1975

Infinite-Dimensional Topology and the Theory of Shape.

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INFINITE-DIMENSIONAL TOPOLOGY AND THE
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The Louisiana State University and Agricultural
and Mechanical College, Ph.D., 1975
Mathematics

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INFINITE-DIMENSIONAL TOPOLOGY AND THE THEORY OF SHAPE

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by

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December, 1975
ACKNOWLEDGMENT

The author wishes to express his appreciation to Professor R. D. Anderson for his advice, criticism and encouragement during the past three years and especially during the preparation of this dissertation. He is also grateful to Professors R. M. Schori and D. W. Curtis at L.S.U. for their friendly care and encouragement, and J. E. West at Cornell and T. A. Chapman at the University of Kentucky for their hospitality during his stay there.

I also wish to acknowledge my gratitude to Professor G. Kozlowski who read the manuscript and made several improvements of the text.

Finally, I wish to thank Monica Loftin for her excellent typing job.
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ABSTRACT

The tools of infinite-dimensional topology were frequently used in the development of shape theory. The most remarkable examples of this are Chapman's complement theorems which show:
(a) that the study of shape properties of compacta is equivalent to the study of weak proper homotopy of open subsets $V$ of the Hilbert cube $Q$ for which $Q-V$ is a $Z$-set in $Q$, and
(b) that two $Z$-sets $X, Y \subseteq Q$ have the same shape if and only if their complements $Q-X$ and $Q-Y$ are homeomorphic.

We shall apply results (a) and (b) in the investigation of various shape invariants of compacta. The natural extensions of this simple idea lead us to the study of three different but closely interrelated topics described below.

1° We shall rework a large portion of shape theory of compacta using techniques of infinite-dimensional topology. This is done as follows. Let $\alpha$ be a shape invariant property of compacta (say, movability). Then we can define a property $\tilde{\alpha}$ of all non-compact locally compact metric spaces such that the statement "A $Z$-set $X$ in $Q$ has property $\alpha$ if and
only if the complement \( Q - X \) of \( X \) has property \( \tilde{a} \) holds.

In many instances it is relatively easier to investigate property \( \tilde{a} \) which makes most of our proofs simpler and more geometric than the original ones. In Chapters III, IV, V, VI and VII this program will be carried on for shape dimension, movability, shape absolute neighborhood retracts, shape absolute retracts, and \( UVW^n \)-spaces, respectively.

2° We shall introduce the category \( \mathcal{C}_\infty \) of non-compact locally compact metric spaces and classes of homotopic at \( \infty \) proper maps between them and study the homotopy theory of that category. In the short Chapter VIII, we will show how objects of \( \mathcal{C}_\infty \) can be considered as inverse systems and how classes of homotopic at \( \infty \) proper maps between one-ended locally compact ANR's induce homotopy classes of maps of associated systems. Chapter IX treats the question: "When is a proper map \( f: M \to N \) between non-compact locally compact metric spaces a homotopy equivalence at \( \infty \)?" Our conditions are in terms of two geometric properties of \( f \) detectable by maps of polyhedra. In Chapters III, IV, V, and VI, property \( \tilde{a} \), discussed above, is considered as a part of homotopy theory of \( \mathcal{C}_\infty \).

3° Relationships between 1° and 2° are exploited throughout.

Chapter II contains the proof of the relative version of the Geometric Characterization Theorem of Chapman and
Siebenmann.

In Chapter I, we collect relevant definitions and results from infinite-dimensional topology in order to make our presentation self-contained as much as possible.
CHAPTER I
INTRODUCTION AND PRELIMINARIES

The notion of shape, introduced by Borsuk [5], is a modification of homotopy type designed to allow the study of global properties of locally complicated compacta for which the classical theory does not give much information. Later on, several people including Mardesić-Segal [36], Fox [25], Kozlowski [29], Mardesić [33], and Ball-Sher [4] extended Borsuk's theory of shape to various classes of spaces. In this dissertation, we shall consider shapes of compact metric spaces and of locally compact metric spaces. It is well known [33] that in this category of compacta, in the absolute case, all standard approaches coincide.

The tools of infinite-dimensional topology were frequently used in the development of shape theory. The most remarkable examples of this are Theorems 1 and 2 in [12], where Chapman showed (see later on in this chapter for precise statements): (a) that the study of shape properties of compacta is equivalent to the study of weak proper homo-
topy of open subsets $V$ of the Hilbert cube $Q$ for which $Q-V$ is a $Z$-set in $Q$ (for definitions, see later on in this chapter), and (b) that two $Z$-sets $X, Y \subset Q$ have the same shape if and only if their complements $Q-X$ and $Q-Y$ are homeomorphic.

It was suggested by R. D. Anderson that by applying results (a) and (b) one can get new ways of treating shape invariants of compacta like shape dimension, movability, shape absolute (neighborhood) retracts, and others. The nucleus of the main idea in this dissertation is that suggestion. The natural extensions of it lead us to the study of four different but closely interrelated topics described below.

1° We shall rework a large portion of shape theory of compacta using techniques of infinite-dimensional topology. This is done as follows. Let $\alpha$ be a shape invariant property of compacta (say, movability). Then we can define a property $\bar{\alpha}$ of all non-compact locally compact metric spaces such that the statement "A $Z$-set $X$ in $Q$ has property $\alpha$ if and only if the complement $Q-X$ of $X$ has property $\bar{\alpha}$" holds. In many instances it is relatively easier to investigate property $\bar{\alpha}$ which makes most of our proofs simpler and more geometric than the original ones. In Chapters III, IV, V, VI, and VII, this program will be carried on for shape dimension, movability, shape absolute neighborhood retracts,
shape absolute retracts, and $UVW^n$-spaces, respectively.

2° We shall introduce the category $\mathcal{LC}_\infty$ of non-compact locally compact metric spaces and classes of homotopic-at-$\infty$ proper maps between them and study the homotopy theory of that category. The concept of homotopy-at-$\infty$ is weaker than Chapman's weak proper homotopy (see Chapter I) and it seems more useful. In the short Chapter VIII, we will show how objects of $\mathcal{LC}_\infty$ can be considered as inverse systems and how classes of homotopic-at-$\infty$ proper maps between one-ended locally compact ANR's induce homotopy classes of maps of associated systems. Chapter XI treats the question: "When is a proper map $f:M \to N$ between non-compact locally compact metric spaces a homotopy equivalence at $\infty$?" Our conditions are in terms of two geometric properties of $f$ detectable by maps of polyhedra. In Chapters III, IV, V, and VI, property $\tilde{a}$, discussed above, is considered as a part of homotopy theory of $\mathcal{LC}_\infty$.

3° Relationships between 1° and 2° are exploited throughout.

4° Let us observe that another aspect of our work can be interpreted as a study of proper shape theory of non-compact locally compact metric spaces, due to Ball and Sher [4], by considering Z-sets in $Q\{-\text{point}\}$ instead of in $Q$.

This dissertation is only an initial step in the direction of effective use of ideas 1°-4° and the author
plans to continue research on these questions.

Chapter II contains the proof of the relative version of the Geometric Characterization Theorem of Chapman and Siebenmann [17]. We include the proof for two reasons, first, this theorem has a key role in our program described under 1° so that the understanding of its proof is rather important in later chapters, and second, since it is proved in the relative form, most of our discussion in the remaining chapters holds for pairs of compact metric spaces as well. However, we warn the reader that in this case our results concern a shape of pairs of compact metric spaces in terms of relative ANR-sequences defined by Mardesić and Segal [36].

In this chapter, we collect relevant definitions and results from infinite-dimensional topology in order to make our presentation self-contained as much as possible.

In this dissertation all statements are numbered with two numbers so that the first number indicates a chapter in which it is contained. For example, Theorem 1.3 means the third numbered statement in Chapter I, and Corollary 3.5 is the fifth result in Chapter III.

The most often used definitions and results from infinite dimensional topology will be listed here. We shall assume that the reader is familiar with the shape theory of compacta in the description using ANR-sequences given by Mardesić and Segal [36]. It is known [37] that in the
absolute case this approach is equivalent to Borsuk's original definition [5], but in the relative case they are different [35].

The Hilbert cube $Q$ is represented as $\prod_{i>0} I_i$, where $I_i = [-1,1]$. For each $n > 0$ we can write $Q = I^n \times Q_{n+1}$ with $I^n = I_1 \times \cdots \times I_n$ and $Q_{n+1} = I_{n+1} \times I_{n+2} \times \cdots$. The endslice $\{1\} \times Q_2$ is denoted by $W$.

By a prism neighborhood of a subset $A \subseteq Q$ we mean any neighborhood of the form $P \times Q_{n+1}$ where $P \subseteq I^n$ is a polyhedron. A Q-manifold is a separable metric manifold modeled on $Q$. A closed subset $X$ in a Q-manifold $M$ is called a Z-set in $M$ provided for every non-empty homotopically trivial open subset $U$ of $M$, $U - X$ is also non-empty and homotopically trivial. The notion of Z-set was introduced by Anderson [1] and plays an important role in infinite-dimensional topology primarily because of the following mapping replacement and Z-set unknotting theorems of Anderson and Chapman [2] (a useful relative version of the latter is stated as Lemma 2.3 below and is due to Felt [23]).

Theorem 1.1 (Mapping Replacement). Let $M$ be a Q-manifold, $\mathcal{U}$ an open cover of $M$, $X$ be a locally compact separable metric space, and let $Y$ be a closed subset of $X$. If $f:X \to M$ is any proper map such that $f|_Y$ is an embedding of $Y$ onto a Z-set in $M$, then there is
an embedding \( g: X \rightarrow M \) such that \( g(X) \) is a Z-set, 
\( g|_Y = f|_Y \), and \( g \) is \( \text{St}(\mathcal{U}) \)-close to \( f \).

**Theorem 1.2 (Z-set Unknotting).** Let \( M \) be a Q-manifold, \( \mathcal{U} \) an open cover of \( M \), \( X \) a locally compact separable metric space and \( F: X \times [0,1] \rightarrow M \) a proper homotopy limited by \( \mathcal{U} \) such that \( F_0 \) and \( F_1 \) are embeddings of \( X \) onto Z-sets in \( M \). Then there is an ambient isotopy \( G: M \times I \rightarrow M \) limited by \( \text{St}(\mathcal{U}) \) such that \( G_0 \circ F_0 = F_1 \) (ambient means that each \( G_t \) is a homeomorphism onto).

Recall that a map \( f: X \rightarrow Y \) between locally compact metric spaces is **proper** if \( f^{-1}(A) \) is compact for every compact subset \( A \) of \( Y \).

Following Chapman [12], two proper maps \( f, g: X \rightarrow Y \) are **weakly proper homotopic** if for every compactum \( B \subseteq Y \) there is a compactum \( A \subseteq X \) and a homotopy \( H: X \times I \rightarrow Y \) joining \( f \) and \( g \) with \( H((X-A) \times I) \subseteq Y-B \). This is an equivalence relation on the class of all proper maps \( X \rightarrow Y \) and in a usual way one defines nations of a **weak proper homotopy equivalence** and a **weak proper homotopy domination**.

Now, we can state precisely Chapman's Theorems 1 and 2 from [12].

Let \( \mathcal{J} \) be the shape category of Z-sets in \( Q \) and homotopy classes of fundamental sequences between them, and let \( \mathcal{G} \) be the category whose objects are complements of
Z-sets in \( Q \) and whose morphisms are weak proper homotopy classes of proper maps between them.

**Theorem 1.3.** There is a category isomorphism \( T : \mathcal{C} \rightarrow \mathcal{D} \) such that \( T(X) = Q-X \) for every object \( X \) of \( \mathcal{C} \).

**Theorem 1.4.** Z-sets \( X \) and \( Y \) in \( Q \) have the same shape if and only if \( Q-X \) and \( Q-Y \) are homeomorphic (in notation, \( Q-X \cong Q-Y \)).

We shall mostly work with a larger category \( \mathcal{LC}_\infty \) of locally compact metric spaces and classes of proper maps between them that are homotopic near \( \infty \), defined as follows. Proper maps \( f,g:X \rightarrow Y \) are **homotopic-at-\( \infty \)** provided for any compactum \( B \subseteq Y \) there is a compactum \( A \subseteq X \) and a homotopy \( H: (X-A) \times I \rightarrow Y-B \) joining \( f|_{X-A} \) with \( g|_{X-A} \). Again, this is an equivalence relation on the class of all proper maps \( X \rightarrow Y \) so we can define notions of a **homotopy equivalence-at-\( \infty \)** and a **homotopy domination-at-\( \infty \)**. Let us observe that in case \( Y \) is an absolute retract, if proper maps \( f,g:X \rightarrow Y \) are homotopic-at-\( \infty \), then they are weak proper homotopic. Therefore, in Theorem 1.3, we can replace weak proper homotopy classes of proper maps by classes of homotopic-at-\( \infty \) proper maps.

A **mapping cylinder** \( M(f) \) of a map \( f:X \rightarrow Y \) between compact metric spaces is obtained from the disjoint union of \( X \times [0,1] \) and \( Y \) by identifying \( (x,0) \) with \( f(x) \).
for every $x \in X$. A natural copy of $Y$ in $M(f)$ is the base of $M(f)$ and $X \times \{0\} \subseteq M(f)$ is the top of $M(f)$.

If $\sigma = \{X_i, f_i\}_{i=1}^\infty$ is an inverse sequence of compact metric spaces then the infinite mapping cylinder $\text{Map}(\sigma)$ of $\sigma$ is a result of piecing $M(f_i)$'s together, i.e., $\text{Map}(\sigma) = (X_1 \times \{1\} \cup \bigcup_{i \geq 1} X_i \times [i-1,i])/\sim$, where $(x,i) \sim (f_i(x),i)$ for all $x \in X_{i+1}$ and $i > 0$. All points of $\text{Map}(\sigma)$ with the second coordinate $\geq k$ are denoted by $\text{Map}_k(\sigma)$. It is clear how to define $\text{Map}(\sigma, \sigma_0)$, where $\sigma = \{(X_i, X_{0i}), f_i\}_{i=1}^\infty$ is an inverse sequence of compact pairs $(X_i, X_{0i})$ and maps of pairs $f_i$ and $\sigma_0 = \{X_{0i}, f_i |_{X_{0i+1}}\}_{i=1}^\infty$.

Infinite mapping cylinders have been used often in topology and their usefulness will become evident in Chapters II and III. There we shall repeatedly need Chapman's Cylinder Completion Theorem [17].

**Theorem 1.5 (Cylinder Completion).** If $\sigma = \{X_i, f_i\}_{i=1}^\infty$ is an inverse sequence of compact polyhedra $X_i$ with inverse limit $\lim_i \sigma$, then $(\text{Map}(\sigma) \cup \lim_i \sigma) \times Q$ is a compact $Q$-manifold homeomorphic to $X_1 \times Q$.

Finally, some comments on maps that we shall work with. By a homeomorphism we shall always mean a homeomorphism onto and a homeomorphism into is called an embedding. A CE-map is a proper surjection all point inverses of which have trivial shape.
CHAPTER II

RELATIVE GEOMETRIC CHARACTERIZATION THEOREM

In this chapter, we shall prove the relative version (Theorem 2.1) of the Geometric Characterization Theorem due to Chapman and Siebenmann [17].

By a Q-manifold pair \((M, M_0)\) we mean that \(M\) and \(M_0\) are Q-manifolds with \(M_0\) a Z-set in \(M\). We will say that a pair \((Z, Z_0)\) of compacta is a boundary for a Q-manifold pair \((M, M_0)\) if there exists a compact Q-manifold pair \((N, N_0) \supset (M, M_0)\) such that \((N, N_0) - (M, M_0) \cong (Z, Z_0)\) is a Z-pair in \((N, N_0)\), i.e., \(N - M\) is a Z-set in \(N\) and \(N_0 - M_0\) is a Z-set in \(N_0\).

The above definition of a boundary for a Q-manifold pair is a relativization of a corresponding absolute form in [17]. In fact, using results like Lemmas 2.2 - 2.6 and Theorem 2.7 below, it is possible to extend most of the theorems from [17] to our setting.

**Theorem 2.1.** A Q-manifold pair \((M, M_0)\) admits a boundary if and only if there is a relative inverse...
sequence \( (\sigma, \sigma_0) = \{X_i, Y_i, f_i\} \) of compact polyhedra \( X_i \) and closed subpolyhedra \( Y_i \subseteq X_i \) such that \( \text{Map}(\sigma, \sigma_0) \times Q \cong (M, M_0) \).

If \( (M, M_0) = \text{Map}(\sigma, \sigma_0) \times Q \) then the pair \( (\lim_i \sigma, \lim_i \sigma_0) \times Q \) is a boundary for \( (M, M_0) \) by the Cylinder Completion Theorem 1.5.

The other half of Theorem 2.1 is proved by a method given in [17] with special attention to certain \( Z \)-sets.

We will first state several standard results from infinite-dimensional topology in the relative setting that are needed for a successful relativization of Chapman-Siebenmann's proof.

**Lemma 2.2** (Chapman [13]). Every compact \( Q \)-manifold pair \( (M, M_0) \) can be triangulated, i.e., if \( K_0 \) is a finite simplicial complex and \( \varphi: M_0 \to K_0 \times Q \) is a homeomorphism, then \( \varphi \) can be extended to a homeomorphism \( \varphi \) of \( M \) onto \( K \times Q \), where \( K \) is a finite simplicial complex containing \( K_0 \) as a PL subspace.

**Lemma 2.3** (Felt [23]). Let \( (M, M_0) \) be a \( Q \)-manifold pair and \( X_0 \) a closed subspace of a space \( X \). Let \( \mathcal{U} \) be an open cover of \( M \) and let \( F: (X, X_0) \times [0,1] \to (M, M_0) \) be a proper homotopy limited by \( \mathcal{U} \) such that \( F_0 \) and \( F_1 \) are embeddings of \( (X, X_0) \) onto \( Z \)-pairs of \( (M, M_0) \). Then there exists a homeomorphism \( h: (M, M_0) \to (M, M_0) \) limited
Lemma 2.4. Let \( f: (M, M_0) \to (N, N_0) \) be a CE-mapping between compact Q-manifold pairs such that \( f^{-1}(N_0) = M_0 \) and let \( \varepsilon > 0 \). Then there is a homeomorphism \( h: (M, M_0) \to (N, N_0) \) that is \( \varepsilon \)-close to \( f \).

**Proof.** Let \( \eta, \frac{\varepsilon}{5} > \eta > 0 \), have the property that \( 2\eta \)-close maps into \( N \) are \( \varepsilon/5 \)-homotopic. Since CE-maps of compact Q-manifolds can be uniformly approximated by homeomorphisms [15], there are homeomorphisms \( h': M \to N \) and \( h'': M_0 \to N_0 \) that are \( \eta \)-close to \( f \) and \( f|_{M_0} \), respectively. By Z-set unknotting, there is a homeomorphism \( G: N \to N \) \( \frac{4\varepsilon}{5} \)-close to the identity and such that \( G h'|_{M_0} = h'' \). Put \( h = G h' \). □

Lemma 2.5. Let \( f: (M, M_0) \to (N, N_0) \) be a CE-map between compact Q-manifold pairs such that \( f^{-1}(N_0) = M_0 \). Then there is a homotopy \( H: (M, M_0) \times I \to (N, N_0) \) such that \( H_0 = f \) and \( H_1 \) is a homeomorphism \( (M, M_0) \to (N, N_0) \).

**Proof.** Let \( \mathcal{U} \) be the normal cover for the open subset \( N \times [0,1) \) of \( N \times [0,1] \). Let \( \mathcal{V} \) be a refinement of \( \mathcal{U} \) with the property that \( \text{St}^4(\mathcal{V}) \) also refines \( \mathcal{U} \). Observe that for every \( t \in [0,1) \) there is \( 1 > \varepsilon_t > 0 \) with the property: if \( x, y \in N \) are such that \( d(x, y) < \varepsilon_t \) then \( (x, t) \) and \( (y, t) \) are in the same member of \( \mathcal{V} \).
where $\mathcal{V}$ refines $\mathcal{V}$ and any two $\mathcal{V}$-close maps into $N \times [0,1)$ are $\mathcal{V}$-homotopic. Clearly, we can define a continuous map $\varepsilon : [0,1) \to (0,1)$ such that $\varepsilon(t) < \varepsilon_t$ for every $t \in [0,1)$.

Since groups of homeomorphisms $\mathcal{V}(M,N)$ and $\mathcal{V}(M_0,N_0)$ are locally contractible [14], and CE-maps between compact $Q$-manifolds can be uniformly approximated by homeomorphisms [15], using Lemma 2.4, we can find arcs $h_t$ and $g_t$ such that $h_t = f$, $g_t = f|M_0$, $h_t \in \mathcal{V}(M,N)$ and $g_t \in \mathcal{V}(M_0,N_0)$ for all $t \neq 1$, $d(h_t, f) < \varepsilon(t)/2$, $d(g_t, f|M_0) < \varepsilon(t)/2$, and $g_0 = h_0|M_0$.

Define $H : M \times I \to N \times I$ by $H(m,t) = (h_t(m), t)$ and $G : M_0 \times I \to N_0 \times I$ by $G(m_0, t) = (g_t(m_0), t)$. Observe that $H|_{M_0 \times [0,1)}$ and $G|_{M_0 \times [0,1)}$ are two proper embeddings onto $Z$-sets in $N \times [0,1)$ that are $\mathcal{V}$-close and therefore $\mathcal{V}$-homotopic. If $\mathcal{V}$ is small enough this homotopy will be proper. By $Z$-set unknotting there is a homeomorphism $F : N \times [0,1) \to N \times [0,1)$ limited by $\text{St}^b(\mathcal{V})$ such that $F|_{M_0 \times [0,1)} = G|_{M_0 \times [0,1)}$. Then

$\tilde{H} = \pi_N \circ F^* \circ H : M \times I \to N \times I \to N \times I \to N$

is the required homotopy, where $F^*$ is a natural extension of $F$. \hfill $\Box$

Lemma 2.6. Let $(N, N_0) \subset (M, M_0)$ be a compact
Q-manifold pair in a Q-manifold pair \((M, M_0)\) such that 
\(N\) is a Z-set in \(M\) and \(N_0\) is a Z-set in \(M_0\). Then
there is a collar \(\varphi: N \times (0,1) \to M\) of \(N\), i.e., an open embedding with \(\varphi|_{N \times 0} = \text{id}\), such that \(\varphi(N_0 \times [0,1]) \subseteq M_0\).

**Proof.** Since \(N\) is a Z-set Q-manifold in \(M\), there
is an open embedding \(\varphi': N \times [0,2) \to M\) such that \(\varphi'(n,0) = n\) for every \(n \in N\). Now we are going to change \(\varphi'\) so as to
achieve the required \(\varphi\). Observe that \(N \cup \varphi'(N_0 \times [0,1])\)
is a Z-set in \(M\). Indeed, \(\varphi'(N_0 \times (0,2))\) is a Z-set in
\(\varphi'(N \times (0,2))\) because \(N_0\) is a Z-set in \(N\); hence a closed
subset \(\varphi'(N_0 \times [1/4,1])\) of \(\varphi'(N_0 \times (0,2))\) is a Z-set in
\(\varphi'(N \times (0,2))\) which is an open subset of \(M\) so that
\(\varphi'(N_0 \times [1/4,1])\) is a Z-set in \(M\). Then
\(N \cup \varphi'(N_0 \times [0,1]) = N \cup \bigcup_{i=1}^{p} (\varphi'(N_0 \times [\frac{1}{4},1]))\) is a Z-set in
\(M\).

Since \(N_0\) is a Z-set Q-manifold in \(M_0\) there is an
open embedding \(\varphi'': N_0 \times [0,2) \to M_0\) such that \(\varphi''(n_0,0) = n_0\)
for every \(n_0 \in N_0\). Now, \(i_1 = \text{id}_N \cup (\varphi'|_{N_0 \times [0,1]})\)
and \(i_2 = \text{id}_N \cup (\varphi''|_{N_0 \times [0,1]})\) are two embeddings of
\(N \cup (N_0 \times [0,1])\) onto Z-sets in \(M\) that are homotopic (both
are homotopic to \(\text{id}_N\); simply collapse collars onto
\(N_0 \times \{0\} \subseteq N\)). By Z-set unknotting, there is \(G: M \to M\)
such that \(G \circ i_1 = i_2\). Then \(G \circ \varphi': N \times [0,1) \to M\) is our
\(\varphi\). \(\square\)
Using Lemmas 2.2 - 2.6, we can now prove the other half of Theorem 2.1 following step by step Chapman-Siebenmann's proof in [17].

For the convenience of the reader we will outline this proof in some detail since rather non-trivial technical adjustment must be performed.

Suppose that a Q-manifold pair, \((M,M_0)\), admits a boundary \((B,B_0)\) and let \((N,N_0)\) be a compact Q-manifold pair that compactifies \((M,M_0)\). With the help of Lemmas 2.3 and 2.5 (applied to the projection \((N,N_0) \times I \rightarrow (N,N_0)\)) we may replace \((N,N_0)\) by \((N,N_0) \times I\) and assume that \((B,B_0) \subseteq (N,N_0) \times \{0\}\). With this notation, we must prove that \((N,N_0) \times [0,1] - (B,B_0)\) is homeomorphic to some \(\text{Map}(\sigma,\sigma_0) \times Q\).

Since \((N,N_0)\) can be triangulated (Lemma 2.2), we can write \((B,B_0) = \cap_{i=1}^{\infty} (M_i,M_{0i})\), where

\[(N,N_0) \times \{0\} \supset (M_1,M_{01}) \supset (M_2,M_{02}) \supset \cdots\]

is a basis of compact Q-manifold pair neighborhoods of \((B,B_0)\) in \((N,N_0) \times \{0\}\). Let \(\varphi_i : M_i \times [0,1) \rightarrow N \times [0,1)\) be a collar on \(M_i\) such that \(\varphi_i(M_0i \times [0,1]) \subseteq N_0 \times [0,1)\). If the image of \(\varphi_i\) is close to \(M_i\), then \((N_1,N_{01}) = (\varphi_i(M_1 \times [0,1/2]), \varphi_i(M_0i \times [0,1/2]))\) will be a decreasing sequence of compact Q-manifold pair neighborhoods of \((B,B_0)\) in \((N,N_0) \times I\) such that
\[(B, B_0) = \cap_{i=1}^{\infty}(N_i, N_{0i})\, ,\]

each \((N_i, N_{0i}) = (\partial N_i, \partial N_{0i}) \times [0,1]\), and \((\partial N_i, \partial N_{0i})\) is a compact \(Q\)-manifold pair collared in both \((N_i, N_{0i})\) and \((N, N_0) \times I - \text{int}(N_{i}, N_{0i})\).

Choose compact polyhedral pairs \((X'_0, X'_{00})\), \((X'_1, X'_{01})\), \ldots

and homeomorphisms

- \(h'_0: (X'_0, X'_{00}) \times Q \rightarrow (N, N_0) \times [0,1] - \text{int}(N_{1}, N_{01})\)
- \(h'_1: (X'_1, X'_{01}) \times Q \rightarrow \delta(N_1, N_{01})\)
- \(h'_2: (X'_2, X'_{02}) \times Q \rightarrow \delta(N_2, N_{02})\)

We will construct polyhedral pairs \((X_i, X_{0i})\) and maps \(f_i: (X_{i+1}, X_{0i+1}) \rightarrow (X_i, X_{0i})\) such that if \((\sigma, \sigma_0) = \{X_i, X_{0i}, f_i\}\)
then \(\text{Map}(\sigma, \sigma_0) \times Q = (N, N_0) \times [0,1] - (B, B_0)\).

First, define \(f'_0: (X'_1, X'_{01}) \rightarrow (X'_0, X'_{00})\) and \(f'_i: (X'_{i+1}, X'_{0i+1}) \rightarrow (X'_i, X'_{0i})\) \((i > 0)\) so that the following rectangles commute up to homotopy.

\[
\begin{array}{ccc}
(X'_1, X'_{01}) \times Q & \rightarrow & \delta(N_1, N_{01}) \\
\downarrow f_0 \times \text{id} & & \downarrow h_0 \\
(X'_0, X'_{00}) \times Q & \rightarrow & (N, N_0) \times [0,1] - \text{int}(N_{1}, N_{01}) \\
\end{array}
\]

\[
\begin{array}{ccc}
(X'_{i+1}, X'_{0i+1}) \times Q & \rightarrow & \delta(N_{i+1}, N_{0i+1}) \\
\downarrow f'_i \times \text{id} & & \downarrow h_i \\
(X'_i, X'_{0i}) \times Q & \rightarrow & \delta(N_i, N_{0i}) \\
\end{array}
\]
Here $x_i$ is a composition $\delta(N_{i+1}, N_{0i+1}) \hookrightarrow (N_i, N_{0i})$ -\text{ int}(N_{i+1}, N_{0i+1}) \xrightarrow{\lambda_i} \delta(N_i, N_{0i})$ where $\lambda_i$ is a homotopy inverse of the inclusion $\delta(N_i, N_{0i}) \hookrightarrow (N_i, N_{0i})$ -\text{ int}(N_{i+1}, N_{0i+1})$. This inclusion is in fact relatively homotopic to a homeomorphism as is seen from the commutative diagram below using Lemma 2.5.

\[
\begin{array}{ccc}
\delta(N_i, N_{0i}) & \hookrightarrow & (N_i, N_{0i}) - \text{ int}(N_{i+1}, N_{0i+1}) \\
\downarrow \alpha & & \downarrow \beta \\
(N_i, N_{0i}) & &
\end{array}
\]

The maps $\alpha$ and $\beta$ are natural collapses of closed collars onto bases and thus are CE-maps.

Let $X_i$ be the mapping cylinder of $X_{0i} \hookrightarrow X'_i$ and let $X_{0i}$ be the top of $X_i$. Let $f_i: (X_{i+1} \times X_{0i+1}) \rightarrow (X_i, X_{0i})$ be defined level-wise as $f'_i$. This change has been made to ensure that $\text{Map}(X_{0i}, f_i |_{X_{0i+1}})$ is a Z-set in $\text{Map}(X_i, f_i)$.

Observe that for each $i$, the diagram
commutes ($c_i$'s are collapses of mapping cylinders onto bases) and, by Lemma 2.5, $c_i$ is relatively homotopic to a homeomorphism $h_i':(X_i,X_{0i}) \times Q \to (X_i',X_{0i}') \times Q$ (note that $c_i|_{X_{0i}}:X_{0i} \to X_{0i}'$ is a CE-map so that proof of Lemma 2.5 goes through without the assumption $f^{-1}(N_0) = M_0$).

Now, define $h_i = h_i' \circ h_i''$ for each $i \geq 0$. Then all rectangles commute up to homotopy if we replace primed letters with those without primes.

The rest of the proof is essentially as in [17]. We repeat the key steps here.

Consider the relative infinite mapping cylinder $M(\sigma,\sigma_0)$ and write $\text{Map}(\sigma) = A_0 \cup A_1 \cup \cdots$ and $\text{Map}(\sigma_0) = A_{00} \cup A_{01} \cup \cdots$ as in the picture

Observe that the added $[0,1]$ direction in each $X_i$ makes each $A_{0i}$ a Z-set in $A_i$.

The inclusion $(X_0,X_{00}) \times Q \subseteq (A_0,A_{00}) \times Q$ is rela-
tively homotopic to a homeomorphism because its relative homotopy inverse, a collapse \((A_0, A_{00}) \times Q \rightarrow (X_0, X_{00}) \times Q\), is homotopic to a homeomorphism by Lemma 2.5. Using \(h_0\) we can get a homeomorphism \(g_0:(A_0, A_{00}) \times Q \rightarrow (N, N_0) \times [0,1] - \text{int}(N_1, N_{01})\) which by Lemma 2.3 can be adjusted so that \(g_0|_{X_1 \times \{1/2\} \times Q}\) is given by \(h_1\). In a similar way, we define

\[
g_i:(A_i, A_{0i}) \times Q \rightarrow (N_i, N_{0i}) - \text{int}(N_{i+1}, N_{0i+1})
\]

that match nicely. □

We close this chapter by stating Felt’s generalization [23] of Theorem 1.4.

**Theorem 2.7.** Two Z-set pairs \((X, X_0)\) and \((Y, Y_0)\) in \((Q, W)\) have the same shape in the sense of Mardesić and Segal if and only if \((Q-X, W-X_0)\) and \((Q-Y, W-Y_0)\) are homeomorphic as pairs.
CHAPTER III
SHAPE DIMENSION

In this chapter, we shall prove several properties of shape dimension. Our proofs use the absolute version of Theorem 2.1.

Let \( n \geq 0 \) be an integer. A locally compact separable metric space \( M \) is tame- (n-tame) at \( -\infty \) provided, that for every compactum \( A \subseteq M \), there is a compactum \( B \supseteq A \) and a (an n-dimensional) finite complex \( K \) such that the inclusion \( M-B \hookrightarrow M-A \) factors up to homotopy through \( K \). The notion of tame-at-\( -\infty \) space is due to Chapman and Siebenmann [17].

Lemma 3.1. Let \( M \) and \( N \) be spaces such that \( N \) homotopy dominates \( M \) at \( -\infty \). If \( N \) is tame- (n-tame) at \( -\infty \), then so is \( M \).

Proof. Let \( f: N \to M \) and \( g: M \to N \) be proper maps such that \( f \circ g \) is homotopic-at-\( -\infty \) to \( \text{id}_M \); i.e., for any compactum \( A \subseteq M \), there is a compactum \( A' \supseteq A \) and a homotopy \( H: (M-A') \times I \to M-A \) such that \( H_0 \) is the inclusion
M-A' \hookrightarrow M-A \text{ and } H_1 = f \circ g |_{M-A'} . \text{ Let } B' \supset f^{-1}(A') \text{ be a compactum in } N \text{ for which there is a (an n-dimensional) complex } K \text{ such that for some maps } \alpha', \beta' 

\begin{align*}
\begin{array}{ccc}
N-B' & \xleftarrow{\alpha'} & N-f^{-1}(A') \\
\downarrow & & \downarrow \\
K & & \beta'
\end{array}
\end{align*}

\text{homotopy commutes. Put } B = g^{-1}(B') , \alpha = \alpha' \circ (g |_{M-B}) , \text{ and } \beta = (f |_{N-f^{-1}(A')}) \circ \beta'. \text{ Clearly, the diagram} 

\begin{align*}
\begin{array}{ccc}
M-B & \xleftarrow{\alpha} & M-A \\
\downarrow & & \downarrow \\
K & & \beta
\end{array}
\end{align*}

\text{homotopy commutes, i.e., } M \text{ is tame- (n-tame) at- } \infty \text{. [1]}

A shape dimension \( Sd(X) \) of a compactum \( X \) is defined as \( \min \{ \dim Y | \text{Sh}(Y) \geq \text{Sh}(X) \} \), where \( \text{Sh}(Y) \geq \text{Sh}(X) \) means that \( Y \) shape dominates \( X \), a notion first introduced by Borsuk [5].

Our next theorem gives a new characterization of the inequality \( Sd(X) \leq n \) in terms of homotopy properties of the complement \( M = Q-X \) where \( X \) is considered as a \( Z \)-set in the Hilbert cube \( Q \).
**Theorem 3.2.** For a Z-set \( X \) in \( Q \), \( \text{Sd}(X) \leq n \) if and only if \( M = Q - X \) is n-tame-at-\( \infty \).

**Proof.** Suppose \( X \) is a Z-set in \( Q \) and \( \text{Sd}(X) \leq n \). Then there is an \( n \)-dimensional compactum \( Y \) such that \( \text{Sh}(Y) \geq \text{Sh}(X) \). We can represent \( Y \) as the inverse limit \( \lim_{\rightarrow} \sigma \) of an inverse sequence \( \sigma = \{ P_i, f_i \} \) where \( P_1 = \) point and \( \dim P_i \leq n \), for every \( i > 0 \). The infinite mapping cylinder \( \text{Map}(\sigma) \times Q \) pictured below

![Diagram](image)

when compactified by adding \( (\lim_{\rightarrow} \sigma) \times Q \) is homeomorphic to \( Q \) by the Cylinder Completion Theorem 1.5 and 
\( (\lim_{\rightarrow} \sigma) \times Q \) is a Z-set in \( (\text{Map}(\sigma) \cup \lim_{\rightarrow} \sigma) \times Q \cong Q \). By Theorem 1.2, since \( \text{Sh}(Y) = \text{Sh}(\lim_{\rightarrow} \sigma \times Q) \geq \text{Sh}(X) \), \( \text{Map}(\sigma) \times Q \) weak proper homotopy dominates \( M \). Hence, if we prove that \( \text{Map}(\sigma) \times Q \) is n-tame-at-\( \infty \) it would follow from Lemma 3.1 that \( M \) is n-tame-at-\( \infty \).

Let \( A \subseteq \text{Map}(\sigma) \times Q \) be an arbitrary compactum. Pick \( k \) large enough so that \( A \subseteq \text{Map}(\{ P_1 \leftarrow P_2 \leftarrow P_k \leftarrow P_{k-1} \}) \times Q = B \). We will prove that there is a homotopy \( g_t: V \rightarrow V \), with
\( V = \text{Map}(\sigma) \times Q - B \), such that \( g_0 = \text{id} \) and \( \dim g(V) \leq n \).

\[
\begin{align*}
&d_t \\
&\xymatrix{
PK & PK+1 & PK+2 \\
| & \ar@{.>}[r] & \\
PK & PK+1 & PK+2 \\
| & \ar[r]_{i_2} & \\
PK & PK+1 & PK+2 \\
| & \ar[r]_{i_1} & \\
PK & PK+1 & PK+2 & \lim(a_k) \\
| & \ar[d]_{h} & \\
PK & PK+1 & PK+2 & Q \\
\end{align*}
\]

Let \( d_t \) be the strong deformation retraction of \( V \) onto \( \text{Map}_k(\sigma) \times Q \) that slides \( P_k \times [0,1) \) onto \( P_k \times \{0\} \). In the picture, \( \gamma \) is a homotopy inverse of \( i_1 \). It
exists since \( \lim_\leftarrow (P_k \leftarrow P_{k+1} \leftarrow \cdots) \times Q \) is a Z-set in 

\([\text{Map}_k(\sigma) \cup \lim_\leftarrow (\{P_k \leftarrow P_{k+1} \leftarrow \cdots\})] \times Q \). By \( e_t \) we denoted a homotopy connecting \( \text{id} \) and \( \gamma_0 i_1 \). The map \( h \) is a homeomorphism of the latter space onto \( P_k \times Q \) (again, we used the Cylinder Completion Theorem 1.5) while \( \lambda_t \) is a map \( \lambda_t(x,q) = (x,(1-t)q) \). Then our \( g_t \) is defined by

\[
g_t = \begin{cases} 
    d_{3t} & , 0 \leq t \leq 1/3 \\
    i_2^0 e_3(t-1/3)^0 d_1 & , 1/3 \leq t \leq 2/3 \\
    i_2^0 \gamma^0 h^{-1} \lambda_3(t-2/3)^0 h_0 i_1^0 d_1 & , 2/3 \leq t \leq 1 
\end{cases}
\]

Observe that by the Mapping Replacement Theorem 1.1 we can assume \( \gamma \) is an embedding so that condition \( \dim g_1(V) \leq n \) indeed holds.

Conversely, suppose a Z-set \( X \) has the property that \( M = Q - X \) is \( n \)-tame-at-\( \infty \). Then we can find a sequence 

\( M = V_1 \supset M = V_2 \supset V_3 \supset V_4 \supset \cdots \) of open subsets of \( M \) with compact complements and \( \bigcap_{i>0} V_i = \emptyset \) such that for some sequence \( K_1 = \text{point}, K_2, K_3, K_4, \ldots \), where \( K_2, K_3, \ldots \) are \( n \)-dimensional finite polyhedra we can form a commutative diagram

\[
\begin{array}{ccc}
V_1 & \leftarrow & V_2 & \leftarrow & V_3 & \leftarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
K_1 & \leftarrow & K_2 & \leftarrow & \cdots \\
\end{array}
\]

In Appendix II of [17] it was shown that under these
conditions there is a proper homotopy equivalence
\[ \text{Map}(\sigma) \times Q \to M, \] where \( \sigma = \{K_1 \to K_2 \to \cdots\} \). Since both
\( M \) and \( \text{Map}(\sigma) \times Q \) are contractible Q-manifolds admitting
boundaries (as defined in [17], see §2) by [17, Theorems 7
and 9], \( M \cong \text{Map}(\sigma) \times Q \) and \( \text{Sh}(\lim_\sigma \times Q) = \text{Sh}(\lim_\sigma) = \text{Sh}(X) \). But, \( \dim(\lim_\sigma) \leq n \) so that \( \text{Sd}(X) \leq n \). □

**Note.** It is clear from the above proof that a Z-set
\( X \) in \( Q \) has a shape dimension \( \leq n \) if and only if \( X \) has
arbitrary small Q-manifold nbds of the form \( K \times Q \) where
\( K \) is an at most \( n \)-dimensional finite complex. Therefore,
the intersection \( X \) of Z-sets \( X_1 \supset X_2 \supset \cdots \) in \( Q \) with
\( \text{Sd}(X_k) \leq n \) for every \( k = 1, 2, \cdots \) also satisfies \( \text{Sd}(X) \leq n \)
([39, Theorem (3.1)]).

**Corollary 3.3.** If \( \text{Sd}(X) = n \) then there is an
\( n \)-dimensional compactum \( Y \) such that \( \text{Sh}(X) = \text{Sh}(Y) \).

**Proof.** Consider \( X \) as a Z-set in \( Q \). The equality
\( \text{Sd}(X) = n \) by the above theorem implies that \( M = Q-X \) is
\( n \)-tame-at-\( \infty \). With the notation from the second half of
the proof for Theorem 3.2, since \( \dim K_i = n \) for every
\( i > 1 \), \( \dim(\lim_\sigma) \leq n \). Then \( Y = \lim_\sigma \) is the required
compactum because \( \dim Y < n \) is impossible. □

**Corollary 3.3 was earlier proved by Holsztyński (un-
published) and Nowak [39]. Also included in the proof of
Theorem 3.2 is the following version of a result of Nowak**
Corollary 3.4. Let $X \subset Q$ be a Z-set. (a) If for every neighborhood $U$ of $X$ in $Q$ there is a homotopy $\varphi: X \times I \to U$ such that $\varphi_0$ is the inclusion $X \subset U$ and $\dim \varphi_1(X) \leq n$, then $\text{Sd}(X) \leq n$. (b) If $\text{Sd}(X) \leq n$, then for every neighborhood $U$ of $X$ in $Q$, there is a smaller neighborhood $V$ and a deformation $\varphi: V \times I \to V$ with $\dim \varphi_1(V) \leq n$.

Proof. (a) The method in [7] can be applied to get a closed prism neighborhood $V \subset U$ of $X$ and a homotopy $\psi: V \times I \to U$ such that $\psi_0$ is the inclusion $V \subset U$ and $\dim \psi_1(V) \leq n$. We claim that there is no loss of generality to assume $\psi_1(V)$ is an at most $n$-dimensional finite complex $K$. Indeed, let $\epsilon > 0$ be such that any map $f: \psi_1(V) \to U$ $\epsilon$-close to the inclusion $i: \psi_1(V) \subset U$ is homotopic in $U$ with $i$. Since $\dim \psi_1(V) \leq n$ there is an at most $n$-dimensional finite complex $K$ and an $\epsilon$-map $g$ of $\psi_1(V)$ onto $K$. By [31], we can find an embedding $h: K \to U$ with $d(h \circ g, i) < \epsilon$. The choice of $\epsilon$ gives us a homotopy $\chi: \psi_1(V) \times I \to U$ with $\chi_0 = i$ and $\chi_1(\psi_1(V)) = h \circ g \circ \psi_1(V)$. Then
will fulfill our requirement.

To prove (a), it suffices to see that $M = Q - X$ is $n$-tame-at-$\infty$. Let $A \subseteq M$ be a compactum. The complement $U = Q - A$ is a neighborhood of $X$ in $Q$. Select $V \subseteq U$ and $\psi : V \times I \to U$ as above. Then $Q - \text{int} V$ is a compactum in $M$ and the inclusion $M - B = \text{int} V - X \subseteq M - A = U - X$ factors through $K$. To see this, recall [12], that because $X$ is a Z-set in $Q$, there is a homotopy $g_t : U \to U$ such that $g_0 = \text{id}_U$ and $g_t(U) \cap X = \emptyset$ for all $t > 0$.

If we put $f_t(v) = g_t \circ \psi_t \circ i_1(v)$ for every $v \in \text{int} V - X$, then $f_0 = \text{the inclusion \text{int} V - X \subseteq U - X}$, and $f_1 = (g_1 \circ j \circ (\psi_1 \circ i_1))$.

(b) was demonstrated in the first half of the proof for Theorem 3.2. \[\square\]

For a topological space $X$ by a deformation dimension
d dim X we mean the smallest integer n such that every map f: X \to K of X into a CW complex K is homotopic to a map g: X \to K^n of X into the n-skeleton K^n of K. One can easily verify that the deformation dimension is a shape invariant. Here we give a new proof that, for compactum X, d dim X = Sd(X) (see [18, Corollary 2.4] and [39, Corollary (1.7)]).

**Corollary 3.5.** If X is a compact metric space, then d dim X = Sd(X).

**Proof.** (a) d dim X \leq Sd(X). Assume Sd(X) = n. We must show that, given an arbitrary map f: X \to K of X into any CW-complex K, there is g: X \to K^n with f \simeq g.

Regard X as a Z-set in Q and select a compactum A \subseteq M = Q-X such that there is an extension \tilde{f}: (M-A) \cup X \to K. By Theorem 3.2, we can find a compactum B \supset A in M and an n-dimensional finite complex L such that the diagram below homotopy commutes.

```
\begin{tikzcd}
L \\
M-B \\
(M-B) \cup X \\
X \\
K
\end{tikzcd}
```

\begin{align*}
\alpha & \quad \beta \\
i_4 & \quad i_3 \\
j_4 \quad j_3 \\
(M-B) \cup X & \quad (M-A) \cup X
\end{align*}

\begin{align*}
i_2 & \quad i_1 \\
f & \quad \tilde{f}
\end{align*}
Here, \( j_3 \) and \( j_4 \) are homotopy inverses of \( i_3 \) and \( i_4 \), respectively, (they exist because \( X \) is a \( Z \)-set in \( Q \)). The composition \( F = \tilde{\varphi} \circ i_3 \circ \beta : L \to K \) is a map of an \( n \)-dimensional finite complex into \( K \). It follows from the Cellular Approximation Theorem [32] that there is \( G : L \to K^n \) with \( F \simeq G \). Define \( g : X \to K \) as \( G \circ \alpha \circ j_4 \circ i_2 \).

(b) \( Sd(X) \leq d \dim X \). Assume \( d \dim X = n \). Regarding \( X \) as a \( Z \)-set in \( Q \), we will show that \( M = Q - X \) is \( n \)-tame-at-\( \infty \), which suffices by Theorem 3.2.

Let \( A \subset M = Q - X \) be an arbitrary compactum. Pick a closed prism neighborhood \( P \subset (M - A) \cup X \) of \( X \) and let \( K \) be a finite complex, and \( f : K \to P \), \( g : P \to K \) maps such that \( f \circ g = \text{id}_P \). By assumption, \( g|_X : X \to K \) is homotopic to a map \( \tilde{g} : X \to K^n \). Let \( \varphi : X \times I \to K \) be a homotopy connecting \( g|_X \) and \( \tilde{g} \) and let \( N \subset P \) be an open neighborhood of \( X \) for which there is an extension \( \tilde{g}' : N \to K^n \) of \( \tilde{g} \). Then, on a closed subset \( T = N \times \{0,1\} \cup X \times [0,1] \) of \( N \times [0,1] \) we can define a map \( \hat{\varphi} \) into \( K \) by

\[
\hat{\varphi}(x,t) = \begin{cases} 
  g(x), & \text{if } t = 0 \text{ and } x \in N \\
  \tilde{g}'(x), & \text{if } t = 1 \text{ and } x \in N \\
  \varphi(x,t), & \text{if } t \in [0,1] \text{ and } x \in X.
\end{cases}
\]

Since \( K \) is an ANR, there is a neighborhood \( T' \) of \( T \) in \( N \times [0,1] \) and an extension \( \hat{\varphi} : T' \to K \) of \( \hat{\varphi} \). Let \( B \subset M \) be a compactum with the property
[(M-B) ∪ X] × [0,1] ⊆ T'. Hence, \( \tilde{g}'(M-B) \subseteq K^n \). Finally, if \( Y: (M-A) ∪ X \to M-A \) is a homotopy inverse of the inclusion \( M-A \to (M-A) ∪ X \), then the diagram

\[
\begin{array}{ccc}
(M-A) ∪ X & \xrightarrow{f} & K \\
\downarrow & & \downarrow \tilde{g}' |_{M-B} \\
M-A & \xleftarrow{p} & M-B
\end{array}
\]

homotopy commutes, proving that \( M \) is \( n \)-tame-at-\( \infty \). □

Our Theorem 3.2 can be conveniently formulated as follows. The advantage of this form is that questions about a shape dimension of a \( Z \)-set \( X \subseteq Q \) are reduced to the questions about \( n \)-domination of the complement \( Q-X \) in the proper category.

Recall that every \( Q \)-manifold \( M \) can be written as \( P \times Q \), where \( P \) is a locally finite countable simplicial complex [13]. We will define a formal dimension \( d(M) \) of \( M \) as \( \min \{ \dim P | P \text{ is a polyhedron and } P \times Q \approx M \} \).

**Corollary 3.6.** (a) If for a \( Z \)-set \( X \subseteq Q \), \( \text{Sd}(X) \leq n \), then \( \text{for-d}(Q-X) \leq n+1 \). (b) Equality \( \text{for-d}(Q-X) = n+1 \) implies \( \text{Sd}(X) = n \).

**Proof.** If \( \text{Sd}(X) = n \), then there is, by Corollary 3.3, an \( n \)-dimensional compactum \( Y \) such that \( \text{Sh}(Y) = \text{Sh}(X) \).
We can represent $Y$ as $\lim \sigma$ where $\sigma = \{P_1 \leftarrow P_2 \leftarrow \cdots\}$ is an inverse sequence of compact polyhedra with $P_1 =$ point and $\dim P_i \leq n$, for all $i > 1$. But $\text{Map}(\sigma) \times Q \simeq Q - X$ and, obviously, for $-d(\text{Map}(\sigma) \times Q) \leq n+1$.

On the other hand, if $-d(Q - X) = n+1$, then the assumption $Sd(X) \leq n$ would lead to a contradiction since then $\text{Sh}(X) = \text{Sh}(\lim \sigma)$, where $\sigma$ is an inverse sequence of $(n-1)$-dimensional polyhedra beginning with the point, and $Q - X \simeq \text{Map}(\sigma) \times Q$ so that for $-d(Q - X) \leq (n-1) + 1 = n$. 

Here is a method showing how to use the above result in studying the properties of shape dimension. Take an arbitrary polyhedron $P$ for which $P \times Q \simeq Q - X$. Suppose we established that $P$ is proper homotopy dominated by an $n$-dimensional polyhedron $P'$. We claim that $P \times Q$ is $n$-tame-at-$\infty$. Since $P'$ proper homotopy dominates $P \times Q$ for any compactum $A \subset P \times Q$ there is a compactum $B \supset A$, a subpolyhedron $P''$ of $P$, and maps $f : P'' \to P \times Q - A$, $g : P \times Q - B \to P''$ such that

\[
\begin{array}{ccc}
  P'' & \xrightarrow{f} & P \times Q - A \\
  \downarrow{g} & & \downarrow{g} \\
  P \times Q - B & \xleftarrow{f} & P \times Q - A
\end{array}
\]

homotopy commutes. But we can assume that $B$ is a $Q$-manifold such that $\text{Bd} B$ is a collared $Q$-manifold both
in B and \( P \times Q - \text{int} B \) so that \( P \times Q - B \) is homotopy equivalent to a finite complex \( K \), i.e., there are maps \( \psi: P \times Q - B \to K \) and \( \varphi: K \to P \times Q - B \) with \( \varphi \circ \psi = \text{id}_{P \times Q - B} \). Since \( \varphi \circ \psi(K) \) is compact, there is a finite polyhedron \( P'' \) of \( P' \) containing \( \varphi \circ \psi(K) \). Then the factorization

\[
\begin{array}{ccc}
P'' & \xrightarrow{g \circ \varphi \circ \psi} & K \\
P \times Q - B & \xleftarrow{\text{id}_{P \times Q - B}} & P \times Q - A \\
& f \mid_{P''} & \\
\end{array}
\]

supports our claim.

S. Nowak [39] gave a rather complicated proof of the next proposition. The proof we present shows some merits of our approach to questions of shape.

**Proposition 3.7.** Let \( X_1, X_2 \) be compacta. Then

\[
\text{Sd}(X_1 \cup X_2) \leq \max(\text{Sd}(X_1), \text{Sd}(X_2), \text{Sd}(X_1 \cap X_2) + 1).
\]

**Proof.** Denote the integers appearing in the above inequality by \( k, l, m, n \), respectively. Hence, we must prove \( k \leq \max(l, m, n) \). Consider \( X = X_1 \cup X_2 \) as a Z-set in \( Q \times [-1,1] \) with \( X_1 \subseteq Q \times [-1,0] \), \( X_2 \subseteq Q \times [0,1] \), and \( X \cap (Q \times \{0\}) = X_0 = X_1 \cap X_2 \) being a Z-set in \( Q \times \{0\} \). Since \( \text{Sd}(X_1) = l \), \( \text{Sd}(X_2) = m \), there are, by Theorem 3.2, arbitrary small closed \( Q \)-manifold neighborhoods \( N_1 \) of \( X_1 \) in \( Q \times [-1,0] \) and \( N_2 \) of \( X_2 \) in \( Q \times [0,1] \) such that
$N_1 = K_1 \times Q$, $N_2 = K_2 \times Q$, where $K_1$ and $K_2$ are finite complexes of dimensions $t$ and $m$, respectively. Pick a closed $Q$-manifold neighborhood $N_0 \subset N_1 \cap N_2$ of $X_0$ in $Q \times \{0\}$ that is homeomorphic to $K_0 \times Q$ for some $(n-1)$-dimensional finite complex $K_0$. But $N_0$ is a Z-set submanifold in both $N_1$ and $N_2$ so that, by Chapman's Relative Triangulation Theorem (see Lemma 2.2), we can find complexes and homeomorphisms $h_1 : N_1 \to K_1 \times Q$, $h_2 : N_2 \to K_2 \times Q$ extending a fixed homeomorphism $h_0 : N_0 \to K_0 \times Q$. Moreover, $\dim K_1' = \max(t, n)$ and $\dim K_2' = \max(m, n)$. Let $N \subset N_1 \cup N_2$ be any neighborhood of $X$ for which $N \cap (Q \times \{0\}) = N_0$, and let $K$ be the result of gluing $K_1'$ and $K_2'$ along $K_0$. We claim that the inclusion $N - X \subset (N_1 \cup N_2) - X$ factors through $K$. To see this, let $\lambda_t : Q \times [-1,1] \to Q \times [-1,1]$ move $Q$ off of $X$ with $\lambda_t|{(Q \times [-1,1]) - X} = \text{id}$, for all $t$. Define $\alpha : N - X \to K$ by $\alpha = p \circ h_N|_{N - X}$, where $h_N = (h_1|_{N \cap N_1}) \cup (h_2|_{N \cap N_2})$ and $p$ is the projection $K \times Q \to K$. Also, $\beta : K \to (N_1 \cup N_2) - X$ is given by $\beta = \lambda_1 \circ h_N^{-1} \circ (x0)$. One can easily check that the diagram

\[
\begin{array}{ccc}
N - X & \xrightarrow{\alpha} & (N_1 \cup N_2) - X \\
& \searrow & \downarrow \beta \\
& K & \\
\end{array}
\]

homotopy commutes, which proves that $(Q \times [-1,1]) - X$ is
max(l,m,n)-tame-at-∞, i.e., k ≤ max(l,m,n). □

Remark 3.8. If \( X_0 = X_1 \cap X_2 \) has trivial shape, then we can take \( K_0 \) to be a point and then \( Sd(X_1 \cup X_2) = \max(Sd(X_1), Sd(X_2)) \) ([39, Theorem (4.19)]).
CHAPTER IV
MOVABILITY

In this chapter, we shall first characterize movability of a Z-set $X$ in $Q$ in terms of homotopy invariants at $\infty$ of $Q-X$ as introduced in Chapter I, and then apply this apparatus to get new theorems whose corollaries include new proofs (in most cases simplified) of standard results in shape theory of compacta.

Recall Borsuk's definition of movability. A compactum $X \subset Q$ is (Borsuk) movable provided that for every neighborhood $U$ of $X$ in $Q$, there is a smaller neighborhood $V$ of $X$ such that given an arbitrary neighborhood $W$ of $X$, there is a homotopy $H:V \times I \to U$ with $H_0$ = the inclusion $V \hookrightarrow U$ and $H_1(V) \subset W$.

**Theorem 4.1.** A Z-set $X \subset Q$ is (Borsuk) movable if and only if the $Q$-manifold $M = Q-X$ satisfies the following property:

(7) For every compactum $A \subset M$, there is a compactum $B \supset A$ such that, given any map $\varphi:P \to M-B$ of a finite polyhedron $P$ into $M-B$ and any compactum $C \supset B$ there
is a map $\psi: P \to M-C$ with $\varphi = \psi$ in $M-A$.

**Proof.** Assume a Z-set $X \subset Q$ is (Borsuk) movable.

If $A \subset M = Q-X$ is an arbitrary compactum, $U = Q-A$ is a neighborhood of $X$ in $Q$. Pick an open neighborhood $V \subset U$ as in the above definition, and set $B = Q-V$. Let $\varphi: P \to M-B = V-X$ be a map of a finite complex $P$ into $V-X$ and let $C \supset B$ be a compactum. Let $H: V \times I \to U$ be a homotopy with $H_0 = \text{the inclusion } V \hookrightarrow U$ and $H_1(V) \subset W = Q-C = (M-C) \cup X$. Also, let $G: V \times I \to V$ be a deformation such that $G_t(V) \subset V-X$ for all $t > 0$. It is clear that the map $\psi = G_0 \circ H_1 \circ \varphi: P \to M-C$ is homotopic in $M-A$ to a map $G_0 \circ H_0 \circ \varphi = \varphi$.

Conversely, suppose that condition (M) holds. Let $U$ be an open neighborhood of $X$ in $Q$. For a compactum $A = Q-U$ in $M$, select a compactum $B \supset A$ as in (M) and put $V' = Q-B$. Let $V \subset V'$ be a closed prism neighborhood of $X$ in $Q$. We claim that this $V$ does the job. Indeed, take any open neighborhood $W$ of $X$ in $Q$. We can always suppose $W \subset V'$. Then $C = Q-W$ is a compactum containing $B$. Observe that $V$ can be written as $P \times Q_k$ where $P \subset I_1 \times \cdots \times I_{k-1}$ is a finite polyhedron. Let $\lambda_t: V \to V$ be a deformation for which $\lambda_t(V) \subset V-X$ for all $t > 0$.

A map $\varphi = \lambda_0: P \to V-X \subset M-B$, where $\lambda_0: P \to P \times Q_k = V$ is given by $(p,0) = (p,0)$, is homotopic via $\lambda_t: P \to M-A = U-X$ to a map $\psi = \lambda_t: P \to M-C = W-X$. Then a
homotopy $\xi_t: V \to U$ defined as

$$
\xi_t(p, q_k) = \begin{cases} 
\lambda_{2t}(p, (1-2t)q_k), & 0 \leq t \leq 1/2 \\
\chi_{2t-1}(p), & 1/2 \leq t \leq 1
\end{cases}
$$

satisfies $\xi_0 = \text{the inclusion } V \hookrightarrow U$ and $\xi_1(V) \subseteq W$. □

Definition 4.2. A non-compact locally compact separable metric space $M$ is called movable if $M$ has property $(\mathcal{M})$.

Corollary 4.3. A $Z$-set $X$ in a compact $Q$-manifold $N$ is (Borsuk) movable if and only if $M = N - X$ is movable.

Proof. We can assume $X$ is contained in $N \times \{0\} \subseteq \subseteq N \times [0,1]$. Consider $N \times \{1\}$ as a $Z$-set in $Q$ and let

$Q'$ denote the union of $N \times [0,1]$ and $Q$ along $N \times \{1\}$.

Then $Q' \cong Q$ and $N \times [0,1] - X$ is movable if and only if $Q' - X$ is movable. Now, Proposition 4.1 applies. □

Theorem 4.4. Let $M$ and $N$ be non-compact locally compact separable metric spaces. If $M$ is movable and $M$ homotopy dominates $N$ at $\infty$, then $N$ is also movable.

Proof. Let $f: M \to N$ and $g: N \to N$ be proper maps such that $f \circ g$ is homotopic-at-$\infty$ to $id_N$. If $A$ is an arbitrary compactum in $N$, there is a compactum $A^* \supset A$ and a homotopy $H: (N-A^*) \times I \to N-A$ between $id_{N-A^*}$ and $f \circ g|_{N-A^*}$. Since $M$ is movable, for a compactum $A' = f^{-1}(A^*)$ in $M$
we can select \( B' \supset A' \) such that (7) holds. Let \( B = g^{-1}(B') \cup A \). Assume \( \varphi: P \to N-B \) is a map of a finite complex into \( N-B \). For a composition \( \varphi' = g \circ \varphi: P \to M-B' \) and a compactum \( C' = f^{-1}(C) \cup B' \), where \( C \) is any compactum in \( N \) containing \( B \), by assumption, there is a map \( \psi: P \to M-C' \) homotopic to \( \varphi' \) in \( M-A' \). But, \( \psi = f \circ \psi': P \to N-C \) is then homotopic in \( N-A' \) to a map \( f \circ g \circ \varphi \) and thus to \( \varphi \) in \( N-A \). □

**Corollary 4.5 ([7]).** If \( X \) is a (Borsuk) movable compactum and \( X \) shape dominates compactum \( Y \), then \( Y \) is also (Borsuk) movable.

**Proof.** Regard \( X \) and \( Y \) as Z-sets in \( Q \). It follows from Theorem 4.1 that \( M = Q-X \) is movable and from Theorem 1.2 that \( M \) homotopy dominates \( N = Q-Y \) at \( \infty \). Hence, \( Y \) is (Borsuk) movable by Theorem 4.4 and Theorem 4.1. □

**Definition 4.6.** A locally compact separable metric space \( X \) is in the class \( \text{ANR}(\infty) \), notation: \( X \in \text{ANR}(\infty) \), if there is a compact \( \text{ANR} \), \( Y \), and a Z-set \( \text{ANR} \), \( Z \subset Y \), such that \( Y-Z \cong X \).

**Examples 4.7.** (a) By the main result in [24] a non-compact Q-manifold \( M \) is in \( \text{ANR}(\infty) \) if and only if \( M \) admits a Q-manifold boundary. This class of Q-manifolds
was completely identified in algebraic terms in [16].

(b) Sher [41] introduced the concept "docile-at-∞" for a locally compact \( X \in \text{ANR}(\mathbb{M}) \). His main result characterizes docile-at-∞ spaces \( X \) as those for which the Freudenthal compactification \( \tilde{F}X \) of \( X \) is an ANR and the space \( E_X \) of ends of \( X \) is a Z-set in \( \tilde{F}X \). A similar result was established for so called contractible-at-∞ spaces by Kozlowski [29]. In our terminology, every docile-at-∞ space with finitely many ends is in \( \text{ANR}(∞) \). Since every 0-dimensional compactum is (Borsuk) movable, spaces that are docile-at-∞ are in \( \text{BM}(∞) \) (see note following Corollary 4.8).

**Corollary 4.8.** If \( X \in \text{ANR}(∞) \), then \( X \) is movable.

**Proof.** We will present two proofs. The second is more elementary.

**First proof.** Let \( Y \) be a compact ANR and \( Z \subseteq Y \) a Z-set ANR such that \( X = Y - Z \). By [21], \( Y \times Q \) is a compact Q-manifold so that \( X \times Q \) admits a (Borsuk) movable boundary \( Z \times Q \). Hence, by Corollary 4.3, \( X \times Q \) is movable. Now, Theorem 4.4 implies that \( X \) is movable.

**Second proof.** Let \( A \subseteq X' = Y - Z \) be a compactum. Since \( Z \) is an ANR, there is a neighborhood \( U \subseteq Y - A \) of \( Z \) in \( Y \) and a retraction \( r:U \to Z \). \( Y \) being an ANR
implies the existence of $\varepsilon > 0$ such that every two $\varepsilon$-close maps into $U$ are homotopic in $Y-A$ (we might need to pass to a smaller $U$). A standard argument by contradiction shows that there is a small open neighborhood $U_\varepsilon \subseteq U$ of $Z$ in $Y$ such that the inclusion $i: U_\varepsilon \hookrightarrow U$ and $r|_{U_\varepsilon}$ are $\varepsilon$-close maps and therefore homotopic in $Y-A$ via $h_t$. A compactum $B$ that we are looking for is $Y-U_\varepsilon$. Indeed, suppose a map $\varphi: P \to X-B = U_\varepsilon - Z$ of a finite polyhedron $P$ into $X'-B$ and a compactum $C \supseteq B$ in $X'$ are given. Since $Z$ is a $Z$-set in $Y$ there is a deformation $\lambda_t: Y-C \to Y-C$ such that $\lambda_t(Y-C) \subseteq X'-C$ for all $t > 0$. Then $\psi = \lambda_1 \circ r \circ \varphi: P \to X'-C$ is homotopic in $X'-A$ to $\varphi$ via a homotopy $\lambda_t \circ h_t$. 

**Note.** It is clear from the first proof that a non-compact locally compact ANR, $X$, is movable if $X \in BM(\infty)$, i.e., if there exists a compact ANR, $Y$, and a (Borsuk) movable $Z$-set $Z \subseteq Y$ such that $X = Y-Z$.

By restricting the dimension of a polyhedron $P$ in (M) to be $\leq n$, we will get the concept of an $n$-movable locally compact space. It is easy to verify that results 4.1 - 4.5 hold when movable is replaced by $n$-movable and (Borsuk) movable with (Borsuk) $n$-movable. (This notion was studied in [8], [30].)

**Theorem 4.9.** If $M$ is an $n$-movable and $n$-tame-at-$(\infty)$
locally compact separable metric space, then $M$ is movable.

**Proof.** Let $A$ be a compactum in $M$. Pick a compactum $B' \supset A$ such that ($\mathfrak{M}$) holds for all finite polyhedra $P$ of dimension $\leq n$. Using $n$-tameness of $M$ at $\alpha$, there is a compactum $B \supset B'$ and an $n$-dimensional finite polyhedron $K$ such that the diagram

$$
\begin{array}{ccc}
M-B & \xleftarrow{\alpha} & N-B' \\
\downarrow{\beta} & & \downarrow{\gamma} \\
K & & K
\end{array}
$$

homotopy commutes. Then $B$ is the required compactum.

Indeed, let $\varphi: P \to M-B$ be an arbitrary map of a finite complex $P$ into $M-B$ and $C \supset B$ any compactum. By assumption, there is a map $\beta': K \to M-C$ with $\beta' \sim \beta$ in $M-A$ because $\dim K = n$. The map $\psi = \beta' \circ \alpha \circ \varphi: P \to M-C$ is in $M-A$ homotopic to $\beta \circ \alpha \circ \varphi$ and thus to $\varphi$. $\ □$

Theorem 4.9 trivially implies the following: Every (Borsuk) $n$-movable compactum with a shape dimension $\leq n$ is (Borsuk) movable ([8], [30]).

In [34], Mardesić proved that an $n$-dimensional $LC^{n-1}$ compactum is movable. Later on, simpler proofs of this theorem were presented in [40] and [8]. Applying an elementary lemma from [28] we are able to give a generalization
in the following form.

**Theorem 4.10.** If \( X \) is an \( LC^{n-1} \) compactum of a shape dimension \( \leq n \), then \( X \) is movable.

**Proof.** By Theorem 4.9, it suffices to prove that the complement \( M = Q - X \) is \( n \)-movable, where \( X \) is considered as a \( Z \)-set in \( Q \). Since \( X \) is \( LC^{n-1} \), the inclusion \( i : X \hookrightarrow Q \) is a strong local connection in dimension \( n-1 \) (see [28] for definitions and notation). Let \( A \subset M \) be an arbitrary compactum and \( U \subset Q - A \) an open neighborhood of \( X \) such that \( U = \{ Q - cl U \cup Q - A \} \) is an open cover of \( Q \). Pick \( \mathcal{U} \) such that \( \mathcal{U} \)-close maps into \( Q \) are \( \mathcal{U} \)-homotopic. Let \( \mathcal{V} \) be a refinement of \( \mathcal{U} \) for which the assertion \( E(\mathcal{V}, \mathcal{U}, n) \) holds ([28, Lemma 1]). In other words, given an at most \( n \)-dimensional finite simplicial complex \( K \) and a map \( h : K \to Q \) that maps every closed simplex \( \sigma \) of \( K \) into some member of the collection \( \{ V \in \mathcal{V} \mid V \cap X \neq \emptyset \} \), there is \( h' : K \to X \) such that for every simplex \( \sigma \) of \( K \) we can find \( W \in \mathcal{U} \) with \( i \circ h'(\sigma) \cup h(\sigma) \subseteq W \). The choice of \( \mathcal{U} \) assures that, then \( h \) and \( h' \) are homotopic in \( Q - A \). The compactum \( B = Q - \cup \{ V \in \mathcal{V} \mid V \cap X \neq \emptyset \} \) is the one we are looking for. Indeed, if \( C \) is a compactum in \( M \) containing \( B \), let \( \lambda_t : Q \to Q \) be a deformation with \( \lambda_t(Q) \subseteq Q - X \), \( \lambda_t(Q - A) \subseteq Q - A \), and \( \lambda_t(Q - C) \subseteq Q - C \) for all \( t > 0 \). Suppose \( h : K \to M - B \subseteq \cup \{ V \in \mathcal{V} \mid V \cap X = \emptyset \} \) is any map of an at most
n-dimensional finite simplicial complex $K$ into $M-B$. We can assume by the previous discussion that there is a map $h': K \to X$ homotopic in $M-A$ to $h$. Let $\varphi_t: K \to M-A$ be a homotopy in $Q-A$ connecting $h$ and $\lambda_t \circ h': K \to M-C$. Then $\lambda_t (1-t) \varphi_t$ is a homotopy in $M-A$ between $h$ and $\lambda_1 \circ h'$. Hence, $M$ is $n$-movable. □

**Theorem 4.11.** If $M$ is a movable locally compact separable metric space, then $M \times (0,1)$ and $M \times [0,1)$ are movable.

**Proof.** Let $A \subset M \times (0,1)$ be an arbitrary compactum. Let $B' \supseteq \pi_M(A)$ be selected so that for every map $f: P \to M-B'$ of a finite complex $P$ into $M-B'$ and a compactum $C' \supseteq B'$ there is a map $g: P \to M-C'$ with $f \sim g$ in $M-\pi_M(A)$. Then $B = B' \times B''$, where $B''$ is a compact subinterval containing $\pi_{(0,1)}(A)$, is a compactum that works. Indeed, let $\varphi: P \to M \times (0,1) - B$ be a map of a finite complex $P$ and let $C \supseteq B$ be any compactum. Open sets $V_1 = (M-B') \times (0,1)$, $V_2 = M \times ((0,1) - B'')$ and $V_0 = (M-B') \times ((0,1) - B'')$, cover $M \times (0,1) - B$. In case $\varphi(P)$ is contained completely either in $V_1$ or $V_2$, we can easily homotop $\varphi$ off of $C$ by changing just one coordinate. Hence, we can assume that $\varphi(P)$ intersects each set $V_0$, $V_1-V_0$, $V_2-V_0$. Then there is a fine subdivision $P'$ of $P$ such that for every simplex $\sigma$ of $P'$ some member of $\{V_0, V_1, V_2\}$ contains
\( \varphi(\sigma) \). By \( P'_i \) (\( i = 0,1,2 \)), we will denote a subcomplex of \( P' \) consisting of all simplices \( \sigma \) for which \( \varphi(\sigma) \subseteq V_i \).

If \( P' \) is fine enough, \( P'_1 - P'_0 \) and \( P'_2 - P'_0 \) will be non-empty and \( P'_0 \) will have non-empty interior. Let \( \rho: P \rightarrow [0,1] \) be a continuous function, that is, \( 1 \) on \( P'_1 - P'_0 \) and is \( 0 \) on \( P'_2 - P'_0 \). Let \( \lambda_t: (0,1) - B' \rightarrow (0,1) - B'' \) be a deformation with \( \lambda_1((0,1) - B'') \subseteq (0,1) - \pi(0,1)(C) \). A homotopy \( h_t \) on \( P' \) defined by

\[
\begin{align*}
\text{if } P \in P_2 & \text{ and } \text{ then } h_t(p) = \varphi(p) \quad \text{ for } p \in P_1 - P_0 \\
\text{if } P \in P_1 - P_0 & \text{ and } \text{ then } h_t(p) = \left( \pi_M \varphi(p), \lambda_t \cdot \rho(p) \pi(0,1) \varphi(p) \right)
\end{align*}
\]

Observe that \( \pi_M h_1 \mid P_1 \) maps \( P'_1 \) into \( M - B' \). Let \( g_t: P'_1 \rightarrow M - \pi_M(A) \) be a homotopy of \( \pi_M h_1 \mid P_1 \) to a map of \( P'_1 \) into \( M - \pi_M(C) \). Then \( k_t: P \rightarrow M \times (0,1) - A \) defined as

\[
k_t(p) = \begin{cases} (g_t(1 - \rho(p))) \pi(0,1) h_1(p), & P \in P'_1 \\ h_1(p), & P \in P'_2 - P'_1 \end{cases}
\]

is a homotopy between \( h_1 \) and \( k_1: P \rightarrow M \times (0,1) - C \) in \( M \times (0,1) - A \).

The factor \( (0,1) \) also admits a deformation with the properties of \( \lambda_t \) and hence the same proof applies to this case. (!)

**Corollary 4.12** (Borsuk [9]). If \( X \) is a (Borsuk) movable compactum, then the suspension, \( S(X) \), of \( X \) is also (Borsuk) movable.
Proof. Consider \( X \) a Z-set in \( Q \). It is easy to
see that \( S(X) \subseteq S(Q) \) is a Z-set in
\( S(Q) \) and that \( S(Q) \cong Q \) [27]. Hence, by Theorem 4.1,
\( S(X) \) is (Borsuk) movable provided \( S(Q) - S(X) \cong (Q-X) \times (0,1) \)
is movable. But this is the case, by Theorem 4.11, since
\( Q-X \) was assumed movable. \( \square \)

**Definition 4.13.** Let \( M \) be a locally compact separable
metric space and \( M_0 \subseteq M \) an open subset. We will say that \( M \)
is *isotopable into* \( M_0 \) if for every compactum \( B \subseteq M_0 \) and
every compactum \( D \supset B \) in \( M \) there is an invertible isotopy
\( \lambda_t : M \to M \) such that \( \lambda_1(D) \subseteq M_0 \) and \( \lambda_t|_B = id \), for all \( t \).

**Theorem 4.14.** If a locally compact separable metric
space \( M \) is the union of an increasing sequence of its
movable open subsets \( M_i, i \geq 0 \), such that \( M_i+1 \) is iso-
topable into \( M_i \), for every \( i > 0 \), then \( M \) is movable.

Proof. Let \( A \subseteq M \) be an arbitrary compactum. We can
assume \( A \subseteq M_0 \). Since \( M_0 \) is movable, there is a com-
pactum \( B \supset A \) in \( M_0 \) such that for every map \( \varphi : P - M_0 - B \)
of a finite polyhedron \( P \) into \( M_0 - B \) and a compactum
\( C \supset B \) in \( M_0 \), there is \( \psi : P \to M_0 - C \) with \( \varphi \sim \psi \) in
\( M_0 - A \). Let \( C \supset B \) be any compactum in \( M \) and \( \varphi : P \to M - B \)
a map. Pick an index \( k \) large enough so that
\( D = C \cup \varphi(P) \subseteq M_k \). For each \( k \geq j > 0 \), let \( \lambda^j_t : M_j \to M_j \)
be an invertible isotopy such that \( \lambda^j_t|_B = id \) for all \( t \).
and all \( j \), and \( \lambda^1(D) \subseteq M_{k-1} \), \( \lambda^1(D) \subseteq M_{k-2}, \ldots \), \( \lambda^1(D) \subseteq M_0 \). Now, define a homotopy \( h_t: M_k \to M_k \) by composing homotopies \( \lambda^j_t \) together. Then \( h_0 = id \), \( h_t(M_k-B) \subseteq M_k-B \) for all \( t \), and \( h_1(D) \subseteq M_0 \). Let \( g_t: P \to M_0-A \) be a homotopy between \( h_1 \circ \varphi \) and \( g_1: P \to M_0 - h_1(C) \). Then the join of homotopies \( h_t \circ \varphi \), \( g_t \), and \( h_t^{-1} \circ g_1 \) connects \( \varphi \) with the map into \( M_k-C \subseteq M-C \). □

We will say that a non-compact locally compact metric space \( M \) is flat if for any compactum \( A \subseteq M \) every map \( \varphi: P \to M \) of a finite polyhedron \( P \) into \( M \) is homotopic to a map \( \psi: P \to M-A \).

**Theorem 4.15.** Let, for each \( i > 0 \), \( X_i \) be a compact metric space and \( Y_i \subseteq X_i \) be a closed subset with \( X_i-Y_i \) movable and flat. If \( X = \prod_{i>0} X_i \) and \( Y = \prod_{i>0} Y_i \), then \( X-Y \) is movable.

**Proof.** Let \( A \) be a compactum in \( X-Y \). Then \( X-A \) is a neighborhood of \( Y \) in \( X \). Pick an index \( n \) large enough so that \( U_1 \times \cdots \times U_n \times \prod_{j>n} X_j \subseteq X-A \), where \( U_i \) \((i = 1, \cdots, n)\) is an open neighborhood of \( Y_i \) in \( X_i \).

Since each \( X_i-Y_i \) is movable, there are compacta \( B_i \subseteq X_i-Y_i \) with \( B_i \supseteq X_i-U_i \) and such that for any compactum \( C_i \supseteq B_i \) and a map \( \varphi_i: P \to (X_i-Y_i)-B_i \) of a finite polyhedron \( P \), \( \varphi_i \) is homotopic inside \( U_i-Y_i \) to a map into \( (X_i-Y_i)-C_i \). Let \( B = \bigcup_{i=1}^n (B_i \times \prod_{j \neq i} X_j) \).
Let $C \supset B$ be an arbitrary compactum in $X - Y$ and
$\varphi: P \to (X - Y) - B$ any map of a finite complex $P$. Take
small open neighborhoods $V_i$ ($i = 1, \ldots, k; k \geq n$) of $Y_i$
in $X_i$ such that $V_1 \times \cdots \times V_k \times \Pi_{i>k} X_i \subset X - C$. By
the choice of sets $B_i$, each map $\varphi_i = \pi_i \circ \varphi: P \to (X_i - Y_i) - B_i$
obtained from $\varphi$ by projecting onto $X_i$ coordinate, is,
for $i = 1, \ldots, n$, homotopic in $U_i - Y_i$ to a map
$\psi_i: P \to V_i - Y_i$, and for $i = n+1, \ldots, k$, using flatness of
$X_i - Y_i$, $\varphi_i$ is homotopic to a map $\psi_i: P \to V_i - Y_i$. Hence,$\varphi$
is homotopic outside $A$ to a map
$\psi_1 \times \cdots \times \psi_k \times \varphi_{k+1} \times \varphi_{k+2} \times \cdots : P \to (X - Y) - C$
proving that $X - Y$ is movable. □

**Corollary 4.16** [7]. The product $X = \Pi_{i>0} X_i$ of countably many compacta is (Borsuk) movable if and only if each
factor $X_i$ is (Borsuk) movable.

**Proof.** Clearly $Sh(X) \geq Sh(X_i)$ for every $i > 0$, so
that if $X$ is (Borsuk) movable, by Corollary 4.5, then $X_i$
has the same property.

Conversely, assume each $X_i$ is (Borsuk) movable and is
a $Z$-set in $Q_i \equiv Q$. Applying Theorem 4.1 and Theorem 4.15,
we see that $(\Pi_{i>0} Q_i) - (\Pi_{i>0} X_i)$ is movable. But
$\Pi_{i>0} Q_i \equiv Q$ and $\Pi_{i>0} X_i$ is a $Z$-set in $\Pi_{i>0} Q_i$. Therefore,
$X = \Pi_{i>0} X_i$ is (Borsuk) movable. □

An isolated end $e$ (see [4]) of a locally compact
metric space $X$ will be called movable if for every neighborhood $U$ of $e$, there is another neighborhood $V \subseteq U$ of $e$ such that, for every neighborhood $W$ of $e$ and a map $\varphi: P \to V \cap X$ of a finite polyhedron $P$ into $V \cap X$, there is a homotopy $h_t: P \to U \cap X$ of $\varphi$ with $h_1(P) \subseteq W \cap X$.

**Theorem 4.17.** Let $X$ be a locally compact metric space with finitely many ends $e_1, \ldots, e_n$. Then $X$ is movable if and only if each end $e_i$ is movable.

**Proof.** Let $e_1, \ldots, e_n$ be movable ends of $X$. Let $A \subseteq X$ be an arbitrary compactum. Then $(X-A) \cup e_i = U_i$ is a neighborhood of $e_i$, for every $i = 1, \ldots, n$. Pick disjoint neighborhoods $V_1, \ldots, V_n$ of $e_1, \ldots, e_n$, respectively, as in the above definition. A compactum $B = X - \bigcup_{i=1}^n V_i$ is easily seen to satisfy condition (M).

Conversely, assume $X$ satisfies condition (M). Let $U_1$ be a neighborhood of the end $e_1$ of $X$. We can find disjoint neighborhoods $U'_1, \ldots, U'_n$ of $e_1, \ldots, e_n$, respectively, with $U'_1 \subseteq U_1$ such that $A = X - \bigcup_{i=1}^n U'_i$ is a compactum in $X$. Take $B \supseteq A$ using movability of $X$. Then $V_1 = U'_1 - B$ is a neighborhood of $e_1$ such that any map of a finite polyhedron into $V_1 \cap X$ deforms inside $U_1$ to a map arbitrarily close to $e_1$. Hence, the end $e_1$ is movable. □

**Theorem 4.18.** Let $X_1$ and $X_2$ be compacta of trivial shape and $Z = X_1 \cup X_2$ with $X_0 = X_1 \cap X_2$ (Borsuk) movable.
Then $Z$ is also a (Borsuk) movable compactum.

**Proof.** Consider $K = \mathbb{Q} \times [-1,1]$ and denote $K_1 = \mathbb{Q} \times [-1,0]$, $K_0 = \mathbb{Q} \times \{0\}$, and $K_2 = \mathbb{Q} \times [0,1]$.

Embed $Z$ into $K$ in the following way. First, embed $X_0$ as a Z-set in $K_0$ and $K$. Second, extend this embedding which can be done using the Mapping Replacement Theorem 1.1, to an embedding of $X_1$ onto a Z-set in $K_1$, and finally, to an embedding of $X_2$ onto a Z-set in $K_2$. Then $Z$ will be a Z-set in $K$ and we will prove that $M = K - Z$ is movable.

At this point it is important to observe that the complement of a (Borsuk) movable Z-set in $\mathbb{Q}$ satisfies (7?) for any compactum in place of $P$ because every closed subset of $\mathbb{Q}$ has arbitrary small prism neighborhoods.

Let $A \subset M$ be a compactum and put $A_i = K_i \cap A$ for $i = 0,1,2$. Let $B_1 \supset A_1$ and $B_2 \supset A_2$ be compacta in $K_1 - X_1$ and $K_2 - X_2$, respectively, such that $(K_i - X_i) - B_i$ is contractible ($i = 1,2$). Let $B_0 \supset K_0 \cap (B_1 \cup B_2)$ be a compactum in $K_0 - Z = K_0 - X_0$ that satisfies (7?) with respect to $X_0$, $(K_0 - X_0) \cap (B_1 \cup B_2)$ in $K_0 - X_0$. Define $B$ as $B_0 \cup B_1 \cup B_2$.

Suppose $C \supset B$ is an arbitrary compactum in $M$ and $\varphi : P \rightarrow M - B$ is a map of a finite polyhedron $P$ into $M - B$. Select $D_1 \supset K_1 \cap C$ in $K_1 - X_1$ and $D_2 \supset K_2 \cap C$ in $K_2 - X_2$ so that $(K_i - X_i) - D_i$ ($i = 1,2$) are contractible.
For $i = 0,1,2$ , let $P_i = \varphi^{-1}(K_i)$ . By the above, note there is a homotopy $h_t: P_0 \to (K_0 - X_0) \cap (B_1 \cup B_2)$ with $h_0 = \varphi|_{P_0}$ and $h_1(P_0) \subset K_0 - (D_1 \cup D_2)$ . Now, extend $h_1$ to a map $h^*_{1}: P_1 \to (K_1 - X_1) - D_1$ and to a map $h^{**}_{1}: P_2 \to (K_2 - X_2) - D_2$ . Thus, we have maps of $P_1 \times \{0,1\} \cup P_0 \times I$ into $(K_1 - X_1) - D_1$ and of $P_2 \times \{0,1\} \cup P_0 \times I$ into $(K_2 - X_2) - D_2$ . Extending each of them separately will give us the required homotopy of $\varphi$ .

**Corollary 4.19.** If $X$ is a compactum of trivial shape and $X_0 \subset X$ a (Borsuk) movable closed subset, then the space $X/X_0$ , obtained from $X$ by shrinking $X_0$ to a point, is also (Borsuk) movable.

**Proof.** Embed $X_0$ as a $Z$-set in $Q \times \{0\}$ and extend this to an embedding of $X$ onto a $Z$-set in $Q \times [0,1]$ such that $X \cap (Q \times \{0\}) = X_0$ . Let $C(Q) = \bar{Q} \times [0,1]/\bar{Q} \times \{0\}$ and let $q:Q \times [0,1] \to C(Q)$ be a natural projection. Then $C(Q)$ is homeomorphic to $Q$ and $q(X)$ is a $Z$-set copy of $X/X_0$ in $C(Q)$ . Also, $C(Q) - q(X) \cong (Q \times [0,1]) - (Q \times \{0\} \cup X)$ . But, by Theorem 4.18, $Q \times \{0\} \cup X$ is (Borsuk) movable, so that $C(Q) - q(X)$ is movable. 

**Example 4.20.** Let $\sigma = \{X_i, f_i\}$ be an inverse sequence of finite polyhedra. Let $M^*(\sigma)$ denote the subspace $\text{Map}(\sigma) \times \{0\} \cup (\cup_{i > 0} X_i \times [0,1])$ of $\text{Map}(\sigma) \times [0,1]$ where $X_i \subset \text{Map}(\sigma)$ is a copy of $X_i$ in $\text{Map}(\sigma)$ . Then $M^*(\sigma)$ is movable.
CHAPTER V

SHAPE ABSOLUTE NEIGHBORHOOD RETRACTS

The class of shape absolute neighborhood retracts (SANR's for short) was introduced by Borsuk [6] under the name of fundamental absolute neighborhood retracts (change in terminology was suggested by Kozlowski). It is easy to prove that it agrees with the class of all compacta that are shape dominated by finite complexes. Using tools of infinite-dimensional topology we can characterize SANR's rather simply. First we need a couple of definitions from [16] and a lemma.

A space $X$ (always locally compact, separable and metric) is finitely dominated near $\infty$ if there is a finite complex $K$ and a domination near $\infty$, $f:K \times [0,1) \to X$, i.e., $f$ is a proper map for which there is a compactum $A \subseteq X$ and a proper map $g:X \to \text{int } A \to K \times [0,1)$ such that $f \circ g$ is proper homotopic to the inclusion $X \to \text{int } A \hookrightarrow X$. If $f \circ g$ is only weakly proper homotopic to the inclusion $X \to \text{int } A \hookrightarrow X$ we will say that $X$ is weak finitely dominated near $\infty$. Chapman and Ferry proved in
[16, Proposition 8.1] that if \( L \) is a locally finite CW complex which is weak finitely dominated near \( \infty \), then \( L \) is finitely dominated near \( \infty \). This will also be true for an arbitrary locally compact ANR \( X \), since R. Edwards [21] proved that \( X \times Q \) is a \( Q \)-manifold and by Chapman's Triangulation Theorem [13], \( X \times Q \cong K \times Q \) for some CW complex \( K \).

**Lemma 5.1.** Let \( X \) be a finitely dominated near \( \infty \) locally compact ANR. If \( X \) weak proper homotopy dominates ANR \( Y \), then \( Y \) is also finitely dominated near \( \infty \).

**Proof.** It suffices to see that \( Y \) is weak finitely dominated near \( \infty \). Let \( f:K \times [0,1) \to X \), \( A \subseteq X \), \( g:X - \text{int } A \to K \times [0,1) \) and a proper homotopy \( G:(X - \text{int } A) \times I \to X \) between \( f \circ g \) and \( i:X - \text{int } A \hookrightarrow X \) be given as in the above definition. In addition, let \( \phi:X \to Y \) and \( \psi:Y \to X \) be proper maps such that \( \phi \circ \psi \) is weak proper homotopic to \( \text{id}_Y \). Let \( A' = \psi^{-1}(A) \) and define \( f':K \times [0,1) \to Y \) to be \( \phi \circ f \) and \( g':Y - \text{int } A' \to K \times [0,1) \) to be \( g \circ \psi \mid_{Y - \text{int } A'} \). We must demonstrate that for any compactum \( B' \subseteq Y \), there is a compactum \( C' \subseteq Y - \text{int } A' \) and a homotopy \( H \) between \( f' \circ g' \) and the inclusion \( Y - \text{int } A' \hookrightarrow Y \) with \( H((Y - \text{int } A') - C') \times I \subseteq Y - B' \).

To find our \( C' \), let \( B = \phi^{-1}(B) \), \( D = \pi_X G^{-1}(B) \), where \( \pi_X:X \times I \to X \) is a projection onto the first factor,
\[ D' = \psi^{-1}(D) \]. Finally, there is a compactum \( C'' \) in \( Y \)
and a homotopy \( K: Y \times I \to Y \) between \( \text{id}_Y \) and \( \psi \circ \varphi \) such
that \( K((Y-C'') \times I) \subset Y-D' \). Then put \( C' = (Y - \text{int } A') \cap C'' \).
A homotopy \( H \) is given by
\[
H(y,t) = \begin{cases} 
K_{2t}(y), & 0 \leq t \leq 1/2 \\
\varphi_0 \circ G_{2-2t} \circ \psi(y), & 1/2 \leq t \leq 1
\end{cases}
\]
for every \( y \in Y - \text{int } A' \). \( \square \)

**Theorem 5.2.** A Z-set \( X \) in a compact \( Q \)-manifold \( N \)
is a SANR if and only if \( M = N-X \) is finitely dominated
near \( \infty \).

**Proof.** Using the trick from the proof of Corollary 4.3,
embedding \( N \) in a copy of \( Q \), we can effectively assume
\( N = Q \). Let \( X \subset Q \) be a Z-set SANR. There is a Z-set
and a finite complex \( K \subset Q \) such that \( N = Q-K \) weak proper
homotopy dominates \( M \). By Lemma 5.1, it suffices to prove
that \( N \) is finitely dominated near \( \infty \). But this is
certainly true because \( K \times Q \) is a Z-set \( Q \)-manifold in the
Hilbert cube \( Q \times Q \) [45] so that there is an open collar
around \( K \times Q \) in \( Q \times Q \) (see Lemma 2.6). Hence, \( N \times Q = N \)
[3] is finitely dominated near \( \infty \) by \( K \).

Conversely, suppose \( M = Q-X \) is finitely dominated
near \( \infty \). Then there is a finite complex \( K \), a compactum
\( A \subset M \), proper maps \( f:K \times [0,1) \to M \), \( g:M - \text{int } A \to K \times [0,1) \),
and a proper homotopy $H:(M - \text{int } A) \times I \to M$ connecting the inclusion $M - \text{int } A \hookrightarrow M$ with $f \circ g$. Let $C_0(K)$ denote $K \times [0,1)$ with $K \times \{0\}$ identified to a point and let $\pi:K \times [0,1) \to C_0(K)$ be the projection map. A map $f':C_0(K) \to M$ is defined as follows: $f'(x,t) = f(x,t)$ if $1 > t \geq 1/2$, and $f'|_{K \times [0,1/2]}$ is an extension of $f|_{K \times \{1/2\}}$ (recall, $M$ is contractible). Observe that $f' \circ \pi$ and $f$ are proper homotopic. We claim that $f' \circ g'$ is weak proper homotopic to $\text{id}_M$, where $g'$ is an extension of $\pi \circ g:M - \text{int } A \to C_0(K)$ to a map $g':M \to C_0(K)$. Before we prove this, let us see how this implies that $X$ is a SANR. The statement about $f' \circ g'$ assures that a contractible $Q$-manifold $C_0(K) \times Q$ weak proper homotopy dominates $M$. Since $(C_0(K) \times Q) \cup (K \times \{1\} \times Q) = \text{cone}(K) \times Q \approx Q$ [27] and $K \times \{1\} \times Q$ is a Z-set in $\text{cone}(K) \times Q$, Theorem 1.3 gives $\text{Sh}(K) = \text{Sh}(K \times Q) \geq \text{Sh}(X)$.

Thus, for an arbitrary compactum $B$ in $M$, all we have to do is find a compactum $C$ and a homotopy $G:M \times I \to M$ between $\text{id}_M$ and $f' \circ g'$ that satisfies $G((M-C) \times I) \subseteq M-B$. Let $C' = A \cup \pi_{M-\text{int } A}^{-1}(B)$, where $\pi_{M-\text{int } A}:(M-\text{int } A) \times I \to M - \text{int } A$ is a projection onto the first factor. Let $F:[K \times [0,1]] \times [0,1] \to M$ be a proper homotopy between $f$ and $f' \circ \pi$, and put $C'' = g^{-1}(\pi_{K \times [0,1]}^{-1}(F^{-1}(B)))$. The required $G$ is any extension of the map $G':(M - \text{int } C) \times I \cup M \times \{0,1\} \to M$, where $C = C' \cup C''$,
defined below, to all of \( M \times I \).

\[
G'(x,t) = \begin{cases} 
 x & \text{if } t = 0, \ x \in M \\
 f' o g'(x) & \text{if } t = 1, \ x \in M \\
 H_{2t}(x) & \text{if } 0 \leq t \leq 1/2, \ x \in M - \text{int } C \\
 F_{2t-1}(g(x)) & \text{if } 1/2 \leq t \leq 1, \ x \in M - \text{int } C.
\end{cases}
\]

Closely related to Theorem 5.2 is the following characterization of SANR's from [44].

**Proposition 5.3.** A Z-set \( X \) in a compact Q-manifold \( M \) is an SANR if and only if \( X \) admits \( I \)-regular open neighborhoods in \( M \) or, equivalently, if and only if there is a fundamental sequence \( U_1 \supset U_2 \supset \cdots \) of open neighborhoods of \( X \) in \( M \) and a finitely dominated complex \( K \) such that a diagram

\[
\begin{array}{cccccc}
U_1-X & \leftarrow & U_2-X & \leftarrow & U_3-X & \leftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \cdots \\
K & \leftarrow & K & \leftarrow & K & \leftarrow & \cdots \\
\text{id} & & \text{id} & & \text{id} & & \cdots
\end{array}
\]

homotopy commutes, i.e., \( M \) is docile at \( X \).

Borsuk [10] defined a compactum \( X \subset Q \) to be strongly movable if for every neighborhood \( U \) of \( X \), there exists a neighborhood \( V \) of \( X \) such that for every neighborhood \( W \) of \( X \), there is a homotopy \( \varphi : V \times I \rightarrow U \) satisfying the following two conditions:
(1) \( \varphi_0(v) = v, \varphi_1(v) \in W \) for every \( v \in V \),
(2) \( \varphi_1(x) = x \), for every point \( x \in X \).

In the same paper, he proved the following.

**Proposition 5.4.** A compactum \( X \subseteq Q \) is a SANR if and only if it is (Borsuk) strongly movable.

In the case \( X \) is a \( Z \)-set in \( Q \) we can prove Proposition 5.4 as follows. Assume \( X \) is a SANR. By Proposition 5.3, \( X \) admits \( I \)-regular open neighborhoods in \( Q \). Therefore, the axiom \( I - \text{Comp}(Q, X) \) from [43] holds. In other words, for every neighborhood \( U \) of \( X \), there is \( V \subseteq U \) such that for any neighborhood \( W \subseteq U \), there is an isotopy \( h_t : Q \to Q \) with \( h_t \mid (Q - U) \cup N = \text{id} \), where \( N \) is a small neighborhood of \( X \), and \( h_1(V) \subseteq W \). Clearly, a homotopy \( \varphi_t = h_t \mid V \) satisfies (1) and (2) above. We shall use later the observation that a homotopy \( \varphi_t \) satisfies much stronger condition of keeping a neighborhood \( N \) of \( X \) pointwise fixed at all times.

Conversely, suppose a \( Z \)-set \( X \subseteq Q \) is (Borsuk) strongly movable. We shall demonstrate that \( Q - X \) is finitely dominated near \( \infty \).

Let \( Q = V_0 \supset V = V_1 \supset V_2 \supset \cdots \) be a decreasing sequence of prism neighborhoods of \( X \) in \( Q \) and let, for every \( i > 0 \), \( \varphi_t^i : V_i \times I \to V_{i-1} \) be a homotopy for which \( \varphi_0^i = \text{id} \), \( \varphi_1^i(V_i) \subseteq V_{i+1} \), and \( \varphi_1^i \mid X = \text{id} \). We can select
neighborhoods $V_i$ such that there is a homotopy $\lambda_t : Q \to Q$ with the properties: $\lambda_0 = \text{id}$, $\lambda_t(Q) \subseteq Q - X$ for every $t > 0$, and $\lambda_t(V_{i-1}) \subseteq V_{i-1}$, $\lambda_t(Q - V_{i-2}) \subseteq Q - V_{i-1}$ whenever $t \leq 1/i$. If $r : V \to [0, \infty)$ is a continuous map satisfying $r^{-1}(\infty) = X$, define a proper map $g : V - X \to V \times [0, \infty)$ by $f(v) = (v, r(v))$. Next, construct a proper map $f : V \times [0, \infty) \to Q - X$ setting $f(v, t) = \lambda_t^n \circ \varphi_{t-n}^{1} \cdots \varphi_{t-n}^{1} \circ \varphi_{t-n}^{1} \circ \varphi_{t-n}^{1}(v)$, whenever $n \leq t \leq n+1$. Then it is easy to check that $f \circ g$ is proper homotopic to the inclusion $V - X \hookrightarrow Q - X$.

Note. In the above definition we can, in addition, assume $\varphi_{t-n}^{1} \big|_N = \text{id}$ for some neighborhood $N \subseteq W$ of $X$. Indeed, on the closed subset $T = V \times \{0\} \cup \tilde{W} \times \{1\} \cup X \times [0, 1]$, of $V \times [0, 1]$ (where $\tilde{W}$ is a closed neighborhood of $X$ contained in $W$) we can define a map $\varphi$ into $W$ by

$$\varphi(v, t) = \begin{cases} \varphi(v), & t = 0 \\ v, & t = 1 \text{ or } v \in X. \end{cases}$$

and because $W$ can always be assumed an ANR, there is a neighborhood $\tilde{N}$ of $T$ in $V \times [0, 1]$ and an extension $\tilde{\varphi} : \tilde{N} \to W$ of $\varphi$. Let $N_1 \supset \text{cl } N \supset N$ be two open neighborhoods of $X$ with $N_1 \times I \subseteq \tilde{N}$ and let $r : V \to [0, 1]$ be a continuous function such that $r|_{V - N_1} = 0$ and $r|_N = 1$. Now, define a homotopy $H : V \times [0, 1] \to W$ as
$H(v,t) = \psi(v,t \cdot r(v))$.

Then, $H_0 = \varphi_1$ and $H_1 \mid_N = \text{id}$.

With this simple observation, using techniques of Chapters III and IV, we can establish the following.

**Theorem 5.5.** A Z-set $X \subseteq Q$ is a SANR if and only if the complement $M = Q - X$ has the following property:

(NR) For every compactum $A \subseteq M$ there is a compactum $D \supset C$ so that given any map $\varphi:(K,L) \rightarrow (M - B,M - D)$ of a pair $(K,L)$ of finite polyhedra there is a homotopy $H:K \times I \rightarrow M - A$ with $H_0 = \varphi$, $H_1(K) \subseteq M - C$, and $H_1 \mid_L = \varphi \mid_L$.

**Definition 5.6.** A locally compact metric space $M$ is said to belong to a class $\text{SANR}(\infty)$ if $M$ is an ANR($\mathbb{M}$) and has property (NR). By restricting a dimension of $K$ in (NR) to be $\leq n$ we will get a class $\text{SANR}_n(\infty)$ of locally compact metric ANR's.

**Lemma 5.7.** If an ANR($\mathbb{M}$) $M$ is homotopy dominated at $\infty$ by an $\text{SANR}(\infty)$ $N$, then $M$ is also an $\text{SANR}(\infty)$.

**Proof.** Let $f:N \rightarrow M$ and $g:M \rightarrow N$ be proper maps such that for every compactum $X$ in $M$, there is a compactum $X_0 \supset X$ in $M$ and a homotopy $H_X:(M - X_0) \times I \rightarrow M - X$ between $\text{id}_{M - X_0}$ and $f \circ g \mid_{M - X_0}$. Given a compactum $A \subseteq M$, $f^{-1}(A_0) = A'$ is a compactum in $N$. Pick $B' \supset A'$ using the fact that
N \subseteq \text{SANR}(\infty) \), and put \( B = A \cup g^{-1}(B') \). If \( C \supset B \) is an arbitrary compactum, \( C' = B' \cup f^{-1}(C_0) \) is a compactum in \( N \) so that there is \( D' \supset C' \) satisfying (NR). Let \( D \) be the union of \( C_0 \) and \( g^{-1}(D') \).

Assume \( \varphi: (K,L) \rightarrow (M-B,M-D) \) is a map of a pair \((K,L)\) of finite polyhedra. By assumption, there is a homotopy \( \Phi_t: K \rightarrow N-A' \) with \( \Phi_0 = g \circ \varphi \), \( \Phi_1(K) \subseteq N-C' \) and \( \Phi_1|_L = g \circ \varphi|_L \). Define \( \Upsilon_t: K \rightarrow M-A_0 \) as \( f \circ \Phi_t \). Then \( \Upsilon_0 = f \circ g \circ \varphi \), \( \Upsilon_1(K) \subseteq M-C_0 \) and \( \Upsilon_1|_L = f \circ g \circ \varphi|_L \). A homotopy \( H_{A_0 \circ \varphi} \) connects \( \varphi \) with \( f \circ g \circ \varphi \) and \( H_{C_0 \circ (\varphi|_L)} \) joins \( \varphi|_L \) with \( f \circ g \circ (\varphi|_L) \).

Since \( M-A \) is an ANR\((\mathcal{M})\) we can apply the Homotopy Extension Theorem to arrive to a required homotopy.

We will say that a compactum \( X \) is a \( \text{SANR}_n \)-space provided a complement of a Z-set copy of \( X \) in \( Q \) is in the class \( \text{SANR}_n(\infty) \). The proof of the previous lemma shows that this is a shape invariant property.

**Theorem 5.8.** If \( X \) is an LC\(^{n-1}\) compactum and a Z-set in \( Q \), then \( M = Q-X \in \text{SANR}_n(\infty) \).

**Proof.** We will use the full strength of [28] already applied in the proof of Proposition 4.10. Let \( A \subseteq M \) be an arbitrary compactum. Select \( B \supset A \) as in Proposition 4.10. If \( C \) is a compactum in \( M \) containing \( B \), we can find \( D \supset C \) in the analogous way to how \( B \) was chosen with respect to \( A \).
Suppose \( \varphi: (K, L) \to (M-B, M-D) \) is a map of a pair 
\((K, L)\) of finite polyhedra, \( \dim K \leq n \), into \((M-B, M-D)\).
A restriction \( \varphi|_L \) is homotopic in \( M-C \), to a map 
\( \tilde{\varphi}: L \to X \) by the choice of \( D \). Hence, by a homotopy ex-
tension theorem, \( \varphi \) is homotopic in \((Q-B, Q-C)\) to a map 
\( \psi: (K, L) \to (Q-B, X) \). Then \( \psi \) is homotopic in \( Q-A \) to a map 
\( \psi': K \to Q-C \), where \( \psi'|_L = \psi|_L \). As in Proposition 4.10,
using small homotopies that slide \( Q \) off of \( X \) we will get
a required homotopy. \( \Box \)

**Lemma 5.9.** If \( X \) is a Z-set SANR in \( Q \), then \( Q/X \)
is an ANR.

**Proof.** This follows immediately from [26] with the use
of Proposition 5.3. \( \Box \)

**Theorem 5.10.** Let \( X \) be an SANR and \( A \subset X \) a closed
subset and an SANR. Then \( X/A \) is also an SANR.

**Proof.** Consider \( X \) as a Z-set in \( Q \). Combining
Lemma 5.9 and a recent Edwards' theorem [21], \( X/A \times \{0\} \) is
a Z-set copy of \( X/A \) in a \( Q \)-manifold \( Q/A \times Q \). Since \( Q \)
is docile at \( X \) (see Proposition 5.3), it is easy to verify
that \( Q/A \times Q \) is docile at \( X/A \times \{0\} \), so that, by Propo-
sition 5.3, \( X/A \) is an SANR. \( \Box \)

The following two corollaries give partial answers to
Borsuk's question [11]: "Is it true that \( X, Y, X \cap Y \in \text{SANR} \)

---

...
implies $X \cup Y \in \text{SANR}$ and its converse?

**Corollary 5.11.** Let $Z$ be the union of a SANR $X_1$, and a set of trivial shape $X_2$ such that $X_0 = X_1 \cap X_2$ is an SANR. Then $Z$ is an SANR.

**Proof.** It is well known that $Sh(Z) = Sh(Z/X_2)$. But $Z/X_2 = X_1/X_0$, and we can apply Theorem 5.10. \( \square \)

**Corollary 5.12.** Let $Z$ be the union of compacta $X_1$ and $X_2$ such that $X_0 = X_1 \cap X_2$ has trivial shape. If $Z$ is an SANR, then both $X_1$ and $X_2$ are SANR's.

**Proof.** Again, $Sh(Z) = Sh(Z/X_0)$ and $X_1/X_0$ is a retract of $Z/X_0$ (simply collapse all of $X_2/X_0$ to a point $X_0$). Hence, $X_1/X_0$ is an SANR. Therefore, $X_1$ is an SANR because $Sh(X_1) = Sh(X_1/X_0)$. \( \square \)

Now, we shall describe a method of establishing when a locally finite CW complex $L$ is $n$-dominated near $\infty$, i.e., when we can find an $n$-dimensional finite complex $M$ and a domination near $\infty$, $d:M \times [0,1) \to L$ (Lemma 5.13) and then apply this to get a characterization of shape dimension of SANR's (Theorem 5.14).

Let $f:K \times [0,1) \to L_1 \preceq L$, $g:L_1 \to K \times [0,1)$, and $\alpha_u:K \to K$ be as in the Definition 4.4 of [16] (see also diagram (***) below).

**Lemma 5.13.** $L$ is $n$-dominated near $\infty$ if and only if
there is an n-dimensional finite complex $M$ such that the diagram

$$
\begin{array}{ccc}
K & \xrightarrow{\alpha_u} & K \\
\downarrow \varphi & & \downarrow \psi \\
M & & 
\end{array}
$$

homotopy commutes.

**Proof.** Suppose that there are maps $\varphi$ and $\psi$ making the diagram (*) homotopy commutative. Let $L_2 \subseteq L_1$ be a subcomplex for which $f \circ g |_{L_2}$ is homotopic in $L_1$ to the inclusion $L_2 \hookrightarrow L_1$ and let $u \in [0,1)$ be such that $f(K \times \{u\}) \subseteq L_2$. Put $a = f \circ (xu) \circ \psi \circ p_M : M \times [0,1) \to L_2$ and $b = (xv) \circ \varphi \circ p_K \circ (g |_{L_2}) : L_2 \to M \times [0,1)$.

$$
\begin{array}{ccc}
L_2 & \xrightarrow{g |_{L_2}} & K \\
\downarrow f & & \downarrow g \\
K \times [0,1) & \xrightarrow{\alpha_u} & K \\
\downarrow \varphi & & \downarrow \psi \\
M & \xrightarrow{p_M} & M \times [0,1) \\
\end{array}
$$

We claim that $a \circ b$ is properly homotopic to the inclusion $L_2 \hookrightarrow L$. Note that $a \circ b = f \circ (\alpha_u \times c_u) \circ (g |_{L_2}) = \gamma$. 

where \( c_u: [0,1) \to [0,1) \) is a constant map into \( u \). Hence, the value \( \gamma(t) \), for any \( t \in L_2 \), is obtained as follows:

\[
l \mapsto (p_Kg(t), p_{[0,1)}g(t)) \mapsto ((p_Kg(p_Kg(t), u), u) \mapsto f((p_Kg(p_Kg(t), u)), u).
\]

Using homotopies only on \([0,1)\) direction we can easily see that \( \gamma \) is proper homotopic in \( L \) to a composition \( f \circ g \circ f \circ (g|_{L_2}) \). But this later can be written as \( (f \circ g) \circ (f \circ (g|_{L_2})) \) and by assumption is proper homotopic in \( L \) to the inclusion \( L_2 \hookrightarrow L \).

Conversely, if \( L \) is \( n \)-dominated near \( \infty \) by \( M^n \) and \( f: K \times [0,1) \to L_1 \hookrightarrow L \) is an arbitrary domination near \( \infty \) with the associated map \( \alpha_u: K \to K \), we claim that \( \alpha_u \) factors through \( M \). Indeed, without loss of generality, we can assume that there are proper maps \( f_1: M \times [0,1) \to L_1 \) and \( g_1: L_1 \to M \times [0,1) \) such that \( f_1 \circ g_1 \) is proper homotopic to the inclusion \( L_1 \hookrightarrow L \). Then \( \varphi = p_M \circ g_1 \circ (xu): K \to M \) and \( \psi = p_K \circ g \circ f_1 \circ (xv): M \to K \) satisfy \( \psi \circ \varphi = \alpha_u \). \( \square \)

**Theorem 5.14.** A \( Z \)-set and an \( SANR \), \( X \), in \( Q \) has a shape dimension \( \leq n \) if and only if \( M = Q - X \) is dominated near \( \infty \) by an \( n \)-dimensional finite complex.

**Proof.** If \( M \) is \( n \)-dominated near \( \infty \), then it is easy to verify that \( M \) is \( n \)-tame at \( \infty \), and therefore that \( \text{Sd}(X) \leq n \).

On the other hand, if a \( Z \)-set and an \( SANR \), \( X \), in \( Q \) has a shape dimension \( \leq n \), we will prove that the map
associated to, a domination near \( \infty \), \( f: K \times [0,1) \to M \) as defined above can be factored through an \( n \)-dimensional finite complex. By Lemma 5.13, it would follow that \( M \) is \( n \)-dominated near \( \infty \). Let \( A \subset M \) be a compactum for which there is a proper map \( g:M - \text{int} \ A \to K \times [0,1) \) with \( f \circ g \) proper homotopic to the inclusion \( M - \text{int} \ A \hookrightarrow M \). Since \( M \) is \( n \)-tame at \( \infty \), we can find an \( n \)-complex \( L \) and a compactum \( B \supset A \) in \( M \) such that \( M - \text{int} \ B \hookrightarrow M - \text{int} \ A \) factors through \( L \). Pick a real number \( u \in [0,1) \) for which \( f(K \times \{u\}) \subset M - \text{int} \ B \). Then \( \varphi = \alpha \circ f \circ (xu): K \to L \) and \( \psi = P_K \circ g \circ \beta: L \to K \) give a factorization of \( \alpha_u \) through \( L \), where \( u \) and \( \beta \) make \( \psi \circ \varphi = f \). □

**Corollary 5.15.** Let \( L \) be a locally finite CW complex and let \( L \) be finitely dominated near \( \infty \). Then \( L \) is \( n \)-tame at \( \infty \) if and only if \( L \) is \( n \)-dominated near \( \infty \).

**Proof.** Assume \( L \) is dominated near \( \infty \) by an \( n \)-dimensional finite complex \( K \). Let \( f: K \times [0,1) \to L_1 \hookrightarrow L \) and \( g:L_1 \to K \times [0,1) \) be as above. Then, with the notation from
the proof of Lemma 5.13,

\[
\begin{array}{ccc}
L_1 & \xleftarrow{p_K \circ g} & K \\
& f_0(xu) \searrow & \\
& \downarrow \swarrow & \searrow \\
& K & \xrightarrow{f_0(xu) \circ p} & L
\end{array}
\]

is a factorization of \( L_1 \hookrightarrow L \) through \( K \). It is clear that this can be accomplished in any neighborhood of \( \infty \). Hence, \( L \) is \( n \)-tame at \( \infty \).

Conversely, suppose \( L \) is dominated near \( \infty \) by a finite complex \( K \) and also that \( L \) is \( n \)-tame at \( \infty \). We shall show that \( \alpha_u \) factors through an \( n \)-dimensional finite complex. Lemma 5.13 implies, then, \( L \) is \( n \)-dominated near \( \infty \).

In the diagram (**), select \( L_2 \) in such a way that

\[
\begin{array}{ccc}
L_2 & \xleftarrow{\alpha} & L_1 \\
& \downarrow \swarrow & \downarrow \searrow \\
& \downarrow & \searrow \\
& \alpha_u & \xrightarrow{f_0(xu) \circ p} & K \\
& \downarrow \swarrow & \searrow \\
& \downarrow & \searrow \\
& p & \xrightarrow{p_K \circ g \circ \beta} & K
\end{array}
\]

homotopy commutes for some \( n \)-dimensional finite complex \( P \).

Let \( u \in (0,1) \) be so close to \( 1 \) that \( f(K \times \{u\}) \subseteq L_2 \).

Then

\[
\begin{array}{ccc}
K & \xrightarrow{\alpha_u} & K \\
& \downarrow \swarrow & \searrow \\
& \downarrow \swarrow & \searrow \\
& \downarrow & \searrow \\
& \alpha_0 f_0(xu) & \xrightarrow{p_K \circ g \circ \beta} & K \\
& \downarrow \swarrow & \searrow \\
& \downarrow & \searrow \\
& p & \xrightarrow{p_K \circ g \circ \beta} & K
\end{array}
\]

is a required factorization of \( \alpha_u \). \( \square \)
CHAPTER VI
COMPACTA OF TRIVIAL SHAPE

In this chapter, we will consider locally compact non-compact spaces trivial at \( \infty \) and compacta with the shape of a point, i.e., shape absolute retracts (SAR's).

Lemma 6.1. A Z-set \( X \subseteq Q \) is an SAR if and only if the complement \( M = Q - X \) is trivial at \( \infty \), i.e., for any compactum \( A \subseteq M \), there is a compactum \( B \supseteq A \) such that every map \( \varphi: S^k \to M - B \) of a \( k \)-sphere \( S^k \) \( (k \geq 0) \) is null-homotopic in \( M - A \).

Proof. If \( X \) is an SAR, \( Q - X \cong Q \times [0,1) \) [12] is clearly trivial at \( \infty \). Conversely, if \( M \) is trivial at \( \infty \), then, by a technique of Chapters III - V, we can see that \( X \) is contractible inside each of its neighborhoods, and thus is an SAR [6, 26].

Restricting the dimension \( k \) in Lemma 6.1 to be \( \leq n \), we get the notion of an \( n \)-trivial at \( \infty \) locally compact metric space. This concept can be used for characterizing \( UV^n \)-compacta [46], or, equivalently, approximatively
n-connected compacta [10].

**Theorem 6.2.** If $M$ is $n$-tame at $\infty$ and $n$-trivial at $\infty$, then $M$ is trivial at $\infty$.

**Proof.** Take any compactum $A$ in $M$ and pick $B \supset A$ using $n$-triviality of $M$ at $\infty$. Let $C \supset B$ such that $\text{homotopy commutes for some } n\text{-dimensional finite complex } K$. Given a map $\varphi: P \to M-C$, we see that $\varphi$ is null-homotopic in $M-A$ since $\beta$ is and $\varphi \sim \beta \circ \alpha \varphi$. □

**Theorem 6.3.** If $M$ is movable and $n$-trivial at $\infty$, for every $n > 0$, then $M$ is trivial at $\infty$.

**Proof.** Let $A$ be a compactum in $M$. Pick $B \supset A$ so that the condition (M) of Chapter IV holds. One can easily check that a compactum $B$ suffices since every map into $M-B$ can be homotoped outside $A$ to a map whose image is arbitrarily close to $\infty$. □

**Corollary 6.4 ([10]).** A compactum $X$ has trivial shape if and only if $X$ is movable and approximatively $n$-connected for all $n = 0,1,2,\cdots$. 

Theorem 6.5. A Z-set \( X \) in \( Q \) has trivial shape if and only if

(a) \( Sd(X) < \infty \),

(b) \( X \) is approximatively 1-connected, i.e., the complement \( M = Q-X \) is 1-trivial at \( \infty \), and

(c) the reduced Čech cohomology \( \check{\mathcal{H}}^*(X) \) of \( X \) is zero.

Proof. It is clear that every Z-set of trivial shape in \( Q \) has properties (a) - (c). Conversely, by (b), \( X \) is connected so that \( M \) has exactly one end. Observe that \( M \) is contractible and in light of (a), proper homotopy equivalent to a locally finite polyhedron of finite dimension. Let \( f:X \to \{pt\} \) be a constant fundamental sequence. We will prove that \( f \) is a fundamental equivalence. It follows from Theorem 1.3 that \( f \) determines a proper map \( f:M \to Q-\{pt\} \). But \( f \) induces isomorphisms

\[
H_*(M) \to H_*(Q-\{pt\}) , \quad H_C^*(M) \to H_C^*(Q-\{pt\}) ,
\]

where \( H_* \) denotes a singular homology and \( H_C^* \) a singular cohomology with compact supports, since \( H_C^*(M) \) is canonically isomorphic to \( \check{H}_C^*(M) \), the Alexander cohomology with compact support ([47, p. 341]), and \( \check{H}_C^*(M) \cong \check{H}_*(Q,X) \cong \check{H}_*(Q,X) \cong \check{H}_*(X) \) ([47, p. 321]). Now, we can apply Corollary 4.9 in [22] to conclude that \( f \) is a proper homotopy equivalence and, therefore that \( f \) is a fundamental equivalence by Theorem 1.3 again. \( \square \)
Corollary 6.6. Let \( X \subset Q \) be an approximatively 1-connected \( Z \)-set of a finite shape dimension. If \( Q/X \) has zero reduced Čech cohomology, then \( X \) has trivial shape.

**Proof.** This follows immediately from the isomorphisms \( \tilde{H}^*(Q/X) \cong \tilde{H}^*(Q/X) \) and \( \tilde{H}^*(Q/X) \cong \tilde{H}^*_C(Q-X) \) \cite{47} and Theorem 6.5. \( \square \)

**Proposition 6.7** \cite{48}. Let \( X \) be a compactum with \( \text{Sh}(X) < \text{Sh}(S^n) \). Then \( X \) has a trivial shape.

**Proof.** Consider \( S^n \) and \( X \) as \( Z \)-sets in \( Q \). Observe that \( \tilde{H}^*_C(Q-S^n) \cong \tilde{H}^*(Q,S^n) \cong \tilde{H}^*(S^n) \). If \( \text{Sh}(X) < \text{Sh}(S^n) \), then there is a fundamental sequence \( f:S^n \to X \) that is not a fundamental equivalence, but for which there is \( g:X \to S^n \) such that \( f \circ g = \text{id}_X \). By Theorem 1.3, we can find proper maps \( f:Q-S^n \to Q-X \) and \( g:Q-X \to Q-S^n \) such that \( f \) is not a weak proper homotopy equivalence but \( f \circ g \) is weakly proper homotopic to \( \text{id}_{Q-X} \). Note that \( Q-S^n \) is finitely dominated near \( \infty \). Hence \( Q-X \) is also finitely dominated near \( \infty \). Therefore, we can assume that \( f \circ g \) is properly homotopic to \( \text{id}_{Q-X} \) \cite[Section 3]{16}. This means that \( g \circ f = (f \circ g) \circ \text{id}_{Q-X} \) is the identity, i.e., \( \tilde{H}^*_C(Q-X) \) is a direct summand of \( \tilde{H}^*_C(S^n) \) given by

\[
\tilde{H}^*_C(S^n) = \begin{cases} 
\mathbb{Z}, & q = n, 0 \\
0, & q \neq n, 0.
\end{cases}
\]
If $\tilde{H}_C^n(Q-X) = 0$, then we are done by Theorem 6.5. The other possibility is $\tilde{H}_C^n(Q-X) = \mathbb{Z}$. We will prove that this is impossible. First, note that, by Corollary 4.9 in [22], neither $f_n^*$ nor $g_n^*$ can be isomorphisms. Let $g_n^*(1) = \theta$ and $f_n^*(1) = \eta$ where 1 denotes unit in $\tilde{H}_n^n(Q-\Sigma^n)$ and $\tilde{H}_n^n(Q-X)$, respectively. Note $|\theta| > 1$ since $g_n^*$ is not an isomorphism. Then $g_n^*$ cannot be onto as a relation $g_n^* \circ f_n^* = \text{id}$ requires. □
CHAPTER VII
UVW\^n-SPACES

Armentrout [49], introduced property UVW\^n of compact spaces in order to extend classical theorems about decompositions of ANR's. He asked if this property is a shape invariant and how it relates to SANR's. In this chapter, we will answer two of his questions affirmatively (see Corollary 7.2 and Theorem 7.3).

If \( n \geq 0 \) is an integer, a closed subset \( X \) of \( Q \) has property UVW\(^n\) if and only if, for each open neighborhood \( U \) of \( X \), there is an open neighborhood \( V \) of \( X \) such that, for each open neighborhood \( W \) of \( X \), there is an open neighborhood \( Z \) of \( X \) such that, for each \( k \), with \( 0 \leq k \leq n \) and each map \( f: (B^k, \text{Bd } B^k) \rightarrow (V, Z) \), there is a homotopy \( H: (B^k \times I, (\text{Bd } B^k) \times I) \rightarrow (U, W) \) connecting \( f \) with a map \( H_1 \) for which \( (H_1(B^k), H_1(\text{Bd } B^k)) \subseteq (W, Z) \). Let UVW\(^\infty\) mean UVW\(^n\) for all \( n \geq 0 \).

**Lemma 7.1.** A Z-set \( X \) in \( Q \) has property UVW\(^n\) if and only if the complement \( M = Q - X \) satisfies:

\((\mathcal{Q})\) for each compactum \( A \) in \( M \), there is a
compactum $B \supset A$ such that, for each compactum $C \supset A$ in $M$, we can find a compactum $D \supset C$ such that, for each $k$ with $0 \leq k \leq n$, and each map $f:(B^k,Bd B^k) \rightarrow (M-B,M-D)$, there is a homotopy $H:(B^k \times I,(Bd B^k) \times I) \rightarrow (M-A,M-C)$ of $f$ such that $(H_1(B^k),H_1(Bd B^k)) \subset (M-C,M-D)$.

**Proof.** Let $X$ be a $Z$-set in $Q$ and assume $X$ has property $UVW^n$. If $A$ and $C$, $A \subset C$, are arbitrary compacta in $M = Q-X$, $U = Q-A$ and $W = Q-C$ are open neighborhoods of $X$ in $Q$. Pick open neighborhoods $V$ and $Z$ as in the above definition and put $B = Q-V$ and $D = Q-Z$. Compacta $A,B,C,D$ will satisfy $(\sigma)$ as is easily verified using Chapman's trick [12] of deforming $Q$ off of $X$ moving points only inside $Z$.

The converse is equally routine and therefore left to the reader. □

It is clear that if a space $M$ has property $(\sigma)$ and $h:M \rightarrow N$ is a homeomorphism, then $N$ also has property $(\sigma)$. Hence, since two shape equivalent $Z$-sets in $Q$ have homeomorphic complements (Theorem 1.4), we immediately get

**Corollary 7.2.** The property $UVW^n$ of compacta is a shape invariant. □

**Theorem 7.3.** Every $SANR$, $X$, has property $UVW^\infty$.

**Proof.** Consider $X$ as a $Z$-set in $Q$. By Theorem 5.2,
the complement $M = Q - X$ is finitely dominated near $\infty$.

Let $f : K \times [0, 1) \to M$ be a domination near $\infty$, and select a compactum $A' \subset M$ for which there is a proper map $g : M - \text{int} A' \to K \times [0, 1)$ such that $f \circ g$ is properly homotopic to the inclusion $M - \text{int} A' \hookrightarrow M$ via a proper homotopy $h_t$. If $A \subset C$ are arbitrary compacta in $M$, we can find $B \supseteq A \cup A'$ and $D \supseteq C \cup A'$ with $h_t(M - D) \subseteq M - C$ and $h_t(M - B) \subseteq M - A$. Now, we will show that $f \circ g |_{M - B}$ is homotopic to a map $(M - B, M - D) \to (M - C, M - D)$ by a homotopy in $(M - A, M - C)$. To see this, pick numbers $t_A < t_B$ and $t_C < t_D$ with $t_C > t_A$ and $t_D > t_B$ in $[0, 1)$ such that $f^{-1}(Y) \subseteq K \times [0, t_Y]$ for each $Y \in \{A, B, C, D\}$. Let $\lambda_t : [0, 1) \to [0, 1)$ be a homotopy such that $\lambda_0 = \text{id}$, $\lambda_1([t_B, 1)) \subseteq [t_C, 1)$, $\lambda_1([t_C, 1)) \subseteq [t_D, 1)$, $\lambda_t|_{[0, t_A]} = \text{id}$, $\lambda_t([t_C, 1)) \subseteq [t_C, 1)$. Then $f \circ \lambda_t \circ g |_{M - B}$ is the required homotopy. Hence, we have proved that, for every compactum $A$ in $M$, there is a compactum $B \supseteq A$ such that, for any compactum $C \supseteq A$, we can find a compactum $D \supseteq C \cup B$ and a homotopy $h_t : M - B \to M - A$ with $h_0$ - the inclusion $M - B \hookrightarrow M - A$, $h_t(M - D) \subseteq M - C$, and $(h_1(M - B), h_1(M - D)) \subseteq \subseteq (M - C, M - D)$. It is clear that this implies that $M$ has property ($\mathcal{C}$). 

**Theorem 7.4.** If a compactum $X$ is approximatively $k$-connected for all $0 \leq k \leq n$, then $X$ is a UVW$^n$-space.
Proof. Considering $X$ as a $Z$-set in $Q$, we will prove that $M = Q - X$ has the property $(\mathcal{Q})$. Let $A$ be a compactum in $M$. Pick $B \supset A$ such that every map $S^k \to M - B$, $0 \leq k \leq n$, is null-homotopic in $M - A$. If $C \supset A$ is a compactum, select $D \supset C$ so that every map $S^k \to M - D$, $0 \leq k \leq n$, is null-homotopic in $M - (B \cup C)$.

Now, let $f : (B^k, \partial B^k) \to (M - B, M - D)$, $0 \leq k \leq n$, be given. By assumption, there is a homotopy $h_t : \partial B^k \to M - C$ with $h_0 = f|_{\partial B^k}$ and $h_1(\partial B^k) = p = \text{point} \in M - D$ (observe that $h_t(\partial B^k)$ remains in the same path-component $K$ of $M - C$ and $K \cup (\text{path-component of } M - D \text{ containing } h_0(\partial B^k))$ is path-connected). Define a map $\varphi$ on the boundary of $B^k \times I$ by

$$
\varphi(x, t) = \begin{cases} 
  p, & t = 1 \\
  h_t(x), & x \in \partial B^k, \quad t \in I \\
  f(x), & t = 0.
\end{cases}
$$

Again, $\varphi$ extends to a map $\tilde{\varphi} : B^k \times I \to M - A$ that gives a required homotopy of $f$. □

We leave the question if every compactum $X$ which has property $UVW^\infty$ is an SANR unanswered and only make some comments on how that question is related to the following (converse of Lemma 5.9).

Conjecture 7.5. Let $A$ be a $Z$-set in $Q$ such that $Q/A$ is an ANR. Then $A$ is an SANR.
Let $X$ be an $n$-dimensional ($n < \infty$) compactum with property $UVW^{2n+1}$. Consider $X$ as a subset of a $(2n+1)$-dimensional closed cell $B^{2n+1}$. It follows from Theorem 2 in [49] that $B^{2n+1}/X$ is an ANR. Applying the main result of [26], we see that $Q/X'$ is an ANR, where $X'$ is a copy of $X$ in $Q$. Therefore, if Conjecture 7.5 is a true statement, then at least all finite-dimensional $UVW^\infty$-compacta are SANR's. On the other hand, the existence of a finite-dimensional $UVW^\infty$-compactum that is not an SANR gives a counterexample for Conjecture 7.5.
CHAPTER VIII

HOMOLOGY AND HOMOTOPIE PRO-GROUPS

In this chapter, we shall briefly describe a method of introducing algebraic techniques in our study of homotopy properties of non-compact locally compact metric spaces. In a natural way, we shall define certain inverse systems of groups associated with every locally compact non-compact metric space. The language of pro-categories will be used to accomplish this. We shall assume that the reader is familiar with results and terminology employed in this chapter (see [38] and [20] for references).

Let $\mathcal{M}$ be a locally compact metric space. We can think about $\mathcal{M}$ as an element $\mathcal{M}$ of pro-$\mathcal{V}$, where $\mathcal{V}$ is a category of topological spaces and homotopy classes of continuous maps. Indeed, let $\mathcal{C}_M$ be the set of all compact subsets of $\mathcal{M}$ directed by inclusions. Then $\mathcal{M} = \{ M_A = M - A, [i_B, A] : \mathcal{C}_M \}$ is an inverse system with bonding maps homotopy classes of inclusions $i_{B, A} : M - B \rightarrow M - A$. Every proper map $f : \mathcal{M} \rightarrow \mathcal{N}$ induces a morphism $f : \mathcal{M} \rightarrow \mathcal{N}$ in pro-$\mathcal{V}$. The index function of $f$ is a map $\varphi : \mathcal{C}_N \rightarrow \mathcal{C}_M$. 

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given by \( \phi(B) = f^{-1}(B) \), and coordinate maps \( f_B : M \phi(B) \to N_B \) are simply homotopy classes of restrictions of \( f \) from \( M-f^{-1}(B) \) into \( N-B \).

**Lemma 8.1.** Proper maps \( f, g : M \to N \) are homotopic at \( \infty \) iff \( f, g \) are homotopic maps of systems.

**Proof.** Assume \( f \sim g \). Given a compactum \( B \) in \( N \), we must find a compactum \( A \supset f^{-1}(B) \cup g^{-1}(B) \) in \( M \) such that

\[
foi_{A,f^{-1}(B)} \sim goj_{A,g^{-1}(B)}
\]

in \( N-B \). \( A \) is just a compactum containing \( f^{-1}(B) \) and \( g^{-1}(B) \) so that \( f|_{M-A} \) is homotopic in \( N-B \) to \( g|_{M-A} \).

Conversely, if \( f \sim g \), then clearly for any compactum \( B \subset N \), we can find \( A \subset M \) so that \( f|_{M-A} \) is homotopic in \( N-B \) to \( g|_{M-A} \). □

Let \( G \) be an abelian group, and \( k \geq 0 \) an integer. Then

\[
H_k(M;G) = \{H_k(M_A;G), (i_{B,A})_{k*} C_M\}
\]

is an inverse system of groups, i.e., an object of the category \( \text{pro-}\mathcal{G} \) of pro-groups called the \( k \)-th homology pro-group of \( M \) with coefficient in \( G \). A proper map \( f : M \to N \) induces a morphism of pro-groups \( f_{k*} : H_k(M;G) \to H_k(N;G) \) in a natural way. It is readily seen that \((gf)_{k*} = g_{k*} \circ f_{k*} \) and
id_{k*} = id. Therefore, two spaces that are homotopy equivalent at $\infty$ have isomorphic homology pro-groups.

In order to define analogues of homotopy groups, we shall consider only one-ended locally compact metric spaces $M$. Then $C_M$ has a cofinal subset $C_M^*$ consisting of all compacta $A \subseteq M$ for which $M-A$ is connected. For every $A \in C_M^*$, pick a point $x_A \in M-A$ and if $B$ and $A$, $B \supset A$, are both in $C_M^*$, connect points $x_A$, $x_B$ by a path $\alpha_{B,A}:I \to M-A$. Now, define an inverse system

$$
\pi_k(M,\{x_A\},\{\alpha_{B,A}\}) = \{\pi_k(M_A, x_A), (\alpha_{B,A}) \circ (i_{B,A})_{k*}^* C_M^* \}
$$

of groups, where $(\alpha_{B,A})_k$ is an isomorphism $\pi_k(M_A, x_B) \to \pi_k(M_A, x_A)$ induced by a path $\alpha_{B,A}$ [50, p. 126]. Of course, systems $\pi_k(M,\{x_A\},\{\alpha_{B,A}\})$ depend on the choice of points $\{x_A\}$ and paths $\{\alpha_{B,A}\}$. In case $M$ is l-trivial at $\infty$, all such choices lead to systems that have the same homotopy type [36].

Suppose now that $M$ is an ANR. If $B$ and $A$, $B \supset A$, are compacta in $C_M^*$, by the Homotopy Extension Theorem, there is a map $P_{B,A}:M_B \to M_A$ homotopic to $i_{B,A}$ and such that $P_{B,A}(x_B) = x_A$. Hence, the inverse system $\mu = \{M_A, [P_{B,A}], C_M^*\}$ is equivalent to $M$ and bonding maps of $\mu$ map base points into base points. Then we can define $k$-th homotopy pro-group of $M$ to be the $k$-th homotopy pro-group of the inverse system $\mu$ as defined in [38], i.e.,
For a proper map \( f : M \to N \) between one-ended ANR's, we can select base points \( \{ y_B \} \) in \( N \) and \( \{ x_A \} \) in \( M \) so that \( f \) induces in a natural way a homomorphism \( f_* : \pi_k(M, \{ x_A \}) \to \pi_k(N, \{ y_B \}) \); in order to accomplish this we might need to pass to a cofinal subset of \( C^*_M \).

**Theorem 3.2.** A one-ended ANR \( M \) is \( n \)-trivial at \( \infty \) iff \( \pi_k(M, \{ x_A \}) = 0 \) for some set of base points \( \{ x_A \} \) and all \( 1 \leq k \leq n \).

**Proof.** Let \( \pi_k(M, \{ x_A \}) = 0 \), \( 1 \leq k \leq n \), and let \( A \in C^*_M \). Pick \( B \supset A \) in \( C^*_M \) so that \( (P_{B,A})^* \) is a zero map, for all \( 1 \leq k \leq n \). Take any map \( \varphi : S^k \to M_B \). By the homotopy extension theorem, \( \varphi \) is homotopic in \( M_B \) to a map \( \psi : (S^k, s_0) \to (M_B, x_B) \), where \( s_0 \) is a north pole of \( S^k \). But,

\[
\text{constant map } \sim p_{B,A} \circ \psi = i_{B,A} \circ \psi = \psi \sim \varphi,
\]

which shows that \( M \) is \( n \)-trivial at \( \infty \).

Conversely, assume \( M \) is \( n \)-trivial at \( \infty \) and take any compactum \( A \in C^*_M \). Let \( B \supset B_1 \supset A ; B, B_1 \in C^*_M \), be such that every map \( \varphi : S^k \to M_B \) (\( 1 \leq k \leq n \)) is null-homotopic in \( M_{B_1} \) and every loop in \( M_{B_1} \) is trivial in \( M_A \). We shall prove that, for a pointed map \( \psi : (S^k, s_0) \to (M_B, x_B) \),
\( \chi = p_{B,B_1} \circ \psi \) is homotopic in \( M_A \) to a constant map into \( x_{B_1} \) by a homotopy that maps \( s_0 \) into \( x_{B_1} \) at all times.

Let \( H: S^K \times I \to M_{B_1} \) be a null-homotopy of \( \chi \). The existence of \( H \) follows from \( \chi = p_{B,B_1} \circ \psi = i_{B,B_1} \circ \psi = \psi \).

Let \( \alpha: I \to M_{B_1} \) be a path connecting points \( H_1(s_0) \) and \( \chi(s_0) \). On the boundary of \( \{s_0\} \times I \times I \) define a map \( g \) into \( M_{B_1} \) by

\[
g(s_0,a,b) = \begin{cases} 
\chi(s_0), & b = 1 \text{ or } a = 0 \\
H_a(s_0), & b = 0 \\
\alpha(b), & a = 1
\end{cases}
\]

The choice of \( B \) gives us an extension \( G: \{s_0\} \times I \times I \to M_B \) of \( g \). Define a partial homotopy on \( (S^K \times I) \times \{0\} \cup S^K \times \{0\} \times I \cup S^K \times \{1\} \times I \cup \{s_0\} \times I \times I \) by \( H \) on \( (S^K \times I) \times \{0\} \), by \( G \) on \( \{s_0\} \times I \times I \), by \( \chi \) level-wise on \( S^K \times \{0\} \times I \), and by \( \alpha \) level-wise on \( S^K \times \{1\} \times I \). Applying the homotopy extension theorem, we will get a map \( S^K \times I \times I \) whose restriction to \( S^K \times I \times \{1\} \) is the required homotopy. Now, it is clear that \( (p_{B,A})_{k*} \) is a zero map. \( \square \)

In this way, to every one-ended ANR \( M \), we can associate an element \( \{(M_A, x_A), [p_{B,A}], C_M^*\} \) of \( \text{pro-} \mathcal{W}_0 \), where \( \mathcal{W}_0 \) is a homotopy category of pointed spaces having the homotopy type of a pointed CW complex. Hence, we can get analogues of Hurewicz's isomorphism theorem and Whitehead's theorem since they have been proved (see [20]) for any object in \( \text{pro-} \mathcal{W}_0 \).
CHAPTER IX
HOMOTOPY EQUIVALENCES AT $\infty$

In this chapter, we shall define certain classes of proper maps and investigate their properties from two points of view. We shall first look for properties preserved under these maps and then try to decide under what additional assumptions they will be homotopy equivalences at $\infty$.

**Definition 9.1.** A proper map $f: M \to N$ between locally compact metric spaces is in the class $C$ provided for any compactum $A \subseteq N$, there is a larger $B \subseteq N$ such that, given a map $\varphi: P \to N-B$ of a finite polyhedron $P$ into $N-B$, there is $\tilde{\varphi}: P \to f^{-1}(N-A)$ with $f \circ \tilde{\varphi} = \varphi$ in $N-A$. By restricting $P$ to have a dimension $\leq n$, we will get a class $C_n$. $C_\infty$ will denote the intersection $\bigcap_{i>0} C_i$.

One easily verifies that every homotopy domination at $\infty$ is in the class $C$ and that the inclusion $\mathbb{R} \hookrightarrow \mathbb{R}^2$ is not in $C$.

**Lemma 9.2.** If $f: M \to N$ is in $C$ ($C_n$) and $M$ is movable ($n$-movable), then $N$ is also movable ($n$-movable).
Proof. Let a compactum $A \subset N$ be given. Since $M$ is movable and $f^{-1}(A)$ is a compactum, there is $B' \supset f^{-1}(A)$ satisfying $(\mathfrak{M})$ of Chapter IV. Pick $B \supset f(B')$ in $N$ using the fact $f \in C$. Then $B$ is a required compactum. Indeed, let $C \supset B$ be an arbitrary compactum in $N$ and $\varphi: P \to N - B$ a map of a finite polyhedron $P$. There is $\varphi: P \to f^{-1}(N - B) \subset M - B'$ with $f \varphi = \varphi$ in $N - B$. But $\varphi$ is in $M - f^{-1}(A)$ homotopic to a map $\varphi^*: P \to M - f^{-1}(C)$. Therefore, $\varphi$ is in $N - A$ homotopic to $f \varphi^*: P \to N - C$. □

Lemma 9.3. If $f: M \to N$ is in $C_n$ and $M$ is trivial (n-trivial) at $\infty$, then $N$ is also trivial (n-trivial) at $\infty$.

Proof. Let $A \subset N$ be any compactum, and take a compactum $B' \supset f^{-1}(A)$ such that every map of a sphere $S^k$ (of dimension $k \leq n$) into $M - B'$ is null-homotopic in $M - f^{-1}(A)$. Select $B \supset f(B')$ as in the previous lemma. Clearly, every map of a sphere $S^k$ (of dimension $k \leq n$) into $N - B$ is null-homotopic in $N - A$. □

Lemma 9.4. If $f: M \to N$ is in $C_n$ and $N$ is n-tame at $\infty$, then $f$ is in $C$.

Proof. Let $A \subset N$ be a compactum. Since $f \in C_n$, there is $B' \supset A$ such that for any map $\varphi: P \to N - B'$ of an at most $n$-dimensional finite polyhedron, we can find
$\varphi: P \to f^{-1}(N-A)$ with $\varphi = f \circ \tilde{\varphi}$ in $N-A$. Let $B \supseteq B'$ have the property that the inclusion $N-B \hookrightarrow N-B'$ factors through a finite $n$-complex $K$, i.e., a diagram

\[
\begin{array}{ccc}
N-B & \hookrightarrow & N-B' \\
\alpha & & \beta \\
& K & \\
\end{array}
\]

homotopy commutes. Take $\tilde{\beta}: K \to f^{-1}(N-A)$ for which $f \circ \tilde{\beta} = \beta$ in $N-A$. If $\varphi: P \to N-B$ is a map of a finite polyhedron $P$ into $N-B$, let $\tilde{\varphi} = \tilde{\beta} \circ \alpha \circ \varphi: P \to f^{-1}(N-A)$. Then $f \circ \tilde{\varphi} = \varphi$ in $N-A$. \[
\]

**Definition 9.5.** A proper map $f: M \to N$ is in a class $\mathcal{B}$ provided, for every compactum $A \subseteq M$, there is a compactum $\tilde{A}$ in $N$ with $f^{-1}(N-\tilde{A}) \subseteq M-A$ and such that, for any two maps $\varphi, \psi: P \to f^{-1}(N-\tilde{A})$ of a finite polyhedron $P$, the relation $f \circ \varphi = f \circ \psi$ in $N-\tilde{A}$ implies $\varphi = \psi$ in $M-A$. If only polyhedra of dimension $\leq n$ are allowed, we will get a class $\mathcal{B}_n$. $\mathcal{B}_\infty = \bigcap_{i=0}^{\infty} \mathcal{B}_i$.

**Lemma 9.6.** If $f: M \to N$ is in $\mathcal{B}$ ($\mathcal{B}_n$) and $N$ is trivial ($n$-trivial) at $\infty$, then $M$ is also trivial ($n$-trivial) at $\infty$.

**Proof.** Let $A \subseteq M$ be a compactum. Pick $\tilde{A} \subseteq N$ as in the above definition, and, using triviality of $N$ at $\infty$, let $\tilde{A}' \supseteq \tilde{A}$ be such that every map of a sphere $S^k$ (of
dimension \( k \leq n \) into \( N-\tilde{A}' \) is null-homotopic in \( N-\tilde{A} \). Then \( B = f^{-1}(\tilde{A}') \) contains \( A \) and every map \( \varphi:S^k \to M-B \) is null-homotopic in \( M-A \) because \( \varphi \) and a constant map \( \psi:S^k \to f^{-1}(N-\tilde{A}') \) satisfy \( f \circ \varphi \sim f \circ \psi \) in \( N-\tilde{A} \) so that \( \varphi \sim \psi \) in \( M-A \). □

Let us observe that if, for a proper map \( f:M \to N \), there is a proper map \( g:N \to M \) with \( g \circ f \) homotopic to \( \text{id}_M \) at \( \infty \), then \( f \) is in the class \( \mathcal{B} \). Also, the inclusion \( \mathbb{R} \hookrightarrow \mathbb{R}^2 \) is not in \( \mathcal{B} \).

**Definition 9.7.** The class of proper maps that are both in \( \mathcal{C} \) and \( \mathcal{B} \) (\( \mathcal{C}_n \) and \( \mathcal{B}_n \)) will be denoted by \( \mathcal{B} \) (\( \mathcal{B}_n \)).

\[ \mathcal{B}_\infty = \bigcap_{i>0} \mathcal{B}_i. \]

Note that every homotopy equivalence at \( \infty \) is in the class \( \mathcal{B} \). Let \( f:X \to Y \) be a CE map between \( Z \)-sets \( X, Y \) in \( Q \). Take an extension \( F:Q \to Q \) of \( f \) such that \( F^{-1}(Y) = X \). In [19] (see also [29]), it was proved that \( F|_{Q-X} : Q-X \to Q-Y \) is in the class \( \mathcal{B}_\infty \). The converse statement is clearly false as there are maps that are shape equivalences but not CE.

**Theorem 9.8.** If \( f:M \to N \) is in the class \( \mathcal{B} \) (\( \mathcal{B}_n \)) and \( N \) is movable (n-movable), then \( M \) is movable (n-movable).

**Proof.** Given a compactum \( A \subset M \), take \( \tilde{A} \subset N \) according to Definition 9.5. Since \( N \) is movable, there
is \( \widetilde{B} \supset \widetilde{A} \) that satisfies \((\mathcal{M})\) of Chapter IV. We claim that \( B = f^{-1}(\widetilde{B}) \) will suffice.

Indeed, \( f^{-1}(N-\widetilde{A}) \subset M-A \) implies \( f^{-1}(\widetilde{A}) \supset A \) so \( B \supseteq A \). Let \( C \supset B \) be a compactum and \( \varphi:P \to M-B \) a map of a finite polyhedron \( P \) into \( M-B \). Since \( f \in C \) and \( f(C) \supset \widetilde{B} \) is a compactum we can find \( \widetilde{C} \supset f(C) \) such that every map \( \mu:K \to N-\widetilde{C} \) is homotopic in \( N-f(C) \subset N-\widetilde{A} \) to a map \( f \circ \mu \), where \( \widetilde{\mu}:K \to f^{-1}(N-f(C)) \subset N-C \) and \( K \) is a finite polyhedron.

Now, \( f \circ \varphi:P \to N-\widetilde{B} \) is homotopic in \( N-\widetilde{A} \) to a map \( \widetilde{\psi}:P \to N-\widetilde{C} \). But there is \( \psi:P \to f^{-1}(N-f(C)) \subset N-C \) with \( f \circ \psi \sim \widetilde{\psi} \) in \( N-f(C) \subset N-\widetilde{A} \). Hence, \( f \circ \varphi \sim f \circ \psi \) in \( N-A \) so that \( \varphi \sim \psi \) in \( M-A \) by the choice of \( \widetilde{A} \). \( \square \)

**Theorem 9.9.** Let \( f:M \to N \) be a homotopy domination at \( \infty \). If \( f \) in the class \( \mathcal{L}_\infty \) and \( M \) is both movable and tame at \( \infty \), then \( f \) is a homotopy equivalence at \( \infty \).

**Proof.** Take a proper map \( g:N \to M \) such that \( f \circ g \) is homotopic at \( \infty \) to \( \text{id}_N \). We will show that \( g \circ f \) is homotopic at \( \infty \) with \( \text{id}_M \). Let \( A \subset M \) be a compactum. Pick a compactum \( B_1 \) in \( N \) such that \( f^{-1}(N-B_1) \subset M-A \) and \( g(N-B_1) \subset M-A \). Since \( M \) is movable, there is a compactum \( D \supset A_1 \) that satisfies \((\mathcal{M})\) of Chapter IV with respect to \( A_1 \). \( M \), being tame at \( \infty \), gives us a compactum \( E \supset D \) so that the diagram
homotopy commutes for some finite complex $K$. Since $f \in C_\infty$, there is a compactum $B_2$ with $f^{-1}(N-B_2) \subset M-E$ and any two maps $\varphi, \psi: K \to f^{-1}(N-B_2)$ satisfying $f \circ \varphi = f \circ \psi$ in $N-B_2$ are homotopic in $M-E$. Take a compactum $B_3 \supset B_2$ so that $f \circ g |_{N-B_3}$ is homotopic in $N-B_2$ to the inclusion $i: N-B_3 \hookrightarrow N-B_2$. Then $B = f^{-1}(B_3)$ is our required compactum. Indeed, $\beta: K \to M-D$ is homotopic in $M-A$ to a map $\beta': K \to M-B$. Observe that $f \circ (g \circ f) \circ \beta' = (f \circ g) \circ (f \circ \beta')$ is homotopic in $N-B_2$ to the map $f \circ \beta'$. Hence, the choice of $B_2$ gives us that $g \circ f \circ \beta'$ and $\beta'$ are homotopic in $M-E$. But, then $g \circ f \circ \beta'$ and $g \circ f \circ \beta$ are homotopic in $g \circ f(M-D) \subset C \subset M-A$, and therefore $g \circ f \circ \beta$ and $\beta$ are homotopic in $M-A$. The compositions $g \circ f \circ \beta \circ \alpha$ and $\beta \circ \alpha$ are still homotopic in $M-A$. From here, one easily sees that $g \circ f |_{M-B}$ is homotopic to the inclusion $M-B \subset M-A$ inside $M-A$. □

Theorem 9.10. Let $f: M \to N$ be in $C_n \cap B$. Suppose $M$ is tame at $\infty$ and $N$ is $n$-tame at $\infty$. Then $M$ is also $n$-tame at $\infty$.

Proof. Take an arbitrary compactum $A \subset M$ and select $A_1 \subset N$ satisfying 9.5, $A_2 \supset A_1$ that satisfies 9.1 with respect to $A_1$, and $A_3 \supset A_2$ so that
homotopy commutes for some finite \( n \)-complex \( K \). Put \( \tilde{B} = f^{-1}(A_3) \supset A \). Then our required compactum \( B \supset \tilde{B} \) is chosen in such a way that

\[
\begin{array}{ccc}
M-B & \xleftarrow{j} & M-\tilde{B} \\
\pi & \searrow & \rho \\
& L & \\
\end{array}
\]

homotopy commutes for some finite complex \( L \). Let \( \tilde{\beta}:K \rightarrow f^{-1}(N-A_1) \) have the property \( f_0\tilde{\beta} = \beta \) in \( N-A_1 \). Consider maps \( \tilde{u} = a_0f_0j \) and \( \tilde{\beta} \). It suffices to prove that \( \tilde{\beta}_0\tilde{u} = j \) in \( M-A \). To this end, observe that \( \rho \) and \( \beta_0a_0f_0\rho \) are two maps of \( L \) into \( f^{-1}(N-A_1) \) with \( f_0\rho \) and \( f_0\tilde{\beta}_0a_0f_0\rho \) homotopic in \( N-A_1 \) because \( f_0\tilde{\beta} = \beta \) in \( N-A_1 \) and \( \beta_0a = i \) in \( N-A_2 \subset N-A_1 \). The choice of \( A_1 \) implies \( \rho = \tilde{\beta}_0a_0f_0\rho \) in \( M-A \). But, then \( \rho_0\pi = \tilde{\beta}_0(a_0f_0)(\rho_0\pi) \) in \( M-A \). Now, it is easy to see that this gives \( \tilde{\beta}_0(a_0f_0j) = j \). □

**Lemma 9.11.** Let \( f:M \rightarrow N \) be in \( C \) and assume \( M \) is \( n \)-tame at \( \infty \) and \( N \) is tame at \( \infty \). Then \( N \) is \( n \)-tame at \( \infty \).

**Proof.** Let \( A \subset N \) be a compactum. For \( A' = f^{-1}(A) \) find a compactum \( B' \supset A' \) and a finite \( n \)-complex \( K \) so that
homotopy commutes. Let $B_1$ be such that $f^{-1}(N-B_1) \subseteq M-B'$, $B_2 \supset B_1$ is picked using $f \in C$, and a required $B \supset B_2$ makes

$$
\begin{array}{ccc}
N-B_2 & \leftarrow & N-B \\
\beta & \downarrow & \alpha \\
L & & \\
\end{array}
$$

homotopy commutative for some finite complex $L$. Now, take a map $\tilde{\beta}:L \rightarrow f^{-1}(N-B_2)$ for which $f \circ \tilde{\beta} = \beta$ in $N-B_1$. Consider maps $\alpha' = \rho \circ \tilde{\beta} \circ \alpha: N-B \rightarrow K$ and $\beta' = f \circ \pi: K \rightarrow N-A$. We will show, $\beta' \circ \alpha' = \iota$, the inclusion $N-B \hookrightarrow N-A$, in $N-A$. Indeed, $\beta' \circ \alpha' = f \circ (\pi \circ \rho) \circ (\tilde{\beta} \circ \alpha)$ is homotopic in $f(M-A') \subseteq N-A$ to $f \circ j \circ (\tilde{\beta} \circ \alpha)$, where $j$ is the inclusion $M-B' \hookrightarrow M-A'$. But $f \circ j \circ \tilde{\beta} \circ \alpha = f \circ \tilde{\beta} \circ \alpha$ is homotopic in $N-B_1$ to $\tilde{\beta} \circ \alpha$ and, therefore, to $\iota$ as claimed. □

**Corollary 9.12.** Let $f:M \rightarrow N$ be an $\mathcal{G}$-map between locally compact metric spaces that are tame at $\infty$. Then $M$ is n-tame at $\infty$ if and only if $N$ is n-tame at $\infty$. □

**Theorem 9.13.** Let $f:M \rightarrow N$ be in $C_\infty$ and suppose $N$ is both movable and tame at $\infty$ while $M$ is n-tame at $\infty$. 

```
Then $N$ is $n$-tame at $\infty$.

**Proof.** Let $A \subseteq N$ be a compactum. For $A' = f^{-1}(A)$ find a compactum $B' \supset A'$ and a finite $n$-complex $K$ so that

$$
\begin{array}{ccc}
M-B' & \overset{j_1}{\longrightarrow} & M-A' \\
\alpha & \downarrow & \beta \\
K & & \\
\end{array}
$$

homotopy commutes. Since $N$ is movable, there is a compactum $C \supset A$ so that condition (M) of Chapter IV holds. Let $D \supset C$ and a finite complex $L^k$ be such that

$$
\begin{array}{ccc}
N-D & \overset{i_1}{\longrightarrow} & N-C \\
\pi & \downarrow & \rho \\
L & & \\
\end{array}
$$

commutes up to homotopy and $f^{-1}(N-D) \subseteq M-B'$. Finally, our required compactum $B \supset D$ is picked using the fact $f \in C^k$. We will show that the inclusion $N-B \hookrightarrow N-A$ factors through the complex $K$. Take a map $\rho':L \rightarrow N-B$ homotopic in $N-A$ to $\rho$. The choice of $B$ gives us a map $\beta':L \rightarrow f^{-1}(N-D) \subseteq M-B'$ with $f \circ \beta' \sim \beta'$ in $N-D$.

Define $a:N-B \rightarrow K$ by $a = \alpha \circ \rho' \circ (\pi|_{N-B})$ and $b:K \rightarrow N-A$ by $b = f \circ \beta$. One can easily check that $b \circ a \sim i$ in $N-A$. $\square$

**Theorem 9.14.** Let $f:M \rightarrow N$ be in $C_n \cap B_\infty$ and assume $M$ is movable and tame at $\infty$ while $N$ is $n$-tame
at $\infty$. Then $M$ is also $n$-tame at $\infty$. 

**Proof.** Let $A \subseteq M$ be an arbitrary compactum and take $C \supseteq A$ so that the condition (M) from Chapter IV holds. Next, pick a compactum $D \supseteq C$ and a finite complex $L^k$ for which the diagram

\[
\begin{array}{ccc}
M-C & \xrightarrow{i} & M-D \\
\downarrow{\rho} & & \downarrow{\pi} \\
L & & \\
\end{array}
\]

homotopy commutes. Now, we select $A_1 \subseteq N$ using $f \in B_k$ (with respect to $D$) and then $A_2 \supseteq A_1$ using $f \in C_n$. Finally, $A_3 \supseteq A_2$ is such that

\[
\begin{array}{ccc}
N-A_2 & \xrightarrow{j} & N-A_3 \\
\downarrow{\beta} & & \downarrow{\alpha} \\
K & & \\
\end{array}
\]

commutes for some finite $n$-complex $K$, and $B = f^{-1}(A_3)$ is the required compactum. Indeed, let $\tilde{\beta}: K \to f^{-1}(N-A_1)$ satisfy $f \circ \tilde{\beta} \sim \beta$ in $N-A_1$. Take $(\tilde{\beta})': K \to M-B$ and $\rho': L \to M-B$ where $(\tilde{\beta})' = \tilde{\beta}$ and $\rho' \sim \rho$ with both homotopies in $M-A$. Define $\alpha: M-B \to K$ as $\alpha = \alpha \circ (f|_{M-B}) \circ (\rho' \circ \pi|_{M-B})$. We will show that $\tilde{\beta} \circ \alpha$ is homotopic to the inclusion $i: M-B \supseteq M-A$ in $M-A$. Observe that $\tilde{\beta} \circ f' \circ \rho'$ and $\rho'$ are two maps of $L$ into $f^{-1}(N-A_1)$ with
Lemma 9.15. Classes $C, C_n, \mathcal{J}, \mathcal{J}_n$ are closed under composition of maps.

Proof. Let $f:M \to N$ and $g:N \to P$ be proper maps between locally compact metric spaces. We will prove that
(a) $f, g \in C$ implies $gof \in C$, and (b) $f, g \in \mathcal{J}$ implies $gof \in \mathcal{J}$.

(a) Let $A \subset P$ be a compactum. Since $g^{-1}(A)$ is a compactum in $N$ and $f \in C$, there is $C \supset g^{-1}(A)$ so that for any map $\tilde{\varphi}:K \to N-C$ of a finite complex $K$ there is a map $\tilde{\varphi}:K \to f^{-1}(N-g^{-1}(A)) = M - (gof)^{-1}(A)$ with $f \circ \tilde{\varphi} = \tilde{\varphi}$ in $N-C$. Since $g \in C$ and $g(C)$ is a compactum in $P$, there is $B \supset g(C) \supset A$ so that for any map $\varphi:K \to P-B$ of a finite complex $K$ we can find $\tilde{\varphi}:K \to g^{-1}(P-g(C)) \subset N-C$ such that $g \circ \tilde{\varphi} = \varphi$ in $P-g(C) \subset P-A$. Hence, $(gof) \circ \tilde{\varphi} = g \circ \varphi = \varphi$ in $P-A$.

(b) Since $f \in \mathcal{J}$, there is a compactum $\tilde{\mathcal{A}} \subset N$ satisfying 9.5 with respect an arbitrary compactum $\mathcal{A}$ in $M$. Then use $g \in \mathcal{J}$ and pick $\tilde{\mathcal{A}} \subset P$ with respect to $\tilde{\mathcal{A}}$. Observe that $g^{-1}(P-\tilde{\mathcal{A}}) \subset N-\tilde{\mathcal{A}}$ and $f^{-1}(N-\tilde{\mathcal{A}}) \subset M-\mathcal{A}$, hence,
$f^{-1}g^{-1}(P\sim) \subset M\sim$ and given $\varphi,\psi:K \rightarrow f^{-1}g^{-1}(P\sim)$ with $g\circ f\circ \varphi = g\circ f\circ \psi$ in $P\sim$, then $f\circ \varphi = f\circ \psi$ in $N\sim$ and, finally, $\varphi \sim \psi$ in $M\sim$. □

We will say that a locally compact metric space $M$ has type $n$ at $\infty$ provided for any compactum $A \subset M$ there is a larger compactum $B$ so that every two maps $\varphi,\psi:X \rightarrow M-B$ that are $n$-homotopic in $M-B$ (i.e., for every map $\alpha:K \rightarrow X$ of an at most $n$-dimensional finite complex $K$, $\varphi\alpha = \psi\alpha$ in $M-B$) are homotopic in $M-A$.

Note 9.16. If $f:M \rightarrow N$ is in $\mathcal{D}_n$ and $M$ has type $n$ at $\infty$, then $f$ is in $\mathcal{D}$.

Proof. Let $A$ be a compactum in $M$ and take $B \supset A$ according to above definition. Now, choose a compactum $\tilde{B}$ in $N$ using $f \in \mathcal{D}_n$. Assume $\varphi,\psi:P \rightarrow f^{-1}(N\sim)$ satisfy $f\circ \varphi = f\circ \psi$ in $N\sim$. If $\alpha:K \rightarrow P$ is an arbitrary map of an at most $n$-dimensional finite complex $K$ into $P$. Then $f\circ \varphi\alpha = f\circ \psi\alpha$ in $N\sim$, so the choice of $\tilde{B}$ gives $\varphi\alpha = \psi\alpha$ in $M-B$. In other words, $\varphi$ and $\psi$ are $n$-homotopic in $M-B$. Therefore, $\varphi \sim \psi$ in $M-A$. □

Lemma 9.17. Let $f:M \rightarrow N$ be a proper map between locally compact metric spaces. Then the natural projection $\pi:Z_f \rightarrow N$ of a mapping cylinder $Z_f$ of $f$ onto $N$ is in the class $\mathcal{D}$. 
Proof. (a) $\pi$ is in $\mathcal{C}$. If $A$ is a compactum in $N$ and $\varphi: P \to N-A$ is a map of a finite complex into $N-A$, let $\tilde{\varphi}: P \to Z_f^{-1}(A) = \pi^{-1}(N-A)$ be the composition $j \circ \varphi$, where $j: N \to Z_f$ is the natural inclusion. Clearly, $\pi \circ j = \text{id}_N$.

(b) $\pi$ is in $\mathcal{B}$. Let $A \subset Z_f$ be a compactum. Put $\tilde{A} = \pi(A) \subset N$. If $\varphi, \psi: P \to \pi^{-1}(N-\tilde{A})$ are two maps of a finite complex $P$ such that $\pi \circ \varphi = \pi \circ \psi$ in $N-\tilde{A}$, then $\varphi = \psi$ in $Z_f-A$ because $\varphi = j \circ \varphi$ in $Z_f-A$ and $\psi = j \circ \varphi$ in $Z_f-A$. □

Lemma 9.18. Let $f: M \to N$ and $g: N \to P$ be proper maps between locally compact metric spaces. If $g$ is in $\mathcal{B}$ ($\mathcal{B}_n$) and $h = g \circ f$ is in $\mathcal{C}$ ($\mathcal{C}_n$), then $f$ is in $\mathcal{C}$ ($\mathcal{C}_n$).

Proof. Let a compactum $A \subset N$ be given. Since $g \in \mathcal{B}$ we can find a compactum $\tilde{A} \subset P$ such that $g^{-1}(P-\tilde{A}) \subset N-A$ and whenever $\varphi, \psi: K \to g^{-1}(P-\tilde{A})$ are two maps of a finite complex with $g \circ \varphi = g \circ \psi$ in $P-\tilde{A}$, then $\varphi = \psi$ in $N-A$. Note that $g(A) \subset \tilde{A}$. Since $h \in \mathcal{C}$, there is $B' \supset \tilde{A}$ such that every map $\psi: K \to P-B'$ can be up to homotopy in $P-\tilde{A}$ represented as $h \circ \tilde{\psi}$, where $\tilde{\psi}: K \to h^{-1}(P-\tilde{A}) \subset f^{-1}(N-A)$. Let $B = g^{-1}(B') \supset A$. If $\varphi: K \to N-B$ is an arbitrary map of a finite complex, then $\psi = g \circ \varphi: K \to P-B'$ is homotopic in $P-\tilde{A}$ to $g \circ f \circ \tilde{\psi}$, where $\tilde{\psi}: K \to h^{-1}(P-\tilde{A})$. By the choice of $\tilde{A}$, $f \circ \tilde{\psi} = \varphi$ in $N-A$. □

Lemma 9.19. Let $f: M \to N$ and $g: N \to P$ be proper
maps between locally compact metric spaces. If \( h = g \circ f \) is in the class \( \mathcal{B}_n \), then \( f \in \mathcal{B}_n \) (\( f \in \mathcal{B}_n \)).

**Proof.** Take a compactum \( A \subset M \). Since \( h \in \mathcal{B}_n \), there is \( A' \subset P \) such that \( h^{-1}(P-A') \subset M-A \) and for any two maps \( \varphi, \psi : K \to h^{-1}(P-A') \) of a finite complex with \( h \circ \varphi = h \circ \psi \) in \( P-A' \) it follows \( \varphi \simeq \psi \) in \( M-A \). Let \( \widetilde{A} = g^{-1}(A') \). One easily verifies that 9.5 holds. \( \square \)


**Theorem 9.20.** A proper map \( f : M \to N \) is in a class \( \mathfrak{X} \) \((\mathfrak{X} = \mathcal{C}_n, \mathcal{C}_n, \mathcal{B}_n, \mathcal{B}_n, \mathcal{B}_n, \mathcal{B}_n)\) iff the inclusion \( i : M \to Z_f \) is in a class \( \mathfrak{X} \).

**Theorem 9.21.** Let \( f : M \to N \) be a proper map between finite-dimensional locally compact ANR's. If \( f \in \mathfrak{B}_n \), then \( f \) is a homotopy equivalence at \( \infty \).

**Proof.** There is no loss of generality if we assume that \( M, N \) are locally finite CW complexes [51]. Now, the idea is to verify that conditions \( (\pi_1)_\infty \) and \( (\pi_n)_\infty \) in [42] hold for the inclusion \( X \subset Y \), where \( X \) is a copy of \( M \) and \( Y \) is a mapping cylinder of \( f \).

Since \( X \subset Y \) is of class \( \mathfrak{B}_n \) (Corollary 9.20) for any cofinite subcomplex \( Y' \) of \( Y \) (i.e., \( Y-Y' \) has compact closure) there exists a cofinite \( Y'' \subset Y \) so that (a) every
map \( \varphi:K \rightarrow Y'' \) is homotopic in \( Y' \) to a map \( \tilde{\varphi}:K^0 \rightarrow X' \), where \( K \) is any finite complex and \( X' = X \cap Y' \), and (b) any two maps \( \varphi, \psi:K \rightarrow X'' = X \cap Y'' \) that are homotopic in \( Y'' \) are homotopic in \( X' \).

One can easily check that (a) and (b) imply that \( (\pi_1)_\infty \) holds. Some diagram chasing will be necessary to see that the condition

\[
(\pi_\ast)_\infty: \text{For some cofinite subcomplex } Y' \text{ in each neighborhood of } \infty \text{ in } Y, \text{ there is a cofinite } Y'' \subset Y' \text{ so that the map } \pi_\ast(Y'',X'') \rightarrow \pi_\ast(Y',X') \text{ is zero is satisfied.}
\]

Let \( Y' \) be given inside an arbitrary neighborhood of \( \infty \). Take cofinite subcomplexes \( Y_0 \supset Y_1 \supset Y_2 \supset Y_3 \supset Y_4 = Y'' \) such that (a) and (b) hold for each pair \((Y_i,Y_{i+1})\) (\( i = 0,1,2,3 \)). Consider homotopy ladders for pairs \( (Y_i,X_i = X \cap Y_i) \).

\[
\begin{array}{cccccccccc}
\pi_n(X_4) & \xrightarrow{k_4} & \pi_n(Y_4) & \xrightarrow{l_4} & \pi_n(Y_4,X_4) & \xrightarrow{\partial} & \pi_{n-1}(X_4) & \xrightarrow{k_4} & \pi_{n-1}(Y_4) \\
\downarrow i_{43} & & \downarrow j_{43} & & \downarrow l_3 & & \downarrow m_{43} & & \downarrow i_{43} & & \downarrow j_{43} \\
\pi_n(X_3) & \xrightarrow{k_3} & \pi_n(Y_3) & \xrightarrow{l_3} & \pi_n(Y_3,X_3) & \xrightarrow{\partial} & \pi_{n-1}(X_3) & \xrightarrow{k_3} & \pi_{n-1}(Y_3) \\
\downarrow i_{32} & & \downarrow j_{32} & & \downarrow l_2 & & \downarrow m_{32} & & \downarrow i_{32} & & \downarrow j_{32} \\
\pi_n(X_2) & \xrightarrow{k_2} & \pi_n(Y_2) & \xrightarrow{l_2} & \pi_n(Y_2,X_2) & \xrightarrow{\partial} & \pi_{n-1}(X_2) & \xrightarrow{k_2} & \pi_{n-1}(Y_2) \\
\downarrow i_{21} & & \downarrow j_{21} & & \downarrow l_1 & & \downarrow m_{21} & & \downarrow i_{21} & & \downarrow j_{21} \\
\pi_n(X_1) & \xrightarrow{k_1} & \pi_n(Y_1) & \xrightarrow{l_1} & \pi_n(Y_1,X_1) & \xrightarrow{\partial} & \pi_{n-1}(X_1) & \xrightarrow{k_1} & \pi_{n-1}(Y_1) \\
\downarrow i_{10} & & \downarrow j_{10} & & \downarrow l_0 & & \downarrow m_{10} & & \downarrow i_{10} & & \downarrow j_{10} \\
\pi_n(X_0) & \xrightarrow{k_0} & \pi_n(Y_0) & \xrightarrow{l_0} & \pi_n(Y_0,X_0) & \xrightarrow{\partial} & \pi_{n-1}(X_0) & \xrightarrow{k_0} & \pi_{n-1}(Y_0). \\
\end{array}
\]
Here i's, j's and k's are induced by inclusions. We will show that \( m_{40}: \pi_n(Y',X') \to \pi_n(Y'',X'') \) is a zero map, for every \( n \). To this end, first define a function \( \beta: \pi_n(Y_2) \to \pi_n(X_0) \) such that \( k_0 \circ \beta = j_{20} \) as follows: if \( [\varphi] \in \pi_n(Y_2) \) has a representative \( \varphi \), by (a), there is a map \( \psi \) into \( X_1 \) homotopic in \( Y_1 \) to \( \varphi \); put \( \beta([\varphi]) = i_{10}(\psi) \). Now, take \( [\varphi] \in \pi_n(Y_4,X_4) \). Since \( k_4 \circ (\varphi) = 0 = k_4(0) \), by (b), \( i_{43} \circ \beta([\varphi]) = i_{43} \circ \beta(0) \) or \( \partial \circ m_{43}([\varphi]) = 0 \). Hence, \( i_{32} \circ \partial \circ m_{43}([\varphi]) = 0 \). By commutativity, \( \partial \circ m_{42}([\varphi]) = 0 \) which means that \( \text{Im}(m_{42}) \subseteq \ker \delta \). But \( \ker \delta = \text{Im} \ l_2 \) so that \( m_{43}([\varphi]) = l_2([\psi]) \), for some element \( [\psi] \in \pi_n(Y_2) \). Now, \( m_{40}([\varphi]) = m_{20} l_2([\psi]) = l_0 \circ j_{20}([\psi]) = l_0 k_0 \circ \beta([\varphi]) = 0 \) since the last row is exact.

**Theorem 9.22.** Let \( f:M \to N \) be a proper map for which there is a proper map \( g:N \to M \) with \( g \circ f \) homotopic at \( \infty \) to \( \text{id}_M \). If \( f \in \mathcal{C} \) and \( N \) is tame at \( \infty \), then \( f \) is a homotopy equivalence at \( \infty \).

**Proof.** We will show that \( f \circ g \) is homotopic at \( \infty \) to \( \text{id}_N \). Let \( A \subset N \) be an arbitrary compactum. \( A' = f^{-1}(A) \) is a compactum in \( M \). Pick \( B' \supset A' \) so that \( g \circ f|_{M-B'} \) is homotopic in \( M-A' \) to the inclusion \( i:M-B' \to M-A' \). Let \( B_1 = g^{-1}(B') \). \( B_2 \supset B_1 \) is chosen in such a way that every map \( \varphi:P \to N-B_2 \) of a finite complex \( P \) is homotopic in \( N-B_1 \) to \( f \circ \tilde{\varphi} \), where \( \tilde{\varphi}:P \to f^{-1}(N-B_1) \). Then our required
compactum $B$ is selected so that the inclusion $j: N-B \hookrightarrow N-B_2$ factors up to homotopy through a finite complex $K$

\[ \begin{array}{ccc}
N-B & \xrightarrow{j} & N-B_2 \\
\downarrow{\alpha} & & \downarrow{\beta} \\
K & & 
\end{array} \]

By assumption, there is $\tilde{\beta}: K \to f^{-1}(N-B_1)$ such that $f \circ \tilde{\beta} \sim \beta$ in $N-B_1$. Then $g \circ f \circ \tilde{\beta} \sim g \circ \beta$ in $M-B'$. But $g \circ f|_{M-B'} \sim i$ in $M-A'$ so that $f \circ g \circ \beta = f \circ \tilde{\beta} \sim \beta$ in $N-A$. Therefore, $f \circ g \circ \beta \circ \alpha \sim \beta \circ \alpha$ in $N-A$ or $f \circ g|_{N-B} = \text{the inclusion } N-B \hookrightarrow N-A$ in $N-A$. □

**Lemma 9.23.** Let $f: M \to N$ be of class $C_\infty$ with $N$ both movable and tame at $\infty$. Then $f$ is in $C$.

**Proof.** Take an arbitrary compactum $A \subset N$ and pick compacta $B_1$ and $B_2$ such that $B_1$ satisfies condition $(M)$ from Chapter IV with respect to $A$ while

\[ \begin{array}{ccc}
N-B_1 & \xleftarrow{\beta} & N-B_2 \\
\downarrow{\beta} & & \downarrow{\alpha} \\
K & & 
\end{array} \]

homotopy commutes, for some finite complex $K$. Since $f \in C_{\dim K}$ we can select our required $B$ for which 9.1 holds with respect to $B_2$. Assume $\phi: P \to N-B$ is a map of
an arbitrary finite complex \( P \). Let \( \beta':K \to N-B \) be homotopic in \( N-A \) to \( \beta \), and \( \beta':K \to f^{-1}(N-B_2) \subset f^{-1}(N-A) \) a map with \( f\circ \beta' \sim \beta \) in \( N-B_2 \). Then one easily sees that \( \phi = \beta' \circ (\alpha|_{N-B}) \circ \phi \) has the property \( f\circ \phi \sim \phi \) in \( N-A \). 

**Theorem 9.24.** Let \( f:M \to N \) be a domination at \( \infty \) and suppose \( f \in B \). If \( M \) is tame at \( \infty \), then \( f \) is an equivalence at \( \infty \).

**Proof.** Take a proper map \( g:N \to M \) such that \( f\circ g \) is homotopic at \( \infty \) to \( \text{id}_N \). We will demonstrate that \( g\circ f \) is homotopic at \( \infty \) to \( \text{id}_M \).

Let \( A \subset M \) be any compactum. Pick a compactum \( \tilde{A} \) in \( N \) using \( f \in B \), and a compactum \( \tilde{B} \supset \tilde{A} \) for which \( f\circ g|_{N-\tilde{B}} \) is homotopic in \( N-\tilde{A} \) to the inclusion \( i:N-\tilde{B} \subset N-\tilde{A} \). Let \( B_1 \supset A \) satisfy \( f(M-B_1) \subset N-\tilde{B} \) and select our required compactum \( B \supset B_1 \) in such a way that the diagram

\[
\begin{array}{ccc}
M-B & \xrightarrow{j} & M-B_1 \\
\downarrow{\alpha} & & \downarrow{\beta} \\
K & & \\
\end{array}
\]

homotopy commutes for some finite complex \( K \). Since 
\( f \circ (g\circ f\beta) = (f\circ g) \circ (f\circ \beta) \) is homotopic in \( N-\tilde{A} \) to \( f\circ \beta \), by the choice of \( \tilde{A} \) we have, \( g\circ f\circ \beta \sim \beta \) in \( M-A \). Hence \( g\circ f\circ \beta \circ \alpha \sim \beta \circ \alpha \) in \( M-A \). But \( \beta \circ \alpha \sim j \) in \( M-B_1 \) and 
\( g \circ f(M-B_1) \subset M-A \) so that \( g \circ f \sim i \) in \( M-A \).
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Major Field: Mathematics

Title of Thesis: Infinite-Dimensional Topology and the Theory of Shape

Approved:

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Date of Examination: December 1, 1975