Fourier-Stieltjes Transforms of Measures With a Certain Continuity Property.

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The Louisiana State University and Agricultural
and Mechanical College, Ph.D., 1975
Mathematics

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FOURIER-STIELTJES TRANSFORMS OF MEASURES
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A Dissertation
Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in
The Department of Mathematics

by
Laurence Thomas Ramsey
B.A., University of North Dakota, 1970
May, 1975
ACKNOWLEDGMENT

The author wishes to acknowledge with gratitude the determining role played in this research by his friend and advisor, Dr. Carruth McGehee, under whose supervision this work was done. He also wishes to thank Professors Gordon Woodward, Lenny Richardson and Lynn Williams for their patient and considerate reading of successive manuscripts. To Professor Gustav Hedlund of Yale University who with a prompt and complete answer to queries about sequence spaces steered me away from a dead end in the proof of Lemma 1, a special "thank you."
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>SECTION</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENT</td>
<td>ii</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>iv</td>
</tr>
<tr>
<td>I INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II THEOREMS</td>
<td>3</td>
</tr>
<tr>
<td>III THEOREM 1 FOR G = T</td>
<td>8</td>
</tr>
<tr>
<td>IV RANDOM WALKS IN Z^n</td>
<td>10</td>
</tr>
<tr>
<td>V THEOREM 1 FOR G = T^n</td>
<td>19</td>
</tr>
<tr>
<td>VI THE GENERAL CASE</td>
<td>22</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>24</td>
</tr>
<tr>
<td>VITA</td>
<td>25</td>
</tr>
</tbody>
</table>
ABSTRACT

Let \( G \) be a compact abelian group whose dual group \( \Gamma \) has a finite torsion subgroup. Let \( \mu \in M(G) \) such that \(|\mu|\) assigns no mass to any coset of any closed subgroup of \( G \) whose index is infinite. Then there is \( d > 0 \), dependent only on \( ||\mu|| \), such that if for each \( \gamma \in \Gamma \), \(|\hat{\mu}(\gamma)| \geq 1\) or \(|\hat{\mu}(\gamma)| \leq d\), then the set \( \{\gamma : |\hat{\mu}(\gamma)| \geq 1\} \) is finite. An upper bound on the cardinality of this set is obtained in terms of \( ||\mu|| \) and the cardinality of the torsion subgroup of \( \Gamma \).
SECTION I
INTRODUCTION

G will denote a compact abelian group, \( \Gamma \) its dual group and \( M(G) \) the measure algebra of finite Borel measures on \( G \). We shall assume that \( \Gamma \) has a finite torsion subgroup. Let \( \mu \in M(G) \) such that \( |\mu| \) assigns no mass to any coset of any closed subgroup of \( G \) whose index is infinite.

We prove that there is a number \( d > 0 \), dependent only on \( ||\mu|| \), such that if for each \( \gamma \in \Gamma \), \( |\hat{\mu}(\gamma)| \geq 1 \) or \( |\hat{\mu}(\gamma)| < d \) then the set of \( \gamma \) such that \( |\hat{\mu}(\gamma)| \geq 1 \) is finite. We obtain an upper bound on the cardinality of this set in terms of \( ||\mu|| \) and the cardinality of the torsion subgroup of \( \Gamma \).

De Leeuw and Katzenelson first proved this theorem for the circle group \( T \) [1, Lemma 2]. They proved that, for any \( C > 0 \), there is \( d = d(C) < 10^{-2} \) satisfying the following: Suppose that \( \mu \in M(T) \) is a continuous measure with \( ||\mu|| \leq C \) and, for \( |n| \) sufficiently large, \( |\hat{\mu}(n)| < d \) or \( \text{Re}(\hat{\mu}(n)) > 1-d \); then \( \{n : |\hat{\mu}(n)| \geq d \} \) is finite. Without a numerical bound on the cardinality of \( \{n : |\hat{\mu}(n)| \geq d \} \)
their method does not seem to generalize. Such a bound can be obtained by imitating Davenport's procedure in [2], if $\alpha$ is small enough and $|\hat{a}(n)| < \alpha$ or $|\hat{a}(n)| > 1-\alpha$ for all integers $n$. In [2] Davenport proves that if a trigonometric polynomial $p(x) = \Sigma a(n)\exp(2\pi inx)$ has $N$ coefficients of modulus at least one and all other coefficients equal to zero, then the $L^1$-norm of $p$ is at least $8^{-1}(\log N)^{1/4}(\log \log N)^{-1/4}$.

For an arbitrary locally compact abelian group $G$, Glicksberg proved in [3] that if $\mu \in M(G)$ and 0 is isolated in $\{0\} \cup \hat{\mu}(\hat{G})$ then there is a compact subgroup $H$ of $G$ for which $\mu_H$, the part of $\mu$ carried by the cosets of $H$, is the convolution of a non-zero idempotent and an invertible. He proved that $\mu_H = [\left( \Sigma_{\gamma \in \Lambda} m_H \right) * \lambda$ where $\Lambda$ is a finite subset of $\hat{G}$, $m_H$ is the Haar measure on $H$, and $\lambda \in M(G)^{-1}$.

For measures $\mu$ such that $\mu_H = 0$ when $H$ is a closed subgroup of $G$ of infinite index, the hypothesis that 0 is isolated in $\{0\} \cup \hat{\mu}(\hat{G})$ yields the conclusion that $\mu$ is a trigonometric polynomial. When $G$ is compact and $\hat{G}$ has a finite torsion subgroup, one can then use Hewitt and Zuckerman's generalization [4] of Davenport's result [2] to estimate the cardinality of the set of $\gamma \in \Gamma$ such that $|\hat{\mu}(\gamma)| > 0$. 
SECTION II
THEOREMS

In this section we state our theorems precisely and prove
Theorem 2 assuming Theorem 1. In what follows \( B = B(\mu) = \{ \gamma \in \Gamma : |\hat{\mu}(\gamma)| \geq 1 \} \).

**Theorem 1.** Let \( G \) be a compact abelian group whose dual group \( \Gamma \) has at most \( K \) torsion elements. Let \( \mu \in \text{M}(G) \) such that \( |\mu| \) assigns no mass to any coset of any closed subgroup of \( G \) whose index is infinite. If \( \text{card}(B(\mu)) > K(r+1)^{3r^2} \), then there exist \( \gamma_0 \) and \( \gamma_k, j, 1 \leq k \leq r^2, 1 \leq j \leq r \), in \( B(\mu) \) such that if \( P_0 = \{ \gamma_0 \} \) and

\[
P_{k+1} = P_k \cup \{ \gamma_{k+1, j} : 1 \leq j \leq r \} \cup \bigcup_{1 < j} \left( P_k + \gamma_{k+1, i} - \gamma_{k+1, j} \right),
\]

then

for \( \gamma \in P_{k-1} \) and \( i < j \), we have \( \gamma + \gamma_{k, i} - \gamma_{k, j} \notin B \). (1)

The proof of Theorem 1 requires several reductions which
will be postponed to later sections. Theorem 1 was suggested by Theorem 1' which for the case of \( G = T \) can be found in Davenport's paper [2].

**Theorem 1'.** Let \( G \) be a compact abelian group whose dual group \( \Gamma \) is an ordered group. Suppose that \( \mu \in M(G) \) such that \((r+1)^{3r^2} < \text{card}(B(\mu)) < \infty \). Then there exist \( Y_0 \) and \( Y_{k,j}, 1 \leq k \leq r^2, 1 \leq j \leq r \), in \( B(\mu) \) such that if \( P_0 = \{Y_0\} \) and

\[
P_{k+1} = P_k \cup \{Y_{k+1,j} : 1 \leq j \leq r\} \cup \bigcup_{i<j} (P_k + Y_{k+1,i} - Y_{k+1,j}) ,
\]

then for \( \gamma \in P_{k-1} \) and \( i < j \) we have \( \gamma + Y_{k,i} - Y_{k,j} \notin B \).

Since the proof of Theorem 1' found in [2] for \( G = T \) works without change for the general case, we omit the proof of Theorem 1'.

**Theorem 2.** Let \( G \) be a compact abelian group whose dual group \( \Gamma \) has at most \( K \) torsion elements. Let \( \mu \in M(G) \) such that \( |\mu| \) assigns no mass to any coset of any closed subgroup of \( G \) whose index is infinite. Let \( r \) be a positive integer greater than 2 such that \( 4^{-1}(1-e^{-2})r^{1/2} > ||\mu|| \).

If
\[ |\hat{\mu}(\gamma)| \geq 1 \text{ or } |\hat{\mu}(\gamma)| \leq 2^{-1}r^{3/2}r^{-2r^2} \] (2)

for all \( \gamma \in \Gamma \), then the cardinality of \( B(\mu) \) is at most \( K(r+1)^{3r^2} \).

The proof of Theorem 2 is adapted from [2]. Originally condition (2) read

\[ |\hat{\mu}(\gamma)| \geq 1 \text{ or } |\hat{\mu}(\gamma)| \leq 2^{-1}r^{3/2}(r+1)^{-3r^2} . \]

Gordon Woodward suggested the improvement.

Proof of Theorem 2. Suppose \( \text{card}(B(\mu)) > K(r+1)^{3r^2} \). Using \( \gamma_0 \) and \( \gamma_{k,j}, 1 \leq k \leq r^2, 1 \leq j \leq r \), as given by Theorem 1, we define trigonometric polynomials \( \varphi_0, \cdots, \varphi_{r^2} \) inductively as follows:

\[ \varphi_0 = \overline{\sigma}(\hat{\mu}(\gamma_0))(\gamma_0, \cdot) \]

where \( \sigma(x) = x|x|^{-1} \) for \( x \neq 0 \) and \( \overline{\sigma}(x) = \overline{x}|x|^{-1} \).

\[ \varphi_k = \varphi_{k-1}\{1 - 2r^{-2} - r^{-3} \sum_{i<j} \overline{\sigma}(\hat{\mu}(\gamma_{k,i}))\sigma(\hat{\mu}(\gamma_{k,j}))(\gamma_{k,i} - \gamma_{k,j}, \cdot) \}
+ r^{-5/2} \sum_j \overline{\sigma}(\hat{\mu}(\gamma_{k,j}))(\gamma_{k,j}, \cdot) . \]

Note that if \( P_0, \cdots, P_{r^2} \) are defined as in the statement of
Theorem 1, each \( \varphi_k \) is a \( P_k \)-polynomial. By [2, Lemmas 1 and 2], \( |\varphi_k(\xi)| \leq 1 \) for all \( \xi \in G \). Let \( I_k = \int_G \varphi_k(\xi) d\mu(-\xi) \).

Then \( I_0 = |\hat{\mu}(Y_0)| \geq 1 \). Moreover

\[
\text{Re}(I_k) \geq (1-2r^{-2})\text{Re}(I_{k-1}) + \frac{1}{2} r^{-3/2} .
\] (3)

To compute (3) we write

\[
I_k = (1-2r^{-2})I_{k-1} + r^{-5/2} \sum_j |\hat{\mu}(Y_k,j)|
- r^{-3} \sum_{Y \in P_{k-1}} \sum_{1<j} \varphi_{k-1}(Y)\overline{\sigma(\hat{\mu}(Y_k,i))}\sigma(\hat{\mu}(Y_k,j))\hat{\mu}(Y + Y_k,i - Y_k,j)
= (1-2r^{-2})I_{k-1} + r^{-5/2} \sum_j |\hat{\mu}(Y_k,j)| - r^{-3}A .
\]

Thus,

\[
\text{Re}(I_k) \geq (1-2r^{-2})\text{Re}(I_{k-1}) + r^{-3/2} - r^{-3}|A| .
\]

Observe that each term of \( A \) is bounded in modulus by \( 2^{-1}r^{3/2}r^{-2r^2} \) by (1) and (2) and that the number of terms in \( A \) is at most \( \frac{1}{2} r(r-1)\cdot\text{card}(P_{k-1}) \leq r^2 \cdot \text{card} P_{k-1} \leq r^{2k} .
\)

Note that \( \text{card}(P_0) = 1 \leq r^{2.0} \) and that

\[
\text{card}(P_{k+1})
= \text{card}(P_k \cup \{Y_{k+1,j} : 1 \leq j \leq r\}) \cup \bigcup_{1<j} P_k + Y_{k+1,i} - Y_{k+1,j})
\leq \text{card} P_k + r + \frac{1}{2} r(r-1)\text{card} P_k \leq (1+r+\frac{1}{2}(r)(r-1))\text{card}(P_k)
\leq r^2 \text{card}(P_k) ,
\]
hence $\text{card}(P_k) \leq r^{2k}$. It follows from (3) using induction that

$$\text{Re}(I_k) \geq \frac{1}{4} r^{1/2} - (1-r^{-2})^k \left( \frac{1}{4} r^{1/2} - 1 \right).$$

For $k = r^2$ we conclude that

$$|I_k| \geq \text{Re}(I_k) \geq \frac{1}{4} r^{1/2} - (1-r^{-2})^2 \left( \frac{1}{4} r^{1/2} - 1 \right) \geq \frac{1}{4} r^{1/2} - e^{-2} \left( \frac{1}{4} r^{1/2} - 1 \right) \geq \frac{1}{4} r^{1/2}(1-e^{-2}) > \|\mu\|$$

although $|\varphi_k(g)| \leq 1$ for all $g \in G$. This contradiction establishes Theorem 1.
SECTION III
THEOREM 1 FOR G = T

In this section we prove Theorem 1 for G = T.

Proof. Let μ ∈ M(T) and \text{card}(B(μ)) > (r+1)^3r^2. We must exhibit \( γ_0 \) and \( γ_{k,j}, 1 \leq k \leq r^2, 1 \leq j \leq r \), in B(μ) such that if \( P_0 = \{γ_0\} \) and

\[
P_{k+1} = P_k \cup \{γ_{k+1,j} : 1 \leq j \leq r\} \cup \bigcup_{i<j} P_k + γ_{k+1,i} - γ_{k+1,j},
\]

then for \( γ \in P_{k-1} \) and \( i < j \) we have \( γ + γ_{k,i} - γ_{k,j} \not\in B \). By Theorem 1' we may assume \text{card}(B(μ)) = \( \infty \). We suppose that B(μ) ∩ Z⁺ is infinite.

Let \( γ_0 \) be any member of B(μ). Suppose that \( γ_{k,j} \) in B(μ) have been chosen for \( 1 \leq j \leq r, 1 \leq k \leq m-1 \) (\( m \geq 1 \)) consistent with (1). Let \( γ_{m,r} \) be any element of B such that

\[
γ_{m,r} > |γ| \text{ for } γ \in P_{m-1}. \quad (4)
\]

We suppose that \( γ_{m,j} \) have been chosen in B(μ) for
i+1 ≤ j ≤ r consistent with (1) and satisfying (4) in the place of γ_{m,r}. Suppose that no ρ ∈ B can be chosen as γ_{m,i} to satisfy (4) in the role of γ_{m,r}. Then for large ρ ∈ B there are γ ∈ P_{m-1} and i+1 ≤ j ≤ r such that ρ+γ-γ_{m,j} ∈ B. If ρ is large enough ρ+γ-γ_{m,j} will satisfy (4) in the place of γ_{m,r}. There exist γ' ∈ P_{m-1} and i+1 ≤ j' ≤ r so that (ρ+γ-γ_{m,j}) + γ' - γ_{m,j} ∈ B. Let M be 2 \max(γ_{m,j} : i+1 ≤ j ≤ r). If LM ≤ ρ < (L+1)M, then (L-1)M < ρ+γ-γ_{m,j} < ρ; thus there are at least L points in B ∩ [M, (L+1)M). We conclude that

\lim \inf_{R \to \infty} (2R+1)^{-1} \sum_{|n| ≤ R} |\hat{\mu}(n)|^2 ≥ (2M)^{-1} > 0

which implies that μ is not continuous, a contradiction.

Thus some ρ ∈ B satisfying (4) in the place of γ_{m,r} can be chosen as γ_{m,i}. Inductively we obtain γ_0 and γ_{k,j}, 1 ≤ k ≤ r^2, 1 ≤ j ≤ r, as required.
SECTION IV
RANDOM WALKS IN $\mathbb{Z}^n$

We shall prove Theorem 1 for groups $G = \mathbb{T}^n$, $n > 1$, by induction on $n$. We require some geometrical lemmas concerning random walks in $\mathbb{Z}^n$. In what follows, a hyperplane $H$ in $\mathbb{R}^n$ will be called rational if for some $z \in \mathbb{Z}^n$, $z + H$ is a subspace of $\mathbb{R}^n$ containing $n-1$ linearly independent vectors from $\mathbb{Z}^n$. This is equivalent to saying that for some $z$ in $\mathbb{Z}^n$, $(z + H) \cap \mathbb{Z}^n$ is isomorphic to $\mathbb{Z}^{n-1}$.

Lemma 1. Let $n > 1$, $\{p_i\}$ be a sequence in $\mathbb{Z}^n$ and $S$ be a finite subset of $\mathbb{Z}^n$ such that $p_{i+1} - p_i \in S$ for all $i$. Then for each positive integer $N$ there are $N$ integers $j$ and a rational hyperplane $H$ such that $p_j \in H$.

Before we prove Lemma 1, consider an example in $\mathbb{Z}^2$. We assume that $p_i \neq 0$ for all $i$ and that $\theta = (\theta_1, \theta_2)$ is a cluster point of $\{\|p_i\|^{-1}p_i\}$ such that $\theta_1$ and $\theta_2$ are rational. Let $H$ be the line through 0 and $\theta$. Since $\theta_1$ and $\theta_2$ are rational there is a minimum distance $d > 0$
between translates of $H$ by elements of $\mathbb{Z}^2$. We can enumerate such translates of $H$ as $H_1$ so that $H_1$ is a distance $d|i|$ from $H$. Suppose the lemma false for some $N$. Fix a point $p_J$. Among the first $(2k+1)(N-1) + 1$ successors of $p_J$ at least one, say $p^*_j$, occurs on an $H_i$ with $|i| > k$. Let $M$ be the maximum of $|<s,\theta>|$ for $s \in S$, where $<\cdot,\cdot>$ denotes the usual inner product in $\mathbb{R}^2$. Consider the angle $A$ formed between two lines, $H$ and the line through $0$ and $p^*_j$. We have

$$|\tan(A)| \geq (kd)(|<p^*_j,\theta>| + (2k+1)(N-1)M)^{-1}$$

If $k$ is large enough,

$$|\tan(A)| \geq \frac{1}{2} a(2MN)^{-1} = a(4MN)^{-1}.$$  

Let $H'$ and $H''$ be lines through $0$ with rational slopes forming angles with $H$ that are less than $\arctan a(4MN)^{-1}$, but on opposite sides of $H$. Since a subsequence of $\{\|p_i\|^{-1}p_i\}$ converges to $\theta$, we have infinitely many choices for $p_j$ in the same region between $H'$ and $H''$ as $H$ is. For each such $p_j$ there is a successor $p_j$ on the opposite side of $H'$ or $H''$. We conclude that the broken-line path traced by the sequence $\{p_i\}$ crosses $H'$ or $H''$ infinitely often. Since $H'$ and $H''$ have rational slopes, a finite
number of translates of them cover all the points in $\mathbb{Z}^n$ within a certain fixed distance. If we choose that distance to be the maximum of $\|s\|$ for $s \in S$, one of the translates contains $p_i$ for infinitely many $i$.

This example suggested how to handle the general case. When $\theta_1$ and $\theta_2$ could not both be rational, we chose $\theta'_1$ and $\theta'_2$ close to $\theta_1$ and $\theta_2$ and attempted a similar argument. It became important to control the least common denominator $Q'$ of $\theta'_1$ and $\theta'_2$ because our lower estimate for $d$ was $(Q')^{-1}$. We were led to invoke the diophantine approximations given by Theorem VII of [5, p. 14]: If $\theta_1, \ldots, \theta_n$ are real numbers, then there are integers $Q, q_1, \ldots, q_n$ with $Q$ arbitrarily large such that

$$Q^{1/n} \max\{|Q\theta_i - q_i| : 1 \leq i \leq n\} < n/(n+1).$$

Proof of Lemma 1. We shall argue by contradiction to obtain a rational hyperplane $H$ which the broken-line path traced by the sequence $\{p_i\}$ crosses infinitely often. A finite number of translates of any rational hyperplane $H$ covers all the points in $\mathbb{Z}^n$ whose distance from $H$ is bounded by a certain number. In our case, if we choose that number to be the maximum of $\|s\|$ for $s \in S$, some translate of $H$ will contain $p_i$ for infinitely many integers $i$, because there will be that many points $p_i$ for which $p_i$ and $p_{i+1}$ are on
opposite sides of $H$.

Let $\theta = (\theta_1, \ldots, \theta_n)$ be a cluster point of $\{\|p_i\|^{-1}p_i\}$. Note that if $p_i = 0$ infinitely often, the lemma follows. We may therefore assume that $p_i \neq 0$ for all $i$. Since $\theta \neq 0$, we may assume $\theta_1 \neq 0$. By Theorem VII of [5, p. 14] there are integers $Q > 0$ and $q_1, \ldots, q_n$ such that

(a) $Q^{1/n}|q_{1} - q_1| < 1$ for $1 \leq i \leq n$;
(b) $Q^{1/n} > 64MNn^{1/2}$, where $M > 1 + \|s\|$ for all $s \in S$;
(c) $|q_i| \geq (1/2)|\theta_1 q_i|$ for $1 \leq i \leq n$;
(d) $Q^{-1}(q_1^2 + q_2^2)^{1/2} \leq 2\|\theta\| = 2$.

Let $q$ be the vector $(q_1, \ldots, q_n)$ and $w$ the vector $(-q_2, q_1, 0, \ldots, 0)$. Choose a rational number $r$ so that $16n^{1/2}Q^{-(n+1)/n} < r < (4MNQ)^{-1}$, by (b). Let $H'$ and $H''$ be the subspaces of $R^n$ orthogonal to $rq\cdot w$ and $rq+w$, respectively.

Assuming the lemma false, we shall show that the path traced by the sequence $\{p_i\}$ crosses either $H'$ or $H''$ infinitely often. We shall estimate the ratio $|\langle p, w \rangle \langle p, q \rangle^{-1}|$ for some points $p$ from the sequence. We shall show that the inequalities $|\langle p_i, w \rangle \langle p_i, q \rangle^{-1}| < r$ and $|\langle p_i, w \rangle \langle p_i, q \rangle^{-1}| \geq (4MNQ)^{-1}$ each have infinitely many solutions for the index. It then suffices to show that points satisfying the first inequality are separated from points satisfying the second by $H'$ or $H''$. Note that $H'$ and $H''$ are the points where that ratio is $r$. For example, suppose that
\[ |\langle p_1, w \rangle \langle p_1, q \rangle^{-1}| < r \] and that \( (\langle p_1, q \rangle) > 0 \), but that \( \langle p_j, w \rangle \langle p_j, q \rangle^{-1} \geq (4MNQ)^{-1} \) and \( (\langle p_j, q \rangle) > 0 \). Then
\[
\langle p_1, r q \rangle - \langle p_1, w \rangle = \langle p_1, q \rangle (r - \langle p_1, w \rangle \langle p_1, q \rangle^{-1}) > 0
\]
but
\[
\langle p_j, r q \rangle - \langle p_j, w \rangle = \langle p_j, q \rangle (r - \langle p_j, w \rangle \langle p_j, q \rangle^{-1}) < 0.
\]
Thus \( p_1 \) and \( p_j \) are on opposite sides of \( H' \). The other cases are handled similarly.

To see that \( |\langle p_1, w \rangle \langle p_1, q \rangle^{-1}| < r \) infinitely often, we need only show that \( |\langle \theta, w \rangle \langle \theta, q \rangle^{-1}| \leq 16 n^{1/2} q^{-(n+1)/n} \), since a subsequence of \( \{\|p_1\|^{-1} p_i\} \) converges to \( \theta \) and \( 16 n^{1/2} q^{-(n+1)/n} < r \). Since \( q \) is orthogonal to \( w \),
\[
|\langle \theta, w \rangle| = |\langle \theta - q^{-1} q, w \rangle| \leq \|\theta - q^{-1} q\|\|w\|
\]
\[
\leq n^{1/2} q^{-(n+1)/n} (q_1^2 + q_2^2)^{1/2} \leq 2 n^{1/2} q^{-1/n}.
\]
On the other hand,
\[
|\langle \theta, q \rangle| = q^{-1} |\langle q, \theta - q, q \rangle| = q^{-1} |\langle q, \theta - q, q \rangle + \langle q, q \rangle|
\]
\[
\geq q^{-1} (\|q\|^2 - \|q\| \|q\|) \geq q^{-1} \|q\| (\|q\| - n^{1/2} q^{-1/n})
\]
Since $|q_i| \geq \frac{1}{2} |\theta_i q|$ for all $i$, $\|q\| \geq \frac{1}{2} Q$. Moreover, by (b), $\frac{1}{4} Q > n^{1/2}Q^{-1/n}$. Thus

$$Q^{-1}\|q\|(\|q\| - n^{1/2}Q^{-1/n}) \geq 1/2(\frac{1}{2} Q - n^{1/2}Q^{-1/n}) \geq (1/8)Q.$$ 

Thus

$$|\langle \theta, w \rangle \langle \theta, q \rangle^{-1}| \leq 2n^{1/2}Q^{-1/n}[(1/8)Q]^{-1} = 16n^{1/2}Q^{-(n+1)/n}.$$ 

To argue that $|\langle p_i, w \rangle \langle p_i, q \rangle^{-1}| \geq (4MNQ)^{-1}$ infinitely often we need to assume that the lemma is false. Let $F$ be the subspace of $\mathbb{R}^n$ generated by the vector $q$ and the vectors $e_i = (\delta_{k,i})$, for $3 \leq i \leq n$. Note that $F$ is a rational hyperplane and that there is a minimum distance $d$ between translates of $F$ by elements of $\mathbb{Z}^n$. Enumerate these translates as $F_i$ for $i \in \mathbb{Z}$ in such a manner that $F_i$ is in distance $|i|d$ from $F$. Fix some integer $J$. If the lemma is false, among the points $p_{J+i}$, $1 \leq i \leq (2k+1)(N-1) + 1$, at least one, say $p_{J+j}$, occurs on an $F_i$ with $|i| > k$. Since $w$ is orthogonal to $F$ and $F$ is $n-1$-dimensional, the distance from $p_{J+j}$ to $F$ is equal to $|\langle p_{J+j}, \|w\|^{-1}w \rangle|$. Thus

$$|\langle p_{J+j}, w \rangle| > kd\|w\|.$$ 

Estimating $|\langle p_{J+j}, q \rangle|$, we obtain
\[ |<p_{j+1}, q>| \leq |<p_j, q>| + [(2k+1)(N-1)+1] \max\{|<s, q>| : s \in S\}. \]

Since

\[ |<s, q>| \leq |<s, q-Q\theta>| + |<s, \theta>| \]

\[ \leq \|s\|\|q-Q\theta\| + Q\|s\| \leq \|s\|n^{1/2}Q^{-(n+1)/n} + Q\|s\| \]

\[ \leq 1 + Q\|s\| < QM, \text{ by (b)}, \]

we have

\[ |<p_{j+1}, q>| \leq |<p_j, q>| + (2k+1)QM. \]

Thus

\[ |<p_{j+1}, w, p_{j+1}, q>^{-1}| \geq kd\|w\|(|<p_j, q>| + (2k+1)QM)^{-1}. \]

If \( k \) is large enough

\[ |<p_{j+1}, w, p_{j+1}, q>^{-1}| \geq d\|w\|(4QM)^{-1}. \]

All we have left to show is that \( d\|w\| \geq 1 \).

Let \( u = (q_1, q_2, 0, \ldots, 0) \). For \( z = (z_1, z_2, \ldots, z_n) \in \mathbb{Z}^n \)
let \( Q(z) \) be the set of vectors \( v \) in \( \mathbb{R}^n \) such that

(i) \( z_1 < |<v, e_1>| \leq z_1 + 1 \) for \( 3 \leq i \leq n \);

(ii) \( z_2\|u\|^2 < |<v, u>| \leq (z_2+1)\|u\|^2 \);
(iii) $z_1\|w\|^2 < \langle v, w \rangle \leq (z_1+1)\|w\|^2$.

The sets $Q(z)$ partition $\mathbb{R}^n$ and each has Lebesgue measure $\|w\||u|| = q_1^2 + q_2^2$. Since the cardinality of $Q(z) \cap \mathbb{Z}^n$ is independent of $z$, each $Q(z)$ has $q_1^2 + q_2^2$ points of $\mathbb{Z}^n$. For every hyperplane $z+F$ with $z \in \mathbb{Z}^n$ and $0 < \langle z, w \rangle < \|w\|^2$ there is a point $z'$ in $\mathbb{Z}^n \cap Q(0,0,0,\ldots,0)$ such that $z'+F = z+F$. For example, if $z$ is in $Q(0,v_2,\ldots,v_n)$, then $z' = z - (v_2e_2 + \cdots + v_ne_n)$ is in $Q(0,\ldots,0)$. Since $z'-z \in F$, $z'+F = z+F$. Thus the cardinality of $\mathbb{Z}^n \cap Q(0,\ldots,0)$ is an upper bound on the number of hyperplanes $z+F$ such that $0 < \langle z, w \rangle \leq \|w\|^2$ and $z \in \mathbb{Z}^n$. The distance of a point $v$ in $Q(0,\ldots,0)$ from $F$ is given by $|\langle v, w \rangle|$. Since $|\langle v, w \rangle| \leq \|w\|^2$, that distance is bounded by $\|w\|$. Since the distance between two translates of $F$ by elements of $\mathbb{Z}^n$ is always an integer multiple of the minimum distance $d$, we have

$$d(q_1^2 + q_2^2) \geq \|w\|$$

but since $\|w\|^2 = q_1^2 + q_2^2$, we have $d\|w\| \geq 1$, and we are done.

Lemma 2 is a restatement of Lemma 1 in the form that will be used in the next section.

Lemma 2. Let $n > 1$ and $S$ a finite subset of $\mathbb{Z}^n$. Let $N \in \mathbb{Z}^+$ be given. Then there is an $N' \in \mathbb{Z}^+$ such that if
\{p_i\} is a finite sequence in \( \mathbb{Z}^n \) of length \( N' \) and 
p_{i+1} - p_i \in S \quad \text{for all} \quad i < N', \quad \text{then there are} \quad N \quad \text{distinct integers} \quad j \quad \text{and a hyperplane} \quad H \quad \text{such that} \quad p_j \in H.

\textbf{Proof.} \quad \text{Suppose Lemma 2 is false for some} \quad N; \quad \text{then choose for each} \quad N' \in \mathbb{Z}^+ \text{ a sequence} \quad \{p_{i}',\} \text{ of length} \quad N' \text{ which meets no rational hyperplane more than} \quad N-1 \quad \text{times. We shall inductively define an infinite sequence} \quad \{p_i\} \text{ such that} 
p_{i+1} - p_i \in S \quad \text{for all} \quad i \quad \text{and such that} \quad \{p_i\} \text{ meets no hyperplane more than} \quad N-1 \quad \text{times, a contradiction of Lemma 1. We may assume that} \quad p_{i}', = 0 \quad \text{for all} \quad N'; \quad \text{let} \quad p_1 = 0. \quad \text{Then} 
p_{i}', \in S \quad \text{for all} \quad N'. \quad \text{Since} \quad S \quad \text{is finite there is a} \quad p_2 \in S \quad \text{such that for infinitely many} \quad N', \quad p_{i}', = p_1 \quad \text{and} \quad p_{i}', = p_2. 

\text{Suppose} \quad p_1, \ldots, p_k \quad \text{have been chosen so that for arbitrarily large choices of} \quad N' \quad \text{we have} \quad p_{i}', = p_i \quad \text{for} \quad 1 \leq i \leq k.

Then among those sequences we have \( p_{i}', \in p_k + S \). \quad \text{Since} \quad p_k + S \quad \text{is finite, there is a} \quad p_{k+1} \in p_k + S \quad \text{such that for infinitely many} \quad N', \quad p_{i}', = p_1 \quad \text{for} \quad 1 \leq i \leq k+1. \quad \text{The infinite sequence} \quad \{p_i\} \quad \text{meets no rational hyperplane more than} \quad N-1 \quad \text{times because no initial segment of it can.
SECTION V

THEOREM 1 FOR $G = T^n$

In this section we prove Theorem 1 for $G = T^n$ by induction on $n$.

Proof. We may assume that $n > 1$ and that the theorem is true for $G = T^{n-1}$. If $\text{card}(B(\mu)) < \infty$ we are done by Theorem 1', since $Z^n$, the dual group of $T^n$, is an ordered group under the lexicographic ordering. We therefore suppose that $\text{card}(B(\mu)) = \infty$. Let $\pi$ be the projection onto a coordinate such that $\pi(B(\mu))$ is unbounded. We may suppose that there exists $\{Y_1\} \subseteq B(\mu)$ such that $\lim \pi(Y_1) = +\infty$. Let $\gamma_0 \in B(\mu)$ be arbitrary and suppose that $\gamma_{k,j}$, $1 \leq j \leq r$, $1 \leq k \leq m-1$ ($m \geq 1$), have been found in $B(\mu)$. It is consistent with (1) to let $\gamma_{m,r}$ be any member of $B(\mu)$ such that

$$\pi(\gamma_{m,r}) > \pi(\gamma) \text{ for all } \gamma \in P_{m-1}.$$  \hfill (5)

We may suppose that $\gamma_{m,j}$ for $i+1 \leq j \leq r$ have been found
to satisfy (5) in the role of $\gamma_{m,r}$ and (1), but that no $\gamma_{m,i}$ can be found to satisfy (5) in the role of $\gamma_{m,r}$. Let $M$ be $2\max\{\pi(\gamma_{m,j}) : i+1 \leq j \leq r\}$. Let $N'$ be the integer given by Lemma 2 for $S = \{\gamma - \gamma_{m,j} : \gamma \in P_{m-1}$ and $i+1 \leq j \leq r\}$ and $N = (r+1)^3r^2 + 1$. Consider any $\rho \in B$ such that

$$\pi(\rho) > N'M.$$ 

Since $\rho$ satisfies (5) in the place $\gamma_{m,r}$, the reason $\rho$ cannot serve as a $\gamma_{m,i}$ satisfying (5) is that for some $\gamma \in P_{m-1}$ and $i+1 \leq j \leq r$ we have $\rho + \gamma - \gamma_{m,j} \in B$. Note that $(N'-1)M < \pi(\rho + \gamma - \gamma_{m,j}) < \pi(\rho)$. Since $\rho' = \rho + \gamma - \gamma_{m,j}$ satisfies (5) also, there must be $\gamma' \in P_{m-1}$ and $i+1 \leq j' \leq r$ such that $\rho' + \gamma' - \gamma_{m,j'} \in B$. Also $(N'-2)M < \pi(\rho' + \gamma' - \gamma_{m,j'}) < \pi(\rho')$. If we let $\rho_1 = \rho$, and $\rho_2 = \rho_1 + \gamma - \gamma_{m,j}$, we can continue in this way to obtain a sequence $\{\rho_i\}$ of length $N'$ such that $\rho_{i+1} - \rho_i \in S$ for $i < N'$. Note that the sequence is composed of distinct points. By Lemma 2 there are a rational hyperplane $H$ and $N = (r+1)^3r^2 + 1$ integers $j$ such that $\rho_j \in H$. Thus, since the $\rho_i$'s are distinct, $\text{card}(H \cap B(\mu)) > (r+1)^3r^2$.

Let $z \in \mathbb{Z}^n$ such that $(z+H) \cap \mathbb{Z}^n$ is isomorphic to $\mathbb{Z}^{n-1}$. Let $\psi$ be the quotient map from $T^n$ to $T^n/[(z+H) \cap \mathbb{Z}^n]$. Let $\nu = \psi(z \cdot u)$ be the measure on $\psi(T^n)$ such that $\nu(E) = (z \cdot u)(\psi^{-1}(E))$ for all Borel sets $E \subseteq \psi(T^n)$. Equivalently, $\nu$ is that measure in $\mathcal{M}(\psi(T^n))$ such that for $\gamma \in (z+H) \cap \mathbb{Z}^n$

$$\hat{\nu}(\gamma) = [\psi(z \cdot u)]^\wedge(\gamma) = (z \cdot u)^\wedge(\gamma) = \hat{\mu}(\gamma - z). \quad (6)$$
Then \( v \) satisfies the hypotheses of Theorem 1 for \( \psi(T^n) \) isomorphic to \( T^{n-1} \). Let \( \gamma_0', \gamma_{k,j}', 1 \leq k \leq r^2, 1 \leq j \leq r \), be given in \( B(v) = z + (B(\mu) \cap H) \) satisfying condition (1) of Theorem 1. Then redefine \( \gamma_0 \) to be \( \gamma_0' - z \), \( \gamma_{k,j} \) to be \( \gamma_{k,j}' - z \) for \( 1 \leq k \leq m-1, 1 \leq j \leq r \), and \( \gamma_{m,j} \) to be \( \gamma_{m,j}' - z \) for \( i+1 \leq j \leq r \). If we let \( \gamma_{k,j} = \gamma_{k,j}' - z \) for the remaining indices, we are done.
SECTION VI
THE GENERAL CASE

In this section we finish the proof of Theorem 1.

Proof. Let us assume that $\Gamma$ is finitely generated. By [6, p. 49], $\Gamma = \Lambda \oplus \mathbb{Z}^n$ for some non-negative integer $n$, where $\Lambda$ is the torsion subgroup of $\Gamma$ and hence by assumption $\text{card}(\Lambda) \leq K$. Since $\text{card}(B(\mu)) > K(r+1)^3 r^2$ there is a $\lambda \in \Lambda$ such that $\text{card}((\lambda+\mathbb{Z}^n) \cap B(\mu)) > (r+1)^3 r^2$. Let $\psi$ be the quotient map $G$ to $G/(\mathbb{Z}^n)^1$. Then $\nu = \psi(\lambda \cdot \mu)$ is the measure on $\psi(G)$ satisfying

$$\hat{\nu}(z) = (\overline{\lambda \mu})^\wedge(z) = \hat{\mu}(\lambda+z), \text{ for } z \in \mathbb{Z}^n.$$ 

Since $\psi(G)$ is isomorphic to $T^n$ and since $\nu$ satisfies the hypotheses of Theorem 1, there exist $\gamma'_0$ and $\gamma'_{k,j}$, $1 \leq k \leq r^2$, $1 \leq j \leq r$, in $B(\nu) = -\lambda + [B(\mu) \cap (\lambda+\mathbb{Z}^n)]$ satisfying (1), with $\gamma'_0$, $\gamma'_{k,j}$ replacing $\gamma'_0$ and $\gamma'_{k,j}$, respectively. Then $\gamma'_0 = \lambda + \gamma'_0$, $\gamma'_{k,j} = \lambda + \gamma'_{k,j}$, $1 \leq k \leq r^2$, $1 \leq j \leq r$. 

22
satisfy (1) for \( \mu \).

In the fully general case, let \( S \) be a subset of \( B(\mu) \) of cardinality \( K(r+1)^{3r^2} + 1 \). Let \( \Lambda \) be the subgroup of \( \Gamma \) generated by \( S \). Let \( \psi \) be the quotient map from \( G \) to \( G/\Lambda \). Let \( \nu = \psi(\mu) \) be the measure on \( G/\Lambda \) such that \( \hat{\nu}(\lambda) = \hat{\psi}(\lambda) \) for all \( \lambda \in \Lambda \). \( \nu \) satisfies the hypotheses of Theorem 1 and \([\psi(G)]^\Lambda = \Lambda \) which is finitely generated. As we have already proven, for \( \nu \) there are \( \gamma_0, \gamma_k, j \), \( 1 \leq k \leq r^2, 1 \leq j \leq r \), satisfying (1). Since \( B(\mu) \cap \Lambda = B(\nu) \) and \( \Lambda \) is a subgroup, the same \( \gamma_0 \) and \( \gamma_k, j \)'s will work for \( \mu \).
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April 11, 1975