Path-Integral Evaluation of the Time-Evolution Propagator for Quadratic Hamiltonian Systems With Application to the Lee Model.

John Lee Pell
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Path-Integral Evaluation of the Time-Evolution Propagator for Quadratic Hamiltonian Systems with Application to the Lee Model

A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy in

The Department of Physics and Astronomy

by

John Lee Pell
B.A., Memphis State University, 1967
M.S., Louisiana State University, 1969
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ABSTRACT

Path-integral methods are used to derive an exact expression for the time-evolution propagator for a broad class of systems with quadratic Hamiltonians. For a certain sub-class of such systems, the result is reduced to a simplified closed form. The propagators are exhibited for several illustrative elementary cases and for the Lee Model for a single heavy particle, with two isotopic states of equal mass, interacting with a Bosonic field. The propagator for the latter is used to calculate the survival probability of the unstable state of the heavy particle with no Bosons present.
(1.1) Orientation to the Problem

The quantum mechanical motion of many physical systems of interest may be derived from Hamiltonian operators which are expressible in the Schrödinger picture, either exactly or as a useful approximation, by the quadratic form

\[ H(q, p, t) = \sum_{j=1}^{N} \sum_{j'=1}^{N} [a_{jj'} \hat{p}_j \hat{p}_{j'} + b_{jj'} \hat{q}_j \hat{q}_{j'}] + \sum_{j=1}^{N} \frac{1}{2} c_{jj'} (\hat{p}_j \hat{q}_{j'} + \hat{q}_j \hat{p}_{j'}) + \sum_{j=1}^{N} [d_j(t) \hat{p}_j + e_j(t) \hat{q}_j] + f(t), \]  

where \( \hat{q}_j \) is the operator for multiplication by the coordinate \( q_j \) associated with the \( j \)-th degree of freedom, presumed to be a real variable ranging from \( -\infty \) to \( +\infty \), and where

\[ \hat{p}_j = -\hbar \frac{\partial}{\partial q_j}. \]  

The coefficient matrices \( a, b, c, d(t), e(t), \) and \( f(t) \) occurring in Eq. (1) are taken to be real; \( a, b, \) and \( c \) are assumed to be independent of time, \( t \); \( a \) is assumed to be non-singular; and, as a notational convenience, both \( a \) and \( b \) are presumed to have been symmetrized. It is the purpose of this paper to present a unified exact explicit description of the quantum behavior of the fairly broad class of systems.
characterized by Eq. (1) and to apply them to the Lee Model, as an example. Henceforth, the term "quadratic systems" will be limited to systems characterized by Eq. (1).

A complete description of the quantum mechanical behavior of every quadratic system can be achieved, in a general manner which is independent of the system's initial state, by exhibiting explicitly the system's space-time propagator, \( K(q'', t''; q', t') \), which is defined by

\[
\psi(q'', t'') = \int K(q'', t''; q', t') \psi(q', t') dV',
\]

where \( \psi(q,t) \) is the system's coordinate-space, wavefunction and where \( dV' \) is the element of volume in coordinate space at point \( q' \). \(^1\) Propagator theory is reviewed in Section 2 of the present chapter, with emphasis on methods for obtaining detailed information concerning specific aspects of the behavior of the system. In Chapter 2 a procedure is developed and carried to conclusion for the evaluation of the propagator for quadratic systems, with the general solution for the propagator expressed in the coordinate representation as defined in Eq. (3). The propagator is given also in a simplified form for a fairly broad subclass of quadratic systems. In Chapter 3 the solution for the propagator is applied to some familiar one- and three-dimensional systems, yielding exact descriptions of their behavior. As a further application, Chapter 4 contains a treatment of the Lee Model for a single heavy particle, with two isotopic states of equal mass, interacting with a
Bosonic field. The exact propagator for this model is obtained and used to calculate some quantities of special physical interest. Chapter 5 summarizes the accomplishments of the paper, with a few suggestions for further research.

The general expression derived in Chapter 2 for the space-time propagator for quadratic systems is potentially useful in several respects: (1) It explicates some of the general features common to the dynamical behavior of quadratic systems. Such features may be sufficient in some applications to make some specific predictions of interest, without the need for a complete quantum treatment. An example of such a feature is provided by the well-known fact that for quadratic systems the dependence of \( K(q'', t''; q', t') \) upon the coordinates, \( q'' \) and \( q' \), is contained wholly within the factor \( \exp(iS_c(q'', t''; q', t')/\hbar) \), where \( S_c(q'', t''; q', t') \) is the classical action function. (2) In some applications, the Hamiltonian for the system may be adequately approximated by Eq. (1), in which case the solution for \( K \) as expressed in detail in Chapter 2 may be employed without further approximations to obtain predictions of interest in formats which are explicit and practical for numerical computation. If such an approximation is not in itself entirely adequate, it still may serve as a "zero-th" order approximation in a systematic perturbation calculation in which results are obtained in successively higher orders in the difference between the exact Hamiltonian and its quadratic approximation.

To select a suitable quadratic approximate Hamiltonian of the
form of Eq. (1), Feynman's path-integral variational method may be helpful.  

(3) A prospectively powerful use of the result for $K$ arises in connection with the problem of obtaining the propagator for a system with many degrees of freedom such that the Hamiltonian may be expressed in the form of Eq. (1) with the $q$'s and $p$'s representing some (but not all) of the degrees of freedom and with the time-dependent functions, $d_1(t)$, $e_1(t)$, and $f(t)$, representing functions of the motion of the remaining degrees of freedom. In such a case the propagator for the entire system may be expressed as a path integral over all of the degrees of freedom and the integration over some of the degrees of freedom then performed by application of the result obtained for $K$ in Chapter 2. The outcome is an exact path-integral expression requiring path integration of only the remaining degrees of freedom. Exact reduction of the number of degrees of freedom of a system can be a significant step in the exact or approximate analysis of the system. According to the theory of interactions there are many instances of widely used models of physical systems of great interest for which a set of particles (or a field) interacts directly only with some Bosonic field in such a way that when the Hamiltonian is expressed in terms of the Bosonic field oscillator coordinates and momenta, the result is of the form for which the reduction process just described is possible. Examples of such applications occur in Feynman's treatment of quantum electrodynamics. Another example is
(1.2) Review of Propagator Theory

(1.2.1) General properties

In Section (1.1) the propagator in the coordinate representation, \( K(q'', t''; q', t') \), was defined by Eq. (3) in terms of the wavefunction, \( \psi(q, t) \). Since \( \psi(q, t) \) must satisfy the Schrödinger equation,

\[
\frac{i\hbar}{\partial t} \psi(q, t) = \hat{H}(q, p, t) \psi(q, t),
\]

(4)

substitution of Eq. (3) into Eq. (4) implies

\[
\frac{i\hbar}{\partial t} K(q'', t''; q', t') = \hat{H}(q'', p'', t'') K(q'', t''; q', t').
\]

(5)

Continuity of the wavefunction as the time interval

\[
T = t'' - t'
\]

(6)

approaches zero in Eq. (3) requires that

\[
\lim_{t'' \to t'} K(q'', t''; q', t') = \delta(q'' - q').
\]

(7)

Since Eq. (5) is first order in \( t'' \), then Eq. (5) and the initial condition specified by Eq. (7) determine \( K \) uniquely.

An immediate consequence of Eq. (3) is the superposition principle,

\[
K(q'', t''; q', t') = \int K(q'', t''; q, t) K(q, t; q', t') dq,
\]

(8)

where the time \( t \) partitions the interval \([t', t'']\) into the subintervals \([t', t]\) and \([t, t'']\). Note that Eq. (8) may be
readily generalized to the case of arbitrarily fine partitions, \( t_1, t_2, \ldots, t_N \), of \([t', t'']\):

\[
K(q'', t''; q', t') = \int dq_1 dq_2 \cdots dq_N K(q'', t''; q_1, t_N) \cdots \cdots K(q_2, t_2; q_1, t_1) K(q_1, t_1; q', t').
\]

(1.2.2) Transition probability amplitudes

It is often useful to express the propagator in terms of arguments other than coordinates. Let \( \alpha \) be a parameter which indexes a complete set of states, \( \psi_\alpha(q) \), of the system. The propagator for the system in the \( \alpha \)-representation is defined to be

\[
K_{\alpha'', \alpha'}(t'', t') = \int \psi_\alpha^\ast(q'') \psi_\alpha(q'', t'') \, dq''
\]

where \( \psi_\alpha(q'', t'') \) is the wavefunction for the system at time \( t'' \), given that it was initially in state \( \alpha' \) at time \( t' \). Use of Eq. (3) then yields

\[
K_{\alpha'', \alpha'}(t'', t') = \int \psi_\alpha^\ast(q'') K(q'', t''); q', t') \psi_\alpha(q') \, dq' \, dq''.
\]

This expression, when absolute squared, may be identified as the probability for transition from state \( \psi_\alpha' \), to state \( \psi_\alpha'' \), during the time interval from the initial time \( t' \) to the final time \( t'' \). Such transition probabilities are of central importance in scattering theory, particularly if evaluated for \( (t'' - t') \to \infty \). In such an application, normally the coordinates represented by \( q' \)'s in Eq. (11) would not be taken to stand for the spatial coordinates of the particles.
involved in the scattering process; but rather the states \( \psi_a \) and \( \psi_a'' \) would represent specific states of a system for which a Bosonic field (of particles of some particular type, \( \tau \), present in the system) is occupied by a specified number of particles with specified definite momenta. In the limit \( (t'' - t') \to \infty \), \( K_{a''a'} \) would then be called the scattering matrix (S-matrix) element for the scattering process in which the system is initially in the state \( \psi_a' \), with the \( \tau \) particles having specified initial momenta, and finally after a sufficiently long time is found to have all of the measurable properties which characterize the specified final state \( \psi_a'' \) (including the specified momenta of the \( \tau \) particles).

(1.2.3) Ground-state energies

Suppose that the Hamiltonian for the system happens to be time-independent and that \( \phi_0(q), \phi_1(q), \phi_2(q), \ldots \) form a complete orthonormal set of energy eigenfunctions with corresponding eigenvalues \( E_0, E_1, E_2, \ldots \). Then the coordinate propagator may be expressed as

\[
K(q'', t''; q', t') = \sum_{n=0}^{\infty} \phi_n(q'') \phi_n^*(q') e^{-i(t'' - t')E_n/K}. \tag{12}
\]

The validity of this expression may be seen by observing that it satisfies Eqs. (5) and (7). Suppose that the ground-state energy, \( E_0 \), is non-degenerate. Then examination of Eq. (12) yields the following behavior of the propagator for large imaginary times,
Thus

$$\beta = 1(t'' - t');$$ (13)

$$K(q'',t'-1\beta; q',t') = \sum_{n=0}^{\infty} \phi_n(q'')\phi_n^*(q')e^{-\beta E_n/h}.$$ (14)

$$\lim_{\beta \to \infty} \phi_0(q'')\phi_0^*(q')e^{-\beta E_0/h}.$$. (15)

Thus

$$\ln[K(q'',t'-1\beta; q',t')] \to (\ln(-\beta E_0/h) + \ln[\phi_0(q'')\phi_0^*(q')]) (16)$$

$$\ln[K(q'',t'-i\beta; q',t')] \to -\beta E_0/h.$$ (17)

Therefore the ground-state energy may be written in terms of the propagator as:

$$E_0 = -\hbar \lim_{\beta \to \infty} \{\ln[K(q'',t'-i\beta; q',t')]\}/\beta.$$ (18)
(2.1) Path-Integral Expression for the Propagator

The last section of Chapter 1 demonstrates the usefulness of the propagator for a given quadratic system with regard to obtaining specific information about that system, thus providing motivation for evaluating the propagator. The phase-space path-integral language of Garrod is used here to formulate the propagator conveniently for evaluation in the next section of this chapter.

Garrod's expression for the propagator for quadratic systems can be constructed by the following procedure, as verified in Appendix A:

1. Let \( t_1, t_2, \ldots, t_{M-1} \) be a partition of the time interval \([t', t'']\) with
\[
t_0 = t' \quad \text{and} \quad t_M = t''.
\]

For convenience let
\[
t_\ell - t_{\ell-1} = \epsilon = (t'' - t') / M \tag{20}
\]
for all \( \ell = 1, 2, \ldots, M \). Equation (9) gives the propagator for the complete time interval as
\[
K(q'', t''; q', t') = \int dq_1 \int dq_2 \int_{-\infty}^{\infty} dq_M \prod_{\ell=1}^{M} K(q_\ell, t_\ell; q_{\ell-1}, t_{\ell-1}), \tag{21}
\]

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where $K(q_{\ell},t_{\ell};q_{\ell-1},t_{\ell-1})$ is the propagator for the $\ell$th subinterval, and

$$q_o = q' \text{ and } q_M = q''.$$  \hfill (22)

(2) Let

$$K(q_{\ell},t_{\ell};q_{\ell-1},t_{\ell-1}) =$$

$$\int \frac{dp_{\ell}}{\hbar} \exp\left\{ \frac{1}{\hbar} \int_{t_{\ell-1}}^{t_{\ell}} [p(t)q(t) - H(q(t),p(t),t)] dt \right\},$$  \hfill (23)

where

$$q(t) = \frac{[q_{\ell-1}(t_{\ell}-t) + q_{\ell}(t-t_{\ell-1})]}{\epsilon} \text{ for } t_{\ell-1} \leq t \leq t_{\ell},$$  \hfill (24)

$$p(t) = p_{\ell} \text{ for } t_{\ell-1} < t < t_{\ell},$$  \hfill (25)

and $H(q,p,t)$ is the classical Hamiltonian function defined by Eq. (1). The functions $q(t)$ and $p(t)$ are shown graphically in Figs. 1 and 2 respectively.

(3) Take the limit as $M$ approaches infinity. Note that, in this limit, $q(t)$ and $p(t)$ characterize a large class of paths in phase space as the variables $q_{1}, q_{2}, \ldots, q_{M-1}$ and $p_{1}, p_{2}, \ldots, p_{M}$ vary independently. The result generalized from one to $N$ degrees of freedom is

$$K(q'',t'';q',t') = \lim_{M \to \infty} \int_{-\infty}^{\infty} \prod_{j=1}^{N} dq_{j} \prod_{j=1}^{M-N} dp_{j}$$

$$\times \exp\left\{ \frac{1}{\hbar} \int_{t'}^{t''} \sum_{j=1}^{N} p_j(t)q_j(t) - H(q(t),p(t),t) \right\},$$

$$= \int D^{N}[q(t),p(t)] \exp\left\{ \frac{1}{\hbar} \int_{t'}^{t''} \sum_{j=1}^{N} p_j(t)q_j(t) - H(q(t),p(t),t) \right\}.$$  \hfill (26)

$$- H(q(t),p(t),t) dt.$$  \hfill (27)
Fig. 1

Fig. 2
Feynman's coordinate-space representation of the quadratic-system propagator can be reproduced from Eq. (26) by execution of the indicated integration of the $p_{j,k}'$'s. This is easily accomplished with the aid of two momentum transformations. First let

$$p_{j}(t) = \sum_{k=1}^{N} \mu_{jk}^{-1} p_{k}'(t)$$

and

$$p_{j,k} = \sum_{j'=1}^{N} \nu_{j'j}^{-1} p_{j'}', k$$

where $\mu$ is a symmetric matrix which satisfies the equation

$$\mu^2 = a.$$ 

Second, let

$$p''_{j,k} = p_{j,k} + \frac{1}{2} \sum_{j'=1}^{N} \nu_{j'j}^{-1} \left[ \bar{d}_{k,k} - \bar{q}_{k,k} + \sum_{j'=1}^{N} \nu_{j'j}^{-1} \bar{q}'_{j,k} \right]$$

where $\bar{d}_{k,k}$, $\bar{q}_{k,k}$, and $\bar{q}'_{k,k}$ represent respectively the mean values of $d_k(t)$, $\frac{dq_k(t)}{dt}$, and $q_k(t)$ in the interval $t_{k-1} < t < t_k$. Application of these two transformations to Eq. (26) with use of Eqs. (1), (24), and (25) yields

$$K(q'', t'' ; q', t') = \lim_{M \to \infty} \left[ \ldots \int \left[ \prod_{j=1}^{N} dq_j, \ldots dq_{j,M-1} \right] \sum_{j=1}^{M} \exp \left[ \frac{-1}{\gamma_{j}} \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{1}{\nu_{jk} \gamma_{l}^{-1}} \left( \sum_{j'=1}^{N} \gamma_{j'j}^{-1} \bar{q}'_{j,k} \right) \right] \right]$$

$$\exp \left[ \frac{-1}{\gamma_{j}} \sum_{j'=1}^{N} \sum_{l=1}^{N} \frac{1}{\nu_{jk} \gamma_{l}^{-1}} \left( \sum_{j'=1}^{N} \gamma_{j'j}^{-1} \bar{q}'_{j,k} \right) \right] \right] \right]$$

$$x \exp \left[ \frac{-1}{\gamma_{j}} \sum_{j'=1}^{N} \sum_{l=1}^{N} \frac{1}{\nu_{jk} \gamma_{l}^{-1}} \left( \sum_{j'=1}^{N} \gamma_{j'j}^{-1} \bar{q}'_{j,k} \right) \right] \right]$$

The final result, obtained by integration of the $p''_{j,k}$'s, is
\[ K(q'',t'';q',t') = \lim_{M \to \infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} [a|2\hbar T/M|^N]^{-M/2} \]

\[ x \prod_{j=1}^{N} \prod_{l=1}^{M-1} dq_j \exp[\frac{i}{\hbar} \int t'' \mathcal{L}(q(t),\dot{q}(t),t)dt], \]

where

\[ \mathcal{L}(q,\dot{q},t) = \sum_{j=1}^{N} p_j \dot{q}_j - H(q,p,t) \]

\[ = \sum_{j=1}^{N} \sum_{j'=1}^{N} \frac{1}{\hbar} a_j^{-1} (q_j - \dot{q}_j(t))(\dot{q}_j - \dot{q}_{j'}(t)) + q_j \dot{q}_j - \rho_j(t)q_j - f(t) \]

is the Lagrangian corresponding to the Hamiltonian \( H(q,p,t) \), where

\[ Q = -\frac{1}{2}a^{-1}c, \]

\[ g = b - \frac{1}{4}a^{-1}c, \]

and

\[ \rho(t) = e(t) + \tilde{q} \mathcal{C}(t), \]

and where the limit \( M \to \infty \) was taken in the argument of the second exponential function occurring in Eq. (32). The tilde in Eqs. (37) and (38) denotes transposition. The square root of any complex number is to be understood to represent the root with an argument, \( \theta \), in the range \(-\pi/2 < \theta \leq +\pi/2\). Since the coordinate space representation of the propagator, given in Eq. (33), involves a system-dependent normalization factor, \([a|2\hbar T/M|^N]^{-M/2}\), the phase-space expression, which does not, is more readily generalizable.
(2.2) General Derivation

A convenient first step in evaluating the propagator given by Eq. (33) is the extraction of its dependence on the endpoint values \( q' \) and \( q'' \). This can be accomplished by writing the path as

\[
q(t) = q^c(t) + z(t)
\]  

(39)

and integrating \( \int_{t'}^{t''} \mathcal{L}(q(t), \dot{q}(t), t) dt \) by parts to get

\[
\int_{t'}^{t''} \mathcal{L}(q(t), \dot{q}(t), t) dt = \int_{t'}^{t''} \mathcal{L}(q^c(t), \dot{q}^c(t), t) dt + \int_{t'}^{t''} \mathcal{L}'(z(t), \dot{z}(t)) dt,
\]

(40)

where \( q^c(t) \) is the classical trajectory determined by the boundary conditions

\[
q^c(t') = q(t') = q',
\]

(41)

and

\[
q^c(t'') = q(t'') = q'',
\]

(42)

and the classical equation of motion

\[
\frac{1}{2}a^{-1}(q^c(t)-\dot{d}(t)) + 2Q^a q^c(t) + 2gq^c(t) + p(t) = 0;
\]

(43)

where

\[
z(t) = q(t) - q^c(t);
\]

(44)

and where

\[
\mathcal{L}'(z, \dot{z}) = \sum_{j=1}^{N} \sum_{j'=1}^{N} \left[ \frac{1}{2}a_{jj'} \dot{z}_j \dot{z}_{j'} - Q^a_{jj'} z_j \dot{z}_{j'} - g_{jj'} z_j z_{j'}, \right]
\]

(45)

is just the quadratic part of \( \mathcal{L}(z, \dot{z}, t) \), with \( Q^a = \frac{1}{2}(Q - Q) \)

denoting the antisymmetric part of the matrix \( Q \).
Substitution of Eq. (40) into Eq. (33) yields

\[ K(q'', t''; q', t') = \exp[i S_c(q'', t''; q', t')/\hbar] F(T), \]  

(46)

where

\[ S_c(q'', t''; q', t') = \int_{t'}^{t''} \mathcal{L}(q^c(t), \dot{q}^c(t), t) \, dt \]  

(47)

is the classical action associated with the Lagrangian \( \mathcal{L} \)

and where

\[ F(T) = \lim_{M \to \infty} \left( \int_{-\infty}^{\infty} \left[ \frac{1}{(2\pi \hbar T/M)^N} \right] \prod_{j=1}^{N} \prod_{\ell=1}^{M-1} dq_{j, \ell} \right)^{1/2} \]

\[ \times \exp \left[ \frac{1}{\hbar} \int_{t'}^{t''} \mathcal{L}'(z(t), \dot{z}(t)) \, dt \right] \]  

(48)

\[ = K'(0, t''; 0, t'), \]  

(49)

with \( K' \) defined by replacement of \( \mathcal{L} \) by \( \mathcal{L}' \) in Eq. (33).

Equation (46) reveals the coordinate dependence of

\( K(q'', t''; q', t') \) to be as stated in Section (1.1).

In order to evaluate \( F(T) \) let \( S_c' \) be defined by

replacement of \( \mathcal{L} \) by \( \mathcal{L}' \) in Eq. (47), as \( K' \) is similarly

defined in Eq. (49). Since the form of \( \mathcal{L}' \) is a special

case of the form of \( \mathcal{L} \), then according to Eq. (46)

\[ K'(q'', t''; q', t') = \exp[i S_c'(q'', t''; q', t')/\hbar] F(T). \]  

(50)

By setting \( q'' = q' \) in Eq. (50) and integrating the

result with respect to \( q' \), one obtains

\[ F(T) = A(T)/B(T), \]  

(51)

where
\[ B(T) = \int \ldots \int \exp[iS_c(q',q')/\hbar] \prod dq'_j \]  
\[ A(T) = \int \ldots \int K'(q',t'';q',t') \prod dq'_j \] 
\[ = \lim_{M \to \infty} \int \ldots \int |a|^{-M/2} \exp[i\int L'(q(t),\dot{q}(t))dt/\hbar] \] 
\[ \times \prod_{j=1}^{N} \left[ \prod_{\ell=1}^{M-1} \frac{dq_j, \ell}{\sqrt{(2\hbar T/M)}} \frac{dq'_j}{\sqrt{(2\hbar T/M)}} \right] \] 

where \( q(t) \) is given by Eq. (24) with
\[ q'_j = q_j(t') = q_j, 0 = q_j, M = q_j(t''). \] 

The propagator \( K(q'',t'';q',t') \) is given by Eq. (46) in terms of the classical action \( S_c(q'',t'',q',t') \) and the function \( F(t) \) given by Eqs. (51)-(55) and (24).

To evaluate the integrals occurring in Eq. (54), it is helpful to introduce new variables of integration, \( \beta_j(n)'s \), defined by
\[ \dot{q}_j(t) = \sum_{j'=1}^{N} \mu_{jj'}(2\hbar/T)^{1/2} \{ \beta_{j'}(0)\phi_o(t) \] 
\[ + \sum_{n=1}^{\infty} [\beta^c_j(n)\phi^c_n(t) + \beta^s_j(n)\phi^s_n(t)] \}, \]

where \( \mu \) is a symmetric matrix which satisfies Eq. (30), and the set of functions
\[ \phi_o(t) = 1, \] 
\[ \phi^c_n(t) = \sqrt{2}\cos(2\omega_n(t-t')), \] 
\[ \phi^s_n(t) = \sqrt{2}\sin(2\omega_n(t-t')). \]
and

$$\phi_n(t) = \sqrt{2}\sin(2\omega_n(t-t')),$$

(59)

with

$$\omega_n = n \pi/T$$

(60)

and \(n = 1, 2, 3, \ldots\), form a complete orthonormal set of functions of \(\tau = (t-t')/T\) on the interval \(0 \leq \tau \leq 1\). Since

the relation between the \(q^j, l\)'s occurring as variables of integration in Eq. (54) and the \(\beta^j(n)'s\), as given by Eq. (56), is independent of the matrices \(Q\) and \(g\) in Eq. (45),

then, according to a proof by Davison,\(^4\) the Jacobian for a transformation from the \(q^j, l\)'s to the \(\beta^j(n)'s\) in the limit \(M \to \infty\) is

$$\lim_{M \to \infty} \{(|a|)^2 \left[ \frac{2hT}{M} \right]^{M-1} \frac{N}{2} \}. \quad (61)$$

Application of this transformation to Eq. (54) yields

$$A(T) = |a|^{-2} \int_{-\infty}^{t'} \int_{-\infty}^{t''} \exp \left[ i \int_{t'}^{t''} \gamma'(q(t), \dot{q}(t)) dt' / h \right]$$

$$\times \prod_{j=1}^{N} \frac{d\beta^j(n)}{\sqrt{(+1)}} \frac{d\phi^j(n)}{\sqrt{(+1)}} \frac{d\dot{q}^j}{\sqrt{(+2i\hbar T)}} \}, \quad (62)$$

where the components of \(\dot{q}(t)\) are given by Eq. (56) and where

\[ q(t) = \int_{t'}^{t} \dot{q}(\tau) d\tau + q'. \quad (63) \]

Note that Eqs. (55)-(60) imply that

\[ \beta^j(0) = 0. \quad (64) \]

Let new variables of integration be defined by
\[ y_j = q_j - \sum_{n=1}^{\infty} \frac{\eta_j(n)}{\omega_n} \]  

and by the real and imaginary parts of

\[ \xi_j(n) = \xi_j(n) + i\eta_j(n) \]

\[ \Xi_j(n) = \frac{1}{2} \sum_{j'=1}^{N} \Omega_{j'j}(\omega_n) \xi_j^*(n) \xi_j(n) - g_{jj'} y_j y_{j'} \]  

where

\[ \Omega(\omega_n) = a^{-1} - 2i \frac{Q}{\omega_n} - g/\omega_n^2. \]  

Substitution of Eqs. (65)-(68) into Eq. (62) yields

\[ A(T) = |a|^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp \left( \frac{iS'(q(t))/\hbar}{2} \right) \prod_{n=1}^{N} \left( \frac{a_j}{h\alpha_j} \right) d\xi_j(n) d\eta_j(n) \exp(1S'(q(t))/\hbar) \]  

\[ = [2iT]^N |a|^2 |g|^2 \prod_{n=1}^{\infty} (|a| |\Omega(\omega_n)|)^{-1}, \]  

where the \( a_j \)'s are eigenvalues of a.

In order to evaluate B(T), defined by Eq. (52), the classical action \( S_c'(q',q') \) needed may be determined as the
minimization of $S'(q(t))$ (given by Eqs. (68) and (65)) with respect to $\zeta$ with $q'$ held constant. Minimization requires that

$$\zeta(n) = \frac{2}{i\omega_n} \Omega^{-1}(\omega_n) g y,$$

(72)
in which case Eqs. (65) and (68) become

$$y = \Lambda^{-1} g q'$$

(73) and

$$S'(q',q') = \min S'(q(t)) = -\sum_{j=1}^{N} \sum_{j'=1}^{N} \Lambda_{jj'} y_j y_{j'},$$

(74)

respectively, where

$$\Lambda = g - g x g$$

(75)
and

$$x = \sum_{n=1}^{\infty} \frac{1}{2} (\Omega^{-1}(\omega_n) + \Omega^{-1}(\omega_n)).$$

(76)

Substitution of Eq. (74) into Eq. (52) and transformation of integration variables from $q_j$'s to the $y_j$'s given by Eq.(73) yield

$$B(T) = \left| g^{-1} \Lambda \right| \prod_{n=1}^{\infty} \sum_{j=1}^{N} \sum_{j'=1}^{N} \Lambda_{jj'} y_j y_{j'}, dy_j$$

(77)

$$= \left| g^{-1} (h/2iT)^N \right| |g^{-1} \Lambda|^{1/2}.$$ (78)

Equation (51), with $A(T)$ given by Eq. (71) and $B(T)$ given by Eq. (78) gives

$$F(T) = \left\{ [(2iT)^N |a||I-xg|] \prod_{n=1}^{\infty} (|a||\Omega(\omega_n)|)^{-1} \right\}^{1/2}.$$ (79)

In summary, the propagator for quadratic systems is given by
\[ K(q'',t'';q',t') = \exp(iS_c(q'',t'';q',t')/\hbar)F(T) \]  

where

\[ F(T) = \left\{ (2\pi \hbar)^N |a| |\phi| \right\}^{1/2} \]

\[ r = \prod_{n=1}^{\infty} \left| a_n |\Omega(\omega_n) \right| \]

\[ \phi = 1 - \sum_{n=1}^{\infty} \left[ \Omega^{-1}(\omega_n) + \Omega^{-1}(\omega_n) \right] g/\omega_n^2 \]

\[ \Omega(\omega_n) = a^{-1} - 2iQa/\omega_n - g/\omega_n^2 \]

\[ g = b - \frac{1}{\Omega} ca^{-1}c \]

\[ Qa = -\frac{1}{\Omega}(a^{-1}c - ca^{-1}) \]

\[ \omega_n = n\pi/T \]

\[ T = t'' - t' \]

and \( S_c(q'',t'';q',t') \) is the classical action function.

\[ (2.3) \text{ Special Case} \]

A simplified expression for the propagator \( K(q'',t'';q',t') \) may be obtained if \( a \) is positive definite, so that \( a \) has a unique positive definite square root, \( \mu \), and if

\[ g'Q' = Q'g', \]

where

\[ g' = \mu g \mu \]

and

\[ Q' = \mu Q^a \mu. \]

In this case the infinite product in Eq. (82) and the
infinite sum in Eq. (83) can be performed.

Consider the function

$$L = \prod_{n=1}^{\infty} (\Omega'(\omega_n)\Omega'^*(\omega_n)),$$

where

$$\Omega'(\omega_n) = \mu\Omega(\omega_n)^*$$
$$= I - 21Q'\omega_n - g'/\omega_n^2.$$  (94)

Note that according to Eqs. (82) and (92), the determinant of $L$ is equal to $r^2$ since $\Omega'(\omega_n)$ is Hermitian and the determinant of a matrix is invariant under transposition.

According to Eqs. (92) and (94),

$$\frac{\partial}{\partial g'}|_{Q',\omega_n} (\ln L) = -\sum_{n=1}^{\infty} \frac{1}{\omega_n^2} (\Omega'^{-1}(\omega_n) + \Omega'^{*-1}(\omega_n)).$$  (95)

Comparison of Eqs. (83) and (95) indicates that

$$\phi' = \mu^{-1}\phi\mu$$  (96)

$$= I + \frac{\partial}{\partial g'}|_{Q',\omega_n} (\ln L)|g'.$$  (97)

Note that the values of the determinants of $\phi$ and $\phi'$ are equal. Therefore, Eq. (81) may be written as

$$F(T) = \{(21Th)^N|a||\phi'||L|\}^{-1/2}$$  (98)

since, as already noted, the determinant of $L$ is equal to $r^2$.

Substitution of Eq. (94) into Eq. (92), completion of the square in both factors, and factorization of the result yield
\[ L = \prod_{n=1}^{\infty} \left( I + \frac{Q_n^2}{2} \right) \frac{2}{(\omega_n I - 1 i Q')^2} \left[ I - \frac{g' - Q_n^2}{(\omega_n I + 1 i Q')^2} \right]. \quad (99) \]

Application of the identity
\[
\frac{\cos(2x) - \cos(2y)}{2(y^2 - x^2)} = \prod_{k=1}^{\infty} \left( 1 - \frac{y^2}{k^2 \pi^2} \right) \frac{x^2}{(k\pi + y)^2} \frac{x^2}{(k\pi - y)^2} \quad (100)
\]
yields
\[
L = \frac{\cos(2iTQ') - \cos(2T\sqrt{g' - Q'^2})}{2g'T^2}. \quad (101)
\]

After substitution of Eqs. (101) and (97) into Eq. (98), one obtains the result that the propagator for the special case is given by Eq. (80) with
\[
F(T) = \{(2iTh)^N|a||G[4T^2(g' - Q'^2)]|\}^{-1/2} \quad (102)
\]
where
\[
G(x^2) = \frac{\sin(x)}{x}. \quad (103)
\]
CHAPTER 3
SIMPLE APPLICATIONS

(3.1) One Degree of Freedom

In one degree of freedom the Hamiltonian function defined by Eq. (1) reduces to the form

\[ H(q, p, t) = \frac{1}{2m} p^2 + \frac{1}{2} mbq^2 + \frac{1}{2}\omega_0(pq + qp) + d(t)p + e(t)q + f(t), \]  

(104)

where \( m, b, \omega_0 \) are given constants and \( d(t), e(t), f(t) \) are specified functions of time. This Hamiltonian describes a forced harmonic oscillator of mass, \( m \), for which the classical action is

\[
S_c(q", q') = \frac{1}{\sin(\omega T)} \left( \frac{1}{2} \omega [\omega^2 (q"^2 + q'^2) \cos(\omega T) - 2q"q'] 
+ \int_{t'}^{t''} E(t) \sin(\omega (t-t')) \, dt + q' \int_{t'}^{t''} E(t) \sin(\omega (t'-t)) \, dt 
- \frac{1}{\omega} \int_{t'}^{t''} \int_{t'}^{t''} E(t)E(\tau) \sin(\omega (t-\tau)) \sin(\omega (\tau-t')) \, d\tau \right) 
- \frac{1}{2} \omega_0 (q"^2 - q'^2) + \int_{t'}^{t''} \left[ \frac{1}{2} md^2(t) - f(t) \right] \, dt - m[q"d(t") - q'd(t')] ,
\]

(105)

where

\[ \omega = (b-\omega_0^2)^{1/2} \]

(106)

is the angular frequency of oscillation and
\[ E(t) = m \dot{\mathbf{a}}(t) - \mathbf{e}(t) + m_0 \mathbf{d}(t) \] (107)

is the driving force on the oscillator. Substitution of Eqs. (85), (86), (90), (91), and (30) into Eq. (102), and use of Eq. (103), give

\[ F(T) = \left[ \frac{m_0}{\hbar \sin(\omega T)} \right]^{1/2}. \] (108)

According to Eqs. (80) and (108) the propagator is

\[ K(q'', t''; q', t') = \left( \frac{m_0}{\hbar \sin(\omega T)} \right)^{1/2} \exp(iS_c(q'', q')/\hbar) \] (109)

where \( S_c(q'', q') \) is given by Eq. (105).

In the special case of Eq. (104) for which

\[ d(t) = e(t) = f(t) = 0 = \omega_0, \] (110)

the propagator for a simple harmonic oscillator is recovered as given by

\[ K(q'', t''; q', t') = \left( \frac{m_0}{\hbar \sin(\omega T)} \right)^{1/2} \exp\left\{ \frac{i m_0}{2 \hbar \sin(\omega T)} \left[ (q''^2 + q'^2) \cos(\omega t) - 2 q'' q' \right] \right\}. \] (111)

In the limit \( \omega \to 0 \), this reduces to the free particle propagator

\[ K(q'', t''; q', t') = \left( \frac{m}{\hbar T} \right)^{1/2} \exp[i (q'' - q')^2 / 2 \hbar T]. \] (112)

(3.2) Particle in a Magnetic Field

The Hamiltonian for a particle in a constant magnetic field may be expressed in the form
\[ H(x,p) = \frac{3}{2} \sum_{j=1}^{3} \frac{1}{2m} p_j^2 + \frac{1}{2m} \left( \frac{qB}{c} \right)^2 x_1^2 - \frac{gB}{mc} p_2 x_1, \]  

where \( q \) is the charge of the particle, \( m \) is its mass, and \( B \) is the magnitude of the magnetic field, which is taken to be in the \( x_3 \)-direction. Substitution of the coefficients of Eq. (113) into Eqs. (85), (86), (90), (91), and (30) gives

\[ g' = 0 \]  

and

\[ Q' = \frac{i}{\hbar} \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]  

where

\[ \omega = \frac{1}{2} qB/mc. \]  

Substitution of Eqs. (114), (115), and (103) into Eq. (102) and use of Eq. (80) yield

\[ K(x'',t'';x',t') = \left( \frac{m}{i\hbar T} \right)^{1/2} \frac{m\omega}{i\hbar \sin(\omega T)} \exp(iS_c(x'',x')/\hbar) \]  

where \( S_c(x'',x') \) is the classical action for a particle in a constant magnetic field, which is given by

\[ S_c(x'',x') = \frac{1}{2} m\omega \left[ (x_1'' - x_1')^2 + (x_2'' - x_2')^2 \right] \cot(\omega T) \]

\[ - 2(x''_1x'_2 - x''_2x'_1) + \frac{m}{cT} (x''_3 - x'_3)^2. \]  

Glasser\textsuperscript{5} has performed the path-integral evaluation of this propagator.
CHAPTER 4
LEE MODEL

(4.1) Description of the Lee Model

The Lee Model, devised by T. D. Lee, describes a Fermi field of particles (called heavy particles) with two isotopic states (called V-particle states and N-particle states) interacting with a Boson field of particles (called light particles, or θ particles). The Hamiltonian describing the system is

\[ H = \sum_{p, \lambda} E_{\lambda}(p) \hat{b}_{\lambda}^+(p) \hat{b}_{\lambda}(p) + \sum_{k} \frac{-g_{0}}{\sqrt{V}} \frac{1}{f(\omega_k/\sqrt{2\omega_k})} \left[ a_k^+ b_V^+(p') b_N(p) a_k + a_k^+ a_k^+ b_V(p) b_V(p') \right] \] (119)

where \( \lambda \) is an isotope index with domain \((V,N)\), \( \hat{b}_{\lambda}(p) \) and \( \hat{b}_{\lambda}^+(p) \) are Fermionic creation and destruction operators for a heavy particle of isotopic type \( \lambda \) and momentum \( p \), \( \hat{a}_k \) and \( \hat{a}_k^+ \) are Bosonic creation and destruction operators for a θ particle of momentum \( k \), \( E_{\lambda}(p) \) is the energy of a heavy particle of type \( \lambda \) and momentum \( p \), \( \omega_k \) is the energy of a θ particle of momentum \( k \), \( g_{0} \) is a coupling constant, \( V \) is the normalization volume (which ultimately is to be taken \( \to \infty \), in which limit summations over momenta are to be evaluated by the rule
\[ \varepsilon(\varepsilon, p) \rightarrow (\gamma/h^{3}) \int d^{3}p \ldots \] (120)

and \( f(\omega_{k}) \) is a "cut-off" function which will be presumed here to have some form such that all integrals over \( \theta \)-particle momenta will be finite. The first two terms in Eq. (119) represent the energies of the heavy particle field and the light particle field respectively without interaction and the third term characterizes the interaction of these two fields. This interaction induces processes consisting of successions of the basic reaction process \( V + N \rightarrow \theta \), which conserves not only the momentum of the system, but also the two quantities

\[ Q_{1} = \sum_{p, \lambda} b_{\lambda}^{\dagger}(p) b_{\lambda}(p) \] (121)

and

\[ Q_{2} = \sum_{p} b_{V}^{\dagger}(p) b_{V}(p) + \sum_{k} a_{k}^{\dagger} a_{k} \] (122)

which represent respectively the number of heavy particles and the sum of the number of \( V \) particles and the number of \( \theta \) particles.

The Lee Model is a simplistic, but non-trivial, model for the interaction of nucleons (heavy particles) with mesons (light particles). Much of the original interest in the model pertained to renormalization methods.\(^7\) The Lee Model can be handled mathematically fairly easily only in the special case for which \( Q_{1} = Q_{2} = 1 \), in which case a complete set of states consists only of those states for which
either there is exactly one V particle present or there are present exactly one N particle and one θ particle.

There is a somewhat simplified, widely used, version of the model (which retains most of the elements for which the original model is of interest) obtained by regarding the heavy particles as being equally massive and so heavy that one may neglect their momenta (which are necessarily present in the momentum-conserving original model due to their recoil upon absorption and emission of light particles). In this version

$$E(p) = \hbar \omega = \text{constant.} \quad (123)$$

It is this simplified version of the Lee Model which will be used in the remainder of this work.

In order to illustrate the power of the path-integral method within the scope of quadratic Hamiltonian systems, it is the purpose of this chapter to use this method to obtain a complete description of the dynamics of the Lee Model for the case $Q_1 = 1$. In this case the number of heavy particles is restricted to one, which has only two independent states:

the V-particle state $\equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (124)$

and

the N-particle state $\equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (125)$

but the number of light particles is unrestricted. The Hamiltonian is
where
\[ H = \mathcal{H} \{ [\omega + \sum_k \omega_k \hat{a}_k^+] \hat{a}_k ] - \sum_k f_k' [\sigma_+ \hat{a}_k + \hat{a}_k^+ \sigma] \}, \]
(126)
and
\[ f_k' = g_o f(\omega_k) / [(\mathcal{H}^2 + \omega_k) \mathcal{V}]. \]
(128)

Since the (time dependent) propagator characterizes explicitly the time evolution of a system validly for all possible initial states of the system, then a satisfactory mode of complete description is obtainable by evaluating exactly the propagator in some convenient known basis of representation. For this purpose, the occupation-number representation is a natural choice. The basis set of the occupation-number representation to be used is the set of all states characterized by specifying whether the heavy particle present is of the V type or the N type and for each possible momentum value specifying the number of light particles present with that momentum. These states are eigenstates of the Hamiltonian with the interaction term omitted.

(4.2) Evaluation of the Propagator

Although the occupation-number representation is a natural choice for a meaningfully descriptive display of the propagator, a more convenient representation for computational purposes is in terms of a complete continuous set of
states, called "coherent" states, which are eigenstates of the Bosonic destruction operator. One solution to the equation

\[ \hat{a}_k |a_k\rangle = a_k |a_k\rangle, \]  

(129)

where \( |a_k\rangle \) is the eigenstate of \( \hat{a}_k \) which corresponds to the eigenvalue \( a_k \), is

\[ |a_k\rangle = e^{-\frac{1}{2} |a_k|^2} \sum_{n_k=0}^{\infty} \left( a_k/\sqrt{n_k} \right) |n_k\rangle \]  

(130)

where \( |n_k\rangle \) is the state containing exactly \( n_k \) particles having momentum \( k \). That the coherent states defined by Eq. (130) satisfy the completeness relationship

\[ \int\int |a\rangle \frac{da_r da_\perp}{\pi} \langle a| = 1, \]  

(131)

where \( a_r \) and \( a_\perp \) are the real and imaginary parts of \( a \) respectively, is easily demonstrated by substitution of Eq. (130) into the left-hand side of Eq. (131), transformation of integration variables to polar coordinates, and use of the orthonormality and completeness of the basis states, \( |n_k\rangle \), of the occupation-number representation.9

As a first step in the construction of the propagator for a finite time interval, \( t' \) to \( t'' \), consider the familiar solution of Eq. (4),

\[ \psi(q,t+\Delta t) = e^{-i(\Delta t)H/\hbar} \psi(q,t), \]  

\( \Delta t \to 0 \)  

(132)

for an infinitesimal time interval, \( t \) to \( t + \Delta t \). The
coherent-state representation of the propagator for the
time interval, \( t \) to \( t + \Delta t \), can be obtained from Eq. (10)
by substitution of Eq. (132) for \( \psi \):

\[
K_{a''}^{a'}(t + \Delta t, t) = \langle a'' | e^{-\frac{i(\Delta t)H}{\hbar}} | a' \rangle. \quad (133)
\]

Because of the completeness of the coherent states, expressed
by Eq. (131), the superposition principle, stated by Eq. (9)
for the coordinate representation, is valid also in coherent-
state language. Therefore the complete propagator for the
time interval, \( t' \) to \( t'' \), is

\[
K_{a''}^{a'}(t'', t') = \lim_{M \to \infty} \left\{ \prod_{k=1}^{M-1} \left[ \langle a(2) | e^{-\frac{i\epsilon H}{\hbar}} | a(1) \rangle \langle a(1) | e^{-\frac{i\epsilon H}{\hbar}} | a' \rangle \right] \prod_{k,l=1}^{M-1} d^{B}_{a_{k}}(\epsilon) \right\}
\]

where

\[
d^{B}_{a_{k}}(\epsilon) = d(a_{k}(\epsilon))_{r}d(a_{k}(\epsilon))_{l}/\pi. \quad (135)
\]

As a notational convenience for future use, let

\[
a(M) \equiv a'', \quad a(0) \equiv a'. \quad (136)
\]

Expansion of the exponential functions in Eq. (134) to first
order only in \( \epsilon \) yields

\[
K_{a''}^{a'}(t'', t') = \lim_{M \to \infty} \left\{ \prod_{k=1}^{M} \left[ \langle a(2) | a(1) \rangle \langle a(1) | a' \rangle \right] \prod_{k,l=1}^{M-1} d^{B}_{a_{k}}(\epsilon) \right\}
\]

where the arrow under the product symbol indicates the
direction in which the index increases in the time ordered
product and where

$$\Omega(\ell) \equiv \langle a(\ell) | H | a(\ell-1) \rangle / [\Phi(a(\ell)) | a(\ell-1) \rangle]$$

$$= \left[ \omega + \Sigma_{k} \omega_{k} a_{k}^{\dagger}(\ell) a_{k}(\ell-1) \right] \Gamma_{-} \Phi \left[ a_{k}^{\dagger}(\ell) \sigma_{+} + a_{k}(\ell-1) \sigma_{+} \right]$$

and

$$\langle a(\ell) | a(\ell-1) \rangle = \exp\left\{ -\frac{1}{2} \Sigma_{k} |a_{k}(\ell)|^{2} \right\}$$

$$+ |a_{k}(\ell-1)|^{2} - 2a_{k}^{\dagger}(\ell)a_{k}(\ell-1)\Gamma_{-} \Phi \left[ a_{k}^{\dagger}(\ell) \sigma_{+} + a_{k}(\ell-1) \sigma_{+} \right]$$

have been evaluated by using Eqs. (126) and (130). To first order in $\epsilon$, the quantity $(1-\epsilon \Omega(\ell))$ in Eq. (137), with $\Omega(\ell)$ given by Eq. (139), can be factored to yield

$$(1-\epsilon \Omega(\ell)) = [1-\epsilon \Sigma_{k} \omega_{k} a_{k}^{\dagger}(\ell) a_{k}(\ell-1) \right]$$

$$\times \left[ 1 + \epsilon \Xi \Phi a_{k}^{\dagger}(\ell) \sigma_{+} + a_{k}(\ell-1) \sigma_{+} \right]$$

$$= \exp\left\{ -\epsilon \Sigma_{k} \omega_{k} a_{k}(\ell) a_{k}(\ell-1) \right\}$$

$$\times \left[ 1 + (\epsilon \Xi \Phi a_{k}^{\dagger}(\ell)) \sigma_{+} + (\epsilon \Xi \Phi a_{k}(\ell-1)) \sigma_{+} \right].$$

After substitution of Eqs. (140) and (142) into Eq. (137), the expression for the propagator becomes

$$K_{a^{\nu},a^{\prime}(t',t')} = \lim_{M \to \infty} \left[ \epsilon^{-1} \omega_{k}^{T} \right] \left[ 1 + (\epsilon \Xi \Phi a_{k}(\ell)) \sigma_{+} \right]_{l=1}^{M}$$

$$+ (\epsilon \Xi \Phi a_{k}(\ell-1)) \sigma_{+} \exp\left\{ -\frac{1}{2} \Sigma_{k} \omega_{k} |a_{k}(\ell)|^{2} \right\}$$

$$+ |a_{k}(\ell-1)|^{2} - 2(1-\epsilon \omega_{k}) a_{k}^{\dagger}(\ell) a_{k}(\ell-1) \right\} \prod_{k} \left[ a_{k}(\ell) \right].$$

(143)
Expansion of the time-ordered product in powers of \( \varepsilon \) proceeds as follows:

\[
M \prod_{k=1}^{M} \left[ 1 + \frac{(1 \varepsilon \Sigma f_i a_i^*(\ell)) \sigma_+ + (i \varepsilon \Sigma f_i a_i(\ell-1)) \sigma_+}{k k k k} \right] = \sum \sum' \prod_{j=0}^{\ell_1 > \ell_j \nu = 1} \frac{(i \varepsilon \Sigma f_j a_j(\ell_j \nu)) (i \varepsilon \Sigma f_j a_j(\ell_j \nu - 1)) \sigma_+}{k k k k} (144)
\]

\[
\sum M^\frac{1}{2} M^\frac{1}{2} M^\mu \mu = \sum \sum' \prod_{\mu = 0}^{\ell_1 > \ell_2 \mu = 1} \frac{(i \varepsilon \Sigma f_j a_j(\ell_2 \nu)) (i \varepsilon \Sigma f_j a_j(\ell_2 \nu - 1))}{k k k k} \sigma_+ \sigma_+ (145)
\]

where

\[
\sigma_V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma_N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]

and

\[
\begin{align*}
\sum'_{\ell_1 \ell_1} &= \begin{cases} 
\beta \; \ell_{1-1} \; \ell_{j-1} \; \ell_j \\
\ell_{1} = 1 \; \ell_{1+1} = 1 \; \ell_j = 1
\end{cases} & \text{if } j \geq 1 \\
\sum'_{\ell_1 \ell_j} &= \begin{cases} 
\beta \; \ell_{1} \; \ell_{j+1} = \ell_{j+1} \; \ell_j = \ell_j + 1 \\
\beta \; \ell_{1} = \ell_{1+1} \; \ell_j = \ell_j + 1
\end{cases} & \text{if } i \geq j \geq 1.
\end{align*}
\]

The square brackets used in four summation limits in Eq. (145) indicate that the greatest integer less than or equal to the enclosed quantity is to be taken as the summation
limit. Substitution of Eq. (145) into Eq. (143) expresses the propagator \( K_{a'' a'}(t'', t') \) in the form of a two-by-two matrix whose elements require multifold integration for evaluation.

An alternative to evaluation of the propagator, \( K_{a'' a'}(t'', t') \), by direct integration of each matrix element is to generate all four matrix elements from the function,

\[
K_{a'' a'}^M(t'', t') = \left\{ \text{exp}\left( \sum_{\ell=1}^{M} \frac{1}{2} \left( |a_k(\ell)|^2 + |a_k(\ell-1)|^2 \right) \right) \right\} (l-1) \epsilon_k a_k(\ell-1) - i \epsilon f_k' [\lambda(\ell) a_k(\ell) + \lambda'(\ell) a_k(\ell-1)] \prod_{\ell=1}^{M-1} d a_k(\ell),
\]

in accordance with the prescription

\[
K_{a'' a'}(t'', t') = e^{-i \omega T} \lim_{M \to \infty} \lim_{\lambda(\ell), \lambda'(\ell) \to 0} \left[ \frac{M}{2} \right]_{\ell=1, \ldots, M}^{M} \left\{ \prod_{\mu=0}^{\frac{M}{2}} \left( \prod_{\nu=1}^{\frac{M}{2}} \frac{\partial}{\partial \lambda(\ell_2 \nu)} \frac{\partial}{\partial \lambda'(\ell_2 \nu-1)} \right) \right\} \left[ \prod_{\mu=0}^{\frac{M-1}{2}} \left( \prod_{\nu=1}^{\frac{M-1}{2}} \frac{\partial}{\partial \lambda(\ell_2 \nu)} \frac{\partial}{\partial \lambda'(\ell_2 \nu-1)} \right) \right] \left[ \prod_{\mu=0}^{\frac{M-1}{2}} \left( \prod_{\nu=1}^{\frac{M-1}{2}} \frac{\partial}{\partial \lambda(\ell_2 \nu+1)} \frac{\partial}{\partial \lambda'(\ell_2 \nu-1)} \right) \right] \left[ \prod_{\mu=0}^{\frac{M-1}{2}} \left( \prod_{\nu=1}^{\frac{M-1}{2}} \frac{\partial}{\partial \lambda(\ell_2 \nu+1)} \frac{\partial}{\partial \lambda'(\ell_2 \nu-1)} \right) \right] \prod_{\ell=1}^{M} K_{a'' a'}(t'', t').
\]
In the limit $M \to \infty$, $K'_M^{a''_k, a'_k}(t'', t')$ conforms to the path-integral expression for the propagator of a system which has a quadratic Hamiltonian of the form

$$H'_k = \hbar \sum_k \left( \omega_k^{a'_k a'_k} - f'_k (g(t) a'_k + g'(t) a'_k) \right),$$

where

$$g(t) = \lambda(\epsilon), \quad g'(t) = \lambda'(\epsilon) \quad \text{for} \quad t'' + (l-1)\epsilon < t < t' + \epsilon \epsilon. \quad (151)$$

One possible method of evaluating the propagator $K'$ would be to transform Eq. (109) into coherent state notation according to the prescription given by Eq. (11) and apply the result to $K'$. It is more convenient in this case however to integrate directly in Eq. (148) to obtain

$$K'_M^{a''_k, a'_k}(t'', t') = \exp \left( -\frac{1}{2} (|a''_k|^2 + |a'_k|^2) + a''_k a'_k (1-\epsilon \omega_k)^M \right)
- \epsilon f'_k a''_k \sum_{k=1}^{M} \left( (1-\epsilon \omega_k)^{M-k} \lambda(\epsilon) - \epsilon f'_k a'_k \sum_{k=1}^{M} \left( (1-\epsilon \omega_k)^{k-1} \lambda'(\epsilon) \right) \right)
- \epsilon^2 f'_k a''_k a'_k \sum_{k=1}^{M} \left( (1-\epsilon \omega_k)^{k-1} \lambda(\epsilon) \lambda'(\epsilon') \right). \quad (152)$$

Substitution of Eq. (152) into Eq. (149) yields

$$K^{a''_k, a'_k}(t'', t') = \exp \left[ -\omega T \left( |a''_k|^2 + |a'_k|^2 - 2a''_k a'_k e^{-\epsilon \omega k T} \right) \right]
\times \lim_{M \to \infty} \lim_{M \to \infty} \{ \sigma_{2\nu} \sum_{\lambda(\epsilon) \to 0} \sum_{\lambda'(\epsilon)} \prod_{\mu=0}^{2\nu-1} \left[ A_{2\nu+1}^{\ell_1 \lambda(\epsilon) \epsilon} B_{2\nu+1}^{\ell_1 \lambda(\epsilon) \epsilon} \right] \}.$$
\[
\begin{align*}
\mu = \frac{1}{2} & \quad \ell = 2v, -1 \\
\Pi & [B_{2v}, + \sum_{\ell=1}^{M} C(\ell, 2v, \ell) \lambda(\ell)] + \sigma \sum_{m=0}^{\mu} (A_{2v+1} + \frac{3}{\lambda(\ell, 2v+1)}) \Pi [B_{2v}, \\
\ell = 1 & \quad \ell = 2v, -1 \\
+ \sum_{\ell=1}^{M} C(\ell, 2v, \ell) \lambda(\ell)]},
\end{align*}
\]

\[
= \exp[-i\omega T - \frac{1}{2} \Sigma (|a_k|^2 + |a_k|^2 - 2a^*_k a_k e^{-i\omega k^T})] .
\]

\[
\begin{align*}
\lim_{M \to \infty} & \quad \mu = 0 \quad m = 0 \quad n > n_1 \quad n_m \quad \ell > \ell_2 \mu + 1 \quad j = 1 \\
A_{2v} & \quad \Pi B_{2v} \\
\end{align*}
\]

\[
\begin{align*}
\lim_{M \to \infty} & \quad \mu = 0 \quad m = 0 \quad n > n_1 \quad n_m \quad \ell > \ell_2 \mu + 1 \\
A_{2v} & \quad \Pi B_{2v} \\
\end{align*}
\]

\[
\begin{align*}
\lim_{M \to \infty} & \quad \mu = 0 \quad m = 0 \quad n > n_1 \quad n_m \quad \ell > \ell_2 \mu + 1 \\
A_{2v} & \quad \Pi B_{2v} \\
\end{align*}
\]

\[
\begin{align*}
\lim_{M \to \infty} & \quad \mu = 0 \quad m = 0 \quad n > n_1 \quad n_m \quad \ell > \ell_2 \mu + 1 \\
A_{2v} & \quad \Pi B_{2v} \\
\end{align*}
\]

where
In Eq. (154), all of the differentiations with respect to \( \lambda \) and \( \lambda' \) indicated in Eq. (149) have been performed and the required limits \( \lambda \rightarrow 0 \) and \( \lambda' \rightarrow 0 \) have been taken. After the limit \( M \rightarrow \infty \) is taken and the required time integrals are evaluated, the coherent-state expression for the propagator is in its final form:
\[
K_{a''}(a'(t'', t')) = \exp[-i\omega T - \frac{1}{2}\sum_k \left( |a''_k|^2 + |a'_k|^2 - 2a''_ka'_ke^{-i\omega_k T} \right)]
\]

\[
x \left\{ \sum_{v=0}^{\infty} \sum_{m=0}^{n_1} \left( \prod_{j=1}^{\mu} f_j \right) \sum_{m>n_1} \sum_{m} \left( \prod_{j=1}^{\mu} \delta_{k_2n_j-k_2n_j'} \right) \right\}
\]

\[
x \left\{ \sum_{v=0}^{\infty} \sum_{m=0}^{n_1} \left( \prod_{j=1}^{\mu} f_j \right) \sum_{m>n_1} \sum_{m} \left( \prod_{j=1}^{\mu} \delta_{k_2n_j-k_2n_j'} \right) \right\} \]

where

\[
F_r(\omega_1, \omega_2, \ldots, \omega_r, T) \equiv (1)^r \int_{t}^{t''} e^{-i\omega_1(t_1-t')} dt_1 \int_{t'}^{t} e^{-i\omega_2(t_2-t')} dt_2 \ldots \int_{t_{r-1}}^{t_{r-1}} e^{-i\omega_r(t_r-t')} dt_r,
\]
and where \( i = 1, \ldots, m \) wherever \( n_i \) and \( n'_i \) occur. The function \( F_r \) has been evaluated in Appendix B as

\[
F_r(\omega_1, \ldots, \omega_r, T) = \sum_{n=r}^{\infty} \left( \frac{iT}{n} \right)^n \sum_{r} \prod_{\gamma_1=0}^{\gamma_r} \frac{(-1)^{\alpha}}{\gamma_1^{\alpha}} \frac{\gamma_1!}{\prod_{k=\alpha}^{\gamma_r} (\gamma_k+1)!} \frac{1}{\gamma_1^{n-r}}
\]

and possesses the useful property

\[
F_r(-\omega_1, -\omega_2, \ldots, -\omega_r, T) = (-1)^r F_r(\omega_1, \omega_2, \ldots, \omega_r, -T)
\]

which has been used in obtaining Eq. (163).

The occupation-number representation of the propagator for the simplified Lee Model, the primary objective of this chapter, can be obtained from the coherent-state expression, given by Eq. (163), by the procedure indicated in Eq. (11), which in the present context reads

\[
K_{N^n, N^i}(t^n, t^i) = \int \langle N^n | a^n \rangle K_{a^n, a'}(t^n, t^i) \langle a' | N^i \rangle d^B a^n d^B a',
\]

where each of the indices \( N^i \) and \( N^n \) denotes a set of (integer) values, \( \{N_k\} \), specifying the number of \( \theta \) particles with each possible momentum value, \( k \). This requires the evaluation of integrals of the form
\[
\int \text{d}e \frac{e^{i \omega \cdot T} - |a_k|^2 - |a_k'|^2 + a_k'^* a_k e^{-i \omega \cdot T}}{\sqrt{N_k''}} \frac{a_k'' a_k''^* a_k' e^{i \omega \cdot T}}{\sqrt{N_k'}} 
\]

\[
= \frac{-i(N_k'' - \alpha) \omega \cdot T}{(N_k'' - \alpha)!} e^{\delta_{N_k''-\alpha,N_k'-\beta}} 
\]

which is exhibited in Appendix C. Substitution of Eqs. (163) and (130) into Eq. (167), with application of Eq. (168) yields the propagator in its occupation-number representation:

\[
K_{N'',N'}(t'',t') = e^{-i \omega \cdot T - 1 \text{SE} N_k'' \omega \cdot T} 
\]

\[
\times \left\{ \sigma \left[ \sum_{\mu=0}^{\infty} \sum_{k_1}^{\infty} \sum_{k_2}^{\infty} \left( \sum_{j=1}^{2 \mu} \prod_{k_j} f_{j}^r \right)^{\mu} \sum_{m}^{\mu} \left( \sum_{n_m}^{\mu} \sum_{n_1}^{m} \sum_{n_1}^{m} \sum_{j=1}^{1} k_{2n_j-1, k_{2n_j}} \Delta' \right) \right] \right. 
\]

\[
\times \left\{ \prod_{v=1}^{2 \mu} \prod_{k_{2v}'} \prod_{k_{2v}''} \prod_{k_{2v}'''} \delta_{N_k''-1,N_k'-1} \delta_{N_k''-1,N_k'-1} \right\} 
\]

\[
\times \left\{ \prod_{j=1}^{m} \delta_{k_{2n_j}, k_{2n_j+1}} \Delta' \right\} \left\{ 1 \sum_{m=0}^{\mu-1} \sum_{n_m}^{\mu-1} \sum_{n_1}^{\mu-1} \sum_{j=1}^{1} k_{2n_j-1, k_{2n_j}} \Delta' \right\} 
\]

\[
\times \left\{ F_{2 \mu}(\omega_{k_1}, \ldots, \omega_{k_{2\mu}}, -T) \right\} 
\]

\[
\left\{ \sigma \left[ \sum_{\mu=0}^{\infty} \sum_{k_1}^{\infty} \sum_{k_2}^{\infty} \left( \sum_{j=1}^{2 \mu} \prod_{k_j} f_{j}^r \right)^{\mu} \sum_{m}^{\mu} \left( \sum_{n_m}^{\mu} \sum_{n_1}^{m} \sum_{n_1}^{m} \sum_{j=1}^{1} k_{2n_j-1, k_{2n_j}} \Delta' \right) \right] \right. 
\]

\[
\times \left\{ \prod_{v=1}^{2 \mu} \prod_{k_{2v}'} \prod_{k_{2v}''} \prod_{k_{2v}'''} \delta_{N_k''-1,N_k'-1} \delta_{N_k''-1,N_k'-1} \right\} 
\]

\[
\times \left\{ \prod_{j=1}^{m} \delta_{k_{2n_j}, k_{2n_j+1}} \Delta' \right\} \left\{ 1 \sum_{m=0}^{\mu-1} \sum_{n_m}^{\mu-1} \sum_{n_1}^{\mu-1} \sum_{j=1}^{1} k_{2n_j-1, k_{2n_j}} \Delta' \right\} 
\]

\[
\left\{ F_{2 \mu}(\omega_{k_1}, \ldots, \omega_{k_{2\mu}}, -T) \right\} 
\]
\[
x \left( \prod_{v'=1}^{\mu+1} \sqrt{N_{k_2v'}^{n'_{2v'}} - 1} \right) \left( \prod_{v'=1}^{\mu+1} \sqrt{N_{k_2v'}^{n'_{2v'}} - 1} \right) = \sum_{m=0}^{\infty} \left( \sum_{l=1}^{k_{2v} + 1} \delta_{k_{2v}^{n_m}, k_{2v}^{l_1}} \right) \left( \sum_{l=1}^{k_{2v} + 1} \delta_{k_{2v}^{n_m}, k_{2v}^{l_1}} \right)
\]

where

\[
\Delta = \delta_{k_{2v}^{n_m}, k_{2v}^{l_1}}, \quad \Delta' = \delta_{k_{2v}^{n_m}, k_{2v}^{l_1}}, \quad \Delta'' = \delta_{k_{2v}^{n_m}, k_{2v}^{l_1}}.
\]

and

\[
D = \prod_{k \neq k_1}^{k_{2v} + 1} \delta_{k_{2v}^{n_m}, k_{2v}^{l_1}}, \quad D' = \prod_{k \neq k_1}^{k_{2v} + 1} \delta_{k_{2v}^{n_m}, k_{2v}^{l_1}}.
\]

4.3 Application to the N + 0 Sector

In the literature concerning the Lee Model, the case for which \(Q_1 = Q_2 = 1\) (see Eqs. (121) and (122)), commonly called the "N + 0 sector", has received considerable attention. As stated earlier, this is the only case which can be treated with any mathematical ease. The propagator for this case, obtained from Eq. (169) with substitution of Eq. (165) and use of Eq. (B9) in Appendix B, is
\[ K_{N^m, N^l}(t^n, t^l) = e^{-i\omega T - i} \sum_{k} N^n_k \omega^T \]

\[ x \{ \sigma_N[1+ \sum_{n=2}^{\infty} \frac{(-iT)^n}{n!} \left( \sum_{\mu=1}^{n-2\mu} \gamma_1 = 0, \gamma_\mu = 0, \alpha = 1 \right) \left( \sum_{k} f^\prime k^2 \omega^k \right) \prod_{k} \delta_{N^m_k, N^l_k} \}^{\mu} \sum_{i=1}^{n-2\mu} \gamma_1 = n-2\mu \]

\[ + \sigma_N[1+ \sum_{n=2}^{\infty} \frac{(-iT)^n}{n!} \left( \sum_{\mu=1}^{n-2\mu} \gamma_1 = 0, \gamma_\mu + 1 = 0 \right) \left( -1 \right) \sum_{k} f^\prime k^2 \omega^k \prod_{k} \delta_{N^m_k, N^l_k} \}^{\mu+1} \sum_{i=1}^{n-2\mu} \gamma_1 = n-2\mu \]

\[ x \left( f^\prime k^2 \omega^k \right) \prod_{\alpha=2}^{\mu} \left( \sum_{\gamma_1 = 0, \gamma_\mu = 0, \alpha = 2} \gamma_1 = 0, \gamma_\mu = 0 \right) \left( -1 \right) \sum_{k} f^\prime k^2 \omega^k \prod_{k} \delta_{N^m_k, N^l_k} \}

where the \( N_k \)'s satisfy the conditions
\[ \Sigma N'_k \leq 1, \quad \Sigma N''_k \leq 1 \quad (173) \]

in accordance with the condition \( Q_2 = 1 \).

Of particular interest is the matrix element, \( K_{V+V}(T) \), representing the probability amplitude for finding exactly one \( V \) particle (i.e., a "bare" \( V \) particle with no \( \theta \) particles present) at a given time \( T \) after the system was initialized to possess exactly one \( V \) particle. The absolute square of \( K_{V+V}(T) \) is called the (time dependent) survival probability of a (bare) \( V \) particle. The expression for \( K_{V+V}(T) \) given by the coefficient of \( \sigma_V \) in Eq. (172) is

\[
K_{V+V}(T) = e^{-\omega T} \{ 1 + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \sum_{\mu=1}^{H} \sum_{\sigma_1=0}^{n-2\mu} \sum_{\sigma_\mu=0}^{n-2\mu} \prod_{v=1}^{n} \{ \Sigma k \omega_k \} \}.
\]

(174)

In order to obtain an independent check on the validity of Eq. (174), the function \( K_{V+V}(T) \) may also be expressed for comparison as

\[
K_{V+V}(T) = \langle V | e^{\frac{-iTH}{\hbar}} | V \rangle \quad (175)
\]

\[
= e^{-\omega T} \langle V | e^{\frac{-iT}{\hbar}(H - \hbar \omega)} | V \rangle \quad (176)
\]

\[
= e^{-\omega T} \sum_{n=0}^{\infty} \frac{(-iT)^n}{n!} \langle V | H_1^n | V \rangle, \quad (177)
\]

where

\[
\hat{H}_1 = (H - \hbar \omega)/\hbar, \quad (178)
\]
where $H$ is given by Eq. (126), and where

$$|V\rangle, \{|N; \theta_k\rangle\}$$

(179)

form a complete orthonormal set of states for the $N + \theta$ sector. Here $|V\rangle$ denotes the state with only one $V$ particle and $|N; \theta_k\rangle$ denotes a state with one $N$ particle and with one $\theta$ particle of momentum $k$. The results of the operator, $\hat{H}_1$, acting on the states $|V\rangle$ and $|N; \theta_k\rangle$ are

$$\hat{H}_1 |V\rangle = - \sum_k f'_k |N; \theta_k\rangle$$

(180)

and

$$\hat{H}_1 |N; \theta_k\rangle = -f'_k |V\rangle + \omega_k |N; \theta_k\rangle.$$  

(181)

When $\hat{H}_1$ operates on the $V$-state $n$ times the result is

$$\hat{H}_1^n |V\rangle = \sum_{\ell=0}^{n-2} \langle V|\hat{H}_1^{n-2-\ell}|V\rangle \left( \sum_k f'_k \omega_k^\ell \right) |V\rangle$$

$$- \sum_{\ell=0}^{n-1} \langle V|\hat{H}_1^{n-1-\ell}|V\rangle \sum_k f'_k \omega_k^\ell |N; \theta_k\rangle,$$  

(182)

which implies that

$$\langle V|\hat{H}_1^n |V\rangle = \sum_{\ell=0}^{n-2} \langle V|\hat{H}_1^{n-2-\ell}|V\rangle \left( \sum_k f'_k \omega_k^\ell \right).$$  

(183)

The proof of Eq. (182) is an inductive one in which $\hat{H}_1$ is applied to Eq. (182), with substitution of Eq. (183), to yield

$$H_1^{n+1} |V\rangle = -\langle V|\hat{H}_1^n |V\rangle \sum_k f'_k |N; \theta_k\rangle + \sum_{\ell=0}^{n-1} \langle V|\hat{H}_1^{n-1-\ell}|V\rangle \left( \sum_k f'_k \omega_k^\ell \right) |V\rangle$$

$$- \sum_{\ell=0}^{n-1} \langle V|\hat{H}_1^{n-1-\ell}|V\rangle \sum_k f'_k \omega_k^{\ell+1} |N; \theta_k\rangle.$$  

(184)
\[
\text{(185)}
\]

Since Eq. (185) is of the same form as Eq. (182), which correctly predicts
\[
\hat{H}_1^2 |V\rangle = \left[ \sum_k f_k^r \sigma_k \right] |V\rangle - \sum_k f_k^r \omega_k |N, \theta_k\rangle,
\]
the proof of Eq. (182), and therefore Eq. (183), is complete.

Equation (183) can be used to prove that
\[
\text{(186)}
\]
is true for \( n' \geq 2 \), which will verify that Eqs. (174) and (177) are equivalent. The proof is again inductive with substitution of Eq. (187) into Eq. (185) for \( n' \leq n - 2 \)
\[
\text{(187)}
\]
\[
\text{(188)}
\]
\[
\text{(189)}
\]
where $\mu' = \mu + 1$ and the summations over $\xi$ and $\mu'$ have been interchanged. Because of the condition

$$\xi + \sum_{i=2}^{\mu'} \sigma_i = n - 2\mu', \quad (190)$$

Eq. (189) may be rewritten as

$$\langle V|H_1^n|V\rangle = \sum_{\mu'=1}^{[n/2]} \sum_{\sigma_1=0}^{n-2\mu'} \sigma_{\mu'} \prod_{\nu=1}^{\mu'} \left( \sum_{k=1}^{f_k} \omega_k \right), \quad (191)$$

where $\xi$ has been relabeled $\sigma_1$. Equation (191) confirms that Eq. (187) holds for $n' = n$ if it holds for $n' \leq n - 2$. In view of Eqs. (180), (181), and (186), it is easy to see that Eq. (187) holds for $n' = 2, 3$, and therefore for all $n' \geq 2$.

The equivalence of Eqs. (174) and (177), thus established, demonstrates that the propagator for the Lee Model derived generally in Section (4.2) reduces to the correct probability amplitude for the survival of the (bare) V-particle in the $N + \theta$ sector.
(5.1) Summary of Results

The space-time propagator $K(q'',t'';q',t')$ for a given system, defined by Eq. (3), represents the probability density amplitude for transition of the system from a given initial configuration, $q'$, at an initial time, $t'$, to a given final configuration, $q''$, at a final time, $t''$. Equation (27) is an exact phase-space path-integral expression for $K(q'',t'';q',t')$ for systems, called "quadratic systems", whose Hamiltonians have the form specified by Eq. (1). This path integral has been performed exactly to yield the result that $K(q'',t'';q',t') = \mathcal{F}(t''-t') \times \exp(iS_c(q'',t'';q',t')/\hbar)$, where $S_c(q'',t'';q',t')$ is the classical action function connecting the initial and final space-time points $(q',t')$ and $(q'',t'')$, and where $\mathcal{F}(t''-t')$ is independent of $q'$ and $q''$ and is given explicitly and in detail by Eqs. (81)-(88) as a function of $(t''-t')$ and the coefficients occurring in only the quadratic part of the Hamiltonian of the system. Under the frequently encountered special conditions stated in the sentence containing Eq. (89), the infinite sums and products contained in the expression for $\mathcal{F}$ may be evaluated and the result for $\mathcal{F}$ simplified to the closed form given by Eqs. (102) and (103).
An example application - one that has attracted wide interest during the last two decades - is the Lee Model discussed in Chapter 4. The Lee Model describes interactions between a Bosonic field of (light) particles, called Θ particles, and a Fermionic field of (heavy) particles with two isotopic states, the V state and the N state. The Lee Model is a simplistic simulation of the interaction of nucleons with mesons. The complete propagator in the occupation number representation for the one-heavy-particle Lee Model has been obtained exactly (Eq. (169)) and has been shown, by means of an independent validity check, to correctly predict the time-dependent survival probability (Eq. (174)) of the unstable V state of the heavy particle with no Bosons present.

(5.2) Discussion of Results

The primary significance of the present work is that it provides a unified, exact, complete treatment of the quantum dynamics of a fairly broad class of physical systems with arbitrarily many degrees of freedom. As discussed in Section (1.1), the results obtained for such systems may be useful in the quantum analysis of any system whose Hamiltonian is expressible (exactly or approximately) in such a form that Eq. (1) describes the dependence of the Hamiltonian upon some or all of the system's coordinates and their conjugate momenta. The analysis of the Lee Model in
Chapter 4 provides an example, non-simple application. The result obtained for the Lee Model is significant apart from its illustrative value because it provides, for the first time, a complete exact solution for the time evolution of the one-heavy-particle Lee-Model system valid in all sectors. The general expression obtained for the Lee Model, unfortunately, is complicated (although completely explicit). This expression readily simplifies greatly for sectors spanned by states containing either the N particle and n 0 particles or the V particle and (n-1) 0 particles, provided n is small—the smaller n, the greater the simplification. This simplification is illustrated in the case of the simplest, non-trivial sector for which n = 1.

To further clarify the significance of the present work, some comparisons of it with other closely related publications may be helpful and will be given in the remainder of this section. Firstly the treatment of quadratic Hamiltonian systems will be compared with two other path-integral investigations in each of which the propagators are sought for systems which are quadratic in a more general sense than defined by Eq. (1).

One of these investigations in Feynman's derivation of the form for the propagator for systems with one coordinate, y, and with a quadratic action of the form
\[ S(y(t)) = \frac{1}{2} \int_{t'}^{t''} y(t) \int_{t'}^{t''} A(t,s) y(s) \, ds \, dt + \int_{t'}^{t''} B(t) y(t) \, dt, \]  

(192)

where \(A(t,s)\) and \(B(t)\) are independent of path, \(y(t)\). His result for the propagator is the same as the form of Eq. (80) where the factor \(F\) is independent of \(q', q'', \) and \(B\) and is to be determined apart from a factor independent of \(A\) by the functional differential equation

\[ \frac{\delta F}{\delta A(t,s)} = -\frac{1}{2} N(t,s)F, \]  

(193)

where \(N(t,s)\) is the reciprocal kernel to \(A(t,s)\) subject to appropriate boundary conditions. For some cases this equation may be solved easily.\(^{11}\) Equations (192) and (193) may be generalized readily to the case of many degrees of freedom, but the functional differential equation remains to be solved for \(F\) in the general case. The present work may be viewed as providing the required solution for \(F\) in a certain special case (of fairly broad interest) for which the kernel \(A\) is a superposition of \(\delta\)-functions and derivatives of \(\delta\)-functions.\(^{12}\)

The other path-integral investigation of more general quadratic systems is the treatment by DeWitt\(^{13}\) of a system whose Lagrangian has the form

\[ \mathcal{L}(q,q,t) = \frac{1}{2} G_{jk}(q,t)q_j q_k + a_j(q,t)q_j - v(q,t). \]  

(194)

This Lagrangian may be interpreted as describing the motion of a particle moving in a curved multi-dimensional space.
DeWitt uses path-integral methods to derive an expression for the infinitesimal propagator which remains to be iterated to obtain the propagator connecting finitely separated space-time points. Equations (80)-(88) give the result of such an iteration for the special case for which the space is flat (i.e. \( G_{jk} \) is constant), and \( a_j \) and \( v \) are of the forms

\[
a_j(q,t) = Q_{jk} q^k + d_j(t) \tag{195}
\]

and

\[
v(q,t) = g_{jk} q^j q^k + \rho_j(t) q^j + f(t) \tag{196}
\]

respectively. The most interesting aspects and applications\(^{14}\) of DeWitt's paper, however, lie outside of the range of this special case.

The second aspect of the present paper which must be viewed in perspective relative to other published work is the treatment of the Lee Model. Of the many papers spawned by the Lee Model in its twenty year history, a substantial portion have dealt with renormalization considerations regarding the model. These treatments have no direct bearing on the present paper in which renormalization has been ignored under the assumption that the cut-off function \( f(\omega_k) \) can be chosen suitably to prevent the divergence of any integrals over \( k\)-space. A number of publications have studied the spectral properties of the Lee Model,\(^ {15}\) sometimes focusing attention on eigenvalues in specific, minimally-complicated sectors.\(^ {16} \) The relationship between
the propagator and the energy eigenspectrum is stated in Eq. (12) followed by a prescription (Eq. (18)) for obtaining the ground-state energy. Other papers have been primarily interested in scattering processes in the Lee Model, concentrating on the S-matrix (the limit of the propagator as

\[ T = (t'' - t') + \infty \]  

for specified sectors.\(^{17,18}\)

The only paper on the Lee Model which has a close relation to the present work is one by Fried.\(^{18}\) He obtains an explicit, though formal, functional expression for the propagator of the Lee Model, starting from a more general relativistic theory. The propagator obtained in Section (4.2) of the present paper is more detailed, however, leaving only sums and products to be evaluated.

(5.3) Suggestions for Future Research

As long as the minds of men remain active, any newly acquired knowledge will become a useful tool in the quest for additional information, even to the point of answering questions which had not been conceived previously. In fact the present research directed toward the evaluation of the propagator for quadratic systems was motivated, at least in part, by the need for such a propagator as a tool in an attempt to calculate the ground-state energy of liquid helium. The Hamiltonian characterizing the liquid helium system, expressed in terms of Bosonic field operators, is quartic but can be approximated quadratically. As noted in Section (1.1), the propagator obtained from a quadratic
approximate Hamiltonian would be useful to approximate the ground-state energy by a Feynman variational calculation in which the coefficients of the quadratic approximate Hamiltonian would be variational parameters to be determined by the variational method.

Two extensions of the Lee Model which have been discussed in recent publications appear to be approachable using the treatment of the Lee Model in Chapter 4 as a springboard. The Bronzan-Lee Model\(^1\) adds to the Lee Model a third Fermionic field of (heavy) particles, called "U-particles", and the additional reaction, \(U \leftrightarrow V + \theta\). Consideration of the U-particle as a third isotopic state of the heavy particle considered in the present treatment of the Lee Model suggests a reasonable extension of the present work to obtain a similar treatment of the Bronzan-Lee Model. The Nonlinear Lee Model\(^2\) uses a Hamiltonian of the form
\[
H = H_0 + f(H_I),
\]
where \(H_0\) is the free particle Hamiltonian, \(H_I\) is the usual Lee Model interaction term, and \(f(x)\) is a largely arbitrary real function. For some nonlinear choices of \(f(x)\), the propagator for the model could be at least approximated using the results of Section (4.2) as a starting point.

An interesting possible extension of the present evaluation of the propagator for quadratic Hamiltonian systems would be the evaluation of the propagator for a system of indistinguishable Fermions whose Hamiltonian is
quadratic in the creation and destruction operators for Fermions in specified single particle states. Such systems of Fermions are very similar to the quadratic systems considered in this work in the respect that Eq. (1) gives the Hamiltonian for a system of indistinguishable Bosons if the coordinates and momenta in Eq. (1) are regarded as field-oscillator coordinates and momenta expressed in terms of the corresponding Boson creation and destruction operators. The Hamiltonians for both the Boson system and the Fermion system are the same when expressed in terms of creation and destruction operators. The distinction between the Boson system and the Fermion system is only that the commutation rules for Boson creation and destruction operators are replaced by anti-commutation rules for the Fermion case. The path-integral expression to be evaluated for the propagator in the coherent-state representation in the Fermion case is exactly the same as for the boson case except that the class of phase-space paths to be integrated over are unrestricted in the Boson case, but, in the Fermion case, are restricted to the interior of the unit circle of the complex phase-space plane for every single-particle state.9

As a final suggestion for further research, it seems that a very challenging project which would contribute significantly to progress in theoretical physics would be
the iteration of de Witt's infinitesimal propagator mentioned in the previous section. This might involve a series of projects broadening the class of quadratic systems for which a finite space-time propagator can be obtained.
APPENDIX A
VALIDITY OF THE PATH-INTEGRAL AS AN EXPRESSION
FOR THE PROPAGATOR

The path-integral expression for \(K(q'', t'', q', t')\) stated in Eq. (27) is a valid representation of a propagator for a quadratic system if and only if \(K(q'', t'', q', t')\), thus defined, satisfies the conditions imposed by Eqs. (5) and (7).

In order to show that Eq. (5) is satisfied, consider the definition
\[
\frac{\partial K(q'', t'', q', t')}{\partial t''} \equiv \lim_{\epsilon \to 0} \{[K(q'', t'' + \epsilon; q', t') - K(q'', t''; q', t')]/\epsilon\}.
\]

According to Eq. (8),
\[
K(q'', t'' + \epsilon; q', t') = \int \cdots \int K(q'', t'' + \epsilon, q, t'') K(q, t''; q', t') \prod_{j=1}^{N} dq_{j}.
\]

Since the limit \(\epsilon \to 0\) is to be taken, the function \(K(q'', t'' + \epsilon; q, t')\) may be evaluated from Eq. (23) as
\[
K(q'', t'' + \epsilon; q, t') = \int \cdots \int \exp\left[\frac{i}{\hbar} \int_{t''}^{t'' + \epsilon} \left[ \sum_{j=1}^{N} p_{j}(t)q_{j}(t) - H(q(t), p(t), t) \right] dt \right] \prod_{j=1}^{N} \frac{dp_{j}}{\hbar}
\]
\[
= \int \cdots \int \exp\left[\frac{i}{\hbar} \sum_{j=1}^{N} (q'' - q_{j}) p_{j} \right] e^{-\frac{1}{\hbar} \mathcal{H}(q'', q', p)} \prod_{j=1}^{N} \frac{dp_{j}}{\hbar}, \quad (A3)
\]

where

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\[ p_j(t) = p_j, \quad (A5) \]
\[ q_j(t) = \left( \frac{t''+\varepsilon-t}{\varepsilon} \right) q_j + \left( \frac{t-t''}{\varepsilon} \right) q''_j, \quad (A6) \]
and
\[ H(q,q'',p) = \int_{t''}^{t''+\varepsilon} H(q(t),p(t),t) dt/\varepsilon \quad (A7) \]
\[ = \sum_{j=1}^{N} \sum_{j'=1}^{N} \left\{ a_{jj'} p_j p_j + \frac{1}{3} b_{jj'} (q''_j q''_j + q''_j q''_{jj'} + q''_{jj'} q''_j) \right\} + \frac{1}{\varepsilon} c_{jj'} p_j (q''_j + q''_{jj'}) \]
\[ + \int_{t''}^{t''+\varepsilon} \left\{ \sum_{j=1}^{N} [p_j q_j + \left( \frac{t''+\varepsilon-t}{\varepsilon} \right) q_j + \left( \frac{t-t''}{\varepsilon} \right) q''_j] e_j(t) \right\} dt/\varepsilon. \quad (A8) \]

Equation (A8) follows from Eq. (A7) after substitution of Eq. (1) as the definition of \( H(q,p,t) \). Expansion of the second exponential function in Eq. (A7) to first order in \( \varepsilon \) followed by integration of the \( p_j \)'s yields
\[ K(q'',t''+\varepsilon;q,t'') = \prod_{j=1}^{N} \delta(q''_j-q_j) - \frac{ie^{\gamma}}{h} \prod_{j=1}^{N} \delta(q''_j-q_j), \quad (A9) \]

since
\[ \int_{-\infty}^{\infty} e^{\frac{1}{h}(q''-q)p} dp/h = ie^{\gamma} \int_{-\infty}^{\infty} e^{\frac{1}{h}(q''-q)p} dp/h. \quad (A10) \]

Substitution of Eqs. (A9) and (A8) into Eq. (A2) yields
\[ K(q'', t''; q', t') = K(q'', t''; q', t') \]

\[ - \frac{i \epsilon}{\hbar} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \{ \Sigma_{j=1}^{N} \Sigma_{j'=1}^{N} \left[ -\hbar^2 a_{jj'} \frac{\partial}{\partial q_{jj'}} \frac{\partial}{\partial q_{jj''}} \right] 

+ \frac{1}{3} \b_{jj'}, (q''_j q_{j''} + q''_j q_{j''} + q''_j q_{j''}) + \frac{1}{2} \kappa c_{jj'}, (q''_j \frac{\partial}{\partial q_{jj'}} + \frac{\partial}{\partial q_{jj''}} q_{j''}) \}

+ \int_{t''}^{t'' + \epsilon} \left[ \Sigma_{j=1}^{N} \left[ i \hbar d_j(t) \frac{\partial}{\partial q_j} \right] + \left[ \left( \frac{t'' + \epsilon - t}{\epsilon} \right) q_j'' + \left( \frac{t'' - t''}{\epsilon} \right) q_j'' \right] e_j(t) \right] \]

\[ + \int_{t''}^{t'' + \epsilon} \left[ \Sigma_{j=1}^{N} \left[ i \hbar d_j(t) \frac{\partial}{\partial q_j} \right] + \left( \frac{t'' + \epsilon}{\epsilon} - t'' \right) e_j(t) \right] \frac{(q''_j - q_{j'} q_{j''}) dq_j}{(A11)} \]

With Eq. (A11) inserted into Eq. (A1), the resulting expression for the time derivative of the propagator is

\[ \frac{\partial K(q'', t''; q', t')}{\partial t''} = - \frac{i}{\hbar} \left[ \sum_{j=1}^{N} \sum_{j'=1}^{N} \left[ a_{jj'} (i \hbar \frac{\partial}{\partial q_{jj'}}) (i \hbar \frac{\partial}{\partial q_{jj''}}) \right) 

+ b_{jj'}, (q''_j \frac{\partial}{\partial q_{jj'}} + \frac{\partial}{\partial q_{jj''}} q_{j''}) \right] 

+ \sum_{j=1}^{N} \left[ i \hbar (\lim_{t'' \to t''} \int_{t''}^{t'' + \epsilon} d_j(t) dt') \frac{\partial}{\partial q_j} \right] + \left( \lim_{t'' \to t''} \int_{t''}^{t'' + \epsilon} e_j(t) dt' \right) K(q''_j, t''; q', t') \]

\[ + \left( \lim_{t'' \to t''} \int_{t''}^{t'' + \epsilon} f(t) dt' \right) K(q''_j, t''; q', t') \] (A12)

\[ = - \frac{i}{\hbar} H(\hat{q}''_j, \hat{p}''_j, t'') K(q''_j, t''; q', t'), \] (A13)

which is identical to Eq. (5).

The continuity condition given by Eq. (7) follows quickly from Eq. (A9) in the limit \( \epsilon \to 0 \):
\[ \lim_{\varepsilon \to 0} K(q'',t' + \varepsilon; q', t') = \prod_{j=1}^{N} \delta(q''_j - q'_j). \] (A14)

Therefore Eq. (27) is correct.
APPENDIX B
EVALUATION OF $F_r$

The definition of $F_r$, Eq. (164), can be rewritten in the form of a recursion relation:

$$F_r(\omega_1, \ldots, \omega_r, T) = \int_t t^n e^{-i\omega_1(t-t')} dt F_{r-1}(-\omega_2, \ldots, -\omega_r, t_1-t')$$  \quad (B1)

for $r > 1$, where

$$F_1(\omega, T) = \int_t t^n e^{-i\omega(t-t')} dt$$  \quad (B2)

$$= \frac{(e^{-i\omega T} - 1)}{(-\omega)}$$  \quad (B3)

$$= \sum_{n=1}^{\infty} \left( \frac{iT}{n} \right)^n (-\omega)^{n-1} \frac{1}{n!}$$  \quad (B4)

according to the definition. Since Eq. (B4) is precisely the result predicted by Eq. (165) for the case $r = 1$, an inductive proof of the validity of Eq. (165) for any value of $r$ can be completed by using Eq. (B1) to show that if Eq. (165) is correct for $r = r' - 1$ then it is correct for $r = r'$. Therefore assume that

$$F_{r'}(\omega_1, \ldots, \omega_r, T) = \int_t t^n e^{-i\omega_1(t-t')} dt$$

$$\times \sum_{n=r'-1}^{\infty} (i(t_{1-t'}))^n \frac{\sum_{\gamma_2=0}^{r_2} \frac{\sum_{\gamma_2=0}^{r_2} (\gamma_2+1) \gamma_2}{r_2} \sum_{\gamma_1=0}^{r_1} \frac{\sum_{\gamma_2=0}^{r_2} (\gamma_2+1) \gamma_2}{r_2}}{\sum_{\gamma_1=0}^{r_1} \frac{\sum_{\gamma_2=0}^{r_2} (\gamma_2+1) \gamma_2}{r_2}}$$  \quad (B5)
Substitution of
\[ e^{-i\omega_1(t_1-t')} = \sum_{\gamma_1=0}^{\infty} \frac{[-i\omega_1(t_1-t')]^{\gamma_1}}{\gamma_1!} \] (B6)

into Eq. (B5), replacement of the summation over \( n \) by a summation over
\[ n' \equiv n + \gamma_1 + 1, \] (B7)

and execution of the indicated integration over \( t_1 \) yield
\[
F_{r_1} = \sum_{n'=r_1}^{\infty} (\Sigma \sum_{r'=0}^{n'-r'} \sum_{r'=0}^{n'-r'} \prod_{a=1}^{r} \frac{[(-1)^a \omega_a \gamma_a]}{\gamma_a! \sum_{\gamma_{a+1}=0}^{\gamma_a}}) \] (B8)

Note that Eq. (B7) and the restriction on the summation indices in Eq. (B5) imply that \( n' = \sum_{i=1}^{r} (\gamma_i + 1) \). Equation (B8) matches Eq. (165), completing the proof of its validity.

Simplification of the expression for \( F_r \) given by
Eq. (B8) is possible when some \( \omega \)'s are equal. As an illustration consider the case \( \omega_j = \omega_{j+1} \) for \( 1 \leq j \leq r - 1 \). The reduction

\[
= \sum_{\gamma_j=0}^{m} \sum_{\gamma_{j+1}=0}^{m-l} \prod_{a=1}^{l} \frac{[(-1)^a \omega_{j+a} \gamma_{j+a}]}{\gamma_{j+a}! \sum_{\gamma_{a+1}=0}^{\gamma_a}} \] (B9)

\[
= \sum_{\gamma_j=0}^{m} \sum_{\gamma_{j+1}=0}^{m-l} \prod_{a=1}^{l} \frac{[\sum_{\gamma_{j+1}=0}^{\gamma_j} \gamma_{j+1}]!}{\gamma_{j+1}! \sum_{\gamma_{a+1}=0}^{\gamma_a}} \]
is possible for this case because of the identity

\[
\sum_{\gamma=0}^{\gamma'} (-1)^\gamma \frac{(N+1+\gamma)(\gamma'-\gamma)!\gamma!}{(N+1+\gamma')!} = \frac{N!}{(B10)}
\]
APPENDIX C
EVALUATION OF A USEFUL INTEGRAL

It is desirable to evaluate the integral

\[ I = \iint e^{-|a_k'|^2 - |a_k''|^2 + a_k'^* a_k'' e^{-i\omega_k T}} \frac{a_k'^* a_k''^* \beta_{a_k'} \beta_{a_k''}}{\sqrt{N_k''}} \frac{a_k'^* a_k''^* \beta_{a_k'} \beta_{a_k''}}{\sqrt{N_k'}} x \times \frac{N_k''}{N_k'} \]  \hspace{1cm} (C1)

where both \( a_k' \) and \( a_k'' \) are integrated over the entire complex plane, and where

\[ d_{a_k} = d(a_k)_r d(a_k)_i / \pi. \]  \hspace{1cm} (C2)

After a change of variables from the real and imaginary parts of \( a_k'' \) to

\[ x = (a_k'')_r - \frac{1}{2} a_k'^* e^{-i\omega_k T} \]  \hspace{1cm} (C3)

and

\[ y = (a_k'')_i + \frac{1}{2} i a_k'^* e^{-i\omega_k T}, \]  \hspace{1cm} (C4)

Eq. (C1) becomes

\[ I = \iint e^{-|a_k'|^2} \frac{a_k'^* a_k''^* \beta_{a_k'} \beta_{a_k''}}{\sqrt{N_k'}} x \times \frac{N_k''}{N_k'} \int e^{-x^2} e^{-y^2} \left\{ x + iy + a_k'^* e^{-i\omega_k T} \right\} \frac{N_k''}{\sqrt{N_k'}} (x-iy)^\alpha \frac{dx dy}{\pi}. \]  \hspace{1cm} (C5)

A binomial expansion of the factor, \( \left\{ x + iy + a_k'^* e^{-i\omega_k T} \right\} \),
and conversion to polar variables of integration through the transformation,

$$x + iy = re^{i\theta}, \quad (C6)$$

yield

$$I = \int d^2a_k e^{-|a_k'|^2} a_k^* \frac{N_k' \sqrt{N_k''!}}{\sqrt{N_k''!}} \sum_{\sigma=0}^{N_k''-\sigma} \frac{e^{N_k''+\beta-\sigma - i(N_k''-\sigma)\omega_k T}}{(N_k''-\sigma)!} \int_0^\infty \int_0^{2\pi} e^{-r^2} r^{\alpha+\sigma} e^{i(\sigma-\alpha)\theta} \frac{rd\theta}{\pi}$$

$$= \frac{\sqrt{N_k''!}}{\sqrt{N_k''!}} \frac{-i(N_k''-\alpha)\omega_k T}{(N_k''-\alpha)!} \int e^{-|a_k'|^2} a_k^* a_k' \frac{N_k''-\alpha+\beta}{N_k''!} d^2a_k'. \quad (C8)$$

Similarly the transformation

$$a_k' = r'e^{i\theta'} \quad (C9)$$

facilitates the evaluation of the remaining integral, with the result

$$I = \frac{\frac{1}{2}}{(N_k''!N_k'!)^2} e^{-\frac{1}{2}i(N_k''-\alpha)\omega_k T} \frac{\delta_{N_k''-\alpha, N_k'-'\beta}}{(N_k''-\alpha)!} \quad (C10)$$
REFERENCES


VITA

John Lee Pell was born in Jackson, Tennessee, on January 18, 1945. He attended public schools in Texas, Arkansas, South Carolina, and Louisiana before graduating from Central High School in Memphis, Tennessee, in 1963. He received the Bachelor of Arts degree with majors in both physics and mathematics in 1967 from Memphis State University, where he was elected to several honor societies, participated in campus activities, and, upon graduation, received an award for achieving the highest academic average among men in his graduating class. During his senior year he held a teaching assistantship in the Department of Mathematics. In 1969 he received the Master of Science degree in physics from Louisiana State University where he held a National Defense Education Act Fellowship. He is now a candidate for the degree of Doctor of Philosophy in the Department of Physics and Astronomy.
EXAMINATION AND THESIS REPORT

Candidate: John Lee Pell

Major Field: Physics

Title of Thesis: Path-Integral Evaluation of the Time-Evolution Propagator for Quadratic Hamiltonian Systems with Application to the Lee Model

Approved:

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EXAMINING COMMITTEE:

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Date of Examination:

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