta(t), \phi \rangle = \langle \theta(0), \phi \rangle + \int_0^t \langle \theta(s), b(\cdot, \theta(s)) \phi + \mathcal{L}(\theta(s)) \phi \rangle ds \]
\[ + \int_0^t \int_G \langle \theta(s), \beta(\cdot, \theta(s), z) \phi + \alpha(\cdot, z) \nabla \phi \rangle W(dz ds) \]  \hspace{1cm} (6.36)

where \( \theta \) is the distributional limit of \( U^n(t) = \frac{1}{n} \sum_{j=1}^n A^n_j(t) \delta_{X^n_j(t)} \) with \( (X^n_j, A^n_j, U^n) \) governed by the system (6.23)-(6.24)-(6.25), and \( \mathcal{L}(\theta) = \mathcal{L}_t(\theta) \).

\[ \langle \vartheta(t), \phi \rangle = \langle \vartheta(0), \phi \rangle + \int_0^t \langle \vartheta(s), d(\cdot, \vartheta(s)) \phi + \tilde{\mathcal{L}}(\vartheta(s)) \phi \rangle ds \]
\[ + \int_0^t \int_G \sum_{k=1}^N p_k \langle \vartheta(s), \vartheta(\cdot, \vartheta(s), z) \phi + \alpha(\cdot, z) \nabla \phi \rangle W(dz ds) \]  \hspace{1cm} (6.37)

where \( \vartheta \) is the distributional limit of \( V^n(t) = \frac{1}{n} \sum_{j=1}^n C^n_j(t) \delta_{X^n_j(t)} \) with \( (X^n_j, C^n_j, V^n) \) governed by the system (6.23)-(6.26)-(6.27), and \( \tilde{\mathcal{L}}(\theta) = \tilde{\mathcal{L}}_t(\theta) \).

We have:

**Theorem 6.4.** Under the assumptions (C1), (C2) and (C3), \( \theta(t) \geq \vartheta(t) \) a.s. for any \( t \in [0, \infty) \).
6.3 Compactness of Support for the Solutions

We consider the interacting stochastic system (6.1)-(6.2)-(6.3) again. In this section, our interest is to investigate when the values of the location processes $X^n_j(t)$, $j = 1, ..., n$ can stay in a compact set $[l, r] \in \mathbb{R}$. Consider any pair of real numbers $a, b \in (l, r)$ such that $l < a < b < r$, and define a set of stopping times below:

$$ T_{a,b}^j = \inf \{ t \geq 0; \ X^n_j(t) \notin (a, b) \}, \quad T_{a,b} = \bigwedge_{j=1}^n T_{a,b}^j, $$

$$ T_{l,r}^j = \inf \{ t \geq 0; \ X^n_j(t) \notin (l, r) \}, \quad T_{l,r} = \bigwedge_{j=1}^n T_{l,r}^j. $$

For a fixed number $c \in (l, r)$ and for any $x \in \mathbb{R}$, define the scale function $p$ by:

$$ p(x) = \int_c^x \exp \left\{ -2 \int_c^z \frac{c(y)}{\sigma^2(y) + \int_G \alpha^2(y, v) \mu(dv)} \cdot dy \right\} \cdot dz \quad (6.38) $$

It is not hard to see that the function $p(x)$ is continuous and increasing on $\mathbb{R}$ and satisfies:

$$ p''(x) = -\frac{2c(x)}{\sigma^2(x) + \int_G \alpha^2(x, v) \mu(dv)} \cdot p'(x). $$

**Lemma 6.5.** Suppose $X^n_j(t)$ be the solution of the system (6.1)-(6.2)-(6.3) with $X^n_j(t) \in [a, b]$ and $\sigma(x)^2 + \int_G \alpha^2(x, v) \mu(dv) > 0$ for all $x \in [a, b]$. Then

$$ P\{X^n_j(T_{a,b}) = a\} = \frac{p(b) - p(\xi_j)}{p(b) - p(a)}, \quad P\{X^n_j(T_{a,b}) = b\} = \frac{p(\xi_j) - p(a)}{p(b) - p(a)}. $$

**Proof.** By the definition in (6.38),

$$ p(X^n_j(t \wedge T_{a,b})) = \int_c^{X^n_j(t \wedge T_{a,b})} \exp \left\{ -2 \int_c^z \frac{c(y)}{\sigma^2(y) + \int_G \alpha^2(y, v) \mu(dv)} \cdot dy \right\} \cdot dz $$

$$ p'(X^n_j(t \wedge T_{a,b})) = \exp \left\{ -2 \int_c^{X^n_j(t \wedge T_{a,b})} \frac{c(y)}{\sigma^2(y) + \int_G \alpha^2(y, v) \mu(dv)} \cdot dy \right\} $$

$$ p''(X^n_j(t \wedge T_{a,b})) = p'(X^n_j(t \wedge T_{a,b})) \cdot \left\{ -\frac{2c(X^n_j(t \wedge T_{a,b}))}{\sigma^2(X^n_j(t \wedge T_{a,b})) + \int_G \alpha^2(X^n_j(t \wedge T_{a,b}), v) \mu(dv)} \right\}. $$

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Applying the Itô formula to $p(X^a_n(t \wedge T_{a,b}))$,

$$
p(X^a_n(t \wedge T_{a,b})) = p(\xi_j) + \int_0^{t\wedge T_{a,b}} p'(X^a_n(s))c(X^a_n(s))ds
+ \int_0^{t\wedge T_{a,b}} p'(X^a_n(s))\sigma(X^a_n(s))dB_j(s)
+ \int_0^{t\wedge T_{a,b}} p'(X^a_n(s))\int G\alpha(X^a_n(s), v)W(dvds)
+ \frac{1}{2} \int_0^{t\wedge T_{a,b}} p''(X^a_n(s))\sigma^2(X^a_n(s))ds
+ \frac{1}{2} \int_0^{t\wedge T_{a,b}} p''(X^a_n(s))\int G\alpha^2(X^a_n(s), v)\mu(dv)ds
$$  \hspace{1cm} (6.39)

Substituting $p''(X^a_n(s))$ in the above equation (6.39),

$$
p(X^a_n(t \wedge T_{a,b})) = p(\xi_j) + \int_0^{t\wedge T_{a,b}} p'(X^a_n(s))\sigma(X^a_n(s))dB_j(s)
+ \int_0^{t\wedge T_{a,b}} p'(X^a_n(s))\int G\alpha(X^a_n(s), v)W(dvds)
$$  \hspace{1cm} (6.40)

Taking expectation on the both sides of (6.40), we have:

$$
\mathbb{E}\{p(X^a_n(t \wedge T_{a,b}))\} = p(\xi_j).
$$

Then let $t$ goes to $\infty$, therefore $\mathbb{E}\{p(X^a_n(T_{a,b}))\} = p(\xi_j)$ for each $j = 1, ..., n$.

$$
p(\xi_j) = \mathbb{E}\{p(X^a_n(T_{a,b}))\} = p(a) \cdot P\{X^a_n(T_{a,b}) = a\} + p(b) \cdot P\{X^a_n(T_{a,b}) = b\}. \hspace{1cm} (6.41)
$$

By the proof in Theorem 2 and Theorem 4 in [12], we have:

$$
P\{X^a_n(T_{a,b}) = a\} + P\{X^a_n(T_{a,b}) = b\} = 1,
$$

then (6.41) leads to:

$$
P\{X^a_n(T_{a,b}) = a\} = \frac{p(b) - p(\xi_j)}{p(b) - p(a)}, \hspace{0.5cm} P\{X^a_n(T_{a,b}) = b\} = \frac{p(\xi_j) - p(a)}{p(b) - p(a)}.
$$

□
Theorem 6.6. Let $l$ and $r$ be any two real numbers such that, $r < l$, $p(l+) = -\infty$ and $p(r-) = \infty$. Assume that for any $x \in (l, r)$ there exists $\epsilon > 0$ such that

$$\int_{x-\epsilon}^{x+\epsilon} \frac{c(y)}{\sigma^2(y) + \int_G \alpha^2(y, v) \mu(dv)} \cdot dy < \infty,$$

and

$$\sigma^2(x) + \int_G \alpha^2(x, v) \mu(dv) > 0$$

Let $X^n_j$ is the weak solution of the equation (1)-(2)-(3) ($j = 1, \ldots, n$). Then we have:

$$P\{T_{l,r} = \infty\} = P\left\{ \sup_{0 \leq t < T_{l,r}} X^n_j(t) = r \right\} = P\left\{ \inf_{0 \leq t < T_{l,r}} X^n_j(t) = l \right\} = 1$$

Proof.

Because $\{\omega; \inf_{0 \leq t < T_{l,r}} X^n_j(t, \omega) \leq a\}$ contains the set $\{\omega; X^n_j(T_{a,b}, \omega) = a\}$ and $T_{a,b} \leq T_{l,r}$, then from Lemma 6.5 we have:

$$1 \geq P\left\{ \inf_{0 \leq t < T_{l,r}} X^n_j(t) \leq a \right\} \geq P\{X^n_j(T_{a,b}) = a\} = \frac{p(b) - p(\xi_j)}{p(b) - p(a)} = \frac{1 - \frac{p(\xi_j)}{p(b)}}{1 - \frac{p(a)}{p(b)}} \quad (6.42)$$

Letting $b \nearrow r$ in (6.42), then $p(b) \to p(r-) = \infty$. Therefore,

$$1 \geq \lim_{b \nearrow r} P\left\{ \inf_{0 \leq t < T_{l,r}} X^n_j(t) \leq a \right\} \geq \lim_{b \nearrow r} \frac{1 - \frac{p(\xi_j)}{p(b)}}{1 - \frac{p(a)}{p(b)}} = 1$$

that is,

$$P\left\{ \inf_{0 \leq t < T_{l,r}} X^n_j(t) \leq a \right\} = 1$$

for each $j = 1, \ldots, n$ and for any $a \in (l, r)$.

By letting $a \searrow l$ in the above equation, we have for each $j = 1, \ldots, n$,

$$\lim_{a \searrow l} P\left\{ \inf_{0 \leq t < T_{l,r}} X^n_j(t) \leq a \right\} = \lim_{a \searrow l} P\left\{ \inf_{0 \leq t < T_{l,r}} l \leq X^n_j(t) \leq a \right\} = P\left\{ \inf_{0 \leq t < T_{l,r}} X^n_j(t) = l \right\} = 1.$$
On the other hand,

\[ \{ \omega; \sup_{0 \leq t < T_{l,r}} X_j^n(t, \omega) \geq b \} \text{ contains } \{ \omega; X_j^n(T_{a,b}, \omega) = b \} \]

and \( T_{a,b} \leq T_{l,r} \), therefore from Lemma 6.5,

\[
1 \geq P \left\{ \sup_{0 \leq t < T_{l,r}} X_j^n(t) \geq b \right\} \geq P \left\{ X_j^n(T_{a,b}) = b \right\} = \frac{p(\xi_j) - p(a)}{p(b) - p(a)} = \frac{p(\xi_j)}{p(b)} - 1
\]  

(6.43)

Letting \( a \searrow l \) in (6.43), \( p(a) \rightarrow p(l+) = -\infty \) and we have:

\[
1 \geq \lim_{a \searrow l} P \left\{ \sup_{0 \leq t < T_{l,r}} X_j^n(t) \geq b \right\} \geq \lim_{a \searrow l} \frac{p(\xi_j)}{p(a)} - 1 = 1
\]

that is, for each \( j = 1, \ldots, n \)

\[
P \left\{ \sup_{0 \leq t < T_{l,r}} X_j^n(t) \geq b \right\} = 1, \quad \forall b \in (l, r).
\]

Letting \( b \nearrow r \) in the above equation, we have for each \( j = 1, \ldots, n \):

\[
\lim_{b \nearrow r} P \left\{ \sup_{0 \leq t < T_{l,r}} X_j^n(t) \geq b \right\} = \lim_{b \nearrow r} P \left\{ \sup_{0 \leq t < T_{l,r}} r \geq X_j^n(t) \geq b \right\} = P \left\{ \sup_{0 \leq t < T_{l,r}} X_j^n(t) = r \right\} = 1.
\]

Next we show that \( P \{ T_{l,r} = \infty \} = 1 \). Assume \( P \{ T_{l,r} < \infty \} > 0 \). We first notice that this assumption means that the exit time of the interval \((l, r)\) is finite with a nonzero probability. In the other word,

\[
\lim_{t \nearrow T_{l,r}} X_j^n(t) \text{ exists and } \lim_{t \nearrow T_{l,r}} X_j^n(t) = l \text{ or } r
\]

is an event of positive probability. But this implies that both of the following two probabilities can’t be 1.

\[
P \left\{ \sup_{0 \leq t < T_{l,r}} X_j^n(t) = r \right\}, \quad P \left\{ \inf_{0 \leq t < T_{l,r}} X_j^n(t) = l \right\}.
\]
That is,
\[
P\left\{ \sup_{0 \leq t < T_{l,r}} X^n_j(t) = r \right\} < 1, \quad P\left\{ \inf_{0 \leq t < T_{l,r}} X^n_j(t) = l \right\} < 1.
\]
which is a contradiction. Therefore \( P\{T_{l,r} = \infty\} = 1 \).

\[\square\]

**Corollary 6.7.** Under the same conditions in Theorem 6.6, \( P\{l \leq X^n_j(t) \leq r\} = 1 \) for each \( j = 1, \ldots, n \) and any \( t \in [0, \infty) \).

By Theorem 6.6 and Corollary 6.7, we have:

**Theorem 6.8.** Let \( (X^n_j, A^n_j, U^n) \) be the solutions of the interacting system (3.1)-(3.3)-(3.4). Under the conditions of Theorem 6.6, the solution \( \theta \) of the following SPDE (written in the weak formulation) has a nonempty compact support in \((l, r)\).

\[
\left\langle \theta(t), \phi \right\rangle = \left\langle \theta(0), \phi \right\rangle + \int_0^t \left\langle \theta(s), b(\cdot, \theta(s)) \phi + \mathcal{L}(\theta(s)) \phi \right\rangle ds
\]
\[+ \int_0^t \int_G \left\langle \theta(s), \beta(\cdot, \theta(s), z) \phi + \alpha(\cdot, z) \nabla \phi \right\rangle W(dz ds)
\]

(6.21)
References


Vita

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