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Fourier-Stieltjes Transforms of Measures With a Certain Continuity Property.

Laurence Thomas Ramsey
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FOURIER-STIELTJES TRANSFORMS OF MEASURES

WITH A CERTAIN CONTINUITY PROPERTY

A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College
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The Department of Mathematics

by

Laurence Thomas Ramsey
B.A., University of North Dakota, 1970
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ABSTRACT

Let $G$ be a compact abelian group whose dual group $\Gamma$ has a finite torsion subgroup. Let $\mu \in M(G)$ such that $|\mu|$ assigns no mass to any coset of any closed subgroup of $G$ whose index is infinite. Then there is $d > 0$, dependent only on $||\mu||$, such that if for each $\gamma \in \Gamma$, $|\hat{\mu}(\gamma)| \geq 1$ or $|\hat{\mu}(\gamma)| \leq d$, then the set $\{\gamma : |\hat{\mu}(\gamma)| \geq 1\}$ is finite. An upper bound on the cardinality of this set is obtained in terms of $||\mu||$ and the cardinality of the torsion subgroup of $\Gamma$. 
SECTION I
INTRODUCTION

G will denote a compact abelian group, \( \Gamma \) its dual group and \( M(G) \) the measure algebra of finite Borel measures on \( G \). We shall assume that \( \Gamma \) has a finite torsion subgroup. Let \( \mu \in M(G) \) such that \( |\mu| \) assigns no mass to any coset of any closed subgroup of \( G \) whose index is infinite. We prove that there is a number \( d > 0 \), dependent only on \( ||\mu|| \), such that if for each \( \gamma \in \Gamma \), \( |\hat{\mu}(\gamma)| \geq 1 \) or \( |\hat{\mu}(\gamma)| \leq d \) then the set of \( \gamma \) such that \( |\hat{\mu}(\gamma)| \geq 1 \) is finite. We obtain an upper bound on the cardinality of this set in terms of \( ||\mu|| \) and the cardinality of the torsion subgroup of \( \Gamma \).

De Leeuw and Katznelson first proved this theorem for the circle group \( T \) [1, Lemma 2]. They proved that, for any \( C > 0 \), there is \( d = d(C) < 10^{-2} \) satisfying the following: Suppose that \( \mu \in M(T) \) is a continuous measure with \( ||\mu|| \leq C \) and, for \( |n| \) sufficiently large, \( |\hat{\mu}(n)| < d \) or \( \text{Re}(\hat{\mu}(n)) > 1-d \); then \( \{n : |\hat{\mu}(n)| \geq d \} \) is finite. Without a numerical bound on the cardinality of \( \{n : |\hat{\mu}(n)| \geq d \} \)
their method does not seem to generalize. Such a bound can be obtained by imitating Davenport's procedure in [2], if \( \alpha \) is small enough and \( |\hat{\alpha}(n)| < \alpha \) or \( |\hat{\alpha}(n)| > 1-\alpha \) for all integers \( n \). In [2] Davenport proves that if a trigonometric polynomial \( p(x) = \sum a(n)\exp(2\pi inx) \) has \( N \) coefficients of modulus at least one and all other coefficients equal to zero, then the \( L^1 \)-norm of \( p \) is at least
\[
8^{-1}(\log N)^{1/4}(\log \log N)^{-1/4}.
\]

For an arbitrary locally compact abelian group \( G \), Glicksberg proved in [3] that if \( \mu \in \mathcal{M}(G) \) and \( 0 \) is isolated in \( \{0\} \cup \hat{\mu}(\hat{G}) \) then there is a compact subgroup \( H \) of \( G \) for which \( \mu_H \), the part of \( \mu \) carried by the cosets of \( H \), is the convolution of a non-zero idempotent and an invertible. He proved that
\[
\mu_H = [\sum_{\gamma \in \Lambda} \gamma \hat{\mu}] * \lambda
\]
where \( \Lambda \) is a finite subset of \( \hat{G} \), \( m_H \) is the Haar measure on \( H \), and \( \lambda \in \mathcal{M}(G)^{-1} \). For measures \( \mu \) such that \( \mu_H = 0 \) when \( H \) is a closed subgroup of \( G \) of infinite index, the hypothesis that \( 0 \) is isolated in \( \{0\} \cup \hat{\mu}(\hat{G}) \) yields the conclusion that \( \mu \) is a trigonometric polynomial. When \( G \) is compact and \( \hat{G} \) has a finite torsion subgroup, one can then use Hewitt and Zuckerman's generalization [4] of Davenport's result [2] to estimate the cardinality of the set of \( \gamma \in \Gamma \) such that \( |\hat{\mu}(\gamma)| > 0 \).
SECTION II
THEOREMS

In this section we state our theorems precisely and prove Theorem 2 assuming Theorem 1. In what follows $B = B(\mu) = \{ \gamma \in \Gamma : |\hat{\mu}(\gamma)| \geq 1 \}$.

**Theorem 1.** Let $G$ be a compact abelian group whose dual group $\Gamma$ has at most $K$ torsion elements. Let $\mu \in M(G)$ such that $|\mu|$ assigns no mass to any coset of any closed subgroup of $G$ whose index is infinite. If $\text{card}(B(\mu)) > K(r+1)^{3r^2}$, then there exist $\gamma_0$ and $\gamma_k, j, 1 \leq k \leq r^2, 1 \leq j \leq r$, in $B(\mu)$ such that if $P_0 = \{\gamma_0\}$ and

$$P_{k+1} = P_k \cup \{\gamma_{k+1}, j : 1 \leq j \leq r\} \cup \bigcup_{i<j} P_k + \gamma_{k+1}, i - \gamma_{k+1}, j ,$$

then

for $\gamma \in P_{k-1}$ and $i < j$, we have $\gamma + \gamma_{k, i} - \gamma_{k, j} \not\in B$. (1)

The proof of Theorem 1 requires several reductions which
will be postponed to later sections. Theorem 1 was suggested by Theorem 1' which for the case of $G = T$ can be found in Davenport's paper [2].

**Theorem 1'**. Let $G$ be a compact abelian group whose dual group $\Gamma$ is an ordered group. Suppose that $\mu \in M(G)$ such that $(r+1)^{3r^2} < \text{card}(B(\mu)) < \infty$. Then there exist $Y_0$ and $Y_{k,j}$, $1 \leq k \leq r^2$, $1 \leq j \leq r$, in $B(\mu)$ such that if $P_0 = \{Y_0\}$ and

$$P_{k+1} = P_k \cup \{Y_{k+1,j} : 1 \leq j \leq r\} \cup \left[ \bigcup_{1 \leq j} P_k + Y_{k+1,i} - Y_{k+1,j} \right],$$

then

for $\gamma \in P_{k-1}$ and $i < j$ we have $\gamma + Y_{k,i} - Y_{k,j} \not\in B$.

Since the proof of Theorem 1' found in [2] for $G = T$ works without change for the general case, we omit the proof of Theorem 1'.

**Theorem 2**. Let $G$ be a compact abelian group whose dual group $\Gamma$ has at most $K$ torsion elements. Let $\mu \in M(G)$ such that $|\mu|$ assigns no mass to any coset of any closed subgroup of $G$ whose index is infinite. Let $r$ be a positive integer greater than 2 such that $4^{-1}(1-e^{-2})r^{1/2} > ||\mu||$.

If
\begin{equation}
|\hat{\mu}(\gamma)| \geq 1 \quad \text{or} \quad |\hat{\mu}(\gamma)| \leq 2^{-1}r^{3/2}r^{-2r^2}
\end{equation}

for all $\gamma \in \Gamma$, then the cardinality of $B(\mu)$ is at most $K(r+1)^{3r^2}$.

The proof of Theorem 2 is adapted from [2]. Originally condition (2) read

\begin{equation}
|\hat{\mu}(\gamma)| \geq 1 \quad \text{or} \quad |\hat{\mu}(\gamma)| \leq 2^{-1}r^{3/2}(r+1)^{-3r^2}.
\end{equation}

Gordon Woodward suggested the improvement.

\underline{Proof of Theorem 2.} Suppose $\text{card}(B(\mu)) > K(r+1)^{3r^2}$. Using $\gamma_0$ and $\gamma_{k,j}$, $1 \leq k \leq r^2$, $1 \leq j \leq r$, as given by Theorem 1, we define trigonometric polynomials $\varphi_0, \ldots, \varphi_{r^2}$ inductively as follows:

\[ \varphi_0 = \overline{\sigma}(\hat{\mu}(\gamma_0))(\gamma_0, \cdot) \]

where $\sigma(x) = x|x|^{-1}$ for $x \neq 0$ and $\overline{\sigma}(x) = \overline{x}|x|^{-1}$.

\[ \varphi_k = \varphi_{k-1}[1 - 2r^{-2} - r^{-3} \sum_{i<j} \overline{\sigma}(\hat{\mu}(\gamma_{k,i}))\sigma(\hat{\mu}(\gamma_{k,j}))(\gamma_{k,i} - \gamma_{k,j}, \cdot)] \]

\[ + r^{-5/2} \sum_{j} \overline{\sigma}(\hat{\mu}(\gamma_{k,j}))(\gamma_{k,j}, \cdot). \]

Note that if $P_0, \ldots, P_{r^2}$ are defined as in the statement of
Theorem 1, each $\varphi_k$ is a $P_k$-polynomial. By [2, Lemmas 1 and 2], $|\varphi_k(g)| \leq 1$ for all $g \in G$. Let $I_k = \int_G \varphi_k(g) d\mu(-g)$. Then $I_0 = |\hat{\mu}(\gamma_0)| \geq 1$. Moreover

$$\text{Re}(I_k) \geq (1-2r^{-2})\text{Re}(I_{k-1}) + \frac{1}{2} r^{-3/2}. \quad (3)$$

To compute (3) we write

$$I_k = (1-2r^{-2})I_{k-1} + r^{-5/2} \sum_j |\hat{\mu}(\gamma_{k,j})|$$

$$- r^{-3} \sum_{\gamma \in P_{k-1}} \sum_{1<j} \varphi_{k-1}(\gamma) \overline{\sigma(\hat{\mu}(\gamma_{k,i}))} \sigma(\hat{\mu}(\gamma_{k,j})) \hat{\mu}(\gamma + \gamma_{k,i} - \gamma_{k,j})$$

$$= (1-2r^{-2})I_{k-1} + r^{-5/2} \sum_j |\hat{\mu}(\gamma_{k,j})| - r^{-3} A.$$

Thus,

$$\text{Re}(I_k) \geq (1-2r^{-2})\text{Re}(I_{k-1}) + r^{-3/2} - r^{-3} |A|.$$

Observe that each term of $A$ is bounded in modulus by $2^{-1} r^{3/2} r^{-2r^2}$ by (1) and (2) and that the number of terms in $A$ is at most $\frac{1}{2} r (r-1) \cdot \text{card}(P_{k-1}) \leq r^2 \text{card} P_{k-1} \leq r^{2k}$.

Note that $\text{card}(P_0) = 1 \leq r^{2.0}$ and that

$$\text{card}(P_{k+1})$$

$$= \text{card}(P_k \cup \{\gamma_{k+1,j} : 1 \leq j \leq r\} \cup \{ \cup_{i<j} P_k + \gamma_{k+1,i} - \gamma_{k+1,j}\})$$

$$\leq \text{card} P_k + r + \frac{1}{2} r (r-1) \text{card} P_k \leq (1+r+\frac{1}{2} r (r-1)) \text{card}(P_k)$$

$$\leq r^2 \text{card}(P_k).$$
hence \( \text{card}(P_k) \leq r^{2k} \). It follows from (3) using induction that

\[
\text{Re}(I_k) \geq \frac{1}{4} r^{1/2} - (1 - r^{-2})^k \left( \frac{1}{4} r^{1/2} - 1 \right).
\]

For \( k = r^2 \) we conclude that

\[
|I_k| \geq \text{Re}(I_k) \geq \frac{1}{4} r^{1/2} - (1 - r^{-2}) r^2 \left( \frac{1}{4} r^{1/2} - 1 \right)
\]

\[
\geq \frac{1}{4} r^{1/2} - e^{-2} \left( \frac{1}{4} r^{1/2} - 1 \right) \geq \frac{1}{4} r^{1/2} (1 - e^{-2}) > \|\mu\|
\]

although \( |\varphi_k(g)| \leq 1 \) for all \( g \in G \). This contradiction establishes Theorem 1.
SECTION III
THEOREM 1 FOR $G = T$

In this section we prove Theorem 1 for $G = T$.

Proof. Let $\mu \in M(T)$ and $\text{card}(B(\mu)) > (r+1)^{3r^2}$. We must exhibit $\gamma_0$ and $\gamma_{k,j}, 1 \leq k \leq r^2, 1 \leq j \leq r$, in $B(\mu)$ such that if $P_0 = \{\gamma_0\}$ and

$$P_{k+1} = P_k \cup \{\gamma_{k+1,j} : 1 \leq j \leq r\} \cup \bigcup_{1 < j} P_k + \gamma_{k+1,i} - \gamma_{k+1,j},$$

then for $\gamma \in P_{k-1}$ and $i < j$ we have $\gamma + \gamma_{k,i} - \gamma_{k,j} \not\in B$. By Theorem 1', we may assume $\text{card}(B(\mu)) = \infty$. We suppose that $B(\mu) \cap Z^+$ is infinite.

Let $\gamma_0$ be any member of $B(\mu)$. Suppose that $\gamma_{k,j}$ in $B(\mu)$ have been chosen for $1 \leq j \leq r, 1 \leq k \leq m-1$ ($m \geq 1$) consistent with (1). Let $\gamma_{m,r}$ be any element of $B$ such that

$$\gamma_{m,r} > |\gamma| \text{ for } \gamma \in P_{m-1}. \quad (4)$$

We suppose that $\gamma_{m,j}$ have been chosen in $B(\mu)$ for
\[ i+1 \leq j \leq r \] consistent with (1) and satisfying (4) in the place of \( \gamma_{m,r} \). Suppose that no \( \rho \in B \) can be chosen as \( \gamma_{m,i} \) to satisfy (4) in the role of \( \gamma_{m,r} \). Then for large \( \rho \in B \) there are \( \gamma \in P_{m-1} \) and \( i+1 \leq j \leq r \) such that \( \rho + \gamma - \gamma_{m,j} \in B \). If \( \rho \) is large enough \( \rho + \gamma - \gamma_{m,j} \) will satisfy (4) in the place of \( \gamma_{m,r} \). There exist \( \gamma' \in P_{m-1} \) and \( i+1 \leq j' \leq r \) so that \( (\rho + \gamma - \gamma_{m,j}) + \gamma' - \gamma_{m,j} \in B \). Let \( M \) be \( 2 \max\{\gamma_{m,j} : i+1 \leq j \leq r\} \). If \( LM < \rho < (L+1)M \), then \( (L-1)M < \rho + \gamma - \gamma_{m,j} < \rho \); thus there are at least \( L \) points in \( B \cap [M, (L+1)M) \). We conclude that

\[
\lim \inf_{R \to \infty} (2R+1)^{-1} \sum_{|n| \leq R} |\hat{\mu}(n)|^2 \geq (2M)^{-1} > 0
\]

which implies that \( \mu \) is not continuous, a contradiction.

Thus some \( \rho \in B \) satisfying (4) in the place of \( \gamma_{m,r} \) can be chosen as \( \gamma_{m,i} \). Inductively we obtain \( \gamma_0 \) and \( \gamma_{k,j} \), \( 1 \leq k \leq r^2 \), \( 1 \leq j \leq r \), as required.
SECTION IV
RANDOM WALKS IN $\mathbb{Z}^n$

We shall prove Theorem 1 for groups $G = \mathbb{T}^n$, $n > 1$, by induction on $n$. We require some geometrical lemmas concerning random walks in $\mathbb{Z}^n$. In what follows, a hyperplane $H$ in $\mathbb{R}^n$ will be called rational if for some $z \in \mathbb{Z}^n$, $z + H$ is a subspace of $\mathbb{R}^n$ containing $n-1$ linearly independent vectors from $\mathbb{Z}^n$. This is equivalent to saying that for some $z$ in $\mathbb{Z}^n$, $(z + H) \cap \mathbb{Z}^n$ is isomorphic to $\mathbb{Z}^{n-1}$.

**Lemma 1.** Let $n > 1$, $\{p_i\}$ be a sequence in $\mathbb{Z}^n$ and $S$ be a finite subset of $\mathbb{Z}^n$ such that $p_{i+1} - p_i \in S$ for all $i$. Then for each positive integer $N$ there are $N$ integers $j$ and a rational hyperplane $H$ such that $p_j \in H$.

Before we prove Lemma 1, consider an example in $\mathbb{Z}^2$. We assume that $p_i \neq 0$ for all $i$ and that $\theta = (\theta_1, \theta_2)$ is a cluster point of $\{\|p_i\|^{-1}p_i\}$ such that $\theta_1$ and $\theta_2$ are rational. Let $H$ be the line through 0 and $\theta$. Since $\theta_1$ and $\theta_2$ are rational there is a minimum distance $d > 0$. 
between translates of \( H \) by elements of \( \mathbb{Z}^2 \). We can enumerate such translates of \( H \) as \( H_i \) so that \( H_i \) is a distance \( d|i| \) from \( H \). Suppose the lemma false for some \( N \).

Fix a point \( P_j \). Among the first \( (2k+1)(N-1) + 1 \) successors of \( P_j \) at least one, say \( P_j \), occurs on an \( H_i \) with \( |i| > k \). Let \( M \) be the maximum of \( \langle s, \theta \rangle \) for \( s \in S \), where \( \langle \cdot, \cdot \rangle \) denotes the usual inner product in \( \mathbb{R}^2 \). Consider the angle \( A \) formed between two lines, \( H \) and the line through \( 0 \) and \( P_j \).

We have

\[
|\tan(A)| \geq (kd)(\langle P_j, \theta \rangle + (2k+1)(N-1)M)^{-1}
\]

If \( k \) is large enough,

\[
|\tan(A)| \geq \frac{1}{2} d(2MN)^{-1} = d(4MN)^{-1}
\]

Let \( H' \) and \( H'' \) be lines through \( 0 \) with rational slopes forming angles with \( H \) that are less than \( \arctan d(4MN)^{-1} \), but on opposite sides of \( H \). Since a subsequence of \( \{||p_i||^{-1}p_i\} \) converges to \( \theta \), we have infinitely many choices for \( P_j \) in the same region between \( H' \) and \( H'' \) as \( H \) is. For each such \( P_j \) there is a successor \( P_j \) on the opposite side of \( H' \) or \( H'' \). We conclude that the broken-line path traced by the sequence \( \{P_i\} \) crosses \( H' \) or \( H'' \) infinitely often. Since \( H' \) and \( H'' \) have rational slopes, a finite
number of translates of them cover all the points in \( \mathbb{Z}^n \) within a certain fixed distance. If we choose that distance to be the maximum of \( \|s\| \) for \( s \in S \), one of the translates contains \( p_i \) for infinitely many \( i \).

This example suggested how to handle the general case. When \( \theta_1 \) and \( \theta_2 \) could not both be rational, we chose \( \theta'_1 \) and \( \theta'_2 \) close to \( \theta_1 \) and \( \theta_2 \) and attempted a similar argument. It became important to control the least common denominator \( Q' \) of \( \theta'_1 \) and \( \theta'_2 \) because our lower estimate for \( d \) was \((Q')^{-1}\). We were led to invoke the diophantine approximations given by Theorem VII of [5, p. 14]: If \( \theta_1, \ldots, \theta_n \) are real numbers, then there are integers \( Q, q_1, \ldots, q_n \) with \( Q \) arbitrarily large such that

\[
Q^{-1/n} \max[|Q \theta_i - q_i| : 1 \leq i \leq n] < n/(n+1).
\]

Proof of Lemma 1. We shall argue by contradiction to obtain a rational hyperplane \( H \) which the broken-line path traced by the sequence \( \{p_i\} \) crosses infinitely often. A finite number of translates of any rational hyperplane \( H \) covers all the points in \( \mathbb{Z}^n \) whose distance from \( H \) is bounded by a certain number. In our case, if we choose that number to be the maximum of \( \|s\| \) for \( s \in S \), some translate of \( H \) will contain \( p_i \) for infinitely many integers \( i \), because there will be that many points \( p_i \) for which \( p_i \) and \( p_{i+1} \) are on
Let $\theta = (\theta_1, \ldots, \theta_n)$ be a cluster point of $\left(\|p_i\|^{-1}p_i\right)$. Note that if $p_i = 0$ infinitely often, the lemma follows. We may therefore assume that $p_i \neq 0$ for all $i$. Since $\theta \neq 0$, we may assume $\theta_1 \neq 0$. By Theorem VII of [5, p. 14] there are integers $Q > 0$ and $q_1, \ldots, q_n$ such that

(a) $Q^{1/n}|q_{1i} - q_{11}| < 1$ for $1 \leq i \leq n$;
(b) $Q^{1/n} > 64MN^{-1/2}$, where $M > 1 + \|s\|$ for all $s \in S$;
(c) $|q_{11}| \geq (1/2)|\theta_1|Q$ for $1 \leq i \leq n$;
(d) $Q^{-1}(q_{11}^2 + q_{12}^2)^{1/2} \leq 2\|\theta\| = 2$.

Let $q$ be the vector $(q_1, \ldots, q_n)$ and $w$ the vector $(-q_2, q_1, 0, \ldots, 0)$. Choose a rational number $r$ so that

$16n^{1/2}Q^{-1}(n+1)/n < r < (4MNQ)^{-1}$, by (b). Let $H'$ and $H''$ be the subspaces of $\mathbb{R}^n$ orthogonal to $rq-w$ and $rq+w$, respectively.

Assuming the lemma false, we shall show that the path traced by the sequence $\{p_i\}$ crosses either $H'$ or $H''$ infinitely often. We shall estimate the ratio $|\langle p, w \rangle \langle p, q \rangle^{-1}|$ for some points $p$ from the sequence. We shall show that the inequalities $|\langle p_1, w \rangle \langle p_1, q \rangle^{-1}| < r$ and $|\langle p_1, w \rangle \langle p_1, q \rangle^{-1}| \geq (4MNQ)^{-1}$ each have infinitely many solutions for the index. It then suffices to show that points satisfying the first inequality are separated from points satisfying the second by $H'$ or $H''$. Note that $H'$ and $H''$ are the points where that ratio is $r$. For example, suppose that
\[ |\langle p_1, w \rangle \langle p_1, q \rangle^{-1} | < r \text{ and that } (\langle p_1, q \rangle) > 0, \text{ but that} \]
\[ \langle p_j, w \rangle \langle p_j, q \rangle^{-1} \geq (4MNQ)^{-1} \text{ and } (\langle p_j, q \rangle) > 0. \text{ Then} \]
\[ \langle p_1, rq - w \rangle = r \langle p_1, q \rangle - \langle p_1, w \rangle = \langle p_1, q \rangle (r - \langle p_1, w \rangle \langle p_1, q \rangle^{-1}) > 0 \]
\[ \text{but} \]
\[ \langle p_j, rq - w \rangle = \langle p_j, q \rangle (r - \langle p_j, w \rangle \langle p_j, q \rangle^{-1}) < 0. \]

Thus \( p_1 \) and \( p_j \) are on opposite sides of \( H' \). The other cases are handled similarly.

To see that \( |\langle p_1, w \rangle \langle p_1, q \rangle^{-1} | < r \) infinitely often, we need only show that \( |\langle \theta, w \rangle \langle \theta, q \rangle^{-1} | \leq 16n^{-1/2}q^{-(n+1)/n} \), since a subsequence of \( \left\langle \|p_1\|^{-1}p_i \right\rangle \) converges to \( \theta \) and \( 16n^{-1/2}q^{-(n+1)/n} < r \). Since \( q \) is orthogonal to \( w \),

\[ |\langle \theta, w \rangle| = |\langle \theta - Q^{-1}q, w \rangle| \leq \|\theta - Q^{-1}q\| \|w\| \]
\[ \leq n^{1/2}q^{-(n+1)/n}(q_1^2 + q_2^2)^{1/2} \leq 2n^{1/2}q^{-1/n}. \]

On the other hand,

\[ |\langle \theta, q \rangle| = Q^{-1}|\langle Q_{\theta}, q \rangle| = Q^{-1}|\langle Q_{\theta} - q, q \rangle + \langle q, q \rangle| \]
\[ \geq Q^{-1}(\|q\|^2 - \|Q_{\theta} - q\| \|q\|) \geq Q^{-1}\|q\|(\|q\| - n^{1/2}q^{-1/n}). \]
Since \(|a_i| \geq \frac{1}{2} |\theta_1 q|\) for all \(i\), \(|q| \geq \frac{1}{2} Q\). Moreover, by (b), \(\frac{1}{4} Q > n^{1/2} Q^{-1/n}\). Thus

\[Q^{-1}||q||(|q| - n^{1/2} Q^{-1/n}) \geq 1/2 \left(\frac{1}{2} Q - n^{1/2} Q^{-1/n}\right) > (1/8) Q.\]

Thus

\[|<\theta, w><\theta, q>^{-1}| \leq 2n^{1/2} Q^{-1/n}[1/(8) Q]^{-1} = 16n^{1/2} Q^{-(n+1)/n}.\]

To argue that \(|<p_1, w><p_1, q>^{-1}| \geq (4 MNQ)^{-1}\) infinitely often we need to assume that the lemma is false. Let \(F\) be the subspace of \(\mathbb{R}^n\) generated by the vector \(q\) and the vectors \(e_i = (\delta_{ki})\), for \(3 \leq i \leq n\). Note that \(F\) is a rational hyperplane and that there is a minimum distance \(d\) between translates of \(F\) by elements of \(\mathbb{Z}^n\). Enumerate these translates as \(F_i\) for \(i \in \mathbb{Z}\) in such a manner that \(F_i\) is in distance \(|i|d\) from \(F\). Fix some integer \(J\). If the lemma is false, among the points \(p_{j+i}, 1 \leq i \leq (2k+1)(N-1) + 1\), at least one, say \(p_{j+i}\), occurs on an \(F_i\) with \(|i| > k\).

Since \(w\) is orthogonal to \(F\) and \(F\) is \(n-1\)-dimensional, the distance from \(p_{j+i}\) to \(F\) is equal to \(|<p_{j+i}, ||w||^{-1}w>|\).

Thus

\[|<p_{j+i}, w>| > k d ||w||.\]

Estimating \(|<p_{j+i}, q>|\), we obtain
\[ |<p_{j+j}, q>| \leq |<p_j, q>| + [(2k+1)(N-1)+1] \max\{|<s, q>| : s \in S\}. \]

Since
\[ |<s, q>| \leq |<s, q-Q\theta>| + |<s, Q\theta>| \]
\[ \leq \|s\| \|q-Q\theta\| + Q\|s\| \leq \|s\|^2/2Q^{-(n+1)/n} + Q\|s\| \]
\[ \leq 1 + Q\|s\| < QM, \text{ by (b)}, \]

we have
\[ |<p_{j+j}, q>| \leq |<p_j, q>| + (2k+1)QNM. \]

Thus
\[ |<p_{j+j}, w><p_{j+j}, q>^{-1}| \geq kd\|w\|(|<p_j, q>| + (2k+1)QNM)^{-1}. \]

If \( k \) is large enough
\[ |<p_{j+j}, w><p_{j+j}, q>^{-1}| \geq d\|w\|(4QNM)^{-1}. \]

All we have left to show is that \( d\|w\| \geq 1 \).

Let \( u = (q_1, q_2, 0, \ldots, 0) \). For \( z = (z_1, z_2, \ldots, z_n) \in \mathbb{Z}^n \)
let \( Q(z) \) be the set of vectors \( v \) in \( \mathbb{R}^n \) such that
\begin{enumerate}
  \item \( z_1 < \langle v, e_1 \rangle \leq z_1 + 1 \) for \( 3 \leq i \leq n \);
  \item \( z_2\|u\|^2 < \langle v, u \rangle \leq (z_2+1)\|u\|^2 \);
\end{enumerate}
\[(\text{i}) z_1 \|w\|^2 < (\langle v, w \rangle) \leq (z_1 + 1) \|w\|^2.\]

The sets \(Q(z)\) partition \(\mathbb{R}^n\) and each has Lebesgue measure \(\|w\|\|u\| = q_1^2 + q_2^2\). Since the cardinality of \(Q(z) \cap \mathbb{Z}^n\) is independent of \(z\), each \(Q(z)\) has \(q_1^2 + q_2^2\) points of \(\mathbb{Z}^n\). For every hyperplane \(z + F\) with \(z \in \mathbb{Z}^n\) and \(0 < \langle z, w \rangle \leq \|w\|^2\) there is a point \(z'\) in \(\mathbb{Z}^n \cap Q(0, 0, 0, \ldots, 0)\) such that \(z'+F = z+F\). For example, if \(z\) is in \(Q(0, v_2, \ldots, v_n)\), then \(z' = z - (v_2 u + v_3 e_3 + \cdots + v_n e_n)\) is in \(Q(0, \ldots, 0)\). Since \(z' - z \in F\), \(z' + F = z + F\). Thus the cardinality of \(\mathbb{Z}^n \cap Q(0, \ldots, 0)\) is an upper bound on the number of hyperplanes \(z + F\) such that \(0 < \langle z, w \rangle \leq \|w\|^2\) and \(z \in \mathbb{Z}^n\). The distance of a point \(v\) in \(Q(0, \ldots, 0)\) from \(F\) is given by \(\|v, \|w\|^{-1}w\|\). Since \(\|v, w\| \leq \|w\|^2\), that distance is bounded by \(\|w\|\). Since the distance between two translates of \(F\) by elements of \(\mathbb{Z}^n\) is always an integer multiple of the minimum distance \(d\), we have
\[d(q_1^2 + q_2^2) \geq \|w\|\]
but since \(\|w\|^2 = q_1^2 + q_2^2\), we have \(d\|w\| \geq 1\), and we are done.

Lemma 2 is a restatement of Lemma 1 in the form that will be used in the next section.

**Lemma 2.** Let \(n > 1\) and \(S\) a finite subset of \(\mathbb{Z}^n\). Let \(N \in \mathbb{Z}^+\) be given. Then there is an \(N' \in \mathbb{Z}^+\) such that if
\( \{p_i\} \) is a finite sequence in \( \mathbb{Z}^n \) of length \( N' \) and
\[ p_{i+1} - p_i \in S \quad \text{for all} \quad i < N' , \]
then there are \( N \) distinct integers \( j \) and a hyperplane \( H \) such that \( p_j \in H \).

**Proof.** Suppose Lemma 2 is false for some \( N \); then choose
for each \( N' \in \mathbb{Z}^+ \) a sequence \( \{p_{N',i}\} \) of length \( N' \) which
meets no rational hyperplane more than \( N-1 \) times. We shall
inductively define an infinite sequence \( \{p_i\} \) such that
\[ p_{i+1} - p_i \in S \quad \text{for all} \quad i \]
and such that \( \{p_i\} \) meets no hyperplane more than \( N-1 \) times, a contradiction of Lemma 1. We
may assume that \( p_{N',1} = 0 \) for all \( N' \); let \( p_1 = 0 \). Then
\[ p_{N',i} \in S \quad \text{for all} \quad N' . \]
Since \( S \) is finite there is a \( p_2 \in S \)
such that for infinitely many \( N' \), \( p_{N',1} = p_1 \) and \( p_{N',2} = p_2 \).

Suppose \( p_1, \ldots, p_k \) have been chosen so that for arbitrarily
large choices of \( N' \) we have \( p_{N',i} = p_i \) for \( 1 \leq i \leq k \).
Then among those sequences we have \( p_{N',i} \in p_k + S \). Since
\[ p_k + S \quad \text{is finite, there is a} \quad p_{k+1} \in p_k + S \quad \text{such that for} \]
infinitely many \( N' \), \( p_{N',i} = p_i \) for \( 1 \leq i \leq k+1 \). The in­
finite sequence \( \{p_i\} \) meets no rational hyperplane more than
\( N-1 \) times because no initial segment of it can.
SECTION V

THEOREM 1 FOR G = T^n

In this section we prove Theorem 1 for G = T^n by induction on n.

Proof. We may assume that n > 1 and that the theorem is true for G = T^{n-1}. If \text{card}(B(\mu)) < \infty we are done by Theorem 1', since \mathbb{Z}^n, the dual group of T^n, is an ordered group under the lexicographic ordering. We therefore suppose that \text{card}(B(\mu)) = \infty. Let \pi be the projection onto a coordinate such that \pi(B(\mu)) is unbounded. We may suppose that there exists \{Y_1\} \subseteq B(\mu) such that \lim_{t \to \infty} \pi(Y_t) = +\infty. Let Y_0 \in B(\mu) be arbitrary and suppose that Y_{k,j}, 1 \leq j \leq r, 1 \leq k \leq m-1 (m \geq 1), have been found in B(\mu). It is consistent with (1) to let Y_{m,r} be any member of B(\mu) such that

$$\pi(Y_{m,r}) > \pi(\gamma) \text{ for all } \gamma \in P_{m-1}. \quad (5)$$

We may suppose that Y_{m,j} for i+1 \leq j \leq r have been found.
to satisfy (5) in the role of $\gamma_{m,r}$ and (1), but that no $\gamma_{m,1}$ can be found to satisfy (5) in the role of $\gamma_{m,r}$. Let $M$ be $2 \max\{\pi(\gamma_{m,j}) : i+1 \leq j \leq r\}$. Let $N'$ be the integer given by Lemma 2 for $S = \{\gamma-Y_{m,j} : \gamma \in P_{m-1}$ and $i+1 \leq j \leq r\}$ and $N = (r+1)^{3r^2} + 1$. Consider any $\rho \in B$ such that $\pi(\rho) > N'M$. Since $\rho$ satisfies (5) in the place $\gamma_{m,r}$, the reason $\rho$ cannot serve as a $\gamma_{m,1}$ satisfying (5) is that for some $\gamma \in P_{m-1}$ and $i+1 \leq j \leq r$ we have $\rho + \gamma - \gamma_{m,j} \in B$. Note that $(N'-1)M < \pi(\rho + \gamma - \gamma_{m,j}) < \pi(\rho)$. Since $\rho' = \rho + \gamma - \gamma_{m,j}$ satisfies (5) also, there must be $\gamma \in P_{m-1}$ and $i+1 \leq j \leq r$ such that $\rho' + \gamma - \gamma_{m,j} \in B$. Also $(N'-2)M < \pi(\rho' + \gamma - \gamma_{m,j}) < \pi(\rho')$. If we let $\rho = \rho_1$, and $\rho' = \rho_2$, we can continue in this way to obtain a sequence $[\rho_1]$ of length $N'$ such that $\rho_{i+1} - \rho_i \in S$ for $1 < N'$. Note that the sequence is composed of distinct points. By Lemma 2 there are a rational hyperplane $H$ and $N = (r+1)^{3r^2} + 1$ integers $j$ such that $\rho_j \in H$. Thus, since the $\rho_i$'s are distinct, $\text{card}(H \cap B(\mu)) > (r+1)^{3r^2}$.

Let $z \in \mathbb{Z}^n$ such that $(z+H) \cap \mathbb{Z}^n$ is isomorphic to $\mathbb{Z}^{n-1}$. Let $\psi$ be the quotient map from $T^n$ to $T^n/[(z+H) \cap \mathbb{Z}^n]$. Let $\nu = \psi(z \cdot u)$ be the measure on $\psi(T^n)$ such that $\nu(E) = (z \cdot u)(\psi^{-1}(E))$ for all Borel sets $E \subseteq \psi(T^n)$. Equivalently, $\nu$ is the measure in $M(\psi(T^n))$ such that for $\gamma \in (z+H) \cap \mathbb{Z}^n$

$$\hat{\nu}(\gamma) = [\psi(z \cdot u)]^\gamma(\gamma) = (z \cdot u)^\gamma(\gamma) = \hat{\mu}(\gamma-z). \quad (6)$$
Then $v$ satisfies the hypotheses of Theorem 1 for $\psi(T^n)$ isomorphic to $T^{n-1}$. Let $Y_0', Y_{k,j}', 1 \leq k \leq r^2, 1 \leq j \leq r$, be given in $\mathcal{B}(v) = z + (\mathcal{B}(\mu) \cap H)$ satisfying condition (1) of Theorem 1. Then redefine $Y_0$ to be $Y_0' - z$, $Y_{k,j}$ to be $Y_{k,j}' - z$ for $1 \leq k \leq m-1, 1 \leq j \leq r$, and $Y_{m,j}$ to be $Y_{m,j}' - z$ for $i+1 \leq j \leq r$. If we let $Y_{k,j} = Y_{k,j}' - z$ for the remaining indices, we are done.
SECTION VI

THE GENERAL CASE

In this section we finish the proof of Theorem 1.

Proof. Let us assume that $\Gamma$ is finitely generated. By [6, p. 49], $\Gamma = \Lambda \oplus \mathbb{Z}^n$ for some non-negative integer $n$, where $\Lambda$ is the torsion subgroup of $\Gamma$ and hence by assumption $\text{card}(\Lambda) \leq K$. Since $\text{card}(B(\mu)) > K(r+1)^{3r^2}$ there is a $\lambda \in \Lambda$ such that $\text{card}(\langle \lambda + \mathbb{Z}^n \rangle \cap B(\mu)) > (r+1)^{3r^2}$. Let $\psi$ be the quotient map $G$ to $G/(\mathbb{Z}^n)^1$. Then $\nu = \psi(\lambda \cdot \mu)$ is the measure on $\psi(G)$ satisfying

$$\hat{\nu}(z) = (\lambda \mu)^\Lambda(z) = \hat{\mu}(\lambda + z), \text{ for } z \in \mathbb{Z}^n.$$ 

Since $\psi(G)$ is isomorphic to $\mathbb{T}^n$ and since $\nu$ satisfies the hypotheses of Theorem 1, there exist $Y_0'$ and $Y_{k,j}', 1 \leq k \leq r^2, 1 \leq j \leq r$, in $B(\nu) = -\lambda + [B(\mu) \cap (\lambda + \mathbb{Z}^n)]$ satisfying (1), with $Y_0', Y_{k,j}'$ replacing $Y_0$ and $Y_{k,j}$, respectively. Then $Y_0 = \lambda + Y_0', Y_{k,j} = \lambda + Y_{k,j}'$ for $1 \leq k \leq r^2, 1 \leq j \leq r$.
satisfy (1) for $\mu$.

In the fully general case, let $S$ be a subset of $B(\mu)$ of cardinality $K(r+1)^3r^2 + 1$. Let $\Lambda$ be the subgroup of $\Gamma$ generated by $S$. Let $\psi$ be the quotient map from $G$ to $G/\Lambda^\perp$. Let $\nu = \psi(\mu)$ be the measure on $G/\Lambda^\perp$ such that $\hat{\nu}(\lambda) = \hat{\mu}(\lambda)$ for all $\lambda \in \Lambda$. $\nu$ satisfies the hypotheses of Theorem 1 and $[\psi(G)]^\perp = \Lambda$, which is finitely generated. As we have already proven, for $\nu$ there are $\gamma_0, \gamma_k,j$, $1 \leq k \leq r^2$, $1 \leq j \leq r$, satisfying (1). Since $B(\mu) \cap \Lambda = B(\nu)$ and $\Lambda$ is a subgroup, the same $\gamma_0$ and $\gamma_k,j$'s will work for $\mu$. 
BIBLIOGRAPHY


VITA

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